Cotorsion Theories in Contramodule Categories

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Cotorsion theories in exact categories

Let $\mathcal{A}$ be an exact category (in Quillen's sense). Given a class of objects $\mathcal{F} \subset \mathcal{A}$, its right orthogonal complement $\mathcal{F}^{\perp}$ is the class of objects

$$\mathcal{F}^{\perp} = \{X \in \mathcal{A} : \operatorname{Ext}^1_{\mathcal{A}}(F, X) = 0 \ \ \forall\, F \in \mathcal{F}\}.$$

Given a class of objects $\mathcal{C} \subset \mathcal{A}$, its left orthogonal complement ${}^{\perp}\mathcal{C}$ is the class of objects

$${}^{\perp}\mathcal{C} = \{X \in \mathcal{A} : \operatorname{Ext}^1_{\mathcal{C}}(X, C) = 0 \ \ \forall\, C \in \mathcal{C}\}.$$

A *cotorsion theory* in an exact category $\mathcal{A}$ is a pair of classes of objects $\mathcal{F}$, $\mathcal{C} \subset \mathcal{A}$ such that $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$.

We will call the class of objects $\mathcal{F} \subset \mathcal{A}$ the *flat class* and the class $\mathcal{C} \subset \mathcal{A}$ the *cotorsion class*.

Cotorsion theories in exact categories

A cotorsion theory $(\mathcal{F}, \mathcal{C})$ in an exact category $\mathcal{A}$ is called *complete* if for every object $X \in \mathcal{A}$ there exist two short exact sequences

$$0 \longrightarrow X \longrightarrow C \longrightarrow F' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow F \longrightarrow X \longrightarrow 0$$

in the category $\mathcal{A}$ with F, $F' \in \mathcal{F}$ and C, $C' \in \mathcal{C}$.

Theorem (Eklof–Trlifaj 2001)

Let A be an associative ring and $\mathcal{A} = A\text{-}\mathrm{mod}$ be the abelian category of left A-modules. Let $\mathcal{S} \subset \mathcal{A}$ be a set of objects. Then the classes $\mathcal{C} = \mathcal{S}^{\perp}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$ form a complete cotorsion theory in the category $\mathcal{A}$.

Cotorsion theories in exact categories

A complete cotorsion theory $(\mathcal{F}, \mathcal{C})$ in an exact category $\mathcal{A}$ is called *hereditary* if any of the following equivalent conditions holds:

- the class $\mathcal{F}$ is closed under kernels of admissible epimorphisms, i.e. for any short exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

  in $\mathcal{A}$ with G, $H \in \mathcal{F}$ the object F also belongs to $\mathcal{F}$; or
- the class $\mathcal{C}$ is closed under cokernels of admissible monomorphisms, i.e. for any short exact sequence

$$0 \longrightarrow E \longrightarrow D \longrightarrow C \longrightarrow 0$$

  in $\mathcal{A}$ with E, $D \in \mathcal{C}$ the object C also belongs to $\mathcal{C}$; or
- one has $\mathrm{Ext}^2_{\mathcal{A}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$; or
- one has $Ext^n_{\mathcal{A}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$ and all $n \geqslant 1$.

Cotorsion theories in exact categories

In a hereditary complete cotorsion theory $(\mathcal{F}, \mathcal{C})$, the class $\mathcal{F}$ can be characterized as consisting of all the objects $F \in \mathcal{A}$ for which the functor $\mathrm{Hom}_{\mathcal{A}}(F, -)$ takes short exact sequences

$$0 \longrightarrow E \longrightarrow D \longrightarrow C \longrightarrow 0$$

of objects E, D, $C \in \mathcal{C}$ to short exact sequences of abelian groups $0 \to \mathrm{Hom}_{\mathcal{A}}(F, E) \to \mathrm{Hom}_{\mathcal{A}}(F, D) \to \mathrm{Hom}_{\mathcal{A}}(F, C) \to 0$.

The class $\mathcal{C}$ can be characterized as consisting of all the objects $C \in \mathcal{A}$ for which the functor $\mathrm{Hom}_{\mathcal{A}}(-, C)$ takes short exact sequences

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

of objects F, G, $H \in \mathcal{F}$ to short exact sequences of abelian groups $0 \to \mathrm{Hom}_{\mathcal{A}}(H, C) \to \mathrm{Hom}_{\mathcal{A}}(G, C) \to \mathrm{Hom}_{\mathcal{A}}(F, C) \to 0$.

Flat and very flat cotorsion theories

Let A be an associative ring. A left A-module C is called *cotorsion* if one has $\mathrm{Ext}^1_A(F, C) = 0$ for any flat left A-module F. Denote the class of flat left A-modules by $\mathcal{F}_A$ and the class of cotorsion left A-modules by $\mathcal{C}_A$.

Let R be a commutative ring. An R-module C is called *contraadjusted* if one has $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ for all elements $s \in R$. An R-module F is called *very flat* if one has $\mathrm{Ext}^1_R(F, C) = 0$ for any contraadjusted R-module C. Denote the class of very flat R-modules by $\mathcal{VF}_R$ and the class of contraadjusted R-modules by $\mathcal{CA}_R$.

Corollary (of the Eklof–Trlifaj Theorem)

(a) *The classes $\mathcal{F}_A$ and $\mathcal{C}_A$ form a hereditary complete cotorsion theory in the abelian category $A\text{-}\mathrm{mod}$.*
(b) *The classes $\mathcal{VF}_R$ and $\mathcal{CA}_R$ form a hereditary complete cotorsion theory in the abelian category $R\text{-}\mathrm{mod}$.*

In this talk, we are only interested in hereditary complete cotorsion theories.

The flat and very flat cotorsion theories play a key role in the theory of contraherent cosheaves over schemes and stacks. Development of the theory of contraherent cosheaves of contramodules over formal schemes and ind-schemes requires a construction of the flat and/or very flat (hereditary complete) cotorsion theories in the abelian categories of contramodules.

The proof of the Eklof–Trlifaj theorem is based on the facts that the category of modules is a Grothendieck abelian category, and also that it has enough projective objects. In the categories of contramodules there are also enough projective objects, but the infinite direct sums are not exact. Thus the argument of Eklof and Trlifaj does not seem to be applicable.

The categories of contramodules are locally presentable, and λ-filtered inductive limits in them are exact for big enough cardinals λ, but this does not seem to help, as non-λ-filtered filtered inductive limits are hard to control.

I will explain how to construct the flat and very flat hereditary complete cotorsion theories in the abelian categories of contramodules

- using the fact of hereditary completeness of such theories in the category of modules as a black box,
- or otherwise, using old-style explicit constructions of flat/cotorsion resolutions of modules.

Contramodules

Contramodules are module-like objects endowed with infinite summation (or, occasionally, integration) operations, understood algebraically as infinitary (linear) operations subject to natural axioms. Contramodules carry no underlying topologies on them, but feel like being in some sense "complete". For about every class of "discrete" or "torsion" modules, there is an much less familiar, but no less interesting accompanying class of contramodules.

"Discrete" or "torsion" module categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

Contramodules over a Commutative Ring with an Ideal

An abelian group P with an additive operator $s\colon P \longrightarrow P$ is said to be *s-contraadjusted* if for any sequence $p_0, p_1, p_2, \ldots \in P$ the infinite system of nonhomogeneous linear equations

$$q_n = sq_{n+1} + p_n \quad \text{for all } n \geqslant 0$$

has a solution $q_0, q_1, q_2, \ldots \in P$.

An abelian group P with an additive operator s is said to be an *s-contramodule* if for any $p_0, p_1, p_2, \ldots$ the system of equations $q_n = sq_{n+1} + p_n$ has a *unique* solution in P.

The infinite summation operation with s-power coefficients in an s-contramodule P is defined by the rule

$$\sum_{n=0}^{\infty} s^n p_n = q_0.$$

Contramodules over a commutative ring with an ideal

Conversely, given an additive, associative, and unital s-power infinite summation operation

$$(p_n)_{n=0}^{\infty} \longmapsto \sum\nolimits_{n=0}^{\infty} s^n p_n$$

in an abelian group P, one can uniquely solve the system of equations $q_n = sq_{n+1} + p_n$ in P by setting

$$q_n = \sum\nolimits_{i=0}^{\infty} s^i p_{n+i}.$$

A module P over a commutative ring R with an element $s \in R$ is s-contraadjusted (i.e., contraadjusted with respect to the operator of multiplication with s) if and only if $\operatorname{Ext}_R^1(R[s^{-1}], P) = 0$.

An R-module P is an s-contramodule if and only if $\operatorname{Ext}_R^i(R[s^{-1}], P) = 0$ for $i = 0$ and 1. (Notice that the R-module $R[s^{-1}]$ has projective dimension at most 1.)

Contramodules over a commutative ring with an ideal

Any quotient R-module of an s-contraadjusted R-module is s-contraadjusted. Any extenstion of two s-contraadjusted R-modules is s-contraadjusted. An infinite product of s-contraadjusted R-modules is s-contraadjusted.

The kernel and cokernel of any morphism of s-contramodule R-modules are s-contramodule R-modules. Any extension of two s-contramodule R-modules is an s-contramodule R-module.

Let $I \subset R$ be an ideal. An R-module P is called an I-contramodule if it is an s-contramodule for every $s \in I$. It suffices to impose this condition for a set of generators s_j of the ideal I.

The category of I-contramodule R-modules $R\text{-}\mathrm{mod}_{I\text{-ctra}}$ is abelian with exact functors of infinite product. The embedding functor $R\text{-}\mathrm{mod}_{I\text{-ctra}} \longrightarrow R\text{-}\mathrm{mod}$ is exact and preserves infinite products.

Flat and very flat cotorsion theories for I-contramodules

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Denote by $\mathcal{C}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-ctra}}$ the class of all I-contramodule R-modules that are at the same time cotorsion R-modules. Denote by $\mathcal{F}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-ctra}}$ the class of all I-contramodule R-modules that are at the same time flat R-modules.

Denote by $\mathcal{CA}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-ctra}}$ the class of all I-contramodule R-modules that are at the same time contraadjusted R-modules.

An I-contramodule R-module P is said to be very flat if any of the following equivalent conditions holds:

- P is a flat R-module and P/IP is a very flat R/I-module; or
- the R/I^n-module P/I^nP is very flat for every $n \geqslant 1$.

Denote the class of all very flat I-contramodule R-modules by $\mathcal{VF}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-ctra}}$.

Flat and Very Flat Cotorsion Theories for I-Contramodules

Theorem

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Then
(a) *the classes $\mathcal{F}_{R,I}$ and $\mathcal{C}_{R,I}$ form a hereditary complete cotorsion theory in the abelian category $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}}$;*
(b) *the classes $\mathcal{VF}_{R,I}$ and $\mathcal{CA}_{R,I}$ form a hereditary complete cotorsion theory in the abelian category $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}}$.*

[L.P., "Contraherent cosheaves", `arXiv:1209.2995 [math.CT]`, Sections C.2–C.3]

Sketch of proof of part (a).

Let $I = (s_j) \subset R$. Any R-module L that is s_j-contraadjusted for every j has a unique maximal quotient R-module that is an I-contramodule. Denote it by $L/L(I)$.

Let P be an I-contramodule R-module, and let $0 \longrightarrow P \longrightarrow C \longrightarrow F' \longrightarrow 0$ be a short exact sequence of R-modules, where C is a cotorsion R-module and F' is a flat R-module. Then $0 \longrightarrow P \longrightarrow C/C(I) \longrightarrow F'/F'(I) \longrightarrow 0$ is a short exact sequence of I-contramodule R-modules, while $C/C(I)$ is a cotorsion R-module and $F'/F'(I)$ is a flat R-module.

Let $0 \longrightarrow C' \longrightarrow F \longrightarrow P \longrightarrow 0$ be a short exact sequence of R-modules, where C' is a cotorsion R-module and F' is a flat R-module. Then $0 \longrightarrow C'/C'(I) \longrightarrow F/F(I) \longrightarrow P \longrightarrow 0$ is a short exact sequence of I-contramodule R-modules, while $C'/C'(I)$ is a cotorsion R-module and $F/F(I)$ is a flat R-module. $\square$

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X \colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X \colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m \colon R[M] \longrightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

- and the unity equation $m \circ \varepsilon_X = \mathrm{id}_M$

$$M \longrightarrow R[M] \longrightarrow M.$$

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U}\subset\mathfrak{R}$ the set $\{x\mid r_x\notin\mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]]\longrightarrow\mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X\colon X\longrightarrow\mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X\longmapsto\mathfrak{R}[[X]]\colon \mathrm{Sets}\longrightarrow\mathrm{Sets}$.

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{P}.$$

The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite product and enough projectives (which are the direct summands of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$).
The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

Let R be a Noetherian commutative ring with an ideal $I \subset R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology). Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the I-contramodule R-modules.

In particular, $\mathbb{Z}_p$-contramodules $=$ weakly p-complete (Ext-p-complete) abelian groups [Bousfield–Kan '72, Jannsen '88].

Flat and very flat cotorsion theories for $\mathfrak{R}$-contramodules

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of commutative rings and surjective homomorphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called flat if

- the R_n-module $\overline{\mathfrak{F}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{F} \longrightarrow \varprojlim_n \overline{\mathfrak{F}}_n$ is an isomorphism.

The class $\mathcal{F}_{\mathfrak{R}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-}\mathrm{contra}$. Projective $\mathfrak{R}$-contramodules are flat.

Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

Assume that the ideals $\ker(R_{n+1} \to R_n) \subset R_{n+1}$ are finitely generated. Fix a commutative ring R endowed with a ring homomorphism $R \longrightarrow \mathfrak{R}$ such that the compositions $R \longrightarrow \mathfrak{R} \longrightarrow R_n$ are surjective maps.

Denote by $\mathcal{CA}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$ the class of all $\mathfrak{R}$-contramodules that are contraadjusted as R-modules. (We will see that this class does not in fact depend on the choice of a ring R.)

A flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called very flat if the R_n-module $\overline{\mathfrak{F}}_n$ is very flat for every $n \geqslant 0$. Denote the class of all very flat $\mathfrak{R}$-contramodules by $\mathcal{VF}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

An $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be cotorsion if the functor of $\mathfrak{R}$-contramodule homomorphisms $\operatorname{Hom}^{\mathfrak{R}}(-, \mathfrak{Q})$ takes short exact sequences of flat $\mathfrak{R}$-contramodules to short exact sequences of abelian groups. Denote the class of all cotorsion $\mathfrak{R}$-contramodules by $\mathcal{C}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

Theorem

(a) *For any projective system of commutative rings and surjective homomorphisms* $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ *with finitely generated kernel ideals* $\ker(R_{n+1} \to R_n)$, *the classes* $\mathcal{VF}_{\mathfrak{R}}$ *and* $\mathcal{CA}_{\mathfrak{R}}$ *form a hereditary complete cotorsion theory in the abelian category* $\mathfrak{R}\text{-}\mathrm{contra}$.

(b) *When the rings* R_n *are Noetherian and their Krull dimensions are uniformly bounded by a constant, the classes* $\mathcal{F}_{\mathfrak{R}}$ *and* $\mathcal{C}_{\mathfrak{R}}$ *form a hereditary complete cotorsion theory in the abelian category* $\mathfrak{R}\text{-}\mathrm{contra}$.

[“Contraherent cosheaves”, Sections D.3–D.4]

Brief sketch of proof of part (a).

Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module and embed in into a contraadjusted R-module K with a very flat quotient R-module $F = K/\mathfrak{P}$.

Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$. Furthermore, there are surjective homomorphisms of R_n-modules $R_n \otimes_R \mathfrak{P} \longrightarrow \overline{\mathfrak{P}}_n$, and the induced short exact sequences $0 \longrightarrow \overline{\mathfrak{P}}_n \longrightarrow Q_n \longrightarrow R_n \otimes_R F \longrightarrow 0$.

Put $\mathfrak{Q} = \varprojlim_n Q_n$ and $\mathfrak{F} = \varprojlim_n R_n \otimes_R F$. Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$, the R-module $\mathfrak{Q}$ is contraadjusted, and the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is very flat.

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

Sketch of proof of part (a) — final comment.

One still has to prove that $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{Q})=0$ when $\mathfrak{F}$ is a very flat $\mathfrak{R}$-contramodule and $\mathfrak{Q}$ is an R-contraadjusted $\mathfrak{R}$-contramodule. $\square$

Sketch of proof of part (b).

This argument is not based on using the Eklof–Trlifaj theorem as a black box, but rather on an explicit construction of cotorsion/flat resolutions that people used in pre-ET times [Xu '96].

Let T be a Noetherian commutative ring and H be a flat T-module. For any prime ideal $\mathfrak{q} \subset T$, consider the localization $H_\mathfrak{q} = T_\mathfrak{q} \otimes_T H$ of the T-module H at $\mathfrak{q}$, and take its $\mathfrak{q}$-adic completion $\widehat{H_\mathfrak{q}} = \varprojlim_n H_\mathfrak{q}/\mathfrak{q}^n H_\mathfrak{q}$.

Set $FC_T(H) = \prod_{\mathfrak{q} \in \operatorname{Spec} T} \widehat{H_\mathfrak{q}}$. Then the T-module $FC_T(H)$ is flat cotorsion, and the natural map $H \longrightarrow FC_T(H)$ is injective with a flat cokernel.

Given a surjective ring homomorphism $T \longrightarrow S$, there is a natural isomorphism $S \otimes_T FC_T(H) \simeq FC_S(S \otimes_T H)$.

Sketch of proof of part (b) — cont'd.

Let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule. So $\mathfrak{G}$ is the projective limit of its quotient R_n-modules $\overline{\mathfrak{G}}_n$. Set $\mathfrak{C} = \varprojlim_n FC_{R_n}(\overline{\mathfrak{G}}_n)$ and $\mathfrak{F} = \varprojlim_n (FC_{R_n}(\overline{\mathfrak{G}}_n)/\overline{\mathfrak{G}}_n)$.

Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{F} \longrightarrow 0$, the $\mathfrak{R}$-contramodule $\mathfrak{C}$ is flat cotorsion, and the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat.

When the Krull dimensions of the rings R_n are uniformly bounded, any flat $\mathfrak{R}$-contramodule has finite projective dimension, and any $\mathfrak{R}$-contramodule has finite cotorsion dimension.

This allows to deduce the existence of cotorsion/flat resolutions for arbitrary $\mathfrak{R}$-contramodules from their existence for flat $\mathfrak{R}$-contramodules. $\square$

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