Comodule-Contramodule Correspondence

Leonid Positselski – Moscow

Třešť, Czech Republic

April 11–13, 2014

The Comodule-Contramodule Correspondence is

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]
- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]
- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory

  [Iyengar–Krause '06, Neeman–Murfet '07–'08, L.P. '11–'14]

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]
- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory

  [Iyengar–Krause '06, Neeman–Murfet '07–'08, L.P. '11–'14]

Maximal natural generality not found yet

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality
- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]
- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory

  [Iyengar–Krause '06, Neeman–Murfet '07–'08, L.P. '11–'14]

Maximal natural generality not found yet (maybe does not exist

The Comodule-Contramodule Correspondence is

- a fundamental homological phenomenon on par with, e.g., the Koszul Duality

- known in Representation Theory as the duality betweeen representations of infinite-dimensional (Virasoro, Kac–Moody) Lie algebras on the complementary central charge levels (c and $26 - c$ for the Virasoro)

  [Feigin–Fuchs '83, Rocha-Caridi — Wallach '84, Arkhipov '96–'99, L.P. '02–'10]

- known in Algebraic Geometry as the Covariant Serre–Grothendieck Duality Theory

  [Iyengar–Krause '06, Neeman–Murfet '07–'08, L.P. '11–'14]

Maximal natural generality not found yet (maybe does not exist because the phenomenon is too general)

The co-contra correspondence is known to happen for or over

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,
  separated Noetherian schemes with dualizing complexes;

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,
  separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,
  separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension
  (= noncommutative smooth semi-separated stacks);

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,
  separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension
  (= noncommutative smooth semi-separated stacks);
- corings over rings with dualizing complexes
  (= noncommutative semi-separated stacks with dualizing complexes);

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields, curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes, separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension ($=$ noncommutative smooth semi-separated stacks);
- corings over rings with dualizing complexes ($=$ noncommutative semi-separated stacks with dualizing complexes);
- curved DG-rings with Gorenstein underlying graded rings;

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields,
  curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes,
  separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension
  (= noncommutative smooth semi-separated stacks);
- corings over rings with dualizing complexes
  (= noncommutative semi-separated stacks with dualizing complexes);
- curved DG-rings with Gorenstein underlying graded rings;
- complete Noetherian rings in the adic topology (= affine Noetherian formal schemes) with dualizing complexes;

The co-contra correspondence is known to happen for or over

- (coassociative) coalgebras over fields, curved DG-coalgebras over fields;
- (associative) rings with dualizing complexes, separated Noetherian schemes with dualizing complexes;
- corings over rings of finite homological dimension (= noncommutative smooth semi-separated stacks);
- corings over rings with dualizing complexes (= noncommutative semi-separated stacks with dualizing complexes);
- curved DG-rings with Gorenstein underlying graded rings;
- complete Noetherian rings in the adic topology (= affine Noetherian formal schemes) with dualizing complexes;
- pro-Noetherian rings (= ind-affine ind-Noetherian ind-schemes) with dualizing complexes;

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The latter two versions are a bit different from the previous ones

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The latter two versions are a bit different from the previous ones (in that a different kind of derived category construction is used; this is called the *semimodule-semicontramodule correspondence*).

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The latter two versions are a bit different from the previous ones (in that a different kind of derived category construction is used; this is called the *semimodule-semicontramodule correspondence*).

The version for quasi-compact semi-separated schemes also differs a bit

The co-contra correspondence happens for/over

- quasi-coherent sheaves and contraherent cosheaves over quasi-compact semi-separated schemes;
- quasi-coherent sheaves and contraherent cosheaves over Noetherian schemes with dualizing complexes;
- semimodules and semicontramodules over semialgebras over coalgebras over fields (including in particular infinite-dimensional algebraic Harish-Chandra pairs);
- semimodules and semicontramodules over semialgebras over corings over rings of finite homological dimension.

A semialgebra is an algebra over a coalgebra or coring.

The latter two versions are a bit different from the previous ones (in that a different kind of derived category construction is used; this is called the *semimodule-semicontramodule correspondence*).

The version for quasi-compact semi-separated schemes also differs a bit (the conventional derived category is used here).

Main ingredients:

Main ingredients:

- curved DG-structures;

Main ingredients:

- curved DG-structures;
- comodules and contramodules;

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;
- of the second kind (exotic) better behaved for coalgebras.

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;
- of the second kind (exotic) better behaved for coalgebras.

Taking derived category of the second kind along algebra variables,

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;
- of the second kind (exotic) better behaved for coalgebras.

Taking derived category of the second kind along algebra variables, a dualizing complex is needed for the co-contra correspondence.

Main ingredients:

- curved DG-structures;
- comodules and contramodules;
- derived categories of the first and second kind.

For relative situations (mixing algebra and coalgebra features):

- dualizing complexes (for coalgebras over algebras);
- semiderived categories (for algebras over coalgebras).

As a general rule — derived categories

- of the first kind (conventional) better behaved for algebras;
- of the second kind (exotic) better behaved for coalgebras.

Taking derived category of the second kind along algebra variables, a dualizing complex is needed for the co-contra correspondence.

The coalgebra plays the role of a dualizing complex over itself.

Curved DG-structures

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d \colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

h is called the *curvature element*.

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

h is called the *curvature element*.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $\;n = 1, 2, \ldots$

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

h is called the *curvature element*.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n = 1, 2, \dots$

A CDG-algebra has $m_0 = h$, $m_1 = d$, and m_2.

Curved DG-structures

A CDG-ring $B = (B, d, h)$ is

- a graded ring $B = \bigoplus_{i=-\infty}^{\infty} B^i$ endowed with
- an odd derivation $d\colon B^i \longrightarrow B^{i+1}$,
  $d(ab) = d(a)b + (-1)^{|a|} a d(b)$ for all a, $b \in B$
- and an element $h \in B^2$ such that
- $d^2(b) = [h, b]$ for all $b \in B$
- and $d(h) = 0$.

h is called the *curvature element*.

An A_∞-algebra is a graded vector space with the operations $m_n\colon A^{\otimes n} \longrightarrow A[2-n]$, $n = 1, 2, \ldots$

A CDG-algebra has $m_0 = h$, $m_1 = d$, and m_2.

[Getzler–Jones '90, L.P. '93]

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \ \ m \in M$

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M\colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \ \ m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \quad m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B, \quad n \in N$

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M\colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N\colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B, \;\; m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B, \;\; n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring (B, d, h) is naturally neither a left, nor a right CDG-module over itself.

Curved DG-structures

A left CDG-module $M = (M, d_M)$ over a CDG-ring (B, d_B, h) is

- a graded left B-module $M = \bigoplus_{i=-\infty}^{\infty} M^i$ endowed with
- an d_B-derivation $d_M \colon M^i \longrightarrow M^{i+1}$,
  $d_M(bm) = d_B(b)m + (-1)^{|b|} b d_M(m)$ for all $b \in B$, $m \in M$
- such that $d_M^2(m) = hm$ for all $m \in M$.

A right CDG-module $N = (N, d_N)$ over a CDG-ring (B, d_B, h) is

- a graded right B-module $N = \bigoplus_{i=-\infty}^{\infty} N^i$ endowed with
- an d_B-derivation $d_N \colon N^i \longrightarrow N^{i+1}$,
  $d_N(nb) = d_N(n)b + (-1)^{|n|} n d_B(b)$ for all $b \in B$, $n \in N$
- such that $d_N^2(n) = -nh$ for all $n \in N$.

A CDG-ring (B, d, h) is naturally neither a left, nor a right CDG-module over itself. But it has a natural structure of CDG-bimodule over itself.

Curved DG-structures occur in connection with

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations

Curved DG-structures occur in connection with

- nonhomogeneous Koszul duality: the bar-construction of a nonaugmented algebra is a CDG-coalgebra;
- vector bundles with nonflat connections: if M is a smooth variety, $\mathcal{E}$ is a vector bundle on M, and $\nabla_{\mathcal{E}}$ is a connection in $\mathcal{E}$, then the ring $\Omega(M, \mathcal{E}nd(\mathcal{E}))$ of differential forms with coefficients in the bundle of endomorphisms of $\mathcal{E}$ is a CDG-ring with the de Rham differential $d = d_{\nabla_{\mathcal{E}nd(\mathcal{E})}}$ and the curvature element $h = h_{\nabla_{\mathcal{E}}} \in \Omega^2(M, \mathcal{E}nd(\mathcal{E}))$, while $(\Omega(M, \mathcal{E}), d_{\nabla_{\mathcal{E}}})$ is a CDG-module over $\Omega(M, \mathcal{E}nd(\mathcal{E}))$;
- matrix factorizations, which are the CDG-modules over the $\mathbb{Z}/2$-graded CDG-ring $(B = B^0,\ d = 0,\ h = w)$, where B^0 is an associative ring and $w \in B^0$ is a central element.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ (the supercommutator)

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings: CDG-modules over a CDG-ring form a DG-category.

Curved DG-structures

A morphism of CDG-rings $(B, d_B, h_B) \longrightarrow (A, d_A, h_A)$ is a pair (f, a), where

- $f : B \longrightarrow A$ is a homomorphism of graded rings
- and $a \in A^1$ is an element such that
- $f(d_B(b)) = d_A(f(b)) + [a, b]$ for all $b \in B$
- and $f(h_B) = h_A + d_A(a) + a^2$.

a is called the *change-of-connection element*.

The embedding functor $\mathrm{DG\text{-}rings} \longrightarrow \mathrm{CDG\text{-}rings}$ is faithful but not fully faithful: nonisomorphic DG-rings may be isomorphic as CDG-rings.

The construction of the DG-category of DG-modules over a DG-ring extends to CDG-rings: CDG-modules over a CDG-ring form a DG-category. (In particular, the DG-categories of DG-modules over CDG-isomorphic DG-rings are isomorphic.)

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-mod})$

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

In particular, the conventional derived categories of DG-modules over two CDG-isomorphic DG-rings can be very different

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

In particular, the conventional derived categories of DG-modules over two CDG-isomorphic DG-rings can be very different and entirely unrelated to each other.

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

In particular, the conventional derived categories of DG-modules over two CDG-isomorphic DG-rings can be very different and entirely unrelated to each other.

Except in the so-called "weakly curved" case,

Derived categories for curved DG-structures

Thus the construction of the triangulated homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ works perfectly well for CDG-modules over a CDG-ring $B = (B, d, h)$.

However, the conventional derived category construction does not make sense for CDG-modules, because CDG-modules have no cohomology groups, hence no conventional notion of quasi-isomorphism.

In particular, the conventional derived categories of DG-modules over two CDG-isomorphic DG-rings can be very different and entirely unrelated to each other.

Except in the so-called "weakly curved" case, it is generally only the derived categories of the second kind that are well-defined for CDG-modules.

Derived categories of the first and second kind

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^*) \Longrightarrow \mathbb{H}^{p+q}(C^*);$$
$$''E_2^{pq} = H^p(R^qF(C^*)) \Longrightarrow \mathbb{H}^{p+q}(C^*).$$

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^*) \Longrightarrow \mathbb{H}^{p+q}(C^*);$$
$$''E_2^{pq} = H^p(R^qF(C^*)) \Longrightarrow \mathbb{H}^{p+q}(C^*).$$

For unbounded complexes C^*, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*.

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^*) \Longrightarrow \mathbb{H}^{p+q}(C^*);$$
$$''E_2^{pq} = H^p(R^qF(C^*)) \Longrightarrow \mathbb{H}^{p+q}(C^*).$$

For unbounded complexes C^*, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*. The same problem occurs for (even totally finite-dimensional) DG-modules.

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^*) \Longrightarrow \mathbb{H}^{p+q}(C^*);$$
$$''E_2^{pq} = H^p(R^qF(C^*)) \Longrightarrow \mathbb{H}^{p+q}(C^*).$$

For unbounded complexes C^*, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*. The same problem occurs for (even totally finite-dimensional) DG-modules.

Hence differential derived functors of the first and the second kind [Husemoller–Moore–Stasheff '74].

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives.

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A})$

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Not true for unbounded complexes.

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Not true for unbounded complexes.

Example: let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (the ring of dual numbers) over a field k.

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Not true for unbounded complexes.

Example: let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (the ring of dual numbers) over a field k. Then

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

is an unbounded complex of projective, injective Λ-modules.

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Not true for unbounded complexes.

Example: let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (the ring of dual numbers) over a field k. Then

$$\cdots \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \cdots$$

is an unbounded complex of projective, injective Λ-modules. It is acyclic, but not contractible.

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not “projective” or “injective” (not suitable for computing the derived functors)

- “projective” and/or “injective” (adjusted for computing derived functors)

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)

- "projective" and/or "injective" (adjusted for computing derived functors), representing a nontrivial object in the derived category

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)
  derived category of the first kind

- "projective" and/or "injective" (adjusted for computing derived functors),
  representing a nontrivial object in the derived category

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)
  derived category of the first kind

- "projective" and/or "injective" (adjusted for computing derived functors),
  representing a nontrivial object in the derived category
  derived category of the second kind

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)
  derived category of the first kind (conventional)

- "projective" and/or "injective" (adjusted for computing derived functors),
  representing a nontrivial object in the derived category
  derived category of the second kind

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category, not "projective" or "injective" (not suitable for computing the derived functors)
  derived category of the first kind (conventional)

- "projective" and/or "injective" (adjusted for computing derived functors),
  representing a nontrivial object in the derived category
  derived category of the second kind (exotic)

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, . . . '60s –]

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, ... '60s–]

Classical homological algebra can be defined as encompassing all the settings

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, … '60s–]

Classical homological algebra can be defined as encompassing all the settings in which there is no difference between the theories of the first and of the second kind.

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, . . . ’60s –]

Classical homological algebra can be defined as encompassing all the settings in which there is no difference between the theories of the first and of the second kind.

This includes

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, ... '60s –]

Classical homological algebra can be defined as encompassing all the settings in which there is no difference between the theories of the first and of the second kind.

This includes

- bounded or unbounded complexes over abelian or exact categories of finite homological dimension;

Derived categories of the first and second kind

Classical homological algebra: both the equivalence relation on complexes and the classes of resolutions simply described.

[Grothendieck, Verdier, Deligne, . . . '60s –]

Classical homological algebra can be defined as encompassing all the settings in which there is no difference between the theories of the first and of the second kind.

This includes

- bounded or unbounded complexes over abelian or exact categories of finite homological dimension;
- appropriately bounded above or below complexes over arbitrary abelian or exact categories;

Derived categories of the first and second kind

Classical homological algebra settings include

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i,\ d\colon A^i \longrightarrow A^{i+1}$);

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i$, $d\colon A^i \longrightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d\colon A^i \longrightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i$, $d\colon A^i \longrightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d\colon A^i \longrightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

In most other situations (including, e.g., DG-modules over the de Rham DG-algebra

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i$, $d\colon A^i \longrightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d\colon A^i \longrightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

In most other situations (including, e.g., DG-modules over the de Rham DG-algebra or the standard cohomological complex of a Lie algebra, etc.)

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i$, $d\colon A^i \longrightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d\colon A^i \longrightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

In most other situations (including, e.g., DG-modules over the de Rham DG-algebra or the standard cohomological complex of a Lie algebra, etc.) one has to choose between derived categories of the first and of the second kind.

Derived categories of the first and second kind

Classical homological algebra settings include

- appropriately bounded DG-modules over nonpositively graded DG-rings ($A = \bigoplus_{i=-\infty}^{0} A^i$, $d\colon A^i \longrightarrow A^{i+1}$);
- appropriately bounded DG-modules over connected, simply connected nonnegatively graded DG-rings $A = \bigoplus_{i=0}^{\infty} A^i$, $d\colon A^i \longrightarrow A^{i+1}$, A^0 is a semisimple ring, $A^1 = 0$.

In most other situations (including, e.g., DG-modules over the de Rham DG-algebra or the standard cohomological complex of a Lie algebra, etc.) one has to choose between derived categories of the first and of the second kind.

Sometimes one wants to use their mixtures—the semiderived categories.

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described (depending only on the underlying graded module structure, irrespective of the differentials on complexes)

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described (depending only on the underlying graded module structure, irrespective of the differentials on complexes)
- complicated descriptions of equivalence relations on complexes (more delicate than the conventional quasi-isomorphism)

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described (depending only on the underlying graded module structure, irrespective of the differentials on complexes)
- complicated descriptions of equivalence relations on complexes (more delicate than the conventional quasi-isomorphism)

[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, . . . '98 –]

Derived categories of the first and second kind

Philosophical conclusion:

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind,
a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as
a deformation of itself endowed with the zero differential.

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Warning: derived category of the second kind comes in several versions.

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Warning: derived category of the second kind comes in several versions. The largest one is called the absolute derived category.

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Warning: derived category of the second kind comes in several versions. The largest one is called the absolute derived category. The two most important definitions, dual to each other, are called the coderived and the contraderived category.

Coderived and contraderived categories of CDG-modules

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring.

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring. Suppose $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ is a short exact sequence of left CDG-modules over B

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring. Suppose $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ is a short exact sequence of left CDG-modules over B:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \xrightarrow{d} & K^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow\partial & & \downarrow & & \\
\cdots & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \xrightarrow{d} & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow\partial & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring. Suppose $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ is a short exact sequence of left CDG-modules over B:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^{0} & \longrightarrow & K^{1} & \xrightarrow{d} & K^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow\partial & & \downarrow & & \\
\cdots & \longrightarrow & L^{-1} & \longrightarrow & L^{0} & \xrightarrow{d} & L^{1} & \longrightarrow & L^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow\partial & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^{0} & \longrightarrow & M^{1} & \longrightarrow & M^{2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Form the total CDG-module $\mathrm{Tot}(K \to L \to M)$ by taking direct sums along the diagonals,

Coderived and contraderived categories of CDG-modules

Let $B = (B, d, h)$ be a CDG-ring. Suppose $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ is a short exact sequence of left CDG-modules over B:

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \xrightarrow{d} & K^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow\partial & & \downarrow & & \\
\cdots & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \xrightarrow{d} & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow\partial & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

Form the total CDG-module $\mathrm{Tot}(K \to L \to M)$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ containing the CDG-modules $\mathrm{Tot}(K \to L \to M)$

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ containing the CDG-modules $\mathrm{Tot}(K \to L \to M)$ for all the short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$:

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathrm{Tot}(K \to L \to M)$ for all the short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$:

$$\mathrm{Acycl}^{\mathrm{abs}}(B\text{-mod}) = \langle \mathrm{Tot}(K \to L \to M) \rangle \subset \mathrm{Hot}(B\text{-mod}).$$

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathrm{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathrm{Tot}(K \to L \to M)$ for all the short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$:

$$\mathrm{Acycl}^{\mathrm{abs}}(B\text{-mod}) = \langle \mathrm{Tot}(K \to L \to M) \rangle \subset \mathrm{Hot}(B\text{-mod}).$$

The triangulated quotient category

$$\mathrm{D}^{\mathrm{abs}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl}^{\mathrm{abs}}(B\text{-mod})$$

Coderived and Contraderived Categories of CDG-modules

The totalization $\mathsf{Tot}(K \to L \to M)$ of a short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B is indeed again a CDG-module, as one has $D^2 = (\partial \pm d)^2 = \partial^2 + d^2 = d^2 = [h, -]$.

A left CDG-module over B is said to be absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ for all the short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$:

$$\mathrm{Acycl}^{\mathrm{abs}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle \subset \mathrm{Hot}(B\text{-mod}).$$

The triangulated quotient category

$$\mathrm{D}^{\mathrm{abs}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl}^{\mathrm{abs}}(B\text{-mod})$$

is called the absolute derived category of left CDG-modules over B.

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-}\mathrm{mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite direct sums:

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite direct sums:

$$\mathrm{Acycl}^{\mathrm{co}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle_{\oplus} \subset \mathrm{Hot}(B\text{-mod}).$$

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite direct sums:

$$\mathrm{Acycl}^{\mathrm{co}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle_{\oplus} \subset \mathrm{Hot}(B\text{-mod}).$$

A left CDG-module over B is called contraacyclic

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite direct sums:

$$\mathrm{Acycl}^{\mathrm{co}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle_{\oplus} \subset \mathrm{Hot}(B\text{-mod}).$$

A left CDG-module over B is called contraacyclic if it belongs to the minimal triangulated subcategory of $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite products:

Coderived and Contraderived Categories of CDG-modules

A left CDG-module over B is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite direct sums:

$$\mathrm{Acycl}^{\mathrm{co}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle_{\oplus} \subset \mathrm{Hot}(B\text{-mod}).$$

A left CDG-module over B is called contraacyclic if it belongs to the minimal triangulated subcategory of $\mathrm{Hot}(B\text{-mod})$ containing the CDG-modules $\mathsf{Tot}(K \to L \to M)$ and closed under infinite products:

$$\mathrm{Acycl}^{\mathrm{ctr}}(B\text{-mod}) = \langle \mathsf{Tot}(K \to L \to M) \rangle_{\prod} \subset \mathrm{Hot}(B\text{-mod}).$$

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D^{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D^{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

The absolute derived category is perfectly well-defined for any (Quillen) exact category,

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D}^{\mathrm{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl}^{\mathrm{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D}^{\mathrm{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl}^{\mathrm{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

The absolute derived category is perfectly well-defined for any (Quillen) exact category, and in fact even for an "exact DG-category" (like that of CDG-modules).

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D^{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

The absolute derived category is perfectly well-defined for any (Quillen) exact category, and in fact even for an "exact DG-category" (like that of CDG-modules).

The coderived (respectively, contraderived) category is defined for any exact (DG-) category with exact functors of infinite direct sum (resp., exact functors of infinite product).

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D^{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

The absolute derived category is perfectly well-defined for any (Quillen) exact category, and in fact even for an "exact DG-category" (like that of CDG-modules).

The coderived (respectively, contraderived) category is defined for any exact (DG-) category with exact functors of infinite direct sum (resp., exact functors of infinite product).

For unbounded complexes of modules already, these categories can differ from the conventional derived category

Coderived and contraderived categories

The triangulated quotient category

$$\mathrm{D^{co}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{co}}(B\text{-mod})$$

is called the coderived category of left CDG-modules over B.
The quotient category

$$\mathrm{D^{ctr}}(B\text{-mod}) = \mathrm{Hot}(B\text{-mod})/\mathrm{Acycl^{ctr}}(B\text{-mod})$$

is called the contraderived category of left CDG-modules over B.

The absolute derived category is perfectly well-defined for any (Quillen) exact category, and in fact even for an "exact DG-category" (like that of CDG-modules).

The coderived (respectively, contraderived) category is defined for any exact (DG-) category with exact functors of infinite direct sum (resp., exact functors of infinite product).

For unbounded complexes of modules already, these categories can differ from the conventional derived category *and from each other*.

Comodules and contramodules

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists.

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more “comodule-like” abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more “comodule-like” abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or “smooth” modules over topological groups; discrete modules over topological Lie algebras;

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or "smooth" modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs,

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or "smooth" modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs, modules of the Bernstein–Gelfand–Gelfand category "$\mathcal{O}$";

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or "smooth" modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs, modules of the Bernstein–Gelfand–Gelfand category "$\mathcal{O}$";
- sheaves generally

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or "smooth" modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs, modules of the Bernstein–Gelfand–Gelfand category "$\mathcal{O}$";
- sheaves generally, or at least certainly quasi-coherent sheaves on schemes or algebraic stacks.

Comodules and contramodules

Comodules over coalgebras or corings are familiar to many algebraists. In a sense relevant to the comodule-contramodule correspondence theory, there are many more "comodule-like" abelian categories in algebra, including

- torsion abelian groups or torsion modules;
- discrete modules over topological rings;
- discrete or "smooth" modules over topological groups; discrete modules over topological Lie algebras;
- modules over algebraic groups, algebraic Harish-Chandra pairs, modules of the Bernstein–Gelfand–Gelfand category "$\mathcal{O}$";
- sheaves generally, or at least certainly quasi-coherent sheaves on schemes or algebraic stacks.

Every category of comodules is typically accompanied by a closely related (but much less familiar) category of contramodules.

Comodules and Contramodules

While comodules are “torsion” modules,

Comodules and Contramodules

While comodules are “torsion” modules, contramodules are defined as modules with infinite summation operations

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodules and Contramodules

While comodules are “torsion” modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense “complete”.

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects,

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

While comodules are “torsion” modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense “complete”.

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects,

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

The comodule-contramodule correspondences are covariant equivalences of (exact or triangulated) categories.

Comodules and Contramodules

While comodules are "torsion" modules, contramodules are defined as modules with infinite summation operations and feel like being in some sense "complete".

Still contramodules carry no underlying topologies on them.

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

The derived comodule-contramodule correspondences are covariant equivalences of triangulated categories.

Comodules and contramodules over corings

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

- and the counity equations $(\varepsilon \otimes \mathrm{id}) \circ \mu = \mathrm{id}_{\mathcal{C}} = (\mathrm{id} \otimes \varepsilon) \circ \mu$

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

- and the counity equations $(\varepsilon \otimes \mathrm{id}) \circ \mu = \mathrm{id}_{\mathcal{C}} = (\mathrm{id} \otimes \varepsilon) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C}.$$

Comodules and contramodules over corings

Let A be an associative ring (with unit). A *coring* $\mathcal{C}$ over A is

- an A-A-bimodule endowed with
- a comultiplication map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$
- and a counit map $\varepsilon\colon \mathcal{C} \longrightarrow A$,
- which must be morphisms of A-A-bimodules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \mu = (\mathrm{id} \otimes \mu) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$$

- and the counity equations $(\varepsilon \otimes \mathrm{id}) \circ \mu = \mathrm{id}_{\mathcal{C}} = (\mathrm{id} \otimes \varepsilon) \circ \mu$

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C}.$$

A *coalgebra* over a commutative ring A (most typically over a field) is a coring whose left and right A-module structures coincide.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with
- a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$,

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with
- a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$,
- which must be a morphism of left A-modules

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with
- a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$,
- which must be a morphism of left A-modules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \nu = (\mathrm{id} \otimes \nu) \circ \nu$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with
- a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$,
- which must be a morphism of left A-modules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \nu = (\mathrm{id} \otimes \nu) \circ \nu$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$$

- and the counity equation $(\varepsilon \otimes \mathrm{id}) \circ \nu = \mathrm{id}_{\mathcal{M}}$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{M}.$$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A left $\mathcal{C}$-comodule $\mathcal{M}$ is

- a left A-module endowed with
- a coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$,
- which must be a morphism of left A-modules
- satisfying the coassociativity equation $(\mu \otimes \mathrm{id}) \circ \nu = (\mathrm{id} \otimes \nu) \circ \nu$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$$

- and the counity equation $(\varepsilon \otimes \mathrm{id}) \circ \nu = \mathrm{id}_{\mathcal{M}}$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{M}.$$

A right $\mathcal{C}$-comodule $\mathcal{N}$ is a right A-module endowed with a coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C}$ satisfying the similar conditions.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left* $\mathcal{C}$*-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$,

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \mathrm{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules
- satisfying the contraassociativity equation $\pi\circ\mathrm{Hom}(\mu,\mathrm{id}) = \pi\circ\mathrm{Hom}(\mathrm{id},\pi)$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules
- satisfying the contraassociativity equation $\pi\circ\operatorname{Hom}(\mu,\mathrm{id}) = \pi\circ\operatorname{Hom}(\mathrm{id},\pi)$

$$\operatorname{Hom}_A(\mathcal{C}\otimes_A\mathcal{C},\ \mathfrak{P}) \simeq \operatorname{Hom}_A(\mathcal{C},\operatorname{Hom}_A(\mathcal{C},\mathfrak{P})) \rightrightarrows \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules
- satisfying the contraassociativity equation $\pi \circ \operatorname{Hom}(\mu, \mathrm{id}) = \pi \circ \operatorname{Hom}(\mathrm{id}, \pi)$

$$\operatorname{Hom}_A(\mathcal{C} \otimes_A \mathcal{C},\ \mathfrak{P}) \simeq \operatorname{Hom}_A(\mathcal{C}, \operatorname{Hom}_A(\mathcal{C},\mathfrak{P})) \rightrightarrows \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$$

- and the counity equation $\pi \circ \operatorname{Hom}(\varepsilon, \mathrm{id}) = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}.$$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules
- satisfying the contraassociativity equation $\pi\circ\operatorname{Hom}(\mu,\mathrm{id}) = \pi\circ\operatorname{Hom}(\mathrm{id},\pi)$

$$\operatorname{Hom}_A(\mathcal{C}\otimes_A\mathcal{C},\ \mathfrak{P}) \simeq \\ \operatorname{Hom}_A(\mathcal{C},\operatorname{Hom}_A(\mathcal{C},\mathfrak{P})) \rightrightarrows \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$$

- and the counity equation $\pi\circ\operatorname{Hom}(\varepsilon,\mathrm{id}) = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}.$$

[Eilenberg–Moore '65]

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. A *left $\mathcal{C}$-contramodule* $\mathfrak{P}$ is

- a left A-module endowed with
- a contraaction map $\pi\colon \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$,
- which must be a morphism of left A-modules
- satisfying the contraassociativity equation $\pi \circ \operatorname{Hom}(\mu, \mathrm{id}) = \pi \circ \operatorname{Hom}(\mathrm{id}, \pi)$

$$\operatorname{Hom}_A(\mathcal{C} \otimes_A \mathcal{C},\, \mathfrak{P}) \simeq \operatorname{Hom}_A(\mathcal{C}, \operatorname{Hom}_A(\mathcal{C},\mathfrak{P})) \rightrightarrows \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$$

- and the counity equation $\pi \circ \operatorname{Hom}(\varepsilon, \mathrm{id}) = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \operatorname{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}.$$

[Eilenberg–Moore '65] (almost forgotten between 1970–2000)

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms.

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module.

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\operatorname{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\operatorname{Hom}_A(\mathcal{C}, \operatorname{Hom}_B(\mathcal{N}, U)) \simeq \operatorname{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \operatorname{Hom}_B(\mathcal{N}, U).$$

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, U)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \mathrm{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k. Then the structure of a left B-module on a k-vector space L

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, U)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \mathrm{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k. Then the structure of a left B-module on a k-vector space L can be defined alternatively as a map $B \otimes_k L \longrightarrow L$

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, U)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \mathrm{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k. Then the structure of a left B-module on a k-vector space L can be defined alternatively as a map $B \otimes_k L \longrightarrow L$ or a map $L \longrightarrow \mathrm{Hom}_k(B, L)$.

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, U)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \mathrm{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k. Then the structure of a left B-module on a k-vector space L can be defined alternatively as a map $B \otimes_k L \longrightarrow L$ or a map $L \longrightarrow \mathrm{Hom}_k(B, L)$.

For a coalgebra $\mathcal{C}$ over k, the datum of a map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$

Comodules and contramodules over corings

Example: let $\mathcal{N}$ be a right $\mathcal{C}$-comodule endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms. Let U be a left B-module. Then the left A-module $\mathrm{Hom}_B(\mathcal{N}, U)$ has a natural left $\mathcal{C}$-contramodule structure:

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, U)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ U) \xrightarrow{\nu^*} \mathrm{Hom}_B(\mathcal{N}, U).$$

Remark: let B be an algebra over a field k. Then the structure of a left B-module on a k-vector space L can be defined alternatively as a map $B \otimes_k L \longrightarrow L$ or a map $L \longrightarrow \mathrm{Hom}_k(B, L)$.

For a coalgebra $\mathcal{C}$ over k, the datum of a map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ is quite different from that of a map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators,

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-}\mathrm{comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-}\mathrm{comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits,

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and (consequently) enough injectives.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

The category $\mathcal{C}\text{-contra}$ of left $\mathcal{C}$-contramodules is abelian provided that $\mathcal{C}$ is a projective left A-module.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-}\mathrm{comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-}\mathrm{comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

The category $\mathcal{C}\text{-}\mathrm{contra}$ of left $\mathcal{C}$-contramodules is abelian provided that $\mathcal{C}$ is a projective left A-module. In this case, the category $\mathcal{C}\text{-}\mathrm{contra}$ has exact functors of infinite product and enough projectives.

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-}\mathrm{comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-}\mathrm{comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

The category $\mathcal{C}\text{-}\mathrm{contra}$ of left $\mathcal{C}$-contramodules is abelian provided that $\mathcal{C}$ is a projective left A-module. In this case, the category $\mathcal{C}\text{-}\mathrm{contra}$ has exact functors of infinite product and enough projectives.

The projective left $\mathcal{C}$-contramodules are the direct summands of the $\mathcal{C}$-contramodules $\mathrm{Hom}_A(\mathcal{C}, F)$

Comodules and contramodules over corings

Let $\mathcal{C}$ be a coring over a ring A. Then the category $\mathcal{C}\text{-comod}$ of left $\mathcal{C}$-comodules is abelian provided that $\mathcal{C}$ is a flat right A-module. In this case, $\mathcal{C}\text{-comod}$ is a Grothendieck abelian category, i.e., it has a set of generators, exact functors of filtered inductive limits, and enough injectives.

The injective left $\mathcal{C}$-comodules are the direct summands of the $\mathcal{C}$-comodules $\mathcal{C} \otimes_A J$ coinduced from injective left A-modules J.

The category $\mathcal{C}\text{-contra}$ of left $\mathcal{C}$-contramodules is abelian provided that $\mathcal{C}$ is a projective left A-module. In this case, the category $\mathcal{C}\text{-contra}$ has exact functors of infinite product and enough projectives.

The projective left $\mathcal{C}$-contramodules are the direct summands of the $\mathcal{C}$-contramodules $\mathrm{Hom}_A(\mathcal{C}, F)$ induced from free (or projective) left A-modules F.

Comodules and contramodules over coalgebras (over fields)

Example:

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, ...

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, ... endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^{n*}) = 0$ for $n \geqslant 1$.

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^{n*}) = 0$ for $n \geqslant 1$.

Then a $\mathcal{C}$-comodule $\mathcal{M}$ is

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, ... endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^{n*}) = 0$ for $n \geqslant 1$.

Then a $\mathcal{C}$-comodule $\mathcal{M}$ is

- a k-vector space with a linear operator $t\colon \mathcal{M} \longrightarrow \mathcal{M}$

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^{n*}) = 0$ for $n \geqslant 1$.

Then a $\mathcal{C}$-comodule $\mathcal{M}$ is

- a k-vector space with a linear operator $t\colon \mathcal{M} \longrightarrow \mathcal{M}$
- which must be locally nilpotent,

Comodules and contramodules over coalgebras (over fields)

Example: let $\mathcal{C}$ be the coalgebra over a field k whose dual topological algebra $\mathcal{C}^*$ is the algebra of formal power series $k[[t]]$ in one variable.

Explicitly, $\mathcal{C}$ is a k-vector space with the basis 1^*, t^*, t^{2*}, t^{3*}, … endowed with the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$ and the counit $\varepsilon(1^*) = 1$, $\varepsilon(t^{n*}) = 0$ for $n \geqslant 1$.

Then a $\mathcal{C}$-comodule $\mathcal{M}$ is

- a k-vector space with a linear operator $t\colon \mathcal{M} \longrightarrow \mathcal{M}$
- which must be locally nilpotent, i.e., for every $m \in \mathcal{M}$ there exists an integer $n > 0$ such that $t^n m = 0$.

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, $\ldots$ and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, $\ldots \in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i$$

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} =$$

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n(ap_n + bq_n) = a\sum_{n=0}^{\infty} t^n p_n + b\sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n$$

Comodules and contramodules over coalgebras (over fields)

Example: $\mathcal{C}$ is a k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

A $\mathcal{C}$-contramodule $\mathfrak{P}$ is

- a k-vector space endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, $\ldots \in \mathfrak{P}$ an element denoted formally by $\sum_{n=0}^{\infty} t^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^{\infty} t^n (ap_n + bq_n) = a \sum_{n=0}^{\infty} t^n p_n + b \sum_{n=0}^{\infty} t^n q_n,$$

- unitality: $\sum_{n=0}^{\infty} t^n p_n = p_0$ when $p_i = 0$ for all $i \geqslant 1$,
- and contraassociativity:

$$\sum_{i=0}^{\infty} t^i \sum_{j=0}^{\infty} t^j p_{ij} = \sum_{n=0}^{\infty} t^n \sum_{i+j=n} p_{ij}.$$

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p =$

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, $\ldots$ and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 $\ldots$ $\in \mathfrak{P}$ such that

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 … $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$,

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 … $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, … and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 … $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$

Comodules and contramodules over coalgebras (over fields)

Counterexample:

Let $\mathcal{C}$ be the k-coalgebra with a basis 1^*, t^*, t^{2*}, t^{3*}, ... and the comultiplication $\mu(t^{n*}) = \sum_{i+j=n} t^{i*} \otimes t^{j*}$, so that $\mathcal{C}^* = k[[t]]$.

For any $\mathcal{C}$-contramodule $\mathfrak{P}$, an element $p \in \mathfrak{P}$, and an integer $n \geqslant 0$, one can define
$t^n p = 1 \cdot 0 + \cdots + t^{n-1} \cdot 0 + t^n p + t^{n+1} \cdot 0 + \cdots \in \mathfrak{P}$.

Then there exists a $\mathcal{C}$-contramodule $\mathfrak{P}$ and a sequence of elements p_0, p_1, p_2 ... $\in \mathfrak{P}$ such that $t^n p_n = 0$ for every $n \geqslant 0$, but $\sum_{n=0}^{\infty} t^n p_n \neq 0$.

In particular, the element $\sum_{n=0}^{\infty} t^n p_n$ belongs to $t^m \mathfrak{P}$ for every $m \geqslant 0$, so the t-adic topology on $\mathfrak{P}$ is not separated.

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k.

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{comod}$

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{comod}$ are exactly the direct summands of the *coinduced* $\mathcal{C}$*-comodules* $\mathcal{C} \otimes_k U$

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{comod}$ are exactly the direct summands of the *coinduced* $\mathcal{C}$*-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-}\mathrm{vect}$.

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\mathrm{Hom}_k(\mathcal{C}, U)$ and back

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\mathrm{Hom}_k(\mathcal{C}, U)$ and back:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k U,\, \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{C} \otimes_k U,\ V) \simeq$$
$$\mathrm{Hom}_k(U, \mathrm{Hom}_k(C, V)) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_k(\mathcal{C}, U), \mathrm{Hom}_k(\mathcal{C}, V)).$$

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\mathrm{Hom}_k(\mathcal{C}, U)$ and back:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k U,\, \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{C} \otimes_k U,\ V) \simeq$$
$$\mathrm{Hom}_k(U, \mathrm{Hom}_k(C, V)) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_k(\mathcal{C}, U), \mathrm{Hom}_k(\mathcal{C}, V)).$$

This generalizes to comodules and contramodules over any coring,

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\mathrm{Hom}_k(\mathcal{C}, U)$ and back:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k U,\, \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{C} \otimes_k U,\ V) \simeq$$
$$\mathrm{Hom}_k(U, \mathrm{Hom}_k(C, V)) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_k(\mathcal{C}, U), \mathrm{Hom}_k(\mathcal{C}, V)).$$

This generalizes to comodules and contramodules over any coring, and is a particular case of the abstract-categorical *equivalence of Kleisli categories* for an adjoint monad and comonad

Co-contra correspondence for coalgebras over fields

Let $\mathcal{C}$ be a coalgebra over a field k. Then the injective objects of the category of $\mathcal{C}$-comodules $\mathcal{C}\text{-}\mathrm{comod}$ are exactly the direct summands of the *coinduced $\mathcal{C}$-comodules* $\mathcal{C} \otimes_k U$ with $U \in k\text{-}\mathrm{vect}$.

Similarly, the projective objects of the category of $\mathcal{C}$-contramodules $\mathcal{C}\text{-}\mathrm{contra}$ are the direct summands of the *induced $\mathcal{C}$-contramodules* $\mathrm{Hom}_k(C, U)$ with $U \in k\text{-}\mathrm{vect}$.

The additive categories of coinduced left $\mathcal{C}$-comodules and induced left $\mathcal{C}$-contramodules are equivalent, with the equivalence taking $\mathcal{C} \otimes_k U$ to $\mathrm{Hom}_k(\mathcal{C}, U)$ and back:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k U,\, \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{C} \otimes_k U,\ V) \simeq$$
$$\mathrm{Hom}_k(U, \mathrm{Hom}_k(C, V)) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_k(\mathcal{C}, U), \mathrm{Hom}_k(\mathcal{C}, V)).$$

This generalizes to comodules and contramodules over any coring, and is a particular case of the abstract-categorical *equivalence of Kleisli categories* for an adjoint monad and comonad [connection noticed by Böhm–Brzeziński–Wisbauer '09].

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k,

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod})$;

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\mathcal{C}\text{-comod}_{\text{inj}} \simeq \mathcal{C}\text{-contra}_{\text{proj}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}_{\text{inj}}) \simeq \mathrm{D}^{\text{co}}(\mathcal{C}\text{-comod})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}_{\text{proj}}) \simeq \mathrm{D}^{\text{ctr}}(\mathcal{C}\text{-contra})$.

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\Psi_{\mathcal{C}} : \mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}} : \Phi_{\mathcal{C}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

Co-contra correspondence for coalgebras over fields

Hence an equivalence of additive categories

$$\Psi_{\mathcal{C}} \colon \mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}} \colon \Phi_{\mathcal{C}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

Corollary

For any coassociative coalgebra $\mathcal{C}$ over a field k, there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}} \colon \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}) \colon \mathbb{L}\Phi_{\mathcal{C}}.$$

Hence an equivalence of additive categories

$$\Psi_{\mathcal{C}} : \mathcal{C}\text{-comod}_{\mathrm{inj}} \simeq \mathcal{C}\text{-contra}_{\mathrm{proj}} : \Phi_{\mathcal{C}}$$

for any coassociative coalgebra $\mathcal{C}$ over a field k.

Theorem

For any coassociative coalgebra $\mathcal{C}$ over k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

Corollary

For any coassociative coalgebra $\mathcal{C}$ over a field k, there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}} : \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}) : \mathbb{L}\Phi_{\mathcal{C}}. \quad \square$$

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim*

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs)

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-comod}$ and $\mathcal{C}\text{-contra}$ abelian).

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-}\mathrm{comod}$ and $\mathcal{C}\text{-}\mathrm{contra}$ abelian).

There are further generalizations to

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-comod}$ and $\mathcal{C}\text{-contra}$ abelian).

There are further generalizations to

- corings over rings with dualizing complexes,

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-comod}$ and $\mathcal{C}\text{-contra}$ abelian).

There are further generalizations to

- corings over rings with dualizing complexes, endowed a lifting of the ring's dualizing complex to (a complex of bicomodules over) the coring;

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-comod}$ and $\mathcal{C}\text{-contra}$ abelian).

There are further generalizations to

- corings over rings with dualizing complexes, endowed a lifting of the ring's dualizing complex to (a complex of bicomodules over) the coring;
- corings $\mathcal{C}$ that are only flat left and right A-modules,

Co-contra correspondence for corings

The assertions of the previous Theorem and Corollary hold true *verbatim* (though with more complicated proofs) for

- all corings $\mathcal{C}$ over associative rings A of finite homological dimension;
- all corings $\mathcal{C}$ over Gorenstein associative rings A (i.e., such that the classes of left A-modules of finite projective dimension and of finite injective dimension coincide)

assuming only that $\mathcal{C}$ is a projective left and a flat right A-module (to make the categories $\mathcal{C}\text{-}\mathrm{comod}$ and $\mathcal{C}\text{-}\mathrm{contra}$ abelian).

There are further generalizations to

- corings over rings with dualizing complexes, endowed a lifting of the ring's dualizing complex to (a complex of bicomodules over) the coring;
- corings $\mathcal{C}$ that are only flat left and right A-modules, when one has to restrict the class of contramodules under consideration.

Co-contra correspondence for coalgebras over fields

Example:

Co-contra correspondence for coalgebras over fields

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables.

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Consider the one-dimensional trivial $\mathcal{C}$-comodule k; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Consider the one-dimensional trivial $\mathcal{C}$-comodule k; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$. Have a right injective (Koszul) resolution

$$0 \longrightarrow k \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \textstyle\bigwedge_k^2 W \longrightarrow \cdots$$

in the category of $\mathcal{C}$-comodules.

Co-contra correspondence for coalgebras over fields

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Consider the one-dimensional trivial $\mathcal{C}$-comodule k; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$. Have a right injective (Koszul) resolution

$$0 \longrightarrow k \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \textstyle\bigwedge_k^2 W \longrightarrow \cdots$$

in the category of $\mathcal{C}$-comodules. Applying $\Psi_{\mathcal{C}}$, obtain the complex of projective $\mathcal{C}$-contramodules

$$\mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W)$$
$$\longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^{m-1} W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^m W).$$

Co-contra correspondence for coalgebras over fields

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Consider the one-dimensional trivial $\mathcal{C}$-comodule k; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$. Have a right injective (Koszul) resolution

$$0 \longrightarrow k \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \textstyle\bigwedge_k^2 W \longrightarrow \cdots$$

in the category of $\mathcal{C}$-comodules. Applying $\Psi_{\mathcal{C}}$, obtain the complex of projective $\mathcal{C}$-contramodules

$$\mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W)$$
$$\longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^{m-1} W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^m W).$$

This is a left projective $\mathcal{C}$-contramodule resolution of the one-dimensional trivial $\mathcal{C}$-contramodule $\bigwedge_k^m W$ placed at the cohomological degree m.

Example: let $\mathcal{C}$ be the coalgebra for which $\mathcal{C}^* \simeq k[[x_1, \cdots, x_m]]$ is the algebra of formal power series in m variables. In other words, $\mathcal{C} = \mathrm{Sym}(W)$ is the symmetric coalgebra of a vector space W such that $x_1, \ldots, x_m$ is a basis in W^*.

Consider the one-dimensional trivial $\mathcal{C}$-comodule k; let us compute $\mathbb{R}\Psi_{\mathcal{C}}(k)$. Have a right injective (Koszul) resolution

$$0 \longrightarrow k \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \textstyle\bigwedge_k^2 W \longrightarrow \cdots$$

in the category of $\mathcal{C}$-comodules. Applying $\Psi_{\mathcal{C}}$, obtain the complex of projective $\mathcal{C}$-contramodules

$$\mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \\ \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^{m-1} W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^m W).$$

This is a left projective $\mathcal{C}$-contramodule resolution of the one-dimensional trivial $\mathcal{C}$-contramodule $\bigwedge_k^m W$ placed at the cohomological degree m. So we get $\mathbb{R}\Psi_{\mathcal{C}}(k) \simeq k[-m]$.

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$;

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space.

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be
an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k

Now set $m = \infty$; in other words, take W to be
an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$.

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-}\mathrm{comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be
an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex.

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-}\mathrm{comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-}\mathrm{contra})$.

This complex is known as

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-}\mathrm{comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-}\mathrm{contra})$.

This complex is known as "a left projective resolution of the one-dimensional trivial $\mathcal{C}$-contramodule $\bigwedge_k^{\infty} W$ placed at the cohomological degree $+\infty$".

Co-contra correspondence for coalgebras over fields

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra})$.

This complex is known as "a left projective resolution of the one-dimensional trivial $\mathcal{C}$-contramodule $\bigwedge_k^\infty W$ placed at the cohomological degree $+\infty$".

This phenomenon appears in the representation theory of infinite-dimensional Lie algebras such as the Virasoro

Now set $m = \infty$; in other words, take W to be an infinite-dimensional discrete k-vector space. Then our Koszul resolution of the $\mathcal{C}$-comodule k is still a coacyclic complex in $\mathcal{C}\text{-}\mathrm{comod}$. Thus the infinite complex of $\mathcal{C}$-contramodules

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \textstyle\bigwedge_k^2 W) \longrightarrow \cdots$$

represents the object $\mathbb{R}\Psi_{\mathcal{C}}(k)$.

This is an acyclic complex. It is not contraacyclic; so it is a nontrivial object of the contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-}\mathrm{contra})$.

This complex is known as "a left projective resolution of the one-dimensional trivial $\mathcal{C}$-contramodule $\bigwedge_k^{\infty} W$ placed at the cohomological degree $+\infty$".

This phenomenon appears in the representation theory of infinite-dimensional Lie algebras such as the Virasoro [B. Feigin, mid-'80s].

Co-contra correspondence for CDG-coalgebras

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon\colon \mathcal{C}^0 \longrightarrow k$

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon\colon \mathcal{C}^0 \longrightarrow k$
- endowed with an odd coderivation $d\colon \mathcal{C} \longrightarrow \mathcal{C}$ with the components $d^i\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon\colon \mathcal{C}^0 \longrightarrow k$
- endowed with an odd coderivation $d\colon \mathcal{C} \longrightarrow \mathcal{C}$ with the components $d^i\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a curvature linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon\colon \mathcal{C}^0 \longrightarrow k$
- endowed with an odd coderivation $d\colon \mathcal{C} \longrightarrow \mathcal{C}$ with the components $d^i\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a curvature linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$
- satisfying the equations dual to those of a CDG-algebra over k.

Co-contra correspondence for CDG-coalgebras

A CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k is

- a graded k-coalgebra $\mathcal{C} = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^i$
- with a comultiplication map μ with the components $\mathcal{C}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{C}^j$
- and a counit map $\varepsilon\colon \mathcal{C}^0 \longrightarrow k$
- endowed with an odd coderivation $d\colon \mathcal{C} \longrightarrow \mathcal{C}$ with the components $d^i\colon \mathcal{C}^i \longrightarrow \mathcal{C}^{i+1}$
- and a curvature linear function $h\colon \mathcal{C}^{-2} \longrightarrow k$
- satisfying the equations dual to those of a CDG-algebra over k.

The graded dual vector space $\mathcal{C}^* = \bigoplus_{i=-\infty}^{\infty} \mathcal{C}^{-i*}$ to a CDG-coalgebra over k is a CDG-algebra.

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$
- endowed with a $d_{\mathcal{C}}$-coderivation $d_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d_{\mathcal{M}}^i \colon \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_\mathcal{M})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_\mathcal{C}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$
- endowed with a $d_\mathcal{C}$-coderivation $d_\mathcal{M} \colon \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d^i_\mathcal{M} \colon \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$
- satisfying the equations dual to those of a CDG-module over a CDG-algebra over k

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$
- endowed with a $d_{\mathcal{C}}$-coderivation $d_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d_{\mathcal{M}}^i \colon \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$
- satisfying the equations dual to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_{\mathcal{M}}^2 \colon \mathcal{M} \longrightarrow \mathcal{M}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathcal{M}$.

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_{\mathcal{M}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$
- endowed with a $d_{\mathcal{C}}$-coderivation $d_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d_{\mathcal{M}}^i \colon \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$
- satisfying the equations dual to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_{\mathcal{M}}^2 \colon \mathcal{M} \longrightarrow \mathcal{M}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathcal{M}$.

Any graded left $\mathcal{C}$-comodule has a natural (induced) structure of a graded left $\mathcal{C}^*$-module.

Co-contra correspondence for CDG-coalgebras

A left CDG-comodule $\mathcal{M} = (\mathcal{M}, d_\mathcal{M})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_\mathcal{C}, h)$ over a field k is

- a graded left $\mathcal{C}$-comodule $\mathcal{M} = \bigoplus_{i=-\infty}^{\infty} \mathcal{M}^i$
- with a coaction map ν with the components $\mathcal{M}^n \longrightarrow \bigoplus_{i+j=n} \mathcal{C}^i \otimes_k \mathcal{M}^j$
- endowed with a $d_\mathcal{C}$-coderivation $d_\mathcal{M} \colon \mathcal{M} \longrightarrow \mathcal{M}$ with the components $d_\mathcal{M}^i \colon \mathcal{M}^i \longrightarrow \mathcal{M}^{i+1}$
- satisfying the equations dual to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_\mathcal{M}^2 \colon \mathcal{M} \longrightarrow \mathcal{M}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathcal{M}$.

Any graded left $\mathcal{C}$-comodule has a natural (induced) structure of a graded left $\mathcal{C}^*$-module. The mentioned equations can be rewritten as saying that $(\mathcal{M}, d_\mathcal{M})$ is a CDG-module over $\mathcal{C}^*$.

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \operatorname{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \operatorname{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$
- endowed with a $d_{\mathcal{C}}$-contraderivation $d_{\mathfrak{P}} \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ with the components $d_{\mathfrak{P}}^i \colon \mathfrak{P}^i \longrightarrow \mathfrak{P}^{i+1}$

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_\mathfrak{P})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_\mathcal{C}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \mathrm{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$
- endowed with a $d_\mathcal{C}$-contraderivation $d_\mathfrak{P} \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ with the components $d_\mathfrak{P}^i \colon \mathfrak{P}^i \longrightarrow \mathfrak{P}^{i+1}$
- satisfying the equations similar to those of a CDG-module over a CDG-algebra over k

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \mathrm{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$
- endowed with a $d_{\mathcal{C}}$-contraderivation $d_{\mathfrak{P}} \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ with the components $d_{\mathfrak{P}}^i \colon \mathfrak{P}^i \longrightarrow \mathfrak{P}^{i+1}$
- satisfying the equations similar to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_{\mathfrak{P}}^2 \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathfrak{P}$.

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \mathrm{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$
- endowed with a $d_{\mathcal{C}}$-contraderivation $d_{\mathfrak{P}} \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ with the components $d_{\mathfrak{P}}^i \colon \mathfrak{P}^i \longrightarrow \mathfrak{P}^{i+1}$
- satisfying the equations similar to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_{\mathfrak{P}}^2 \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathfrak{P}$.

Any graded left $\mathcal{C}$-contramodule has a natural (underlying) structure of a graded left $\mathcal{C}^*$-module.

Co-contra correspondence for CDG-coalgebras

A left CDG-contramodule $\mathfrak{P} = (\mathfrak{P}, d_{\mathfrak{P}})$ over a CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d_{\mathcal{C}}, h)$ over a field k is

- a graded left $\mathcal{C}$-contramodule $\mathfrak{P} = \prod_{i=-\infty}^{\infty} \mathfrak{P}^i$
- with a contraaction map π with the components $\prod_{j-i=n} \operatorname{Hom}_k(\mathcal{C}^i, \mathfrak{P}^j) \longrightarrow \mathfrak{P}^n$
- endowed with a $d_{\mathcal{C}}$-contraderivation $d_{\mathfrak{P}} \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ with the components $d_{\mathfrak{P}}^i \colon \mathfrak{P}^i \longrightarrow \mathfrak{P}^{i+1}$
- satisfying the equations similar to those of a CDG-module over a CDG-algebra over k,
- i.e., in particular, the operator $d_{\mathfrak{P}}^2 \colon \mathfrak{P} \longrightarrow \mathfrak{P}$ should be equal to the action of the element $h \in \mathcal{C}^*$ in $\mathfrak{P}$.

Any graded left $\mathcal{C}$-contramodule has a natural (underlying) structure of a graded left $\mathcal{C}^*$-module. The mentioned equations can be rewritten as saying that $(\mathfrak{P}, d_{\mathfrak{P}})$ is a CDG-module over $\mathcal{C}^*$.

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules.

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Denote by $\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}} \subset \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}} \subset \mathcal{C}\text{-contra}^{\mathrm{cdg}}$

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Denote by $\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}} \subset \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}} \subset \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ the DG-subcategories of CDG-comodules with injective underlying graded $\mathcal{C}$-comodules

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Denote by $\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}} \subset \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}} \subset \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ the DG-subcategories of CDG-comodules with injective underlying graded $\mathcal{C}$-comodules and CDG-contramodules with projective underlying graded $\mathcal{C}$-contramodules.

Co-contra correspondence for CDG-coalgebras

CDG-comodules and CDG-contramodules over $\mathcal{C} = (\mathcal{C}, d, h)$ form DG-categories, and even exact DG-categories, $\mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}$, so the construction of the absolute derived category is applicable to them.

The definition of the coderived category is sensibly applicable to CDG-comodules and that of the contraderived category to CDG-contramodules. We denote these triangulated categories by $\mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Denote by $\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}} \subset \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ and $\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}} \subset \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ the DG-subcategories of CDG-comodules with injective underlying graded $\mathcal{C}$-comodules and CDG-contramodules with projective underlying graded $\mathcal{C}$-contramodules.

The homotopy (H^0) categories of these DG-subcategories are denoted by $\mathrm{Hot}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}})$ and $\mathrm{Hot}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}})$, as usually.

Co-contra correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k,

Co-contra correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k, the natural functors induce equivalences of triangulated categories

Co-contra correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$;

Co-contra correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Co-contra correspondence for CDG-coalgebras

Theorem

For any CDG-coalgebra $\mathcal{C} = (\mathcal{C}, d, h)$ over a field k, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}_{\mathrm{inj}}) \simeq \mathrm{D^{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}})$;
- $\mathrm{Hot}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}_{\mathrm{proj}}) \simeq \mathrm{D^{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}})$.

Corollary

For any CDG-coalgebra $\mathcal{C}$ over a field k, there is a natural equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{C}} \colon \mathrm{D^{co}}(\mathcal{C}\text{-comod}^{\mathrm{cdg}}) \simeq \mathrm{D^{ctr}}(\mathcal{C}\text{-contra}^{\mathrm{cdg}}) : \mathbb{L}\Phi_{\mathcal{C}}.$$

Co-contra correspondence for CDG-coalgebras

Co-contra correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$

Co-contra correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ between the whole abelian DG-categories of left CDG-comodules and CDG-contramodules over $\mathcal{C}$.

Co-contra correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ between the whole abelian DG-categories of left CDG-comodules and CDG-contramodules over $\mathcal{C}$.

The right adjoint functor $\Psi_{\mathcal{C}}$ is simply

$$\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, -),$$

Co-contra correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ between the whole abelian DG-categories of left CDG-comodules and CDG-contramodules over $\mathcal{C}$.

The right adjoint functor $\Psi_{\mathcal{C}}$ is simply

$$\Psi_{\mathcal{C}}(\mathcal{M}) = \mathsf{Hom}_{\mathcal{C}}(\mathcal{C}, -),$$

while the left adjoint functor $\Phi_{\mathcal{C}}$ is the *contratensor product*

$$\Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}.$$

Co-contra correspondence for CDG-coalgebras

The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ actually exist as a pair of adjoint DG-functors $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{-comod}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-contra}^{\mathrm{cdg}}$ and $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{-contra}^{\mathrm{cdg}} \longrightarrow \mathcal{C}\text{-comod}^{\mathrm{cdg}}$ between the whole abelian DG-categories of left CDG-comodules and CDG-contramodules over $\mathcal{C}$.

The right adjoint functor $\Psi_{\mathcal{C}}$ is simply

$$\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, -),$$

while the left adjoint functor $\Phi_{\mathcal{C}}$ is the *contratensor product*

$$\Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}.$$

One restricts these functors to graded-injective CDG-comodules and graded-projective CDG-contramodules in order to construct the derived functors.

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m \colon R[M] \longrightarrow M$

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X\colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X\colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m\colon R[M] \longrightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m \colon R[M] \longrightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

- and the unity equation $m \circ \varepsilon_X = \mathrm{id}_M$

$$M \longrightarrow R[M] \longrightarrow M.$$

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ the set $\{x \mid r_x \notin \mathfrak{U}\}$ must be finite.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U}\subset\mathfrak{R}$ the set $\{x \mid r_x\notin\mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U}\subset\mathfrak{R}$ the set $\{x \mid r_x\notin\mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients.

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U}\subset\mathfrak{R}$ the set $\{x \mid r_x\notin\mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X\colon X\longrightarrow\mathfrak{R}[[X]]$.

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ the set $\{x \mid r_x \notin \mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X : \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X : X \longrightarrow \mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X \longmapsto \mathfrak{R}[[X]] : \mathrm{Sets} \longrightarrow \mathrm{Sets}$.

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{P}.$$

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{P}.$$

The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{P}.$$

The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projectives.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projectives. The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projectives. The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

A right $\mathfrak{R}$-module $\mathcal{N}$ is called *discrete* if the action map $\mathcal{N} \times \mathfrak{R} \longrightarrow \mathcal{N}$ is continuous in the given topology of $\mathfrak{R}$ and the discrete topology of $\mathcal{N}$

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projectives. The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

A right $\mathfrak{R}$-module $\mathcal{N}$ is called *discrete* if the action map $\mathcal{N} \times \mathfrak{R} \longrightarrow \mathcal{N}$ is continuous in the given topology of $\mathfrak{R}$ and the discrete topology of $\mathcal{N}$, i.e., if the annihilator of any element of $\mathcal{N}$ is open in $\mathfrak{R}$.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite products and enough projectives. The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

A right $\mathfrak{R}$-module $\mathcal{N}$ is called *discrete* if the action map $\mathcal{N} \times \mathfrak{R} \longrightarrow \mathcal{N}$ is continuous in the given topology of $\mathfrak{R}$ and the discrete topology of $\mathcal{N}$, i.e., if the annihilator of any element of $\mathcal{N}$ is open in $\mathfrak{R}$.

For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any abelian group U, the left $\mathfrak{R}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, U)$ has a natural left $\mathfrak{R}$-contramodule structure.

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers.

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^\infty \ell^n(ap_n + bq_n) = a\sum_{n=0}^\infty \ell^n p_n + b\sum_{n=0}^\infty \ell^n q_n,$$

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^\infty \ell^n(ap_n + bq_n) = a\sum_{n=0}^\infty \ell^n p_n + b\sum_{n=0}^\infty \ell^n q_n,$$

- unitality + compatibility: $\sum_{n=0}^\infty \ell^n p_n = p_0 + \ell p_1$ when $p_i = 0$ for all $i \geqslant 2$,

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^\infty \ell^n(ap_n + bq_n) = a\sum_{n=0}^\infty \ell^n p_n + b\sum_{n=0}^\infty \ell^n q_n,$$

- unitality + compatibility: $\sum_{n=0}^\infty \ell^n p_n = p_0 + \ell p_1$ when $p_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^\infty \ell^i \sum_{j=0}^\infty \ell^j p_{ij} =$$

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, ... $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^\infty \ell^n(ap_n + bq_n) = a\sum_{n=0}^\infty \ell^n p_n + b\sum_{n=0}^\infty \ell^n q_n,$$

- unitality + compatibility: $\sum_{n=0}^\infty \ell^n p_n = p_0 + \ell p_1$ when $p_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^\infty \ell^i \sum_{j=0}^\infty \ell^j p_{ij} = \sum_{n=0}^\infty \ell^n \sum_{i+j=n} p_{ij}.$$

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent,

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule.

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$,

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that $\operatorname{Ext}^*_R(R[s_i^{-1}], P) = 0$ for all $i = 1, \ldots, m$.

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that $\operatorname{Ext}^*_R(R[s_i^{-1}], P) = 0$ for all $i = 1, \ldots, m$.

In particular, $\mathbb{Z}_\ell$-contramodules = weakly ℓ-complete (Ext-ℓ-complete) abelian groups

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neiborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that it topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that $\mathrm{Ext}_R^*(R[s_i^{-1}], P) = 0$ for all $i = 1, \ldots, m$.

In particular, $\mathbb{Z}_\ell$-contramodules = weakly ℓ-complete (Ext-ℓ-complete) abelian groups [Bousfield-Kan '72, Jannsen '88].

Covariant Serre–Grothendieck duality (affine case)

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(R\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(R\text{-mod})$;

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(R\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(R\text{-mod})$;
- $\mathrm{Hot}(R\text{-mod}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(R\text{-mod}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod})$.

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(R\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(R\text{-mod})$;
- $\mathrm{Hot}(R\text{-mod}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(R\text{-mod}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod})$.

Corollary

The choice of a dualizing complex $D_R^\bullet$ for a Noetherian commutative ring R induces an equivalence of triangulated categories $\mathbb{R}\Psi_{D_R^\bullet}\colon \mathrm{D}^{\mathrm{co}}(R\text{-mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod}) \colon \mathbb{L}\Phi_{D_R^\bullet}$.

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(R\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(R\text{-mod})$;
- $\mathrm{Hot}(R\text{-mod}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(R\text{-mod}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod})$.

Corollary

The choice of a dualizing complex $D_R^\bullet$ for a Noetherian commutative ring R induces an equivalence of triangulated categories $\mathbb{R}\Psi_{D_R^\bullet} : \mathrm{D}^{\mathrm{co}}(R\text{-mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod}) : \mathbb{L}\Phi_{D_R^\bullet}$.

Here the functors to be derived are just
$\Psi_{D_R^\bullet}(M^\bullet) = \mathrm{Hom}_R(D_R^\bullet, M^\bullet)$ and $\Phi_{D_R^\bullet}(P^\bullet) = D_R^\bullet \otimes_R P^\bullet$.

Covariant Serre–Grothendieck duality (affine case)

Theorem

For any Noetherian commutative ring R of finite Krull dimension, the natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(R\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(R\text{-mod})$;
- $\mathrm{Hot}(R\text{-mod}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(R\text{-mod}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod})$.

Corollary

The choice of a dualizing complex $D_R^\bullet$ for a Noetherian commutative ring R induces an equivalence of triangulated categories $\mathbb{R}\Psi_{D_R^\bullet} : \mathrm{D}^{\mathrm{co}}(R\text{-mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(R\text{-mod}) : \mathbb{L}\Phi_{D_R^\bullet}$.

Here the functors to be derived are just
$\Psi_{D_R^\bullet}(M^\bullet) = \mathrm{Hom}_R(D_R^\bullet, M^\bullet)$ and $\Phi_{D_R^\bullet}(P^\bullet) = D_R^\bullet \otimes_R P^\bullet$.

[Jørgensen, Krause, Iyengar–Krause, '05-'06]

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them.

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure.

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

- the R_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

- the R_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

- the R_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

The class $\mathfrak{R}\text{-}\mathrm{contra}_{\mathrm{flat}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-}\mathrm{contra}$,

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

- the R_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

The class $\mathfrak{R}\text{-contra}_{\text{flat}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-contra}$, so in particular $\mathfrak{R}\text{-contra}_{\text{flat}}$ inherits an exact category structure from $\mathfrak{R}\text{-contra}$.

Co-contra correspondence over a pro-Noetherian ring

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{P}$ is called *flat* if

- the R_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

The class $\mathfrak{R}\text{-contra}_{\text{flat}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-contra}$, so in particular $\mathfrak{R}\text{-contra}_{\text{flat}}$ inherits an exact category structure from $\mathfrak{R}\text{-contra}$.

Denote by $\mathfrak{R}\text{-discr}$ the abelian category of discrete $\mathfrak{R}$-modules.

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{R}\text{-discr})$;

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{R}\text{-discr})$;
- $\mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$.

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{R}\text{-discr})$;
- $\mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$.

When the Krull dimensions of the rings R_n are uniformly bounded,

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{R}\text{-discr})$;
- $\mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$.

When the Krull dimensions of the rings R_n are uniformly bounded, one has $\mathrm{Hot}(\mathfrak{R}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$.

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{R}\text{-discr})$;
- $\mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$.

When the Krull dimensions of the rings R_n are uniformly bounded, one has $\mathrm{Hot}(\mathfrak{R}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{R}\text{-contra})$. This is not necessary for the following corollary.

Co-contra correspondence over a pro-Noetherian ring

Let $\mathfrak{R} = \varprojlim_n R_n$ be a commutative pro-Noetherian ring.

Theorem

The natural functors induce equivalences of triangulated categories

- $\mathrm{Hot}(\mathfrak{R}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D^{co}}(\mathfrak{R}\text{-discr})$;
- $\mathrm{D^{ctr}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D^{ctr}}(\mathfrak{R}\text{-contra})$.

When the Krull dimensions of the rings R_n are uniformly bounded, one has $\mathrm{Hot}(\mathfrak{R}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D^{abs}}(\mathfrak{R}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D^{ctr}}(\mathfrak{R}\text{-contra})$. This is not necessary for the following corollary.

Corollary

Any compatible system $\mathcal{D}^\bullet_{\mathfrak{R}}$ of choices of dualizing complexes $D^\bullet_{R_n}$ for the Noetherian rings R_n, $n \geqslant 0$, induces an equivalence of triangulated categories

$$\mathbb{R}\Psi_{\mathcal{D}^\bullet_{\mathfrak{R}}} : \mathrm{D^{co}}(\mathfrak{R}\text{-discr}) \simeq \mathrm{D^{ctr}}(\mathfrak{R}\text{-contra}) : \mathbb{L}\Phi_{\mathcal{D}^\bullet_{\mathfrak{R}}}.$$

- H. Becker. Models for singularity categories. *Advances in Math.* **254**, p. 187–232, 2014. `arXiv:1205.4473 [math.CT]`
- J. Bernstein, V. Lunts. Equivariant sheaves and functors. *Lecture Notes in Math.* **1578**, Springer, Berlin, 1994.
- A. K. Bousfield, D. M. Kan. Homotopy limits, completions and localizations. *Lecture Notes in Math.* **304**, Springer, 1972–1987.
- G. Böhm, T. Brzeziński, R. Wisbauer. Monads and comonads in module categories. *Journ. of Algebra* **233**, #5, p. 1719–1747, 2009. `arXiv:0804.1460 [math.RA]`
- E. Getzler, J. D. S. Jones. A_∞-algebras and the cyclic bar complex. *Illinois Journ. of Math.* **34**, #2, 1990.
- S. Eilenberg, J. C. Moore. Limits and spectral sequences. *Topology* **1**, p. 1–23, 1962.
- S. Eilenberg, J. C. Moore. Foundations of relative homological algebra. *Memoirs of the American Math. Society* **55**, 1965.

B. L. Feigin, D. B. Fuchs. Verma modules over the Virasoro algebra. Topology (Leningrad, 1982), p. 230–245, *Lecture Notes in Math.* **1060**, Springer-Verlag, Berlin, 1984.

D. Gaitsgory. Ind-coherent sheaves. Electronic preprint `arXiv:1105.4857 [math.AG]`.

V. Hinich. DG coalgebras as formal stacks. *Journ. of Pure and Appl. Algebra* **162**, #2–3, p. 209–250, 2001. `arXiv:math.AG/9812034`

D. Husemoller, J. C. Moore, J. Stasheff. Differential homological algebra and homogeneous spaces. *Journ. of Pure and Appl. Algebra* **5**, p. 113–185, 1974.

S. Iyengar, H. Krause. Acyclicity versus total acyclicity for complexes over noetherian rings. *Documenta Math.* **11**, p. 207–240, 2006.

U. Jannsen. Continuous étale cohomology. *Mathematische Annalen* **280**, #2, p. 207–245, 1988.

P. Jørgensen. The homotopy category of complexes of projective modules. *Advances in Math.* **193**, #1, p. 223–232, 2005. arXiv:math.RA/0312088

B. Keller. Deriving DG-categories. *Ann. Sci. École Norm. Sup. (4)* **27**, #1, p. 63–102, 1994.

B. Keller. Koszul duality and coderived categories (after K. Lefèvre). October 2003. Available from http://www.math.jussieu.fr/~keller/publ/index.html .

B. Keller, W. Lowen, P. Nicolás. On the (non)vanishing of some "derived" categories of curved dg algebras. *Journ. of Pure and Appl. Algebra* **214**, #7, p. 1271–1284, 2010. arXiv:0905.3845 [math.KT]

H. Krause. The stable derived category of a Noetherian scheme. *Compositio Math.* **141**, #5, p. 1128–1162, 2005. arXiv:math.AG/0403526

K. Lefèvre-Hasegawa. Sur les A_∞-catégories. Thèse de doctorat, Université Denis Diderot – Paris 7, November 2003. arXiv:math.CT/0310337 . Corrections, by B. Keller. Available from http://people.math.jussieu.fr/~keller/lefevre/publ.html .

D. Murfet. The mock homotopy category of projectives and Grothendieck duality. Ph. D. Thesis, Australian National University, September 2007. Available from http://www.therisingsea. org/thesis.pdf .

A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Inventiones Math.* **174**, p. 225–308, 2008.

L. Positselski. Nonhomogeneous quadratic duality and curvature. *Funct. Anal. Appl.* **27**, #3, p. 197–204, 1993.

L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne, vol. 70, Birkhäuser/Springer Basel, 2010, xxiv+349 pp. arXiv:0708.3398 [math.CT]

- L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs Amer. Math. Soc.* **212**, #996, 2011, vi+133 pp. arXiv:0905.2621 [math.CT]
- L. Positselski. Coherent analogues of matrix factorizations and relative singularity categories. Electronic preprint arXiv:1102.0261 [math.CT] .
- L. Positselski. Weakly curved A_∞-algebras over a topological local ring. Electronic preprint arXiv:1202.2697 [math.CT] .
- L. Positselski. Contraherent cosheaves. Electronic preprint arXiv:1209.2995 [math.CT] .

- A. Rocha-Caridi, N. Wallach. Characters of irreducible representations of the Virasoro algebra. *Math. Zeitschrift* **185**, #1, p. 1–21, 1984.
- N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.* **65**, #2, p.121–154, 1988.