Cotorsion Theories in Contramodule Categories

Leonid Positselski – Haifa & Brno

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We will call the class of objects $\mathcal{F} \subset \mathcal{A}$ the *flat class*

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Let A be an associative ring and $\mathcal{A} = A\text{-}\mathrm{mod}$ be the abelian category of left A-modules.

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$$0 \longrightarrow E \longrightarrow D \longrightarrow C \longrightarrow 0$$

of objects E, D, $C \in \mathcal{C}$ to short exact sequences of abelian groups $0 \to \mathrm{Hom}_{\mathcal{A}}(F, E) \to \mathrm{Hom}_{\mathcal{A}}(F, D) \to \mathrm{Hom}_{\mathcal{A}}(F, C) \to 0$.

The class $\mathcal{C}$ can be characterized as consisting of all the objects $C \in \mathcal{A}$ for which the functor $\mathrm{Hom}_{\mathcal{A}}(-, C)$ takes short exact sequences

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Flat and very flat cotorsion theories

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Corollary (of the Eklof–Trlifaj Theorem)

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Corollary (of the Eklof–Trlifaj Theorem)

(a) *The classes $\mathcal{F}_A$ and $\mathcal{C}_A$ form a hereditary complete cotorsion theory in the abelian category $A\text{-}\mathrm{mod}$.*
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The proof of the Eklof–Trlifaj theorem is based on the facts that the category of modules is a Grothendieck abelian category, and also that it has enough projective objects. In the categories of contramodules there are also enough projective objects, but the infinite direct sums are not exact. Thus the argument of Eklof and Trlifaj does not seem to be applicable.

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The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

Contramodules over a Commutative Ring with an Ideal

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The infinite summation operation with s-power coefficients in an s-contramodule P

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An abelian group P with an additive operator s is said to be an *s-contramodule* if for any $p_0, p_1, p_2, \ldots$ the system of equations $q_n = sq_{n+1} + p_n$ has a *unique* solution in P.

The infinite summation operation with s-power coefficients in an s-contramodule P is defined by the rule

$$\sum_{n=0}^{\infty} s^n p_n = q_0.$$

Contramodules over a commutative ring with an ideal

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The category of I-contramodule R-modules $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}}$ is abelian with exact functors of infinite product. The embedding functor $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}} \longrightarrow R\text{-}\mathrm{mod}$ is exact

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Any quotient R-module of an s-contraadjusted R-module is s-contraadjusted. Any extenstion of two s-contraadjusted R-modules is s-contraadjusted. An infinite product of s-contraadjusted R-modules is s-contraadjusted.

The kernel and cokernel of any morphism of s-contramodule R-modules are s-contramodule R-modules. Any extension of two s-contramodule R-modules is an s-contramodule R-module.

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The category of I-contramodule R-modules $R\text{-mod}_{I\text{-ctra}}$ is abelian with exact functors of infinite product. The embedding functor $R\text{-mod}_{I\text{-ctra}} \longrightarrow R\text{-mod}$ is exact and preserves infinite products.

Flat and very flat cotorsion theories for I-contramodules

Let R be a Noetherian commutative ring and $I \subset R$ be an ideal.

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Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Denote by $\mathcal{C}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}}$ the class of all I-contramodule R-modules

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- P is a flat R-module and P/IP is a very flat R/I-module; or
- the R/I^n-module P/I^nP is very flat for every $n \geqslant 1$.

Denote the class of all very flat I-contramodule R-modules by $\mathcal{VF}_{R,I} \subset R\text{-}\mathrm{mod}_{I\text{-ctra}}$.

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(b) *the classes $\mathcal{VF}_{R,I}$ and $\mathcal{CA}_{R,I}$ form a hereditary complete cotorsion theory in the abelian category $R\text{-}\mathrm{mod}_{I\text{-}\mathrm{ctra}}$.*

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[L.P., "Contraherent cosheaves", `arXiv:1209.2995 [math.CT]`, Sections C.2–C.3]

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The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

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The class $\mathcal{F}_{\mathfrak{R}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-}\mathrm{contra}$.

Flat and very flat cotorsion theories for $\mathfrak{R}$-contramodules

Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ be a projective system of commutative rings and surjective homomorphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$, and endow it with the projective limit topology.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{R}$-contramodule structure comes from an R_n-module structure. An $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called flat if

- the R_n-module $\overline{\mathfrak{F}}_n$ is flat for every $n \geqslant 0$,
- and the natural map $\mathfrak{F} \longrightarrow \varprojlim_n \overline{\mathfrak{F}}_n$ is an isomorphism.

The class $\mathcal{F}_{\mathfrak{R}}$ of flat $\mathfrak{R}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{R}\text{-}\mathrm{contra}$. Projective $\mathfrak{R}$-contramodules are flat.

Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

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Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

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Denote by $\mathcal{CA}_{\mathfrak{R}} \subset \mathfrak{R}\text{-}\mathrm{contra}$ the class of all $\mathfrak{R}$-contramodules

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Denote by $\mathcal{CA}_{\mathfrak{R}} \subset \mathfrak{R}\text{-}\mathrm{contra}$ the class of all $\mathfrak{R}$-contramodules that are contraadjusted as R-modules.

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A flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called very flat if the R_n-module $\overline{\mathfrak{F}}_n$ is very flat for every $n \geqslant 0$. Denote the class of all very flat $\mathfrak{R}$-contramodules by $\mathcal{VF}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

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Assume that the ideals $\ker(R_{n+1} \to R_n) \subset R_{n+1}$ are finitely generated. Fix a commutative ring R endowed with a ring homomorphism $R \longrightarrow \mathfrak{R}$ such that the compositions $R \longrightarrow \mathfrak{R} \longrightarrow R_n$ are surjective maps.

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Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

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A flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called very flat if the R_n-module $\overline{\mathfrak{F}}_n$ is very flat for every $n \geqslant 0$. Denote the class of all very flat $\mathfrak{R}$-contramodules by $\mathcal{VF}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

An $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be cotorsion if the functor of $\mathfrak{R}$-contramodule homomorphisms $\mathrm{Hom}^{\mathfrak{R}}(-, \mathfrak{Q})$

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A flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called very flat if the R_n-module $\overline{\mathfrak{F}}_n$ is very flat for every $n \geqslant 0$. Denote the class of all very flat $\mathfrak{R}$-contramodules by $\mathcal{VF}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

An $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be cotorsion if the functor of $\mathfrak{R}$-contramodule homomorphisms $\mathrm{Hom}^{\mathfrak{R}}(-, \mathfrak{Q})$ takes short exact sequences of flat $\mathfrak{R}$-contramodules to short exact sequences of abelian groups.

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Assume that the ideals $\ker(R_{n+1} \to R_n) \subset R_{n+1}$ are finitely generated. Fix a commutative ring R endowed with a ring homomorphism $R \longrightarrow \mathfrak{R}$ such that the compositions $R \longrightarrow \mathfrak{R} \longrightarrow R_n$ are surjective maps.

Denote by $\mathcal{CA}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$ the class of all $\mathfrak{R}$-contramodules that are contraadjusted as R-modules. (We will see that this class does not in fact depend on the choice of a ring R.)

A flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called very flat if the R_n-module $\overline{\mathfrak{F}}_n$ is very flat for every $n \geqslant 0$. Denote the class of all very flat $\mathfrak{R}$-contramodules by $\mathcal{VF}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

An $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be cotorsion if the functor of $\mathfrak{R}$-contramodule homomorphisms $\operatorname{Hom}^{\mathfrak{R}}(-, \mathfrak{Q})$ takes short exact sequences of flat $\mathfrak{R}$-contramodules to short exact sequences of abelian groups. Denote the class of all cotorsion $\mathfrak{R}$-contramodules by $\mathcal{C}_{\mathfrak{R}} \subset \mathfrak{R}\text{-contra}$.

Flat and Very Flat Cotorsion Theories for $\mathfrak{R}$-Contramodules

Theorem

(a) *For any projective system of commutative rings and surjective homomorphisms* $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$

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Theorem

(a) *For any projective system of commutative rings and surjective homomorphisms* $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ *with finitely generated kernel ideals* $\ker(R_{n+1} \to R_n)$,

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["Contraherent cosheaves", Sections D.3–D.4]

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module

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Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module and embed in into a contraadjusted R-module K with a very flat quotient R-module $F = K/\mathfrak{P}$.

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Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module and embed in into a contraadjusted R-module K with a very flat quotient R-module $F = K/\mathfrak{P}$.

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Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$.

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Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$. Furthermore, there are surjective homomorphisms of R_n-modules $R_n \otimes_R \mathfrak{P} \longrightarrow \overline{\mathfrak{P}}_n$

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Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$. Furthermore, there are surjective homomorphisms of R_n-modules $R_n \otimes_R \mathfrak{P} \longrightarrow \overline{\mathfrak{P}}_n$, and the induced short exact sequences

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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Put $\mathfrak{Q} = \varprojlim_n Q_n$ and $\mathfrak{F} = \varprojlim_n R_n \otimes_R F$.

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Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module and embed in into a contraadjusted R-module K with a very flat quotient R-module $F = K/\mathfrak{P}$.

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Put $\mathfrak{Q} = \varprojlim_n Q_n$ and $\mathfrak{F} = \varprojlim_n R_n \otimes_R F$. Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$,

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Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$. Furthermore, there are surjective homomorphisms of R_n-modules $R_n \otimes_R \mathfrak{P} \longrightarrow \overline{\mathfrak{P}}_n$, and the induced short exact sequences $0 \longrightarrow \overline{\mathfrak{P}}_n \longrightarrow Q_n \longrightarrow R_n \otimes_R F \longrightarrow 0$.

Put $\mathfrak{Q} = \varprojlim_n Q_n$ and $\mathfrak{F} = \varprojlim_n R_n \otimes_R F$. Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$, the R-module $\mathfrak{Q}$ is contraadjusted,

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

Brief sketch of proof of part (a).

Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Suppose that the map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism (the general case will follow). Consider $\mathfrak{P}$ as an R-module and embed in into a contraadjusted R-module K with a very flat quotient R-module $F = K/\mathfrak{P}$.

Then there are short exact sequences of R_n-modules $0 \longrightarrow R_n \otimes_R \mathfrak{P} \longrightarrow R_n \otimes_R K \longrightarrow R_n \otimes_R F \longrightarrow 0$. Furthermore, there are surjective homomorphisms of R_n-modules $R_n \otimes_R \mathfrak{P} \longrightarrow \overline{\mathfrak{P}}_n$, and the induced short exact sequences $0 \longrightarrow \overline{\mathfrak{P}}_n \longrightarrow Q_n \longrightarrow R_n \otimes_R F \longrightarrow 0$.

Put $\mathfrak{Q} = \varprojlim_n Q_n$ and $\mathfrak{F} = \varprojlim_n R_n \otimes_R F$. Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$, the R-module $\mathfrak{Q}$ is contraadjusted, and the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is very flat.

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

Sketch of proof of part (a) — final comment.

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Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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One still has to prove that $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{Q}) = 0$ when $\mathfrak{F}$ is a very flat $\mathfrak{R}$-contramodule and $\mathfrak{Q}$ is an R-contraadjusted $\mathfrak{R}$-contramodule.

Very Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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Flat Cotorsion Theory for $\mathfrak{R}$-Contramodules

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Let T be a Noetherian commutative ring and H be a flat T-module.

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Let T be a Noetherian commutative ring and H be a flat T-module. For any prime ideal $\mathfrak{q} \subset T$, consider the localization $H_\mathfrak{q} = T_\mathfrak{q} \otimes_T H$ of the T-module H at $\mathfrak{q}$

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Let T be a Noetherian commutative ring and H be a flat T-module. For any prime ideal $\mathfrak{q} \subset T$, consider the localization $H_\mathfrak{q} = T_\mathfrak{q} \otimes_T H$ of the T-module H at $\mathfrak{q}$, and take its $\mathfrak{q}$-adic completion $\widehat{H_\mathfrak{q}} = \varprojlim_n H_\mathfrak{q}/\mathfrak{q}^n H_\mathfrak{q}$.

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Given a surjective ring homomorphism $T \longrightarrow S$, there is a natural isomorphism $S \otimes_T FC_T(H) \simeq FC_S(S \otimes_T H)$.

Sketch of proof of part (b) — cont'd.

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Sketch of proof of part (b) — cont'd.

Let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule.

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Let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule. So $\mathfrak{G}$ is the projective limit of its quotient R_n-modules $\overline{\mathfrak{G}}_n$.

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Let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule. So $\mathfrak{G}$ is the projective limit of its quotient R_n-modules $\overline{\mathfrak{G}}_n$. Set $\mathfrak{C} = \varprojlim_n FC_{R_n}(\overline{\mathfrak{G}}_n)$

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Let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule. So $\mathfrak{G}$ is the projective limit of its quotient R_n-modules $\overline{\mathfrak{G}}_n$. Set $\mathfrak{C} = \varprojlim_n FC_{R_n}(\overline{\mathfrak{G}}_n)$ and $\mathfrak{F} = \varprojlim_n (FC_{R_n}(\overline{\mathfrak{G}}_n)/\overline{\mathfrak{G}}_n)$.

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Then there is a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{F} \longrightarrow 0$

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Sketch of proof of part (b) — cont'd.

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When the Krull dimensions of the rings R_n are uniformly bounded,

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When the Krull dimensions of the rings R_n are uniformly bounded, any flat $\mathfrak{R}$-contramodule has finite projective dimension, and any $\mathfrak{R}$-contramodule has finite cotorsion dimension.

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When the Krull dimensions of the rings R_n are uniformly bounded, any flat $\mathfrak{R}$-contramodule has finite projective dimension, and any $\mathfrak{R}$-contramodule has finite cotorsion dimension.

This allows to deduce the existence of cotorsion/flat resolutions

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This allows to deduce the existence of cotorsion/flat resolutions for arbitrary $\mathfrak{R}$-contramodules

Sketch of proof of part (b) — cont'd.

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When the Krull dimensions of the rings R_n are uniformly bounded, any flat $\mathfrak{R}$-contramodule has finite projective dimension, and any $\mathfrak{R}$-contramodule has finite cotorsion dimension.

This allows to deduce the existence of cotorsion/flat resolutions for arbitrary $\mathfrak{R}$-contramodules from their existence for flat $\mathfrak{R}$-contramodules.

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