CONTRAHERENT COSHEAVES

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Abstract. Contraherent cosheaves are globalizations of cotorsion (or similar) modules over commutative rings obtained by gluing together over a scheme. The category of contraherent cosheaves over a scheme is a Quillen exact category with exact functors of infinite product. Over a quasi-compact semi-separated scheme or a Noetherian scheme of finite Krull dimension (in a different version—over any locally Noetherian scheme), it also has enough projectives. We construct the derived co-contra correspondence, meaning an equivalence between appropriate derived categories of quasi-coherent sheaves and contraherent cosheaves, over a quasi-compact semi-separated scheme and, in a different form, over a Noetherian scheme with a dualizing complex. The former point of view allows us to obtain an explicit construction of Neeman's extraordinary inverse image functor $f^!$ for a morphism of quasi-compact semi-separated schemes $f\colon X \longrightarrow Y$. The latter approach provides an expanded version of the covariant Serre–Grothendieck duality theory and leads to Deligne's extraordinary inverse image functor $f^!$ (which we denote by f^+) for a morphism of finite type f between Noetherian schemes. Semi-separated Noetherian stacks, affine Noetherian formal schemes, and ind-affine ind-schemes (together with the noncommutative analogues) are briefly discussed in the appendices.

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Introduction

Quasi-coherent sheaves resemble comodules. Both form abelian categories with exact functors of infinite direct sum (and in fact, even of filtered inductive limit) and with enough injectives. When one restricts to quasi-coherent sheaves over Noetherian schemes and comodules over (flat) corings over Noetherian rings, both abelian categories are locally Noetherian. Neither has projective objects or exact functors of infinite product, in general.

In fact, quasi-coherent sheaves *are* comodules. Let X be a quasi-compact semi-separated scheme and $\{U_\alpha\}$ be its finite affine open covering. Denote by T the disconnected union of the schemes U_α; so T is also an affine scheme and the natural morphism $T \longrightarrow X$ is affine. Then quasi-coherent sheaves over X can be described as quasi-coherent sheaves $\mathcal{F}$ over T endowed with an isomorphism $\phi\colon p_1^*(\mathcal{F}) \simeq p_2^*(\mathcal{F})$ between the two inverse images under the natural maps p_1, $p_2\colon T \times_X T \rightrightarrows T$. The isomorphism ϕ has to satisfy a natural associativity constraint.

In other words, this means that the ring of functions $\mathcal{C} = \mathcal{O}(T \times_X T)$ has a natural structure of a coring over the ring $A = \mathcal{O}(T)$. The quasi-coherent sheaves over X are the same thing as (left or right) comodules over this coring. The quasi-coherent sheaves over a (good enough) stack can be also described in such way [37].

There are two kinds of module categories over a coalgebra or coring: in addition to the more familiar comodules, there are also *contramodules* [56]. Introduced originally by Eilenberg and Moore [17] in 1965 (see also the notable paper [4]), they were all but forgotten for four decades, until the author's preprint and then monograph [52] attracted some new attention to them towards the end of 2000's.

Assuming a coring $\mathcal{C}$ over a ring A is a projective left A-module, the category of left $\mathcal{C}$-contramodules is abelian with exact functors of infinite products and enough projectives. Generally, contramodules are "dual-analogous" to comodules in most respects, i. e., they behave as though they formed two opposite categories—which in fact they don't (or otherwise it wouldn't be interesting).

On the other hand, there is an important homological phenomenon of *comodule-contramodule correspondence*, or a *covariant* equivalence between appropriately defined ("exotic") derived categories of left comodules and left contramodules over the same coring. This equivalence is typically obtained by deriving certain adjoint functors which act between the abelian categories of comodules and contramodules and induce a covariant equivalence between their appropriately picked exact or additive subcategories (e. g., the equivalence of *Kleisli categories*, which was empasized in the application to comodules and contramodules in the paper [7]).

Contraherent cosheaves are geometric module objects over a scheme that are similar to (and, sometimes, particular cases of) contramodules in the same way as quasi-coherent sheaves are similar to (or particular cases of) comodules. Thus the simplest way to define contramodules would be to assume one's scheme X to be quasi-compact and semi-separated, pick its finite affine covering $\{U_\alpha\}$, and consider contramodules over the related coring $\mathcal{C}$ over the ring A as constructed above.

This indeed largely agrees with our approach in this paper, but there are several problems to be dealt with. First of all, $\mathcal{C}$ is not a projective A-module, but only a flat one. The most immediate consequence is that one cannot hope for an abelian category of $\mathcal{C}$-contramodules, but at best for an exact category. This is where the *cotorsion modules* [67, 20] (or their generalizations which we call the *contraadjusted* modules) come into play. Secondly, it turns out that the exact category of $\mathcal{C}$-contramodules, however defined, depends on the choice of an affine covering $\{U_\alpha\}$.

In the exposition below, we strive to make our theory as similar (or rather, dual-analogous) to the classical theory of quasi-coherent sheaves as possible, while refraining from the choice of a covering to the extent that it remains practicable. We start with defining cosheaves of modules over a sheaf of rings on a topological space, and proceed to introducing the exact subcategory of contraherent cosheaves in the exact category of cosheaves of $\mathcal{O}_X$-modules on an arbitrary scheme X.

Several attempts to develop a theory of cosheaves have been made in the literature over the years (see, e. g., [9, 62, 11]). The main difficulty arising in this connection is that the conventional theory of sheaves depends on the exactness property of filtered inductive limits in the categories of abelian groups or sets in an essential way. E. g., the most popular approach is based on the wide use of the construction of stalks, which are defined as filtered inductive limits. It is important that the functors of stalks are exact on both the categories of presheaves and sheaves.

The problem is that the costalks of a co(pre)sheaf would be constructed as filtered projective limits, and filtered projective limits of abelian groups are not exact. One possible way around this difficulty is to restrict oneself to (co)constructible cosheaves, for which the projective limits defining the costalks may be stabilizing and hence exact. This is apparently the approach taken in the notable recent dissertation [13] (see the discussion with further references in [13, Section 2.5]). The present work is essentially based on the observation that one does not really need the (co)stalks in the quasi-coherent/contraherent (co)sheaf theory, as *the functors of (co)sections over affine open subschemes are already exact on the (co)sheaf categories.*

Let us explain the main definition in some detail. A quasi-coherent sheaf $\mathcal{F}$ on a scheme X can be simply defined as a correspondence assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ and to every pair of embedded affine open subschemes $V \subset U \subset X$ an isomorphism of $\mathcal{O}_X(V)$-modules

$$\mathcal{F}(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U).$$

The obvious compatibility condition for three embedded affine open subschemes $W \subset V \subset U \subset X$ needs to be imposed.

Analogously, a contraherent cosheaf $\mathfrak{P}$ on a scheme X is a correspondence assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ and to every pair of embedded affine open subschemes $V \subset U \subset X$ an isomorphism of $\mathcal{O}_X(V)$-modules

$$\mathfrak{P}[V] \simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]).$$

The difference with the quasi-coherent case is that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is always flat, but not necessarily projective. So to make one's contraherent cosheaves well-behaved, one has to impose the additional Ext-vanishing condition

$$\operatorname{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$$

for all affine open subschemes $V \subset U \subset X$. Notice that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ always has projective dimension not exceeding 1, so the condition on Ext^1 is sufficient.

This nonprojectivity problem is the reason why one does not have an abelian category of contraherent cosheaves. Imposing the Ext-vanishing requirement allows to obtain, at least, an exact one. Given a rule assigning to affine open subschemes $U \subset X$ the $\mathcal{O}_X(U)$-modules $\mathfrak{P}[U]$ together with the isomorphisms for embedded affine open subschemes $V \subset U$ as above, and assuming that the Ext^1-vanishing condition holds, one can show that $\mathfrak{P}$ satisfies the cosheaf axioms for coverings of affine open subschemes of X by other affine open subschemes. Then a general result from [26] says that $\mathfrak{P}$ extends uniquely from the base of affine open subschemes to a cosheaf of $\mathcal{O}_X$-modules defined, as it should be, on all the open subsets of X.

A module P over a commutative ring R is called *contraadjusted* if $\mathrm{Ext}^1_R(R[s^{-1}],P) = 0$ for all $s \in R$. This is equivalent to the vanishing of $\mathrm{Ext}^1_R(S,P)$ for all the R-algebras S of functions on the affine open subschemes of $\operatorname{Spec} R$. More generally, a left module P over a (not necessarily commutative) ring R is said to be *cotorsion* if $\mathrm{Ext}^1_R(F,P) = 0$ (or equivalently, $\mathrm{Ext}^{>0}_R(F,P) = 0$) for any flat left R-module F. It has been proven that cotorsion modules are "numerous enough" (see [18, 6]); one can prove the same for contraadjusted modules in the similar way.

Any quotient module of a contraadjusted module is contraadjusted, so the "contra-adjusted dimension" of any module does not exceed 1; while the cotorsion dimension of a module may be infinite if the projective dimensions of flat modules are. This makes the contraadjusted modules useful when working with schemes that are not necessarily Noetherian of finite Krull dimension.

Modules of the complementary class to the contraadjusted ones (in the same way as the flats are complementary to the cotorsion modules) we call *very flat*. All very flat modules have projective dimensions not exceeding 1. The wide applicability of contraadjusted and very flat modules (cf. [15, Remark 2.6]) implies the importance of very flat morphisms of schemes, and we initiate the study of these (though our results in this direction are still far from what one would hope for).

The above-discussed phenomenon of dependence of the category of contramodules over the coring $\mathcal{C} = \mathcal{O}(T \times_X T)$ over the ring $A = \mathcal{O}(T)$ on the affine covering $T \longrightarrow X$ used to construct it manifests itself in our approach in the unexpected predicament of the *contraherence property* of a cosheaf of $\mathcal{O}_X$-modules *being not local*. So we have to deal with the *locally contraherent cosheaves*, and the necessity to control the extension of this locality brings the coverings back.

Once an open covering in restriction to which our cosheaf becomes contraherent is safely fixed, though, many other cosheaf properties that we consider in this paper become indeed local. And any locally contraherent cosheaf on a quasi-compact semi-separated scheme has a finite left Čech resolution by contraherent cosheaves.

One difference between homological theories developed in the settings of exact and abelian categories is that whenever a functor between abelian categories isn't exact, a similar functor between exact categories will tend to have a shrinked domain. Any functor between abelian categories that has an everywhere defined left or right derived functor will tend to be everywhere defined itself, if only because one can always pass to the degree-zero cohomology of the derived category objects. Not so with exact categories, in which complexes may have no cohomology objects in general.

Hence the (sometimes annoying) necessity to deal with multitudes of domains of definitions of various functors in our exposition. On the other hand, a functor with the domain consisting of adjusted objects is typically exact on the exact subcategory where it is defined.

Another difference between the theory of comodules and contramodules over corings as developed in [52] and our present setting is that in *loc. cit.* we considered corings $\mathcal{C}$ over base rings A of finite homological dimension. On the other hand, the ring $A = \mathcal{O}(T)$ constructed above has infinite homological dimension in most cases, while the coring $\mathcal{C} = \mathcal{O}(T \times_X T)$ can be said to have "finite homological dimension relative to A". For this reason, while the comodule-contramodule correspondence theorem [52, Theorem 5.4] was stated for the derived categories of the second kind, one of the most general of our co-contra correspondence results in this paper features an equivalence of the conventional derived categories.

One application of this equivalence is a new construction of the extraordinary inverse image functor $f^!$ on the derived categories of quasi-coherent sheaves for any morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$. To be more precise, this is a construction of what we call *Neeman's extraordinary inverse image functor*, that is the right adjoint functor $f^!$ to the derived direct image functor $\mathbb{R}\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ on the derived categories of quasi-coherent sheaves (see discussion below). In fact, we construct a *right derived* functor $\mathbb{R}f^!$, rather than just a triangulated functor $f^!$, as it was usually done before [30, 47].

To a morphism f one assigns the direct and inverse image functors $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ and $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ between the abelian categories of quasi-coherent sheaves on X and Y; the functor f^* is left adjoint to the functor f_*. To the same morphism, one also assigns the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ between the exact categories of colocally projective contraherent cosheaves and the inverse image functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ between the exact categories of locally injective locally contraherent cosheaves on X and Y. The functor $f^!$ is "partially" right adjoint to the functor $f_!$.

Passing to the derived functors, one obtains the adjoint functors $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ between the (conventional unbounded) derived categories of quasi-coherent cosheaves. One also obtains the adjoint functors $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ between the derived categories of contraherent cosheaves on X and Y.

The derived co-contra correspondence (for the conventional derived categories) provides equivalences of triangulated categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh})$ transforming the direct image functor $\mathbb{R}f_*$ into the direct image functor $\mathbb{L}f_!$. So the two inverse image functors $\mathbb{L}f^*$ and $\mathbb{R}f^!$ can be viewed as the adjoints on the two sides to the same triangulated functor of direct image. This finishes our construction of the triangulated functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ right adjoint to $\mathbb{R}f_*$.

As usually in homological algebra, the tensor product and Hom-type operations on the quasi-coherent sheaves and contraherent cosheaves play an important role in our theory. First of all, under appropriate adjustness assumptions one can assign a contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F}, \mathfrak{P})$ to a quasi-coherent sheaf $\mathcal{F}$ and a contraherent cosheaf $\mathfrak{P}$ over a scheme X. This operation is the analogue of the tensor product of quasi-coherent sheaves in the contraherent world.

Secondly, to a quasi-coherent sheaf $\mathcal{F}$ and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ over a scheme X one can assign a cosheaf of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{\mathcal{O}_X} \mathfrak{P}$. Under our duality-analogy, this corresponds to taking the sheaf of $\mathcal{H}om$ from a quasi-coherent sheaf to a sheaf of $\mathcal{O}_X$-modules. When the scheme X is Noetherian, the sheaf $\mathcal{F}$ is coherent, and the cosheaf $\mathfrak{P}$ is contraherent, the cosheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathfrak{P}$ is contraherent, too. Under some other assumptions one can apply the (derived or underived) *contraherator* functor $\mathbb{L}\,\mathfrak{C}$ or $\mathfrak{C}$ to the cosheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathfrak{P}$ to obtain the (complex of) contraherent cosheaves $\mathcal{F} \otimes^{\mathbb{L}}_{X\text{-ct}} \mathfrak{P}$ or $\mathcal{F} \otimes_{X\text{-ct}} \mathfrak{P}$. These are the analogues of the quasi-coherent internal Hom functor $\mathcal{H}om_{X\text{-qc}}$ on the quasi-coherent sheaves, which can be obtained by applying the coherator functor $\mathcal{Q}$ [65, Appendix B] to the $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ sheaf.

The remaining two operations are harder to come by. Modelled after the comodule-contramodule correspondence functors $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ from [52], they play a similarly crucial role in our present co-contra correspondence theory. Given a quasi-coherent sheaf $\mathcal{F}$ and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a quasi-separated scheme X, one constructs a quasi-coherent sheaf $\mathcal{F} \odot_X \mathfrak{P}$ on X. Given two quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{P}$ on a quasi-separated scheme, under certain adjustness assuptions one can constuct a contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F}, \mathcal{P})$.

Derived categories of the second kind, whose roots go back to the work of Husemoller, Moore, and Stasheff on two kinds of differential derived functors [32] and Hinich's paper about DG-coalgebras [31], were introduced in their present form in the author's monograph [52] and memoir [53]. The most important representatives of this class of derived category constructions are known as the *coderived* and the *contraderived* categories; the difference between them consists in the use of the closure with respect to infinite direct sums in one case and with respect to infinite products in the other. Here is a typical example of how they occur.

According to Iyengar and Krause [34], the homotopy category of complexes of projective modules over a Noetherian commutative ring with a dualizing complex is equivalent to the homotopy category of complexes of injective modules. This theorem was extended to semi-separated Noetherian schemes with dualizing complexes by Neeman [49] and Murfet [44] in the following form: the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ of the exact category of flat quasi-coherent sheaves on such a scheme X is equivalent to the homotopy category of injective quasi-coherent sheaves $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$. These results are known as the *covariant Serre–Grothendieck duality* theory.

One would like to reformulate this equivalence so that it connects certain derived categories of modules/sheaves, rather than just subcategories of resolutions. In other words, it would be nice to have some procedure assigning complexes of projective, flat, and/or injective modules/sheaves to arbitrary complexes.

In the case of modules, the homotopy category of projectives is identified with the contraderived category of the abelian category of modules, while the homotopy category of injectives is equivalent to the coderived category of the same abelian category. Hence the Iyengar–Krause result is interpreted as an instance of the "co-contra correspondence"—in this case, an equivalence between the coderived and contraderived categories of the same abelian category.

In the case of quasi-coherent sheaves, however, only a half of the above picture remains true. The homotopy category of injectives $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ is still equivalent to the coderived category of quasi-coherent sheaves $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$. But the attempt to similarly describe the derived category of flats runs into the problem that the infinite products of quasi-coherent sheaves are not exact, so the contraderived category construction does not make sense for them.

This is where the contraherent cosheaves come into play. The covariant Serre–Grothendieck duality for a nonaffine (but semi-separated) scheme with a dualizing complex is an equivalence of *four* triangulated categories, rather than just two. In addition to the derived category of flat quasi-coherent sheaves $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ and the homotopy category of injective quasi-coherent sheaves $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$, there are also the homotopy category of projective contraherent cosheaves $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ and the derived category of locally injective contraherent cosheaves $\mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$.

Just as the homotopy category of injectives $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ is equivalent to the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, the homotopy category of projectives $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ is identified with the contraderived category $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ of contraherent cosheaves. The equivalence between the two "injective" categories $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ does not depend on the dualizing complex, and neither does the equivalence between the two "projective" (or "flat") categories $\mathsf{D}(X\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$. The equivalences connecting the "injective" categories with the "projective" ones do.

So far, our discussion in this introduction was essentially limited to semi-separated schemes. Such a restriction of generality is not really that natural or desirable. Indeed, the affine plane with a double point (which is not semi-separated) is no less a worthy object of study than the line with a double point (which is). There is a problem, however, that appears to stand in the way of a substantial development of the theory of contraherent cosheaves on the more general kinds of schemes.

The problem is that even when the quasi-coherent sheaves on schemes remain well-behaved objects, the complexes of such sheaves may be no longer so. More precisely, the trouble is with the derived functors of direct image of complexes of quasi-coherent sheaves, which do not always have good properties (such as locality along the base, etc.) Hence the common wisdom that for the more complicated schemes X one is supposed to consider complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves, rather than complexes of quasi-coherent sheaves as such (see, e. g., [59]). As we do not know what is supposed to be either a second kind or a contraherent analogue of the construction of the derived category of complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves, we have to restrict our exposition to, approximately, those situations where the derived category of the abelian category of quasi-coherent sheaves on X is still a good category to work in.

There are, basically, two such situations [65, Appendix B]: (1) the quasi-compact semi-separated schemes and (2) the Noetherian or, sometimes, locally Noetherian schemes (which, while always quasi-separated, do not have to be semi-separated). Accordingly, our exposition largely splits in two streams corresponding to the situations (1) and (2), where different techniques are applicable. The main difference in the generality level with the quasi-coherent case is that in the contraherent context one often needs also to assume one's schemes to have finite Krull dimension, in order to use Raynaud and Gruson's homological dimension results [58].

In particular, we prove the equivalence of the conventional derived categories of quasi-coherent sheaves and contraherent cosheaves not only for quasi-compact semi-separated schemes, but also, separately, for all Noetherian schemes of finite Krull dimension. As to the covariant Serre–Grothendieck duality theorem, it remains valid in the case of a non-semi-separated Noetherian scheme with a dualizing complex in the form of an equivalence between two derived categories of the second kind: the coderived category of quasi-coherent sheaves $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ and the contraderived category of contraherent cosheaves $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$.

Hartshorne's theory of injective quasi-coherent sheaves on locally Noetherian schemes [30], including the assertions that injectivity of quasi-coherent sheaves on locally Noetherian schemes is a local property and such sheaves are flasque, as well as their classification as direct sums of direct images from the embeddings of the scheme points, is an important technical tool of the quasi-coherent sheaf theory. We obtain dual-analogous results for projective locally cotorsion contraherent cosheaves on locally Noetherian schemes, proving that projectivity of locally cotorsion contraherent cosheaves on such schemes is a local property, that such projective cosheaves are coflasque, and classifying them as products of direct images from the scheme points. Just as Hartshorne's theory is based on Matlis' classification of injective modules over Noetherian rings [41] (cf., however, [15, Section A.3]), our results on projective locally cotorsion contraherent cosheaves are based on Enochs' classification of flat cotorsion modules over commutative Noetherian rings [19].

Before describing another application of our theory, let us have a more detailed discussion of the extraordinary inverse images of quasi-coherent sheaves. There are, in fact, *two* different functors going by the name of "the functor $f^!$" in the literature. One of them, which we name after Neeman, is simply the functor right adjoint to the derived direct image $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$, which we were discussing above. Neeman proved its existence for an arbitrary morphism of quasi-compact semi-separated schemes, using the techniques of compact generators and Brown representability [47]. Similar arguments apply in the case of an arbitrary morphism of Noetherian schemes $f\colon Y \longrightarrow X$.

The other functor, which we call *Deligne's extraordinary inverse image* and denote (to avoid ambiguity) by f^+, coincides with Neeman's functor in the case of a proper morphism f. In the case when f is an open embedding, on the other hand, the functor f^+ coincides with the conventional restriction (inverse image) functor f^* (which is left adjoint to $\mathbb{R}f_*$, rather than right adjoint). More generally, in the case of a smooth morphism f the functor f^+ only differs from f^* by a dimensional shift and a top form bundle twist. This is the functor that was constructed in Hartshorne's book [30] and Deligne's appendix to it [14] (hence the name). It is Deligne's, rather than Neeman's, extraordinary inverse image functor that takes a dualizing complex on X to a dualizing complex on Y, i. e., $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$.

It is an important idea apparently due to Gaitsgory [22] that Deligne's extraordinary inverse image functor actually acts between the coderived categories of quasi-coherent sheaves, rather than between their conventional derived categories. To be sure, Hartshorne and Deligne only construct their functor for bounded below complexes (for which there is no difference between the coderived and derived categories). Gaitsgory's "ind-coherent sheaves" are closely related to our coderived category of quasi-coherent sheaves.

Concerning Neeman's right adjoint functor $f^!$, it can be shown to exist on both the derived and (in the case of Noetherian schemes) the coderived categories. The problem arises when one attempts to define the functor f^+ by decomposing a morphism $f\colon Y \longrightarrow X$ into proper morphisms g and open embeddings h and subsequently composing the functors $g^!$ for the former with the functors h^* for the latter. It just so happens that the functor $\mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ obtained in this way depends on the chosen decomposition of the morphism f. This was demonstrated by Neeman in his counterexample [47, Example 6.5].

The arguments of Deligne [14], if extended to the conventional unbounded derived categories, break down on a rather subtle point: while it is true that the restriction of an injective quasi-coherent sheaf to an open subscheme of a Noetherian scheme remains injective, the restriction of a homotopy injective complex of quasi-coherent sheaves to such a subscheme may no longer be homotopy injective. On the other hand, Deligne computes the Hom into the object produced by his extraordinary inverse image functor from an arbitrary bounded complex of coherent sheaves, which is essentially sufficient to make a functor between the coderived categories well-defined, as these are compactly generated by bounded complexes of coherents.

The above discussion of the functor f^+ is to be compared with the remark that the conventional derived inverse image functor $\mathbb{L}f^*$ is not defined on the coderived categories of quasi-coherent sheaves (but only on their derived categories), except in the case of a morphism f of finite flat dimension [15, 22]. On the other hand, the conventional inverse image f^* is perfectly well defined on the derived categories of flat quasi-coherent sheaves (where one does not even need to resolve anything in order to construct a triangulated functor). We show that the functor $f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is transformed into the functor $f^+\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ by the above-described equivalence of triangulated categories.

Let us finally turn to the connection between Deligne's extrardinary inverse image functor and our contraherent cosheaves. Given a morphism of Noetherian schemes $f\colon Y \longrightarrow X$, just as the direct image functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ has a right adjoint functor $f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$, so does the direct image functor $\mathbb{L}f_!\colon \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ has a left adjoint functor $f^*\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh})$. This is the contraherent analogue of Neeman's extraordinary inverse image functor for quasi-coherent sheaves.

Now assume that f is a morphism of finite type and the scheme X has a dualizing complex $\mathcal{D}_X^\bullet$; set $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$. As we mentioned above, the choice of the dualizing complexes induces equivalences of triangulated categories $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ and similarly for Y. Those equivalences of categories transform the functor $f^*\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh})$ into a certain functor $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow$

$\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$. It is the latter functor that turns out to be isomorphic to Deligne's extraordinary inverse image functor (which we denote here by f^+).

Two proofs of this result, working on slightly different generality levels, are given in this paper. One applies to "compactifiable" separated morphisms of finite type between semi-separated Noetherian schemes and presumes comparison with a Deligne-style construction of the functor f^+, involving a decomposition of the morphism f into an open embedding followed by a proper morphism [14]. The other one is designed for "embeddable" morphisms of finite type between Noetherian schemes and the comparison with a Hartshorne-style construction of f^+ based on a factorization of f into a finite morphism followed by a smooth one [30].

To end, let us describe some prospects for future research and applications of contraherent cosheaves. One of such expected applications is related to the $\mathfrak{D}$–$\mathbf{\Omega}$ duality theory. Here $\mathfrak{D}$ stands for the sheaf of rings of differential operators on a smooth scheme and $\mathbf{\Omega}$ denotes the de Rham DG-algebra. The derived $\mathfrak{D}$–$\mathbf{\Omega}$ duality, as formulated in [53, Appendix B] (see also [60] for a further development), happens on two sides. On the "co" side, the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathbf{\Omega},-)$ takes right $\mathfrak{D}$-modules to right DG-modules over $\mathbf{\Omega}$, and there is the adjoint functor $-\otimes_{\mathcal{O}_X}\mathfrak{D}$. These functors induce an equivalence between the derived category of $\mathfrak{D}$-modules and the coderived category of DG-modules over $\mathbf{\Omega}$.

On the "contra" side, over an affine scheme X the functor $\mathbf{\Omega}(X)\otimes_{\mathcal{O}(X)}-$ takes left $\mathfrak{D}$-modules to left DG-modules over $\mathbf{\Omega}$, and the adjoint functor is $\operatorname{Hom}_{\mathcal{O}(X)}(\mathfrak{D}(X),-)$. The induced equivalence is between the derived category of $\mathfrak{D}(X)$-modules and the contraderived category of DG-modules over $\mathbf{\Omega}(X)$. One would like to extend the "contra" side of the story to nonaffine schemes using the contraherent cosheafification, as opposed to the quasi-coherent sheafification on the "co" side. The need for the contraherent cosheaves arises, once again, because the contraderived category construction does not make sense for quasi-coherent sheaves. The contraherent cosheaves are designed for being plugged into it.

Another direction in which we would like to extend the theory presented below is that of Noetherian formal schemes, and more generally, ind-schemes of ind-finite or even ind-infinite type. The idea is to define an exact category of contraherent cosheaves of contramodules over a formal scheme that would serve as the natural "contra"-side counterpart to the abelian category of quasi-coherent torsion sheaves on the "co" side. (For a taste of the contramodule theory over complete Noetherian rings, the reader is referred to [55, Appendix B]; see also [56].)

Moreover, one would like to have a "semi-infinite" version of the homological formalism of quasi-coherent sheaves and contraherent cosheaves (see Preface to [52] for the related speculations). A possible setting might be that of an ind-scheme flatly fibered over an ind-Noetherian ind-scheme with quasi-compact schemes as the fibers. One would define the semiderived categories of quasi-coherent sheaves and contraherent cosheaves on such an ind-scheme as mixtures of the co/contraderived categories along the base ind-scheme and the conventional derived categories along the fibers.

The idea is to put two our (present) co-contra correspondence theorems "on top of" one another. That is, join the equivalence of conventional derived categories of quasi-coherent sheaves and contraherent cosheaves on a quasi-compact semi-separated scheme together with the equivalence between the coderived category of quasi-coherent (torsion) sheaves and the contraderived category of contraherent cosheaves (of contramodules) on an (ind-)Noetherian (ind-)scheme with a dualizing complex into a single "semimodule-semicontramodule correspondence" theorem claiming an equivalence between the two semiderived categories.

What we have in mind here is a theory that might bear the name of a "semi-infinite homological theory of doubly infinite-dimensional algebraic varieties", or *semi-infinite algebraic geometry*. In addition to the above-described correspondence between the complexes of sheaves and cosheaves, this theory would also naturally feature a double-sided derived functor of semitensor (or "mixed tensor-cotensor") product of complexes of quasi-coherent torsion sheaves (cf. the discussion of cotensor products of quasi-coherent sheaves in [15, Section B.2]), together with its contraherent analogue. We refer to the introduction to the paper [57] for a further duscussion.

Furthermore, one might hope to join the de Rham DG-module and the contramodule stories together in a single theory by considering contraherent cosheaves of DG-contramodules over the de Rham–Witt complex, with an eye to applications to crystalline sheaves and the p-adic Hodge theory. As compared so such high hopes, our real advances in this paper are quite modest.

The very possibility of having a meaningful theory of contraherent cosheaves on a scheme rests on there being enough contraadjusted or cotorsion modules over a commutative or associative ring. The contemporary-style set-theoretical proofs of these results [18, 6, 20, 61, 5] only seem to work in abelian (or exact) categories with exact functors of filtered inductive limits. Thus the first problem one encounters when trying to build up the theories of contraherent cosheaves of contramodules is the necessity of developing applicable techniques for constructing flat, cotorsion, contraadjusted, and very flat contramodules to be used in the resolutions.

In the present version of the paper, we partially overcome this obstacle by constructing enough flat, cotorsion, contraadjusted, and very flat contramodules over a Noetherian ring in the adic topology, enough flat and cotorsion contramodules over a pro-Noetherian topological ring of totally finite Krull dimension, and also enough contraadjusted and very flat contramodules over the projective limit of a sequence of commutative rings and surjective morphisms between them with finitely generated kernel ideals. The constructions are based on the existence of enough objects of the respective classes in the conventional categories of modules, which is used as a given fact, and also on the older, more explicit construction [67] of cotorsion resolutions of modules over commutative Noetherian rings of finite Krull dimension.

This provides the necessary background for possible definitions of locally contraadjusted or locally cotorsion contraherent cosheaves of contramodules over Noetherian formal schemes or ind-Noetherian ind-schemes of totally finite Krull dimension, and also of locally contraadjusted contraherent cosheaves over ind-schemes of a more general nature, such as, e. g., the projectivization of an infinite-dimensional discrete vector space, or the total space of such projective space's cotangent bundle. The latter might be a typical kind of example for which one would like to have the "semi-infinite algebraic geometry" worked out. And among the ind-Noetherian ind-schemes of totally finite Krull dimension there are, e. g., the spectra of the (sheaves of) rings of Witt vectors of algebraic varieties in finite characteristic.

This kind of work, intended to prepare ground for future theory building, is presently relegated to appendices. There we also construct the derived co-contra correspondence (an equivalence between the coderived category of torsion modules and the contraderived category of contramodules) over affine Noetherian formal schemes and ind-affine ind-Noetherian ind-schemes with dualizing complexes. Another appendix is devoted to the co-contra correspondence over noncommutative semi-separated stacks (otherwise known as flat corings [37]).

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1. Contraadjusted and Cotorsion Modules

1.1. **Contraadjusted and very flat modules.** Let R be a commutative ring. We will say that an R-module P is *contraadjusted* if the R-module $\operatorname{Ext}^1_R(R[r^{-1}],P)$ vanishes for every element $r\in R$. An R-module F is called *very flat* if one has $\operatorname{Ext}^1_R(F,P)=0$ for every contraadjusted R-module P.

By the definition, any injective R-module is contraadjusted and any projective R-module is very flat. Notice that the projective dimension of the R-module $R[r^{-1}]$ never exceeds 1, as it has a natural two-term free resolution $0\longrightarrow\bigoplus_{n=0}^\infty R\longrightarrow\bigoplus_{n=0}^\infty R\longrightarrow R[r^{-1}]\longrightarrow 0$. It follows that any quotient module of a contraadjusted module is contraadjusted, and one has $\operatorname{Ext}^{>0}_R(F,P)=0$ for any very flat R-module F and contraadjusted R-module P.

Computing the Ext^1 in terms of the above resolution, one can more explicitly characterize contraadjusted R-modules as follows. An R-module P is contraadjusted if and only if for any sequence of elements p_0, p_1, p_2, … $\in P$ and $r\in R$ there exists a (not necessarily unique) sequence of elements q_0, q_1, q_2, … $\in P$ such that $q_i=p_i+rq_{i+1}$ for all $i\geqslant 0$.

Furthermore, the projective dimension of any very flat module is equal to 1 or less. Indeed, any R-module M has a two-term right resolution by contraadjusted modules, which can be used to compute $\mathrm{Ext}^*_R(F, M)$ for a very flat R-module F. (The converse assertion is *not* true, however; see Example 1.7.7 below.)

It is also clear that the classes of contraadjusted and very flat modules are closed under extensions. Besides, the class of very flat modules is closed under the passage to the kernel of a surjective morphism (i. e., the kernel of a surjective morphism of very flat R-modules is very flat). In addition, we notice that the class of contraadjusted R-modules is closed under infinite products, while the class of very flat R-modules is closed under infinite direct sums.

Theorem 1.1.1. (a) *Any R-module can be embedded into a contraadjusted R-module in such a way that the quotient module is very flat.*

(b) *Any R-module admits a surjective map onto it from a very flat R-module such that the kernel is contraadjusted.*

Proof. Both assertions follow from the results of Eklof and Trlifaj [18, Theorem 10]. It suffices to point out that all the R-modules of the form $R[r^{-1}]$ form a set rather than a proper class. For the reader's convenience and our future use, parts of the argument from [18] are reproduced below. $\square$

The following definition will be used in the sequel. Let A be an abelian category with exact functors of inductive limit, and let $\mathsf{C} \subset \mathsf{A}$ be a class of objects. An object $X \in \mathsf{A}$ is said to be a *transfinitely iterated extension* of objects from C if there exist a well-ordered set Γ and a family of subobjects $X_\gamma \subset X$, $\gamma \in \Gamma$, such that $X_\delta \subset X_\gamma$ whenever $\delta < \gamma$, the union (inductive limit) $\varinjlim_{\gamma\in\Gamma} X_\gamma$ of all X_γ coincides with X, and the quotient objects $X_\gamma / \varinjlim_{\delta<\gamma} X_\delta$ belong to C for all $\gamma \in \Gamma$ (cf. Section 4.1).

By [18, Lemma 1], any transfinitely iterated extension of the R-modules $R[r^{-1}]$, with arbitrary $r \in R$, is a very flat R-module. Proving a converse assertion will be one of our goals. The following lemma is a particular case of [18, Theorem 2].

Lemma 1.1.2. *Any R-module can be embedded into a contraadjusted R-module in such a way that the quotient module is a transfinitely iterated extension of the R-modules $R[r^{-1}]$.*

Proof. The proof is a set-theoretic argument based on the fact that the Cartesian square of any infinite cardinality λ is equicardinal to λ. In our case, let λ be any infinite cardinality no smaller than the cardinality of the ring R. For an R-module L of the cardinality not exceeding λ and an R-module M of the cardinality μ, the set $\mathrm{Ext}^1_R(L, M)$ has the cardinality at most μ^λ, as one can see by computing the Ext^1 in terms of a projective resolution of the first argument. In particular; set $\aleph = 2^\lambda$; then for any R-module M of the cardinality not exceeding $\aleph$ and any $r \in R$ the cardinality of the set $\mathrm{Ext}^1_R(R[r^{-1}], M)$ does not exceed $\aleph$, either.

Let $\daleth$ be the smallest cardinality that is larger than $\aleph$ and let Δ be the smallest ordinal of the cardinality $\daleth$. Notice that the natural map $\varinjlim_{\delta\in\Delta} \mathrm{Ext}^*_R(L, Q_\delta) \longrightarrow \mathrm{Ext}^*_R(L, \varinjlim_{\delta\in\Delta} Q_\delta)$ is an isomorphism for any R-module L of the cardinality not exceeding λ (or even $\aleph$) and any inductive system of R-modules Q_δ indexed by Δ. Indeed, the functor of filtered inductive limit of abelian groups is exact and the natural map $\varinjlim_{\delta\in\Delta} \operatorname{Hom}_R(L, Q_\delta) \longrightarrow \operatorname{Hom}_R(L, \varinjlim_{\delta\in\Delta} Q_\delta)$ is an isomorphism for any R-module L of the cardinality not exceeding $\aleph$. The latter assertion holds because the image of any map of sets $L \longrightarrow \Delta$ is contained in a proper initial segment $\{\delta' \mid \delta' < \delta\} \subset \Delta$ for some $\delta \in \Delta$.

We proceed by induction on Δ constructing for every element $\delta \in \Delta$ an R-module P_δ and a well-ordered set Γ_δ. For any $\delta' < \delta$, we will have an embedding of R-modules $P_{\delta'} \longrightarrow P_\delta$ (such that the three embeddings form a commutative diagram for any three elements $\delta'' < \delta' < \delta$) and an ordered embedding $\Gamma_{\delta'} \subset \Gamma_\delta$ (making $\Gamma_{\delta'}$ an initial segment of Γ_δ). Furthermore, for every element $\gamma \in \Gamma_\delta$, a particular extension class $c(\gamma, \delta) \in \operatorname{Ext}^1_R(R[r(\gamma)^{-1}], P_\delta)$, where $r(\gamma) \in R$, will be defined. For every $\delta' < \delta$ and $\gamma \in \Gamma_{\delta'}$, the class $c(\gamma, \delta)$ will be equal to the image of the class $c(\gamma, \delta')$ with respect to the natural map $\operatorname{Ext}^1_R(R[r(\gamma)^{-1}], P_{\delta'}) \longrightarrow \operatorname{Ext}^1_R(R[r(\gamma)^{-1}], P_\delta)$ induced by the embedding $P_{\delta'} \longrightarrow P_\delta$.

At the starting point $0 \in \Delta$, the module P_0 is our original R-module M and the set Γ_0 is the disjoint union of all the sets $\operatorname{Ext}^1_R(R[r^{-1}], M)$ with $r \in R$, endowed with an arbitrary well-ordering. The elements $r(\gamma)$ and the classes $c(\gamma, 0)$ for $\gamma \in \Gamma_0$ are defined in the obvious way.

Given $\delta = \delta' + 1 \in \Delta$ and assuming that the R-module $P_{\delta'}$ and the set $\Gamma_{\delta'}$ have been constructed already, we produce the module P_δ and the set Γ_δ as follows. It will be clear from the construction below that δ' is always smaller than the well-ordering type of the set $\Gamma_{\delta'}$. So there is a unique element $\gamma_{\delta'} \in \Gamma_{\delta'}$ corresponding to the ordinal δ' (i. e., such that the well-ordering type of the subset all the elements in $\Gamma_{\delta'}$ that are smaller than $\gamma_{\delta'}$ is equivalent to δ').

Define the R-module P_δ as the middle term of the extension corresponding to the class $c(\gamma_{\delta'}, \delta') \in \operatorname{Ext}^1_R(R[r(\gamma_{\delta'})^{-1}], P_{\delta'})$. There is a natural embedding $P_{\delta'} \longrightarrow P_\delta$, as required. Set Γ_δ to be the disjoint union of $\Gamma_{\delta'}$ and the sets $\operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ with $r \in R$, well-ordered so that $\Gamma_{\delta'}$ is an initial segment, while the well-ordering of the remaining elements is chosen arbitrarily. The elements $r(\gamma)$ for $\gamma \in \Gamma_{\delta'}$ have been defined already on the previous steps and the classes $c(\gamma, \delta)$ for such γ are defined in the unique way consistent with the previous step, while for the remaining $\gamma \in \Gamma_\delta \setminus \Gamma_{\delta'}$ these elements and classes are defined in the obvious way.

When δ is a limit ordinal, set $P_\delta = \varinjlim_{\delta'<\delta} P'_\delta$. Let Γ_δ be the disjoint union of $\bigcup_{\delta'<\delta} \Gamma_{\delta'}$ and the sets $\operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ with $r \in R$, well-ordered so that $\bigcup_{\delta'<\delta} \Gamma_{\delta'}$ is an initial segment. The elements $r(\gamma)$ and $c(\gamma, \delta)$ for $\gamma \in \Gamma_\delta$ are defined as above.

Arguing by transfinite induction, one easily concludes that the cardinality of the R-module P_δ never exceeds $\aleph$ for $\delta \in \Delta$, and neither does the cardinality of the set Γ_δ. It follows that the well-ordering type of the set $\Gamma = \bigcup_{\delta\in\Delta} \Gamma_\delta$ is equal to Δ. So for every $\gamma \in \Gamma$ there exists $\delta \in \Delta$ such that $\gamma = \gamma_\delta$.

Set $P = \varinjlim_{\delta\in\Delta} P_\delta$. By construction, there is a natural embedding of R-modules $M \longrightarrow P$ and the cokernel is a transfinitely iterated extension of the R-modules $R[r^{-1}]$. As every class $c \in \operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ corresponds to an element $\gamma \in \Gamma_\delta$, has the corresponding ordinal $\delta' \in \Delta$ such that $\gamma = \gamma_{\delta'}$, and dies in $\operatorname{Ext}^1_R(R[r^{-1}], P_{\delta'+1})$, we conclude that $\operatorname{Ext}^1_R(R[r^{-1}], P) = 0$. ☐

Lemma 1.1.3. *Any R-module admits a surjective map onto it from a transfinitely iterated extension of the R-modules $R[r^{-1}]$ such that the kernel is contraadjusted.*

Proof. The proof follows the second half of the proof of Theorem 10 in [18]. Specifically, given an R-module M, pick a surjective map onto it from a free R-module L. Denote the kernel by K and embed it into a contraadjusted R-module P so that the quotient module Q is a transfinitely iterated extension of the R-modules $R[r^{-1}]$. Then the fibered coproduct F of the R-modules L and P over K is an extension of the R-modules Q and L. It also maps onto M surjectively with the kernel P. □

Both assertions of Theorem 1.1.1 are now proven.

Corollary 1.1.4. *An R-module is very flat if and only if it is a direct summand of a transfinitely iterated extension of the R-modules $R[r^{-1}]$.*

Proof. The "if" part has been explained already; let us prove "only if". Given a very flat R-module F, present it as the quotient module of a transfinitely iterated extension E of the R-modules $R[r^{-1}]$ by a contraadjusted R-module P. Since $\operatorname{Ext}^1_R(F,P)=0$, we can conclude that F is a direct summand of E. □

In particular, we have proven that any very flat R-module is flat.

Corollary 1.1.5. (a) *Any very flat R-module can be embedded into a contraadjusted very flat R-module in such a way that the quotient module is very flat.*

(b) *Any contraadjusted R-module admits a surjective map onto it from a very flat contraadjusted R-module such that the kernel is contraadjusted.*

Proof. Follows from Theorem 1.1.1 and the fact that the classes of contraadjusted and very flat R-modules are closed under extensions. □

1.2. **Affine geometry of contraadjusted and very flat modules.** The results of this section form the module-theoretic background of our main definitions and constructions in Sections 2–3.

Lemma 1.2.1. (a) *The class of very flat R-modules is closed with respect to the tensor products over R.*

(b) *For any very flat R-module F and contraadjusted R-module P, the R-module* $\operatorname{Hom}_R(F,P)$ *is contraadjusted.*

Proof. One approach is to prove both assertions simultaneously using the adjunction isomorphism $\operatorname{Ext}^1_R(F\otimes_R G,\ P)\simeq \operatorname{Ext}^1_R(G,\operatorname{Hom}_R(F,P))$, which clearly holds for any R-module G, any very flat R-module F, and contraadjusted R-module P, and raising the generality step by step. Since $R[r^{-1}]\otimes_R R[s^{-1}]\simeq R[(rs)^{-1}]$, it follows that the R-module $\operatorname{Hom}_R(R[s^{-1}],P)$ is contraadjusted for any contraadjusted R-module P and $s\in R$. Using the same adjunction isomorphism, one then concludes that the R-module $R[s^{-1}]\otimes_R G$ is very flat for any very flat R-module G. From this one can deduce in full generality the assertion (b), and then the assertion (a).

Alternatively, one can use the full strength of Corollary 1.1.4 and check that the tensor product of two transfinitely iterated extensions of flat modules is a transfinitely iterated extension of the pairwise tensor products. Then deduce (b) from (a). □

Lemma 1.2.2. *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings. Then*

(a) *any contraadjusted S-module is also a contraadjusted R-module in the R-module structure obtained by the restriction of scalars via f;*

(b) *if F is a very flat R-module, then the S-module $S\otimes_R F$ obtained by the extension of scalars via f is also very flat;*

(c) *if F is a very flat R-module and Q is a contraadjusted S-module, then $\operatorname{Hom}_R(F,Q)$ is also a contraadjusted S-module;*

(d) *if F is a very flat R-module and G is a very flat S-module, then $F\otimes_R G$ is also a very flat S-module.*

Proof. Part (a): one has $\operatorname{Ext}^*_R(R[r^{-1}],P)\simeq \operatorname{Ext}^*_S(S[f(r)^{-1}],P)$ for any R-module P and $r\in R$. Part (b) follows from part (a), or alternatively, from Corollary 1.1.4. To prove part (c), notice that $\operatorname{Hom}_R(F,Q)\simeq \operatorname{Hom}_S(S\otimes_R F,\,Q)$ and use part (b) together with Lemma 1.2.1(b) (applied to the ring S). Similarly, part (d) follows from part (b) and Lemma 1.2.1(a). ☐

Lemma 1.2.3. *Let $f\colon R\longrightarrow S$ be a homomorphism of commutative rings such that the localization $S[s^{-1}]$ is a very flat R-module for any element $s\in S$. Then*

(a) *the S-module $\operatorname{Hom}_R(S,P)$ obtained by the coextension of scalars via f is contraadjusted for any contraadjusted R-module P;*

(b) *any very flat S-module is also a very flat R-module (in the R-module structure obtained by the restriction of scalars via f);*

(c) *the S-module $\operatorname{Hom}_R(G,P)$ is contraadjusted for any very flat S-module G and contraadjusted R-module P.*

Proof. Part (a): one has $\operatorname{Ext}^1_S(S[s^{-1}],\operatorname{Hom}_R(S,P))\simeq \operatorname{Ext}^1_R(S[s^{-1}],P)$ for any R-module P such that $\operatorname{Ext}^1_R(S,P)=0$ and any $s\in S$. Part (b) follows from part (a), or alternatively, from Corollary 1.1.4. Part (c): one has $\operatorname{Ext}^1_S(S[s^{-1}],\operatorname{Hom}_R(G,P))\simeq \operatorname{Ext}^1_R(G\otimes_S S[s^{-1}],\,P)$. By Lemma 1.2.1(a), the S-module $G\otimes_S S[s^{-1}]$ is very flat; by part (b), it is also a very flat R-module; so the desired vanishing follows. ☐

Lemma 1.2.4. *Let $R\longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes $\operatorname{Spec} S\longrightarrow \operatorname{Spec} R$ is an open embedding. Then S is a very flat R-module.*

Proof. The open subset $\operatorname{Spec} S\subset \operatorname{Spec} R$, being quasi-compact, can be covered by a finite number of principal affine open subsets $\operatorname{Spec} R[r_\alpha^{-1}]\subset \operatorname{Spec} R$, where $\alpha = 1,\dots,N$. The Čech sequence

$$(1)\quad 0\longrightarrow S\longrightarrow \bigoplus\nolimits_\alpha R[r_\alpha^{-1}]\longrightarrow \bigoplus\nolimits_{\alpha<\beta} R[(r_\alpha r_\beta)^{-1}] \longrightarrow\cdots\longrightarrow R[(r_1\cdots r_N)^{-1}]\longrightarrow 0$$

is an exact sequence of S-modules, since is its localization by every element r_α is exact. It remains to recall that the class of very flat R-modules is closed under the passage to the kernels of surjective morphisms. ☐

Corollary 1.2.5. *The following assertions hold in the assumptions of Lemma* 1.2.4.

(a) *The S-module $S \otimes_R F$ is very flat for any very flat R-module F.*

(b) *An S-module G is very flat if and only if it is very flat as an R-module.*

(c) *The S-module* $\operatorname{Hom}_R(S, P)$ *is contraadjusted for any contraadjusted R-module P.*

(d) *An S-module Q is contraadjusted if and only if it is contraadjusted as an R-module.*

Proof. Part (a) is a particular case of Lemma 1.2.2(b). Part (b): if G is a very flat S-module, then it is also very flat as an R-module by Lemma 1.2.3(b) and Lemma 1.2.4. Conversely, if G is very flat as an R-module, then $G \simeq S \otimes_R G$ is also a very flat S-module by part (a).

Part (c) follows from Lemma 1.2.3(a) and Lemma 1.2.4. Part (d): if Q is a contraadjusted S-module, then it is also contraadjusted as an R-module by Lemma 1.2.2(a). Conversely, for any S-module Q there are natural isomorphisms of S-modules $Q \simeq \operatorname{Hom}_S(S, Q) \simeq \operatorname{Hom}_S(S \otimes_R S,\ Q) \simeq \operatorname{Hom}_R(S, Q)$; and if Q is contraadjusted as an R-module, then it is also a contraadjusted S-module by part (c). □

Lemma 1.2.6. *Let $R \longrightarrow S_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is a finite open covering. Then*

(a) *an R-module F is very flat if and only if all the S_α-modules $S_\alpha \otimes_R F$ are very flat;*

(b) *for any contraadjusted R-module P, the Čech sequence*

$$(2)\quad 0 \longrightarrow \operatorname{Hom}_R(S_1 \otimes_R \cdots \otimes_R S_N,\ P) \longrightarrow \cdots \\ \longrightarrow \textstyle\bigoplus_{\alpha<\beta} \operatorname{Hom}_R(S_\alpha \otimes_R S_\beta,\ P) \longrightarrow \bigoplus_\alpha \operatorname{Hom}_R(S_\alpha,\ P) \longrightarrow P \longrightarrow 0$$

is an exact sequence of R-modules.

Proof. Part (a): by Corollary 1.2.5(a-b), all the R-modules $S_{\alpha_1} \otimes_R \cdots \otimes_R S_{\alpha_k} \otimes_R F$ are very flat whenever the S_α-modules $S_\alpha \otimes_R F$ are very flat. For any R-module F the Čech sequence

$$(3)\quad 0 \longrightarrow F \longrightarrow \textstyle\bigoplus_\alpha S_\alpha \otimes_R F \longrightarrow \bigoplus_{\alpha<\beta} S_\alpha \otimes_R S_\beta \otimes_R F \\ \longrightarrow \cdots \longrightarrow S_1 \otimes_R \cdots \otimes_R S_N \otimes_R F \longrightarrow 0$$

is an exact sequence of R-modules (since its localization at any prime ideal of R is). It remains to recall that the class of very flat R-modules is closed with respect to the passage to the kernels of surjections.

Part (b): the exact sequence of R-modules

$$(4)\quad 0 \longrightarrow R \longrightarrow \textstyle\bigoplus_\alpha S_\alpha \longrightarrow \bigoplus_{\alpha<\beta} S_\alpha \otimes_R S_\beta \\ \longrightarrow \cdots \longrightarrow S_1 \otimes_R \cdots \otimes_R S_N \longrightarrow 0$$

is composed from short exact sequences of very flat R-modules, so the functor $\operatorname{Hom}_R(-, P)$ into a contraadjusted R-module P preserves its exactness. □

Lemma 1.2.7. *Let $f_\alpha\colon S \longrightarrow T_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} T_\alpha \longrightarrow \operatorname{Spec} S$ *is a finite open covering, and let $R \longrightarrow S$ be a homomorphism of commutative rings. Then*

(a) *if all the R-modules $T_{\alpha_1} \otimes_S \dots \otimes_S T_{\alpha_k}$ are very flat, then the R-module S is very flat;*

(b) *the R-modules $T_\alpha[t_\alpha^{-1}]$ are very flat for all $t_\alpha \in T_\alpha$, $1 \leqslant \alpha \leqslant N$, if and only if the R-module $S[s^{-1}]$ is very flat for all $s \in S$.*

Proof. Part (a) follows from the Čech exact sequence (cf. (4))

$$0 \longrightarrow S \longrightarrow \bigoplus\nolimits_\alpha T_\alpha \longrightarrow \bigoplus\nolimits_{\alpha<\beta} T_\alpha \otimes_S T_\beta \\ \longrightarrow \cdots \longrightarrow T_1 \otimes_S \dots \otimes_S T_N \longrightarrow 0.$$

To prove the "if" assertion in (b), notice that $\operatorname{Spec} T_\alpha[t_\alpha^{-1}]$ as an open subscheme in $\operatorname{Spec} S$ can be covered by a finite number of principal affine open subschemes $\operatorname{Spec} S[s^{-1}]$, and the intersections of these are also principal open affines. The "only if" assertion follows from part (a) applied to the covering of the affine scheme $\operatorname{Spec} S[s^{-1}]$ by the affine open subschemes $\operatorname{Spec} T_\alpha[f_\alpha(s)^{-1}]$, together with the fact that the intersection of every subset of these open affines can be covered by a finite number of principal affine open subschemes of one of the schemes $\operatorname{Spec} T_\alpha$. ☐

Lemma 1.2.8. *Let $f_\alpha\colon S \longrightarrow T_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} T_\alpha \longrightarrow \operatorname{Spec} S$ *is a finite open covering, and let $R \longrightarrow S$ be a homomorphism of commutative rings. Let F be an S-module. Then*

(a) *if all the R-modules $T_{\alpha_1} \otimes_S \dots \otimes_S T_{\alpha_k} \otimes_S F$ are very flat, then the R-module F is very flat;*

(b) *the R-modules $T_\alpha[t_\alpha^{-1}] \otimes_S F$ are very flat for all $t_\alpha \in T_\alpha$, $1 \leqslant \alpha \leqslant N$, if and only if the R-module $F[s^{-1}]$ is very flat for all $s \in S$.*

Proof. Just as in the previous lemma, part (a) follows from the Čech exact sequence (3) constructed for the collection of morphisms of rings $S \longrightarrow T_\alpha$ and the S-module F. The proof of part (b) is similar to that of Lemma 1.2.7(b). ☐

1.3. **Cotorsion modules.** Let R be an associative ring. A left R-module P is said to be *cotorsion* [67, 20] if $\operatorname{Ext}_R^1(F, P) = 0$ for any flat left R-module F, or equivalently, $Ext_R^{>0}(F, P) = 0$ for any flat left R-module F. Clearly, the class of cotorsion left R-modules is closed under extensions and the passage to the cokernels of embeddings, and also under infinite products.

The following theorem, previously known essentially as the "flat cover conjecture", was proven by Eklof–Trlifaj [18] and Bican–Bashir–Enochs [6] (cf. our Theorem 1.1.1). The case of a Noetherian commutative ring R of finite Krull dimension was previously treated by Xu [67] (cf. Lemma 1.3.9 below).

Theorem 1.3.1. (a) *Any R-module can be embedded into a cotorsion R-module in such a way that the quotient module is flat.*

(b) *Any R-module admits a surjective map onto it from a flat R-module such that the kernel is cotorsion.* ☐

The following results concerning cotorsion (and injective) modules are similar to the results about contraadjusted modules presented in Section 1.2. With a possible exception of the last lemma, all of these are very well known.

Lemma 1.3.2. *Let R be a commutative ring. Then*

(a) *for any flat R-module F and cotorsion R-module P, the R-module* $\mathrm{Hom}_R(F,P)$ *is cotorsion;*

(b) *for any R-module M and any injective R-module J, the R-module* $\mathrm{Hom}_R(M,J)$ *is cotorsion;*

(c) *for any flat R-module M and any injective R-module J, the R-module* $\mathrm{Hom}_R(F,J)$ *is injective.*

Proof. One has $\mathrm{Ext}^1_R(G,\mathrm{Hom}_R(F,P))\simeq\mathrm{Ext}^1_R(F\otimes_R G,\ P)$ for any R-modules F, G, and P such that $\mathrm{Ext}^1_R(F,P)=0=\mathrm{Tor}_1^R(F,G)$. All the three assertions follow from this simple observation. ☐

Our next lemma is a generalization of Lemma 1.3.2 to the noncommutative case.

Lemma 1.3.3. *Let R and S be associative rings. Then*

(a) *for any R-flat R-S-bimodule F and any cotorsion left R-module P, the left S-module* $\mathrm{Hom}_R(F,P)$ *is cotorsion;*

(b) *for any R-S-bimodule M and injective left R-module J, the left S-module* $\mathrm{Hom}_R(M,J)$ *is cotorsion;*

(c) *for any S-flat R-S-bimodule F and any injective left R-module J, the left S-module* $\mathrm{Hom}_R(F,J)$ *is injective.*

Proof. One has $\mathrm{Ext}^1_S(G,\mathrm{Hom}_R(F,P))\simeq\mathrm{Ext}^1_R(F\otimes_S G,\ P)$ for any R-S-bimodule F, left S-module G, and left R-module P such that $\mathrm{Ext}^1_R(F,P)=0=\mathrm{Tor}_1^S(F,G)$. Besides, the tensor product $F\otimes_S G$ is flat over R if F is flat over R and G is flat over S. This proves (a); and (b-c) are even easier. ☐

Lemma 1.3.4. *Let $f\colon R\longrightarrow S$ be a homomorphism of associative rings. Then*

(a) *any cotorsion left S-module is also a cotorsion left R-module in the R-module structure obtained by the restriction of scalars via f;*

(b) *the left S-module* $\mathrm{Hom}_R(S,J)$ *obtained by coextension of scalars via f is injective for any injective left S-module J.*

Proof. Part (a): one has $\mathrm{Ext}^1_R(F,Q)\simeq\mathrm{Ext}^1_S(S\otimes_R F,\ Q)$ for any flat left R-module F and any left S-module Q. Part (b) is left to reader. ☐

Lemma 1.3.5. *Let $f\colon R\longrightarrow S$ be an associative ring homomorphism such that S is a flat left R-module in the induced R-module structure. Then*

(a) *the left S-module* $\mathrm{Hom}_R(S,P)$ *obtained by coextension of scalars via f is cotorsion for any cotorsion left R-module P;*

(b) *any injective right S-module is also an injective right R-module in the R-module structure obtained by the restriction of scalars via f.*

Proof. Part (a): one has $\operatorname{Ext}^1_S(F, \operatorname{Hom}_R(S,P)) \simeq \operatorname{Ext}^1_R(F,P)$ for any left R-module P such that $\operatorname{Ext}^1_R(S,P) = 0$ and any left S-module F. In addition, in the assumptions of Lemma any flat left S-module F is also a flat left R-module. □

Lemma 1.3.6. *Let $R \longrightarrow S_\alpha$ be a collection of commutative ring homomorphisms such that the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is an open covering. Then*

(a) *a contraadjusted R-module P is cotorsion if and only if all the contraadjusted S_α-modules* $\operatorname{Hom}_R(S_\alpha, P)$ *are cotorsion;*

(b) *a contraadjusted R-module J is injective if and only if all the contraadjusted S_α-modules* $\operatorname{Hom}_R(S_\alpha, J)$ *are injective.*

Proof. Part (a): the assertion "only if" follows from Lemma 1.3.5(a). To prove "if", use the Čech exact sequence (2) from Lemma 1.2.6(b). By Lemmas 1.3.4(a) and 1.3.5(a), all the terms of the sequence, except perhaps the rightmost one, are cotorsion R-modules, and since the class of cotorsion R-modules is closed under the cokernels of embeddings, it follows that the rightmost term is cotorsion as well.

Part (b) is proven in the similar way using parts (b) of Lemmas 1.3.4–1.3.5. □

Let R be a Noetherian commutative ring, $\mathfrak{p} \subset R$ be a prime ideal, $R_\mathfrak{p}$ denote the localization of R at $\mathfrak{p}$, and $\widehat{R}_\mathfrak{p}$ be the completion of the local ring $R_\mathfrak{p}$. We recall the notion of a *contramodule* over a topological ring, and in particular over a complete Noetherian local ring, defined in [55, Section 1 and Appendix B], and denote the abelian category of contramodules over a topological ring T by T–contra.

The restriction of scalars with respect to the natural ring homomorphism $R \longrightarrow \widehat{R}_\mathfrak{p}$ provides an exact forgetful functor $\widehat{R}_\mathfrak{p}$–$\mathsf{contra} \longrightarrow R$–$\mathsf{mod}$. It follows from [55, Theorem B.1.1(1)] that this functor is fully faithful. The following proposition is a particular case of the assertions of [55, Propositions B.10.1 and B.9.1].

Proposition 1.3.7. (a) *Any $\widehat{R}_\mathfrak{p}$-contramodule is a cotorsion R-module.*

(b) *Any free/projective $\widehat{R}_\mathfrak{p}$-contramodule is a flat cotorsion R-module.*

Proof. In addition to the cited results from [55], take into account Lemma 1.3.4(a) and the fact that any flat $R_\mathfrak{p}$-module is also a flat R-module. □

The following theorem is a restatement of the main result of Enochs' paper [19].

Theorem 1.3.8. *Let R be a Noetherian commutative ring. Then an R-module is flat and cotorsion if and only if it is isomorphic to an infinite product $\prod_\mathfrak{p} F_\mathfrak{p}$ of free contramodules $F_\mathfrak{p} = \widehat{R}_\mathfrak{p}[[X_\mathfrak{p}]] = \varprojlim_n \widehat{R}_\mathfrak{p}/\mathfrak{p}^n[X]$ over the complete local rings $\widehat{R}_\mathfrak{p}$. Here $X_\mathfrak{p}$ are some sets, and the direct product is taken over all prime ideals $\mathfrak{p} \subset R$.* □

The following explicit construction of an injective morphism with flat cokernel from a flat R-module G to a flat cotorsion R-module $\mathrm{FC}_R(G)$ was given in the book [67]. For any prime ideal $\mathfrak{p} \subset R$, consider the localization $G_\mathfrak{p} = R_\mathfrak{p} \otimes_R G$ of the R-module G at $\mathfrak{p}$, and take its $\mathfrak{p}$-adic completion $\widehat{G}_\mathfrak{p} = \varprojlim_n G_\mathfrak{p}/\mathfrak{p}^n G_\mathfrak{p}$. By [55, Lemma 1.3.2 and

Corollary B.8.2(b)], the R-module $\widehat{G}_{\mathfrak p}$ is a free $\widehat{R}_{\mathfrak p}$-contramodule. Furthermore, there is a natural R-module map $G \longrightarrow \widehat{G}_{\mathfrak p}$. Set $\mathrm{FC}_R(G) = \prod_{\mathfrak p} \widehat{G}_{\mathfrak p}$.

Lemma 1.3.9. *The natural R-module morphism $G \longrightarrow \mathrm{FC}_R(G)$ is injective, and its cokernel $\mathrm{FC}_R(G)/G$ is a flat R-module.*

Proof. The following argument can be found in [67, Proposition 4.2.2 and Lemma 3.1.6]. Any R-module morphism from G to an $R_{\mathfrak p}/\mathfrak p^n$-module, and hence also to a projective limit of such modules, factorizes through the morphism $G \longrightarrow \widehat{G}_{\mathfrak p}$. Consequently, in view of Theorem 1.3.8 any morphism from G to a flat cotorsion R-module factorizes through the morphism $G \longrightarrow \mathrm{FC}_R(G)$.

Now one could apply Theorem 1.3.1(a), but it is more instructive to argue directly as follows. Let E be an injective cogenerator of the abelian category of R-modules; e. g., $E = \mathrm{Hom}_{\mathbb Z}(R, \mathbb Q/\mathbb Z)$. The R-module $\mathrm{Hom}_R(G, E)$ being injective by Lemma 1.3.2(c), the R-module $\mathrm{Hom}_R(\mathrm{Hom}_R(G, E), E)$ is flat and cotorsion by part (b) of the same lemma and by Lemma 1.6.1(b) below.

The natural morphism $G \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, E), E)$ is injective, and moreover, for any finitely generated/presented R-module M the induced map $G \otimes_R M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, E), E) \otimes_R M \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(G \otimes_R M,\, E),\, E)$ is injective, too. The map $G \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, E), E)$ factorizes through the map $G \longrightarrow \mathrm{FC}_R(G)$, and it follows that the map $G \otimes_R M \longrightarrow \mathrm{FC}_R(G) \otimes_R M$ is injective as well. ☐

1.4. **Exact categories of contraadjusted and cotorsion modules.** Let R be a commutative ring. As full subcategories of the abelian category of R-modules closed under extensions, the categories of contraadjusted and very flat R-modules have natural exact category structures. In the exact category of contraadjusted R-modules every morphism has a cokernel, which is, in addition, an admissible epimorphism.

In the exact category of contraadjusted R-modules the functors of infinite product are everywhere defined and exact; they also agree with the infinite products in the abelian category of R-modules. In the exact category of very flat R-modules, the functors of infinite direct sum are everywhere defined and exact, and agree with the infinite direct sums in the abelian category of R-modules.

It is clear from Corollary 1.1.5(b) that there are enough projective objects in the exact category of very flat R-modules; these are precisely the very flat contraadjusted R-modules. Similarly, by Corollary 1.1.5(a) in the exact category of very flat R-modules there are enough injective objects; these are also precisely the very flat contraadjusted modules.

Denote the exact category of contraadjusted R-modules by $R\text{–}\mathsf{mod}^{\mathsf{cta}}$ and the exact category of very flat R-modules by $R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The tensor product of two very flat R-modules is an exact functor of two arguments $R\text{–}\mathsf{mod}^{\mathsf{vfl}} \times R\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The Hom_R from a very flat R-module into a contraadjusted R-module is an exact functor of two arguments $(R\text{–}\mathsf{mod}^{\mathsf{vfl}})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$ (where C^{op} denotes the opposite category to a category C).

For any homomorphism of commutative rings $f\colon R \longrightarrow S$, the restriction of scalars with respect to f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$. The extension of scalars $F \longmapsto S \otimes_R F$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{vfl}}$.

For any homomorphism of commutative rings $f\colon R \longrightarrow S$ satisfying the condition of Lemma 1.2.3, the restriction of scalars with respect to f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The coextension of scalars $P \longmapsto \operatorname{Hom}_R(S, P)$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{cta}}$. In particular, these assertions hold for any homomorphism of commutative rings $R \longrightarrow S$ satysfying the assumption of Lemma 1.2.4.

Lemma 1.4.1. *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ is an open covering. Then*

(a) *a pair of homomorphisms of contraadjusted R-modules $K \longrightarrow L \longrightarrow M$ is a short exact sequence if and only if such are the induced sequences of contraadjusted S_α-modules $\operatorname{Hom}_R(S_\alpha, K) \longrightarrow \operatorname{Hom}_R(S_\alpha, L) \longrightarrow \operatorname{Hom}_R(S_\alpha, M)$ for all α;*

(b) *a homomorphism of contraadjusted R-modules $P \longrightarrow Q$ is an admissible epimorphism in $R\text{–}\mathsf{mod}^{\mathsf{cta}}$ if and only if the induced homomorphisms of contraadjusted S_α-modules $\operatorname{Hom}_R(S_\alpha, P) \longrightarrow \operatorname{Hom}_R(S_\alpha, Q)$ are admissible epimorphisms in $S_\alpha\text{–}\mathsf{mod}^{\mathsf{cta}}$ for all α.*

Proof. Part (a): the "only if" assertion follows from Lemma 1.2.4. For the same reason, if the sequences $0 \longrightarrow \operatorname{Hom}_R(S_\alpha, K) \longrightarrow \operatorname{Hom}_R(S_\alpha, L) \longrightarrow \operatorname{Hom}_R(S_\alpha, M) \longrightarrow 0$ are exact, then so are the sequences obtained by applying the functors $\operatorname{Hom}_R(S_{\alpha_1} \otimes_R \cdots \otimes_R S_{\alpha_k}, -)$, $k \geqslant 1$, to the sequence $K \longrightarrow L \longrightarrow M$. Now it remains to make use of Lemma 1.2.6(b) in order to deduce exactness of the original sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$.

Part (b): it is clear from the very right segment of the exact sequence (2) that surjectivity of the maps $\operatorname{Hom}_R(S_\alpha, P) \longrightarrow \operatorname{Hom}_R(S_\alpha, Q)$ implies surjectivity of the map $P \longrightarrow Q$. It remains to check that the kernel of the latter morphism is a contraadjusted R-module. Denote this kernel by K. Since the morphisms $\operatorname{Hom}_R(S_\alpha, P) \longrightarrow \operatorname{Hom}_R(S_\alpha, Q)$ are admissible epimorphisms, so are all the morphisms obtained by applying the coextension of scalars with respect to the ring homomorphisms $R \longrightarrow S_{\alpha_1} \otimes_R \cdots \otimes_R S_{\alpha_k}$, $k \geqslant 1$, to the morphism $P \longrightarrow Q$.

Now Lemma 1.2.6(b) applied to both sides of the morphism $P \longrightarrow Q$ provides a termwise surjective morphism of finite exact sequences of R-modules. The corresponding exact sequence of kernels has K as its rightmost nontrivial term, while by Lemma 1.2.2(a) all the other terms are contraadjusted R-modules. It follows that the R-module K is also contraadjusted. □

Let R be an associative ring. As a full subcategory of the abelian category of R-modules closed under extensions, the category of cotorsion left R-modules has a natural exact category structure.

The functors of infinite product are everywhere defined and exact in this exact category, and agree with the infinite products in the abelian category of R-modules.

Similarly, the category of flat R-modules has a natural exact category structure with exact functors of infinite direct sum.

It follows from Theorem 1.3.1 that there are enough projective objects in the exact category of cotorsion R-modules; these are precisely the flat cotorsion R-modules. Similarly, there are enough injective objects in the exact category of flat R-modules, and these are also precisely the flat cotorsion R-modules.

Denote the exact category of cotorsion left R-modules by $R\text{–}\mathsf{mod}^{\mathsf{cot}}$ and the exact category of flat left R-modules by $R\text{–}\mathsf{mod}^{\mathsf{fl}}$. The abelian category of left R-modules will be denoted simply by $R\text{–}\mathsf{mod}$, and the additive category of injective R-modules (with the trivial exact category structure) by $R\text{–}\mathsf{mod}^{\mathsf{inj}}$.

For any commutative ring R, the Hom_R from a flat R-module into a cotorsion R-module is an exact functor of two arguments $(R\text{–}\mathsf{mod}^{\mathsf{fl}})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. Analogously, the Hom_R from an arbitrary R-module into an injective R-module is an exact functor $(R\text{–}\mathsf{mod})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{inj}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. The functors Hom over a noncommutative ring R mentioned in Lemma 1.3.3 have similar exactness properties.

For any associative ring homomorphism $f\colon R \longrightarrow S$, the restriction of scalars via f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. For any associative ring homomorphism $f\colon R \longrightarrow S$ making S a flat left R-module, the coextension of scalars $P \longmapsto \operatorname{Hom}_R(S, P)$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{cot}}$.

Lemma 1.4.2. *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is an open covering. Then*

(a) *a pair of morphisms of cotorsion R-modules $K \longrightarrow L \longrightarrow M$ is a short exact sequence if and only if such are the sequences of cotorsion S_α-modules* $\operatorname{Hom}(S_\alpha, K) \longrightarrow \operatorname{Hom}(S_\alpha, L) \longrightarrow \operatorname{Hom}(S_\alpha, M)$ *for all α;*

(b) *a morphism of cotorsion R-modules $P \longrightarrow Q$ is an admissible epimorphism if and only if such are the morphisms of cotorsion S_α-modules* $\operatorname{Hom}_R(S_\alpha, P) \longrightarrow \operatorname{Hom}_R(S_\alpha, Q)$ *for all α.*

Proof. Part (a) follows from Lemma 1.4.1(a); part (b) can be proven in the way similar to Lemma 1.4.1(b). □

1.5. **Very flat and cotorsion dimensions.** Let R be a commutative ring. By analogy with the definition of the flat dimension of a module, define the *very flat dimension* of an R-module M as the minimal length of its very flat left resolution.

Clearly, the very flat dimension of an R-module M is equal to the supremum of the set of all integers d for which there exists a contraadjusted R-module P such that $\operatorname{Ext}_R^d(M, P) \neq 0$. The very flat dimension of a module cannot differ from its projective dimension by more than 1.

Similarly, the *cotorsion dimension* of a left module M over an associative ring R is conventionally defined as the minimal length of its right resolution by cotorsion R-modules. The cotorsion dimension of a left R-module M is equal to the supremum of the set of all integers d for which there exists a flat left R-module F such that $\mathrm{Ext}_R^d(F, M) \neq 0$.

Both the very flat and the cotorsion dimensions of a module do not depend on the choice of a particular very flat/cotorsion resolution in the same sense as the familiar projective, flat, and injective dimensions do not (see Corollary A.5.2 for the general assertion of this kind).

Lemma 1.5.1. *Let $R \longrightarrow S$ be a morphism of associative rings. Then any left S-module Q of cotorsion dimension $\leqslant d$ over S has cotorsion dimension $\leqslant d$ over R.*

Proof. Follows from Lemma 1.3.4(a). □

Lemma 1.5.2. *Let $R \longrightarrow S$ be a morphism of associative rings such that S is a left R-module of flat dimension $\leqslant D$. Then*

(a) *any left S-module G of flat dimension $\leqslant d$ over S has flat dimension $\leqslant d + D$ over R;*

(b) *any right S-module Q of injective dimension $\leqslant d$ over S has injective dimension $\leqslant d + D$ over R.*

Proof. Part (a) follows from the spectral sequence $\mathrm{Tor}_p^S(\mathrm{Tor}_q^R(M, S), G) \Longrightarrow \mathrm{Tor}_{p+q}^R(M, G)$, which holds for any right R-module M. Part (b) follows from the spectral sequence $\mathrm{Ext}_{S^{\mathrm{op}}}^p(\mathrm{Tor}_q^R(N, S), Q) \Longrightarrow \mathrm{Ext}_{R^{\mathrm{op}}}^{p+q}(N, Q)$, which holds for any right R-module N (where S^{op} and R^{op} denote the rings opposite to S and R). □

Lemma 1.5.3. (a) *Let $R \longrightarrow S$ be a morphism of commutative rings such that S is an R-module of very flat dimension $\leqslant D$. Then any S-module G of very flat dimension $\leqslant d$ over S has very flat dimension $\leqslant d + 1 + D$ over R.*

(b) *Let $R \longrightarrow S$ be a morphism of commutative rings such that $S[s^{-1}]$ is an R-module of very flat dimension $\leqslant D$ for any element $s \in S$. Then any S-module G of very flat dimension $\leqslant d$ over S has very flat dimension $\leqslant d + D$ over R.*

Proof. Part (a): the S-module G has a left projective resolution of length $\leqslant d + 1$, and any projective S-module has very flat dimension $\leqslant D$ over R, which implies the desired assertion (see Corollary A.5.5(a)).

Part (b): by Corollary 1.1.4, the S-module G has a left resolution of length $\leqslant d$ by direct summands of transfinitely iterated extensions of the S-modules $S[s^{-1}]$. Hence it suffices to show that the very flat dimension of R-modules is not raised by the transfinitely iterated extension.

More generally, we claim that one has $\mathrm{Ext}_R^n(M, P) = 0$ whenever a module M over an associative ring R is a transfinitely iterated extension of R-modules M_α and $\mathrm{Ext}_R^n(M_\alpha, P) = 0$ for all α. The case $n = 0$ is easy; the case $n = 1$ is the result of [18, Lemma 1]; and the case $n > 1$ is reduced to $n = 1$ by replacing the R-module P with an R-module Q occuring at the rightmost end of a resolution $0 \longrightarrow P \longrightarrow J^0 \longrightarrow \cdots \longrightarrow J^{n-2} \longrightarrow Q \longrightarrow 0$ with injective R-modules J^i. □

Lemma 1.5.4. *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes is a finite open covering. Then*

(a) *the flat dimension of an R-module F is equal to the supremum of the flat dimensions of the S_α-modules $S_\alpha \otimes_R F$;*

(b) *the very flat dimension of an R-module F is equal to the supremum of the very flat dimensions of the S_α-modules $S_\alpha \otimes_R F$;*

(c) *the cotorsion dimension of a contraadjusted R-module P is equal to the supremum of the cotorsion dimensions of the contraadjusted S_α-modules $\mathrm{Hom}_R(S_\alpha, P)$;*

(d) *the injective dimension of a contraadjusted R-module P is equal to the supremum of the injective dimensions of the contraadjusted S_α-modules $\mathrm{Hom}_R(S_\alpha, P)$.*

Proof. Part (b) follows easily from Lemma 1.2.6(a), and the proof of part (a) is similar. Parts (c-d) analogously follow from Lemma 1.3.6(a-b). □

The following lemma will be needed in Section 4.10.

Lemma 1.5.5. (a) *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings and P be an R-module such that $\mathrm{Ext}^1_R(S[s^{-1}], P) = 0$ for all elements $s \in S$. Then the S-module $\mathrm{Hom}_R(S, P)$ is contraadjusted.*

(b) *Let $f\colon R \longrightarrow S$ be a homomorphism of associative rings and P be a left R-module such that $\mathrm{Ext}^1_R(G, P) = 0$ for all flat left S-modules G. Then the S-module $\mathrm{Hom}_R(S, P)$ is cotorsion.*

Proof. See the proofs of Lemmas 1.2.3(a) and 1.3.5(a). □

The following theorem is due to Raynaud and Gruson [58, Corollaire II.3.2.7].

Theorem 1.5.6. *Let R be a commutative Noetherian ring of Krull dimension D. Then the projective dimension of any flat R-module does not exceed D. Consequently, the very flat dimension of any flat R-module also does not exceed D.* □

Corollary 1.5.7. *Let R be a commutative Noetherian ring of Krull dimension D. Then the cotorsion dimension of any R-module does not exceed D.*

Proof. For any associative ring R, the supremum of the projective dimensions of flat left R-modules and the supremum of the cotorsion dimensions of arbitrary left R-modules are equal to each other. Indeed, both numbers are equal to the supremum of the set of all integers d for which there exist a flat left R-module F and a left R-module P such that $\mathrm{Ext}^d_R(F, P) \neq 0$. □

1.6. **Coherent rings, finite morphisms, and coadjusted modules.** Recall that an associative ring R is called *left coherent* if all its finitely generated left ideals are finitely presented. Finitely presented left modules over a left coherent ring R form an abelian subcategory in R–mod closed under kernels, cokernes, and extensions. The definition of a right coherent associative ring is similar.

Lemma 1.6.1. *Let R and S be associative rings. Then*

(a) *Assuming that the ring R is left Noetherian, for any R-injective R-S-bimodule J and any flat left S-module F the left R-module $J \otimes_S F$ is injective.*

(b) *Assuming that the ring S is right coherent, for any S-injective R-S-bimodule I and any injective left R-module J the left S-module $\mathrm{Hom}_R(I, J)$ is flat.*

Proof. Part (a) holds due to the natural isomorphism $\operatorname{Hom}_R(M,\ J\otimes_S F)\simeq\operatorname{Hom}_R(M,J)\otimes_S F$ for any finitely presented left module M over an associative ring R, any R-S-bimodule J, and any flat left S-module F. Part (b) follows from the natural isomorphism $N\otimes_S\operatorname{Hom}_R(I,J)\simeq\operatorname{Hom}_R(\operatorname{Hom}_{S^{\mathrm{op}}}(N,I),J))$ for any finitely presented right module N over an associative ring S, any R-S-bimodule I, and any injective left R-module J (where S^{op} denotes the ring opposite to S).

Here we use the facts that injectivity of a left module I over a left Noetherian ring R is equivalent to exactness of the functor $\operatorname{Hom}_R(-,I)$ on the category of finitely generated left R-modules, while flatness of a left module F over a right coherent ring S is equivalent to exactness of the functor $-\otimes_S F$ on the category of finitely presented right S-modules (cf. the proof of the next Lemma 1.6.2). ☐

Lemma 1.6.2. *Let R and S be associative rings such that S is left coherent. Let F be a left R-module of finite projective dimension, P be an S-flat R-S-bimodule such that* $\operatorname{Ext}_R^{>0}(F,P)=0$*, and M be a finitely presented left S-module. Then one has* $\operatorname{Ext}_R^{>0}(F,\ P\otimes_S M)=0$*, the natural map of abelian groups* $\operatorname{Hom}_R(F,P)\otimes_S M\longrightarrow\operatorname{Hom}_R(F,\ P\otimes_S M)$ *is an isomorphism, and the right S-module* $\operatorname{Hom}_R(F,P)$ *is flat.*

Proof. Let $L_\bullet\longrightarrow M$ be a left resolution of M by finitely generated projective S-modules. Then $P\otimes_S L_\bullet\longrightarrow P\otimes_S M$ is a left resolution of the R-module $P\otimes_S M$ by R-modules annihilated by $\operatorname{Ext}^{>0}(F,-)$. Since the R-module F has finite projective dimension, it follows that $\operatorname{Ext}^{>0}(F,\ P\otimes_S M)=0$.

Consequently, the functor $M\longmapsto\operatorname{Hom}_R(F,\ P\otimes_S M)$ is exact on the abelian category of finitely presented left S-modules M. Obviously, the functor $M\longmapsto\operatorname{Hom}_R(F,P)\otimes_R M$ is right exact. Since the morphism of functors $\operatorname{Hom}_R(F,P)\otimes_S M\longrightarrow\operatorname{Hom}_R(F,\ P\otimes_S M)$ is an isomorphism for finitely generated projective S-modules M, we can conclude that it is an isomorphism for all finitely presented left S-modules.

Now we have proven that the functor $M\longmapsto\operatorname{Hom}_R(F,P)\otimes_S M$ is exact on the abelian category of finitely presented left S-modules. Since any left S-module is a filtered inductive limit of finitely presented ones and the inductive limits commute with tensor products, it follows that the S-module $\operatorname{Hom}_R(F,P)$ is flat. ☐

Corollary 1.6.3. *Let R be a commutative ring. Then*

(a) *for any finitely generated R-module M and any contraadjusted R-module P, the R-module $M\otimes_R P$ is contraadjusted;*

(b) *if the ring R is coherent, then for any very flat R-module F and any flat contraadjusted R-module P, the R-module* $\operatorname{Hom}_R(F,P)$ *is flat and contraadjusted;*

(c) *in the situation of* (b)*, for any finitely presented R-module M the natural morphism of R-modules* $\operatorname{Hom}_R(F,P)\otimes_R M\longrightarrow\operatorname{Hom}_R(F,\ P\otimes_R M)$ *is an isomorphism.*

Proof. Part (a) immediately follows from the facts that the class of contraadjusted R-modules is closed under finite direct sums and quotients. Part (b) is provided Lemma 1.6.2 together with Lemma 1.2.1(b), and part (c) is also Lemma 1.6.2. ☐

Corollary 1.6.4. *Let R be either a coherent commutative ring such that any flat R-module has finite projective dimension, or a Noetherian commutative ring. Then*

(a) *for any finitely presented R-module M and any flat cotorsion R-module P, the R-module $M \otimes_R P$ is cotorsion;*

(b) *for any flat R-module F and flat cotorsion R-module P, the R-module $\operatorname{Hom}_R(F, P)$ is flat and cotorsion;*

(c) *in the situation of* (a) *and* (b), *the natural morphism of R-modules* $\operatorname{Hom}_R(F, P) \otimes_R M \longrightarrow \operatorname{Hom}_R(F,\ P \otimes_R M)$ *is an isomorphism.*

Proof. In the former set of assumptions about the ring R, parts (a) and (c) follow from Lemma 1.6.2, and part (b) is provided by the same Lemma together with Lemma 1.3.2(a). In the latter case, the argument is based on Proposition 1.3.7 and Theorem 1.3.8. Part (a): since the functor $M \otimes_R -$ preserves infinite products, it suffices to show that the R-module $M \otimes_R Q_{\mathfrak p}$ has an $\widehat{R}_{\mathfrak p}$-contramodule structure for any $\widehat{R}_{\mathfrak p}$-contramodule $Q_{\mathfrak p}$. Indeed, the full subcategory $\widehat{R}_{\mathfrak p}\text{–}\mathsf{contra} \subset R\text{–}\mathsf{mod}$ is closed with respect to finite direct sums and cokernels.

Part (c): the morphism in question is clearly an isomorphism for a finitely generated projective R-module M. Hence it suffices to show that the functor $M \longmapsto \operatorname{Hom}_R(F,\ P \otimes_R M)$ is right exact. In fact, it is exact, since the functor $\operatorname{Hom}_R(F, -)$ preserves exactness of short sequences of $\widehat{R}_{\mathfrak p}$-contramodules. Therefore, the functor $M \longmapsto \operatorname{Hom}_R(F, P) \otimes_R M$ is also exact, and we have proven part (b) as well. □

Corollary 1.6.5. (a) *Let $R \longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes* $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ *is an open embedding. Assume that the ring R is coherent. Then the S-module* $\operatorname{Hom}_R(S, P)$ *is flat and contraadjusted for any flat contraadjusted R-module P.*

(b) *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is a finite open covering. Assume that either the ring R is Noetherian and an R-module P is cotorsion, or the ring R is Noetherian of finite Krull dimension and an R-module P is contraadjusted. Then the R-module P is flat if and only if all the S_α-modules* $\operatorname{Hom}_R(S_\alpha, P)$ *are flat.*

Proof. Part (a): the S-module $\operatorname{Hom}_R(S, P)$ is contraadjusted by Corollary 1.2.5(c). The R-module $\operatorname{Hom}_R(S, P)$ is flat by Lemma 1.2.4 and Corollary 1.6.3(b). Since $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is an open embedding, it follows that $\operatorname{Hom}_R(S, P)$ is also flat as an S-module (cf. Corollary 1.2.5(b)).

The "only if" assertion in part (b) is provided by part (a). The proof of the "if" is postponed to Section 5. The cotorsion case will follow from Corollary 5.1.4, while the finite Krull dimension case will be covered by Corollary 5.2.2(b). □

Let $R\text{–}\mathsf{mod}_{\mathsf{fp}}$ denote the abelian category of finitely presented left modules over a left coherent ring R. For a coherent commutative ring R, the tensor product of a finitely presented R-module with a flat contraadjusted R-module is an exact functor $R\text{–}\mathsf{mod}_{\mathsf{fp}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$ (where the exact category structure on $R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$ is induced from $R\text{–}\mathsf{mod}$). The Hom_R from a very flat R-module into a flat contraadjusted R-module is an exact functor $(R\text{–}\mathsf{mod}^{\mathsf{vfl}})^{\mathsf{op}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Let R be either a coherent commutative ring such that any flat R-module has finite projective dimension, or a Noetherian commutative ring. Then the tensor product of a finitely presented R-module with a flat cotorsion R-module is an exact functor $R\text{–}\mathsf{mod}_{\mathsf{fp}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cot}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. The Hom_R from a flat R-module to a flat cotorsion R-module is an exact functor $(R\text{–}\mathsf{mod}^{\mathsf{fl}})^{\mathsf{op}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cot}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cot}}$. Here the additive category of flat cotorsion R-modules $R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cot}}$ is endowed with a trivial exact category structure.

Lemma 1.6.6. *Let R be a commutative ring and $I \subset R$ be an ideal. Then*

(a) *an R/I-module Q is a contraadjusted R/I-module if and only if it is a contraadjusted R-module;*

(b) *the R/I-module P/IP is contraadjusted for any contraadjusted R-module P;*

(c) *assuming that the ring R is coherent and the ideal I is finitely generated, for any very flat R-module F and flat contraadjusted R-module P the natural morphism of R/I-modules $\mathrm{Hom}_R(F,P)/I\,\mathrm{Hom}_R(F,P) \longrightarrow \mathrm{Hom}_{R/I}(F/IF, P/IP)$ is an isomorphism.*

Proof. Part (a): the characterization of contraadjusted modules given in the beginning of Section 1.1 shows that the contraadjustness property of a module depends only on its abelian group structure and the operators by which the ring acts in it (rather than on the ring indexing such operators). Alternatively, the "only if" assertion is a particular case of Lemma 1.2.2(a), and one can deduce the "if" from the observation that any element $\bar r \in R/I$ can be lifted to an element $r \in R$ so that one has an isomorphism of R/I-modules $R/I[\bar r^{-1}] \simeq R/I \otimes_R R[r^{-1}]$.

Part (b): the R-module P/IP is contraadjusted as a quotient module of a contraadjusted R-module. By part (a), P/IP is also a contraadjusted R/I-module. Part (c): by Corollary 1.6.3(c), the natural morphism of R-modules $\mathrm{Hom}_R(F,P)/I\,\mathrm{Hom}_R(F,P) \longrightarrow \mathrm{Hom}_R(F,\ P/IP)$ is an isomorphism. ☐

Recall that a morphism of Noetherian rings $R \longrightarrow S$ is called *finite* if S is a finitely generated R-module in the induced R-module structure.

Lemma 1.6.7. *Let $R \longrightarrow S$ be a finite morphism of Noetherian commutative rings. Then*

(a) *the S-module $S \otimes_R P$ is flat and cotorsion for any flat cotorsion R-module P;*

(b) *assuming that the ring S has finite Krull dimension, an S-module Q is a cotorsion S-module if and only if it is a cotorsion R-module;*

(c) *for any flat R-module F and flat cotorsion R-module P, the natural morphism of S-modules $S \otimes_R \mathrm{Hom}_R(F,P) \longrightarrow \mathrm{Hom}_S(S \otimes_R F,\ S \otimes_R P)$ is an isomorphism.*

Proof. The proof of parts (a-b) is based on Theorem 1.3.8. Given a prime ideal $\mathfrak{p} \subset R$, consider all the prime ideals $\mathfrak{q} \subset S$ whose full preimage in R coincides with $\mathfrak{p}$. Such ideals form a nonempty finite set, and there are no inclusions between them [42, Theorems 9.1 and 9.3, and Exercise 9.3]. Let us denote these ideals by $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$. By [42, Theorems 9.4(i), 8.7, and 8.15], we have $S \otimes_R \widehat{R}_\mathfrak{p} \simeq \widehat{S}_{\mathfrak{q}_1} \oplus \cdots \oplus \widehat{S}_{\mathfrak{q}_m}$.

Since the functor $S \otimes_R -$ preserves infinite products, in order to prove (a) it suffices to show that the S-module $S \otimes_R F_\mathfrak{p}$ is a finite direct sum of certain free $\widehat{S}_{\mathfrak{q}_i}$-contramodules $F_{\mathfrak{q}_i}$. This can be done either by noticing that $F_\mathfrak{p}$ is a direct summand of an infinite product of copies of $\widehat{R}_\mathfrak{p}$ (see [55, Section 1.3]), or by showing that the natural map $S \otimes_R \widehat{R}_\mathfrak{p}[[X]] \longrightarrow \bigoplus_{i=1}^m \widehat{S}_{\mathfrak{q}_i}[[X]]$ is an isomorphism for any set X (see [55, proof of the first assertion of Proposition B.9.1]). In addition to the assertion of part (a), we have also proven that any flat cotorsion S-module Q is a direct summand of an S-module $S \otimes_R P$ for a certain flat cotorsion R-module P.

The "only if" assertion in part (b) is a particular case of Lemma 1.3.4(a). Let us prove the "if". According to Corollary 1.5.7, the cotorsion dimension of any R-module is finite. By Theorem 1.3.1(a), it follows that any flat S-module admits a finite right resolution by flat cotorsion S-modules (cf. the dual version of Corollary A.5.3). Hence it suffices to prove that $\mathrm{Ext}_S^{>0}(G, Q) = 0$ for a flat cotorsion S-module G. This allows us to assume that $G = S \otimes_R F$, where F is a flat (cotorsion) R-module. It remains to recall the Ext isomorphism from the proof of Lemma 1.3.4(a).

Part (c): by Corollary 1.6.4(c), the natural morphism of R-modules $S \otimes_R \mathrm{Hom}_R(F, P) \longrightarrow \mathrm{Hom}_R(F, S \otimes_R P)$ is an isomorphism. ☐

Lemma 1.6.8. *Let R be a commutative ring and $I \subset R$ be a finitely generated nilpotent ideal. Then*

(a) *an R-module P is contraadjusted if and only if the R/I-module P/IP is contraadjusted;*

(b) *a flat R-module F is very flat if and only if the R/I-module F/IF is very flat;*

(c) *assuming that the ring R is Noetherian, a flat R-module P is cotorsion if and only if the R/I-module P/IP is cotorsion;*

(d) *assuming that the ring R is coherent and any flat R-module has finite projective dimension, a flat R-module P is cotorsion whenever the R/I-module P/IP is cotorsion.*

Proof. Part (a): the "only if" assertion is a particular case of Lemma 1.6.6(b). To prove the "if", notice that in our assumptions about I the R-module P has a finite decreasing filtration by its submodules I^nP. Furthermore, the successive quotients $I^nP/I^{n+1}P$ are the targets of the natural surjective homomorphisms of R/I-modules $I^n/I^{n+1} \otimes_{R/I} P/IP \longrightarrow I^nP/I^{n+1}P$. Since the R/I-module I^n/I^{n+1} is finitely generated, the R/I-module $I^nP/I^{n+1}P$ is a quotient module of a finite direct sum of copies of the R/I-module P/IP. It remains to use the facts that the class of contraadjusted modules over a given commutative ring is closed under extensions and quotients, together with the result of Lemma 1.6.6(a).

Part (b): the “only if” is a particular case of Lemma 1.2.2(b); let us prove the “if”. Let P be a contraadjusted R-module; we have to show that $\operatorname{Ext}^1_R(F,P)=0$. According to the above proof of part (a), the R-module P has a finite filtration whose successive quotients are contraadjusted R/I-modules with the R-module structures obtained by restriction of scalars. So it suffices to check that $\operatorname{Ext}^1_R(F,Q)=0$ for any contraadjusted R/I-module Q. Now, the R-module F being flat, one has $\operatorname{Ext}^1_R(F,Q)=\operatorname{Ext}^1_{R/I}(F/IF,\,Q)$ for any R/I-module Q (see the proof of Lemma 1.3.4(a)).

Parts (c-d): in “only if” assertion in (c) is a particular case of Lemma 1.6.7(a). To prove the “if”, notice the isomorphisms of R/I-modules $I^nP/I^{n+1}P\simeq I^n/I^{n+1}\otimes_R P\simeq I^n/I^{n+1}\otimes_{R/I}P/IP$ (the former of which holds, since the R-module P is flat). The R/I-module P/IP being flat and cotorsion, the R/I-modules $I^n/I^{n+1}\otimes_{R/I}P/IP$ are cotorsion by Corollary 1.6.4(a), the R-modules $I^n/I^{n+1}\otimes_{R/I}P/IP$ are cotorsion by Lemma 1.3.4(a), and the R-module P is cotorsion, since the class of cotorsion R-modules is closed under extensions. ☐

Let R be a commutative ring and $I\subset R$ be an ideal. Then the reduction $P\longmapsto P/IP$ of a flat contraadjusted R-module P modulo I is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{fl}}\cap R\text{–}\mathsf{mod}^{\mathsf{cta}}\longrightarrow R/I\text{–}\mathsf{mod}^{\mathsf{fl}}\cap R/I\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Let $f\colon R\longrightarrow S$ be a finite morphism of Noetherian commutative rings. Then the extension of scalars $P\longmapsto S\otimes_R P$ of a flat cotorsion R-module P with respect to f is an additive functor $R\text{–}\mathsf{mod}^{\mathsf{fl}}\cap R\text{–}\mathsf{mod}^{\mathsf{cot}}\longrightarrow S\text{–}\mathsf{mod}^{\mathsf{fl}}\cap S\text{–}\mathsf{mod}^{\mathsf{cot}}$ (between additive categories naturally endowed with trivial exact category structures).

Let R be a commutative ring. We will say that an R-module K is *coadjusted* if the functor of tensor product with K over R preserves the class of contraadjusted R-modules. By Corollary 1.6.3(a), any finitely generated R-module is coadjusted.

An R-module K is coadjusted if and only if the R-module $K\otimes_R P$ is contraadjusted for every flat (or very flat) contraadjusted R-module P. Indeed, by Corollary 1.1.5(b), any contraajusted R-module is a quotient module of a very flat contraadjusted R-module; so it remains to recall that any quotient module of a contraadjusted R-module is contraadjusted.

Clearly, any quotient module of a coadjusted R-module is coadjusted. Furthermore, the class of coadjusted R-modules is closed under extensions. One can see this either by applying the above criterion of coadjustness in terms of tensor products with flat contraadjusted R-modules, or straightforwardly from the right exactness property of the functor of tensor product together with the facts that the class of contraadjusted R-modules is closed under quotients and extensions.

Consequently, there is the induced exact category structure on the full subcategory of coadjusted R-modules in the abelian category $R\text{–}\mathsf{mod}$. We denote this exact category by $R\text{–}\mathsf{mod}^{\mathsf{coa}}$. The tensor product of a coadjusted R-module with a flat contraadjusted R-module is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{coa}}\times(R\text{–}\mathsf{mod}^{\mathsf{fl}}\cap R\text{–}\mathsf{mod}^{\mathsf{cta}})\longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Over a Noetherian commutative ring R, any injective module J is coadjusted. Indeed, for any R-module P, the tensor product $J\otimes_R P$ is a quotient module of an infinite direct sum of copies of J, which means a quotient module of an injective module, which is contraadjusted. Hence any quotient module of an injective module is coadjusted, too, as is any extension of such modules.

Lemma 1.6.9. (a) *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes* $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ *is an open embedding. Then the S-module $S \otimes_R K$ obtained by the extension of scalars via f is coadjusted for any coadjusted R-module K.*

(b) *Let $f_\alpha\colon R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is a finite open covering. Then an R-module K is coadjusted if and only if all the S_α-modules $S_\alpha \otimes_R K$ are coadjusted.*

Proof. Part (a): any contraadjusted S-module Q is also contraadjusted as an R-module, so the tensor product $(S \otimes_R K) \otimes_S Q \simeq K \otimes_R Q$ is a contraadjusted R-module. By Corollary 1.2.5(d), it is also a contraadjusted S-module.

Part (b): the "only if" assertion is provided by part (a); let us prove "if". Let P be a contraadjusted R-module. Applying the functor $K \otimes_R -$ to the Čech exact sequence (2) from Lemma 1.2.6(b), we obtain a sequence of R-modules that is exact at its rightmost nontrivial term. So it suffices to show that the R-modules $K \otimes_R \operatorname{Hom}_R(S_\alpha, P)$ are contraadjusted.

Now one has $K \otimes_R \operatorname{Hom}_R(S_\alpha, P) \simeq (S_\alpha \otimes_R K) \otimes_{S_\alpha} \operatorname{Hom}_R(S_\alpha, P)$, the S_α-module $\operatorname{Hom}_R(S_\alpha, P)$ is contraadjusted by Corollary 1.2.5(c), and the restriction of scalars from S_α to R preserves contraadjustness by Lemma 1.2.2(a). □

1.7. **Very flat morphisms of schemes.** A morphism of schemes $f\colon Y \longrightarrow X$ is called *very flat* if for any two affine open subschemes $V \subset Y$ and $U \subset X$ such that $f(V) \subset U$ the ring of regular functions $\mathcal{O}(V)$ is a very flat module over the ring $\mathcal{O}(U)$. By Lemma 1.2.4, any embedding of an open subscheme is a very flat morphism.

According to Lemmas 1.2.6(a) and 1.2.7(b), the property of a morphism to be very flat is local in both the source and the target schemes. A morphism of affine schemes $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is very flat if and only if the morphism of commutative rings $R \longrightarrow S$ satisfies the condition of Lemma 1.2.3. By Lemma 1.2.3(b), the composition of very flat morphisms of schemes is a very flat morphism.

It does not seem to follow from anything that the base change of a very flat morphism of schemes should be a very flat morphism. Here is a partial result in this direction.

Lemma 1.7.1. *The base change of a very flat morphism with respect to any locally closed embedding or universal homeomorphism of schemes is a very flat morphism.*

Proof. Essentially, given a very flat morphism $f\colon Y \longrightarrow X$ and a morphism of schemes $g\colon x \longrightarrow X$, in order to conclude that the morphism $f'\colon y = x \times_x Y \longrightarrow x$ is very flat it suffices to know that the morphism $g'\colon y \longrightarrow Y$ is injective and the topology of y is induced from the topology of Y.

Indeed, the very flatness condition being local, one can assume all the four schemes to be affine. Now any affine open subscheme $v \subset y$ is the full preimage of a certain open subscheme $V \subset Y$. Covering V with open affines if necessary and using the locality again, one can assume that V is affine, too. Finally, if the $\mathcal{O}(X)$-module $\mathcal{O}(V)$ is very flat, then by Lemma 1.2.2(b) so is the $\mathcal{O}(x)$-module $\mathcal{O}(v) = \mathcal{O}(x \times_X V)$. □

A quasi-coherent sheaf $\mathcal{F}$ over a scheme X is called *very flat* if the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is very flat for any affine open subscheme $U \subset X$. According to Lemma 1.2.6(a), very flatness of a quasi-coherent sheaf over a scheme is a local property. By Lemma 1.2.2(b), the inverse image of a very flat quasi-coherent sheaf under any morphism of schemes is very flat. By Lemma 1.2.3(b), the direct image of a very flat quasi-coherent sheaf under a very flat affine morphism of schemes is very flat.

More generally, given a morphism of schemes $f: Y \longrightarrow X$, a quasi-coherent sheaf $\mathcal{F}$ on Y is said to be *very flat over* X if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ the module of sections $\mathcal{F}(V)$ is very flat over the ring $\mathcal{O}_X(U)$. According to Lemmas 1.2.6(a) and 1.2.8(b), the property of very flatness of $\mathcal{F}$ over X is local in both X and Y. By Lemma 1.2.3(b), if the scheme Y is very flat over X and a quasi-coherent sheaf $\mathcal{F}$ is very flat on Y, then $\mathcal{F}$ is also very flat over X.

The following conjecture looks natural. Some evidence in its support is gathered below in this section.

Conjecture 1.7.2. *Any flat morphism of finite type between Noetherian schemes is very flat.*

It is well-known that any flat morphism of finite type between Noetherian schemes (or of finite presentation between arbitrary schemes) is an open map [29, Théorème 2.4.6]. We will see below that the similar result about very flat morphisms does not require any finiteness conditions at all.

Given a commutative ring R and an R-module M, we define its support $\operatorname{Supp} M \subset \operatorname{Spec} R$ as the set of all prime ideals $\mathfrak{p} \subset R$ for which the tensor product $k(\mathfrak{p}) \otimes_R M$, where $k(\mathfrak{p})$ denotes the residue field of the ideal $\mathfrak{p}$, does not vanish.

Lemma 1.7.3. *Let R be a commutative ring without nilpotent elements and F be a nonzero very flat R-module. Then the support of F contains a nonempty open subset in* $\operatorname{Spec} R$.

Proof. By Corollary 1.1.4, any very flat R-module F is a direct summand of a transfinitely iterated extension $M = \varinjlim_\alpha M_\alpha$ of certain R-modules of the form $R[s_\alpha^{-1}]$, where $s_\alpha \in R$. In particular, F is an R-submodule in M; consider the minimal index α for which the intersection $F \cap M_\alpha \subset M$ is nonzero. Denote this intersection by G and set $s = s_\alpha$; then G is a nonzero submodule in F and in $R[s^{-1}]$ simultaneously.

Hence the localization $G[s^{-1}]$ is a nonzero ideal in $R[s^{-1}]$. The support $\operatorname{Supp} G[s^{-1}] \subset \operatorname{Spec} R[s^{-1}]$ of the $R[s^{-1}]$-module $G[s^{-1}]$ is equal to the intersection of $\operatorname{Supp} G \subset \operatorname{Spec} R$ with $\operatorname{Spec} R[s^{-1}] \subset \operatorname{Spec} R$. By right exactness of the tensor product functor, the subset $\operatorname{Supp} G[s^{-1}]$ contains the complement $V(s, G) \subset \operatorname{Spec} R[s^{-1}]$ to the support of the quotient module $R[s^{-1}]/G[s^{-1}]$ in $\operatorname{Spec} R[s^{-1}]$. The latter is a closed subset in $\operatorname{Spec} R[s^{-1}]$ corresponding to the ideal $G[s^{-1}]$; if there are no nilpotents in R then the open subset $V(s, G) \subset \operatorname{Spec} R[s^{-1}]$ is nonempty.

Let us show that the support of F contains $V(s,G)$. Let $\mathfrak{p} \in V(s,G) \subset \operatorname{Spec} R$ be a prime ideal in R and $k(\mathfrak{p})$ be its residue field. Then the map $k(\mathfrak{p}) \otimes_R G = k(\mathfrak{p}) \otimes_R G[s^{-1}] \longrightarrow k(\mathfrak{p}) \otimes_R R[s^{-1}] = k(\mathfrak{p})$ is surjective by the above argument, the map $k(\mathfrak{p}) \otimes_R M_\alpha \longrightarrow k(\mathfrak{p}) \otimes_R M$ is injective because the R-module M/M_α is flat, and the map $k(\mathfrak{p}) \otimes_R F \longrightarrow k(\mathfrak{p}) \otimes_R M$ is injective since F is a direct summand in M (we do not seem to really use the latter observation). Finally, both the composition $G \longrightarrow F \longrightarrow M$ and the map $G \longrightarrow R[s^{-1}]$ factorize through the same R-module morphism $G \longrightarrow M_\alpha$.

Now it follows from the commutativity of the triangle $G \longrightarrow M_\alpha \longrightarrow R[s^{-1}]$ that the map $k(\mathfrak{p}) \otimes_R G \longrightarrow k(\mathfrak{p}) \otimes_R M_\alpha$ is nonzero, it is clear from the diagram $G \longrightarrow M_\alpha \longrightarrow M$ that the composition $k(\mathfrak{p}) \otimes_R G \longrightarrow k(\mathfrak{p}) \otimes_R M$ is nonzero, and it follows from commutativity of the diagram $G \longrightarrow F \longrightarrow M$ that the map $k(\mathfrak{p}) \otimes_R G \longrightarrow k(\mathfrak{p}) \otimes_R F$ is nonzero. The assertion of Lemma is proven. ☐

Proposition 1.7.4. *Let F be a very flat module over a commutative ring R and $Z \subset \operatorname{Spec} R$ be a closed subset. Suppose that the intersection $\operatorname{Supp} F \cap Z \subset \operatorname{Spec} R$ is nonempty. Then $\operatorname{Supp} F$ contains a nonempty open subset in Z.*

Proof. Endow Z with the structure of a reduced closed subscheme in $\operatorname{Spec} R$ and set $S = \mathcal{O}(Z)$. Then the intersection $Z \cap \operatorname{Supp} F$ coincides with the support of the S-module $S \otimes_R F$ in $\operatorname{Spec} S = Z \subset \operatorname{Spec} R$. By Lemma 1.2.2(b), the S-module $S \otimes_R F$ is very flat. Now if this S-module vanishes, then the intersection $Z \cap \operatorname{Supp} F$ is empty, while otherwise it contains a nonempty open subset in $\operatorname{Spec} S$ by Lemma 1.7.3. ☐

Remark 1.7.5. It is clear from the above argument that one can replace the condition that the ring R has no nilpotent elements in Lemma 1.7.3 by the condition of nonvanishing of the tensor product $S \otimes_R F$ of the R-module F with the quotient ring $S = R/J$ of the ring R by its nilradical J. In particular, this condition holds automatically if (the R-module F does not vanish and) the nilradical $J \subset R$ is a nilpotent ideal, i. e., there exists an integer $N \geqslant 1$ such that $J^N = 0$. This includes all Noetherian commutative rings R.

On the other hand, it is not difficult to demonstrate an example of a flat module over a commutative ring that is annihilated by the reduction modulo the nilradical. E. g., take R to be the ring of polynomials in x, $x^{1/2}$, $x^{1/4}$, …, $x^{1/2^N}$, … over a field $k = S$ with the imposed relation $x^r = 0$ for $r > 1$, and $F = J$ to be the nilradical (i. e., the kernel of the augmentation morphism to k) in R. The R-module F is flat as the inductive limit of free R-modules $R \otimes_k kx^{1/2^N}$ for $N \to \infty$, and one clearly has $S \otimes_R F = R/J \otimes_R J = J/J^2 = 0$.

Theorem 1.7.6. *The support of any very flat module over a commutative ring R is an open subset in* $\operatorname{Spec} R$.

Proof. Let F be a very flat R-module. Denote by Z the closure of the complement $\operatorname{Spec} R \setminus \operatorname{Supp} F$ in $\operatorname{Spec} R$. Then, by the definition, $\operatorname{Supp} F$ contains the complement $\operatorname{Spec} R \setminus Z$, and no open subset in (the induced topology of) Z is contained in $\operatorname{Supp} F$. By Proposition 1.7.4, it follows that $\operatorname{Supp} F$ does not intersect Z, i. e., $\operatorname{Supp} F = \operatorname{Spec} R \setminus Z$ is an open subset in $\operatorname{Spec} R$. ☐

Example 1.7.7. In particular, we have shown that the $\mathbb{Z}$-module $\mathbb{Q}$ is *not* very flat, even though it is a flat module of projective dimension 1.

Corollary 1.7.8. *Any very flat morphism of schemes is an open map of their underlying topological spaces.*

Proof. Given a very flat morphism $f\colon Y \longrightarrow X$ and an open subset $W \subset Y$, cover W with open affines $V \subset Y$ for which there exist open affines $U \subset X$ such that $f(V) \subset U$, and apply Theorem 1.7.6 to the very flat $\mathcal{O}(U)$-modules $\mathcal{O}(V)$. $\square$

In the rest of the section we prove several (rather weak) results about morphisms from certain classes being always very flat.

Theorem 1.7.9. *Any finite étale morphism of Noetherian schemes is very flat.*

Proof. The argument is based on the Galois theory of finite étale morphisms of (Noetherian) schemes [45] (see also [40]). Clearly, one can assume both schemes to be affine and connected. Let $\operatorname{Spec} S \to \operatorname{Spec} R$ be our morphism; we have to show that for any $s \in S$ the R-module $S[s^{-1}]$ is very flat.

First let us reduce the question to the case when our morphism is a Galois covering. Let $\operatorname{Spec} T \longrightarrow \operatorname{Spec} S$ be a finite étale morphism from a nonempty scheme $\operatorname{Spec} T$ such that the composition $\operatorname{Spec} T \longrightarrow \operatorname{Spec} R$ is Galois. Notice that the $S[s^{-1}]$-module $T[s^{-1}]$ is flat and finitely presented, and consequently projective. If it is known that the R-module $T[s^{-1}]$ is very flat, then it remains to show that the R-module $S[s^{-1}]$ is a direct summand of a (finite) direct sum of copies of $T[s^{-1}]$.

For this purpose, it suffices to check that the $S[s^{-1}]$-module $T[s^{-1}]$ is a projective generator of the abelian category of $S[s^{-1}]$-modules. In other words, we have to show that there are no nonzero $S[s^{-1}]$-modules M for which any morphism of $S[s^{-1}]$-modules $T[s^{-1}] \longrightarrow M$ vanishes. Since the functor Hom from a finitely presented module over a commutative ring commutes with localizations, and finitely generated projective modules are locally free in the Zariski topology, the desired property follows from the assumption of the scheme $\operatorname{Spec} S$ being connected.

Now let G be the Galois group of $\operatorname{Spec} S$ over $\operatorname{Spec} R$. For any subset $\Gamma \subset G$ consider the element $t_\Gamma = \prod_{g\in\Gamma} g(s) \in S$. We will prove the assertion that $S[t_\Gamma^{-1}]$ is a very flat R-module by decreasing induction in the cardinality of Γ (for a fixed group G, but varying rings R and S). The induction base: if $\Gamma = G$, then the element $t_\Gamma = t_G = \prod_{g\in G} g(s)$ belongs to $R \subset S$, and the ring $S[t_G^{-1}]$, being a projective module over $R[t_G^{-1}]$, is a very flat module over R.

The induction step: let $H \subset G$ be the stabilizer of the subset $\Gamma \subset G$ with respect to the action of G in itself by multiplications on the left, and let $g_1, \dots, g_n \in G$ be some representatives of the left cosets G/H. The union of the open subschemes $\operatorname{Spec} S[t_{g_i(\Gamma)}^{-1}]$, being a G-invariant open subset in $\operatorname{Spec} S$, is the full preimage of a certain open subscheme $U \subset \operatorname{Spec} R$. Replacing the scheme $\operatorname{Spec} R$ by its connected affine open subschemes covering U, we can assume that $\operatorname{Spec} R = U$ and $\operatorname{Spec} S$ is the union of its open subschemes $\operatorname{Spec} S[t_{g_i(\Gamma)}^{-1}]$.

Consider the Čech exact sequence (4) for the covering of the affine scheme $\operatorname{Spec} S$ by its principal affine open subschemes $\operatorname{Spec} S[t_{g_i(\Gamma)}^{-1}]$. The leftmost nontrivial term is the ring S, the next one is the direct sum of the rings $S[t_{g_i(\Gamma)}^{-1}]$, and the further ones are direct sums of the rings $S[t_\Delta^{-1}]$ for subsets $\Delta \subset G$ which, being unions of two or more subsets $g_i(\Gamma) \subset G$, $i = 1, \dots, n$, have cardinality greater than that of Γ. It remains to use the induction assumption together with the facts that the class of very flat R-modules is closed with respect to extensions, the passage to the kernels of surjective morphisms, and direct summands. ☐

Remark 1.7.10. The second half of the above argument essentially proves the following more general result. Suppose that a finite group G acts by automorphisms of a commutative ring S in such a way that S is a finitely generated projective (or, which is the same, a finitely presented flat) module over its subring of G-invariant elements S^G. Then the natural morphism $\operatorname{Spec} S \longrightarrow \operatorname{Spec} S^G$ is very flat. Here, to convince oneself that the above reasoning is applicable, one only needs to notice that G acts transitively in the fibers of the projection $\operatorname{Spec} S \longrightarrow \operatorname{Spec} S^G$ [3, Theorem 5.10 and Exercise 5.13] and the image of any G-invariant open subset in $\operatorname{Spec} S$ is open in $\operatorname{Spec} S^G$ (since any G-invariant ideal in S is contained in the nilradical of the extension in S of its contraction to S^G).

Proposition 1.7.11. *Any finite flat set-theoretically bijective morphism of Noetherian schemes is very flat.*

Proof. A flat morphism of finite presentation is an open map (see above), so any morphism satisfying the assumptions of Proposition is a homeomorphism. Obviously, one can assume both schemes to be affine. Let $\operatorname{Spec} S \to \operatorname{Spec} R$ be our morphism; it suffices to show that an open subset is affine in $\operatorname{Spec} R$ if it is affine in $\operatorname{Spec} S$.

Moreover, we can restrict ourselves to principal affine open subsets in $\operatorname{Spec} S$. So it suffices to check that $\operatorname{Spec} S[s^{-1}] = \operatorname{Spec} S[\operatorname{Norm}_{S/R}(s)^{-1}]$ for any $s \in S$ (where $\operatorname{Norm}_{S/R}(s) \in R$ is the determinant of the R-linear operator of multiplication with s in S). The latter question reduces to the case when R is the spectrum of a field, so S is an Artinian local ring. In this situation, the assertion is obvious (as the norm of an invertible element is invertible, and that of a nilpotent one is nilpotent). ☐

The following lemma, claiming that the very flatness property is local with respect a certain special class of very flat coverings, is to be compared with Lemma 1.7.1. According to Theorem 1.7.9 and Proposition 1.7.11 (and the proof of the former), any surjective finite étale morphism or finite flat set-theoretically bijective morphism of Noetherian schemes $g\colon x \longrightarrow X$ satisfies its conditions.

Lemma 1.7.12. *Let $g\colon x \longrightarrow X$ be a very flat affine morphism of schemes such that for any (small enough) affine open subscheme $U \subset X$ the ring $\mathcal{O}(g^{-1}(U))$ is a projective module over $\mathcal{O}(U)$ and a projective generator of the abelian category of $\mathcal{O}(U)$-modules. Then a morphism of schemes $f\colon Y \longrightarrow X$ is very flat whenever the morphism $f'\colon y = x \times_X Y \longrightarrow x$ is very flat.*

Proof. Let $V \subset Y$ and $U \subset X$ be affine open subschemes such that $f(V) \subset U$. Set $u = g^{-1}(U)$ and assume that the $\mathcal{O}(u)$-module $\mathcal{O}(u) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)$ is very flat. Then, the morphism g being very flat, the ring $\mathcal{O}(u) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)$ is very flat as an $\mathcal{O}(U)$-module, too. Finally, since $\mathcal{O}(u)$ is a projective generator of the category of $\mathcal{O}(U)$-modules, we can conclude that the ring $\mathcal{O}(V)$ is a very flat $\mathcal{O}(U)$-module. $\square$

For any scheme X, we denote by $\mathbb{A}^n_X$ the n-dimensional relative affine space over X; so if $X = \operatorname{Spec} R$ then $\mathbb{A}^n_X = \operatorname{Spec} R[x_1, \dots, x_n] = \mathbb{A}^n_R$.

Theorem 1.7.13. *For any scheme X of finite type over a field k and any $n \geqslant 1$, the natural projection $\mathbb{A}^n_X \longrightarrow X$ is a very flat morphism.*

Proof. Since the very flatness property is local, and the class of very flat morphisms is preserved by compositions and base changes with respect to closed embeddings (see Lemma 1.7.1), it suffices to show that the projection morphisms $\pi_n \colon \mathbb{A}^{n+1}_k \longrightarrow \mathbb{A}^n_k$ are very flat for all $n \geqslant 0$. Furthermore, we will now prove that the field k can be assumed to be algebraically closed.

Indeed, let us check that for any algebraic field extension L/k the morphism $g \colon X_L = \operatorname{Spec} L \times_{\operatorname{Spec} k} X \longrightarrow X$ satisfies the assumptions of Lemma 1.7.12. All the other conditions being obvious, we only have to show that the morphism g is very flat. We can assume the scheme X to be affine. Then any principal affine open subscheme in X_L is defined over some subfield $l \subset L$ finite over k. This reduces the question to the case of a finite field extension l/k, when one can apply Theorem 1.7.9 (in the case of a separable field extension) and Proposition 1.7.11 (for a purely inseparable one).

Assuming the field k to be (at least) infinite, we proceed by induction in n. The case $n = 0$ is obvious. Let U be a principal affine open subscheme in $\mathbb{A}^{n+1}_k$; we have to show that $\mathcal{O}(U)$ is a very flat $k[x_1, \dots, x_n]$-module.

The complement $\mathbb{A}^{n+1}_k \setminus U$ is an affine hypersurface, i. e., the zero locus of a polynomial $f \in k[x_1, \dots, x_{n+1}]$. Let us decompose the polynomial f into a product of irreducible ones and separate those factors which do not depend on x_{n+1}. So the complement $\mathbb{A}^{n+1}_k \setminus U$ is presented as the union $Z \cup \pi_n^{-1}(W)$, where W is a hypersurface in $\mathbb{A}^n_k$ and Z is a hypersurface in $\mathbb{A}^{n+1}_k$ whose projection $\pi_n|_Z \colon Z \longrightarrow \mathbb{A}^n_k$, outside of a full preimage $Y = \pi_n|_Z^{-1}(X) \subset Z$ of a proper closed subvariety $X \subset \mathbb{A}^n_k$, is the composition of a finite flat homeomorphism and a finite étale map. The dimensions of both X and Y do not exceed $n - 1$.

Since the class of very flat R-modules is closed with respect to the tensor products over R, it suffices to show that the $k[x_1, \dots, x_n]$-module $\mathcal{O}(\mathbb{A}^{n+1}_k \setminus Z)$ is very flat; so we can assume W to be empty. Furthermore, we may presume the dimensions of all the irreducible components of the varieties X and Y to be equal to $n - 1$ exactly.

Consider the $(n + 1)$-dimensional vector space k^{n+1} over k with the coordinate linear functions $x_1, \dots, x_{n+1}$. A *line* (one-dimensional vector subspace) in k^{n+1} will be called *vertical* if it contains the vector $(0, \dots, 0, 1)$ and *horizonal* if it is generated by a vector whose last coordinate vanishes. Let us choose a nonvertical line, and draw an affine line through every point in $Y \subset \mathbb{A}^{n+1}_k$ in the chosen direction.

After the passage to the Zariski closure of this whole set of points, we will obtain (the chosen direction being generic) a certain hypersurface $H \subset \mathbb{A}_k^{n+1}$. A linear coordinate change affecting only $x_1, \ldots, x_n$ will transform the chosen nonvertical line into a horizontal one. Let it be the line spanned by the vector $(0, \ldots, 0, 1, 0) \in k^{n+1}$. In these new coordinates, the hypersurface H is the full preimage of a hypersurface in $\operatorname{Spec} k[x_1, \ldots, x_{n-1}, x_{n+1}]$ with respect to the "horizontal" projection $\pi'_n\colon \operatorname{Spec} k[x_1, \ldots, x_{n+1}] \longrightarrow \operatorname{Spec} k[x_1, \ldots, x_{n-1}, x_{n+1}]$ forgetting the coordinate x_n.

Given a fixed point $q \in \mathbb{A}^{n+1}(k) \setminus Y(k)$, the lines in the directions from q to the points in Y form, at most, an $(n-1)$-dimensional subvariety in the n-dimensional projective space of all lines in k^{n+1}. Hence there is a finite set of nonvertical lines $p_i \subset k^{n+1}$ such that the intersection of the corresponding hypersurfaces $H_i \subset \mathbb{A}_k^{n+1}$ coincides with Y. The very flatness property being local, it suffices to show that the $k[x_1, \ldots, x_n]$-modules $\mathcal{O}(\mathbb{A}_{k+1}^n \setminus (H \cup Z))$ are very flat. By the induction assumption, we know that such are the $k[x_1, \ldots, x_n]$-modules $\mathcal{O}(\mathbb{A}_{k+1}^n \setminus H)$.

By Theorem 1.7.9 and Proposition 1.7.11, the ring $\mathcal{O}(Z \setminus H)$ is a very flat $\mathcal{O}(\mathbb{A}_k^n \setminus X)$-module, hence also a very flat $\mathcal{O}(\mathbb{A}_k^n)$-module. To complete the proof, it remains to make use of the following lemma (applied to the rings $R = k[x_1, \ldots, x_n]$ and $S = \mathcal{O}(\mathbb{A}_k^{n+1} \setminus H)$, and the equation of Z in the role of the element s). ☐

Lemma 1.7.14. *Let $R \longrightarrow S$ be a homomorphism of commutative rings, $s \in S$ be an element, and G be an S-module without s-torsion (i. e., s acts in G by an injective operator). Suppose that the R-modules G and G/sG are very flat. Then the R-module $G[s^{-1}]$ is also very flat.*

In particular, if s is a nonzero-divisor in S and the R-modules S and S/sS are very flat, then the R-module $S[s^{-1}]$ is also very flat.

Proof. The formula $G[s^{-1}] = \varinjlim_{n \in \mathbb{N}} s^{-n}G$ makes $G[s^{-1}]$ a transfinitely iterated extension of one copy of G and a countable set of copies of G/sG. ☐

For any scheme X, denote by $\mathbb{A}_X^\infty$ the infinite-dimensional relative affine space (of countable relative dimension) over X. So if $X = \operatorname{Spec} R$, then $\mathbb{A}_X^\infty = \operatorname{Spec} R[x_1, x_2, x_3, \ldots]$.

Corollary 1.7.15. *For any scheme X of finite type over a field k, the natural projection $\mathbb{A}_X^\infty \longrightarrow X$ is a very flat morphism.*

Proof. One can assume X to be affine. Then any principal affine open subscheme in $\mathbb{A}_X^\infty$ is the complement to the subscheme of zeroes of an equation depending on a finite number of variables x_i only. Thus the assertion follows from Theorem 1.7.13. ☐

Proposition 1.7.16. (a) *Any finite flat morphism of one-dimensional schemes of finite type over a field k is very flat.*

(b) *Any flat (or, which is equivalent, quasi-finite) morphism of finite type from a reduced one-dimensional scheme to a smooth one-dimensional scheme of finite type over a field k is very flat.*

Proof. Part (a): let $f\colon Y \longrightarrow X$ be a finite flat morphism of affine one-dimensional schemes over k and $V \subset Y$ be an open subscheme (notice that any open subscheme in an affine one-dimensional scheme is affine).

Assume first that V contains the general points of all the irreducible components of Y, i. e., $V = Y \setminus Z$, where $Z \subset Y$ is a finite set of closed points. Consider the open subscheme $W = Y \setminus (f^{-1}(f(Z)) \setminus Z)$. Then $V \cup W = Y$ and $V \cap W = Y \setminus f^{-1}(f(Z))$, so $V \cup W$ is flat and finite over X, while $V \cap W$ is flat and finite over $U = X \setminus f(Z)$.

Now in the Mayer–Vietoris exact sequence $0 \longrightarrow \mathcal{O}(V \cup W) \longrightarrow \mathcal{O}(V) \oplus \mathcal{O}(W) \longrightarrow \mathcal{O}(V \cap W) \longrightarrow 0$ the left and right terms, being respectively a projective $\mathcal{O}(X)$-module and a projective $\mathcal{O}(U)$-module, are both very flat $\mathcal{O}(X)$-modules. Hence the middle term is also very flat over $\mathcal{O}(X)$, and so is its direct summand $\mathcal{O}(V)$.

In the general case, set $Z = Y \setminus V$ and define an open subscheme $W \subset Y$ as the complement to the Zariski closure of $f^{-1}(f(Z)) \setminus Z$ in Y. Then we have again $V \cap W = Y \setminus f^{-1}(f(Z))$, which is a flat and finite scheme over $U = X \setminus f(Z)$, hence $\mathcal{O}(V \cap W)$ is a very flat $\mathcal{O}(X)$-module. On the other hand, the union $V \cup W$ is an open subscheme containing all the general points of irreducible components in Y, so the $\mathcal{O}(X)$-module $\mathcal{O}(V \cup W)$ is very flat according to the above. It remains to use the very same Mayer–Vietoris sequence once again.

To deduce part (b) from part (a), one can apply Zariski's main theorem, embedding a quasi-finite morphism into a finite one. The equivalence of flatness and quasi-finiteness (a particular case of the general description of flat morphisms from reduced schemes to smooth one-dimensional ones) mentioned in the formulation of part (b) plays a key role in this argument. □

2. Contraherent Cosheaves over a Scheme

2.1. **Cosheaves of modules over a sheaf of rings.** Let X be a topological space. A *copresheaf of abelian groups* on X is a covariant functor from the category of open subsets of X (with the identity embeddings as morphisms) to the category of abelian groups.

Given a copresheaf of abelian groups $\mathfrak{P}$ on X, we will denote the abelian group it assigns to an open subset $U \subset X$ by $\mathfrak{P}[U]$ and call it the group of *cosections* of $\mathfrak{P}$ over U. For a pair of embedded open subsets $V \subset U \subset X$, the map $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ that the copresheaf $\mathfrak{P}$ assigns to $V \subset U$ will be called the *corestriction* map.

A copresheaf of abelian groups $\mathfrak{P}$ on X is called a *cosheaf* if for any open subset $U \subset X$ and its open covering $U = \bigcup_\alpha U_\alpha$ the following sequence of abelian groups is exact

$$\bigoplus\nolimits_{\alpha,\beta} \mathfrak{P}[U_\alpha \cap U_\beta] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{P}[U] \longrightarrow 0. \tag{5}$$

Let $\mathcal{O}$ be a sheaf of associative rings on X. A copresheaf of abelian groups $\mathfrak{P}$ on X is said to be a *copresheaf of* (*left*) $\mathcal{O}$*-modules* if for each open subset $U \subset X$ the abelian group $\mathfrak{P}[U]$ is endowed with the structure of a (left) module over the ring $\mathcal{O}(U)$, and for each pair of embedded open subsets $V \subset U \subset X$ the map of corestriction of cosections $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ in the copresheaf $\mathfrak{P}$ is a homomorphism of $\mathcal{O}(U)$-modules. Here the $\mathcal{O}(U)$-module structure on $\mathfrak{P}[V]$ is obtained from the $\mathcal{O}(V)$-module structure by the restriction of scalars via the ring homomorphism $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$.

A copresheaf of $\mathcal{O}$-modules on X is called a *cosheaf of $\mathcal{O}$-modules* if its underlying copresheaf of abelian groups is a cosheaf of abelian groups.

Remark 2.1.1. One can define copresheaves with values in any category, and cosheaves with values in any category that has coproducts. In particular, one can speak of cosheaves of sets, etc. Notice, however, that, unlike for (pre)sheaves, the underlying copresheaf of sets of a cosheaf of abelian groups is *not* a cosheaf of sets in general, as the forgetful functor from the abelian groups to sets preserves products, but not coproducts. Thus cosheaves of sets (as developed, e. g., in [11]) and cosheaves of abelian groups or modules are two quite distinct theories.

Let $\mathbf{B}$ be a base of open subsets of X. We will consider covariant functors from $\mathbf{B}$ (viewed as a full subcategory of the category of open subsets in X) to the category of abelian groups. We say that such a functor $\mathfrak{Q}$ is *endowed with an $\mathcal{O}$-module structure* if the abelian group $\mathfrak{Q}[U]$ is endowed with an $\mathcal{O}(U)$-module structure for each $U \in \mathbf{B}$ and the above compatibility condition holds for the corestriction maps $\mathfrak{Q}[V] \longrightarrow \mathfrak{Q}[U]$ assigned by the functor $\mathfrak{Q}$ to all V, $U \in \mathbf{B}$ such that $V \subset U$.

The following result is essentially contained in [26, Section 0.3.2], as is its (more familiar) sheaf version, to which we will turn in due order.

Theorem 2.1.2. *A covariant functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure on a base $\mathbf{B}$ of open subsets of X can be extended to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X if and only if the following condition holds. For any open subset $V \in \mathbf{B}$, any its covering $V = \bigcup_\alpha V_\alpha$ by open subsets $V_\alpha \in \mathbf{B}$, and any (or, equivalently, some particular) covering $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ the sequence of abelian groups (or $\mathcal{O}(V)$-modules)*

$$\bigoplus\nolimits_{\alpha,\beta,\gamma} \mathfrak{Q}[W_{\alpha\beta\gamma}] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{Q}[V_\alpha] \longrightarrow \mathfrak{Q}[V] \longrightarrow 0 \tag{6}$$

must be exact. The functor of restriction of cosheaves of $\mathcal{O}$-modules to a base $\mathbf{B}$ is an equivalence between the category of cosheaves of $\mathcal{O}$-modules on X and the category of covariant functors on $\mathbf{B}$, endowed with $\mathcal{O}$-module structures and satisfying (6).

Proof. The elementary approach taken in the exposition below is to pick an appropriate stage at which one can dualize and pass to (pre)sheaves, where our intuitions work better. First we notice that if the functor $\mathfrak{Q}$ (with its $\mathcal{O}$-module structure) has been extended to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X, then for any open subset $U \subset X$ there is an exact sequence of $\mathcal{O}(U)$-modules

$$\bigoplus\nolimits_{W,V',V''} \mathfrak{Q}[W] \longrightarrow \bigoplus\nolimits_V \mathfrak{Q}[V] \longrightarrow \mathfrak{P}[U] \longrightarrow 0, \tag{7}$$

where the summation in the middle term runs over all open subsets $V \in \mathbf{B}$, $V \subset U$, while the summation in the leftmost term is done over all triples of open subsets W, V', $V'' \in \mathbf{B}$, $W \subset V'$, $V'' \subset U$. Conversely, given a functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure one can recover the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ as the cokernel of the left arrow.

Clearly, the modules $\mathfrak{P}[U]$ constructed in this way naturally form a copresheaf of $\mathcal{O}$-modules on X. Before proving that it is a cosheaf, one needs to show that for any open covering $U = \bigcup_\alpha V_\alpha$ of an open subset $U \subset X$ by open subsets $V_\alpha \in \mathbf{B}$ and any open coverings $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ the natural map from the cokernel of the morphism

$$\bigoplus\nolimits_{\alpha,\beta,\gamma} \mathfrak{Q}[W_{\alpha\beta\gamma}] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{Q}[V_\alpha] \tag{8}$$

to the (above-defined) $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is an isomorphism. In particular, it will follow that $\mathfrak{P}[V] \simeq \mathfrak{Q}[V]$ for $V \in \mathbf{B}$.

Notice that it suffices to check both assertions for co(pre)sheaves of abelian groups (though it will not matter in the subsequent argument). Notice also that a copresheaf of $\mathcal{O}$-modules $\mathfrak{P}$ is a cosheaf if and only if the dual presheaf of $\mathcal{O}$-modules $U \longmapsto \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{P}[U], I)$ is a sheaf on X for every abelian group I (or specifically for $I = \mathbb{Q}/\mathbb{Z}$). Similarly, the condition (6) holds for a covariant functor $\mathfrak{Q}$ on a base $\mathbf{B}$ if and only if the dual condition (9) below holds for the contravariant functor $V \longmapsto \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{Q}[V], I)$ on $\mathbf{B}$. So it remains to prove the following Proposition 2.1.3. □

Now we will consider contravariant functors $\mathcal{G}$ from $\mathbf{B}$ to the category of abelian groups, and say that such a functor is endowed with an $\mathcal{O}$-module structure if the abelian group $\mathcal{G}(U)$ is an $\mathcal{O}(U)$-module for every $U \in \mathbf{B}$ and the restriction morphisms $\mathcal{G}(U) \longrightarrow \mathcal{G}(V)$ are morphisms of $\mathcal{O}(U)$-modules for all V, $U \in \mathbf{B}$ such that $V \subset U$.

Proposition 2.1.3. *A contravariant functor $\mathcal{G}$ with an $\mathcal{O}$-module structure on a base $\mathbf{B}$ of open subsets of X can be extended to a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on X if and only if the following condition holds. For any open subset $V \in \mathbf{B}$, any its covering $V = \bigcup_\alpha V_\alpha$ by open subsets $V_\alpha \in \mathbf{B}$, and any (or, equivalently, some particular) covering $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ the sequence of abelian groups (or $\mathcal{O}(V)$-modules)*

$$0 \longrightarrow \mathcal{G}(V) \longrightarrow \prod\nolimits_\alpha \mathcal{G}(V_\alpha) \longrightarrow \prod\nolimits_{\alpha,\beta,\gamma} \mathcal{G}(W_{\alpha\beta\gamma}) \tag{9}$$

must be exact. The functor of restriction of sheaves of $\mathcal{O}$-modules to a base $\mathbf{B}$ is an equivalence between the category of sheaves of $\mathcal{O}$-modules on X and the category of contravariant functors on $\mathbf{B}$, endowed with $\mathcal{O}$-module structures and satisfying (9).

Sketch of proof. As above, we notice that if the functor $\mathcal{G}$ (with its $\mathcal{O}$-module structure) has been extended to a sheaf of $\mathcal{O}$-modules $\mathcal{G}$ on X, then for any open subset $U \subset X$ there is an exact sequence of $\mathcal{O}(U)$-modules

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod\nolimits_V \mathcal{G}(V) \longrightarrow \prod\nolimits_{W,V',V''} \mathcal{G}(W),$$

the summation rules being as in (7). Conversely, given a functor $\mathcal{G}$ with an $\mathcal{O}$-module structure one can recover the $\mathcal{O}(U)$-module $\mathcal{F}(U)$ as the kernel of the right arrow.

The rest is a conventional argument with (pre)sheaves and coverings. Recall that a presheaf $\mathcal{F}$ on X is called *separated* if the map $\mathcal{F}(U) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha)$ is injective for any open covering $U = \bigcup_\alpha U_\alpha$ of an open subset $U \subset X$. Similarly, a contravariant functor $\mathcal{G}$ on a base $\mathbf{B}$ is said to be separated if its sequences (9) are exact at the leftmost nontrivial term.

For any open covering $U = \bigcup_\alpha V_\alpha$ of an open subset $U \subset X$ by open subsets $V_\alpha \in \mathbf{B}$ and any open coverings $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ there is a natural map from the (above-defined) $\mathcal{O}(U)$-module $\mathcal{F}(U)$ to the kernel of the morphism

$$\prod_\alpha \mathcal{G}(V_\alpha) \longrightarrow \prod_{\alpha,\beta,\gamma} \mathcal{G}(W_{\alpha\beta\gamma}). \tag{10}$$

Let us show that this map is an isomorphism provided that $\mathcal{G}$ satisfies (9). In particular, it will follow that $\mathcal{F}(V) = \mathcal{G}(V)$ for $V \in \mathbf{B}$.

Clearly, when $\mathcal{G}$ is separated, the kernel of (10) does not depend on the choice of the open subsets $W_{\alpha\beta\gamma}$. So we can assume that the collection $\{W_{\alpha\beta\gamma}\}$ for fixed α and β consists of all open subsets $W \in \mathbf{B}$ such that $W \subset V_\alpha \cap V_\beta$.

Furthermore, one can easily see that the map from $\mathcal{F}(U)$ to the kernel of (10) is injective whenever $\mathcal{G}$ is separated. To check surjectivity, suppose that we are given a collection of sections $\phi_\alpha \in \mathcal{G}(V_\alpha)$ representing an element of the kernel.

Fix an open subset $V \in \mathbf{B}$, $V \subset U$, and consider its covering by all the open subsets $W \in \mathbf{B}$ such that $W \subset V \cap V_\alpha$ for some α. Set $\psi_W = \phi_\alpha|_W \in \mathcal{G}(W)$ for every such W; by assumption, if $W \subset V_\alpha \cap V_\beta$, then $\phi_\alpha|_W = \phi_\beta|_W$, so the element ψ_W is well-defined. Applying (9), we conclude that there exists a unique element $\phi_V \in \mathcal{G}(V)$ such that $\phi_V|_W = \psi_W$ for any $W \subset V \cap V_\alpha$. The collection of sections ϕ_V represents an element of $\mathcal{F}(U)$ that is a preimage of our original element of the kernel of (10).

Now let us show that $\mathcal{F}$ is a sheaf. Let $U = \bigcup_\alpha U_\alpha$ be a open covering of an open subset $U \subset X$. First let us see that $\mathcal{F}$ is separated provided that $\mathcal{G}$ is. Let $s \in \mathcal{F}(U)$ be a section whose restriction to all the open subsets U_α vanishes. The element s is represented by a collection of sections $\phi_V \in \mathcal{G}(V)$ defined for all open subsets $V \subset U$, $V \in \mathbf{B}$. The condition $s|_{U_\alpha} = 0$ means that $\phi_W = 0$ whenever $W \subset U_\alpha$, $W \in \mathbf{B}$. To check that $\phi_V = 0$ for all V, we notice that open subsets $W \subset V$, $W \in \mathbf{B}$ for which there exists α such that $W \subset U_\alpha$ form an open covering of V.

Finally, let $s_\alpha \in \mathcal{F}(U_\alpha)$ be a collection of sections such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Every element s_α is represented by a collection of sections $\phi_V \in \mathcal{G}(V)$ defined for all open subsets $V \subset U_\alpha$, $V \in \mathbf{B}$. Clearly, the element ϕ_V does not depend on the choice of a particular α for which $V \subset U_\alpha$, so our notation is consistent. All the open subsets $V \subset U$, $V \in \mathbf{B}$ for which there exists some α such that $V \subset U_\alpha$ form an open covering of the open subset $U \subset X$. The collection of sections ϕ_V represents an element of the kernel of the morphism (10) for this covering, hence it corresponds to an element of $\mathcal{F}(U)$. □

Remark 2.1.4. Let X be a topological space with a topology base $\mathbf{B}$ consisting of quasi-compact open subsets (in the induced topology) for which the intersection of any two open subsets from $\mathbf{B}$ that are contained in a third open subset from $\mathbf{B}$ is quasi-compact as well. E. g., any scheme X with the base of all affine open subschemes has these properties. Then it suffices to check both the conditions (6) and (9) for *finite* coverings V_α and $W_{\alpha\beta\gamma}$ only.

Indeed, let us explain the sheaf case. Obviously, injectivity of the left arrow in (9) for any given covering $V = \bigcup_\alpha V_\alpha$ follows from such injectivity for a subcovering

$V = \bigcup_i V_i$, $\{V_i\} \subset \{V_\alpha\}$. Assuming $\mathcal{G}$ is separated, one checks that exactness of the sequence (9) for any given covering follows from the same exactness for a subcovering.

It follows that for a topological space X with a fixed topology base $\mathbf{B}$ satisfying the above condition there is another duality construction relating sheaves to cosheaves in addition to the one we used in the proof of Theorem 2.1.2. Given a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on X, one restricts it to the base $\mathbf{B}$, obtaining a contravariant functor $\mathcal{G}$ with an $\mathcal{O}$-module structure, defines the dual covariant functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure on $\mathbf{B}$ by the rule $\mathfrak{Q}[V] = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{G}(V), I)$, where I is an injective abelian group, and extends the functor $\mathfrak{Q}$ to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X.

It is this second duality functor, rather than the one from the proof of Theorem 2.1.2, that will play a role in the sequel (Section 3.4 being a rare exception).

2.2. **Exact category of contraherent cosheaves.** Let X be a scheme and $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is called *contraherent* if for any pair of embedded affine open subschemes $V \subset U \subset X$

(i) the morphism of $\mathcal{O}_X(V)$-modules $\mathfrak{P}[V] \longrightarrow \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U])$ induced by the corestriction morphism $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ is an isomorphism; and

(ii) one has $\operatorname{Ext}^{>0}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$.

It follows from Lemma 1.2.4 that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ has projective dimension at most 1, so it suffices to require the vanishing of Ext^1 in the condition (ii). We will call (ii) the *contraadjustness condition*, and (i) the *contraherence condition*.

Theorem 2.2.1. *The restriction of cosheaves of $\mathcal{O}_X$-modules to the base of all affine open subschemes of X induces an equivalence between the category of contraherent cosheaves on X and the category of covariant functors $\mathfrak{Q}$ with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X, satisfying the conditions* (i-ii) *for any pair of embedded affine open subschemes $V \subset U \subset X$.*

Proof. According to Theorem 2.1.2, a cosheaf of $\mathcal{O}_X$-modules is determined by its restriction to the base of affine open subsets of X. The contraadjustness and contraherence conditions depend only on this restriction. By Lemma 1.2.4, given any affine scheme U, a module P over $\mathcal{O}(U)$ is contraadjusted if and only if $\operatorname{Ext}^1_{\mathcal{O}(U)}(\mathcal{O}(V), P) = 0$ for all affine open subschemes $V \subset U$. Finally, the key observation is that the contraadjustness and contraherence conditions (i-ii) for a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X imply the cosheaf condition (6). This follows from Lemma 1.2.6(b) and Remark 2.1.4. □

Remark 2.2.2. Of course, one can similarly define quasi-coherent sheaves $\mathcal{F}$ on X as sheaves of $\mathcal{O}_X$-modules such that for any pair of embedded affine open subschemes $V \subset U \subset X$ the morphism of $\mathcal{O}_X(V)$-modules $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ induced by the restriction morphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is an isomorphism. Since the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is always flat, no version of the condition (ii) is needed in this case. The analogue of Theorem 2.2.1 is well-known for quasi-coherent sheaves (and can be proven in the same way).

A short sequence of contraherent cosheaves $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is said to be exact if the sequence of cosection modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact for every affine open subscheme $U \subset X$. Notice that if U_α is an affine open covering of an affine scheme U and $\mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R}$ is a sequence of contraherent cosheaves on U, then the sequence of $\mathcal{O}(U)$-modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact if and only if all the sequences of $\mathcal{O}(U_\alpha)$-modules $0 \longrightarrow \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{Q}[U_\alpha] \longrightarrow \mathfrak{R}[U_\alpha] \longrightarrow 0$ are. This follows from Lemma 1.4.1(a).

We denote the exact category of contraherent cosheaves on a scheme X by $X\text{–}\mathsf{ctrh}$. By the definition, the functors of cosections over affine open subschemes are exact on this exact category. It is also has exact functors of infinite product, which commute with cosections over affine open subschemes (and in fact, over any quasi-compact quasi-separated open subschemes as well). For a more detailed discussion of this exact category structure, we refer the reader to Section 3.1.

Corollary 2.2.3. *The functor assigning the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ to a contraherent cosheaf $\mathfrak{P}$ on an affine scheme U is an equivalence between the exact category $U\text{–}\mathsf{ctrh}$ of contraherent cosheaves on U and the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cta}}$ of contraadjusted modules over the commutative ring $\mathcal{O}(U)$.*

Proof. Clear from the above arguments together with Lemmas 1.2.1(b) and 1.2.4. □

Lemma 2.2.4. *Let $\mathfrak{P}$ be a cosheaf of $\mathcal{O}_U$-modules and $\mathfrak{Q}$ be a contraherent cosheaf on an affine scheme U. Then the group of morphisms of cosheaves of $\mathcal{O}_U$-modules $\mathfrak{P} \longrightarrow \mathfrak{Q}$ is isomorphic to the group of morphisms of $\mathcal{O}(U)$-modules $\mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U]$.*

Proof. Any morphism of cosheaves of $\mathcal{O}_U$-modules $\mathfrak{P} \longrightarrow \mathfrak{Q}$ induces a morphism of the $\mathcal{O}(U)$-modules of global cosections. Conversely, given a morphism of $\mathcal{O}(U)$-modules $\mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U]$ and an affine open subscheme $V \subset U$, the composition $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U]$ is a morphism of $\mathcal{O}(U)$-modules from an $\mathcal{O}(V)$-module $\mathfrak{P}[V]$ to an $\mathcal{O}(U)$-module $\mathfrak{Q}[U]$. It induces, therefore, a morphism of $\mathcal{O}(V)$-modules $\mathfrak{P}[V] \longrightarrow \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V), \mathfrak{Q}[U]) \simeq \mathfrak{Q}[V]$. Now a morphism between the restrictions of two cosheaves of $\mathcal{O}_U$-modules to the base of affine open subschemes of U extends uniquely to a morphism between the whole cosheaves. □

A contraherent cosheaf $\mathfrak{P}$ on a scheme X is said to be *locally cotorsion* if for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathfrak{P}(U)$ is cotorsion. By Lemma 1.3.6(a), the property of a contraherent cosheaf on an affine scheme to be cotorsion is indeed a local, so our terminology is constent.

A contraherent cosheaf $\mathfrak{J}$ on a scheme X is called *locally injective* if for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathfrak{J}(U)$ is injective. By Lemma 1.3.6(b), local injectivity of a contraherent cosheaf is indeed a local property.

Just as above, one defines the exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ of locally cotorsion and locally injective contraherent cosheaves on X. These are full subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in $X\text{–}\mathsf{ctrh}$, with the induced exact category structures.

The exact category $U\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ of locally cotorsion contraherent cosheaves on an affine scheme U is equivalent to the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cot}}$ of cotorsion $\mathcal{O}(U)$-modules.

The exact category $U\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ of locally injective contraherent cosheaves on U is equivalent to the additive category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{inj}}$ of injective $\mathcal{O}(U)$-modules endowed with the trivial exact category structure.

Remark 2.2.5. Notice that a morphism of contraherent cosheaves on X is an admissible monomorphism if and only if it acts injectively on the cosection modules over all the affine open subschemes on X. At the same time, the property of a morphism of contraherent cosheaves on X to be an admissible monomorphism is *not* local in X, and *neither* is the property of a cosheaf of $\mathcal{O}_X$-modules to be contraherent (see Section 3.2 below). The property of a morphism of contraherent cosheaves to be an admissible epimorphism is local, though (see Lemma 1.4.1(b)). All of the above applies to locally cotorsion and locally injective contraherent cosheaves as well.

Notice also that a morphism of locally injective or locally cotorsion contraherent cosheaves that is an admissible epimorphism in $X\text{–}\mathsf{ctrh}$ may *not* be an admissible epimorphism in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$. On the other hand, if a morphism of locally injective or locally cotorsion contraherent cosheaves is an admissible monomorphism in $X\text{–}\mathsf{ctrh}$, then it is also an admissible monomorphism in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$, as it is clear from the above.

2.3. **Direct and inverse images of contraherent cosheaves.** Let $\mathcal{O}_X$ be a sheaf of associative rings on a topological space X and $\mathcal{O}_Y$ be such a sheaf on a topological space Y. Furthermore, let $f\colon Y \longrightarrow X$ be a morphism of ringed spaces, i. e., a continuous map $Y \longrightarrow X$ together with a morphism $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ of sheaves of rings over X. Then for any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$ the rule $(f_!\mathfrak{Q})[W] = \mathfrak{Q}[f^{-1}(W)]$ for all open subsets $W \subset X$ defines a cosheaf of $\mathcal{O}_X$-modules $f_!\mathfrak{Q}$.

Let $\mathcal{O}_X$ be a sheaf of associative rings on a topological space X and $Y \subset X$ be an open subspace. Denote by $\mathcal{O}_Y = \mathcal{O}_X|_Y$ the restriction of the sheaf of rings $\mathcal{O}_X$ onto Y, and by $j\colon Y \longrightarrow X$ the corresponding morphism (open embedding) of ringed spaces. Given a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X, the restriction $\mathfrak{P}|_Y$ of $\mathfrak{P}$ onto Y is a cosheaf of $\mathcal{O}_Y$-modules defined by the rule $\mathfrak{P}|_Y(V) = \mathfrak{P}(V)$ for any open subset $V \subset Y$. One can easily see that the restriction functor $\mathfrak{P} \longmapsto \mathfrak{P}|_Y$ is right adjoint to the direct image functor $\mathfrak{Q} \longmapsto j_!\mathfrak{Q}$ between the categories of cosheaves of $\mathcal{O}_Y$- and $\mathcal{O}_X$-modules, that is the adjunction isomorphism

$$\operatorname{Hom}^{\mathcal{O}_X}(j_!\mathfrak{Q},\mathfrak{P}) \simeq \operatorname{Hom}^{\mathcal{O}_Y}(\mathfrak{Q},\mathfrak{P}|_Y) \tag{11}$$

holds for any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ and cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, where $\operatorname{Hom}^{\mathcal{O}_X}$ and $\operatorname{Hom}^{\mathcal{O}_Y}$ denote the abelian groups of morphisms in the categories of cosheaves of modules over the sheaves of rings $\mathcal{O}_X$ and $\mathcal{O}_Y$. Since one has $(j_!\mathfrak{Q})_Y \simeq \mathfrak{Q}$, it follows, in particular, that the functor $j_!$ is fully faithful.

Let $f\colon Y \longrightarrow X$ be an affine morphism of schemes, and let $\mathfrak{Q}$ be a contraherent cosheaf on Y. Then $f_!\mathfrak{Q}$ is a contraherent cosheaf on X. Indeed, for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $(f_!\mathfrak{Q})[U] = \mathfrak{Q}[(f^{-1}(U)]$ is contraadjusted according to Lemma 1.2.2(a) applied to the morphism of commutative rings $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f^{-1}(U))$. For any pair of embedded affine open subschemes

$V \subset U \subset X$ we have natural isomorphisms of $\mathcal{O}_X(U)$-modules

$$(f_!\mathfrak{Q})[V] = \mathfrak{Q}[f^{-1}(V)] \simeq \operatorname{Hom}_{\mathcal{O}_Y(f^{-1}(U))}(\mathcal{O}_Y(f^{-1}(V)), \mathfrak{Q}[f^{-1}(U)]) \\ \simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{Q}[f^{-1}(U)]) = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), (f_!\mathfrak{Q})[U]),$$

since $\mathcal{O}_Y(f^{-1}(V)) \simeq \mathcal{O}_Y(f^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$.

Recall that a scheme X is called *semi-separated* [65, Appendix B], if it admits an affine open covering with affine pairwise intersections of the open subsets belonging to the covering. Equivalently, a scheme X is semi-separated if and only if the diagonal morphism $X \longrightarrow X \times_{\operatorname{Spec} \mathbb{Z}} X$ is affine, and if and only if the intersection of any two affine open subschemes of X is affine. Any morphism from an affine scheme to a semi-separated scheme is affine, and the fibered product of any two affine schemes over a semi-separated base scheme is an affine scheme.

We will say that a morphism of schemes $f\colon Y \longrightarrow X$ is *coaffine* if for any affine open subscheme $V \subset Y$ there exists an affine open subscheme $U \subset X$ such that $f(V) \subset U$, and for any two such affine open subschemes $f(V) \subset U'$, $U'' \subset X$ there exists a third affine open subscheme $U \subset X$ such that $f(V) \subset U \subset U' \cap U''$. If the scheme X is semi-separated, then the second condition is trivial. (We will see below in Section 3.3 that the second condition is not actually necessary for our constructions.)

Any morphism into an affine scheme is coaffine. Any embedding of an open subscheme is coaffine. The composition of two coaffine morphisms between semi-separated schemes is a coaffine morphism.

Let $f\colon Y \longrightarrow X$ be a very flat coaffine morphism of schemes (see Section 1.7 for the definition and discussion of the former property), and let $\mathfrak{P}$ be a contraherent cosheaf on X. Define a contraherent cosheaf $f^!\mathfrak{P}$ on Y as follows.

Let $V \subset Y$ be an affine open subscheme. Pick an affine open subscheme $U \subset X$ such that $f(V) \subset U$, and set $(f^!\mathfrak{P})[V] = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(V), \mathfrak{P}[U])$. Due to the contraherence condition on $\mathfrak{P}$, this definition of the $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]$ does not depend on the choice of an affine open subscheme $U \subset X$. Since f is a very flat morphism, the $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]$ is contraadjusted by Lemma 1.2.3(a). The contraherence condition obviously holds for $f^!\mathfrak{P}$.

Let $f\colon Y \longrightarrow X$ be a flat coaffine morphism of schemes, and $\mathfrak{P}$ be a locally cotorsion contraherent cosheaf on X. Then the same rule as above defines a locally cotorsion contraherent cosheaf $f^!\mathfrak{P}$ on Y. One just uses Lemma 1.3.5(a) in place of Lemma 1.2.3(a). For any coaffine morphism of schemes $f\colon Y \longrightarrow X$ and a locally injective contraherent cosheaf $\mathfrak{J}$ on X the very same rule defines a locally injective contraherent cosheaf $f^!\mathfrak{J}$ on Y.

For an open embedding of schemes $j\colon Y \longrightarrow X$ and a contraherent cosheaf $\mathfrak{P}$ on X one clearly has $j^!\mathfrak{P} \simeq \mathfrak{P}|_Y$.

If $f\colon Y \longrightarrow X$ is an affine morphism of schemes and $\mathfrak{Q}$ is a locally cotorsion contraherent cosheaf on Y, then $f_!\mathfrak{Q}$ is a locally cotorsion contraherent cosheaf on X. If $f\colon Y \longrightarrow X$ is a flat affine morphism and $\mathfrak{J}$ is a locally injective contraherent cosheaf on Y, then $f_!\mathfrak{J}$ is a locally injective contraherent cosheaf on X.

Let $f\colon Y \longrightarrow X$ be an affine coaffine morphism of schemes. Then for any contraherent cosheaf $\mathfrak{Q}$ on Y and any locally injective contraherent cosheaf $\mathfrak{P}$ on X there is a natural adjunction isomorphism $\operatorname{Hom}^X(f_!\mathfrak{Q}, \mathfrak{P}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$, where Hom^X and Hom^Y denote the abelian groups of morphisms in the categories of contraherent cosheaves on X and Y.

If, in addition, the morphism f is flat, then such an isomorphism holds for any contraherent cosheaf $\mathfrak{Q}$ on Y and any locally cotorsion contraherent cosheaf $\mathfrak{P}$ on X; in particular, $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories of locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ and $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}$. Their restrictions also act as adjoint functors between the exact categories of locally injective contraherent cosheaves $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ and $Y\text{–}\mathsf{ctrh}^{\mathsf{lin}}$.

If the morphism f is very flat, then the functor $f^!\colon X\text{–}\mathsf{ctrh} \longrightarrow Y\text{–}\mathsf{ctrh}$ is right adjoint to the functor $f_!\colon Y\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{ctrh}$. In all the mentioned cases, both abelian groups $\operatorname{Hom}^X(f_!\mathfrak{Q}, \mathfrak{P})$ and $\operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$ are identified with the group whose elements are the collections of homomorphisms of $\mathcal{O}_X(U)$-modules $\mathfrak{Q}(V) \longrightarrow \mathfrak{P}(U)$, defined for all affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ and compatible with the corestriction maps.

All the functors between exact categories of contraherent cosheaves constructed in the above section are exact and preserve infinite products. For a construction of the direct image functor $f_!$ (acting between appropriate exact subcategories of the exact categories of adjusted objects in the exact categories of contraherent cosheaves) for a nonaffine morphism of schemes f, see Section 4.5 below.

2.4. **$\mathfrak{Cohom}$ from a quasi-coherent sheaf to a contraherent cosheaf.** Let X be a scheme over an affine scheme $\operatorname{Spec} R$. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and J be an injective R-module. Then the rule $U \longmapsto \operatorname{Hom}_R(\mathcal{M}(U), J)$ for affine open subschemes $U \subset X$ defines a contraherent cosheaf over X (cf. Remark 2.1.4). We will denote it by $\mathfrak{Cohom}_R(\mathcal{M}, J)$. Since the $\mathcal{O}_X(U)$-module $\operatorname{Hom}_R(\mathcal{M}(U), J)$ is cotorsion by Lemma 1.3.3(b), it is even a locally cotorsion contraherent cosheaf. When $\mathcal{F}$ is a flat quasi-coherent sheaf on X and J is an injective R-module, the contraherent cosheaf $\mathfrak{Cohom}_R(\mathcal{F}, J)$ is locally injective.

We recall the definitions of a very flat morphism of schemes and a very flat quasi-coherent sheaf on a scheme from Section 1.7. If $X \longrightarrow \operatorname{Spec} R$ is a very flat morphism of schemes and $\mathcal{F}$ is a very flat quasi-coherent sheaf on X, then for any contraadjusted R-module P the rule $U \longmapsto \operatorname{Hom}_R(\mathcal{F}(U), P)$ for affine open subschemes $U \subset X$ defines a contraherent cosheaf on X. The contraadjustness condition on the $\mathcal{O}_X(U)$-modules $\operatorname{Hom}_R(\mathcal{F}(U), P)$ holds by Lemma 1.2.3(c). We will denote the cosheaf so constructed by $\mathfrak{Cohom}_R(\mathcal{F}, P)$.

Analogously, if a scheme X is flat over $\operatorname{Spec} R$ and a quasi-coherent sheaf $\mathcal{F}$ on X is flat (or, more generally, the quasi-coherent sheaf $\mathcal{F}$ on X is flat over $\operatorname{Spec} R$, in the obvious sense), then for any cotorsion R-module P the rule $U \longmapsto \operatorname{Hom}_R(\mathcal{F}(U), P)$ for affine open subschemes $U \subset X$ defines a contraherent cosheaf on X. In fact, the $\mathcal{O}_X(U)$-modules $\operatorname{Hom}_R(\mathcal{F}(U), P)$ are cotorsion by Lemma 1.3.3(a), hence the contraherent cosheaf $\mathfrak{Cohom}_R(\mathcal{F}, P)$ constructed in this way is locally cotorsion.

Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on a scheme X and $\mathfrak{P}$ be a contraherent cosheaf on X. Then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the rule $U \longmapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for all affine open subschemes $U \subset X$. For any two embedded affine open subschemes $V \subset U \subset X$ one has

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_X(V)}&(\mathcal{F}(V),\mathfrak{P}[V]) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_X(V)}(\mathcal{O}_X(V)\otimes_{\mathcal{O}_X(U)}\mathcal{F}(U),\ \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\mathfrak{P}[U])) \\ &\qquad\simeq \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])),\end{aligned}$$

so the contraherence condition holds. The contraadjustness condition follows from Lemma 1.2.1(b).

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is locally cotorsion contraherent cosheaf on X, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U \longmapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for all affine open subschemes $U \subset X$. By Lemma 1.3.2(a), $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is a locally cotorsion contraherent cosheaf on X.

Finally, if $\mathcal{M}$ is a quasi-coherent sheaf on X and $\mathfrak{J}$ is a locally injective contraherent cosheaf, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the very same rule. One checks the contraherence condition in the same way as above. By Lemma 1.3.2(b), $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is a locally cotorsion contraherent cosheaf on X. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective contraherent cosheaf on X, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})$ is locally injective.

For any contraadjusted module P over a commutative ring R, denote by $\check{P}$ the corresponding contraherent cosheaf on $\operatorname{Spec} R$. Let $f\colon X \longrightarrow \operatorname{Spec} R$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent sheaf on X. Then whenever $\mathcal{F}$ is a very flat quasi-coherent sheaf and f is a very flat morphism, there is a natural isomorphism of contraherent cosheaves $\mathfrak{Cohom}_R(\mathcal{F},P) \simeq \mathfrak{Cohom}_X(\mathcal{F},f^!\check{P})$ on X. Indeed, for any affine open subscheme $U \subset X$ one has

$$\mathrm{Hom}_R(\mathcal{F}(U),P) \simeq \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathrm{Hom}_R(\mathcal{O}_X(U),P)) \simeq \mathrm{Hom}_R(\mathcal{F}(U),(f^!\check{P})[U]).$$

The same isomorphism holds whenever $\mathcal{F}$ is a flat quasi-coherent sheaf, f is a flat morphism, and P is a cotorsion R-module. Finally, for any quasi-coherent sheaf $\mathcal{M}$ on X, any morphism $f\colon X \longrightarrow \operatorname{Spec} R$, and any injective R-module J there is a natural isomorphism of locally cotorsion contraherent cosheaves $\mathfrak{Cohom}_R(\mathcal{M},J) \simeq \mathfrak{Cohom}_X(\mathcal{M},f^!\check{J})$ on X.

2.5. **Contraherent cosheaves of $\mathfrak{Hom}$ between quasi-coherent sheaves.** A quasi-coherent sheaf $\mathcal{P}$ on a scheme X is said to be *cotorsion* [21] if $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{P})=0$ for any flat quasi-coherent sheaf $\mathcal{F}$ on X. Here Ext_X denotes the Ext groups in the abelian category of quasi-coherent sheaves on X. A quasi-coherent sheaf $\mathcal{P}$ on X is called *contraadjusted* if one has $\mathrm{Ext}^1_X(\mathcal{F},\mathcal{P})=0$ for any very flat quasi-coherent sheaf $\mathcal{F}$ on X (see Section 1.7 for the definition of the latter).

Clearly the two classes of quasi-coherent sheaves on X so defined are closed under extensions, so they form full exact subcategories in the abelian category of quasi-coherent sheaves. Also, these exact subcategories are closed under the passage to direct summands of objects.

For any affine morphism of schemes $f\colon Y \longrightarrow X$, any flat quasi-coherent sheaf $\mathcal{F}$ on X, and any quasi-coherent sheaf $\mathcal{P}$ on Y there is a natural isomorphism of the extension groups $\mathrm{Ext}^1_Y(f^*\mathcal{F}, \mathcal{P}) \simeq \mathrm{Ext}^1_X(\mathcal{F}, f_*\mathcal{P})$. Hence the classes of contraadjusted and cotorsion quasi-coherent sheaves on schemes are preserved by the direct images with respect to affine morphisms.

Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Suppose that an associative ring R acts on X from the right by quasi-coherent sheaf endomorphisms. Let M be a left R-module. Define a contravariant functor $\mathcal{F} \otimes_R M$ from the category of affine open subschemes $U \subset X$ to the category of abelian groups by the rule $(\mathcal{F} \otimes_R M)(U) = \mathcal{F}(U) \otimes_R M$. The natural $\mathcal{O}_X(U)$-module structures on the groups $(\mathcal{F} \otimes_R M)(U)$ are compatible with the restriction maps $(\mathcal{F} \otimes_R M)(U) \longrightarrow (\mathcal{F} \otimes_R M)(V)$ for embedded affine open subschemes $V \subset U \subset X$, and the quasi-coherence condition

$$(\mathcal{F} \otimes_R M)(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} (\mathcal{F} \otimes_R M)(U)$$

holds (see Remark 2.2.2). Therefore, the functor $\mathcal{F} \otimes_R M$ extends uniquely to a quasi-coherent sheaf on X, which we will denote also by $\mathcal{F} \otimes_R M$.

Let $\mathcal{P}$ be a quasi-coherent sheaf on X. Then the abelian group $\mathrm{Hom}_X(\mathcal{F}, \mathcal{P})$ of morphisms in the category of quasi-coherent sheaves on X has a natural left R-module structure. One can easily construct a natural isomorphism of abelian groups $\mathrm{Hom}_X(\mathcal{F} \otimes_R M,\ \mathcal{P}) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_X(\mathcal{F}, \mathcal{P}))$.

Lemma 2.5.1. *Suppose that* $\mathrm{Ext}^i_X(\mathcal{F}, \mathcal{P}) = 0$ *for* $0 < i \leqslant i_0$ *and either*

(a) *M is a flat left R-module, or*

(b) *the right R-modules $\mathcal{F}(U)$ are flat for all affine open subschemes $U \subset X$.*

Then there is a natural isomorphism of abelian groups $\mathrm{Ext}^i_X(\mathcal{F} \otimes_R M,\ \mathcal{P}) \simeq \mathrm{Ext}^i_R(M, \mathrm{Hom}_X(\mathcal{F}, \mathcal{P}))$ *for all* $0 \leqslant i \leqslant i_0$.

Proof. Replace M by its left projective R-module resolution $L_\bullet$. Then $\mathrm{Ext}^i_X(\mathcal{F} \otimes_R L_j,\ \mathcal{P}) = 0$ for all $0 < i \leqslant i_0$ and all j. Due to the flatness condition (a) or (b), the complex of quasi-coherent sheaves $\mathcal{F} \otimes_R L_\bullet$ is a left resolution of the sheaf $\mathcal{F} \otimes_R M$. Hence the complex of abelian groups $\mathrm{Hom}_X(\mathcal{F} \otimes_R L_\bullet,\ \mathcal{P})$ computes $\mathrm{Ext}^i_X(\mathcal{F} \otimes_R M,\ \mathcal{P})$ for $0 \leqslant i \leqslant i_0$. On the other hand, this complex is isomorphic to the complex $\mathrm{Hom}_R(L_\bullet, \mathrm{Hom}_R(\mathcal{F}, \mathcal{P}))$, which computes $\mathrm{Ext}^i_R(M, \mathrm{Hom}_X(\mathcal{F}, \mathcal{P}))$. □

Let $\mathcal{F}$ be a quasi-coherent sheaf with a right action of a ring R on a scheme X, and let $f\colon Y \longrightarrow X$ be a morphism of schemes. Then $f^*\mathcal{F}$ is a quasi-coherent sheaf on Y with a right action of R, and for any left R-module M there is a natural isomorphism of quasi-coherent sheaves $f^*(\mathcal{F} \otimes_R M) \simeq f^*\mathcal{F} \otimes_R M$. Analogously, if $\mathcal{G}$ is a quasi-coherent sheaf on Y with a right action of R and f is a quasi-compact quasi-separated morphism, then $f_*\mathcal{G}$ is a quasi-coherent sheaf on X with a right action of R, and for any left R-module M there is a natural morphism of quasi-coherent sheaves $f_*\mathcal{G}\otimes_R M \longrightarrow f_*(\mathcal{G}\otimes_R M)$ on X. If the morphism f is affine or the R-module M is flat, then this map is an isomorphism of quasi-coherent sheaves on X.

Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on a semi-separated scheme X, and let $\mathcal{P}$ be a contraadjusted quasi-coherent sheaf on X. Define a contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})$ by the rule $U \longmapsto \operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$ for any affine open subscheme $U\subset X$, where $j\colon U\longrightarrow X$ denotes the identity open embedding. Given two embedded affine open subschemes $V\subset U\subset X$ with the identity embeddings $j\colon U\longrightarrow X$ and $k\colon V\longrightarrow X$, the adjunction provides a natural map of quasi-coherent sheaves $j_*j^*\mathcal{F}\longrightarrow k_*k^*\mathcal{F}$. There is also a natural action of the ring $\mathcal{O}_X(U)$ on the quasi-coherent sheaf $j_*j^*\mathcal{F}$. Thus our rule defines a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes in X.

Let us check that the contraadjustness and contraherence conditions are satisfied. For a very flat $\mathcal{O}_X(U)$-module G, we have

$$\operatorname{Ext}^1_{\mathcal{O}_X(U)}(G,\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})) \\ \simeq \operatorname{Ext}^1_X((j_*j^*\mathcal{F})\otimes_{\mathcal{O}_X(U)}G,\ \mathcal{P}) \simeq \operatorname{Ext}^1_X(j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}G),\ \mathcal{P})=0,$$

since $j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}G)$ is a very flat quasi-coherent sheaf on X. For a pair of embedded affine open subschemes $V\subset U\subset X$, we have

$$\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P}))\simeq \operatorname{Hom}_X((j_*j^*\mathcal{F})\otimes_{\mathcal{O}_X(U)}\mathcal{O}_X(V),\ \mathcal{P}) \\ \simeq \operatorname{Hom}_X(j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}\mathcal{O}_X(V)),\ \mathcal{P})\simeq \operatorname{Hom}_X(k_*k^*\mathcal{F},\mathcal{P}).$$

Similarly one defines a locally cotorsion contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})$ for a flat quasi-coherent sheaf $\mathcal{F}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X. When $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{J}$ is an injective quasi-coherent sheaf on X, the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{J})$ is locally injective.

Finally, let $\mathcal{M}$ be a quasi-coherent sheaf on a quasi-separated scheme X, and let $\mathcal{J}$ be an injective quasi-coherent sheaf on X. Then a locally cotorsion contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})$ is defined by the very same rule. The proof of the cotorsion and contraherence conditions is the same as above.

Lemma 2.5.2. *Let $Y\subset X$ be a quasi-compact open subscheme in a semi-separated scheme such that the identity open embedding $j\colon Y\longrightarrow X$ is an affine morphism. Then*

(a) *for any very flat quasi-coherent sheaf $\mathcal{F}$ and contraaadjusted quasi-coherent sheaf $\mathcal{P}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]\simeq \operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$;*

(b) *for any flat quasi-coherent sheaf $\mathcal{F}$ and cotorsion quasi-coherent sheaf $\mathcal{P}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]\simeq \operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$;*

(c) *for any quasi-coherent sheaf $\mathcal{M}$ and injective quasi-coherent sheaf $\mathcal{J}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})[Y]\simeq \operatorname{Hom}_X(j_*j^*\mathcal{M},\mathcal{J})$.*

Now let $Y\subset X$ be any quasi-compact open subscheme in a quasi-separated scheme; let $j\colon Y\longrightarrow X$ denote the identity open embedding. Then

(d) *for any flasque quasi-coherent sheaf* $\mathcal{M}$ *and injective quasi-coherent sheaf* $\mathcal{J}$ *on* X*, there is a natural isomorphism of* $\mathcal{O}(Y)$*-modules* $\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})[Y] \simeq \operatorname{Hom}_X(j_*j^*\mathcal{M},\mathcal{J})$.

Proof. Let $Y=\bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering of a quasi-separated scheme and $\mathcal{G}$ be a quasi-coherent sheaf on Y. Denote by $k_{\alpha_1,\dots,\alpha_i}$ the open embeddings $U_{\alpha_1}\cap\dots\cap U_{\alpha_i}\longrightarrow Y$. Then there is a finite Čech exact sequence

$$(12)\quad 0\longrightarrow \mathcal{G}\longrightarrow \textstyle\bigoplus_\alpha k_{\alpha*}k_\alpha^*\mathcal{G}\longrightarrow \bigoplus_{\alpha<\beta}k_{\alpha,\beta*}k_{\alpha,\beta}^*\mathcal{G} \longrightarrow\dots\longrightarrow k_{1,\dots,N*}k_{1,\dots,N}^*\mathcal{G}\longrightarrow 0$$

of quasi-coherent sheaves on Y (to check the exactness, it suffices to consider the restrictions of this sequence to the open subschemes U_α, over each of which it is contractible). Set $\mathcal{G}=j^*\mathcal{F}$ or $j^*\mathcal{M}$.

When the embedding morphism $j\colon Y\longrightarrow X$ is affine, the functor j_* preserves exactness of sequences of quasi-coherent sheaves. When the sheaf $\mathcal{M}$ is flasque, so are the sheaves constituting the sequence (12), which therefore remains exact after taking the direct images with respect to any morphism. In both cases we obtain a finite exact sequence of quasi-coherent sheaves on X

$$0\longrightarrow j_*j^*\mathcal{F}\longrightarrow \textstyle\bigoplus_\alpha h_{\alpha*}h_\alpha^*\mathcal{F}\longrightarrow \bigoplus_{\alpha<\beta}h_{\alpha,\beta*}h_{\alpha,\beta}^*\mathcal{F} \longrightarrow\dots\longrightarrow h_{1,\dots,N*}h_{1,\dots,N}^*\mathcal{F}\longrightarrow 0$$

or similarly for $\mathcal{M}$, where $h_{\alpha_1,\dots,\alpha_i}$ denote the open embedings $U_{\alpha_1}\cap\dots\cap U_{\alpha_i}\longrightarrow X$.

It is a sequence of very flat quasi-coherent sheaves in the case (a) and a sequence of flat quasi-coherent sheaves in the case (b). The functor $\operatorname{Hom}_X(-,\mathcal{P})$ transforms it into an exact sequence of $\mathcal{O}(Y)$-modules ending in

$$\textstyle\bigoplus_{\alpha<\beta}\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha\cap U_\beta]\longrightarrow \bigoplus_\alpha \mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha] \longrightarrow \operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})\longrightarrow 0$$

and it remains to compare it with the construction (8) of the $\mathcal{O}(Y)$-module $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]$ in terms of the modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha]$ and $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha\cap U_\beta]$. The proofs of parts (c) and (d) are finished in the similar way. ☐

For any affine morphism $f\colon Y\longrightarrow X$ and any quasi-coherent sheaves $\mathcal{M}$ on X and $\mathcal{N}$ on Y there is a natural isomorphism

$$(13)\qquad f_*(f^*\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N})\simeq \mathcal{M}\otimes_{\mathcal{O}_X} f_*\mathcal{N}$$

of quasi-coherent sheaves on X ("the projection formula"). In particular, for any quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ on X there is a natural isomorphism

$$(14)\qquad f_*f^*(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{K})\simeq \mathcal{M}\otimes_{\mathcal{O}_X} f_*f^*\mathcal{K}$$

of quasi-coherent sheaves on X. Assuming that the quasi-coherent sheaf $\mathcal{M}$ on X is flat, the same isomorphisms hold for any quasi-compact quasi-separated morphism of schemes $f\colon Y\longrightarrow X$.

For any embedding $j\colon U \longrightarrow X$ of an affine open subscheme into a semi-separated scheme X and any quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ on X there is a natural isomorphism

$$(15) \qquad j_*j^*(\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{M}) \simeq j_*j^*\mathcal{K}\otimes_{\mathcal{O}_X(U)}\mathcal{M}(U)$$

of quasi-coherent sheaves on X. Assuming that the $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ is flat, the same isomorphism holds a quasi-separated scheme X.

Recall that the *quasi-coherent internal Hom* sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})$ for quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{P}$ on a scheme X is defined as the quasi-coherent sheaf for which there is a natural isomorphism of abelian groups $\operatorname{Hom}_X(\mathcal{K},\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})) \simeq \operatorname{Hom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{K},\,\mathcal{P})$ for any quasi-coherent sheaf $\mathcal{K}$ on X. The sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})$ can be constructed by applying the coherator functor [65, Sections B.12–B.14] to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{P})$.

Lemma 2.5.3. *Let X be a scheme. Then*

(a) *for any very flat quasi-coherent sheaf $\mathcal{F}$ and contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})$ on X is contraadjusted;*

(b) *for any flat quasi-coherent sheaf $\mathcal{F}$ and cotorsion quasi-coherent sheaf $\mathcal{P}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})$ on X is cotorsion;*

(c) *for any quasi-coherent sheaf $\mathcal{M}$ and any injective quasi-coherent sheaf $\mathcal{J}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{J})$ on X is cotorsion;*

(d) *for any flat quasi-coherent sheaf $\mathcal{F}$ and any injective quasi-coherent sheaf $\mathcal{J}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{J})$ is injective.*

Proof. We will prove part (a); the proofs of the other parts are similar. Let $\mathcal{G}$ be a very flat quasi-coherent sheaf on X. We will show that the functor $\operatorname{Hom}_X(-,\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P}))$ transforms any short exact sequence of quasi-coherent sheaves $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{G} \longrightarrow 0$ into a short exact sequence of abelian groups. Indeed, the sequence of quasi-coherent sheaves $0 \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{K} \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L} \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G} \longrightarrow 0$ is exact, because $\mathcal{F}$ is flat (or because $\mathcal{G}$ is flat). Since $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ is very flat by Lemma 1.2.1(a) and $\mathcal{P}$ is contraadjusted, the functor $\operatorname{Hom}_X(-,\mathcal{P})$ transforms the latter sequence of sheaves into a short exact sequence of abelian groups. ☐

It follows from the isomorphism (14) that for any very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a semi-separated scheme X and any contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism of contraherent cosheaves

$$(16) \qquad \mathfrak{Hom}_X(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\,\mathcal{P}) \simeq \mathfrak{Hom}_X(\mathcal{G},\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})).$$

Similarly, for any flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism (16) of locally cotorsion contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{M}$, and injective quasi-coherent sheaf $\mathcal{J}$ on X there are natural isomorphisms of locally cotorsion contraherent cosheaves

$$(17) \quad \mathfrak{Hom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{J}) \simeq \mathfrak{Hom}_X(\mathcal{M},\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{J})) \simeq \mathfrak{Hom}_X(\mathcal{F},\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{J})).$$

The left isomorphism holds over any quasi-separated scheme X.

It follows from the isomorphism (15) that for any very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a contraadjusted quasi-coherent sheaf $\mathcal{P}$ on a semi-separated scheme X there is a natural isomorphism of contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{F},\ \mathcal{P})\simeq\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Hom}_X(\mathcal{G},\mathcal{P})). \tag{18}$$

Similarly, for any flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism (16) of locally cotorsion contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{K}$, and injective quasi-coherent sheaf $\mathcal{J}$ on X there are natural isomorphisms of locally cotorsion contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{F},\ \mathcal{J})\simeq\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Hom}_X(\mathcal{K},\mathcal{J}))\simeq\mathfrak{Cohom}_X(\mathcal{K},\mathfrak{Hom}_X(\mathcal{F},\mathcal{J})). \tag{19}$$

The left isomorphism holds over any quasi-separated scheme X.

Remark 2.5.4. One can slightly generalize the constructions and results of this section by weakening the definitions of contraadjusted and cotorsion quasi-coherent sheaves. Namely, a quasi-coherent sheaf $\mathcal{P}$ on X may be called weakly cotorsion if the functor $\mathrm{Hom}_X(-,\mathcal{P})$ transforms short exact sequences of flat quasi-coherent sheaves on X into short exact sequences of abelian groups. The weakly contraadjusted quasi-coherent sheaves are defined similarly (with the flat quasi-coherent sheaves replaced by very flat ones). Appropriate versions of Lemmas 2.5.1 and 2.5.3 can be proven in this setting, and the contraherent cosheaves $\mathfrak{Hom}$ can be defined.

On a quasi-compact semi-separated scheme X (or more generally, on a scheme where there are enough flat or very flat quasi-coherent sheaves), there is no difference between the weak and ordinary cotorsion/contraadjusted quasi-coherent sheaves (see Section 4.1 below; cf. [52, Sections 5.1.4 and 5.3]). One reason why we chose to use the stronger versions of these conditions here rather that the weaker ones is that it is not immediately clear whether the classes of weakly cotorsion/contraadjusted quasi-coherent sheaves are closed under extensions, or how the exact categories of such sheaves should be defined.

2.6. **Contratensor product of sheaves and cosheaves.** Let X be a quasi-separated scheme and $\mathbf{B}$ be an (initially fixed) base of open subsets of X consisting of some affine open subschemes. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{P}$ be a cosheaf of $\mathcal{O}_X$-modules.

The *contratensor product* $\mathcal{M}\odot_X\mathfrak{P}$ (computed on the base $\mathbf{B}$) is a quasi-coherent sheaf on X defined as the (nonfiltered) inductive limit of the following diagram of quasi-coherent sheaves on X indexed by affine open subschemes $U\in\mathbf{B}$ (cf. [26, Section 0.3.2] and Section 2.1 above).

To any affine open subscheme $U\in\mathbf{B}$ with the identity open embedding $j\colon U\longrightarrow X$ we assign the quasi-coherent sheaf $j_*j^*\mathcal{M}\otimes_{\mathcal{O}_X(U)}\mathfrak{P}[U]$ on X. For any pair of embedded affine open subschemes $V\subset U$, V, $U\in\mathbf{B}$ with the embedding maps $j\colon U\longrightarrow X$ and $k\colon V\longrightarrow X$ there is the morphism of quasi-coherent sheaves

$$k_*k^*\mathcal{M}\otimes_{\mathcal{O}_X(V)}\mathfrak{P}[V]\longrightarrow j_*j^*\mathcal{M}\otimes_{\mathcal{O}_X(U)}\mathfrak{P}[U]$$

defined in terms of the natural isomorphism $k_*k^*\mathcal{M} \simeq j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ of quasi-coherent sheaves on X and the $\mathcal{O}_X(U)$-module morphism $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$.

Let $\mathcal{M}$ and $\mathcal{J}$ be quasi-coherent sheaves on a quasi-separated scheme X for which the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{M}, \mathcal{J})$ is defined (i. e., one of the sufficient conditions given in Section 2.5 for the construction of $\mathfrak{Hom}$ to make sense is satisfied). Then for any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_X(\mathcal{M} \odot_X \mathfrak{P},\ \mathcal{J}) \simeq \mathrm{Hom}^{\mathcal{O}_X}(\mathfrak{P}, \mathfrak{Hom}_X(\mathcal{M}, \mathcal{J})). \tag{20}$$

In other words, the functor $\mathcal{M} \odot_X -$ is left adjoint to the functor $\mathfrak{Hom}_X(\mathcal{M}, -)$ "wherever the latter is defined".

Indeed, both groups of homomorphisms consist of all the compatible collections of morphisms of quasi-coherent sheaves

$$j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U] \longrightarrow \mathcal{J}$$

on X, or equivalently, all the compatible collections of morphisms of $\mathcal{O}_X(U)$-modules

$$\mathfrak{P}[U] \longrightarrow \mathrm{Hom}_X(j_*j^*\mathcal{M}, \mathcal{J})$$

defined for all the identity embeddings $j\colon U \longrightarrow X$ of affine open subschemes $U \in \mathbf{B}$. The compatibility is with respect to the identity embeddings of affine open subschemes $h\colon V \longrightarrow U$, $V, U \in \mathbf{B}$, into one another.

In particular, the adjunction isomorphism (20) holds for any quasi-coherent sheaf $\mathcal{M}$, cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$, and injective quasi-coherent sheaf $\mathcal{J}$. Since there are enough injective quasi-coherent sheaves, it follows that the quasi-coherent sheaf of contratensor product $\mathcal{M} \odot_X \mathfrak{P}$ does not depend on the base of open affines $\mathbf{B}$ that was used to construct it.

More generally, let $\mathbf{D}$ be a partially ordered set endowed with an order-preserving map into the set of all affine open subschemes of X, which we will denote by $a \longmapsto U(a)$, i. e., one has $U(b) \subset U(a)$ whenever $b \leqslant a \in \mathbf{D}$. Suppose that $X = \bigcup_{a \in \mathbf{D}} U(a)$ and for any a, $b \in \mathbf{D}$ the intersection $U(a) \cap U(b) \subset X$ is equal to the union $\bigcup_{c \leqslant a,b} U(c)$. Then the inductive limit of the diagram $j_{a*}j_a^*\mathcal{M} \otimes_{\mathcal{O}_X(U_a)} \mathfrak{P}[U_a]$ indexed by $a \in \mathbf{D}$, where j_a denotes the open embedding $U_a \longrightarrow X$, is naturally isomorphic to the contratensor product $\mathcal{M} \odot_X \mathfrak{P}$.

Indeed, given a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ and a contraherent cosheaf $\mathfrak{Q}$ on X, an arbitrary collection of morphisms of $\mathcal{O}_X(U_a)$-modules $\mathfrak{P}[U_a] \longrightarrow \mathfrak{Q}[U_a]$ compatible with the corestriction maps for $b \leqslant a$ uniquely determines a morphism of cosheaves of $\mathcal{O}_X$-modules $\mathfrak{P} \longrightarrow \mathfrak{Q}$ (see Lemma 2.2.4). In particular, this applies to the case of a contraherent cosheaf $\mathfrak{Q} = \mathfrak{Hom}_X(\mathcal{M}, \mathcal{J})$.

The isomorphism $j_*j^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}) \simeq \mathcal{M} \otimes_{\mathcal{O}_X} j_*j^*\mathcal{K}$ for an embedding of affine open subscheme $j\colon U \longrightarrow X$ and quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ on X (see (14)) allows to construct a natural isomorphism of quasi-coherent sheaves

$$\mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{K} \odot_X \mathfrak{P}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}) \odot_X \mathfrak{P} \tag{21}$$

for any quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ and any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a semi-separated scheme X. The same isomorphism holds over a quasi-separated scheme X, assuming that the quasi-coherent sheaf $\mathcal{M}$ is flat.

3. Locally Contraherent Cosheaves

3.1. **Exact category of locally contraherent cosheaves.** A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X is called *locally contraherent* if every point $x \in X$ has an open neighborhood $x \in W \subset X$ such that the cosheaf of $\mathcal{O}_W$-modules $\mathfrak{P}|_W$ is contraherent.

Given an open covering $\mathbf{W} = \{W\}$ a scheme X, a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is called $\mathbf{W}$*-locally contraherent* if for any open subscheme $W \subset X$ belonging to $\mathbf{W}$ the cosheaf of $\mathcal{O}_W$-modules $\mathfrak{P}|_W$ is contraherent on W. Obviously, a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is locally contraherent if and only if there exists an open covering $\mathbf{W}$ of the scheme X such that $\mathfrak{P}$ is $\mathbf{W}$-locally contraherent.

Let us call an open subscheme of a scheme X *subordinate* to an open covering $\mathbf{W}$ if it is contained in one of the open subsets of X belonging to $\mathbf{W}$. Notice that, by the definition of a contraherent cosheaf, the property of a cosheaf of $\mathcal{O}_X$-modules to be $\mathbf{W}$-locally contraherent only depends on the collection of all affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$.

Theorem 3.1.1. *Let $\mathbf{W}$ be an open covering of a scheme X. Then the restriction of cosheaves of $\mathcal{O}_X$-modules to the base of open subsets of X consisting of all the affine open subschemes subordinate to $\mathbf{W}$ induces an equivalence between the category of $\mathbf{W}$-locally contraherent cosheaves on X and the category of covariant functors with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X subordinate to $\mathbf{W}$, satisfying the contraadjustness and contraherence conditions* (i-ii) *of Section* 2.2 *for all affine open subschemes $V \subset U \subset X$ subordinate to $\mathbf{W}$.*

Proof. The same as in Theorem 2.2.1, except that the base of affine open subschemes of X subordinate to $\mathbf{W}$ is considered throughout. □

Let X be a scheme and $\mathbf{W}$ be its open covering. By Theorem 2.1.2, the category of cosheaves of $\mathcal{O}_X$-modules is a full subcategory of the category of covariant functors with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X subordinate to $\mathbf{W}$. The category of such functors with $\mathcal{O}_X$-module structures is clearly abelian, has exact functors of infinite direct sum and infinite product, and the functors of cosections over a particular affine open subscheme subordinate to $\mathbf{W}$ are exact on it and preserve infinite direct sums and products.

The full subcategory of cosheaves of $\mathcal{O}_X$-modules in this abelian category is closed under extensions, cokernels, and infinite direct sums. For the quasi-compactness reasons explained in Remark 2.1.4, it is also closed under infinite products.

Therefore, the category of cosheaves of $\mathcal{O}_X$-modules acquires the induced exact category structure with exact functors of infinite direct sum and product, and exact functors of cosections on affine open subschemes subordinate to $\mathbf{W}$. Let us denote the category of cosheaves of $\mathcal{O}_X$-modules endowed with this exact category structure (which, of course, depends on the choice of a covering $\mathbf{W}$) by $\mathcal{O}_X\text{–}\mathsf{cosh}_{\mathbf{W}}$. Along the way we have proven that infinite products exist in the additive category of cosheaves of $\mathcal{O}_X$-modules on a scheme X, and the functors of cosections over quasi-compact quasi-separated open subschemes of X preserve them.

The full subcategory of $\mathbf{W}$-locally contraherent cosheaves is closed under extensions, cokernels of admissible monomorphisms, and infinite products in the exact category $\mathcal{O}_X\text{–}\mathsf{cosh}_{\mathbf{W}}$. Thus the category of $\mathbf{W}$-locally contrantraherent cosheaves has the induced exact category structure with exact functors of infinite product, and exact functors of cosections over affine open subschemes subordinate to $\mathbf{W}$. We denote this exact category of $\mathbf{W}$-locally contraherent cosheaves on a scheme X by $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

More explicitly, a short sequence of $\mathbf{W}$-locally contraherent cosheaves $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ if the sequence of cosection modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact for every affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Passing to the inductive limit with respect to refinements of the coverings $\mathbf{W}$, we obtain the exact category structure on the category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$ on the scheme X.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is said to be *locally cotorsion* if for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion. By Lemma 1.3.6(a), this definition can be equivalently rephrased by saying that a locally contraherent cosheaf $\mathfrak{P}$ on X is locally cotorsion if and only if for any affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{P}|_U$ is contraherent on the scheme U the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is cotorsion.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X is called *locally injective* if for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the $\mathcal{O}_X(U)$-module $\mathfrak{J}[U]$ is injective. By Lemma 1.3.6(b), a locally contraherent cosheaf $\mathfrak{J}$ on X is locally injective if and only if for any affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{J}|_U$ is contraherent on the scheme U the $\mathcal{O}(U)$-module $\mathfrak{J}[U]$ is injective.

One defines the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ of locally cotorsion and locally injective $\mathbf{W}$-locally contraherent cosheaves on X in the same way as above. These are full subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, with the induced exact category structures. Passing to the inductive limit with respect to refinements, we obtain the exact categories $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lin}}$ of locally cotorsion and locally injective locally contraherent cosheaves on X.

3.2. **Contraherent and locally contraherent cosheaves.** By Lemma 1.4.1(a), a short sequence of $\mathbf{W}$-locally contraherent cosheaves on X is exact in $X\text{–}\mathsf{lcth}$ (i. e., after some refinement of the covering) if and only if it is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. By Lemma 1.4.1(b), a morphism of $\mathbf{W}$-locally contraherent cosheaves is an admissible epimorphism in $X\text{–}\mathsf{lcth}$ if and only if it is an admissible epimorphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

Analogously, by Lemma 1.4.2(a), a short sequence of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X is exact in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ if and only if it is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. By Lemma 1.4.2(b), a morphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X is an admissible epimorphism in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ if and only if it is an admissible epimorphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. The similar assertions hold for locally injective locally contraherent cosheaves, and they are provable in the same way.

On the other hand, a morphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, or $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ is an admissible monomorphism if and only if it acts injectively on the modules of cosections over all the affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. The following counterexample shows that this condition *does* change when the covering $\mathbf{W}$ is refined.

In other words, the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}$ is closed under the passage to the kernels of admissible epimorphisms, but not to the cokernels of admissible monomorphisms in $X\text{–}\mathsf{lcth}$. Once we show that, it will also follow that there *do* exist locally contraherent cosheaves that are not contraherent. The locally cotorsion and locally injective contraherent cosheaves have all the same problems.

Example 3.2.1. Let R be a commutative ring and f, $g \in R$ be two elements generating the unit ideal. Let M be an R-module containing no f-divisible or g-divisible elements, i. e., $\operatorname{Hom}_R(R[f^{-1}], M) = 0 = \operatorname{Hom}_R(R[g^{-1}], M)$.

Let $M \longrightarrow P$ be an embedding of M into a contraadjusted R-module P, and let Q be the cokernel of this embedding. Then Q is also a contraadjusted R-module. One can take R to be a Dedekind domain, so that it has homological dimension 1; then whenever P is a cotorsion or injective R-module, Q has the same property.

Consider the morphism of contraherent cosheaves $\check{P} \longrightarrow \check{Q}$ on $\operatorname{Spec} R$ related to the surjective morphism of contraadjusted (cotorsion, or injective) R-modules $P \longrightarrow Q$. In restriction to the covering of $\operatorname{Spec} R$ by the two principal affine open subsets $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$, we obtain two morphisms of contraherent cosheaves related to the two morphisms of contraadjusted modules $\operatorname{Hom}_R(R[f^{-1}], P) \longrightarrow \operatorname{Hom}_R(R[f^{-1}], Q)$ and $\operatorname{Hom}_R(R[g^{-1}], P) \longrightarrow \operatorname{Hom}_R(R[g^{-1}], Q)$ over the rings $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$.

Due to the condition imposed on M, the latter two morphisms of contraadjusted modules are injective. On the other hand, the morphism of contraadjusted R-modules $P \longrightarrow Q$ is not. It follows that the cokernel $\mathfrak{R}$ of the morphism of contraherent cosheaves $\check{P} \longrightarrow \check{Q}$ taken in the category of all cosheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules (or equivalently, in the category of copresheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules) is contraherent in restriction to $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$, but not over $\operatorname{Spec} R$. In fact, one has $\mathfrak{R}[\operatorname{Spec} R] = 0$ (since the morphism $P \longrightarrow Q$ is surjective).

Let us point out that for any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X such that the $\mathcal{O}_X(U)$-modules $\mathfrak{P}[U]$ are contraadjusted for all affine open subschemes $U \subset X$ subordinate to a particular open covering $\mathbf{W}$, the $\mathcal{O}_X(U)$-modules $\mathfrak{P}[U]$ are contraadjusted for *all* affine open subschemes $U \subset X$. This is so simply because the class of contraadjusted modules is closed under finite direct sums, restrictions of scalars, and cokernels. So the contraadjustness condition (ii) of Section 2.2 is, in fact, local; it is the contraherence condition (i) that isn't.

In the rest of the section we will explain how to distinguish the contraherent cosheaves among all the locally contraherent ones. Let X be a semi-separated scheme, $\mathbf{W}$ be its open covering, and $\{U_\alpha\}$ be an affine open covering subordinate to $\mathbf{W}$ (i. e., consisting of affine open subschemes subordinate to $\mathbf{W}$).

Let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Consider the homological Čech complex of abelian groups (or $\mathcal{O}(X)$-modules) $C_\bullet(\{U_\alpha\}, \mathfrak{P})$ of the form

$$\cdots \longrightarrow \bigoplus\nolimits_{\alpha<\beta<\gamma} \mathfrak{P}[U_\alpha \cap U_\beta \cap U_\gamma] \longrightarrow \bigoplus\nolimits_{\alpha<\beta} \mathfrak{P}[U_\alpha \cap U_\beta] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{P}[U_\alpha]. \tag{22}$$

Here (as in the sequel) our notation presumes the indices α to be linearly ordered. More generally, the complex (22) can be considered for any open covering U_α of a topological space X and any cosheaf of abelian groups $\mathfrak{P}$ on X. Let $\Delta(X,\mathfrak{P}) = \mathfrak{P}[X]$ denote the functor of global cosections of (locally contraherent) cosheaves on X; then, by the definition, we have $\Delta(X,\mathfrak{P}) \simeq H_0C_\bullet(\{U_\alpha\},\mathfrak{P})$.

Lemma 3.2.2. *Let U be an affine scheme with an open covering $\mathbf{W}$ and a finite affine open covering $\{U_\alpha\}$ subordinate to $\mathbf{W}$. Then a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on U is contraherent if and only if $H_{>0}C_\bullet(\{U_\alpha\},\mathfrak{P}) = 0$.*

Proof. The "only if" part is provided by Lemma 1.2.6(b). Let us prove "if". If the Čech complex $C_\bullet(\{U_\alpha\},\mathfrak{P})$ has no higher homology, then it is a finite left resolution of the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ by contraadjusted $\mathcal{O}(U)$-modules. As we have explained above, the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is contraadjusted, too.

For any affine open subscheme $V \subset U$, consider the Čech complex $C_\bullet(\{V\cap U_\alpha\},\mathfrak{P}|_V)$ related to the restrictions of our cosheaf $\mathfrak{P}$ and our covering U_α to the open subscheme V. The complex $C_\bullet(\{V\cap U_\alpha\},\ \mathfrak{P}|_V)$ can be obtained from the complex $C_\bullet(\{U_\alpha\},\mathfrak{P})$ by applying the functor $\mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V),-)$. We have

$$H_0C_\bullet(\{U_\alpha\},\mathfrak{P}) \simeq \mathfrak{P}[U] \quad \text{and} \quad H_0C_\bullet(\{V\cap U_\alpha\},\ \mathfrak{P}|_V) \simeq \mathfrak{P}[V].$$

Since the functor $\mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V),-)$ preserves exactness of short sequences of contraadjusted $\mathcal{O}(U)$-modules, we conclude that $\mathfrak{P}[V] \simeq \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V),\mathfrak{P}[U])$. Both the contraadjustness and contraherence conditions have been now verified. ☐

Corollary 3.2.3. *If a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Q}$ on an affine scheme U is an extension of two contraherent cosheaves $\mathfrak{P}$ and $\mathfrak{R}$ in the exact category U–$\mathsf{lcth}_{\mathbf{W}}$ (or $\mathcal{O}_U$–$\mathsf{cosh}_{\mathbf{W}}$), then $\mathfrak{Q}$ is also a contraherent cosheaf on U.*

Proof. Pick a finite affine open covering $\{U_\alpha\}$ of the affine scheme U subordinate to the covering $\mathbf{W}$. Then the complex of abelian groups $C_\bullet(\{U_\alpha\},\mathfrak{Q})$ is an extension of the complexes of abelian groups $C_\bullet(\{U_\alpha\},\mathfrak{P})$ and $C_\bullet(\{U_\alpha\},\mathfrak{R})$. Hence whenever the latter two complexes have no higher homology, neither does the former one. ☐

Corollary 3.2.4. *For any scheme X and any its open covering $\mathbf{W}$, the full exact subcategory of $\mathbf{W}$-locally contraherent cosheaves on X is closed under extensions in the exact category of locally contraherent cosheaves on X. In particular, the full exact subcategory of contraherent cosheaves on X is closed under extensions in the exact category of locally contraherent (or $\mathbf{W}$-locally contraherent) cosheaves on X.*

Proof. Follows easily from Corollary 3.2.3. ☐

3.3. **Direct and inverse images of locally contraherent cosheaves.** Let $\mathbf{W}$ be an open covering of a scheme X and $\mathbf{T}$ be an open covering of a scheme Y. A morphism of schemes $f\colon Y \longrightarrow X$ is called $(\mathbf{W},\mathbf{T})$-*affine* if for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the open subscheme $f^{-1}(U) \subset Y$ is affine and subordinate to $\mathbf{T}$. Any $(\mathbf{W},\mathbf{T})$-affine morphism is affine.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine morphism of schemes and $\mathfrak{Q}$ be a $\mathbf{T}$-locally contraherent cosheaf on Y. Then the cosheaf of $\mathcal{O}_X$-modules $f_!\mathfrak{Q}$ on X is $\mathbf{W}$-locally contraherent. The proof of this assertion is similar to that of its global version in Section 2.3. We have constructed an exact functor of direct image $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ between the exact categories of $\mathbf{T}$-locally contraherent cosheaves on Y and $\mathbf{W}$-locally contraherent cosheaves on X.

A morphism of schemes $f\colon Y \longrightarrow X$ is called $(\mathbf{W},\mathbf{T})$-*coaffine* if for any affine open subscheme $V\subset Y$ subordinate to $\mathbf{T}$ there exists an affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$ such that $f(V)\subset U$. Notice that for any fixed open covering $\mathbf{W}$ of a semi-separated scheme X and any morphism of schemes $f\colon Y\longrightarrow X$ the covering $\mathbf{T}$ of the scheme Y consisting of all the full preimages $f^{-1}(U)$ of affine open subschemes $U\subset X$ has the property that the morphism $f\colon Y\longrightarrow X$ is $(\mathbf{W},\mathbf{T})$-coaffine.

If the morphism f is affine, it is also $(\mathbf{W},\mathbf{T})$-affine with respect to the covering $\mathbf{T}$ constructed in this way. A morphism $f\colon Y\longrightarrow X$ is simultaneously $(\mathbf{W},\mathbf{T})$-affine and $(\mathbf{W},\mathbf{T})$-coaffine if and only if it is affine and the set of all affine open subschemes $V\subset Y$ subordinate to $\mathbf{T}$ consists precisely of all affine open subschemes V for which there exists an affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$ such that $f(V)\subset U$.

Let $f\colon Y\longrightarrow X$ be a very flat $(\mathbf{W},\mathbf{T})$-coaffine morphism, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Define a $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{P}$ on Y in the following way. Let $V\subset Y$ be an affine open subscheme subordinate to $\mathbf{T}$. For any affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$ and such that $f(V)\subset U$, we set $(f^!\mathfrak{P})[V]_U = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(V),\mathfrak{P}[U])$.

The $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]_U$ is contraadjusted by Lemma 1.2.3(a). The contraherence isomorphism $(f^!\mathfrak{P})[V']_U\simeq \operatorname{Hom}_{\mathcal{O}_Y(V)}(\mathcal{O}_Y(V'),(f^!\mathfrak{P})[V]_U)$ clearly holds for any affine open subscheme $V'\subset V$. The ($\mathbf{W}$-local) contraherence condition on $\mathfrak{P}$ implies a natural isomorphism of $\mathcal{O}_Y(V)$-modules $(f^!\mathfrak{P})[V]_{U'}\simeq(f^!\mathfrak{P})[V]_{U''}$ for any embedded affine open subschemes $U'\subset U''$ in X subordinate to $\mathbf{W}$ and containing $f(V)$ (cf. Section 2.3). It remains to construct such an isomorphism for any two (not necessarily embedded) affine open subschemes U', $U''\subset X$.

The case of a semi-separated scheme X is clear. In the general case, let $U'\cap U''=\bigcup_\alpha U_\alpha$ be an affine open covering of the intersection. Since V is quasi-compact, the image $f(V)$ is covered by a finite subset of the affine open schemes U_α. Let V_α denote the preimages of U_α with respect to the morphism $V\longrightarrow U'\cap U''$; then $V=\bigcup_\alpha V_\alpha$ is an affine open covering of the affine scheme V.

The restrictions $f_{U'}\colon V\longrightarrow U'$ and $f_{U''}\colon V\longrightarrow U''$ of the morphism f are very flat morphisms of affine schemes, while the restrictions $\mathfrak{P}|_{U'}$ and $\mathfrak{P}|_{U''}$ of the cosheaf $\mathfrak{P}$ are contraherent cosheaves on U' and U''. Consider the contraherent cosheaves $f_{U'}^!\mathfrak{P}|_{U'}$ and $f_{U''}^!\mathfrak{P}_{U''}$ on V (as defined in Section 2.3). Their cosection modules $(f_{U'}^!\mathfrak{P}|_{U'})[V_\alpha]$ and $(f_{U''}^!\mathfrak{P}|_{U''})[V_\alpha]$ are naturally isomorphic for all α, since $f(V_\alpha)\subset U_\alpha\subset U'\cap U''$. Similarly, there are natural isomorphisms $(f_{U'}^!\mathfrak{P}|_{U'})[V_\alpha\cap V_\beta]\simeq(f_{U''}^!\mathfrak{P}|_{U''})[V_\alpha\cap V_\beta]$ forming commutative diagrams with the corestrictions from $V_\alpha\cap V_\beta$ to V_α and V_β, since $f(V_\alpha\cap V_\beta)\subset U_\alpha\cap U_\beta$ and the intersections $U_\alpha\cap U_\beta$ are affine schemes.

Now the cosheaf axiom (5) for contraherent cosheaves $f_{U'}^!\mathfrak{P}|_{U'}$ and $f_{U''}^!\mathfrak{P}|_{U''}$ and the covering $V=\bigcup_\alpha V_\alpha$ provides the desired isomorphism between the $\mathcal{O}_Y(V)$-modules $(f^!\mathfrak{P})[V]_{U'}=(f_{U'}^!\mathfrak{P}|_{U'})[V]$ and $(f^!\mathfrak{P})[V]_{U''}=(f_{U''}^!\mathfrak{P}|_{U''})[V]$. One can easily see that such isomorphisms form a commutative diagram for any three affine open subschemes U', U'', $U''' \subset X$ containing $f(V)$.

The $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{P}$ on Y is constructed. We have obtained an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$.

Let $f\colon Y \longrightarrow X$ be a flat $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes, and let $\mathfrak{P}$ be a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. Then the same procedure as above defines a locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{P}$ on Y. So we obtain an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$. Finally, for any $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes $f\colon Y \longrightarrow X$ and any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X the same rule defines a locally injective $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{J}$ on Y. We obtain an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$.

Passing to the inductive limits of exact categories with respects to the refinements of coverings and taking into account the above remark about $(\mathbf{W},\mathbf{T})$-coaffine morphisms, we obtain an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$ for any very flat morphism of schemes $f\colon Y \longrightarrow X$.

For an open embedding $j\colon Y \longrightarrow X$, the direct image $j^!$ coincides with the restriction functor $\mathfrak{P} \longmapsto \mathfrak{P}|_Y$ on the locally contraherent cosheaves $\mathfrak{P}$ on X. For a flat morphism f, we have an exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$, and for an arbitrary morphism of schemes $f\colon Y \longrightarrow Y$ there is an exact functor of inverse image of locally injective locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$.

If $f\colon Y \longrightarrow X$ is a $(\mathbf{W},\mathbf{T})$-affine morphism and $\mathfrak{Q}$ is a locally cotorsion $\mathbf{T}$-locally contraherent cosheaf on Y, then $f_!\mathfrak{Q}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. So the direct image functor $f_!$ restricts to an exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. If f is a flat $(\mathbf{W},\mathbf{T})$-affine morphism and $\mathfrak{I}$ is a locally injective $\mathbf{T}$-locally contraherent cosheaf on Y, then $f_!\mathfrak{I}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Hence in this case the direct image also restricts to an exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism. Then for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y and locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X there is an adjunction isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{J}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{J}), \tag{23}$$

where Hom^X and Hom^Y denote the abelian groups of morphisms in the categories of locally contraherent cosheaves on X and Y.

If the morphism f is, in addition, flat, then the isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P}) \tag{24}$$

holds for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ and locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X. In particular, $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. When the morphism f is very flat, the functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$ is right adjoint to the functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

Most generally, there is an adjunction isomorphism

$$\operatorname{Hom}^{\mathcal{O}_X}(f_!\mathfrak{Q},\mathfrak{J}) \simeq \operatorname{Hom}^{\mathcal{O}_Y}(\mathfrak{Q}, f^!\mathfrak{J}) \tag{25}$$

for any morphism of schemes f, a cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, and a locally injective locally contraherent cosheaf $\mathfrak{J}$ on X. Similarly, there is an isomorphism

$$\operatorname{Hom}^{\mathcal{O}_X}(f_!\mathfrak{Q},\mathfrak{P}) \simeq \operatorname{Hom}^{\mathcal{O}_Y}(\mathfrak{Q}, f^!\mathfrak{P}) \tag{26}$$

for any flat morphism f, a cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, and a locally cotorsion locally contraherent cosheaf $\mathfrak{P}$ on X, and also for a very flat morphism f, a cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, and a locally contraherent cosheaf $\mathfrak{P}$ on X.

In all the mentioned cases, both abelian groups $\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P})$ or $\operatorname{Hom}^{\mathcal{O}_X}(f_!\mathfrak{Q},\mathfrak{P})$ (etc.) and $\operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$ or $\operatorname{Hom}^{\mathcal{O}_Y}(\mathfrak{Q}, f^!\mathfrak{P})$ (etc.) are identified with the group of all the compatible collections of homomorphisms of $\mathcal{O}_X(U)$-modules $\mathfrak{Q}[V] \longrightarrow \mathfrak{P}[U]$ defined for all affine open subschemes $U \subset X$ and $V \subset Y$ subordinate to, respectively, $\mathbf{W}$ and $\mathbf{T}$, and such that $f(V) \subset U$. In other words, the functor $f^!$ is right adjoint to the functor $f_!$ "wherever the former functor is defined".

All the functors between exact categories of locally contraherent cosheaves constructed above are exact and preserve infinite products. The functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ preserves infinite products whenever a morphism of schemes $f\colon Y \longrightarrow X$ is quasi-compact and quasi-separated.

Let $f\colon Y \longrightarrow X$ be a morphism of schemes and $j\colon U \longrightarrow X$ be an open embedding. Set $V = U\times_X Y$, and denote by $j'\colon V \longrightarrow Y$ and $f'\colon V \longrightarrow U$ the natural morphisms. Then for any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, there is a natural isomorphism of cosheaves of $\mathcal{O}_U$-modules $(f_!\mathfrak{Q})|_U \simeq f'_!(\mathfrak{Q}|_V)$.

In particular, suppose f is a $(\mathbf{W},\mathbf{T})$-affine morphism. Define the open coverings $\mathbf{W}|_U$ and $\mathbf{T}|_V$ as the collections of all intersections of the open subsets $W \in \mathbf{W}$ and $T \in \mathbf{T}$ with U and V, respectively. Then f' is a $(\mathbf{W}|_U,\mathbf{T}|_V)$-affine morphism. For any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y there is a natural isomorphism of $\mathbf{W}|_U$-locally contraherent cosheaves $j^!f_!\mathfrak{Q} \simeq f'_!j'^!\mathfrak{Q}$ on U.

More generally, let $f\colon Y \longrightarrow X$ and $g\colon x \longrightarrow X$ be morphisms of schemes. Set $y = x \times_X Y$; let $f'\colon y \longrightarrow x$ and $g'\colon y \longrightarrow Y$ be the natural morphisms. Let $\mathbf{W}$, $\mathbf{T}$, and $\mathbf{w}$ be open coverings of, respectively, X, Y, and x such that the morphism f is $(\mathbf{W},\mathbf{T})$-affine, while the morphism g is $(\mathbf{W},\mathbf{w})$-coaffine.

Define two coverings $\mathbf{t}'$ and $\mathbf{t}''$ of the scheme y by the rules that $\mathbf{t}'$ consists of all the full preimages of affine open subschemes in x subordinate to $\mathbf{w}$, while $\mathbf{t}''$ is the collection of all the full preimages of affine open subschemes in Y subordinate to $\mathbf{T}$. One can easily see that the covering $\mathbf{t}'$ is subordinate to $\mathbf{t}''$. Let $\mathbf{t}$ be any covering of y such that $\mathbf{t}'$ is subordinate to $\mathbf{t}$ and $\mathbf{t}$ is subordinate to $\mathbf{t}''$. Then the former condition guarantees that the morphism f' is $(\mathbf{w},\mathbf{t})$-affine, while under the latter condition the morphism g' is $(\mathbf{T},\mathbf{t})$-coaffine.

Assume that the morphisms g and g' are very flat. Then for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{P}$ on Y there is a natural isomorphism $g^!f_!\mathfrak{P} \simeq f'_!g'^!\mathfrak{P}$ of $\mathbf{w}$-locally contraherent cosheaves on x.

Alternatively, assume that the morphism g is flat. Then for any locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{P}$ on Y there is a natural isomorphism $g^! f_! \mathfrak{P} \simeq f'_! g'^! \mathfrak{P}$ of locally cotorsion $\mathbf{w}$-locally contraherent cosheaves on x.

As a third alternative, assume that the morphism f is flat (while g may be arbitrary). Then for any locally injective $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{J}$ on Y there is a natural isomorphism $g^! f_! \mathfrak{J} \simeq f'_! g'^! \mathfrak{J}$ of locally injective $\mathbf{w}$-locally contraherent cosheaves on x.

All these isomorphisms of locally contraherent cosheaves are constructed using the natural isomorphism of r-modules $\operatorname{Hom}_R(r, P) \simeq \operatorname{Hom}_S(S \otimes_R r,\ P)$ for any commutative ring homomorphisms $R \longrightarrow S$ and $R \longrightarrow r$, and any S-module P.

In other words, the direct images of $\mathbf{T}$-locally contraherent cosheaves under $(\mathbf{W}, \mathbf{T})$-affine morphisms commute with the inverse images in those base change situations when all the functors involved are defined.

The following particular case will be important for us. Let $\mathbf{W}$ be an open covering of a scheme X and $j\colon Y \longrightarrow X$ be an affine open embedding subordinate to $\mathbf{W}$ (i. e., Y is contained in one of the open subsets of X belonging to $\mathbf{W}$). Then one can endow the scheme Y with the open covering $\mathbf{T} = \{Y\}$ consisting of the only open subset $Y \subset Y$. This makes the embedding j both $(\mathbf{W}, \mathbf{T})$-affine and $(\mathbf{W}, \mathbf{T})$-coaffine. Also, the morphism j, being an open embedding, is very flat.

Therefore, the inverse and direct images $j^!$ and $j_!$ form a pair of adjoint exact functors between the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ of $\mathbf{W}$-locally contraherent cosheaves on X and the exact category $Y\text{–}\mathsf{ctrh}$ of contraherent cosheaves on Y. Moreover, the image of the functor $j_!$ is contained in the full exact subcategory of (globally) contraherent cosheaves $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$. Both functors preserve the subcategories of locally cotorsion and locally injective cosheaves.

Now let $\mathbf{W}$ be an open covering of a quasi-compact semi-separated scheme X and let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine covering of X subordinate to $\mathbf{W}$. Denote by $j_{\alpha_1,\dots,\alpha_k}$ the open embeddings $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \longrightarrow X$. Then for any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X the cosheaf Čech sequence (cf. (12))

$$(27)\quad 0 \longrightarrow j_{1,\dots,N!} j^!_{1,\dots,N} \mathfrak{P} \longrightarrow \cdots \\ \longrightarrow \bigoplus\nolimits_{\alpha<\beta} j_{\alpha,\beta!} j^!_{\alpha,\beta} \mathfrak{P} \longrightarrow \bigoplus\nolimits_{\alpha} j_{\alpha!} j^!_{\alpha} \mathfrak{P} \longrightarrow \mathfrak{P} \longrightarrow 0$$

is exact in the exact category of $\mathbf{W}$-locally contraherent cosheaves on X. Indeed, the corresponding sequence of cosections over every affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ is exact by Lemma 1.2.6(b). We have constructed a finite left resolution of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ by contraherent cosheaves. When $\mathfrak{P}$ is a locally cotorsion or locally injective $\mathbf{W}$-locally contraherent cosheaf, the sequence (27) is exact in the category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ or $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$, respectively.

3.4. **Coflasque contraherent cosheaves.** Let X be a topological space and $\mathfrak{F}$ be a cosheaf of abelian groups on X. A cosheaf $\mathfrak{F}$ is called *coflasque* if for any open subsets $V \subset U \subset X$ the corestriction map $\mathfrak{F}[V] \longrightarrow \mathfrak{F}[U]$ is injective. Obviously, the class of coflasque cosheaves of abelian groups is preserved by the restrictions to open subsets and the direct images with respect to continuous maps.

Lemma 3.4.1. *Let $X = \bigcup_\alpha U_\alpha$ be an open covering. Then*

(a) *a cosheaf $\mathfrak{F}$ on X is coflasque if and only if its restriction $\mathfrak{F}|_{U_\alpha}$ to each open subset U_α is coflasque;*

(b) *if the cosheaf $\mathfrak{F}$ is coflasque, then the Čech complex* (22) *has no higher homology groups, $H_{>0}C_\bullet(\{U_\alpha\}, \mathfrak{F}) = 0$.*

Proof. One can either check these assertions directly or deduce them from the similar results for flasque sheaves of abelian groups using the construction of the sheaf $U \longmapsto \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{F}[U], I)$ from the proof of Theorem 2.1.2. Here I is an injective abelian group; clearly, the sheaf so obtained is flasque for all I if and only if the original cosheaf $\mathfrak{F}$ was coflasque. The sheaf (dual) versions of assertions (a-b) are well-known [23, Section II.3.1 and Théorème II.5.2.3(a)]. □

Corollary 3.4.2. *Let X be a scheme with an open covering $\mathbf{W}$ and $\mathfrak{F}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Suppose that the cosheaf $\mathfrak{F}$ is coflasque. Then $\mathfrak{F}$ is a (globally) contraherent cosheaf on X.*

Proof. Follows from Lemmas 3.2.2 and 3.4.1(b). □

Assume that the topological space X has a base of the topology consisting of quasi-compact open subsets. Then one has $\mathfrak{F}[Y] \simeq \varinjlim_{U\subset Y} \mathfrak{F}[U]$ for any cosheaf of abelian groups $\mathfrak{F}$ on X and any open subset $Y \subset X$, where the filtered inductive limit is taken over all the quasi-compact open subsets $U \subset Y$. It follows easily that a cosheaf $\mathfrak{F}$ is coflasque if and only if the corestriction map $\mathfrak{F}[V] \longrightarrow \mathfrak{F}[U]$ is injective for any pair of embedded quasi-compact open subsets $V \subset U \subset X$.

Moreover, if X is a scheme then it follows from Lemma 3.4.1(a) that a cosheaf of abelian groups $\mathfrak{F}$ on X is coflasque if and only if the corestriction map $\mathfrak{F}[V] \longrightarrow \mathfrak{F}[U]$ is injective for any affine open subscheme $U \subset X$ and quasi-compact open subscheme $V \subset U$. It follows that an infinite product of a family of coflasque contraherent cosheaves on X is coflasque. The following counterexample shows, however, that coflasqueness of contraherent cosheaves on schemes cannot be checked on the pairs of embedded affine open subschemes.

Example 3.4.3. Let X be a Noetherian scheme. It is well-known that any injective quasi-coherent sheaf $\mathcal{J}$ on X is a flasque sheaf of abelian groups. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and $\mathcal{F} \longrightarrow \mathcal{J}$ be an injective morphism. Then one has $(\mathcal{J}/\mathcal{F})(U) \simeq \mathcal{J}(U)/\mathcal{F}(U)$ for any affine open subscheme $U \subset X$, so the map $(\mathcal{J}/\mathcal{F})(U) \longrightarrow (\mathcal{J}/\mathcal{F})(V)$ is surjective for any pair of embedded affine open subschemes $V \subset U$. On the other hand, if the quotient sheaf $\mathcal{J}/\mathcal{F}$ were flasque, one would have $H^{i+1}(X, \mathcal{F}) \simeq H^i(X, \mathcal{J}/\mathcal{F}) = 0$ for $i \geqslant 1$, which is clearly not the case in general.

Now let X be a scheme over an affine scheme $\operatorname{Spec} R$. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and J be an injective R-module. Then for any quasi-compact quasi-separated open subscheme $Y \subset X$ one has $\mathfrak{Cohom}_R(\mathcal{M}, J)[Y] \simeq \operatorname{Hom}_R(\mathcal{M}(U), J)$.

For a Noetherian scheme X, it follows that the cosheaf $\mathfrak{Cohom}_R(\mathcal{M}, J)$ is coflasque if and only if the sheaf $\mathcal{M}$ is flasque (cf. Lemma 3.4.6(c) below).

Corollary 3.4.4. *Let X be a scheme with an open covering $\mathbf{W}$ and $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ be a short exact sequence in $\mathcal{O}_X$–$\mathsf{cosh}_{\mathbf{W}}$ (e. g., a short exact sequence of $\mathbf{W}$-locally contraherent cosheaves on X). Then*

(a) *the cosheaf $\mathfrak{Q}$ is coflasque whenever both the cosheaves $\mathfrak{P}$ and $\mathfrak{R}$ are;*

(b) *the cosheaf $\mathfrak{P}$ is cosflasque whenever both the cosheaves $\mathfrak{Q}$ and $\mathfrak{R}$ are;*

(c) *if the cosheaf $\mathfrak{R}$ is coflasque, then the short sequence $0 \longrightarrow \mathfrak{P}[Y] \longrightarrow \mathfrak{Q}[Y] \longrightarrow \mathfrak{R}[Y] \longrightarrow 0$ is exact for any open subscheme $Y \subset X$.*

Proof. The assertion actually holds for any short exact sequence in the exact category of cosheaves of abelian groups on a scheme X with the exact category structure related to the base of affine open subschemes subordinate to a covering $\mathbf{W}$.

Part (c): let us first consider the case of a semi-separated open subscheme Y. Pick an affine open covering $Y = \bigcup_\alpha U_\alpha$ subordinate to $\mathbf{W}$. The intersection of any nonempty finite subset of U_α being also an affine open subscheme in X, the short sequence of complexes $0 \longrightarrow C_\bullet(\{U_\alpha\}, \mathfrak{P}|_Y) \longrightarrow C_\bullet(\{U_\alpha\}, \mathfrak{Q}|_Y) \longrightarrow C_\bullet(\{U_\alpha\}, \mathfrak{R}|_Y) \longrightarrow 0$ is exact. Recall that $\mathfrak{F}[Y] \simeq C_\bullet(\{U_\alpha\}, \mathfrak{F}|_Y)$ for any cosheaf of abelian groups $\mathfrak{F}$ on X. By Lemma 3.4.1, one has $H_{>0}C_\bullet(\{U_\alpha\}, \mathfrak{R}|_Y) = 0$, and it follows that the short sequence $0 \longrightarrow \mathfrak{P}[Y] \longrightarrow \mathfrak{Q}[Y] \longrightarrow \mathfrak{R}[Y] \longrightarrow 0$ is exact.

Now for any open subscheme $Y \subset X$, pick an open covering $Y = \bigcup_\alpha U_\alpha$ of Y by semi-separated open subschemes U_α. The intersection of any nonempty finite subset of U_α being semi-separated, the short sequence of Čech complexes for the covering U_α and the short exact sequence of cosheaves $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is exact according to the above, and the same argument concludes the proof.

Parts (a-b): let $U \subset V \subset X$ be any pair of embedded open subschemes. Assuming that the cosheaf $\mathfrak{R}$ is coflasque, according to part (c) we have short exact sequences of the modules of cosections $0 \longrightarrow \mathfrak{P}[V] \longrightarrow \mathfrak{Q}[V] \longrightarrow \mathfrak{R}[V] \longrightarrow 0$ and $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$. If the morphism from the former sequence to the latter is injective on the rightmost terms, then it is injective on the middle terms if and only if it is injective on the leftmost terms. ☐

Remark 3.4.5. Let U be an affine scheme. Then a cosheaf of abelian groups $\mathfrak{F}$ on U is coflasque if and only if the following three conditions hold:

(i) for any affine open subscheme $V \subset U$, the corestriction map $\mathfrak{F}[V] \longrightarrow \mathfrak{F}[U]$ is injective;

(ii) for any two affine open subschemes V'', $V'' \subset U$, the image of the map $\mathfrak{F}[V' \cap V''] \longrightarrow \mathfrak{F}[U]$ is equal to the intersection of the images of the maps $\mathfrak{F}[V'] \longrightarrow \mathfrak{F}[U]$ and $\mathfrak{F}[V''] \longrightarrow \mathfrak{F}[U]$;

(iii) for any finite collection of affine open subschemes $V_\alpha \subset U$, the images of the maps $\mathfrak{F}[V_\alpha] \longrightarrow \mathfrak{F}[U]$ generate a distributive sublattice in the lattice of all subgroups of the abelian group $\mathfrak{F}[U]$.

Indeed, the condition (i) being assumed, in view of the exact sequence $\mathfrak{F}[V' \cap V''] \longrightarrow \mathfrak{F}[V'] \oplus gF[V''] \longrightarrow \mathfrak{F}[V' \cup V''] \longrightarrow 0$ the condition (ii) is equivalent to injectivity of the corestriction map $\mathfrak{F}[V' \cup V''] \longrightarrow \mathfrak{F}[U]$.

Furthermore, set $W = \bigcup_\alpha V_\alpha$. The condition (iii) for any proper subcollection of V_α and the conditions (i-ii) being assumed, the condition (iii) for the collection $\{V_\alpha\}$ becomes equivalent to vanishing of the higher homology of the Čech complex $C_\bullet(\{V_\alpha\}, \mathfrak{F}|_W)$ together with injectivity of the map $\mathfrak{F}[W] \longrightarrow \mathfrak{F}[U]$. Indeed, the complex $C_\bullet(\{V_\alpha\}, \mathfrak{F}|_W) \longrightarrow \mathfrak{F}[U]$ can be identified with (the abelian group version of) the "cobar complex" $B^\bullet(\mathfrak{F}[U]; \mathfrak{F}[V_\alpha])$ from [51, Proposition 7.2(c*) of Chapter 1] (see also [52, Lemma 11.4.3.1(c*)]).

Lemma 3.4.6. (a) *Let $\mathcal{M}$ be a flasque quasi-coherent sheaf and $\mathcal{G}$ be a flat quasi-coherent sheaf on a locally Noetherian scheme X. Then the quasi-coherent sheaf $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{G}$ on X is flasque.*

(b) *Let $\mathcal{M}$ be a flasque quasi-coherent sheaf and $\mathfrak{J}$ be a locally injective contraherent cosheaf on a scheme X. Then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J})$ on X is coflasque.*

(c) *Let $\mathcal{M}$ be a flasque quasi-coherent sheaf and $\mathcal{J}$ be an injective quasi-coherent sheaf on an affine Noetherian scheme U. Then the contraherent cosheaf $\mathfrak{Hom}_U(\mathcal{M}, \mathcal{J})$ on U is coflasque.*

(d) *Let $\mathcal{M}$ be a flasque quasi-coherent sheaf and $\mathfrak{G}$ be a flat cosheaf of $\mathcal{O}_U$-modules on an affine scheme U. Then the quasi-coherent sheaf $\mathcal{M} \odot_U \mathfrak{G}$ on U is flasque.*

Proof. The (co)flasqueness of (co)sheaves being a local property, it suffices to consider the case of an affine scheme $X = U$ in all assertions (a-d). Then (c) becomes a restatement of (b) and (d) a restatement of (a). It also follows that one can extend the assertion (c) to locally injective locally contraherent cosheaves $\mathfrak{J}$ (cf. Section 3.6).

To prove part (a) in the affine case, one notices the isomorphism $(\mathcal{M} \otimes_{\mathcal{O}_U} \mathcal{G})(V) \simeq \mathcal{M}(V) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$ holding for any quasi-compact open subscheme V in an affine scheme U, quasi-coherent sheaf $\mathcal{M}$, and a flat quasi-coherent sheaf $\mathcal{G}$ on U. More generally, one has $(\mathcal{M} \otimes_R G)(Y) \simeq \mathcal{M}(Y) \otimes_R G$ for any quasi-coherent sheaf $\mathcal{M}$ with a right R-module structure on a quasi-compact quasi-separated scheme Y and a flat left R-module G. Part (b) follows from the similar isomorphism $\mathfrak{Cohom}_U(\mathcal{M}, \mathfrak{J})[V] \simeq \mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{M}(V), \mathfrak{J}[U])$ holding for any quasi-coherent sheaf $\mathcal{M}$ and a (locally) injective contraherent cosheaf $\mathfrak{J}$ on U. More generally, $\mathfrak{Cohom}_R(\mathcal{M}, J)[Y] \simeq \mathrm{Hom}_R(\mathcal{M}(Y), J)$ for any quasi-compact quasi-separated scheme Y over $\operatorname{Spec} R$, quasi-coherent sheaf $\mathcal{M}$ on Y, and an injective R-module J. ☐

Lemma 3.4.7. (a) *Let X be a Noetherian topological space of finite Krull dimension $\leqslant d+1$, and let $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}^0 \longrightarrow \cdots \longrightarrow \mathcal{F}^d \longrightarrow \mathcal{F} \longrightarrow 0$ be an exact sequence of sheaves of abelian groups on X. Suppose that the sheaves $\mathcal{F}^i$ are flasque. Then the sheaf $\mathcal{F}$ is flasque.*

(b) *Let X be a scheme with an open covering $\mathbf{W}$ and $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{F}_d \longrightarrow \cdots \longrightarrow \mathfrak{F}_0 \longrightarrow \mathfrak{E} \longrightarrow 0$ be an exact sequence in $\mathcal{O}_X\text{–}\mathsf{cosh}_{\mathbf{W}}$. Suppose that the cosheaves $\mathfrak{F}_i$*

are coflasque and the underlying topological space of the scheme X is Noetherian of finite Krull dimension $\leqslant d+1$. Then the cosheaf $\mathfrak{F}$ is coflasque.

Proof. Part (a) is a version of Grothendieck's vanishing theorem [25, Théorème 3.6.5]; it can be either proven directly along the lines of Grothendieck's proof, or, making a slightly stronger assumption that the dimension of X does not exceed d, deduced from the assersion of Grothendieck's theorem. Indeed, let $\mathcal{G}$ denote the image of the morphism of sheaves $\mathcal{F}^{d-1} \longrightarrow \mathcal{F}^d$; by Grothendieck's theorem, one has $H^1(V, \mathcal{G}|_V) \simeq H^{d+1}(V, \mathcal{E}|_V) = 0$ for any open subset $V \subset X$. Hence the map $\mathcal{F}^d(V) \longrightarrow \mathcal{F}(V)$ is surjective, and it follows that the map $\mathcal{F}(X) \longrightarrow \mathcal{F}(V)$ is surjective, too.

Part (b): given an injective abelian group I, the sequence $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{E}[U], I) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}_0[U], I) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}_d[U], I) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}[U], I) \longrightarrow 0$ is exact for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Hence the construction of the sheaf $V \longmapsto \mathrm{Hom}_{\mathbb{Z}}(-[V], I)$, where $V \subset X$ are arbitrary open subschemes, transforms our sequence of cosheaves into an exact sequence of sheaves of $\mathcal{O}_X$-modules $0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{E}, I) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}_0, I) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}_d, I) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}, I) \longrightarrow 0$. Now the sheaves $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}^i, I)$ are flasque, and by part (a) it follows that the sheaf $\mathrm{Hom}_{\mathbb{Z}}(\mathfrak{F}, I)$ is. Therefore, the cosheaf $\mathfrak{F}$ is coflasque. (Cf. Section A.5.) ☐

We have shown, in particular, that coflasque contraherent cosheaves on a scheme X form a full subcategory of $X\text{–}\mathsf{ctrh}$ closed under extensions, kernels of admissible epimorphisms, and infinite products. Hence this full subcategory acquires an induced exact category structure, which we will denote by $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$.

Similarly, coflasque locally cotorsion contraherent cosheaves on X form a full subcategory of $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ closed under extensions, kernels of admissible epimorphisms, and infinite products. We denote the induced exact category structure on this full subcategory by $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$.

Corollary 3.4.8. *Let $f\colon Y \longrightarrow X$ be a quasi-compact quasi-separated morphism of schemes. Then*

(a) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes coflasque contraherent cosheaves on Y to coflasque contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{cfq}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$ between these exact categories;*

(b) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes coflasque locally cotorsion contraherent cosheaves on Y to coflasque locally cotorsion contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$ between these exact categories.*

Proof. Since the contraadjustness/contraherence conditions, local cotorsion, and exactness of short sequences in $X\text{–}\mathsf{ctrh}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ only depend on the restrictions to affine open subschemes, while the coflasqueness is preserved by restrictions to any open subschemes, we can assume that the scheme X is affine. Then the scheme Y is quasi-compact and quasi-separated.

Let $Y = \bigcup_\alpha V_\alpha$ be a finite affine open covering and $\mathfrak{F}$ be a coflasque contraherent cosheaf on Y. Then the Čech complex $C_\bullet(\{V_\alpha\}, \mathfrak{F})$ is a finite left resolution of the

$\mathcal{O}(X)$-module $\mathfrak{F}[Y]$. Let us first consider the case when Y is semi-separated, so the intersection of any nonempty subset of V_α is affine.

By Lemma 1.2.2(a), our resolution consists of contraadjusted $\mathcal{O}(X)$-modules. When $\mathfrak{F}$ is a locally cotorsion contraherent cosheaf, by Lemma 1.3.4(a) this resolution even consists of cotorsion $\mathcal{O}(X)$-modules. It follows that the $\mathcal{O}(X)$-module $(f_!\mathfrak{F})[X] = \mathfrak{F}[Y]$ is contraadjusted in the former case and cotorsion in the latter one.

Now let $U \subset X$ be an affine open subscheme. Then one has

$$\mathfrak{F}[f^{-1}(U) \cap V] \simeq \operatorname{Hom}_{\mathcal{O}_Y(V)}(\mathcal{O}_Y(f^{-1}(U) \cap V),\ \mathfrak{F}[V]) \simeq \operatorname{Hom}_{\mathcal{O}(X)}(\mathcal{O}_X(U), \mathfrak{F}[V])$$

for any affine open subscheme $V \subset Y$; so the complex $C_\bullet(\{f^{-1}(U) \cap V_\alpha\}, \mathfrak{F}|_{f^{-1}(U)})$ can be obtained by applying the functor $\operatorname{Hom}_{\mathcal{O}(X)}(\mathcal{O}_X(U), -)$ to the complex $C_\bullet(\{V_\alpha\}, \mathfrak{F})$. It follows that $(f_!\mathfrak{F})[U] \simeq \mathfrak{F}[f^{-1}(U)] \simeq \operatorname{Hom}_{\mathcal{O}(X)}(\mathcal{O}_X(U), \mathfrak{F}[Y])$ and the contraherence condition holds for $f_!\mathfrak{F}$.

Finally, let us turn to the general case. According to the above, for any semi-separated quasi-compact open subscheme $V \subset Y$ the $\mathcal{O}(X)$-module $\mathfrak{F}[V]$ is contraadjusted (and even cotorsion if $\mathfrak{F}$ is locally cotorsion), and for any affine open subscheme $U \subset X$ one has $\mathfrak{F}[f^{-1}(U) \cap V] \simeq \operatorname{Hom}_{\mathcal{O}(X)}(\mathcal{O}_X(U), \mathfrak{F}[V])$. Given that the intersection of any nonempty subset of V_α is separated (being quasi-affine) and quasi-compact, the same argument as above goes through. ☐

Corollary 3.4.9. *Let $f\colon Y \longrightarrow X$ be a quasi-compact quasi-separated morphism of schemes and $\mathbf{T}$ be an open covering of Y. Then*

(a) *for any complex $\mathcal{F}^\bullet$ of flasque quasi-coherent sheaves on Y that is acyclic as a complex over Y–qcoh, the complex $f_*\mathcal{F}^\bullet$ of flasque quasi-coherent sheaves on X is acyclic as a complex over X–qcoh;*

(b) *for any complex $\mathfrak{F}^\bullet$ of coflasque contraherent cosheaves on Y that is acyclic as a complex over Y–$\mathsf{lcth}_{\mathbf{T}}$, the complex $f_!\mathfrak{F}^\bullet$ of coflasque contraherent cosheaves on X is acyclic as a complex over X–ctrh;*

(c) *for any complex $\mathfrak{F}^\bullet$ of coflasque locally cotorsion contraherent cosheaves on Y that is acyclic as a complex over Y–$\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$, the complex $f_!\mathfrak{F}^\bullet$ of coflasque locally cotorsion contraherent cosheaves on X is acyclic as a complex over X–$\mathsf{ctrh}^{\mathsf{lct}}$.*

Proof. Let us prove part (c), parts (a-b) being analogous. In view of the results of Section 3.2, it suffices to consider the case of an affine scheme X and a quasi-compact quasi-separated scheme Y. We have to show that the complex of cotorsion $\mathcal{O}(X)$-modules $\Delta(Y, \mathfrak{F}^\bullet)$ is acyclic over $\mathcal{O}(X)$–$\mathsf{mod}^{\mathsf{cot}}$. Considering the case of a semi-separated scheme Y first and the general case second, one can assume that Y has a finite open covering $Y = \bigcup_{\alpha=1}^N U_\alpha$ by quasi-compact quasi-separated schemes U_α subordinate to $\mathbf{T}$ such that for any intersection V of a nonempty subset of U_α the complex $\Delta(V, \mathfrak{F}^\bullet)$ is acyclic over $\mathcal{O}(X)$–$\mathsf{mod}^{\mathsf{cot}}$.

Consider the Čech bicomplex $C_\bullet(\{U_\alpha\}, \mathfrak{F}^\bullet)$ of the complex of cosheaves $\mathfrak{F}^\bullet$ on the scheme Y with the open covering U_α. There is a natural morphism of bicomplexes $C_\bullet(\{U_\alpha\}, \mathfrak{F}^\bullet) \longrightarrow \Delta(X, \mathfrak{F}^\bullet)$; and for every degree i the complex $C_\bullet(\{U_\alpha\}, \mathfrak{F}^i) \longrightarrow \Delta(X, \mathfrak{F}^i)$ is a finite (and uniformly bounded) acyclic complex of cotorsion $\mathcal{O}(X)$-modules. It follows that the induced morphism of total complexes is a quasi-isomorphism (in fact, a morphism with an absolutely acyclic cone) of complexes over $\mathcal{O}(X)\text{–}\mathsf{mod}^{\mathsf{cot}}$. On the other hand, for every k the complex $C_k(\{U_\alpha\},\mathfrak{F}^\bullet)$ is by assumption acyclic over over $\mathcal{O}(X)\text{–}\mathsf{mod}^{\mathsf{cot}}$. It follows that the total complex of $C_\bullet(\{U_\alpha\},\mathfrak{F}^\bullet)$, and hence also the complex $\Delta(Y,\mathfrak{F}^\bullet)$, is acyclic over $\mathcal{O}(X)\text{–}\mathsf{mod}^{\mathsf{cot}}$. $\square$

3.5. **Contraherereable cosheaves and the contraherator.** A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X is said to be *derived contraherereable* if

(i°) for any affine open subscheme $U \subset X$ and its finite affine open covering $U = \bigcup_{\alpha=1}^N U_\alpha$ the homological Čech sequence (cf. (22))

$$(28)\quad 0 \longrightarrow \mathfrak{P}[U_1\cap\cdots\cap U_N] \longrightarrow \cdots \longrightarrow \bigoplus\nolimits_{\alpha<\beta}\mathfrak{P}[U_\alpha\cap U_\beta] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{P}[U] \longrightarrow 0$$

is exact; and

(ii) for any affine open subscheme $U \subset X$, the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is contraadjusted.

We will call (i°) the *exactness condition* and (ii) the *contraadjustness condition.*

Notice that the present contraadjustness condition (ii) is an equivalent restatement of the contraadjustness condition (ii) of Section 2.2, while the exactness condition (i°) is weaker than the contraherence condition (i) of Section 2.2 (provided that the condition (ii) is assumed). The condition (i°) is also weaker than the coflasqueness condition on a cosheaf of $\mathcal{O}_X$-modules discussed in Section 3.4.

By Remark 2.1.4, the exactness condition (i°) can be thought of as a strengthening of the cosheaf property (6) of a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X. Any such functor satisfying (i°) can be extended to a cosheaf of $\mathcal{O}_X$-modules in a unique way.

Let $\mathbf{W}$ be an open covering of a scheme X. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X is called $\mathbf{W}$*-locally derived contraherereable* if its restrictions $\mathfrak{P}|_W$ to all the open subschemes $W \in \mathbf{W}$ are derived contraherereable on W. In other words, this means that the conditions (i°) and (ii) must hold for all the affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. A cosheaf of $\mathcal{O}_X$-modules is called locally derived contraherereable if it is $\mathbf{W}$-locally derived contraherereable for some open covering $\mathbf{W}$. Any contraherent cosheaf is derived contraherereable, and any $\mathbf{W}$-locally contraherent cosheaf is $\mathbf{W}$-locally derived contraherereable. Conversely, according to Lemma 3.2.2, if a $\mathbf{W}$-locally contraherent cosheaf is derived contraherereable, then it is contraherent.

A $\mathbf{W}$-locally derived contraherereable cosheaf $\mathfrak{P}$ on X is called *locally cotorsion* (respectively, *locally injective*) if the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion (resp., injective) for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. A locally derived contraherereable cosheaf $\mathfrak{P}$ is locally cotorsion (resp., locally injective) if and only if the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion (resp., injective) for every affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{P}|_U$ is derived contraherereable.

In the exact category of cosheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X\text{–}\mathsf{cosh}_{\mathbf{W}}$ defined in Section 3.1, the $\mathbf{W}$-locally derived contraherereable cosheaves form an exact subcategory closed under extensions, infinite products, and cokernels of admissible monomorphisms.

Hence there is the induced exact category structure on the category of $\mathbf{W}$-locally derived contrahereable cosheaves on X.

Let U be an affine scheme and $\mathfrak{Q}$ be a derived contrahereable cosheaf on U. The *contraherator* $\mathfrak{C}\,\mathfrak{Q}$ of the cosheaf $\mathfrak{Q}$ is defined in this simplest case as the contraherent cosheaf on U corresponding to the contraadjusted $\mathcal{O}(U)$-module $\mathfrak{Q}[U]$, that is $\mathfrak{C}\,\mathfrak{Q} = \widetilde{\mathfrak{Q}[U]}$. There is a natural morphism of derived contrahereable cosheaves $\mathfrak{Q} \longrightarrow \mathfrak{C}\,\mathfrak{Q}$ on U (see Lemma 2.2.4). For any affine open subscheme $V \subset U$ there is a natural morphism of contraherent cosheaves $\mathfrak{C}(\mathfrak{Q}|_V) \longrightarrow (\mathfrak{C}\,\mathfrak{Q})|_V$ on V. Our next goal is to extend this construction to an appropriate class of cosheaves of $\mathcal{O}_X$-modules on quasi-compact semi-separated schemes X (cf. [65, Appendix B]).

Let X be such a scheme with an open covering $\mathbf{W}$ and a finite affine open covering $X = \bigcup_{\alpha=1}^N U_\alpha$ subordinate to $\mathbf{W}$. The *contraherator complex* $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ of a $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{P}$ on X is a finite Čech complex of contraherent cosheaves on X of the form

$$(29)\quad 0 \longrightarrow j_{1,\dots,N!}\,\mathfrak{C}(\mathfrak{P}|_{U_1\cap\dots\cap U_N}) \longrightarrow \cdots \longrightarrow \textstyle\bigoplus_{\alpha<\beta} j_{\alpha,\beta!}\,\mathfrak{C}(\mathfrak{P}|_{U_\alpha\cap U_\beta}) \longrightarrow \bigoplus_\alpha j_{\alpha!}\,\mathfrak{C}(\mathfrak{P}|_{U_\alpha}),$$

where $j_{\alpha_1,\dots,\alpha_k}$ is the open embedding $U_{\alpha_1}\cap\dots\cap U_{\alpha_k} \longrightarrow X$ and the notation $\mathfrak{C}(\mathfrak{Q})$ was explained above. The differentials in this complex are constructed using the adjunction of the direct and inverse images of contraherent cosheaves and the above morphisms $\mathfrak{C}(\mathfrak{Q}|_V) \longrightarrow (\mathfrak{C}\,\mathfrak{Q})|_V$.

Lemma 3.5.1. *Let $\mathfrak{P}$ be a $\mathbf{W}$-locally derived contrahereable cosheaf on a quasi-compact semi-separated scheme X. Then the object of the bounded derived category* $\mathsf{D^b}(X\text{–}\mathsf{ctrh})$ *of the exact category of contraherent cosheaves on X represented by the complex $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ does not depend on the choice of a finite affine open covering $\{U_\alpha\}$ of X subordinate to the covering $\mathbf{W}$.*

Proof. Let us adjoin another affine open subscheme $V \subset X$, subordinate to $\mathbf{W}$, to the covering $\{U_\alpha\}$. Then the complex $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ embeds into the complex $\mathfrak{C}_\bullet(\{V,U_\alpha\},\mathfrak{P})$ by a termwise split morphism of complexes with the cokernel isomorphic to the complex $k_!\,\mathfrak{C}_\bullet(\{V\cap U_\alpha\},\mathfrak{P}|_V) \longrightarrow k_!\,\mathfrak{C}(\mathfrak{P}|_V)$.

The complex of contraherent cosheaves $\mathfrak{C}_\bullet(\{V\cap U_\alpha\},\mathfrak{P}|_V) \longrightarrow \mathfrak{C}(\mathfrak{P}|_V)$ on V corresponds to the acyclic complex of contraadjusted $\mathcal{O}(V)$-modules (28) for the covering of an affine open subscheme $V \subset X$ by the affine open subschemes $V \cap U_\alpha$. Hence the cokernel of the admissible monomorphism of complexes $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P}) \longrightarrow \mathfrak{C}_\bullet(\{V,U_\alpha\},\mathfrak{P})$ is an acyclic complex of contraherent cosheaves on X.

Now, given two affine open coverings $X = \bigcup_\alpha U_\alpha = \bigcup_\beta V_\beta$ subordinate to $\mathbf{W}$, one compares both complexes $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ and $\mathfrak{C}_\bullet(\{V_\beta\},\mathfrak{P})$ with the complex $\mathfrak{C}_\bullet(\{U_\alpha,V_\beta\},\mathfrak{P})$ corresponding to the union of the two coverings $\{U_\alpha,V_\beta\}$. □

A $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{P}$ on a quasi-compact semi-separated scheme X is called $\mathbf{W}$-*locally contrahereable* if the complex $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ for some particular (or equivalently, for any) finite affine open covering $X=\bigcup_\alpha U_\alpha$ is quasi-isomorphic in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ to a $\mathbf{W}$-locally contraherent cosheaf on X (viewed as a complex concentrated in homological degree 0). The $\mathbf{W}$-locally contraherent cosheaf that appears here is called the ($\mathbf{W}$-*local*) *contraherator* of $\mathfrak{P}$ and denoted by $\mathfrak{C}\mathfrak{P}$. A derived contrahereable cosheaf $\mathfrak{P}$ on X is called *contrahereable* if it is locally contrahereable with respect to the covering $\{X\}$.

Any derived contrahereable cosheaf $\mathfrak{Q}$ on an affine scheme U is contrahereable, because the contraherator complex $\mathfrak{C}_\bullet(\{U\},\mathfrak{Q})$ is concentrated in homological degree 0. The contraherator cosheaf $\mathfrak{C}\mathfrak{Q}$ constructed in this way coincides with the one defined above specifically in the affine scheme case; so our notation and terminology is consistent. Any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on a quasi-compact semi-separated scheme X is $\mathbf{W}$-locally contrahereable; the corresponding contraherator complex $\mathfrak{C}_\bullet(\{U_\alpha\},\mathfrak{P})$ is the contraherent Čech resolution (27) of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}=\mathfrak{C}\mathfrak{P}$.

The contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\},-)$ is an exact functor from the exact category of $\mathbf{W}$-locally contrahereable cosheaves to the exact category of finite complexes of contraherent cosheaves on X, or to the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{ctrh})$. The full subcategory of $\mathbf{W}$-locally contrahereable cosheaves in the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves is closed under extensions and infinite products. Hence it acquires the induced exact category structure. The contraherator $\mathfrak{C}$ is an exact functor from the exact category of $\mathbf{W}$-locally contrahereable cosheaves to that of $\mathbf{W}$-locally contraherent ones.

All the above applies to locally cotorsion and locally injective $\mathbf{W}$-locally derived contrahereable cosheaves as well. These form full exact subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves. The contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\},-)$ takes locally cotorsion (resp., locally injective) $\mathbf{W}$-locally derived contrahereable cosheaves to finite complexes of locally cotorsion (resp., locally injective) contraherent cosheaves on X.

A locally cotorsion (resp., locally injective) $\mathbf{W}$-locally derived contrarehreable cosheaf is called $\mathbf{W}$-locally contrahereable if it is $\mathbf{W}$-locally contrahereable as a $\mathbf{W}$-locally derived contrahereable cosheaf. The contraherator functor $\mathfrak{C}$ takes locally cotorsion (resp., locally injective) $\mathbf{W}$-locally contrahereable cosheaves to locally cotorsion (resp., locally injective) $\mathbf{W}$-locally contraherent cosheaves.

Let $f\colon Y\longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine morphism of schemes. Then the direct image functor $f_!$ takes $\mathbf{T}$-locally derived contrahereable cosheaves on Y to $\mathbf{W}$-locally derived contrahereable cosheaves on X. For a $(\mathbf{W},\mathbf{T})$-affine morphism f of quasi-compact semi-separated schemes, the functor $f_!$ also commutes with the contraherator complex construction, as there is a natural isomorphism of complexes of contraherent cosheaves

$$f_!\,\mathfrak{C}_\bullet(\{f^{-1}(U_\alpha)\},\mathfrak{P})\simeq\mathfrak{C}_\bullet(\{U_\alpha\},f_!\mathfrak{P})$$

for any finite affine open covering U_α of X subordinate to $\mathbf{W}$. It follows that the functor $f_!$ takes $\mathbf{T}$-locally contrahereable cosheaves to $\mathbf{W}$-locally contrahereable cosheaves and commutes with the functor $\mathfrak{C}$.

For any $\mathbf{W}$-locally contrahereable cosheaf $\mathfrak{P}$ and $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Q}$ on a quasi-compact semi-separated scheme X, there is a natural isomorphism of the groups of morphisms

$$\mathrm{Hom}^{\mathcal{O}_X}(\mathfrak{P},\mathfrak{Q}) \simeq \mathrm{Hom}^X(\mathfrak{C}\mathfrak{P},\mathfrak{Q}). \tag{30}$$

In other words, the functor $\mathfrak{C}$ is left adjoint to the identity embedding functor of the category of $\mathbf{W}$-locally contrahereable cosheaves into the category of $\mathbf{W}$-locally contraherent ones. Indeed, applying the contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\},-)$ to a morphism $\mathfrak{P}\longrightarrow\mathfrak{Q}$ and passing to the zero homology, we obtain the corresponding morphism $\mathfrak{C}\mathfrak{P}\longrightarrow\mathfrak{Q}$. Conversely, any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is the cokernel of the rightmost arrow of the complex

$$0\longrightarrow j_{1,\dots,N!}(\mathfrak{P}|_{U_1\cap\dots\cap U_N})\longrightarrow\cdots \\ \longrightarrow \textstyle\bigoplus_{\alpha<\beta} j_{\alpha,\beta!}(\mathfrak{P}|_{U_\alpha\cap U_\beta})\longrightarrow \bigoplus_\alpha j_{\alpha!}(\mathfrak{P}|_{U_\alpha}) \tag{31}$$

in the additive category of cosheaves of $\mathcal{O}_X$-modules. A cosheaf $\mathfrak{P}$ satisfying the exactness condition (i°) for affine open subschemes $U\subset X$ subordinate to $\mathbf{W}$ is also quasi-isomorphic to the whole complex (31) in the exact category $\mathcal{O}_X\text{–}\mathsf{cosh}_{\mathbf{W}}$. Passing to the zero homology of the natural morphism between the complexes (31) and (29), we produce the desired adjunction morphism $\mathfrak{P}\longrightarrow\mathfrak{C}\mathfrak{P}$.

For any $\mathbf{W}$-locally contrahereable cosheaf $\mathfrak{P}$ and any quasi-coherent sheaf $\mathcal{M}$ on X the natural morphism of cosheaves of $\mathcal{O}_X$-modules $\mathfrak{P}\longrightarrow\mathfrak{C}\mathfrak{P}$ induces an isomorphism of the contratensor products

$$\mathcal{M}\odot_X\mathfrak{P}\simeq\mathcal{M}\odot_X\mathfrak{C}(\mathfrak{P}). \tag{32}$$

Indeed, for any injective quasi-coherent sheaf $\mathcal{J}$ on X one has

$$\mathrm{Hom}_X(\mathcal{M}\odot_X\mathfrak{P},\ \mathcal{J})\simeq\mathrm{Hom}^{\mathcal{O}_X}(\mathfrak{P},\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})) \\ \simeq\mathrm{Hom}^X(\mathfrak{C}\mathfrak{P},\mathfrak{Hom}_X(\mathcal{M},\mathcal{J}))\simeq\mathrm{Hom}_X(\mathcal{M}\odot_X\mathfrak{C}\mathfrak{P},\ \mathcal{J})$$

in view of the isomorphism (20).

3.6. **$\mathfrak{Cohom}$ into a locally derived contrahereable cosheaf.** We start with discussing the $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a locally contraherent cosheaf.

Let $\mathbf{W}$ be an open covering of a scheme X. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. The $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ on the scheme X is defined by the rule $U\longmapsto\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$. The contraadjustness and $\mathbf{W}$-local contraherence conditions can be verified in the same way as it was done in Section 2.4.

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf

$\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$.

Finally, if $\mathcal{M}$ is a quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the same rule as above. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X, then the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})$ is locally injective.

For any two very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a scheme X and any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X there is a natural isomorphism of $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\,\mathfrak{P}) \simeq \mathfrak{Cohom}_X(\mathcal{G},\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})). \tag{33}$$

Similarly, for any two flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X there is a natural isomorphism (33) of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{M}$, and locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X there are natural isomorphisms of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\begin{multline}\mathfrak{Cohom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F},\,\mathfrak{J}) \\ \simeq \mathfrak{Cohom}_X(\mathcal{M},\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})) \simeq \mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})).\end{multline} \tag{34}$$

More generally, let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally derived contrahereable cosheaf on X. The $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ on the scheme X is defined by the rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. For any pair of embedded affine open subschemes $V \subset U$ the restriction and corestriction morphisms $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ and $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ induce a morphism of $\mathcal{O}_X(U)$-modules

$$\operatorname{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V),\mathfrak{P}[V]) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U]),$$

so our rule defines a covariant functor with $\mathcal{O}_X$-module structure on the category of affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. The contraherence condition clearly holds; and to check the exactness condition (i°) of Section 3.5 for this covariant functor, it suffices to apply the functor $\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),-)$ to the exact sequence of contraadjusted $\mathcal{O}_X(U)$-modules (28) for the cosheaf $\mathfrak{P}$.

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}$ $(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Finally, if $\mathcal{M}$ is a quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the very same rule. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective **W**-locally derived contrahereable cosheaf on X, then the **W**-locally derived cotrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F}, \mathfrak{J})$ is locally injective.

The associativity isomorphisms (33–34) hold for the $\mathfrak{Cohom}$ into **W**-locally derived contrahereable cosheaves under the assumptions similar to the ones made above in the locally contraherent case.

3.7. **Contraherent tensor product.** Let $\mathfrak{P}$ be a cosheaf of $\mathcal{O}_X$-modules on a scheme X and $\mathcal{M}$ be a quasi-coherent cosheaf on X. Define a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X by the rule $U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U]$. To a pair of embedded affine open subschemes $V \subset U \subset X$ this functor assigns the $\mathcal{O}_X(U)$-module homomorphism

$$\mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathfrak{P}[V] \simeq \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[V] \longrightarrow \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U].$$

Obviously, this functor satisfies the condition (6) of Theorem 2.1.2 (since the restriction of the cosheaf $\mathfrak{P}$ to affine open subschemes of X does). Hence the functor $U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U]$ extends uniquely to a cosheaf of $\mathcal{O}_X$-modules on X, which we will denote by $\mathcal{M} \otimes_X \mathfrak{P}$.

For any quasi-coherent sheaf $\mathcal{M}$, cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$, and locally injective **W**-locally derived contrahereable cosheaf $\mathfrak{J}$ on a scheme X there is a natural isomorphism of abelian groups

$$\mathrm{Hom}^{\mathcal{O}_X}(\mathcal{M} \otimes_X \mathfrak{P},\, \mathfrak{J}) \simeq \mathrm{Hom}^{\mathcal{O}_X}(\mathfrak{P}, \mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J})). \tag{35}$$

The analogous adjunction isomorphism holds in the other cases mentioned in Section 3.6 when the functor $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a locally derived contrahereable cosheaf is defined. In other words, the functors $\mathfrak{Cohom}_X(\mathcal{M}, -)$ and $\mathcal{M} \otimes_X -$ between subcategories of the category of cosheaves of $\mathcal{O}_X$-modules are adjoint "wherever the former functor is defined".

For a locally free sheaf of finite rank $\mathcal{E}$ and a **W**-locally contraherent (resp., **W**-locally derived contrahereable) cosheaf $\mathfrak{P}$ on a scheme X, the cosheaf of $\mathcal{O}_X$-modules $\mathcal{E} \otimes_X \mathfrak{P}$ is **W**-locally contraherent (resp., **W**-locally derived contrahereable). There is a natural isomorphism of **W**-locally contraherent (resp., **W**-locally derived contrahereable) cosheaves $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_X \mathfrak{P} \simeq \mathfrak{Cohom}_X(\mathcal{E}, \mathfrak{P})$ on X.

The isomorphism $j_*j^*(\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}) \simeq j_*j^*\mathcal{K} \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)$ (15) for quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ and the embedding of an affine open subscheme $j\colon U \longrightarrow X$ allows to construct a natural isomorphism of quasi-coherent sheaves

$$(\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}) \odot_X \mathfrak{P} \simeq \mathcal{K} \odot_X (\mathcal{M} \otimes_X \mathfrak{P}) \tag{36}$$

for any quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a semi-separated scheme X.

Let **W** be an open covering of a scheme X. We will call a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ **W**-*flat* if the $\mathcal{O}_X(U)$-module $\mathfrak{F}[U]$ is flat for every affine open subscheme $U \subset X$ subordinate to **W**. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ is said to be *flat* if it is $\{X\}$-flat. Clearly, the direct image of a **T**-flat cosheaf of $\mathcal{O}_Y$-modules with respect to a flat $(\mathbf{W}, \mathbf{T})$-affine morphism of schemes $f\colon Y \longrightarrow X$ is **W**-flat.

One can easily see that whenever a cosheaf $\mathfrak{F}$ is $\mathbf{W}$-flat and satisfies the "exactness condition" (i°) of Section 3.5 for finite affine open coverings of affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$, the cosheaf $\mathcal{M} \otimes_X \mathfrak{F}$ also satisfies the condition (i°) for such open affines $U \subset X$. Similarly, whenever a quasi-coherent sheaf $\mathcal{F}$ is flat and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ satisfies the condition (i°), so does the cosheaf $\mathcal{F} \otimes_X \mathfrak{P}$.

The full subcategory of $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves is closed under extensions and kernels of admissible epimorphisms in the exact category of $\mathbf{W}$-locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ on X. Hence it acquires the induced exact category structure, which we will denote by $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}}$. The category $X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{fl}}$ will be denoted by $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. Similarly, there is the exact category structure on the category of $\mathbf{W}$-flat $\mathbf{W}$-locally derived contrahereable cosheaves on X induced from the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves.

A quasi-coherent sheaf $\mathcal{K}$ on a scheme X is called *coadjusted* if the $\mathcal{O}_X(U)$-module $\mathcal{K}(U)$ is coadjusted (see Section 1.6) for every affine open subscheme $U \subset X$. By Lemma 1.6.9, the coadjustness of a quasi-coherent sheaf is a local property. By the definition, if a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X satisfies the contraherence condition (ii) of Section 2.2 or 3.5 and a quasi-coherent sheaf $\mathcal{K}$ on X is coadjusted, then the cosheaf of $\mathcal{O}_X$-modules $\mathcal{K} \otimes_X \mathfrak{P}$ also satisfies the condition (ii).

The full subcategory of coadjusted quasi-coherent sheaves is closed under extensions and the passage to quotient objects in the abelian category of quasi-coherent sheaves $X\text{–}\mathsf{qcoh}$ on X. Hence it acquires the induced exact category structure, which we will denote by $X\text{–}\mathsf{qcoh}^{\mathsf{coa}}$.

Let $\mathcal{K}$ be a coadjusted quasi-coherent sheaf and $\mathfrak{F}$ be a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent (or more generally, $\mathbf{W}$-locally derived contrahereable) cosheaf on a scheme X. Then the tensor product $\mathcal{K} \otimes_X \mathfrak{F}$ satisfies both conditions (i°) and (ii) for affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$, i. e., it is $\mathbf{W}$-locally derived contrahereable. (Of course, the cosheaf $\mathcal{K} \otimes_X \mathfrak{F}$ is *not* in general locally contraherent, even if the cosheaf $\mathfrak{F}$ was $\mathbf{W}$-locally contraherent.)

Assuming the scheme X is quasi-compact and semi-separated, the contraherator complex construction now allows to assign to this cosheaf of $\mathcal{O}_X$-modules a complex of contraherent cosheaves $\mathfrak{C}_\bullet(\{U_\alpha\},\ \mathcal{K} \otimes_X \mathfrak{F})$ on X. To a short exact sequence of coadjusted quasi-coherent sheaves or $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent (or $\mathbf{W}$-locally derived contrahereable) cosheaves on X, the functor $\mathfrak{C}_\bullet(\{U_\alpha\},\ - \otimes_X -)$ assigns a short exact sequence of complexes of contraherent cosheaves.

By Lemma 3.5.1, the corresponding object of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{ctrh})$ does not depend on the choice of a finite affine open covering $\{U_\alpha\}$. We will denote it by $\mathcal{K} \otimes_{X\text{-}\mathsf{ct}}^{\mathbb{L}} \mathfrak{F}$ and call the *derived contraherent tensor product* of a coadjusted quasi-coherent sheaf $\mathcal{K}$ and a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on a quasi-compact semi-separated scheme X.

When the derived category object $\mathcal{K} \otimes_{X\text{-}\mathsf{ct}}^{\mathbb{L}} \mathfrak{F}$, viewed as an object of the derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ via the embedding of exact categories $X\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$, turns out to be isomorphic to an object of the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, we say that the (underived) *contraherent tensor product* of $\mathcal{K}$ and $\mathfrak{F}$ is defined, and denote the corresponding object by $\mathcal{K} \otimes_{X\text{-ct}} \mathfrak{F} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}$. In other words, for a coadjusted quasi-coherent sheaf $\mathcal{K}$ and a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on a quasi-compact semi-separated scheme X one sets $\mathcal{K} \otimes_{X\text{-ct}} \mathfrak{F} = \mathfrak{C}(\mathcal{K} \otimes_X \mathfrak{F})$ whenever the right-hand side is defined (where $\mathfrak{C}$ denotes the $\mathbf{W}$-local contraherator).

Now assume that the scheme X is locally Noetherian. Then, by Corollary 1.6.5(a), a contraherent cosheaf $\mathfrak{F}$ on an affine open subscheme $U \subset X$ is flat if and only if the contraadjusted $\mathcal{O}(U)$-module $\mathfrak{F}[U]$ is flat. Besides, the full subcategory of $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is closed under infinite products. In addition, any coherent sheaf on X is coadjusted, as is any injective quasi-coherent sheaf and any quasi-coherent quotient sheaf of an injective one.

For any injective quasi-coherent sheaf $\mathcal{J}$ and any $\mathbf{W}$-flat $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{F}$ on X, the tensor product $\mathcal{J} \otimes_X \mathfrak{F}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf on X. For any flat quasi-coherent sheaf $\mathcal{F}$ and any locally injective $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{J}$ on X, the tensor product $\mathcal{F} \otimes_X \mathfrak{J}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf. These assertions hold since the tensor product of a flat module and an injective module over a Noetherian ring is injective.

Let $\mathcal{M}$ be a coherent sheaf and $\mathfrak{F}$ be a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf on X. Then the cosheaf of $\mathcal{O}_X$-modules $\mathcal{M} \otimes_X \mathfrak{F}$ is $\mathbf{W}$-locally contraherent. Indeed, by Corollary 1.6.3(a), the $\mathcal{O}_X(U)$-module $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}(U)$ is contraadjusted for any affine open subscheme $U \subset X$. For a pair of embedded affine open subschemes $V \subset U \subset X$ subordinate to the covering $\mathbf{W}$, one has

$$\begin{aligned}\mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathfrak{F}[V] &\simeq \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[V] \\ &\simeq \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{F}[U]) \\ &\simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\, \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U])\end{aligned}$$

according to Corollary 1.6.3(c). If the scheme X is semi-separated and Noetherian, the contraherent tensor product $\mathcal{M} \otimes_{X\text{-ct}} \mathfrak{F}$ is defined and isomorphic to the tensor product $\mathcal{M} \otimes_X \mathfrak{F}$.

Similarly, it follows from Corollary 1.6.4 that for any coherent sheaf $\mathcal{M}$ and $\mathbf{W}$-flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X the tensor product $\mathcal{M} \otimes_X \mathfrak{F}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. (See Sections 5.1–5.2 and Lemma 5.7.2 for further discussion.)

3.8. **Compatibility of direct and inverse images with the tensor operations.** Let $\mathbf{W}$ be an open covering of a scheme X and $\mathbf{T}$ be an open covering of a scheme Y. Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W}, \mathbf{T})$-coaffine morphism.

Let $\mathcal{F}$ be a flat quasi-coherent sheaf and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on the scheme X. Then there is a natural isomorphism of locally injective $\mathbf{T}$-locally contraherent cosheaves

$$f^! \,\mathfrak{Cohom}_X(\mathcal{F}, \mathfrak{J}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{F}, f^!\mathfrak{J}) \tag{37}$$

on the scheme Y.

Assume additionally that f is a flat morphism. Let $\mathcal{M}$ be a quasi-coherent sheaf and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{T}$-locally contraherent cosheaves

$$f^! \mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{M}, f^!\mathfrak{J}) \tag{38}$$

on Y. Analogously, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X, then there is a natural isomorphism of locally cotorsion $\mathbf{T}$-locally contraherent cosheaves

$$f^! \mathfrak{Cohom}_X(\mathcal{F}, \mathfrak{P}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{F}, f^!\mathfrak{P}) \tag{39}$$

on the scheme Y.

Assume that, moreover, f is a very flat morphism. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf and $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Then there is the natural isomorphism (39) of $\mathbf{T}$-locally contraherent cosheaves on Y.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism. Let $\mathcal{N}$ be a quasi-coherent cosheaf on Y and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(f_*\mathcal{N}, \mathfrak{J}) \simeq f_! \mathfrak{Cohom}_Y(\mathcal{N}, f^!\mathfrak{J}) \tag{40}$$

on the scheme X. This is one version of the projection formula for the $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a contraherent cosheaf.

Assume additionally that f is a flat morphism. Let $\mathcal{G}$ be a flat quasi-coherent sheaf on Y and $\mathfrak{P}$ be a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(f_*\mathcal{G}, \mathfrak{P}) \simeq f_! \mathfrak{Cohom}_Y(\mathcal{G}, f^!\mathfrak{P}) \tag{41}$$

on the scheme X.

Assume that, moreover, f is a very flat morphism. Let $\mathcal{G}$ be a very flat quasi-coherent sheaf on Y and $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Then there is the natural isomorphism (41) of $\mathbf{W}$-locally contraherent cosheaves on X.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine morphism. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $\mathfrak{Q}$ be a $\mathbf{T}$-locally contraherent cosheaf on Y. Then there is a natural isomorphism of $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{F}, f_!\mathfrak{Q}) \simeq f_! \mathfrak{Cohom}_Y(f^*\mathcal{F}, \mathfrak{Q}) \tag{42}$$

on the scheme X. The similar isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X holds for any flat quasi-coherent sheaf $\mathcal{F}$ on X and locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y. This is another version of the projection formula for $\mathfrak{Cohom}$.

Assume additionally that f is a flat morphism. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{I}$ be a locally injective $\mathbf{T}$-locally contraherent cosheaf on Y. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{M}, f_!\mathfrak{I}) \simeq f_! \mathfrak{Cohom}_Y(f^*\mathcal{M}, \mathfrak{I}) \tag{43}$$

on the scheme X.

Let $f\colon Y \longrightarrow X$ be either an affine morphism of semi-separated schemes, or a morphism of quasi-compact semi-separated schemes. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $\mathcal{Q}$ be a contraadjusted quasi-coherent sheaf on Y. Then there is a natural isomorphism of contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{F}, f_*\mathcal{Q}) \simeq f_!\,\mathfrak{Hom}_Y(f^*\mathcal{F}, \mathcal{Q}) \tag{44}$$

on X. Here the quasi-coherent sheaf $f_*\mathcal{Q}$ on X is contraadjusted according to Section 2.5 above (if f is affine) or Corollary 4.1.13 below (if X and Y are quasi-compact). The right-hand side is, by construction, a contraherent cosheaf if the morphism f is affine, and a cosheaf of $\mathcal{O}_X$-modules otherwise (see Section 2.3). Both sides are, in fact, contraherent in the general case, because the isomorphism holds and the left-hand side is. This is a version of the projection formula for $\mathfrak{Hom}$.

Indeed, let $j\colon U \longrightarrow X$ be an embedding of an affine open subscheme; set $V = U\times_X Y$. Let $j'\colon V \longrightarrow Y$ and $f'\colon V \longrightarrow U$ be the natural morphisms. Then one has

$$\begin{aligned}\mathfrak{Hom}_X(\mathcal{F}, f_*\mathcal{Q})[U] &\simeq \operatorname{Hom}_X(j_*j^*\mathcal{F}, f_*\mathcal{Q}) \simeq \operatorname{Hom}_Y(f^*j_*j^*\mathcal{F}, \mathcal{Q}) \\ &\simeq \operatorname{Hom}_Y(j'_*f'^*j^*\mathcal{F}, \mathcal{Q}) \simeq \operatorname{Hom}_Y(j'_*j'^*f^*\mathcal{F}, \mathcal{Q}) \simeq \mathfrak{Hom}_Y(f^*\mathcal{F}, \mathcal{Q})[V].\end{aligned}$$

Here we are using the fact that the direct images of quasi-coherent sheaves with respect to affine morphisms of schemes commute with the inverse images in the base change situations. Notice that, when the morphism f is not affine, neither is the scheme V; however, the scheme V is quasi-compact and the open embedding morphism $j'\colon V \longrightarrow Y$ is affine, so Lemma 2.5.2(a) applies. The similar isomorphism of locally cotorsion contraherent cosheaves on X holds for any flat quasi-coherent sheaf $\mathcal{F}$ on X and any cotorsion quasi-coherent sheaf $\mathcal{Q}$ on Y.

Now let f be a flat quasi-compact morphism of semi-separated schemes. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathcal{J}$ be an injective quasi-coherent sheaf on Y. Then there is a natural isomorphism of locally cotorsion contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{M}, f_*\mathcal{J}) \simeq f_!\,\mathfrak{Hom}_Y(f^*\mathcal{M}, \mathcal{J}) \tag{45}$$

on the scheme Y. The proof is similar to the above. For any flat quasi-compact morphism of quasi-separated schemes $f\colon Y \longrightarrow X$, a quasi-coherent sheaf $\mathcal{M}$ on X, and an injective quasi-coherent sheaf $\mathcal{J}$ on Y, there is a natural morphism of cosheaves of $\mathcal{O}_X$-modules from the right-hand side to the left-hand side of (45); this morphism is an isomorphism whenever the morphism f is also affine.

Finally, let $f\colon Y \longrightarrow X$ be a quasi-compact open embedding of quasi-separated schemes. Then the isomorphism (45) holds for any flasque quasi-coherent sheaf $\mathcal{M}$ on X and injective quasi-coherent sheaf $\mathcal{J}$ on Y. In fact, the direct images of quasi-coherent sheaves with respect to quasi-compact quasi-separated morphisms commute with the inverse images with respect to flat morphisms of schemes in the base change situations, while the restrictions to open subschemes also preserve the flasqueness; so one can apply Lemma 2.5.2(d).

Let $f\colon Y \longrightarrow X$ be an affine morphism of semi-separated schemes. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{Q}$ be a cosheaf of $\mathcal{O}_Y$-modules. Then there is a natural isomorphism of quasi-coherent sheaves

$$\mathcal{M} \odot_X f_!\mathfrak{Q} \simeq f_*(f^*\mathcal{M} \odot_Y \mathfrak{Q}) \tag{46}$$

on the scheme X. This is a version of the projection formula for the contratensor product of quasi-coherent sheaves and cosheaves of $\mathcal{O}_X$-modules.

Indeed, in the notation above, for any affine open subscheme $U \subset X$ we have

$$\begin{aligned} j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} (f_!\mathfrak{Q})[U] \simeq j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{Q}[V] \\ \simeq (j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \\ \simeq j_*f'_*f'^*j^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq f_*j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \\ \simeq f_*(j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V]). \end{aligned}$$

As it was explained Section 2.6, the contratensor product $f^*\mathcal{M} \odot_Y \mathfrak{Q}$ can be computed as the inductive limit over the diagram $\mathbf{D}$ formed by the affine open subschemes $V \subset Y$ of the form $V = U \times_X Y$, where U are affine open subschemes in X. It remains to use the fact that the direct image of quasi-coherent sheaves with respect to an affine morphism is an exact functor. The same isomorphism (46) holds for a flat affine morphism f of quasi-separated schemes.

Furthermore, there is a natural morphism from the left-hand side to the right-hand side of (46) for any quasi-compact morphism of quasi-separated schemes $f\colon Y \longrightarrow X$. It is constructed as the composition

$$\begin{aligned} \mathcal{M} \odot_X f_!\mathfrak{Q} = \varinjlim\nolimits_U j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{Q}[f^{-1}(U)] \simeq \varinjlim\nolimits_U j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} (\varinjlim\nolimits_{V \subset f^{-1}(U)} \mathfrak{Q}[V]) \\ \simeq \varinjlim\nolimits_U \varinjlim\nolimits_{f(V) \subset U} j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{Q}[V] \simeq \varinjlim\nolimits_{f(V) \subset U} j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{Q}[V] \longrightarrow \\ \varinjlim\nolimits_{f(V) \subset U} j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq \varinjlim\nolimits_{f(V) \subset U} f_*j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \\ \longrightarrow f_* \varinjlim\nolimits_{f(V) \subset U} j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq f_*(f^*\mathcal{M} \odot_Y \mathfrak{Q}). \end{aligned}$$

Here the inductive limit is taken firstly over affine open subschemes $U \subset X$, then over affine open subschemes $V \subset Y$ such that $f(V) \subset U$, and eventually over pairs of affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$. The open embeddings $U \longrightarrow X$ and $V \longrightarrow Y$ are denoted by j and j', while the morphism $V \longrightarrow U$ is denoted by f'. The final isomorphism holds, since the contratensor product $f^*\mathcal{M} \odot_Y \mathfrak{Q}$ can be computed over the diagram $\mathbf{D}$ formed by all the pairs of affine open subschemes (U, V) such that $f(V) \subset U$.

In particular, the isomorphism

$$\mathcal{M} \odot_X h_!\mathfrak{F} \simeq h_*(h^*\mathcal{M} \odot_Y \mathfrak{F}) \tag{47}$$

holds for any open embedding $h\colon Y \longrightarrow X$ of an affine scheme Y into a quasi-separated scheme X, any quasi-coherent sheaf $\mathcal{M}$ on X, and any flat cosheaf of

$\mathcal{O}_X$-modules $\mathfrak{F}$ on Y. Indeed, in this case one has

$$\varinjlim_{h(V)\subset U} h_*j'_*j'^*h^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{F}[V] \simeq \varinjlim_V h_*j'_*j'^*h^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{F}[V]$$
$$\simeq h_*h^*\mathcal{M} \otimes_{\mathcal{O}(Y)} \mathfrak{F}[Y] \simeq h_*(h^*\mathcal{M} \otimes_{\mathcal{O}(Y)} \mathfrak{F}[Y]),$$

where the $\varinjlim_V$ is taken over all affine open subschemes $V \subset Y$, which is clearly equivalent to considering $V = Y$ only.

Let $f\colon Y \longrightarrow X$ be an affine morphism of schemes. Then for any quasi-coherent sheaf $\mathcal{M}$ on X and any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$ there is a natural isomorphism of cosheaves of $\mathcal{O}_X$-modules

$$f_!(f^*\mathcal{M} \otimes_Y \mathfrak{Q}) \simeq \mathcal{M} \otimes_X f_!\mathfrak{Q}. \tag{48}$$

This is a version of the projection formula for the tensor product of quasi-coherent sheaves and cosheaves of $\mathcal{O}_X$-modules.

4. Quasi-compact Semi-separated Schemes

4.1. **Contraadjusted and cotorsion quasi-coherent sheaves.** Recall that the definition of a very flat quasi-coherent sheaf was given in Section 1.7 and the definition of a contraadjusted quasi-coherent sheaf in Section 2.5 (cf. Remark 2.5.4).

In particular, a quasi-coherent sheaf $\mathcal{P}$ over an affine scheme U is very flat (respectively, contraadjusted) if and only if the $\mathcal{O}(U)$-module $\mathcal{P}(U)$ is very flat (respectively, contraadjusted). The class of very flat quasi-coherent sheaves is preserved by inverse images with respect to arbitrary morphisms of schemes and direct images with respect to very flat affine morphisms (which includes affine open embeddings). The class of contraadjusted quasi-coherent sheaves is preserved by direct images with respect to affine morphisms of schemes.

The class of very flat quasi-coherent sheaves on any scheme X is closed under the passage to the kernel of a surjective morphism. Both the full subcategories of very flat and contraadjusted quasi-coherent sheaves are closed under extensions in the abelian category of quasi-coherent sheaves. Hence they acquire the induced exact category structures, which we denote by $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$, respectively.

Let us introduce one bit of categorical terminology. Given an exact category E and a class of objects $\mathsf{C} \subset \mathsf{E}$, we say that an object $X \in \mathsf{E}$ is a *finitely iterated extension* of objects from C if there exists a nonnegative integer N and a sequence of admissible monomorphisms $0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{N-1} \longrightarrow X_N = X$ in E such that the cokernels of all the morphisms $X_{i-1} \longrightarrow X_i$ belong to C (cf. Section 1.1).

Let X be a quasi-compact semi-separated scheme.

Lemma 4.1.1. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$, where $\mathcal{F}$ is a very flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in X.*

Proof. The proof is based on the construction from [15, Section A.1] and Theorem 1.1.1(b). We argue by a kind of induction in the number of affine open subschemes covering X. Assume that for some open subscheme $h\colon V \longrightarrow X$ there is a short exact sequence $0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow 0$ of quasi-coherent sheaves on X such that the restriction $h^*\mathcal{K}$ of the sheaf $\mathcal{K}$ to the open subscheme V is very flat, while the sheaf $\mathcal{Q}$ is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in X. Let $j\colon U \longrightarrow X$ be an affine open subscheme; we will construct a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$ having the same properties with respect to the open subscheme $U \cup V \subset X$.

Pick an short exact sequence $0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{G} \longrightarrow j^*\mathcal{K} \longrightarrow 0$ of quasi-coherent sheaves on the affine scheme U such that the sheaf $\mathcal{G}$ is very flat and the sheaf $\mathcal{R}$ is contraadjusted. Consider its direct image $0 \longrightarrow j_*\mathcal{R} \longrightarrow j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{K} \longrightarrow 0$ with respect to the affine open embedding j, and take its pull-back with respect to the adjunction morphism $\mathcal{K} \longrightarrow j_*j^*\mathcal{K}$. Let $\mathcal{F}$ denote the middle term of the resulting short exact sequence of quasi-coherent sheaves on X.

By Lemma 1.2.6(a), it suffices to show that the restrictions of $\mathcal{F}$ to U and V are very flat in order to conclude that the restriction to $U \cup V$ is. We have $j^*\mathcal{F} \simeq \mathcal{G}$, which is very flat by the construction. On the other hand, the sheaf $j^*\mathcal{K}$ is very flat over $V \cap U$, hence so is the sheaf $\mathcal{R}$, as the kernel of a surjective map $\mathcal{G} \longrightarrow j^*\mathcal{K}$. The embedding $U \cap V \longrightarrow V$ is a very flat affine morphism, so the sheaf $j_*\mathcal{R}$ is very flat over V. Now it is clear from the short exact sequence $0 \longrightarrow j_*\mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0$ that the sheaf $\mathcal{F}$ is very flat over V.

Finally, the kernel $\mathcal{P}$ of the composition of surjective morphisms $\mathcal{F} \longrightarrow \mathcal{K} \longrightarrow \mathcal{M}$ is an extension of the sheaves $\mathcal{Q}$ and $j_*\mathcal{R}$, the latter of which is the direct image of a contraadjustested quasi-coherent sheaf from an affine open subscheme of X, and the former is a finitely iterated extension of such. ☐

Corollary 4.1.2. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is contraadjusted if and only if the functor* $\operatorname{Hom}_X(-,\mathcal{P})$ *takes short exact sequences of very flat quasi-coherent sheaves on X to short exact sequences of abelian groups.*

(b) *A quasi-coherent sheaf $\mathcal{P}$ on X is contraadjusted if and only if* $\operatorname{Ext}_X^{>0}(\mathcal{F},\mathcal{P}) = 0$ *for any very flat quasi-coherent sheaf $\mathcal{F}$ on X.*

(c) *The class of contraadjusted quasi-coherent sheaves on X is closed with respect to the passage to the cokernels of injective morphisms.*

Proof. While the condition in part (a) is *a priori* weaker and the condition in part (b) is *a priori* stronger than our definition of a contraherent cosheaf $\mathcal{P}$ by the condition $\operatorname{Ext}_X^1(\mathcal{F},\mathcal{P}) = 0$ for any very flat $\mathcal{F}$, all the three conditions are easily seen to be equivalent provided that every quasi-coherent sheaf on X is the quotient sheaf of a very flat one. That much we know from Lemma 4.1.1. The condition in (b) clearly has the property (c). ☐

Lemma 4.1.3. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow 0$, where $\mathcal{F}$ is a very flat quasi-coherent sheaf and*

$\mathcal{P}$ is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in X.

Proof. Any quasi-coherent sheaf on a quasi-compact quasi-separated scheme can be embedded into a finite direct sum of direct images of injective quasi-coherent sheaves from affine open subschemes constituting a finite covering. So an embedding $\mathcal{M} \longrightarrow \mathcal{J}$ of a sheaf $\mathcal{M}$ into a sheaf $\mathcal{J}$ with the desired (an even stronger) properties exists, and it remains to make sure that the quotient sheaf has the desired properties.

One does this using Lemma 4.1.1 and (the dual version of) the procedure used in the second half of the proof of Theorem 10 in [18] (see the proof of Lemma 1.1.3). Present the quotient sheaf $\mathcal{J}/\mathcal{M}$ as the quotient sheaf of a very flat sheaf $\mathcal{F}$ by a subsheaf $\mathcal{Q}$ representable as a finitely iterated extension of the desired kind. Set $\mathcal{P}$ to be the fibered product of $\mathcal{J}$ and $\mathcal{F}$ over $\mathcal{J}/\mathcal{M}$; then $\mathcal{P}$ is an extension of $\mathcal{J}$ and $\mathcal{Q}$, and there is a natural injective morphism $\mathcal{M} \longrightarrow \mathcal{P}$ with the cokernel $\mathcal{F}$. □

Corollary 4.1.4. (a) *Any quasi-coherent sheaf on X admits a surjective map onto it from a very flat quasi-coherent sheaf such that the kernel is contraadjusted.*

(b) *Any quasi-coherent sheaf on X can be embedded into a contraadjusted quasi-coherent sheaf in such a way that the cokernel is very flat.*

(c) *A quasi-coherent sheaf on X is contraadjusted if and only if it is a direct summand of a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes of X.*

Proof. Parts (a) and (b) follow from Lemmas 4.1.1 and 4.1.3, respectively. The proof of part (c) uses (the dual version of) the argument from the proof of Corollary 1.1.4. Given a contraadjusted quasi-coherent sheaf $\mathcal{P}$, use Lemma 4.1.3 to embed it into a finitely iterated extension $\mathcal{Q}$ of the desired kind in such a way that the cokernel $\mathcal{F}$ is a very flat quasi-coherent sheaf. Since $\mathrm{Ext}^1_X(\mathcal{F}, \mathcal{P}) = 0$ by the definition, we can conclude that $\mathcal{P}$ is a direct summand of $\mathcal{Q}$. □

Lemma 4.1.5. *A quasi-coherent sheaf on X is very flat and contraadjusted if and only if it is a direct summand of a finite direct sum of the direct images of very flat contraadjusted quasi-coherent sheaves from affine open subschemes of X.*

Proof. The "if" assertion is clear. To prove "only if", notice that the very flat contraadjusted quasi-coherent sheaves are the injective objects of the exact category of very flat quasi-coherent sheaves (cf. Section 1.4). So it remains to show that there are enough injectives of the kind described in the formulation of Lemma in the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$.

Indeed, let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering. Denote by j_α the identity open embeddings $U_\alpha \longrightarrow X$. For each α, pick an injective morphism $j_\alpha^*\mathcal{F} \longrightarrow \mathcal{G}_\alpha$ from a very flat quasi-coherent sheaf $j_\alpha^*\mathcal{F}$ to a very flat contraadjusted quasi-coherent sheaf $\mathcal{G}_\alpha$ on U_α such that the cokernel $\mathcal{G}_\alpha/j_\alpha^*\mathcal{F}_\alpha$ is a very flat. Then $\bigoplus_\alpha j_{\alpha*}\mathcal{G}_\alpha$ is a very flat contraadjusted quasi-coherent sheaf on X and the cokernel of the natural morphism $\mathcal{F} \longrightarrow \bigoplus_\alpha j_{\alpha*}\mathcal{G}_\alpha$ is very flat (since its restriction to each U_α is). □

Lemma 4.1.6. *A quasi-coherent sheaf on X is flat and contraadjusted if and only if it is a direct summand of a finitely iterated extension of the direct images of flat contraadjusted quasi-coherent sheaves from affine open subschemes of X.*

Proof. Given a flat quasi-coherent sheaf $\mathcal{E}$, we apply the constructions of Lemmas 4.1.1 and 4.1.3 in order to obtain a short exact sequence of quasi-coherent sheaves $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow 0$ with a very flat quasi-coherent sheaf $\mathcal{F}$. One can verify step by step that the whole construction is performed entirely inside the exact category of flat quasi-coherent sheaves on X, so the quasi-coherent sheaf $\mathcal{P}$ it produces is a finitely iterated extension of the direct images of flat contraadjusted quasi-coherent sheaves from affine open subschemes of X. Now if the sheaf $\mathcal{E}$ was also contraadjusted, then the short exact sequence splits by Corollary 4.1.2(b), providing the desired result. ☐

The following corollary provides equivalent definitions of contraadjusted and very flat quasi-coherent sheaves on a quasi-compact semi-separated scheme resembling the corresponding definitions for modules over a ring in Section 1.1.

Corollary 4.1.7. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is contraadjusted if and only if $\operatorname{Ext}_X^{>0}(j_*j^*\mathcal{O}_X, \mathcal{P}) = 0$ for any affine open embedding of schemes $j\colon Y \longrightarrow X$.*

(b) *A quasi-coherent sheaf $\mathcal{F}$ on X is very flat if and only if $\operatorname{Ext}_X^1(\mathcal{F}, \mathcal{P}) = 0$ for any contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X.*

Proof. Part (a): the "only if" assertion follows from Corollary 4.1.2(b). To prove "if", notice that any very flat sheaf $\mathcal{F}$ on X has a finite right Čech resolution by finite direct sums of sheaves of the form $j_*j^*\mathcal{F}$, where $j\colon U \longrightarrow X$ are embeddings of affine open subschemes. Hence the condition $\operatorname{Ext}_X^{>0}(j_*j^*\mathcal{F}, \mathcal{P}) = 0$ for all such j implies $\operatorname{Ext}_X^{>0}(\mathcal{F}, \mathcal{P}) = 0$.

Furthermore, a very flat quasi-coherent sheaf $j^*\mathcal{F}$ on U is a direct summand of a transfinitely iterated extension of the direct images of the structure sheaves of principal affine open subschemes $V \subset U$ (by Corollary 1.1.4). Since the direct images with respect to affine morphisms preserve transfinitely iterated extensions, it remains to use the quasi-coherent sheaf version of the result that Ext^1-orthogonality is preserved by transfinitely iterated extensions in the first argument [18, Lemma 1].

Part (b): "only if" holds by the definition of contraadjusted sheaves, and "if" can be deduced from Corollary 4.1.4(a) by an argument similar to the proof of Corollary 1.1.4 (and dual to that of Corollary 4.1.4(c)). ☐

Now we proceed to formulate the analogues of the above assertions for cotorsion quasi-coherent sheaves. The definition of these was given in Section 2.5. The class of cotorsion quasi-coherent sheaves is closed under extensions in the abelian category of quasi-coherent sheaves on a scheme and under the direct images with respect to affine morphisms of schemes. We denote the induced exact category structure on the category of cotorsion quasi-coherent sheaves on a scheme X by $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$.

As above, in the sequel X denotes a quasi-compact semi-separated scheme.

Lemma 4.1.8. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$, where $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes in X.*

Proof. Similar to that of Lemma 4.1.1, except that Theorem 1.3.1(b) is being used in place of Theorem 1.1.1(b). ☐

Corollary 4.1.9. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is cotorsion if and only if the functor* $\mathrm{Hom}_X(-,\mathcal{P})$ *takes short exact sequences of flat quasi-coherent sheaves on X to short exact sequences of abelian groups.*

(b) *A quasi-coherent sheaf $\mathcal{P}$ on X is cotorsion if and only if* $\mathrm{Ext}_X^{>0}(\mathcal{F},\mathcal{P}) = 0$ *for any flat quasi-coherent sheaf $\mathcal{F}$ on X.*

(c) *The class of cotorsion quasi-coherent sheaves on X is closed with respect to the passage to the cokernels of injective morphisms.*

Proof. Similar to that of Corollary 4.1.2. ☐

Lemma 4.1.10. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow 0$, where $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes in X.*

Proof. Similar to that of Lemma 4.1.3. ☐

Corollary 4.1.11. (a) *Any quasi-coherent sheaf on X admits a surjective map onto it from a flat quasi-coherent sheaf such that the kernel is cotorsion.*

(b) *Any quasi-coherent sheaf on X can be embedded into a cotorsion quasi-coherent sheaf in such a way that the cokernel is flat.*

(c) *A quasi-coherent sheaf on X is cotorsion if and only if it is a direct summand of a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes of X.*

Proof. Similar to that of Corollary 4.1.4. ☐

Lemma 4.1.12. *A quasi-coherent sheaf on X is flat and cotorsion if and only if it is a direct summand of a finite direct sum of the direct images of flat cotorsion quasi-coherent sheaves from affine open subschemes of X.*

Proof. Similar to that of Lemma 4.1.5. ☐

The following result shows that contraadjusted (and in particular, cotorsion) quasi-coherent sheaves are adjusted to direct images with respect to nonaffine morphisms of quasi-compact semi-separated schemes (cf. Corollary 4.5.3 below).

Corollary 4.1.13. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ takes the full exact subcategory $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset Y\text{–}\mathsf{qcoh}$ into the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset X\text{–}\mathsf{qcoh}$, and induces an exact functor between these exact categories;*

(b) *the functor* $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ *takes the full exact subcategory* $Y\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset Y\text{–}\mathsf{qcoh}$ *into the full exact subcategory* $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset X\text{–}\mathsf{qcoh}$*, and induces an exact functor between these exact categories.*

Proof. For any affine morphism $g\colon V \longrightarrow Y$ into a quasi-compact semi-separated scheme Y, the inverse image functor g^* takes quasi-coherent sheaves that can be represented as finitely iterated extensions of the direct images of quasi-coherent sheaves from affine open subschemes in Y to quasi-coherent sheaves of the similar type on V. This follows easily from the fact that direct images of quasi-coherent sheaves with respect to affine morphisms of schemes commute with inverse images in the base change situations. In particular, it follows from Corollary 4.1.4(c) that the functor g^* takes contraadjusted quasi-coherent sheaves on Y to quasi-coherent sheaves that are direct summands of finitely iterated extensions of the direct images of quasi-coherent sheaves from affine open subschemes $W \subset V$.

The quasi-coherent sheaves on V that can be represented as such iterated extensions form a full exact subcategory in the abelian category of quasi-coherent sheaves. The functor of global sections $\Gamma(V,-)$ is exact on this exact category. Indeed, there is a natural isomorphism of the Ext groups $\operatorname{Ext}_V^*(\mathcal{F}, h_*\mathcal{G}) \simeq \operatorname{Ext}_W^*(h^*\mathcal{F}, \mathcal{G})$ for any quasi-coherent sheaves $\mathcal{F}$ on V and $\mathcal{G}$ on W, and a flat affine morphism $h\colon W \longrightarrow V$. Applying this isomorphism in the case when h is the embedding of an affine open subscheme and $\mathcal{F} = \mathcal{O}_V$, one concludes that $\operatorname{Ext}_V^{>0}(\mathcal{O}_V, \mathcal{G}) = 0$ for all quasi-coherent sheaves $\mathcal{G}$ from the exact category in question.

Specializing to the case of the open subschemes $V = U \times_X Y \subset Y$, where U are affine open subschemes in X, we deduce the assertion that the functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}$ is exact. It remains to recall that the direct images of contraadjusted quasi-coherent sheaves with respect to affine morphisms of schemes are contraadjusted in order to show that f_* takes $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$. Since the direct images of cotorsion quasi-coherent sheaves with respect to affine morphisms of schemes are cotorsion, it similarly follows that f_* takes $Y\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ to $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$. ☐

4.2. **Colocally projective contraherent cosheaves.** Let X be a scheme and $\mathbf{W}$ be its open covering. A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is called *colocally projective* if for any short exact sequence $0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{K} \longrightarrow 0$ of locally injective $\mathbf{W}$-locally contraherent cosheaves on X the short sequence of abelian groups $0 \longrightarrow \operatorname{Hom}^X(\mathfrak{P}, \mathfrak{I}) \longrightarrow \operatorname{Hom}^X(\mathfrak{P}, \mathfrak{J}) \longrightarrow \operatorname{Hom}^X(\mathfrak{P}, \mathfrak{K}) \longrightarrow 0$ is exact.

Obviously, the class of colocally projective $\mathbf{W}$-locally contraherent cosheaves on X is closed under direct summands. It follows from the adjunction isomorphism (23) of Section 3.3 that the functor of direct image of $\mathbf{T}$-locally contraherent cosheaves $f_!$ with respect to any $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes $f\colon Y \longrightarrow X$ takes colocally projective $\mathbf{T}$-locally contraherent cosheaves on Y to colocally projective $\mathbf{W}$-locally contraherent cosheaves on X. It is also clear that *any* contraherent cosheaf on an affine scheme U with the covering $\{U\}$ is colocally projective.

Lemma 4.2.1. *On any scheme X with an open covering $\mathbf{W}$, any coflasque contraherent cosheaf is colocally projective.*

Proof. We will prove a somewhat stronger assertion: any short exact sequence $0 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$ in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ with $\mathfrak{F} \in X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$ and $\mathfrak{J} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ splits. It will follow easily that the functor $\mathrm{Hom}^X(\mathfrak{F}, -)$ takes any short exact sequence in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ to a short exact sequence of abelian groups.

We proceed by applying Zorn's lemma to the partially ordered set of sections $\phi_Y\colon \mathfrak{F}|_Y \longrightarrow \mathfrak{Q}|_Y$ of the morphism of cosheaves $\mathfrak{Q} \longrightarrow \mathfrak{F}$ defined over open subsets $Y \subset X$. Since it suffices to define a morphism of cosheaves on the modules of cosections over affine open subschemes, which are quasi-compact, a compatible system of sections ϕ_{Y_i} defined over a linearly ordered family of open subsets $Y_i \subset X$ extends uniquely to a section over the union $\bigcup_i Y_i$.

Now let ϕ_Y be a section over Y and $U \subset X$ be an affine open subscheme subordinate to $\mathbf{W}$. Set $V = Y \cap U$; by assumption, the $\mathcal{O}(U)$-module homomorphism $\mathfrak{F}[V] \longrightarrow \mathfrak{F}[U]$ is injective and the $\mathcal{O}(U)$-module $\mathfrak{J}[U]$ is injective. The short exact sequence of $\mathcal{O}(U)$-modules $0 \longrightarrow \mathfrak{J}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{F}[U] \longrightarrow 0$ splits, and the difference between two such splittings $\mathfrak{F}[U] \rightrightarrows \mathfrak{Q}[U]$ is an arbitrary $\mathcal{O}(U)$-module morphism $\mathfrak{F}[U] \longrightarrow \mathfrak{J}[U]$. The composition $\mathfrak{F}[V] \longrightarrow \mathfrak{Q}[U]$ of the morphism $\phi_Y[V]\colon \mathfrak{F}[V] \longrightarrow \mathfrak{Q}[V]$ and the corestriction morphism $\mathfrak{Q}[V] \longrightarrow \mathfrak{Q}[U]$ can therefore be extended to an $\mathcal{O}(U)$-linear section $\mathfrak{F}[U] \longrightarrow \mathfrak{Q}[U]$ of the surjection $\mathfrak{Q}[U] \longrightarrow \mathfrak{F}[U]$.

We have constructed a morphism of contraherent cosheaves $\phi_U\colon \mathfrak{F}|_U \longrightarrow \mathfrak{Q}|_U$ whose restriction to V coincides with the restriction of the morphism $\phi_Y\colon \mathfrak{F}|_Y \longrightarrow \mathfrak{Q}|_Y$. Set $Z = Y \cup U$; the pair of morphisms of cosheaves ϕ_Y and ϕ_U extends uniquely to a morphism of cosheaves $\phi_Z\colon \mathfrak{F}|_Z \longrightarrow \mathfrak{Q}|_Z$. Since the morphisms ϕ_Y and ϕ_U were some sections of the surjection $\mathfrak{Q} \longrightarrow \mathfrak{F}$ over Y and U, the morphism ϕ_Z is a section of this surjection over Z. □

Generally speaking, according to the above definition the colocal projectivity property of a locally contraherent cosheaf $\mathfrak{P}$ on a scheme X may depend not only on the cosheaf $\mathfrak{P}$ itself, but also on the covering $\mathbf{W}$. No such dependence occurs on quasi-compact semi-separated schemes. Indeed, we will see below in this section that on such a scheme any colocally projective $\mathbf{W}$-locally contraherent cosheaf is (globally) contraherent. Moreover, the class of colocally projective $\mathbf{W}$-locally contraherent cosheaves coincides with the class of colocally projective contraherent cosheaves and does not depend on the covering $\mathbf{W}$.

Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering.

Lemma 4.2.2. *Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$;*

(b) *any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$.*

Proof. The argument is a dual version of the proofs of Lemmas 4.1.1 and 4.1.8. Let us prove part (a); the proof of part (b) is completely analogous.

Arguing by induction in $1 \leqslant \beta \leqslant N$, we consider the open subscheme $V = \bigcup_{\alpha<\beta} U_\alpha$ with the induced covering $\mathbf{W}|_V = \{V \cap W \mid W \in \mathbf{W}\}$ and the identity embedding $h\colon V \longrightarrow X$. Assume that we have constructed an exact triple $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{Q} \longrightarrow 0$ of $\mathbf{W}$-locally contraherent cosheaves on X such that the restriction $h^!\mathfrak{K}$ of the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{K}$ to the open subscheme $V \subset X$ is locally injective, while the cosheaf $\mathfrak{Q}$ on X is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$, $\alpha < \beta$. When $\beta = 1$, it suffices to take $\mathfrak{K} = \mathfrak{M}$ and $\mathfrak{Q} = 0$ for the induction base. Set $U = U_\beta$ and denote by $j\colon U \longrightarrow X$ the identity open embedding.

Let $0 \longrightarrow j^!\mathfrak{K} \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{R} \longrightarrow 0$ be an exact triple of contraherent cosheaves on the affine scheme U such that the contraherent cosheaf $\mathfrak{I}$ is (locally) injective. Consider its direct image $0 \longrightarrow j_!j^!\mathfrak{K} \longrightarrow j_!\mathfrak{I} \longrightarrow j_!\mathfrak{R} \longrightarrow 0$ with respect to the affine open embedding j, and take its push-forward with respect to the adjunction morphism $j_!j^!\mathfrak{K} \longrightarrow \mathfrak{K}$. Let us show that in the resulting exact triple $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{J} \longrightarrow j_!\mathfrak{R} \longrightarrow 0$ the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ is locally injective in the restriction to $U \cup V$. By Lemma 1.3.6(b), it suffices to show that the restrictions of $\mathfrak{J}$ to U and V are locally injective.

Indeed, in the restriction to U we have $j^!j_!j^!\mathfrak{K} \simeq j^!\mathfrak{K}$, hence $j^!\mathfrak{J} \simeq j^!j_!\mathfrak{I} \simeq \mathfrak{I}$ is a (locally) injective contraherent cosheaf. On the other hand, if $j'\colon U \cap V \longrightarrow V$ and $h'\colon U \cap V \longrightarrow U$ denote the embeddings of $U \cap V$, then $h^!j_!\mathfrak{R} \simeq j'_!h'^!\mathfrak{R}$ (as explained in the end of Section 3.3). Notice that the contraherent cosheaf $h'^!j^!\mathfrak{K} \simeq j'^!h^!\mathfrak{K}$ is locally injective, hence the contraherent cosheaf $h'^!\mathfrak{R}$ is locally injective as the cokernel of the admissible monomorphism of locally injective contraherent cosheaves $h'^!j^!\mathfrak{K} \longrightarrow h'^!\mathfrak{I}$. Since the local injectivity of $\mathbf{T}$-locally contraherent cosheaves is preserved by the direct images with respect to flat $(\mathbf{W}, \mathbf{T})$-affine morphisms, the contraherent cosheaf $j'_!h'^!\mathfrak{R}$ is locally injective, too. Now in the exact triple $0 \longrightarrow h^!\mathfrak{K} \longrightarrow h^!\mathfrak{J} \longrightarrow h^!j_!\mathfrak{R} \longrightarrow 0$ of $\mathbf{W}|_V$-locally contraherent cosheaves on V the middle term is locally injective, because so are the other two terms.

Finally, the composition of admissible monomorphisms of $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{M} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{J}$ on X is an admissible monomorphism with the cokernel isomorphic to an extension of the contraherent cosheaves $j_!\mathfrak{R}$ and $\mathfrak{Q}$, hence also a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$, $\alpha \leqslant \beta$. The induction step is finished, and the whole lemma is proven. $\square$

We denote by $\operatorname{Ext}^{X,*}(-,-)$ the Ext groups in the exact category of $\mathbf{W}$-locally contraherent cosheaves on X. Notice that these do not depend on the covering $\mathbf{W}$ and coincide with the Ext groups in the whole category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$. Indeed, the full exact subcategory $X\text{–}\mathsf{lcth}_\mathbf{W}$ is closed under extensions and the passage to kernels of admissible epimorphisms in $X\text{–}\mathsf{lcth}$ (see Section 3.2), and for any object $\mathfrak{P} \in X\text{–}\mathsf{lcth}$ there exists an admissible epimorphism onto $\mathfrak{P}$ from an object of $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_\mathbf{W}$ (see the resolution (27) in Section 3.3).

For the same reasons (up to duality), the Ext groups computed in the exact subcategories of locally cotorsion and locally injective $\mathbf{W}$-locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ agree with those in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (and also in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lin}}$). Indeed, the full exact subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ are closed under extensions and the passage to cokernels of admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see Section 3.1), and we have just constructed in Lemma 4.2.2 an admissible monomorphism from any $\mathbf{W}$-locally contraherent cosheaf to a locally injective one. We refer to Sections A.2–A.3 for further details.

Corollary 4.2.3. (a) *A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is colocally projective if and only if $\operatorname{Ext}^{X,1}(\mathfrak{P},\mathfrak{J})=0$ and if and only if $\operatorname{Ext}^{X,>0}(\mathfrak{P},\mathfrak{J})=0$ for all locally injective $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{J}$ on X.*

(b) *The class of colocally projective $\mathbf{W}$-locally contraherent cosheaves on X is closed under extensions and the passage to kernels of admissible epimorphisms in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.*

Proof. Part (a) follows from the existence of an admissible monomorphism from any $\mathbf{W}$-locally contraherent cosheaf on X into a locally injective $\mathbf{W}$-locally contraherent cosheaf (a weak form of Lemma 4.2.2(a)). Part (b) follows from part (a). ☐

Lemma 4.2.4. *Let $X=\bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0\longrightarrow\mathfrak{J}\longrightarrow\mathfrak{P}\longrightarrow\mathfrak{M}\longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha\subset X$;*

(b) *any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0\longrightarrow\mathfrak{J}\longrightarrow\mathfrak{P}\longrightarrow\mathfrak{M}\longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes $U_\alpha\subset X$.*

Proof. There is an admissible epimorphism $\bigoplus_\alpha j_{\alpha!}j_\alpha^!\mathfrak{M}\longrightarrow\mathfrak{M}$ (see (27) for the notation and explanation) onto any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ from a finite direct sum of the direct images of contraherent cosheaves from the affine open subschemes U_α. When $\mathfrak{M}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf, this is an admissible epimorphism in the category of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves, and $j_\alpha^!\mathfrak{M}$ are locally cotorsion contraherent cosheaves on U_α.

Given that, the desired exact triples in Lemma can be obtained from those of Lemma 4.2.2 by the construction from the second half of the proof of Theorem 10 in [18] (see the proof of Lemma 1.1.3; cf. the proofs of Lemmas 4.1.3 and 4.1.10). ☐

Corollary 4.2.5. (a) *For any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X there exists an admissible monomorphism from $\mathfrak{M}$ into a locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X such that the cokernel $\mathfrak{P}$ is a colocally projective $\mathbf{W}$-locally contraherent cosheaf.*

(b) *For any* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{M}$ *on* X *there exists an admissible epimorphism onto* $\mathfrak{M}$ *from a colocally projective* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{P}$ *on* X *such that the kernel* $\mathfrak{J}$ *is a locally injective* $\mathbf{W}$*-locally contraherent cosheaf.*

(c) *Let* $X = \bigcup_\alpha U_\alpha$ *be a finite affine open covering subordinate to* $\mathbf{W}$*. Then a* $\mathbf{W}$*-locally contraherent cosheaf on* X *is colocally projective if and only if it is (a contraherent cosheaf and) a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes* $U_\alpha \subset X$*.*

Proof. The "if" assertion in part (c) follows from Corollary 4.2.3(b) together with our preliminary remars in the beginning of this section. This having been shown, part (a) follows from Lemma 4.2.2(a) and part (b) from Lemma 4.2.4(a).

The "only if" assertion in (c) follows from Corollary 4.2.3(a) and Lemma 4.2.4(a) by the argument from the proof of Corollary 1.1.4 (cf. Corollaries 4.1.4(c) and 4.1.11(c)). Notice that the functors of direct image with respect to the open embeddings $U_\alpha \longrightarrow X$ take contraherent cosheaves to contraherent cosheaves, and the full subcategory of contraherent cosheaves $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}$ is closed under extensions. ☐

By a colocally projective locally cotorsion $\mathbf{W}$-locally contraherent cosheaf we will mean a $\mathbf{W}$-locally contraherent cosheaf that is simultaneously colocally projective and locally cotorsion.

Corollary 4.2.6. (a) *For any locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{M}$ *on* X *there exists an admissible monomorphism from* $\mathfrak{M}$ *into a locally injective* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{J}$ *on* X *such that the cokernel* $\mathfrak{P}$ *is a colocally projective locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf.*

(b) *For any locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{M}$ *on* X *there exists an admissible epimorphism onto* $\mathfrak{M}$ *from a colocally projective locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{P}$ *on* X *such that the kernel* $\mathfrak{J}$ *is a locally injective* $\mathbf{W}$*-locally contraherent cosheaf.*

(c) *Let* $X = \bigcup_\alpha U_\alpha$ *be a finite affine open covering subordinate to* $\mathbf{W}$*. Then a locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf on* X *is colocally projective if and only if it is (a contraherent cosheaf and) a direct summand of a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes* $U_\alpha \subset X$*.*

Proof. Same as Corollary 4.2.5, except that parts (b) of Lemmas 4.2.2 and 4.2.4 need to be used. Parts (a-b) can be also easily deduced from Corollary 4.2.5(a-b). ☐

Corollary 4.2.7. *The full subcategory of colocally projective* $\mathbf{W}$*-locally contraherent cosheaves in the exact category of all locally contraherent cosheaves on* X *does not depend on the choice of the open covering* $\mathbf{W}$*.*

Proof. Given two open coverings $\mathbf{W}'$ and $\mathbf{W}''$ of the scheme X, pick a finite affine open covering $X = \bigcup_{\alpha=1}^N U_\alpha$ subordinate to both $\mathbf{W}'$ and $\mathbf{W}''$, and apply Corollary 4.2.5(c). ☐

As a full subcategory closed under extensions and kernels of admissible epimorphisms in $X\text{–}\mathsf{ctrh}$, the category of colocally projective contraherent cosheaves on X

acquires the induced exact category structure. We denote this exact category by $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$. The (similarly constructed) exact category of colocally projective locally cotorsion contraherent cosheaves on X is denoted by $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$.

The full subcategory of contraherent sheaves that are simultaneously colocally projective and locally injective will be denoted by $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$. Clearly, any extension of two objects from this subcategory is trivial in $X\text{–}\mathsf{ctrh}$, so the category of colocally projective locally injective contraherent cosheaves is naturally viewed as an additive category endowed with the trivial exact category structure.

It follows from Corollary 4.2.5(a-b) that the additive category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ is simultaneously the category of projective objects in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ and the category of injective objects in $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, and that it has enough of both such projectives and injectives.

Corollary 4.2.8. *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering. Then a contraherent cosheaf on X is colocally projective and locally injective if and only if it is isomorphic to a direct summand of a finite direct sum of the direct images of (locally) injective contraherent cosheaves from the open embeddings $U_\alpha \longrightarrow X$.*

Proof. For any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X, the map $\bigoplus_\alpha j_{\alpha!}j_\alpha^!\mathfrak{J} \longrightarrow \mathfrak{J}$ is an admissible epimorphism in the category of locally injective $\mathbf{W}$-locally contraherent cosheaves. Now if $\mathfrak{J}$ is also colocally projective, then the extension splits. □

Corollary 4.2.9. *The three full subcategories of colocally projective cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$, and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ are closed with respect to infinite products in the category $X\text{–}\mathsf{ctrh}$.*

Proof. The assertions are easily deduced from the descriptions of the full subcategories of colocally projective cosheaves given in Corollaries 4.2.5(c), 4.2.6(c), and 4.2.8 together with the fact that the functor of direct image of contraherent cosheaves with respect to an affine morphism of schemes preserves infinite products. □

Corollary 4.2.10. *Let $f\colon Y \longrightarrow X$ be an affine morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor of inverse image of locally injective locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ takes the full subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ into $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$;*

(b) *assuming that the morphism f is also flat, the functor of inverse image of locally cotorsion locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$ takes the full subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ into $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$;*

(c) *assuming that the morphism f is also very flat, the functor of inverse image of locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$ takes the full subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ into $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.*

Proof. Parts (a-c) follow from Corollaries 4.2.5(c), 4.2.6(c), and 4.2.8, respectively, together with the base change results from the second half of Section 3.3. □

4.3. **Colocally flat contraherent cosheaves.** Let X be a scheme and $\mathbf{W}$ be its open covering. A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X is called *colocally flat* if for any short exact sequence $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X the short sequence of abelian groups $0 \longrightarrow \operatorname{Hom}^X(\mathfrak{F}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^X(\mathfrak{F}, \mathfrak{Q}) \longrightarrow \operatorname{Hom}^X(\mathfrak{F}, \mathfrak{R}) \longrightarrow 0$ is exact.

Let us issue a *warning* that our terminology is misleading: the colocal flatness is, by the definition, a stronger condition that the colocal projectivity. It follows from the adjunction isomorphism (24) that the functor of direct image of $\mathbf{T}$-locally contraherent cosheaves $f_!$ with respect a flat $(\mathbf{W}, \mathbf{T})$-affine $(\mathbf{W}, \mathbf{T})$-coaffine morphism of schemes $f\colon Y \longrightarrow X$ takes colocally flat $\mathbf{T}$-locally contraherent cosheaves on Y to colocally flat $\mathbf{W}$-locally contraherent cosheaves on X. Clearly, a contraherent cosheaf $\mathfrak{F}$ on an affine scheme U with the covering $\{U\}$ is colocally flat whenever the contraadjusted $\mathcal{O}(U)$-module $\mathfrak{F}[U]$ is flat; the converse assertion can be deduced from Theorem 1.3.1(b) (cf. Corollary 4.3.4(c) below).

Let X be a quasi-compact semi-separated scheme. It follows from the results of Section 4.2 that any colocally flat $\mathbf{W}$-locally contraherent cosheaf on X is contraherent. We will see below in this section that the class of colocally flat $\mathbf{W}$-locally contraherent cosheaves on X coincides with the class of colocally flat contraherent cosheaves and does not depend on the covering $\mathbf{W}$.

Lemma 4.3.1. *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$, where $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{F}$ is a finitely iterated extension of the direct images of contraherent cosheaves on U_α corresponding to flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.*

Proof. Similar to the proof of Lemma 4.2.2, except that Theorem 1.3.1(a) needs to be used to resolve a contraherent cosheaf on an affine open subscheme $U \subset X$ (cf. the proof of Lemma 4.1.1). Besides, one has to use Lemma 1.3.6(a) and the fact that the class of locally cotorsion contraherent cosheaves is preserved by direct images with respect to affine morphisms. □

Corollary 4.3.2. (a) *A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X is colocally flat if and only if $\operatorname{Ext}^{X,1}(\mathfrak{F}, \mathfrak{P}) = 0$ and if and only if $\operatorname{Ext}^{X,>0}(\mathfrak{F}, \mathfrak{P}) = 0$ for all locally cotorsion $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{P}$ on X.*

(b) *The class of colocally flat $\mathbf{W}$-locally contraherent cosheaves on X is closed under extensions and the passage to kernels of admissible epimorphisms in the exact category X–$\mathsf{lcth}_{\mathbf{W}}$.*

Proof. Similar to the proof of Corollary 4.2.3. □

Lemma 4.3.3. *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{M} \longrightarrow 0$, where $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{F}$ is a finitely iterated extension of the direct images of contraherent cosheaves on U_α corresponding to flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.*

Proof. The proof is similar to that of Lemma 4.2.4 and based on Lemma 4.3.1. The key is to show that there is an admissible epimorphism in the exact category X–$\mathsf{lcth}_{\mathbf{W}}$

onto any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ from a finitely iterated extension (in fact, even a finite direct sum) of the direct images of contraherent cosheaves on U_α corresponding to flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.

Here it suffices to pick admissible epimorphisms from such contraherent cosheaves $\mathfrak{F}_\alpha$ on U_α onto the restrictions $j_\alpha^!\mathfrak{M}$ of $\mathfrak{M}$ to U_α and consider the corresponding morphism $\bigoplus_\alpha j_{\alpha!}\mathfrak{F}_\alpha \longrightarrow \mathfrak{M}$ of $\mathbf{W}$-locally contraherent cosheaves on X. To check that this is an admissible epimorphism, one can, e. g., notice that it is so in the restriction to each U_α and recall that being an admissible epimorphism of $\mathbf{W}$-locally contraherent cosheaves is a local property (see Section 3.2). □

Corollary 4.3.4. (a) *For any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X there exists an admissible monomorphism from $\mathfrak{M}$ into a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X such that the cokernel $\mathfrak{F}$ is a colocally flat $\mathbf{W}$-locally contraherent cosheaf.*

(b) *For any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X there exists an admissible epimorphism onto $\mathfrak{M}$ from a colocally flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X such that the kernel $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf.*

(c) *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then a $\mathbf{W}$-locally contraherent cosheaf on X is colocally flat if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves on U_α corresponding to flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.*

Proof. Similar to the proof of Corollary 4.2.5 and based on Lemmas 4.3.1, 4.3.3 and Corollary 4.3.2. □

Corollary 4.3.5. *The full subcategory of colocally flat $\mathbf{W}$-locally contraherent cosheaves in the exact category $X{-}\mathsf{lcth}$ does not depend on the open covering $\mathbf{W}$.*

Proof. Similar to the proof of Corollary 4.2.7 and based on Corollary 4.3.4(c). □

As a full subcategory closed under extensions and kernels of admissible epimorphisms in $X{-}\mathsf{ctrh}$, the category of colocally flat contraherent cosheaves on X acquires the induced exact category structure, which we denote by $X{-}\mathsf{ctrh}_{\mathsf{clf}}$.

The following corollary is to be compared with Corollaries 5.2.2(a) and 5.2.9(a).

Corollary 4.3.6. *Any colocally flat contraherent cosheaf over a semi-separated Noetherian scheme is flat.*

Proof. Follows from Corollary 4.3.4(c) together with the remarks about flat contraherent cosheaves over affine Noetherian schemes and the direct images of flat cosheaves of $\mathcal{O}$-modules in Section 3.7. □

Corollary 4.3.7. *Over a semi-separated Noetherian scheme X, the full subcategory $X{-}\mathsf{ctrh}_{\mathsf{clf}}$ is closed with respect to infinite products in $X{-}\mathsf{ctrh}$.*

Proof. In addition to what has been said in the proof of Corollary 4.2.9, it is also important here that infinite products of flat modules over a coherent ring are flat. □

Corollary 4.3.8. *Let X be a semi-separated Noetherian scheme and $j\colon Y \longrightarrow X$ be an affine open embedding. Then the inverse image functor $j^!$ takes colocally flat contraherent cosheaves to colocally flat contraherent cosheaves.*

Proof. Similar to the proof of Corollary 4.2.10 and based on Corollary 4.3.4(c); the only difference is that one also has to use Corollary 1.6.5(a). □

4.4. **Projective contraherent cosheaves.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its affine open covering.

Lemma 4.4.1. (a) *The exact category of $\mathbf{W}$-locally contraherent cosheaves on X has enough projective objects.*

(b) *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then a $\mathbf{W}$-locally contraherent cosheaf on X is projective if and only if it is a direct summand of a direct sum over α of the direct images of contraherent cosheaves on U_α corresponding to very flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.*

Proof. The assertion "if" in part (b) follows from the adjunction of the direct and inverse image functors for the embeddings $U_\alpha \longrightarrow X$ together with the fact that the very flat contraadjusted modules are the projective objects of the exact categories of contraadjusted modules over $\mathcal{O}(U_\alpha)$ (see Section 1.4).

It remains to show that there exists an admissible epimorphism onto any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X from a direct sum of the direct images of contraherent cosheaves on U_α corresponding to very flat contraadjusted modules. The construction is similar to the one used in the proof of Lemma 4.3.3 and based on Theorem 1.1.1(b). One picks admissible epimorphisms from contraherent cosheaves $\mathfrak{F}_\alpha$ of the desired kind on the affine schemes U_α onto the restrictions $j_\alpha^!\mathfrak{M}$ of the cosheaf $\mathfrak{M}$ and considers the corresponding morphism $\bigoplus_\alpha j_{\alpha!}\mathfrak{F}_\alpha \longrightarrow \mathfrak{M}$. □

Corollary 4.4.2. (a) *There are enough projective objects in the exact category X–lcth of locally contraherent cosheaves on X, and all these projective objects belong to the full subcategory of contraherent cosheaves X–$\mathsf{ctrh} \subset X$–lcth.*

(b) *The full subcategories of projective objects in the three exact categories X–$\mathsf{ctrh} \subset X$–$\mathsf{lcth}_{\mathbf{W}} \subset X$–$\mathsf{lcth}$ coincide.* □

Lemma 4.4.3. (a) *The exact category of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X has enough projective objects.*

(b) *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X is projective if and only if it is a direct summand of a direct sum over α of the direct images of locally cotorsion contraherent cosheaves on U_α corresponding to flat cotorsion $\mathcal{O}(U_\alpha)$-modules.*

Proof. Similar to the proof of Lemma 4.4.1 and based on Theorem 1.3.1(b). □

Corollary 4.4.4. (a) *There are enough projective objects in the exact category X–$\mathsf{lcth}^{\mathsf{lct}}$ of locally cotorsion locally contraherent cosheaves on X, and all these projective objects belong to the full subcategory of locally cotorsion contraherent cosheaves X–$\mathsf{ctrh}^{\mathsf{lct}} \subset X$–$\mathsf{lcth}^{\mathsf{lct}}$.*

(b) *The full subcategories of projective objects in the three exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ *coincide.* □

We denote the additive category of projective (objects in the category of) contraherent cosheaves on X by $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$, and the additive category of projective (objects in the category of) locally cotorsion contraherent cosheaves X by $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$.

Let us issue a *warning* that both the terminology and notation are misleading here: a projective locally cotorsion contraherent cosheaf on X does *not* have to be a projective contraherent cosheaf. Indeed, a flat cotorsion module over a commutative ring would not be in general very flat.

On the other hand we notice that, by the definition, both additive categories $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ are contained in the exact category of colocally flat contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ (and consequently also in the exact category of colocally projective contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$). Moreover, by the definition one clearly has $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}} = X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$. Finally, we notice that $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ is the category of *projective* objects in $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$, while $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ is the category of *injective* objects in $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ (and there are enough of both).

A version of part (a) of the following corollary that is valid in a different generality will be obtained in Section 5.2.

Corollary 4.4.5. *Let X be a semi-separated Noetherian scheme. Then*

(a) *any cosheaf from $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ is flat;*

(b) *any cosheaf from $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ is flat.*

Proof. Follows from Corollary 4.3.6. □

More general versions of the next corollary and of part (b) of the previous one will be obtained in Section 5.1. In both cases, we will see that the semi-separatedness and quasi-compactness assumptions can be dropped.

Corollary 4.4.6. *Over a semi-separated Noetherian scheme X, the full subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ of projective locally cotorsion contraherent cosheaves is closed under infinite products in $X\text{–}\mathsf{ctrh}$.*

Proof. Follows from Corollary 4.3.7. □

A more general version of parts (a-b) of the following Corollary will be proven in the next Section 4.5, while more general versions of part (b-c) will be also obtained in Section 5.1. In both cases, we will see that the affineness assumption on the morphism f is unnecessary.

Corollary 4.4.7. *Let $f\colon Y \longrightarrow X$ be an affine morphism of quasi-compact semi-separated schemes. Then*

(a) *if the morphism f is very flat, then the direct image functor $f_!$ takes projective contraherent cosheaves to projective contraherent cosheaves;*

(b) *if the morphism f is flat, then the direct image functor $f_!$ takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves;*

(c) *if the scheme X is Noetherian and the morphism f is an open embedding, then the inverse image functor $f^!$ takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves.*

Proof. Part (a) holds, since in its assumptions the functor $f_!\colon Y\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{ctrh}$ is "parially left adjoint" to the exact functor $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$. The proof of part (b) is similar (alternatively, it can be deduced from the facts that the functor $f_!$ takes $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ to $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ and $Y\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ to $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$). Part (c) follows from Corollary 4.3.8. ☐

4.5. **Homology of locally contraherent cosheaves.** The functor $\Delta(X,-)$ of global cosections of locally contraherent cosheaves on a scheme X, which assigns to a cosheaf $\mathfrak{E}$ the abelian group (or even the $\mathcal{O}(X)$-module) $\mathfrak{E}[X]$, is right exact as a functor on the exact category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$ on X. In other words, if $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{M} \longrightarrow 0$ is a short exact sequence of locally contraherent cosheaves on X, then the sequence of abelian groups

$$\Delta(X,\mathfrak{K}) \longrightarrow \Delta(X,\mathfrak{L}) \longrightarrow \Delta(X,\mathfrak{M}) \longrightarrow 0$$

is exact. Indeed, the procedure recovering the groups of cosections of cosheaves $\mathcal{F}$ on X from their groups of cosections over affine open subschemes $U \subset X$ subordinate to a particular covering $\mathbf{W}$ and the corestriction maps between such groups uses the operations of the infinite direct sum and the cokernel of a morphism (or in other words, the nonfiltered inductive limit) only (see (5), (7), or (22)).

Recall that for any $(\mathbf{W},\mathbf{T})$-affine morphism of schemes $f\colon Y \longrightarrow X$ the functor of direct image $f_!$ takes $\mathbf{T}$-locally contraherent cosheaves on Y to $\mathbf{W}$-locally contraherent cosheaves on X. By the definition, there is a natural isomorphism of $\mathcal{O}(X)$-modules $\mathfrak{E}[Y] \simeq (f_!\mathfrak{E})[X]$ for any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{E}$.

Now let X be a quasi-compact semi-separated scheme. Then the left derived functor of the functor of global cosections of locally contraherent cosheaves on X can be defined in the conventional way using left projective resolutions in the exact category $X\text{–}\mathsf{lcth}$ (see Lemma 4.4.1 and Corollary 4.4.2). Notice that the derived functors of $\Delta(X,-)$ (and in fact, any left derived functors) computed in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ for a particular open covering $\mathbf{W}$ and in the whole category $X\text{–}\mathsf{lcth}$ agree. We denote this derived functor by $\mathbb{L}_*\Delta(X,-)$. The groups $\mathbb{L}_i\Delta(X,\mathfrak{E})$ are called the *homology groups* of a locally contraherent cosheaf $\mathfrak{E}$ on the scheme X.

Let us point out that the functor $\Delta(U,-)$ of global cosections of contraherent cosheaves on an affine scheme U is exact, so the groups $\mathbb{L}_{>0}\Delta(U,\mathfrak{E})$ vanish when U is affine and $\mathfrak{E}$ is contraherent.

By Corollary 4.4.7(a), for any very flat $(\mathbf{W},\mathbf{T})$-affine morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$ the exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ takes projective contraherent cosheaves on Y to projective contraherent cosheaves on X. It also makes a commutative diagram with the restrictions of the functors $\Delta(X,-)$ and $\Delta(Y,-)$ to the categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $Y\text{–}\mathsf{lcth}_{\mathbf{T}}$. Hence one has $\mathbb{L}_*\Delta(Y,\mathfrak{E}) \simeq \mathbb{L}_*\Delta(X,f_!\mathfrak{E})$ for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{E}$ on Y.

In particular, the latter assertion applies to the embeddings of affine open subschemes $j\colon U \longrightarrow X$, so $\mathbb{L}_{>0}\Delta(X, j_!\mathfrak{E}) = 0$ for all contraherent cosheaves $\mathfrak{E}$ on U. Since the derived functor $\mathbb{L}_*\Delta$ takes short exact sequences of locally contraherent cosheaves to long exact sequences of abelian groups, it follows from Corollary 4.2.5(c) that $\mathbb{L}_{>0}\Delta(X, \mathfrak{P}) = 0$ for any colocally projective contraherent cosheaf $\mathfrak{P}$ on X.

Therefore, the derived functor $\mathbb{L}_*\Delta$ can be computed using colocally projective left resolutions. Now we also see that the derived functors $\mathbb{L}_*\Delta$ defined in the theories of arbitrary (i. e., locally contraadjusted) contraherent cosheaves and of locally cotorsion contraherent cosheaves agree.

Let $\mathfrak{E}$ be a $\mathbf{W}$-locally contraherent cosheaf on X, and let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering of X subordinate to $\mathbf{W}$. Then the contraherent Čech resolution (27) for $\mathfrak{E}$ is a colocally projective left resolution of a locally contraherent cosheaf $\mathfrak{E}$, and one can use it to compute the derived functor $\mathbb{L}_*(X, \mathfrak{E})$. In other words, the homology of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{E}$ on a quasi-compact semi-separated scheme X are computed by the homological Čech complex $C_*(\{U_\alpha\}, \mathfrak{E})$ (see (22)) related to any finite affine open covering $X = \bigcup_\alpha U_\alpha$ subordinate to $\mathbf{W}$.

The following result is to be compared with the cohomological criterion of affineness of schemes [27, Théorème 5.2.1].

Corollary 4.5.1. *A locally contraherent cosheaf $\mathfrak{E}$ on an affine scheme U is contraherent if and only if its higher homology $\mathbb{L}_{>0}\Delta(X, \mathfrak{E})$ vanish.*

Proof. See Lemma 3.2.2. □

Corollary 4.5.2. *Let R be a commutative ring, $I \subset R$ be a nilpotent ideal, and $S = R/I$ be the quotient ring. Let $i\colon \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ denote the corresponding homeomorphic closed embedding of affine schemes. Then a locally injective locally contraherent cosheaf $\mathfrak{J}$ on $U = \operatorname{Spec} R$ is contraherent if and only if its inverse image $i^!\mathfrak{J}$ on $V = \operatorname{Spec} S$ is contraherent.*

Proof. Since any morphism into an affine scheme is coaffine, the "only if" assertion is obvious. To prove the "if", we apply again Lemma 3.2.2. Assuming that $\mathfrak{J}$ is $\mathbf{W}$-locally contraherent on U and $U = \bigcup_\alpha U_\alpha$ is a finite affine open covering of U subordinate to $\mathbf{W}$, the homological Čech complex $C_*(\{U_\alpha\}, \mathfrak{J})$ is a finite complex of injective R-modules. Set $V_\alpha = i^{-1}(U_\alpha) \subset V$; then the complex $C_*(\{V_\alpha\}, \mathfrak{J})$ is the maximal subcomplex of R-modules in $C_*(\{U_\alpha\}, \mathfrak{J})$ annihilated by the action of I.

Since the maximal submodule ${}_IM \subset M$ annihilated by I is nonzero for any nonzero R-module M, one easily proves by induction that a finite complex $K^\bullet$ of injective R-modules is acyclic at all its terms except perhaps the rightmost one whenever so is the complex of injective R/I-modules ${}_IK^\bullet$. Notice that this argument does not seem to apply to the maximal reduced closed subscheme $\operatorname{Spec} R/J$ of an arbitrary affine scheme $\operatorname{Spec} R$ in general, as the maximal submodule ${}_JK \subset K$ annihilated by the nilradical $J \subset R$ may well be zero even for a nonzero injective R-module K (set $K = \operatorname{Hom}_k(F, k)$ in the example from Remark 1.7.5). □

The following result is to be compared with Corollaries 3.4.8 and 4.1.13 (for another comparison, see Corollaries 4.2.10 and 4.3.8).

Corollary 4.5.3. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes colocally projective contraherent cosheaves on Y to colocally projective contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ between these exact categories;*

(b) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes colocally projective locally cotorsion contraherent cosheaves on Y to colocally projective locally cotorsion contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ between these exact categories;*

(c) *if the morphism f is flat, then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes colocally flat contraherent cosheaves on Y to colocally flat contraherent cosheaves on X, and induces and exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clf}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ between these exact categories;*

(d) *if the morphism f is flat, then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes colocally projective locally injective contraherent cosheaves on Y to colocally projective locally injective contraherent cosheaves on X.*

Proof. Part (a): by Corollary 4.2.10(a), the inverse image of a colocally projective contraherent cosheaf on Y with respect to an affine open embedding $j\colon V \longrightarrow Y$ is colocally projective. As we have seen above, the global cosections of colocally projective contraherent cosheaves is an exact functor. It follows that the functor $f_!$ takes short exact sequences in $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ to short exact sequences in the exact category of cosheaves of $\mathcal{O}_X$-modules (with the exact category structure $\mathcal{O}_X\text{–}\mathsf{cosh}_{\{X\}}$ related to the covering $\{X\}$ of the scheme X; see Section 3.1).

Since $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ is a full exact subcategory closed under extensions in $X\text{–}\mathsf{ctrh}$, and the latter exact category is such an exact subcategory in $\mathcal{O}_X\text{–}\mathsf{cosh}_{\{X\}}$, in view of Corollary 4.2.5(c) it remains to recall that the direct images of colocally projective contraherent cosheaves with respect to affine morphisms of schemes are colocally projective (see the remarks in the beginning of Section 4.2).

Part (b) is similar; the proof of part (c) is also similar and based on the remarks about direct images in the beginning of Section 4.3 together with Corollary 4.3.8; and to prove part (d) one only needs to recall that the direct images of locally injective contraherent cosheaves with respect to flat affine morphisms of schemes are locally injective and use Corollary 4.2.8. □

Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. By the result of Section 3.3 (see (25)), the adjunction isomorphism (23) holds, in particular, for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally injective locally contraherent cosheaf $\mathfrak{J}$ on X.

If the morphism f is flat then, according to (26), the adjunction isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q}, \mathfrak{M}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{M}) \tag{49}$$

holds for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally cotorsion locally contraherent cosheaf $\mathfrak{M}$ on X. When the morphism f is also affine, the restrictions of $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$. In addition, these functors take the additive categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ into one another.

If the morphism f is very flat, the same adjunction isomorphism (49) holds for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally contraherent cosheaf $\mathfrak{E}$ on X. When the morphism f is also affine, the restrictions of $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

Corollary 4.5.4. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *if the morphism f is very flat, then the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ takes projective contraherent cosheaves to projective contraherent cosheaves;*

(b) *if the morphism f is flat, then the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves.*

Proof. Follows from the above partial adjunctions (49) between the exact functors $f_!$ and $f^!$. Part (b) can be also deduced from Corollary 4.5.3(b-c). ☐

4.6. **The "naïve" co-contra correspondence.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering. We refer to Section A.1 for the definitions of the derived categories mentioned below.

Recall the definition of the left homological dimension $\mathrm{ld}_{\mathsf{F/E}}\,E$ of an object E of an exact category E with respect to a full exact subcategory $\mathsf{F}\subset\mathsf{E}$, given (under a specific set of assumptions on F and E) in Section A.5. The *right homological dimension with respect to an exact subcategory* F (or the right F-homological dimension) $\mathrm{rd}_{\mathsf{F/E}}\,E$ is defined in the dual way (and under the dual set of assumptions).

Lemma 4.6.1. (a) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering, then the right homological dimension of any quasi-coherent sheaf on X with respect to the exact subcategory of contraadjusted quasi-coherent sheaves $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\subset X\text{–}\mathsf{qcoh}$ (is well-defined and) does not exceed N.*

(b) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering subordinate to $\mathbf{W}$, then the left homological dimension of any $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of colocally projective contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ does not exceed $N-1$. Consequently, the same bound holds for the left homological dimension of any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ with respect to the exact subcategory $X\text{–}\mathsf{ctrh}$.*

Proof. Part (a): first of all, the assumptions about the pair of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\subset X\text{–}\mathsf{qcoh}$ making the right homological dimension well-defined hold by Corollaries 4.1.2(c) and 4.1.4(b).

Furthermore, the right homological dimension of any module over a commutative ring R with respect to the exact category of contraadjusted R-modules $R\text{–}\mathsf{mod}^{\mathsf{cta}}\subset R\text{–}\mathsf{mod}$ does not exceed 1. It follows easily that the homological dimension of any quasi-coherent sheaf of the form $j_*\mathcal{G}$, where $j\colon U\longrightarrow X$ is an affine open subscheme, with respect to the exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\subset X\text{–}\mathsf{qcoh}$ does not exceed 1, either.

Now any quasi-coherent sheaf $\mathcal{F}$ on X has a Čech resolution (12) of length $N-1$ by finite direct sums of quasi-coherent sheaves of the above form. It remains to use the dual version of Corollary A.5.5(a).

Part (b): the conditions on the exact categories $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ making the left homological dimension well-defined hold by Corollaries 4.2.3(b) and 4.2.5(b). The pair of exact categories $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ satisfies the same assumptions for the reasons explained in Section 3.2. It remains to recall the resolution (27). □

Lemma 4.6.2. *Let $X = \bigcup_{\alpha=1}^{N} U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *the left homological dimension of any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of colocally projective locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ does not exceed $N-1$. Consequently, the same bound holds for the left homological dimension of any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ with respect to the exact subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$;*

(b) *the left homological dimension of any locally injective $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of colocally projective locally injective contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ does not exceed $N-1$. Consequently, the same bound holds for the left homological dimension of any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ with respect to the exact subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$.*

Proof. Similar to the proof of Lemma 4.6.1(b). □

Corollary 4.6.3. (a) *For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ induced by the embedding of exact categories $X\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is an equivalence of triangulated categories.*

(b) *For any symbol $\star = \mathsf{b}$ or $-$, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth})$ induced by the embedding of exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}$ is an equivalence of triangulated categories.*

The reason why most unbounded derived categories aren't mentioned in part (b) is because one needs a uniform restriction on the extension of locality of locally contraherent cosheaves in order to work simultaneously with infinite collections of these. In particular, infinite products exist in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, but not necessarily in $X\text{–}\mathsf{lcth}$, so the contraderived category of the latter exact category is not well-defined.

Proof of Corollary 4.6.3. Part (a) follows from Proposition A.5.6 together with Lemma 4.6.1(b). Part (b) in the case $\star = \mathsf{b}$ is obtained from part (a) by passing to the inductive limit over refinements of coverings, while in the case $\star = -$ it is provided by Proposition A.3.1(a). □

The following two corollaries are similar to the previous one. The only difference in the proofs is that Lemma 4.6.2 is being used in place of Lemma 4.6.1(b).

Corollary 4.6.4. (a) *For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$ induced by the embedding of exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ *is an equivalence of triangulated categories.* □

Corollary 4.6.5. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lin}}$ *is an equivalence of triangulated categories.* □

The next theorem is the main result of this section.

Theorem 4.6.6. *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs *there is a natural equivalence of triangulated categories* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$. *These equivalences of derived categories form commutative diagrams with the natural functors* $\mathsf{D}^{\mathsf{b}} \longrightarrow \mathsf{D}^{\pm} \longrightarrow \mathsf{D}$, $\mathsf{D}^{\mathsf{b}} \longrightarrow \mathsf{D}^{\mathsf{abs}\pm} \longrightarrow \mathsf{D}^{\mathsf{abs}}$, $\mathsf{D}^{\mathsf{abs}\pm} \longrightarrow \mathsf{D}^{\pm}$, $\mathsf{D}^{\mathsf{abs}} \longrightarrow \mathsf{D}$ *between different versions of derived categories of the same exact category.*

Notice that Theorem 4.6.6 does not say anything about the coderived and contraderived categories D^{co} and $\mathsf{D}^{\mathsf{ctr}}$ of quasi-coherent sheaves and contraherent cosheaves (neither does Corollary 4.6.3 mention the coderived categories). The reason is that infinite products are not exact in the abelian category of quasi-coherent sheaves and infinite direct sums may not exist in the exact category of contraherent cosheaves. So only the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ are well-defined. Comparing these two requires a different approach; the entire Section 5 will be devoted to that.

Proof of Theorem 4.6.6. By Proposition A.5.6 and its dual version, together with Lemma 4.6.1, the functors $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$ induced by the corresponding embedings of exact categories are all equivalences of triangulated categories. Hence it suffices to construct a natural equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ in order to prove all assertions of Theorem.

According to Sections 2.5 and 2.6, there are natural functors

$$\mathfrak{Hom}_X(\mathcal{O}_X, -)\colon X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{ctrh}$$

and

$$\mathcal{O}_X \odot_X -\colon X\text{–}\mathsf{lcth} \longrightarrow X\text{–}\mathsf{qcoh}$$

related by the adjunction isomorphism (20), which holds for those objects for which the former functor is defined. So it remains to prove the following lemma. □

Lemma 4.6.7. *On a quasi-compact semi-separated scheme* X*, the functor* $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ *takes* $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ *to* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$*, the functor* $\mathcal{O}_X \odot_X -$ *takes* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ *to* $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$*, and the restrictions of these functors to these subcategories are mutually inverse equivalences of exact categories.*

Proof. Obviously, on an affine scheme U the functor $\mathfrak{Hom}_U(\mathcal{O}_U, -)$ takes a contraadjusted quasi-coherent sheaf $\mathcal{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global sections $\mathcal{Q}(U)$ to the contraherent cosheaf $\mathfrak{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global cosections $\mathfrak{Q}[U] = \mathcal{Q}(U)$. Furthermore, if $j\colon U \longrightarrow X$ is the embedding of an affine open subscheme, then by the formula (44) of Section 3.8 there is a natural isomorphism $\mathfrak{Hom}_X(\mathcal{O}_X, j_*\mathcal{Q}) \simeq j_!\mathfrak{Q}$ of contraherent cosheaves on X.

Analogously, the functor $\mathcal{O}_U \odot_U -$ takes a contraherent cosheaf $\mathfrak{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global cosections $\mathfrak{Q}[U]$ to the contraadjusted quasi-coherent sheaf $\mathcal{Q}$ with the $\mathcal{O}(U)$-module of global sections $\mathcal{Q}(U)$ on U. If an embedding of affine open subscheme $j\colon U \longrightarrow X$ is given, then by the formula (46) there is a natural isomorphism $\mathcal{O}_X \odot_X j_!\mathfrak{Q} \simeq j_*\mathcal{Q}$ of quasi-coherent sheaves on X.

By Corollary 4.1.4(c), any sheaf from $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ is a direct summand of a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes of X. It is clear from the definition of the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ that it preserves exactness of short sequences of contraadjusted quasi-coherent cosheaves; hence it preserves, in particular, such iterated extensions.

By Corollary 4.2.5(c), any cosheaf from $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from affine open subschemes of X. Let us show that the functor $\mathcal{O}_X \odot_X -$ preserves exactness of short sequences of colocally projective contraherent cosheaves on X, and therefore, in particular, preserves such extensions. Indeed, the adjunction isomorphism

$$\operatorname{Hom}_X(\mathcal{O}_X \odot_X \mathfrak{P},\ \mathcal{F}) \simeq \operatorname{Hom}^X(\mathfrak{P}, \mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{F}))$$

holds for any contraherent cosheaf $\mathfrak{P}$ and contraadjusted quasi-coherent sheaf $\mathcal{F}$. Besides, the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J})$ is locally injective for any injective quasi-coherent sheaf $\mathcal{J}$ on X. By Corollary 4.2.3(a), it follows that the functor $\mathfrak{P} \longmapsto \operatorname{Hom}_X(\mathcal{O}_X \odot_X \mathfrak{P},\ \mathcal{J})$ preserves exactness of short sequences of colocally projective contraherent cosheaves on X, and consequently so does the functor $\mathfrak{P} \longmapsto \mathcal{O}_X \odot_X \mathfrak{P}$.

Now one can easily deduce that the adjunction morphisms

$$\mathfrak{P} \longrightarrow \mathfrak{Hom}_X(\mathcal{O}_X,\ \mathcal{O}_X \odot_X \mathfrak{P}) \quad \text{and} \quad \mathcal{O}_X \odot_X \mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{F}$$

are isomorphisms for any colocally projective contraherent cosheaf $\mathfrak{P}$ and contraadjusted quasi-coherent sheaf $\mathcal{F}$, as a morphism of finitely filtered objects inducing an isomorphism of the associated graded objects is also itself an isomorphism. The proof of Lemma, and hence also of Theorem 4.6.6, is finished. $\square$

Given an exact category E, let $\mathsf{Hot}^\star(\mathsf{E})$ denote the homotopy category $\mathsf{Hot}(\mathsf{E})$ if $\star = \mathsf{abs}$, co, ctr, or $\varnothing$; the category $\mathsf{Hot}^+(\mathsf{E})$, if $\star = \mathsf{abs}+$ or $+$; the category $\mathsf{Hot}^-(\mathsf{E})$, if $\star = \mathsf{abs}-$ or $-$; and the category $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E})$ if $\star = \mathsf{b}$. The following two corollaries provide, essentially, several restricted versions of Theorem 4.6.6.

Corollary 4.6.8. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *there is a natural equivalence of triangulated categories* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cot}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$.

(b) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *there is a natural equivalence of triangulated categories* $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$.

Proof. By Proposition A.5.6 together with Lemma 4.6.2, the functors $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ induced by the corresponding embeddings of exact categories are equivalences of triangulated categories. Hence it suffices to show that the equivalence of exact categories from Lemma 4.6.7 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$. The former of these assertions follows from Corollaries 4.1.11(c) and 4.2.6(c), while the latter one is obtained from Corollary 4.2.8 together with the fact that any injective quasi-coherent sheaf on X is a direct summand of a finite direct sum of the direct images of injective quasi-coherent sheaves from open embeddings $U_\alpha \longrightarrow X$ forming a covering. ☐

Lemma 4.6.9. *Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering. Then*

(a) *the right homological dimension of any very flat quasi-coherent sheaf on X with respect to the exact subcategory of contraadjusted very flat quasi-coherent sheaves $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ does not exceed N;*

(b) *the right homological dimension of any flat quasi-coherent sheaf on X with respect to the exact subcategory of contraadjusted flat quasi-coherent sheaves $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ does not exceed N.*

Proof. The right homological dimension is well-defined due to Corollary 4.1.4(b), so it remains to apply Lemma 4.6.1(a) and the dual version of Corollary A.5.3. ☐

Corollary 4.6.10. (a) *For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, or abs, there is a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$.*

(b) *For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, there is a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clf}})$.*

(c) *There is a natural equivalence of triangulated categories $\mathsf{D}^+(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^+(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$.*

Proof. Part (a): assuming $\star \neq \mathsf{co}$, by Lemma 4.6.9(a) together with the dual version of Proposition A.5.6 the triangulated functor $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ is an equivalence of categories. In particular, we have proven that $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) = \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$, so the previous assertion holds for $\star = \mathsf{co}$ as well. Hence it suffices to show that the equivalence of exact categories from Lemma 4.6.7 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ with $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$. This follows from Lemmas 4.1.5 and 4.4.1(b). The proof of part (b) is similar: in view of Lemma 4.6.9(b), the triangulated functor $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is an equivalence of categories, and it remains to show that the equivalence of exact categories from Lemma 4.6.7 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ with $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$. Here one applies Lemma 4.1.6 and Corollary 4.3.4(c).

To prove part (c), notice that the triangulated functor $\mathsf{Hot}^+(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is an equivalence of categories by Corollary 4.1.11(b) and the dual version of Proposition A.3.1(a). So it remains to show that the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. This follows from Lemmas 4.1.12 and 4.4.3(b). Alternatively, one can prove that the functor $\mathsf{D}^+(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{ctrh}_{\mathsf{clf}})$ is an equivalence of categories by applying directly the dual version of Proposition A.3.1(a) together with Corollaries 4.3.4(a) and 4.3.2(b). ☐

4.7. **Homotopy locally injective complexes.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering. The goal of this section is to construct a full subcategory in the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ that would be equivalent to the unbounded derived category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$. The significance of this construction is best illustrated using the duality-analogy between the contraherent cosheaves and the quasi-coherent sheaves.

As usually, the notation $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ refers to the unbounded derived category of the exact category of flat quasi-coherent sheaves on X. The full triangulated subcategory $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}} \subset \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ of *homotopy flat complexes* of flat quasi-coherent sheaves on X is defined as the minimal triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ containing the objects of $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and closed under infinite direct sums (cf. Section A.4). The following result is essentially due to Spaltenstein [63].

Theorem 4.7.1. (a) *The composition of natural triangulated functors* $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ *is an equivalence of triangulated categories.*

(b) *A complex of flat quasi-coherent sheaves* $\mathcal{F}^\bullet$ *on* X *belongs to* $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ *if and only if its tensor product* $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ *with any acyclic complex of quasi-coherent sheaves* $\mathcal{M}^\bullet$ *on* X *is also an acyclic complex of quasi-coherent sheaves.*

Proof. Part (a) is a particular case of Proposition A.4.3. To prove part (b), let us first show that the tensor product of any complex of quasi-coherent sheaves and a complex of sheaves acyclic with respect to the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ is acyclic. Indeed, any complex in an abelian category is a locally stabilizing inductive limit of finite complexes; so it suffices to notice that the tensor product of any quasi-coherent sheaf with a complex acyclic with respect to $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ is acyclic. Hence the class of all complexes of flat quasi-coherent sheaves satisfying the condition in part (b) can be viewed as a strictly full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$.

Now the "only if" assertion easily follows from the facts that the tensor products of quasi-coherent sheaves preserve infinite direct sums and the tensor product with a flat quasi-coherent sheaf is an exact functor. In view of (the proof of) part (a), it suffices to show that any complex $\mathcal{F}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ satisfying the tensor product condition of part (b) and acyclic with respect to $X\text{–}\mathsf{qcoh}$ is also acyclic with respect to $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ in order to prove "if".

Notice that the tensor product of a bounded above complex of flat quasi-coherent sheaves and an acyclic complex of quasi-coherent sheaves is an acyclic complex. Since any quasi-coherent sheaf $\mathcal{M}$ over X has a flat left resolution, it follows that the complex of quasi-coherent sheaves $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$ is acyclic. One easily concludes that the complex $\mathcal{F}^\bullet$ is acyclic with respect to $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$. ☐

The full triangulated subcategory $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ of *homotopy locally injective complexes* of localy injective $\mathbf{W}$-locally contraherent cosheaves on X is defined as the minimal full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ containing the objects of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$

and closed under infinite products. Given a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ and a complex of $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{P}^\bullet$ on X such that the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M}^i, \mathfrak{P}^j)$ is defined for all i, $j \in \mathbb{Z}$ (see Sections 2.4 and 3.6), we define the complex $\mathfrak{Cohom}_X(\mathcal{M}^\bullet, \mathfrak{P}^\bullet)$ as the total complex of the bicomplex $\mathfrak{Cohom}_X(\mathcal{M}^i, \mathfrak{P}^j)$ constructed by taking infinite products of $\mathbf{W}$-locally contraherent cosheaves along the diagonals.

Theorem 4.7.2. (a) *The composition of natural triangulated functors* $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *is an equivalence of triangulated categories.*

(b) *A complex of locally injective* $\mathbf{W}$*-locally contraherent cosheaves* $\mathfrak{J}^\bullet$ *on* X *belongs to* $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ *if and only if the complex* $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J})$ *into it from any acyclic complex of quasi-coherent sheaves* $\mathcal{M}^\bullet$ *is an acyclic complex in the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *(or, at one's choice, in the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$*).*

Proof. Part (a): the argument goes along the lines of the proof of Theorem 4.7.1(a), but Proposition A.4.3 is not directly applicable, the category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ being not abelian; so there are some complications. First of all, we will need another definition. The full triangulated subcategory $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}} \subset \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ of homotopy locally injective complexes of colocally projective locally injective contraherent cosheaves on X is defined as the minimal full triangulated subcategory containing the objects of $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ and closed under infinite products.

It was shown in Section 4.6 that the natural functor $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories. Analogously one shows (using, e. g., Corollary A.5.3) that the natural functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ is an equivalence of categories, as are the similar functors $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ and $\mathsf{Hot}^+(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$. Therefore, the equivalence $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}) \simeq \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ identifies the subcategories generated by bounded or bounded below complexes. Thus the natural functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ is also an equivalence of triangulated categories, and it remains to show that the functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ is an equivalence of categories.

We will show that any complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ admits a quasi-isomorphism with respect to the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ into a complex belonging to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}$. In particular, by the dual version of [53, Lemma 1.6] applied to the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ with the full triangulated subcategory $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ and the thick subcategory of complexes acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ it will follow that the category $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ is equivalent to the localization of $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ by the thick subcategory of complexes acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$. By the dual version of Corollary A.4.2, the latter subcategory is semiorthogonal to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}$ in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$. In view of the same construction of a quasi-isomorphism with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, these two subcategories form a semiorthogonal decomposition of $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$, which implies the desired assertion.

Lemma 4.7.3. *There exists a positive integer* d *such that for any complex* $\mathfrak{P}^0 \longrightarrow \cdots \longrightarrow \mathfrak{P}^{d+1}$ *over* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ *there exists a complex* $\mathfrak{Q}^0 \longrightarrow \cdots \longrightarrow \mathfrak{Q}^{d+1}$ *over*

X–$\mathsf{ctrh}_{\mathsf{clp}}$ *together with a morphism of complexes* $\mathfrak{P}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ *such that* $\mathfrak{Q}^0 = 0$, *while the morphisms of contraherent cosheaves* $\mathfrak{P}^d \longrightarrow \mathfrak{Q}^d$ *and* $\mathfrak{P}^{d+1} \longrightarrow \mathfrak{Q}^{d+1}$ *are isomorphisms.*

Proof. In view of Lemma 4.6.7, it suffices to prove the assertion of Lemma for a complex $\mathcal{P}^0 \longrightarrow \cdots \longrightarrow \mathcal{P}^{d+1}$ over the category X–$\mathsf{qcoh}^{\mathsf{cta}}$. Set $\mathcal{Q}^0 = 0$. Consider the quasi-coherent sheaf $\mathcal{R}^1 = \operatorname{coker}(\mathcal{P}^0 \to \mathcal{P}^1)$ and embed it into a contraadjusted quasi-coherent sheaf $\mathcal{Q}^1$. Denote by $\mathcal{R}^2$ the fibered coproduct of the quasi-coherent sheaves $\mathcal{Q}^1$ and $\mathcal{P}^2$ over $\mathcal{R}^1$, embed it into a contraadjusted quasi-coherent sheaf $\mathcal{Q}^2$, and proceed by applying the dual version of the construction of Lemma A.3.2(a) up to producing the quasi-coherent sheaves $\mathcal{Q}^{d-2}$ and $\mathcal{R}^{d-1}$.

The sequence $0 \longrightarrow \mathcal{R}^1 \longrightarrow \mathcal{Q}^1 \oplus \mathcal{P}^2 \longrightarrow \mathcal{Q}^2 \oplus \mathcal{P}^3 \longrightarrow \cdots \longrightarrow \mathcal{Q}^{d-2} \oplus \mathcal{P}^{d-1} \longrightarrow \mathcal{R}^{d-1} \longrightarrow 0$ is a right resolution of the quasi-coherent sheaf $\mathcal{R}^1$, all of whose terms, except perhaps the rightmost one, are contraadjusted quasi-coherent sheaves. By Lemma 4.6.1(a) and the dual version of Corollary A.5.2, for d large enough the quasi-coherent sheaf $\mathcal{R}^{d-1}$ will be contraadjusted. It remains to set $\mathcal{Q}^{d-1} = \mathcal{R}^{d-1}$, $\mathcal{Q}^d = \mathcal{P}^d$, and $\mathcal{Q}^{d+1} = \mathcal{P}^{d+1}$. $\square$

Now let $\mathfrak{P}^\bullet$ be a complex over X–$\mathsf{ctrh}_{\mathsf{clp}}$. For each fragment of $d+2$ consequtive terms $\mathfrak{P}^{i-d-1} \longrightarrow \cdots \longrightarrow \mathfrak{P}^i$ in $\mathfrak{P}^\bullet$ we construct the corresponding complex ${}^{(i)}\mathfrak{Q}^{i-d-1} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{Q}^i$ as in Lemma 4.7.3. Pick an admissible monomorphism ${}^{(i)}\mathfrak{Q}^{i-d} \longrightarrow {}^{(i)}\mathfrak{J}^{i-d}$ from a colocally projective contraherent cosheaf ${}^{(i)}\mathfrak{Q}^{i-d}$ into a colocally projective locally injective contraherent cosheaf ${}^{(i)}\mathfrak{J}^{i-d}$ on X (see Corollaries 4.2.5(b) and 4.2.3(a)).

Proceeding with the dual version of the construction of Lemma A.3.2(a) (see also the above proof of Lemma 4.7.3), we obtain a termwise admissible monomorphism with respect to X–$\mathsf{ctrh}_{\mathsf{clp}}$ from the complex ${}^{(i)}\mathfrak{Q}^{i-d} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{Q}^i$ into a complex ${}^{(i)}\mathfrak{J}^{i-d} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{J}^i$ over X–$\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ such that the cone of this morphism is quasi-isomorphic to an object of X–$\mathsf{ctrh}_{\mathsf{clp}}$ placed in the cohomological degree i.

Set ${}^{(i)}\mathfrak{J}^j = 0$ for j outside of the segment $[i-d,\, i]$. We obtain a finite complex ${}^{(i)}\mathfrak{J}^\bullet$ over X–$\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ endowed with a morphism of complexes $\mathfrak{P}^\bullet \longrightarrow {}^{(i)}\mathfrak{J}^\bullet$ with the following property. For any affine open subscheme $U \subset X$, the induced morphism of cohomology modules $H^i(\mathfrak{P}^\bullet[U]) \longrightarrow H^i(\mathfrak{J}^\bullet[U])$ is injective.

Denote by ${}^0\mathfrak{J}^\bullet$ the direct product of all the complexes ${}^{(i)}\mathfrak{J}^\bullet$. The morphism of complexes $\mathfrak{P}^\bullet \longrightarrow {}^0\mathfrak{J}^\bullet$ over the exact category X–$\mathsf{ctrh}_{\mathsf{clp}}$ is a termwise admissible monomorphism. Consider the corresponding complex of cokernels and apply the same procedure to it, constructing a termwise acyclic complex of complexes $0 \longrightarrow \mathfrak{P}^\bullet \longrightarrow {}^0\mathfrak{J}^\bullet \longrightarrow {}^1\mathfrak{J}^\bullet \longrightarrow \cdots$ over the exact category X–$\mathsf{ctrh}_{\mathsf{clp}}$ in which all the complexes ${}^i\mathfrak{J}^\bullet$ are infinite products of finite complexes over X–$\mathsf{ctrh}_{\mathsf{clp}}$.

By Lemma A.3.4 applied to the projective system of quotient complexes of silly filtration with respect to the left index of the bicomplex ${}^\bullet\mathfrak{J}^\bullet$, the total complex of ${}^\bullet\mathfrak{J}^\bullet$ constructed by taking infinite products along the diagonals belongs to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}})^{\mathsf{lin}}$. It remains to show that the cone $\mathfrak{E}^\bullet$ of the morphism from $\mathfrak{P}^\bullet$ to the total complex of ${}^\bullet\mathfrak{J}^\bullet$ is acyclic with respect to X–$\mathsf{ctrh}_{\mathsf{clp}}$.

Indeed, by the dual version of Lemma A.4.4, the complex of cosections $\mathfrak{E}^\bullet[U]$ is an acyclic complex of $\mathcal{O}_X(U)$-modules for any affine open subscheme $U \subset X$. Since the $\mathcal{O}_X(U)$-modules $\mathfrak{E}^i[U]$ are contraadjusted and quotient modules of contraadjusted modules are contraadjusted, the complex $\mathfrak{E}^\bullet[U]$ is also acyclic with respect to the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Since the functor $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), -)$ preserves exactness of short sequences of contraadjusted $\mathcal{O}_X(U)$-modules for any pair of embedded affine open subschemes $V \subset U \subset X$, one easily concludes that the rules $U \longmapsto \mathrm{coker}(\mathfrak{E}^{i-1}[U] \to \mathfrak{E}^i[U])$ define contraherent cosheaves on X. Hence the complex of contrahent cosheaves $\mathfrak{E}^\bullet$ is acyclic over $X\text{–}\mathsf{ctrh}$. Since the contraherent cosheaves $\mathfrak{E}^i$ belong to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, this complex is also acyclic over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ by Lemma 4.6.1(b) and Corollary A.5.2.

Part (a) is proven; let us prove part (b). The argument is similar to the proof of Theorem 4.7.1(b). First we show that $\mathfrak{Cohom}_X$ from any complex of quasi-coherent sheaves into a complex of locally contraherent cosheaves acyclic with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ is acyclic with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$. Indeed, any complex of quasi-coherent sheaves is a locally stabilizing inductive limit of a sequence of finite complexes. So it remains to recall that $\mathfrak{Cohom}_X$ from a quasi-coherent sheaf into a complex acyclic with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ is a complex acyclic with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$, and use Lemma A.3.4 again.

Hence the class of all complexes over $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ satisfying the $\mathfrak{Cohom}$ condition in part (b) can be viewed as a strictly full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$. Now the "only if" assertion follows from the preservation of infinite products in the second argument by the functor $\mathfrak{Cohom}_X$ and its exactness as a functor on the category $X\text{–}\mathsf{qcoh}$ for any fixed locally injective locally contraherent cosheaf in the second argument. In view of (the proof of) part (a), in order to prove "if" it suffices to show that any complex $\mathfrak{J}^\bullet$ over $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ satisfying the $\mathfrak{Cohom}$ condition in (b) and acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is also acyclic with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$.

Notice that the $\mathfrak{Cohom}_X$ from a bounded above complex of very flat quasi-coherent sheaves into an acyclic complex of $\mathbf{W}$-locally contraherent cosheaves is an acyclic complex of $\mathbf{W}$-locally contraherent cosheaves. Since any quasi-coherent sheaf $\mathcal{M}$ on X has a very flat left resolution (see Lemma 4.1.1), it follows that the complex $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J}^\bullet)$ is acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

Now let $U \subset X$ be an affine open subscheme subordinate to $\mathbf{W}$, let N be an $\mathcal{O}(U)$-module, viewed also as a quasi-coherent sheaf on U, and $\mathcal{M}$ be any quasi-coherent extension (e. g., the direct image) of N to X. Then acyclicity of the complex $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J}^\bullet)$ with respect to the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ implies, in particular, exactness of the complex of $\mathcal{O}_X(U)$-modules $\mathrm{Hom}_{\mathcal{O}_X(U)}(N, \mathfrak{J}^\bullet[U])$. Since this holds for all $\mathcal{O}_X(U)$-modules N, it follows that all the $\mathcal{O}_X(U)$-modules of cocycles in the acyclic complex of $\mathcal{O}_X(U)$-modules $\mathfrak{J}^\bullet[U]$ are injective. □

The following lemma will be needed in Section 4.8.

Lemma 4.7.4. *Let $\mathfrak{P}^\bullet$ be a complex over the exact category X–$\mathsf{ctrh}_{\mathsf{clp}}$ and $\mathfrak{J}^\bullet$ be a complex over X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ belonging to $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$. Then the natural morphism of graded abelian groups $H^*\operatorname{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet) \simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})}(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})}(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet)$ is an isomorphism (in other words, the complex $\operatorname{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet)$ computes the groups of morphisms in the derived category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$).*

Proof. Since any complex over X–$\mathsf{lcth}_{\mathbf{W}}$ admits a quasi-isomorphism into it from a complex over X–$\mathsf{ctrh}_{\mathsf{clp}}$ and any complex over X–$\mathsf{ctrh}_{\mathsf{clp}}$ acyclic over X–$\mathsf{lcth}_{\mathbf{W}}$ is also acyclic over X–$\mathsf{ctrh}_{\mathsf{clp}}$, it suffices to show that the complex of abelian groups $\operatorname{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet)$ is acyclic for any complex $\mathfrak{P}^\bullet$ acyclic with respect to X–$\mathsf{ctrh}_{\mathsf{clp}}$ and any complex $\mathfrak{J}^\bullet\in\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$. For a complex $\mathfrak{J}^\bullet$ obtained from objects of X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ by iterating the operations of cone and infinite direct sum the latter assertion is obvious (see Corollary 4.2.3(a)), so it remains to consider the case of a complex $\mathfrak{J}^\bullet$ acyclic with respect to X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$. In this case we will show that the complex $\operatorname{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{J}^\bullet)$ is acyclic for any complex $\mathfrak{P}^\bullet$ over X–$\mathsf{ctrh}_{\mathsf{clp}}$.

Let i be an integer. Applying Lemma 4.7.3 to the fragment $\mathfrak{P}^{i-d-1}\longrightarrow\cdots\longrightarrow\mathfrak{P}^i$ of the complex $\mathfrak{P}^\bullet$, we obtain a morphism of complexes from $\mathfrak{P}^\bullet$ to a finite complex $\mathfrak{Q}^\bullet$ over X–$\mathsf{ctrh}_{\mathsf{clp}}$ such that the morphisms $\mathfrak{P}^{i-1}\longrightarrow\mathfrak{Q}^{i-1}$ and $\mathfrak{P}^i\longrightarrow\mathfrak{Q}^i$ are isomorphisms. The cocone of this morphism splits naturally into a direct sum of two complexes concentrated in cohomological degrees $\leqslant i$ and $\geqslant i$, respectively. We are interested in the former complex. Its subcomplex of silly truncation $\mathfrak{R}(j,i)^\bullet$ is a finite complex over X–$\mathsf{ctrh}_{\mathsf{clp}}$ concentrated in the cohomological degrees between j and i and endowed with a morphism of complexes $\mathfrak{R}(j,i)^\bullet\longrightarrow\mathfrak{P}^\bullet$, which is a termwise isomorphism in the degrees between j and $i-d$.

The complex $\mathfrak{P}^\bullet$ is a termwise stabilizing inductive limit of the sequence of complexes $\mathfrak{R}(j,i)^\bullet$ as the degree j decreases, while the degree i increases (fast enough). It remains to recall that the functor Hom^X from a colocally projective contraherent cosheaf takes acyclic complexes over X–$\mathsf{lcth}_{\mathbf{W}}$ to acyclic complexes of abelian groups, and, e. g., use Lemma A.3.4 once again. ☐

4.8. **Derived functors of direct and inverse image.** For the rest of Section 4, the upper index $\star$ in the notation for derived and homotopy categories stands for one of the symbols b, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs.

Let $f\colon Y\longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then for any symbol $\star\neq\mathsf{ctr}$ the right derived functor of direct image

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh})\longrightarrow\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \tag{50}$$

is constructed in the following way. By Lemma 4.6.1(a) together with the dual version of Proposition A.5.6, the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{cta}})\longrightarrow\mathsf{D}^\star(Y\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories (as is the similar functor for sheaves over X). By Corollary 4.1.13(a), the restriction of the functor of direct image $f_*\colon Y\text{–}\mathsf{qcoh}\longrightarrow X\text{–}\mathsf{qcoh}$ provides an exact functor $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}}\longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$. Now the derived functor $\mathbb{R}f_*$ is defined by by restricting the functor of direct image $f_*\colon\mathsf{Hot}(Y\text{–}\mathsf{qcoh})\longrightarrow$

$\mathsf{Hot}(X\text{–}\mathsf{qcoh})$ to the full subcategory of complexes of contraadjusted quasi-coherent sheaves on Y (with the appropriate boundedness conditions).

For any symbol $\star \neq \mathsf{co}$, the left derived functor of direct image

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \tag{51}$$

is constructed in the following way. By Lemma 4.6.1(b) (for the covering $\{Y\}$ of the scheme Y) together with Proposition A.5.6, the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh})$ is an equivalence of triangulated categories (as is the similar functor for cosheaves over X). By Corollary 4.5.3(a), there is an exact functor of direct image $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$. The derived functor $\mathbb{L}f_!$ is defined as the induced functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$.

Similarly one defines the left derived functor of direct image

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}). \tag{52}$$

Theorem 4.8.1. *For any symbol $\star \neq \mathsf{co}$, ctr, the equivalences of categories $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(Y\text{–}\mathsf{ctrh})$ and $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$ from Theorem 4.6.6 transform the right derived functor $\mathbb{R}f_*$ (50) into the left derived functor $\mathbb{L}f_!$ (51).*

Proof. It suffices to show that the equivalences of exact categories $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ from Lemma 4.6.7 transform the functor f_* into the functor $f_!$. The isomorphism (44) of Section 3.8 proves as much. ☐

Let $f\colon Y \longrightarrow X$ be a morphism of schemes. Let $\mathbf{W}$ and $\mathbf{T}$ be open coverings of the schemes X and Y, respectively, for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine. According to Section 3.3, there is an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$; for a flat morphism f, there is also an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$, and for a very flat morphism f, an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$.

For a quasi-compact semi-separated scheme X, it follows from Corollary 4.1.11(a) and Proposition A.3.1(a) that the natural functor $\mathsf{D}^-(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{qcoh})$ is an equivalence of categories. Similarly, it follows from Corollary 4.2.5(a) and the dual version of Proposition A.3.1(a) that the natural functor $\mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories. This allows to define, for any morphism $f\colon Y \longrightarrow X$ into a quasi-compact semi-separated scheme X and coverings $\mathbf{W}$, $\mathbf{T}$ as above, the derived functors of inverse image

$$\mathbb{L}f^*\colon \mathsf{D}^-(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{qcoh}) \tag{53}$$

and

$$\mathbb{R}f^!\colon \mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{lcth}_{\mathbf{T}})$$

by applying the functors f^* and $f^!$ to (appropriately bounded) complexes of flat sheaves and locally injective cosheaves. When both schemes are quasi-compact and semi-separated, one can take into account the equivalences of categories from Corollary 4.6.3(a) in order to produce the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^+(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{ctrh}), \tag{54}$$

which clearly does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$.

According to Section 4.7, the natural functors $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are equivalences of categories for any quasi-compact semi-separated scheme X with an open covering $\mathbf{W}$. This allows to define, for any morphism $f\colon Y \longrightarrow X$ into a quasi-compact semi-separated scheme X and coverings $\mathbf{W}$, $\mathbf{T}$ as above, the derived functors of inverse image

$$\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh}) \tag{55}$$

and

$$\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{lcth}_{\mathbf{T}})$$

by applying the functors $f^*\colon \mathsf{Hot}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{Hot}(Y\text{–}\mathsf{qcoh})$ and $f^!\colon \mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{Hot}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}})$ to homotopy flat complexes of flat quasi-coherent sheaves and homotopy locally injective complexes of locally injective $\mathbf{W}$-locally contraherent cosheaves, respectively. Of course, this construction is well-known for quasi-coherent sheaves [63, 47]; we discuss here the sheaf and cosheaf situations together in order to emphasize the duality-analogy between them.

When both schemes are quasi-compact and semi-separated, one can use the equivalences of categories from Corollary 4.6.3(a) in order to obtain the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh}) \tag{56}$$

which does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$. Notice also that the restriction of the functor f^* takes $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ into $\mathsf{Hot}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ and the restriction of the functor $f^!$ takes $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ into $\mathsf{Hot}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}})^{\mathsf{lin}}$.

It is easy to see that for any morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$ the functor $\mathbb{L}f^*$ (53) is left adjoint to the functor $\mathbb{R}f_*\colon \mathsf{D}^-(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{qcoh})$ (50) and the functor $\mathbb{R}f^!$ (54) is right adjoint to the functor $\mathbb{L}f_!\colon \mathsf{D}^+(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{ctrh})$ (51). Essentially, one uses the partial adjunctions on the level of exact categories together with the fact that the derived functor constructions are indeed those of the "left" and "right" derived functors, as stated (cf. [52, Lemma 8.3]).

Similarly, one concludes from the construction in the proof of Theorem 4.7.2 that the functor $\mathbb{L}f^*$ (55) is left adjoint to the functor $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ (50). And in order to show that the functor $\mathbb{R}f^!$ (56) is right adjoint to the functor $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ (51), one can use Lemma 4.7.4. So we have obtained a new proof of the following classical result [30, 47].

Corollary 4.8.2. *For any morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$, the derived direct image functor $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ has a right adjoint functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$.*

Proof. We have (more or less) explicitly constructed the functor $f^!$ as the right derived functor $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ (56) of the exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ between exact subcategories of the exact categories of locally contraherent cosheaves on X and Y. The above construction of the functor $f^!$ for bounded below complexes (54) is particularly explicit. In either case, the construction is based on the identification of the functor $\mathbb{R}f_*$ of derived direct image of quasi-coherent sheaves with the functor $\mathbb{L}f_!$ of derived direct image of contraherent cosheaves, which is provided by Theorems 4.6.6 and 4.8.1. □

4.9. **Finite flat and locally injective dimension.** A quasi-coherent sheaf $\mathcal{F}$ on a scheme X is said to have *flat dimension not exceeding* d if the flat dimension of the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ does not exceed d for any affine open subscheme $U \subset X$. If a quasi-coherent sheaf $\mathcal{F}$ on X admits a left resolution by flat quasi-coherent sheaves (e. g., X is quasi-compact and semi-separated), then the flat dimension of $\mathcal{F}$ is equal to the minimal length of such resolution.

The property of a quasi-coherent sheaf to have flat dimension not exceeding d is local, since so is the property of a quasi-coherent sheaf to be flat. Quasi-coherent sheaves of finite flat dimension form a full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}} \subset X\text{–}\mathsf{qcoh}$ closed under extensions and kernels of surjective morphisms; the full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ of quasi-coherent sheaves of flat dimension not exceeding d is closed under the same operations, and also under infinite direct sums.

Let us say that a quasi-coherent sheaf $\mathcal{F}$ on a scheme X has *very flat dimension not exceeding* d if the very flat dimension of the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ does not exceed d for any affine open subscheme $U \subset X$ (see Section 1.5 for the definition). Over a quasi-compact semi-separated scheme X, a quasi-coherent sheaf has very flat dimension $\leqslant d$ if and only if it admits a very flat left resolution of length $\leqslant d$.

Since the property of a quasi-coherent sheaf to be very flat is local, so is its property to have flat dimension not exceeding d (cf. Lemma 1.5.4). Quasi-coherent sheaves of very flat dimension $\leqslant d$ form a full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d} \subset X\text{–}\mathsf{qcoh}$ closed under extensions, kernels of surjective morphisms, and infinite direct sums. We denote the inductive limit of the exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ as $d \to \infty$ by $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}$.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on a scheme X is said to have *locally injective dimension not exceeding* d if the injective dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ does not exceed d for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Over a quasi-compact semi-separated scheme X, a $\mathbf{W}$-locally contraherent cosheaf has locally injective dimension $\leqslant d$ if and only if it admits a locally injective right resolution of length $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

The property of a $\mathbf{W}$-locally contraherent cosheaf to have locally injective dimension not exceeding d is local and refinements of the covering $\mathbf{W}$ do not change it. $\mathbf{W}$-locally contraherent cosheaves of finite locally injective dimension form a full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ closed under extensions and cokernels of admissible monomorphisms; the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ of quasi-coherent sheaves of locally injective dimension not exceeding d is closed under the same operations, and also under infinite products. We set $X\text{–}\mathsf{ctrh}^{\mathsf{flid}} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}\text{–}d}$.

For the rest of the section, let X be a quasi-compact semi-separated scheme with an open covering $\mathbf{W}$.

Corollary 4.9.1. (a) *For any symbol* $\star \neq \mathsf{ctr}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star \neq \mathsf{ctr}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ *is an equivalence of triangulated categories.*

(c) *For any symbol* $\star \neq \mathsf{co}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ *is an equivalence of triangulated categories.*

Proof. Parts (a-b) follow from Proposition A.5.6, while part (c) follows from the dual version of the same. ☐

Corollary 4.9.2. (a) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}$ *is an equivalence of triangulated categories.*

(c) *For any symbol* $\star = \mathsf{b}$ *or* $+$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ *is an equivalence of triangulated categories.*

Proof. The assertions concerning the case $\star = \mathsf{b}$ follow from the respective assertions of Corollary 4.9.1 by passage to the inductive limit as $d \to \infty$. The assertions concerning the case $\star = -$ in parts (a-b) follow from Proposition A.3.1(a), while the assertion about $\star = +$ in part (c) follows from the dual version of it. ☐

Lemma 4.9.3. *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering subordinate to* $\mathbf{W}$*, then the left homological dimension of any object of the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ *with respect to the full exact subcategory* $X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$ *does not exceed* $N-1$.

Proof. In view of Corollary A.5.3, it suffices to show that any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ admits an admissible epimorphism with respect to the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ from an object of $X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$. We will do more and show that the exact sequence (27) is a left resolution of an object $\mathfrak{P} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ by objects of $X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$. Indeed, the functor of inverse image with respect to a very flat $(\mathbf{W},\mathbf{T})$-coaffine morphism $f\colon Y \longrightarrow X$ takes $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ into $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}-d}$, while the functor of direct image with respect to a flat $(\mathbf{W},\mathbf{T})$-affine morphism f takes $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}-d}$ into $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$. The sequence (27) is exact over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$, since it is exact over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ is closed under admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. ☐

Lemma 4.9.4. (a) *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering, then the homological dimension of the exact category* $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ *does not exceed* $N+d$.

(b) *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering subordinate to* $\mathbf{W}$*, then the homological dimension of the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ *does not exceed* $N-1+d$.

Proof. Part (a): in fact, one proves the stronger assertion that $\mathrm{Ext}_X^{\geq N+d}(\mathcal{F},\mathcal{M})=0$ for any quasi-coherent sheaf $\mathcal{M}$ and any quasi-coherent sheaf of very flat dimension $\leqslant d$ over X (also, the Ext groups in the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ agree with those in the abelian category $X\text{–}\mathsf{qcoh}$). Since any object of $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ has a finite left resolution of length $\leqslant d$ by objects $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$, it suffices to consider the case of $\mathcal{F}\in X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$.

The latter can be dealt with using the Čech resolution (12) of the sheaf $\mathcal{M}$ and the adjunction of exact functors j^* and j_* for the embedding of an affine open subscheme $j\colon U\longrightarrow X$, inducing the similar adjunction on the level of Ext groups (cf. the proof of Lemma 5.4.1(b) below). Alternatively, the desired assertion can be deduced from Lemma 4.6.1(a). The proof of part (b) is similar and can be based either on the Čech resolution (27), or on Lemma 4.6.1(b). □

Corollary 4.9.5. (a) *The natural triangulated functors* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})\longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})\longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})$ *and* $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})\longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})$ *are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ *are equivalences of categories.*

(b) *The natural triangulated functors* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})\longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})\longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})$ *and* $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})\longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})$ *are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ *are equivalences of categories.*

Proof. Follows from the respective parts of Lemma 4.9.4 together with the result of [52, Remark 2.1]. □

As a matter of notational convenience, set the triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ as $d\to\infty$ for any symbol $\star\neq\mathsf{ctr}$. Furthermore, set $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d})$ as $d\to\infty$. For any morphism $f\colon Y\longrightarrow X$ into a quasi-compact semi-separated scheme X one constructs the left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})\longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \tag{57}$$

as the functor on the derived categories induced by the exact functor $f^*\colon X\text{–}\mathsf{qcoh}^{\mathsf{fl}}\longrightarrow Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}$. The left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}})\longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}) \tag{58}$$

is constructed in the similar way.

Analogously, set $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ to be the inductive limit of (the equivalences of categories) $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})$ as $d\to\infty$ for any symbol $\star\neq\mathsf{co}$. For any morphism $f\colon Y\longrightarrow X$ into a quasi-compact semi-separated scheme X and any open coverings $\mathbf{W}$ and $\mathbf{T}$ of the schemes Y and X for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine, the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})\longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}})$$

is constructed as the functor on the derived categories induced by the exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$.

As usually, we set $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}) = \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}})$. Now Lemma 4.9.3 together with Proposition A.5.6 provide a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$. For a morphism $f\colon Y \longrightarrow X$ of quasi-compact semi-separated schemes, such equivalences allow to define the derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}}), \tag{59}$$

which clearly does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$.

4.10. **Morphisms of finite flat dimension.** A morphism of schemes $f\colon Y \longrightarrow X$ is said to have *flat dimension not exceeding* D if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ the $\mathcal{O}_X(U)$-module $\mathcal{O}_Y(V)$ has flat dimension not exceeding D. The morphism f has *very flat dimension not exceeding* D if the similar bound holds for the very flat dimension of the $\mathcal{O}_X(U)$-modules $\mathcal{O}_Y(V)$.

For any morphism $f\colon Y \longrightarrow X$ of finite flat dimension $\leqslant D$ into a quasi-compact semi-separated scheme X and any symbol $\star \neq \mathsf{ctr}$, the left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \tag{60}$$

is constructed in the following way. Let us call a quasi-coherent sheaf $\mathcal{F}$ on X *adjusted to* f if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ one has $\mathrm{Tor}_{>0}^{\mathcal{O}_X(U)}(\mathcal{O}_Y(V), \mathcal{F}(U)) = 0$. Quasi-coherent sheaves $\mathcal{F}$ on X adjusted to f form a full subcategory $X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}} \subset X\text{–}\mathsf{qcoh}$ closed under extensions, kernels of surjective morphisms and infinite direct sums, and such that any quasi-coherent sheaf on X has a finite left resolution of length $\leqslant D$ by sheaves from $X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}$. By Proposition A.5.6, it follows that the natural functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories.

The right exact functor $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ restricts to an exact functor $f^*\colon X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}} \longrightarrow Y\text{–}\mathsf{qcoh}$. In view of the above equivalence of categories, the induced functor on the derived categories $f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh})$ provides the desired derived functor $\mathbb{L}f^*$. For any morphism of finite flat dimension $f\colon Y \longrightarrow X$ between quasi-compact semi-separated schemes Y and X, the functor $\mathbb{L}f^*$ is left adjoint to the functor $\mathbb{R}f_*$ (50) from Section 4.8 (cf. [15, Section 1.9]).

For any morphism $f\colon Y \longrightarrow X$ of finite very flat dimension $\leqslant D$ into a quasi-compact semi-separated scheme X, any open coverings $\mathbf{W}$ and $\mathbf{T}$ of the schemes X and Y for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine, and any symbol $\star \neq \mathsf{co}$, the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}_{\mathbf{T}}) \tag{61}$$

is constructed in the following way. Let us call a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X *adjusted to* f if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that U is subordinate to $\mathbf{W}$ and $f(V) \subset U$ one has $\mathrm{Ext}_{\mathcal{O}_X(U)}^{>0}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$. One can easily see that the adjustness condition does not change when restricted to open subschemes V subordinate to $\mathbf{T}$, nor it is changed by a refinement of the covering $\mathbf{W}$.

Locally contraherent cosheaves on X adjusted to f form a full subcategory $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}} \subset X\text{–}\mathsf{lcth}$ closed under extensions and cokernels of admissible monomorphisms; the category $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}} = X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is also closed under infinite products in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and such that any $\mathbf{W}$-locally contraherent cosheaf on X has a finite right resolution of length $\leqslant D$ by objects of $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}$. By the dual version of Proposition A.5.6, the natural functor $\mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories. The construction of the exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{T}}$ from Section 3.3 extends without any changes to the case of cosheaves from $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}$, defining an exact functor

$$f^!\colon X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}.$$

Instead of Lemma 1.2.3(a) used in Sections 2.3 and 3.3, one can use Lemma 1.5.5(a) in order to check that the contraadjustness condition is preserved here.

In view of the above equivalence of triangulated categories, the induced functor $f^!\colon \mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\star}(Y\text{–}\mathsf{lcth}_{\mathbf{T}})$ provides the desired functor $\mathbb{R}f^!$ (61). When both schemes X and Y are quasi-compact and semi-separated, one can use the equivalences of categories from Corollary 4.6.3(a) in order to obtain the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^{\star}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\star}(Y\text{–}\mathsf{ctrh}), \tag{62}$$

which is right adjoint to the functor $\mathbb{L}f_!$ (51) from Section 4.8.

For a morphism $f\colon Y \longrightarrow X$ of flat dimension $\leqslant D$ into a quasi-compact semi-separated scheme X, any open coverings $\mathbf{W}$ and $\mathbf{T}$ of the schemes X and Y for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine, and any symbol $\star \neq \mathsf{co}$, one can similarly construct the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^{\star}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\star}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathbf{T}}). \tag{63}$$

More precisely, a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is said to be *adjusted to* f if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that U is subordinate to $\mathbf{W}$ and $f(V) \subset U$, and for any flat $\mathcal{O}_Y(V)$-module G, one has $\operatorname{Ext}_{\mathcal{O}_X(U)}(G,\mathfrak{P}[U]) = 0$. This condition does not change when restricted to open subschemes $V \subset Y$ subordinate to $\mathbf{T}$, nor is it changed by any refinement of the covering $\mathbf{W}$ of the scheme X.

As above, locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X adjusted to f form a full subcategory $X\text{–}\mathsf{lcth}^{\mathsf{lct},\,f\text{–}\mathsf{adj}}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ closed under extensions, cokernels of admissible monomorphisms, and infinite products. It follows from Lemma 1.5.2(a) that any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X has a finite right resolution of length $\leqslant D$ by objects of $X\text{–}\mathsf{lcth}^{\mathsf{lct},\,f\text{–}\mathsf{adj}}_{\mathbf{W}}$. Hence the natural functor $\mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{\mathsf{lct},\,f\text{–}\mathsf{adj}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ is an equivalence of triangulated categories.

Now Lemma 1.5.5(b) allows to construct an exact functor

$$f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lct},\,f\text{–}\mathsf{adj}}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}},$$

and the induced functor $f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}})$ provides the desired derived functor (63). When both schemes X and Y are quasi-compact and semi-separated, one can use Corollary 4.6.4(a) in order to obtain the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}), \tag{64}$$

which is right adjoint to the functor $\mathbb{L}f_!$ (52).

Let X be a quasi-compact semi-separated scheme with an open covering $\mathbf{W}$.

Lemma 4.10.1. (a) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering, then the right homological dimension of any quasi-coherent sheaf of flat dimension $\leqslant d$ on X with respect to the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}\subset X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ does not exceed N.*

(b) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering, then the right homological dimension of any quasi-coherent sheaf of very flat dimension $\leqslant d$ on X with respect to the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}\subset X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ does not exceed N.*

(c) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering subordinate to $\mathbf{W}$, then the left homological dimension of any $\mathbf{W}$-locally contraherent cosheaf of locally injective dimension $\leqslant d$ on X with respect to the full exact subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}\cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ does not exceed $N-1$.*

Proof. Part (a): in view of Lemma 4.6.1(a) and the dual version of Corollary A.5.3, it suffices to show that there exists an injective morphism from any given quasi-coherent sheaf belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ into a quasi-coherent sheaf belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ with the cokernel belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$. This follows from Corollary 4.1.4(b) or 4.1.11(b). The proof of part (b) is similar.

The proof of part (c) is analogous up to duality, and based on Lemma 4.6.1(b) and Corollary 4.2.5(b) (alternatively, the argument from the proof of Lemma 4.9.3 is sufficient in this case). ☐

Lemma 4.10.2. *Let $X=\bigcup_\alpha U_\alpha$ be a finite affine open covering. Then*

(a) *a quasi-coherent sheaf on X belongs to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of quasi-coherent sheaves from $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$;*

(b) *a quasi-coherent sheaf on X belongs to $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of quasi-coherent sheaves from $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cot}}\cap U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$;*

(c) *a quasi-coherent sheaf on X belongs to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of quasi-coherent sheaves from $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$;*

(d) *a contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}\cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$.*

Proof. One has to repeat the arguments in Sections 4.1 and 4.2 working with, respectively, quasi-coherent sheaves of (very) flat dimension $\leqslant d$ only or locally contraherent cosheaves of locally injective dimension $\leqslant d$ only throughout (cf. Lemma 4.1.6). ☐

Let $f\colon Y \longrightarrow X$ be a morphism of finite flat dimension $\leqslant D$ between quasi-compact semi-separated schemes X and Y.

Corollary 4.10.3. (a) *The exact functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ takes objects of $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ to objects of $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)}$.*

(b) *If the morphism f has very flat dimension not exceeding D, then the exact functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ takes objects of $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ to objects of $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}(d+D)}$.*

(c) *The exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ takes objects of $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$ to objects of $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}(d+D)}$.*

Proof. Part (a) follows from Lemma 4.10.2(a) together with the fact that the direct image with respect to an affine morphism of flat dimension $\leqslant D$ takes quasi-coherent sheaves of flat dimension $\leqslant d$ to quasi-coherent sheaves of flat dimension $\leqslant d + D$. The latter is provided by Lemma 1.5.2(a). The proof of part (b) is similar and based on Lemmas 4.10.2(c) and 1.5.3(b).

Finally, part (c) follows from Lemma 4.10.2(d) together with the fact that the direct image with respect to a $(\mathbf{W}, \mathbf{T})$-affine morphism of flat dimension $\leqslant D$ takes $\mathbf{T}$-locally contraherent cosheaves of locally injective dimension $\leqslant d$ to $\mathbf{W}$-locally contraherent cosheaves of locally injective dimension $\leqslant d+D$. The latter is provided by Lemma 1.5.2(b). ☐

According to Lemma 4.10.1(a) and the dual version of Proposition A.5.6, for any symbol $\star \neq \mathsf{ctr}$ the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d})$ is an equivalence of triangulated categories (as is the similar functor for sheaves over X). So one can construct the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)})$$

as the functor on the derived categories induced by the exact functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)}$ from Corollary 4.10.3(a). Passing to the inductive limits as $d \to \infty$, we obtain the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}), \tag{65}$$

which is right adjoint to the functor $\mathbb{L}f^*$ (57). For a morphism f of finite very flat dimension, the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}) \tag{66}$$

right adjoint to the functor $\mathbb{L}f^*$ (58) is constructed in the similar way.

Analogously, according to Lemma 4.10.1(c) and Proposition A.5.6, for any symbol $\star \neq \mathsf{co}$ the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d})$ is an equivalence of triangulated categories (as is the similar functor for cosheaves over X). Thus one can construct the left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}(d+D)})$$

as the functor on the derived categories induced by the exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}\cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}\longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}\cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}-(d+D)}$ from Corollary 4.10.3(c). Passing to the inductive limits as $d\to\infty$, we obtain the left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}})\longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}), \tag{67}$$

which is left adjoint to the functor $\mathbb{R}f^!$ (59).

4.11. **Finite injective and projective dimension.** For any scheme X, we denote the full subcategory of objects of injective dimension $\leqslant d$ in the abelian category $X\text{–}\mathsf{qcoh}$ by $X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}$. For a quasi-compact semi-separated scheme X, the full subcategory of objects of projective dimension $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is denoted by $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}$ and the full subcategory of objects of projective dimension $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ by $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}}$. Set $X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d}=X\text{–}\mathsf{lcth}_{\{X\},\,\mathsf{fpd}-d}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d}^{\mathsf{lct}}=X\text{–}\mathsf{lcth}_{\{X\},\,\mathsf{fpd}-d}^{\mathsf{lct}}$.

Furthermore, the *colocally flat dimension* of a $\mathbf{W}$-locally contraherent cosheaf on a quasi-compact semi-separated scheme X is defined as its left homological dimension with respect to the exact subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see Section A.5). The colocally flat dimension is well-defined by Corollaries 4.3.2(b) and 4.3.4(b). A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X has colocally flat dimension $\leqslant d$ if and only if $\operatorname{Ext}^{>d}(\mathfrak{F},\mathfrak{P})=0$ for any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X. The full subcategory of objects of colocally flat dimension $\leqslant d$ in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is denoted by $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{clfd}-d}$. We set $X\text{–}\mathsf{ctrh}_{\mathsf{clfd}-d}=X\text{–}\mathsf{lcth}_{\{X\},\,\mathsf{clfd}-d}$.

Since the subcategories of projective objects in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ do not depend on the covering $\mathbf{W}$, and the full subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W}}\subset X\text{–}\mathsf{lcth}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}\subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ are closed under kernels of admissible epimorphisms, the projective dimension of an object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ or $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ does not change when the open covering $\mathbf{W}$ is replaced by its refinement. Similarly, the colocally flat dimension of a $\mathbf{W}$-locally contraherent cosheaf on X does not depend on the covering $\mathbf{W}$.

One can easily see that the full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}\subset X\text{–}\mathsf{qcoh}$ is closed under extensions and cokernels of admissible monomorphisms, while the full subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$, and $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{clfd}-d}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ are closed under extensions and kernels of admissible epimorphisms.

Corollary 4.11.1. (a) *For any scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})\longrightarrow\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})\longrightarrow\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, $\mathsf{Hot}^\pm(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})\longrightarrow\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})\longrightarrow\mathsf{D}^\pm(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})\longrightarrow\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$ *are equivalences of categories.*

(b) *For any quasi-compact semi-separated scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})\longrightarrow\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})\longrightarrow\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, $\mathsf{Hot}^\pm(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})\longrightarrow\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})\longrightarrow\mathsf{D}^\pm(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})\longrightarrow\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

(c) *For any quasi-compact semi-separated scheme X and any symbol $\star\neq\mathsf{co}$, the natural triangulated functors* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clf}})\longrightarrow\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{clfd}-d})$ *are equivalences of categories.*

(d) *For any quasi-compact semi-separated scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}})$ *are equivalences of categories.*

Proof. Parts (b-d) follow from Proposition A.5.6, while part (a) follows from the dual version of it. □

Corollary 4.11.2. (a) *For any scheme X, the natural functor* $\mathsf{Hot}^{+}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{+}(X\text{–}\mathsf{qcoh})$ *is an equivalence of triangulated categories. For any symbol $\star = \mathsf{b}$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, or abs, the natural triangulated functor* $\mathsf{Hot}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{qcoh})$ *is fully faithful.*

(b) *For any quasi-compact semi-separated scheme X, the natural functors* $\mathsf{Hot}^{-}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{lcth})$ *are equivalences of triangulated categories. For any scheme X and any symbol $\star = \mathsf{b}$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the natural triangulated functor* $\mathsf{Hot}^{\star}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{ctrh})$ *is fully faithful.*

(c) *For any quasi-compact semi-separated scheme X, the natural functors* $\mathsf{Hot}^{-}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{-}(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *are equivalences of triangulated categories. For any scheme X and any symbol $\star = \mathsf{b}$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the natural triangulated functor* $\mathsf{Hot}^{\star}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ *is fully faithful.*

Proof. In each part (a-c), the first assertion follows from there being enough injective/projective objects in the respective abelian/exact categories, together with Proposition A.3.1(a) (or the dual version of it). In the second assertions of parts (b-c), the notation $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ or $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ stands for the full additive subcategories of projective objects in the exact categories $X\text{–}\mathsf{ctrh}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ on an arbitrary scheme X. Irrespectively of there being enough such projectives or injectives, these kind of assertions hold in any exact category (or in any exact category with infinite direct sums or products, as appropriate) by Lemma A.1.3. □

Let X be a quasi-compact semi-separated scheme.

Lemma 4.11.3. *The equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ from Lemma 4.6.7 identifies the exact subcategories*

(a) $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ *with* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d} \subset X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$,

(b) $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ *with* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d} \subset X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$,

(c) $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ *with* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{clfd}-d} \subset X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$,

(d) $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ *with* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d}^{\mathsf{lct}} \subset X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

Proof. Part (a): since the functor $\mathcal{O}_X \odot_X -$ takes short exact sequences in $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ to short exact sequences in $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ and commutes with the direct images of contraherent cosheaves and quasi-coherent sheaves from the affine open subschemes of X, it follows from Lemma 4.10.2(d) that this functor takes $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$

into $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}$. To prove the converse, consider a contraadjusted quasi-coherent sheaf $\mathcal{P}$ of injective dimension d on X, and let $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{J}^0 \longrightarrow \cdots \longrightarrow \mathcal{J}^d \longrightarrow 0$ be its right injective resolution of length d in $X\text{–}\mathsf{qcoh}$.

This resolution is an exact sequence over the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$, so it is transformed to an exact sequence over the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ by the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$. The contraherent cosheaves $\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J}^i)$ being locally injective according to the proof of Corollary 4.6.8(b), it follows that the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{P})$ admits a right resolution of length d by locally injective contraherent cosheaves, i. e., has locally injective dimension $\leqslant d$.

Part (d): by the proof of Corollary 4.6.8(a), the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ with $X\text{–}\mathsf{qcoh}^{\mathsf{lct}}_{\mathsf{clp}}$. Furthermore, it follows from Lemma 4.10.2(b) that the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ takes $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ into $X\text{–}\mathsf{qcoh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d}$, since a cotorsion module of flat dimension $\leqslant d$ over a commutative ring R corresponds to a (locally) cotorsion contraherent cosheaf of projective dimension $\leqslant d$ over $\operatorname{Spec} R$. Conversely, a projective resolution of length d of an object of $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$ is transformed by the functor $\mathcal{O}_X \odot_X -$ into a flat resolution of length d of the corresponding cotorsion quasi-coherent sheaf (see the proof of Corolary 4.6.10(c)).

The proof of part (b) is similar and based on Lemma 4.10.2(c) and the proof of Corollary 4.6.10(a), while the proof of part (c) is based on Lemma 4.10.2(a) and the proof of Corollary 4.6.10(b). In both cases it is important that every projective (or, respectively, colocally flat) contraherent cosheaf is colocally projective. ☐

Corollary 4.11.4. *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering. Then*

(a) *a quasi-coherent sheaf on X belongs to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of quasi-coherent sheaves from $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}$;*

(b) *a contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from $U_\alpha\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap U_\alpha\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d}$;*

(c) *a contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}_{\mathsf{clfd}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from $U_\alpha\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap U_\alpha\text{–}\mathsf{ctrh}_{\mathsf{clfd}-d}$;*

(d) *a contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from $U_\alpha\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d}$.*

Proof. Follows from Lemmas 4.10.2 and 4.11.3. ☐

Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes.

Lemma 4.11.5. (a) *Whenever the flat dimension of the morphism f does not exceed D, the functor of direct image f_* takes injective quasi-coherent sheaves on Y to quasi-coherent sheaves of injective dimension $\leqslant D$ on X.*

(b) *Whenever the very flat dimension of the morphism f does not exceed D, the functor of direct image $f_!$ takes projective contraherent cosheaves on Y to contraherent cosheaves of projective dimension $\leqslant D$ on X.*

(c) *Whenever the flat dimension of the morphism f does not exceed D, the functor of direct image $f_!$ takes colocally flat contraherent cosheaves on Y to contraherent cosheaves of colocally flat dimension $\leqslant D$ on X.*

(d) *Whenever the flat dimension of the morphism f does not exceed D, the functor of direct image $f_!$ takes projective locally cotorsion contraherent cosheaves on Y to locally cotorsion contraherent cosheaves of projective dimension $\leqslant D$ on X.*

Proof. In part (a) it actually suffices to assume that the scheme Y is quasi-compact and quasi-separated (while the scheme X has to be quasi-compact and semi-separated). Let us prove part (c), parts (a), (b), (d) being analogous. The functor $f_!$ takes $Y\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ to $X\text{–}\mathsf{ctrh}$ by Corollary 4.5.3(a), so it remains to check that $\mathrm{Ext}^{X,>D}(f_!\mathfrak{F},\mathfrak{P})=0$ for any cosheaves $\mathfrak{F}\in Y\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ and $\mathfrak{P}\in X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$.

According to the adjunction of derived functors $\mathbb{L}f_!$ (51) and $\mathbb{R}f^!$ (62), one has $\mathrm{Ext}^{X,*}(f_!\mathfrak{F},\mathfrak{P})\simeq \mathrm{Hom}_{\mathsf{D^b}(Y\text{–}\mathsf{lcth})}(\mathfrak{F},\ \mathbb{R}f^!(\mathfrak{P})[*])$. Hence it suffices to show that the object $\mathbb{R}f^!(\mathfrak{P})\in\mathsf{D^b}(Y\text{–}\mathsf{lcth}^{\mathsf{lct}})$ can be represented by a finite complex over $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$ concentrated in the cohomological degrees $\leqslant D$. The latter is true because any $\mathbf{W}$-locally contraherent cosheaf on X admits a finite right resolution of length $\leqslant D$ by f-adjusted locally cotorsion $\mathbf{W}$-locally contraherent cosheaves. □

Set the triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fid}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$ as $d\to\infty$ for any $\star\neq\mathsf{co},\mathsf{ctr}$. For any morphism of finite flat dimension $f\colon Y\longrightarrow X$ between quasi-compact semi-separated schemes, one constructs the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{fid}})\longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fid}}) \tag{68}$$

as the functor on the homotopy/derived categories induced by the additive functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}\longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fid}-D}$ from Lemma 4.11.5(a).

Analogously, set $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{fpd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d})$ as $d\to\infty$ for any $\star\neq\mathsf{co},\mathsf{ctr}$. Besides, set $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}_{\mathsf{clfd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clfd}-d})$ as $d\to\infty$, and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d})$ as $d\to\infty$. For any morphism of finite flat dimension $f\colon Y\longrightarrow X$ between quasi-compact semi-separated schemes, one constructs the left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}})\longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}}) \tag{69}$$

as the functor on the homotopy/derived categories induced by the additive functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}\longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-D}$ from Lemma 4.11.5(d). The left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clfd}})\longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clfd}}) \tag{70}$$

is constructed in the similar way. Finally, for a morphism f of finite very flat dimension, one can similarly construct the left derived functor

$$\mathbb{L}f_! \colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{fpd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{fpd}}). \tag{71}$$

The following corollary provides three restricted versions of Theorem 4.8.1.

Corollary 4.11.6. (a) *Assume that the morphism f has finite flat dimension. Then for any symbol $\star \neq \mathsf{co}$ the equivalences of categories $\mathsf{Hot}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ and $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ from Corollary* 4.6.8(b) *transform the right derived functor $\mathbb{R}f_*$* (68) *into the left derived functor $\mathbb{L}f_!$* (67).

(b) *Assume that the morphism f has finite very flat dimension. Then for any symbol $\star \neq \mathsf{ctr}$ the equivalences of categories $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ and $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ from Corollary* 4.6.10(a) *transform the right derived functor $\mathbb{R}f_*$* (66) *into the left derived functor $\mathbb{L}f_!$* (71).

(c) *Assume that the morphism f has finite flat dimension. Then for any symbol $\star \neq \mathsf{co}$, ctr the equivalences of categories $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clf}})$ and $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clf}})$ from Corollary* 4.6.10(b) *transform the right derived functor $\mathbb{R}f_*$* (65) *into the left derived functor $\mathbb{L}f_!$* (70).

Proof. Part (a): Clearly, one can assume $\star \neq \mathsf{ctr}$. The composition of equivalences of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}D}) \simeq \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}D})$ can be constructed directly in terms of the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}D} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{flid}\text{–}D}$ from Lemma 4.11.3(a). Now this equivalence of exact categories together with the equivalence of additive categories $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}} \simeq Y\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ used in the proof of Corollary 4.6.8(b) transform the functor f_* into the functor $f_!$, which implies the desired assertion. Parts (b) and (c) are similarly proved using Lemma 4.11.3(b-c). □

4.12. **Derived tensor operations.** Let X be a quasi-compact semi-separated scheme. The functor of tensor product

$$\otimes_{\mathcal{O}_X} \colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \times \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \tag{72}$$

is well-defined for any symbol $\star = \mathsf{co}$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, since the tensor product of a complex coacyclic (respectively, absolutely acyclic) over the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and any complex over $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ is coacyclic (resp., absolutely acyclic) over $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$. The functor of tensor product

$$\otimes_{\mathcal{O}_X} \colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \times \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \tag{73}$$

is well-defined for the similar reasons (though in this case the situation is actually simpler; see below).

Furthermore, the functor of tensor product

$$\otimes_{\mathcal{O}_X} \colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \times \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \tag{74}$$

is well-defined for $\star = \mathsf{co}$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, since the tensor product of a complex coacyclic (resp., absolutely acyclic) over the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and any complex over $X\text{–}\mathsf{qcoh}$ is coacyclic (resp., absolutely acyclic) over $X\text{–}\mathsf{qcoh}$, as is the tensor product of any complex over $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and a complex coacyclic (resp., absolutely acyclic) over $X\text{–}\mathsf{qcoh}$. In view of Lemma A.1.2, the functors (72–74) are also well-defined for $\star = +$; and it is a standard fact that they are well-defined for $\star = \mathsf{b}$ or $-$. Thus $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ and $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ become tensor triangulated categories for any $\star \neq \varnothing$, ctr, and $\mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ is a triangulated module category over $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$.

Recall the definition of a homotopy flat complex of flat quasi-coherent sheaves from Section 4.7. The left derived functor of tensor product

$$\otimes^{\mathbb{L}}_{\mathcal{O}_X}\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \times \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \tag{75}$$

is constructed for $\star = \varnothing$ by applying the functor $\otimes_{\mathcal{O}_X}$ of tensor product of complexes of quasi-coherent sheaves to homotopy flat complexes of flat quasi-coherent sheaves in one of the arguments and arbitrary complexes of quasi-coherent sheaves in the other one. This derived functor is well-defined by Theorem 4.7.1. In the case of $\star = -$, the derived functor (75) can be constructed by applying the functor $\otimes_{\mathcal{O}_X}$ to bounded above complexes of flat quasi-coherent sheaves in one of the arguments. So $\mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ is a tensor triangulated category for any symbol $\star = -$ or $\varnothing$.

The full triangulated subcategory of *homotopy very flat complexes* of very flat quasi-coherent sheaves $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})^{\mathsf{vfl}}$ in the unbounded derived category of the exact category of very flat quasi-coherent sheaves $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ on X is defined as the minimal triangulated subcategory containing the objects of $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ and closed under infinite direct sums (cf. Section 4.7). By Corollary 4.1.4(a) and Proposition A.4.3, the composition of natural triangulated functors $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})^{\mathsf{vfl}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories.

By Corollary 4.9.5(a), any acyclic complex over $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ is absolutely acyclic over $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$. Hence the composition of functors $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})^{\mathsf{vfl}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ provides a natural functor

$$\mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}), \tag{76}$$

which is clearly a tensor triangulated functor between these tensor triangulated categories. In fact, this functor is also fully faithful, and left adjoint to the composition of Verdier localization functors $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ (see Corollary A.4.8). Composing the functor (76) with the tensor action functor (74), we obtain the left derived functor

$$\otimes^{\mathbb{L}'}_{\mathcal{O}_X}\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \times \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \tag{77}$$

making $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ a triangulated module category over the triangulated tensor category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ (cf. [22, Section 1.4]).

Given a symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs, we set $\star'$ to be the "dual"symbol b, $-$, $+$, $\varnothing$, $\mathsf{abs}-$, $\mathsf{abs}+$, ctr, co, or abs, respectively. Furthermore, let $\mathbf{W}$ be an open covering of the scheme X. Given two complexes $\mathcal{F}^\bullet$ over $X\text{–}\mathsf{qcoh}$ and $\mathfrak{J}^\bullet$ over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ such that the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F}^i, \mathfrak{J}^j)$ is well-defined by the constructions of Section 3.6 for every pair $(i,j) \in \mathbb{Z}^2$, we set $\mathfrak{Cohom}_X(\mathcal{F}^\bullet, \mathfrak{J}^\bullet)$ to be the total complex of the bicomplex $\mathfrak{Cohom}_X(\mathcal{F}^i, \mathfrak{J}^j)$ over

X–$\mathsf{lcth}_{\mathbf{W}}$ constructed by taking infinite products of $\mathbf{W}$-locally contraherent cosheaves along the diagonals of the bicomplex.

The functor of cohomomorphisms

$$\mathfrak{Cohom}_X\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{op}}\times\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})\longrightarrow\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \tag{78}$$

is well-defined for any symbol $\star=\mathsf{co}$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, since the $\mathfrak{Cohom}$ from a complex coacyclic (respectively, absolutely acyclic) over X–$\mathsf{qcoh}^{\mathsf{fl}}$ into any complex over X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ is a contraacyclic (resp., absolutely acyclic) complex over X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, as is the $\mathfrak{Cohom}$ from any complex over X–$\mathsf{qcoh}^{\mathsf{fl}}$ into a complex contraacyclic (resp., absolutely acyclic) over X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. The functor of cohomomorphisms

$$\mathfrak{Cohom}_X\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})^{\mathsf{op}}\times\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}})\longrightarrow\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \tag{79}$$

is well-defined for the similar reasons. In view of Lemma A.1.2, the functors (78–79) are also well-defined for $\star=+$; and one can straightforwardly check that they are well-defined for $\star=\mathsf{b}$ or $-$. Thus the category opposite to $\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$ becomes a triangulated module category over the tensor triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ and the category opposite to $\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is a triangulated module category over the tensor triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ for any symbol $\star\ne\varnothing$, ctr.

Recall the definition of a homotopy locally injective complex of locally injective $\mathbf{W}$-locally contraherent cosheaves from Section 4.7. The right derived functor of cohomomorphisms

$$\mathbb{R}\,\mathfrak{Cohom}_X\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh})^{\mathsf{op}}\times\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}})\longrightarrow\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \tag{80}$$

is constructed for $\star=\varnothing$ by applying the functor $\mathfrak{Cohom}_X$ to homotopy very flat complexes of very flat quasi-coherent sheaves in the first argument and arbitrary complexes of $\mathbf{W}$-locally contraherent cosheaves in the second argument, or alternatively, to arbitrary complexes of quasi-coherent sheaves in the first argument and homotopy locally injective complexes of locally injective $\mathbf{W}$-locally contraherent cosheaves in the second argument. This derived functor is well-defined by (the proof of) Theorem 4.7.2 (cf. [52, Lemma 2.7]). In the case of $\star=-$, the derived functor (80) can be constructed by applying the functor $\mathfrak{Cohom}_X$ to bounded bounded above complexes of very flat quasi-coherent sheaves in the first argument or bounded below complexes of locally injective $\mathbf{W}$-locally contraherent cosheaves in the second argument. So $\mathsf{D}^{\star'}(X\text{–}\mathsf{lcth}_{\mathbf{W}})^{\mathsf{op}}$ is a triangulated module category over the tensor triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ for any symbol $\star=-$ or $\varnothing$.

Composing the functor (76) with the tensor action functor (78), one obtains the right derived functor

$$\mathbb{R}'\,\mathfrak{Cohom}_X\colon \mathsf{D}(X\text{–}\mathsf{qcoh})^{\mathsf{op}}\times\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})\longrightarrow\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \tag{81}$$

making $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})^{\mathsf{op}}$ a triangulated module category over the triangulated tensor category $\mathsf{D}(X\text{–}\mathsf{qcoh})$. Similarly, the composition $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})^{\mathsf{vfl}}\longrightarrow\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})\simeq\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ provides a natural functor

$$\mathsf{D}(X\text{–}\mathsf{qcoh})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}), \tag{82}$$

which is a tensor triangulated functor between these tensor triangulated categories. It is also fully faithful, and left adjoint to the composition of Verdier localization functors $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$. Composing the functor (82) with the tensor action functor (79), one obtains the left derived functor

$$(83)\qquad \mathbb{R}'\,\mathfrak{Cohom}_X\colon \mathsf{D}(X\text{–}\mathsf{qcoh})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$$

making $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})^{\mathsf{op}}$ a triangulated module category over the triangulated tensor category $\mathsf{D}(X\text{–}\mathsf{qcoh})$.

5. Noetherian Schemes

5.1. **Projective locally cotorsion contraherent cosheaves.** Let X be a (not necessarily semi-separated) locally Noetherian scheme and $\mathbf{W}$ be its open covering. The following theorem is to be compared to Hartshorne's classification of injective quasi-coherent sheaves on X [30, Proposition II.7.17] based on Matlis' classification of injective modules over a Noetherian ring [41].

Theorem 5.1.1. (a) *There are enough projective objects in the exact categories of locally cotorsion locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$, and all these projective objects belong to the full subcategory of locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$. The full subcategories of projective objects in the three exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ coincide.*

(b) *For any scheme point $x \in X$, denote by $\widehat{\mathcal{O}}_{x,X}$ the completion of the local ring $\mathcal{O}_{x,X}$ and by $\iota_x\colon \operatorname{Spec}\mathcal{O}_{x,X} \longrightarrow X$ the natural morphism. Then a locally cotorsion contraherent cosheaf $\mathfrak{F}$ on X is projective if and only if it is isomorphic to an infinite product $\prod_{x\in X}\iota_{x!}\check{F}_x$ of the direct images $\iota_{x!}\check{F}_x$ of the contraherent cosheaves $\check{F}_x$ on $\operatorname{Spec}\mathcal{O}_{x,X}$ corresponding to some free contramodules F_x over the complete Noetherian local rings $\widehat{\mathcal{O}}_{x,X}$ (viewed as $\mathcal{O}_{x,X}$-modules via the restriction of scalars).*

Proof. For a quasi-compact semi-separated scheme X, the assertions of part (a) were proven in Section 4.4. In the general case, we will prove parts (a) and (b) simultaneously. The argument is based on Theorem 1.3.8.

First of all, let us show that the cosheaf of $\mathcal{O}_X$-modules $\iota_{x!}\check{P}_x$ is a locally cotorsion contraherent cosheaf on X for any contramodule P_x over $\widehat{\mathcal{O}}_{x,X}$. It suffices to check that the restriction of $\iota_{x!}\check{P}_x$ to any affine open subscheme $U \subset X$ is a (locally) cotorsion contraherent cosheaf. If the point x belongs to U, then the morphism ι_x factorizes through U; denoting the morphism $\operatorname{Spec}\mathcal{O}_{x,X} \longrightarrow U$ by κ_x, one has $(\iota_{x!}\check{P}_x)|_U \simeq \kappa_{x!}\check{P}_x$. The morphism κ_x being affine and flat, and the contraherent cosheaf $\check{P}_x$ over $\operatorname{Spec}\mathcal{O}_{x,X}$ being (locally) cotorsion by Proposition 1.3.7(a), $\kappa_{x!}\check{P}_x$ is a (locally) cotorsion contraherent cosheaf over U.

If the point x does not belong to U, we will show that the $\mathcal{O}_X(U)$-module $(\iota_{x!}\check{P}_x)[U] \simeq \check{P}_x[\iota_x^{-1}(U)]$ vanishes. So it will follow that the restriction of $\iota_{x!}\check{P}_x$ to the open complement of the closure of the point x in X is a zero cosheaf. Indeed, it suffices to check that the module of cosections $\check{P}_x[V]$ vanishes for any principal affine open subscheme $V \subset \operatorname{Spec} \mathcal{O}_{x,X}$ that does not contain the closed point. In other words, it has to be shown that $\operatorname{Hom}_{\mathcal{O}_{x,X}}(\mathcal{O}_{x,X}[s^{-1}], P_x) = 0$ for any element s from the maximal ideal of the local ring $\mathcal{O}_{x,X}$. This holds for any $\widehat{\mathcal{O}}_{x,X}$-contramodule P_x; see [55, Theorem B.1.1(2c)].

It follows that cosheaves of the form $\prod_x \iota_{x!}\check{P}_x$ on X, where P_x are some $\widehat{\mathcal{O}}_{x,X}$-contramodules, belong to $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$. Let us show that cosheaves $\mathfrak{F} = \prod_x \iota_{x!}\check{F}_x$, where F_x denote some free $\widehat{\mathcal{O}}_{x,X}$-contramodules, are projective objects in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. Pick an affine open covering U_α of the scheme X subordinate to the covering $\mathbf{W}$, and choose a well-ordering of the set of indices $\{\alpha\}$.

Given an index α, denote by $\mathfrak{F}_\alpha$ the product of the cosheaves $\iota_{z!}\check{F}_z$ on X taken over all points $z \in S_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. Clearly, one has $\mathfrak{F} \simeq \prod_\alpha \mathfrak{F}_\alpha$. Furthermore, since X is locally Noetherian, for any affine (or even quasi-compact) open subscheme $U \subset X$ one has $U \cap S_\alpha = \varnothing$, and consequently $\mathfrak{F}_\alpha[U] = 0$, for all but a finite number of indices α. We conclude that the cosheaf $\mathfrak{F}$ is also the direct sum of the cosheaves $\mathfrak{F}_\alpha$ (taken in the category of cosheaves of $\mathcal{O}_X$-modules). Hence it suffices to check that each $\mathfrak{F}_\alpha$ is a projective object in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$.

Denoting by j_α the open embedding $U_\alpha \longrightarrow X$ and by κ_z the natural morphisms $\operatorname{Spec} \mathcal{O}_{z,X} \longrightarrow U_\alpha$, we notice that $\mathfrak{F}_\alpha = j_{\alpha!}\mathfrak{G}_\alpha$, where $\mathfrak{G}_\alpha = \prod_{z\in S_\alpha} \kappa_{z!}\check{F}_z$. By Theorem 1.3.8, $\mathfrak{G}_\alpha$ is a projective object in $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}$; and by the adjunction (24) it follows that $\mathfrak{F}_\alpha$ is a projective object in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$.

Now let us construct for any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Q}$ on X an admissible epimorphism onto $\mathfrak{Q}$ from an object $\mathfrak{F} = \prod_{x\in X} \iota_{x!}\check{F}_x$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. Choose an affine open covering U_α as above, and proceed by transfinite induction in α, constructing contraherent cosheaves $\mathfrak{F}_\alpha = \prod_{z\in S_\alpha} \iota_{z!}\check{F}_z$ and morphisms of locally contraherent cosheaves $\mathfrak{F}_\alpha \longrightarrow \mathfrak{Q}$.

Suppose that such cosheaves and morphisms have been constructed for all $\alpha < \beta$; then, as it was explained above, there is the induced morphism of locally contraherent cosheaves $\prod_{\alpha<\beta} \mathfrak{F}_\alpha \longrightarrow \mathfrak{Q}$. Assume that the related morphism of the modules of cosections $\prod_{\alpha<\beta} \mathfrak{F}_\alpha[U] = \bigoplus_{\alpha<\beta} \mathfrak{F}_\alpha[U] \longrightarrow \mathfrak{Q}[U]$ is an admissible epimorphism of cotorsion $\mathcal{O}_X(U)$-modules for any affine open subscheme $U \subset \bigcup_{\alpha<\beta} U_\alpha \subset X$ subordinate to $\mathbf{W}$. We are going to construct a contraherent cosheaf $\mathfrak{F}_\beta = \prod_{z\in S_\beta} \iota_{z!}\check{F}_z$ and a morphism of locally contraherent cosheaves $\mathfrak{F}_\beta \longrightarrow \mathfrak{Q}$ such that the induced morphism $\prod_{\alpha\leqslant\beta} \mathfrak{F}_\alpha \longrightarrow \mathfrak{Q}$ satisfies the above condition for any affine open subscheme $U \subset \bigcup_{\alpha\leqslant\beta} U_\alpha \subset X$ subordinate to $\mathbf{W}$.

Pick an admissible epimorphism in $U_\beta\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ onto the (locally) cotorsion contraherent cosheaf $j_\beta^!\mathfrak{Q}$ from a projective (locally) cotorsion contraherent cosheaf $\mathfrak{G}$ on U_β. By Theorem 1.3.8, the cosheaf $\mathfrak{G}$ decomposes into a direct product $\mathfrak{G} \simeq \prod_{x\in U_\beta} \kappa_{x!}\check{G}_x$, where G_x are some free $\widehat{\mathcal{O}}_{x,X}$-contramodules. Let us rewrite this product as $\mathfrak{G} = \prod_{z\in S_\beta} \kappa_{z!}\breve{G}_z \oplus \prod_{y\in U_\beta\setminus S_\beta} \kappa_{y!}\breve{G}_y$; denote the former direct summand by $\mathfrak{G}_\beta$ and the latter one by $\mathfrak{E}$. Set $\mathfrak{F} = \prod_{\alpha<\beta}\mathfrak{F}_\alpha$ and $\mathfrak{F}_\beta = j_{\beta!}\mathfrak{G}_\beta$.

The property of a morphism of locally cotorsion $\mathbf{T}$-locally contraherent cosheaves to be an admissible epimorphism being local, we only need to show that the natural morphism $\mathfrak{F}\oplus\mathfrak{F}_\beta \longrightarrow \mathfrak{Q}$ becomes an admissible epimorphism in U_β–$\mathsf{ctrh}^{\mathsf{lct}}$ when restricted to U_β. In other words, this means that the morphism $j_\beta^!\mathfrak{F}\oplus\mathfrak{G}_\beta \longrightarrow j_\beta^!\mathfrak{Q}$ should be an admissible epimorphism in U_β–$\mathsf{ctrh}^{\mathsf{lct}}$. It suffices to check that the admissible epimorphism $\mathfrak{G}\longrightarrow j_\beta^!\mathfrak{Q}$ factorizes through the morphism in question, or that the morphism $\mathfrak{E}\longrightarrow j_\beta^!\mathfrak{Q}$ factorizes through the morphism $j_\beta^!\mathfrak{F}\longrightarrow j_\beta^!\mathfrak{Q}$.

Denote by j the open embedding $U_\beta\setminus S_\beta \longrightarrow U_\beta$; then one has $\mathfrak{E}\simeq j_!\mathfrak{L}$, where $\mathfrak{L}$ is a projective object in $(U_\beta\setminus S_\beta)$–$\mathsf{ctrh}^{\mathsf{lct}}$ (as we have proven above). Hence the morphism $\mathfrak{L}\longrightarrow j^!j_\beta^!\mathfrak{Q}$ factorizes through an admissible epimorphism of locally cotorsion contraherent cosheaves $j^!j_\beta^!\mathfrak{F}\longrightarrow j^!j_\beta^!\mathfrak{Q}$, as desired.

We have proven part (a); and to finish the proof of part (b) it remains to show that the class of contraherent cosheaves of the form $\prod_x \iota_{x!}\breve{F}_x$ on X is closed under the passage to direct summands. The argument is based on the following lemma.

Lemma 5.1.2. (a) *Suppose that the set of all scheme points of X is presented as a union of two nonintersecting subsets $X = S\sqcup T$ such that for any points $z\in S$ and $y\in T$ the closure of z in X does not contain y. Then for any cosheaves of $\mathcal{O}$-modules $\mathfrak{P}_y$ over* $\operatorname{Spec}\mathcal{O}_{y,X}$ *and any contramodules P_z over $\widehat{\mathcal{O}}_{z,X}$ one has* $\operatorname{Hom}^{\mathcal{O}_X}(\prod_{y\in T}\iota_{y!}\mathfrak{P}_y,\ \prod_{z\in S}\iota_{z!}\breve{P}_z) = 0$.

(b) *For any scheme point $x\in X$, the functor assigning to an $\widehat{\mathcal{O}}_{x,X}$-contramodule P_x the locally cotorsion contraherent cosheaf $\iota_{x!}\breve{P}_x$ on X is fully faithful.*

Proof. Part (a): by the definition of the infinite product, it suffices to show that $\operatorname{Hom}^{\mathcal{O}_X}(\prod_{y\in T}\iota_{y!}\mathfrak{P}_y,\ \iota_{z!}\breve{P}_z) = 0$ for any $z\in S$. Let Z be the closure of z in X and $Y = X\setminus Z$ be its complement; then one has $T\subset Y$ and Y is an open subscheme in X. Let j denote the open embedding $Y\longrightarrow X$. Given $y\in T$, denote the natural morphism $\operatorname{Spec}\mathcal{O}_{y,X}\longrightarrow Y$ by κ_y, so $\iota_y = j\circ\kappa_y$.

Now we have $\prod_{y\in T}\iota_{y!}\mathfrak{P}_y \simeq j_!\prod_{y\in T}\kappa_{y!}\mathfrak{P}_y$ and, according to the adjunction (26),

$$\operatorname{Hom}^{\mathcal{O}_X}(j_!\prod\nolimits_{y\in T}\kappa_{y!}\mathfrak{P}_y,\ \iota_{z!}\breve{P}_z) \simeq \operatorname{Hom}^{\mathcal{O}_Y}(\prod\nolimits_{y\in T}\kappa_{y!}\mathfrak{P}_y,\ j^!\iota_{z!}\breve{P}_z).$$

It was shown above that $j^!\iota_{z!}\breve{P}_z = 0$, so we are done.

Part (b): it was explained in Section 1.3 that the functor assigning to an $\widehat{\mathcal{O}}_{x,X}$-contramodule P_x the locally cotorsion contraherent cosheaf $\breve{P}_x$ on $\operatorname{Spec}\mathcal{O}_{x,X}$ is fully faithful. The morphism ι_x being flat and coaffine, the adjunction (26) applies and it suffices to show that the adjunction morphism $\breve{P}_x\longrightarrow\iota_x^!\iota_{x!}\breve{P}_x$ is an isomorphism in $\operatorname{Spec}\mathcal{O}_{x,X}$–$\mathsf{ctrh}^{\mathsf{lct}}$. One can replace the scheme X by any affine open subscheme $U\subset X$ containing x, and it remains to use the isomorphism $\mathcal{O}_{x,X}\otimes_{\mathcal{O}(U)}\mathcal{O}_{x,X}\simeq\mathcal{O}_{x,X}$. ☐

Now we can finish the proof of Theorem. Let S be a subset of scheme points of X closed under specialization. It follows from Lemma 5.1.2(a) that any contraherent cosheaf $\mathfrak{F}$ on X isomorphic to $\prod_{x\in X}\iota_{x!}\check{F}_x$ is endowed with a natural projection onto its direct summand $\mathfrak{F}(S)=\prod_{z\in S}\iota_{z!}\check{F}_z$, and such projections commute with any morphisms between contraherent cosheaves $\mathfrak{F}$ of this form. Given two subsets $S'\subset S\subset X$ closed under specialization, there is a natural projection $\mathfrak{F}(S)\longrightarrow\mathfrak{F}(S')$, and the diagram formed by all such projections is commutative.

Any idempotent endomorphism e of the contraherent cosheaf $\mathfrak{F}$ acts on this diagram. When the complement $S\setminus S'$ consists of a single scheme point $x\in X$, the kernel of the projection $\mathfrak{F}(S)\longrightarrow\mathfrak{F}(S')$ is isomorphic to the direct image $\iota_{X!}\check{F}_x$. The endomorphism e induces an idempotent endomorphism of this kernel; by Lemma 5.1.2(b), the latter endomorphism comes from an idempotent endomorphism e_x of the free $\widehat{\mathcal{O}}_{x,X}$-contramodule F_x. By [55, Lemma 1.3.2], the image e_xF_x is also a free $\widehat{\mathcal{O}}_{x,X}$-contramodule. Using appropriate transfinite induction procedures, it is not difficult to show that the infinite product $\prod_{x\in X}\iota_{x!}\widecheck{(e_xF_x)}$ is isomorphic to the image of the idempotent endomorphism e of the contraherent cosheaf $\mathfrak{F}$. $\square$

As in Section 4.4, we denote the category of projective locally cotorsion contraherent cosheaves on X by $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$. The following corollary says that being a projective locally cotorsion contraherent cosheaf on a locally Noetherian scheme is a local property. Besides, all such cosheaves are coflasque (see Section 3.4). The similar properties of injective quasi-coherent sheaves are usually deduced from [30, Theorem II.7.18].

Corollary 5.1.3. (a) *Let $Y\subset X$ be an open subscheme. Then for any cosheaf $\mathfrak{F}\in X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ the cosheaf $\mathfrak{F}|_Y$ belongs to $Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$.*

(b) *In the situation of* (a)*, the corestriction map $\mathfrak{F}[Y]\longrightarrow\mathfrak{F}[X]$ is injective. If the scheme X is affine, then $0\longrightarrow\mathfrak{F}[Y]\longrightarrow\mathfrak{F}[X]\longrightarrow\mathfrak{F}[X]/\mathfrak{F}[Y]\longrightarrow 0$ is a split short exact sequence of flat cotorsion $\mathcal{O}(X)$-modules.*

(c) *Let $X=\bigcup_\alpha Y_\alpha$ be an open covering. Then a locally contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ if and only if its restrictions to Y_α belong to $Y_\alpha\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ for all α.*

Proof. Part (a): let j denote the open embedding $Y\longrightarrow X$. Then by Theorem 5.1.1 one has $j^!\mathfrak{F}\simeq j^!(\prod_{x\in X}\iota_{x!}\check{F}_x)\simeq\prod_{x\in X}j^!\iota_{x!}\check{F}_x$. Furthermore, it was explained in the proof of the same Theorem that $j^!\iota_{z!}\check{F}_z=0$ for any $z\in X\setminus Y$. Denoting by κ_y the natural morphism $\operatorname{Spec}\mathcal{O}_{y,X}\longrightarrow Y$ for a point $y\in Y$, one clearly has $j^!\iota_{y!}\check{F}_y\simeq\kappa_{y!}\check{F}_y$. Hence the isomorphism $j^!\mathfrak{F}\simeq\prod_{y\in Y}\kappa_y\check{F}_y$, proving that $j^!\mathfrak{F}\in Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$.

Part (b): by the definition, one has $(\iota_{x!}\check{F}_x)[X]\simeq F_x$. Since the cosections over a quasi-compact quasi-separated scheme commute with infinite products of contraherent cosheaves, we conclude from the above computation that $\mathfrak{F}[U]\simeq\prod_{x\in U}F_x$ for any quasi-compact open subscheme $U\subset X$. The cosheaf axiom (5) now implies the isomorphism $\mathfrak{F}[X]\simeq\varinjlim_{U\subset X}\mathfrak{F}[U]$, the filtered inductive limit being taken over all quasi-compact open subschemes $U\subset X$, and similarly for $\mathfrak{F}[Y]$. So the corestriction map $\mathfrak{F}[Y]\longrightarrow\mathfrak{F}[X]$ is the embedding of a direct summand.

Part (c): the "only if" assertion follows from part (a); let us prove the "if". A transfinite induction in (any well-ordering of) the set of indices α reduces the question to the following assertion. Suppose X is presented as the union of two open subschemes $W \cup Y$. Furthermore, the restriction of a locally contraherent cosheaf $\mathfrak{F}$ on X onto W is identified with the direct product $\prod_{w \in W} \kappa_{w!} \breve{F}_w$, where F_w are some free $\widehat{\mathcal{O}}_{w,X}$-contramodules, while κ_w are the natural morphisms $\operatorname{Spec} \mathcal{O}_{w,X} \longrightarrow W$. Assume also that the restriction of $\mathfrak{F}$ onto Y is a projective locally cotorsion contraherent cosheaf. Then there exist some free $\widehat{\mathcal{O}}_{x,X}$-contramodules F_z defined for all $z \in X \setminus W$ and an isomorphism of locally contraherent cosheaves $\mathfrak{F} \simeq \prod_{x \in X} \iota_{x!} \breve{F}_x$ whose restriction to W coincides with the given isomorphism $\mathfrak{F}|_W \simeq \prod_{w \in W} \kappa_{w!} \breve{F}_w$.

Indeed, by Theorem 5.1.1 we have $\mathfrak{F}|_Y \simeq \prod_{y \in Y} \iota'_{y!} \breve{F}'_y$, where ι'_y denote the natural morphisms $\operatorname{Spec} \mathcal{O}_{y,X} \longrightarrow Y$ and F'_y are some free contramodules over $\widehat{\mathcal{O}}_{y,X}$. Restricting to the intersection $V = W \cap Y$, we obtain an isomorphism of contraherent cosheaves $\prod_{v \in V} \kappa'_{v!} \breve{F}_v \simeq \prod_{v \in V} \kappa'_{v!} \breve{F}'_v$ on V, where κ'_v denotes the natural morphisms $\operatorname{Spec} \mathcal{O}_{v,X} \longrightarrow V$. It is clear from the arguments in the second half of the proof of Theorem 5.1.1(b) that such an isomorphism of infinite products induces an isomorphism of $\widehat{\mathcal{O}}_{v,X}$-contramodules $F_v \simeq F'_v$. Let us identify F'_v with F_v using this isomorphism, and set $F_y = F'_y$ for $y \in X \setminus W \subset Y$. Then our isomorphism of infinite products can be viewed as an automorphism ϕ of the contraherent cosheaf $\prod_{v \in V} \kappa'_{v!} \breve{F}_v$ on V. We would like to show that the cosheaf $\mathfrak{F}$ is isomorphic to $\prod_{x \in X} \iota_{x!} \breve{F}_x$.

Since a cosheaf of $\mathcal{O}_X$-modules is determined by its restrictions to W and Y together with the induced isomorphism between the restrictions of these to the intersection $V = W \cap Y$, it suffices to check that the automorphism ϕ can be lifted to an automorphism of the contraherent cosheaf $\prod_{v \in V} \iota'_v \breve{F}_v$ on Y. In fact, the rings of endomorphisms of the two contraherent cosheaves are isomorphic. Indeed, the functor of direct image of cosheaves of $\mathcal{O}$-modules $j_!$ with respect to the open embedding $j \colon V \longrightarrow Y$ is fully faithful by (11); the morphism j being quasi-compact and quasi-separated, this functor also preserves infinite products. $\square$

Corollary 5.1.4. *The classes of projective locally cotorsion contraherent cosheaves and* $\mathbf{W}$*-flat locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaves on* X *coincide. In particular, any* $\mathbf{W}$*-flat locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf on* X *is flat, contraherent, and colocally flat.*

Proof. The inclusion in one direction is provided by Corollaries 4.4.5(b) and 5.1.3(a). To prove the converse, pick an affine open covering U_α of the scheme X subordinate to $\mathbf{W}$. Then for any $\mathbf{W}$-flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X the $\mathcal{O}_X(U_\alpha)$-modules $\mathfrak{F}[U_\alpha]$ are, by the definition, flat and cotorsion. Hence the locally cotorsion contraherent cosheaves $\mathfrak{F}|_{U_\alpha}$ are projective, and it remains to apply Corollary 5.1.3(c). $\square$

Corollary 5.1.5. *The full subcategory* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ *is closed with respect to infinite products in* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ *or* $X\text{–}\mathsf{ctrh}$.

Proof. Easily deduced either from Corollary 5.1.4, or directly from Theorem 5.1.1(b) (cf. [55, Lemma 1.3.7]). ☐

Corollary 5.1.6. *Let $f\colon Y \longrightarrow X$ be a quasi-compact morphism of locally Noetherian schemes. Then*

(a) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes projective locally cotorsion contraherent cosheaves on Y to locally cotorsion contraherent cosheaves on X;*

(b) *if the morphism f is flat, then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes projective locally cotorsion contraherent cosheaves on Y to projective locally cotorsion contraherent cosheaves on X.*

Proof. Part (a): let us show that the functor $f_!$ takes locally cotorsion contraherent cosheaves of the form $\mathfrak{P} = \prod_{y\in Y} \kappa_{y!}\check{P}_y$ on Y, where κ_y are the natural morphisms $\operatorname{Spec}\mathcal{O}_{y,Y} \longrightarrow Y$ and P_y are some $\widehat{\mathcal{O}}_{y,Y}$-contramodules, to locally cotorsion contraherent cosheaves of the same form on X. More precisely, one has $f_!\mathfrak{P} \simeq \prod_{x\in X} \iota_{x!}\check{P}_x$, where ι_x are the natural morphisms $\operatorname{Spec}\mathcal{O}_{x,X} \longrightarrow X$ and $P_x = \prod_{f(y)=x} P_y$, the $\widehat{\mathcal{O}}_{x,X}$-contramodule structures on P_y being obtained by the (contra)restriction of scalars with respect to the homomorphisms of complete Noetherian rings $\widehat{\mathcal{O}}_{x,X} \longrightarrow \widehat{\mathcal{O}}_{y,Y}$ (see [55, Section 1.8]).

Indeed, the morphism f being quasi-compact and quasi-separated, the functor $f_!$ preserves infinite products of cosheaves of $\mathcal{O}$-modules. So it suffices to consider the case of a locally cotorsion contraherent cosheaf $\kappa_{y!}\check{P}_y$ on Y. Now the composition of morphisms of schemes $\operatorname{Spec}\mathcal{O}_{y,Y} \longrightarrow Y \longrightarrow X$ is equal to the composition $\operatorname{Spec}\mathcal{O}_{y,Y} \longrightarrow \operatorname{Spec}\mathcal{O}_{x,X} \longrightarrow X$, and it remains to use the compatibility of the direct images of cosheaves of $\mathcal{O}$-modules with the compositions of morphisms of schemes. Alternatively, part (a) is a particular case of Corollary 3.4.8(b).

To deduce part (b), one can apply the assertion that an $\widehat{\mathcal{O}}_{x,X}$-contramodule is projective if and only if it is a flat $\mathcal{O}_{x,X}$-module [55, Corollary B.8.2(c)]. Alternatively, use the adjunction (26) together with exactness of the inverse image of locally cotorsion locally contraherent cosheaves with respect to a flat morphism of schemes. ☐

5.2. **Flat contraherent cosheaves.** In this section we complete our treatment of flat contraherent cosheaves on locally Noetherian schemes, which was started in Sections 1.6 and 3.7 and continued in Sections 4.3 and 5.1.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on a scheme X is said to have *locally cotorsion dimension not exceeding* d if the cotorsion dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{M}[U]$ does not exceed d for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ (cf. Sections 1.5 and 4.9). Clearly, the locally cotorsion dimension of a $\mathbf{W}$-locally contraherent cosheaf does not change when the covering $\mathbf{W}$ is replaced by its refinement (see Lemma 1.5.4). If a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ has a right resolution by locally cotorsion $\mathbf{W}$-locally contraherent cosheaves in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, then its locally cotorsion dimension is equal to the minimal length of such resolution.

Lemma 5.2.1. *Let X be a semi-separated Noetherian scheme of Krull dimension D with an open covering $\mathbf{W}$. Then*

(a) *the right homological dimension of any $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ does not exceed D;*

(b) *the right homological dimension of any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of projective locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}}$ does not exceed D.*

Proof. The locally cotorsion dimension of any locally contraherent cosheaf on a locally Noetherian scheme of Krull dimension D does not exceed D by Corollary 1.5.7. The right homological dimension in part (a) is well-defined due to Corollary 4.2.5(a) or 4.3.4(a) and the results of Section 3.1, hence it is obviously equal to the locally cotorsion dimension. The right homological dimension in part (b) is well-defined by Corollaries 4.3.4(a), 4.3.5–4.3.6, and 5.1.4, so (b) follows from (a) in view of the dual version of Corollary A.5.3. □

Corollary 5.2.2. (a) *On a semi-separated Noetherian scheme X of finite Krull dimension, the classes of $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves and colocally flat contraherent cosheaves coincide.*

(b) *On a locally Noetherian scheme X of finite Krull dimension, any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf is flat and contraherent, so the categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ coincide.*

(c) *On a locally Noetherian scheme X of finite Krull dimension, any flat contraherent cosheaf $\mathfrak{F}$ is coflasque, so one has $X\text{–}\mathsf{ctrh}^{\mathsf{fl}} \subset X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$. If X is affine and $Y \subset X$ is an open subscheme, then $0 \longrightarrow \mathfrak{F}[Y] \longrightarrow \mathfrak{F}[X] \longrightarrow \mathfrak{F}[X]/\mathfrak{F}[Y] \longrightarrow 0$ is a short exact sequence of flat contraadjusted $\mathcal{O}(X)$-modules.*

Proof. Part (a): any colocally flat contraherent cosheaf on X is flat by Corollary 4.3.6 (see also Corollary 4.3.5). On the other hand, by Lemma 5.2.1(b), any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X has a finite right resolution by cosheaves from $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$, which are colocally flat by definition. By Corollary 4.3.2(b), it follows that $\mathfrak{F}$ is colocally flat.

Part (b): given an affine open subscheme $U \subset X$, denote by $\mathbf{W}|_U$ the collection of all open subsets $U \cap W$ with $W \in \mathbf{W}$. Then the restriction $\mathfrak{F}|_U$ of any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X onto U is, by definition, $\mathbf{W}|_U$-flat and $\mathbf{W}|_U$-locally contraherent. Applying part (a), we conclude that the cosheaf $\mathfrak{F}|_U$ is contraherent and flat. This being true for any affine open subscheme $U \subset X$ means precisely that $\mathfrak{F}$ is contraherent and flat on X.

Part (c): coflasqueness being a local property by Lemma 3.4.1(a), it suffices to consider the case of an affine scheme X. Then we have seen above that $\mathfrak{F}$ has a finite right resolution by cosheaves from $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$, which have the properties listed in part (c) by Corollary 5.1.3(b). By Corollary 3.4.4(b), our resolution is an exact sequence in $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$ and the cosheaf $\mathfrak{F}$ is coflasque.

Furthermore, by Corollary 3.4.4(c), the related sequence of cosections over any open subscheme $Y \subset X$ is exact. Passing to the corestriction maps related to the pair of embedded open subschemes $Y \subset X$, we obtain an injective morphism of exact sequences of $\mathcal{O}(X)$-modules; so the related sequence of cokernels is also exact. All of its terms except perhaps the leftmost one being flat, it follows that the lefmost term $\mathfrak{F}[X]/\mathfrak{F}[Y]$ is a flat $\mathcal{O}(X)$-module, too. The $\mathcal{O}(X)$-module $\mathfrak{F}[Y]$ is contraadjusted by Corollary 3.4.8(a) applied to the embedding morphism $Y \longrightarrow X$. $\square$

Lemma 5.2.3. *Let $f\colon Y \longrightarrow X$ be a flat quasi-compact morphism from a semi-separated locally Noetherian scheme Y of finite Krull dimension to a locally Noetherian scheme X. Then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes flat contraherent cosheaves on Y to flat contraherent cosheaves on X, and induces an exact functor $Y\text{–}\mathsf{ctrh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$.*

Proof. Clearly, it suffices to consider the case of a Noetherian affine scheme X. Then the scheme Y is Noetherian and semi-separated. By Corollary 5.2.2(a), any flat contraherent cosheaf on Y is colocally flat. By Corollary 4.5.3(c), the functor $f_!$ restricts to an exact functor $Y\text{–}\mathsf{ctrh}_{\mathsf{clf}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$. By Corollary 4.3.6, any colocally flat contraherent cosheaf on X is flat. (A proof working in a greater generality will be given below in Corollary 5.2.10(b).) $\square$

Let X be a Noetherian scheme of finite Krull dimension D with an open covering $\mathbf{W}$ and a finite affine open covering U_α subordinate to $\mathbf{W}$.

Corollary 5.2.4. (a) *There are enough projective objects in the exact categories of locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}$ on X, and all these projective objects belong to the full subcategory of contraherent cosheaves $X\text{–}\mathsf{ctrh}$. The full subcategories of projective objects in the three exact categories $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}$ coincide.*

(b) *A contraherent cosheaf on X is projective if and only if it is isomorphic to a finite direct sum of the direct images of projective contraherent cosheaves from U_α. In particular, any projective contraherent cosheaf on X is flat (and consequently, coflasque).*

Proof. The argument is similar to the proofs of Lemmas 4.3.3, 4.4.1, and 4.4.3; the only difference is that, the scheme X being not necessarily semi-separated, one has to also use Lemma 5.2.3. Let $j_\alpha\colon U_\alpha \longrightarrow X$ denote the open embedding morphisms. According to Corollary 4.4.5(a) and Lemma 5.2.3, the cosheaf of $\mathcal{O}_X$-modules $j_{\alpha!}\mathfrak{F}_\alpha$ is flat and contraherent for any projective contraherent cosheaf $\mathfrak{F}_\alpha$ on U_α. The adjunction (11) or (26) shows that it is also a projective object in $X\text{–}\mathsf{lcth}$.

It remains to show that there are enough projective objects of the form $\bigoplus_\alpha j_{\alpha!}\mathfrak{F}_\alpha$ in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. Let $\mathfrak{Q}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. For every α, pick an admissible epimorphism onto the contraherent cosheaf $j_\alpha^!\mathfrak{Q}$ from a projective contraherent cosheaf $\mathfrak{F}_\alpha$ on U_α. Then the same adjunction provides a natural morphism $\bigoplus_\alpha j_{\alpha!}\mathfrak{F}_\alpha \longrightarrow \mathfrak{Q}$. This morphism is an admissible epimorphism of locally contraherent cosheaves, because it is so in restriction to each open subscheme $U_\alpha \subset X$. $\square$

As in Section 4.4, we denote the category of projective contraherent cosheaves on X by $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$. Clearly, the objects of $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ are the projective objects of the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ (and there are enough of them). We will see below in this section that the objects of $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ are the *injective* objects of $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ (and there are enough of them).

Lemma 5.2.5. (a) *Any coflasque contraherent cosheaf* $\mathfrak{E}$ *on* X *can be included in an exact triple* $0 \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$ *in* $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$*, where* $\mathfrak{P}$ *is a coflasque locally cotorsion contraherent cosheaf on* X *and* $\mathfrak{F}$ *is a finitely iterated extension of the direct images of flat contraherent cosheaves from* U_α*.*

(b) *The right homological dimension of any coflasque contraherent cosheaf on* X *with respect to the exact subcategory of coflasque locally cotorsion contraherent cosheaves* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \subset X\text{–}\mathsf{ctrh}^{\mathsf{cfq}}$ *does not exceed* D*.*

Proof. Part (a): the proof is similar to that of Lemma 4.3.1; the only difference is that, the scheme X being not necessarily semi-separated, one has to also use Lemma 3.4.8. Notice that flat contraherent cosheaves on U_α are coflasque by Lemma 5.2.2(c) and finitely iterated extensions of coflasque contraherent cosheaves in $X\text{–}\mathsf{ctrh}$ are coflasque by Lemma 3.4.4(a).

One proceeds by induction in a linear ordering of the indices α, considering the open subscheme $V = \bigcup_{\alpha<\beta} U_\alpha$. Assume that we have constructed an exact triple $\mathfrak{E} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{L}$ of coflasque contraherent cosheaves on X such that the contraherent cosheaf $\mathfrak{K}|_V$ is locally cotorsion, while the cosheaf $\mathfrak{L}$ on X is a finitely iterated extension of the direct images of flat contraherent cosheaves from the affine open subschemes U_α with $\alpha < \beta$. Let $j\colon U = U_\beta \longrightarrow X$ be the identity open embedding.

Pick an exact triple $j^!\mathfrak{K} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{G}$ of contraherent cosheaves on U such that $\mathfrak{G}$ is a flat contraherent cosheaf (see Theorem 1.3.1(a)); then the cosheaf $\mathfrak{G}$ is coflasque by Lemma 5.2.2(c) and the cosheaf $\mathfrak{Q}$ is coflasque by Lemma 3.4.4(a). By Lemma 3.4.8(a), the related sequence of direct images $j_!j^!\mathfrak{K} \longrightarrow j_!\mathfrak{Q} \longrightarrow j_!\mathfrak{G}$ is an exact triple of coflasque contraherent cosheaves on X; by part (b) of the same lemma, the cosheaf $j_!\mathfrak{Q}$ belongs to $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$.

Let $\mathfrak{K} \longrightarrow \mathfrak{N} \longrightarrow j_!\mathfrak{G}$ denote the push-forward of the exact triple $j_!j^!\mathfrak{K} \longrightarrow j_!\mathfrak{Q} \longrightarrow j_!\mathfrak{G}$ with respect to the natural morphism $j_!j^!\mathfrak{K} \longrightarrow \mathfrak{K}$. We will show that the coflasque contraherent cosheaf $\mathfrak{N}$ on X is locally cotorsion in restriction to $U \cup V$.

Indeed, in the restriction to U one has $j^!\mathfrak{N} \simeq \mathfrak{Q}$. On the other hand, denoting by j' the embedding $U \cap V \longrightarrow V$, one has $(j_!\mathfrak{G})|_V \simeq j'_!(\mathfrak{G}|_{U\cap V})$. The contraherent cosheaf $\mathfrak{K}|_{U\cap V}$ being locally cotorsion, so is the cokernel $\mathfrak{G}|_{U\cap V}$ of the admissible monomorphism of locally cotorsion contraherent cosheaves $\mathfrak{K}|_{U\cap V} \longrightarrow \mathfrak{Q}|_{U\cap V}$.

By Lemma 3.4.8(b), $j'_!(\mathfrak{G}|_{U\cap V})$ is a coflasque locally cotorsion contaherent cosheaf on V. Now in the exact triple $\mathfrak{K}|_V \longrightarrow \mathfrak{N}|_V \longrightarrow (j_!\mathfrak{G})_V$ the middle term is locally cotorsion, since the two other terms are.

Finally, the composition $\mathfrak{E} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{N}$ of admissible monomorphisms in $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$ is again an admissible monomorphism with the cokernel isomorphic to an extension of the flat contraherent cosheaves $j_!\mathfrak{G}$ and $\mathfrak{L}$.

Part (b): the right homological dimension is well-defined by part (a) and does not exceed D for the reasons explained in the proof of Lemma 5.2.1. $\square$

Corollary 5.2.6. (a) *Any flat contraherent cosheaf* $\mathfrak{G}$ *on* X *can be included in an exact triple* $0 \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$ *in* $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$, *where* $\mathfrak{P}$ *is a projective locally cotorsion contraherent cosheaf on* X *and* $\mathfrak{F}$ *is a finitely iterated extension of the direct images of flat contraherent cosheaves from* U_α.

(b) *The right homological dimension of any flat contraherent cosheaf on* X *with respect to the exact subcategory of projective locally cotorsion contraherent cosheaves* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}} \subset X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ *does not exceed* D*; the homological dimension of the exact category* $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ *does not exceed* D*; and the left homological dimension of any flat contraherent cosheaf on* X *with respect to the exact category of projective contraherent cosheaves* $X\text{–}\mathsf{ctrh}_{\mathsf{prj}} \subset X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ *does not exceed* D.

Proof. Part (a) follows from Lemma 5.2.5(a) together with Corollaries 5.2.2(c) and 5.1.4. Part (b): the right homological dimension is well-defined by part (a) and does not exceed D by Lemma 5.2.5(b) and the dual version of Corollary A.5.3. By the dual version of Proposition A.3.1(a) or A.5.6, it follows that the natural functor $\mathsf{Hot}^+(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ is fully faithful (the exact category structure on $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ being trivial). Applying the first assertion of part (b) again, we conclude that the homological dimension of the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ does not exceed D, and consequently that the left homological dimension of any object of $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ with respect to its subcategory of projective objects $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ does not exceed D. ☐

Lemma 5.2.7. *Any flat contraherent cosheaf* $\mathfrak{G}$ *on* X *can be included in an exact triple* $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow 0$ *in* $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$, *where* $\mathfrak{P}$ *is a projective locally cotorsion contraherent cosheaf on* X *and* $\mathfrak{F}$ *is a finitely iterated extension of the direct images of flat contraherent cosheaves from* U_α.

Proof. According to Corollary 5.2.4, there is an admissible epimorphism $\mathfrak{E} \longrightarrow \mathfrak{G}$ in the exact category $X\text{–}\mathsf{ctrh}$ onto any given contraherent cosheaf $\mathfrak{G}$ from a finite direct sum $\mathfrak{E}$ of the direct images of flat (and even projective) contraherent cosheaves from U_α. Furthermore, any admissible epimorphism in $X\text{–}\mathsf{ctrh}$ between objects from $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ is also an admissible epimorphism in $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. The rest of the argument is similar to the proofs of Lemmas 4.3.3 and 4.2.4, and based on Corollary 5.2.6(a) (applied to the kernel of the morphism $\mathfrak{E} \longrightarrow \mathfrak{G}$). ☐

The following corollary is to be compared with Corollary 4.11.2 above; see also Corollaries 5.3.3, 5.4.4, and Theorem 5.4.10 below.

Corollary 5.2.8. (a) *For any Noetherian scheme* X *of finite Krull dimension, the natural functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$, $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ *are equivalences of triangulated categories, as are the natural functors* $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$, $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$.

(b) *For any locally Noetherian scheme* X *with an open covering* $\mathbf{W}$, *the natural functors* $\mathsf{Hot}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *are equivalences of triangulated categories.*

(c) *For any Noetherian scheme X of finite Krull dimension with an open covering $\mathbf{W}$, the natural functors* $\mathsf{D}^-(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth})$ *are equivalences of triangulated categories.*

(d) *For any Noetherian scheme X of finite Krull dimension with an open covering $\mathbf{W}$, the natural functors* $\mathsf{D}^-(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *and* $\mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth})$ *are equivalences of triangulated categories.*

Proof. In view of Corollary 5.2.6(b), all assertions of part (a) follow from Proposition A.5.6 (together with its dual version) and [52, Remark 2.1]. Part (b) is provided by Proposition A.3.1(a) together with Theorem 5.1.1(a), while part (c) follows from the same Proposition together with Corollary 5.2.4. Finally, part (d) is obtained by comparing parts (a-c). ☐

It follows from Corollary 5.2.8 that the Ext groups computed in the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ agree with each other and with the Ext groups computed in the exact categories $X\text{–}\mathsf{lcth}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$. Besides, these also agree with the Ext groups computed in the exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ (the latter being endowed with the trivial exact category structure). As in Section 4.2, we denote these Ext groups by $\mathrm{Ext}^{X,*}(-,-)$.

Corollary 5.2.9. (a) *Let X be a Noetherian scheme of finite Krull dimension with an open covering $\mathbf{W}$. Then one has $\mathrm{Ext}^{X,>0}(\mathfrak{F},\mathfrak{Q}) = 0$ for any flat contraherent cosheaf $\mathfrak{F}$ and locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Q}$ on X. Consequently, any flat contraherent cosheaf on X is colocally flat.*

(b) *Let X be a Noetherian scheme of finite Krull dimension with a finite affine open covering $X = \bigcup_\alpha U_\alpha$. Then a contraherent cosheaf on X is flat if and only if it is a direct summand of a finitely iterated extension of the direct images of flat contraherent cosheaves from U_α.*

Proof. According to Corollary 5.2.6(b), any flat contraherent cosheaf on X has a finite right resolution by projective locally cotorsion contraherent cosheaves. Since the Ext groups in the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ agree, the assertion (a) follows. Now "only if" assertion in part (b) is easily deduced from Lemma 5.2.7 together with part (a), while the "if" is provided by Lemma 5.2.3. ☐

The following corollary is to be compared with Corollaries 3.4.8, 4.4.7, 4.5.3, 4.5.4, and 5.1.6.

Corollary 5.2.10. *Let $f\colon Y \longrightarrow X$ be a quasi-compact morphism of locally Noetherian schemes such that the scheme Y has finite Krull dimension. Then*

(a) *the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes flat contraherent cosheaves on Y to contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{ctrh}$ between these exact categories;*

(b) *if the morphism f is flat, then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes flat contraherent cosheaves on Y to flat contraherent cosheaves on X, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ between these exact categories;*

(c) *if the scheme Y is Noetherian and the morphism f is very flat, then the functor of direct image of cosheaves of $\mathcal{O}$-modules $f_!$ takes projective contraherent cosheaves on Y to projective contraherent cosheaves on X.*

Proof. Part (a) is a particular case of Corollary 3.4.8(a). In part (b), one can assume that X is an affine scheme, so the scheme Y is Noetherian of finite Krull dimension. It suffices to show that the $\mathcal{O}(X)$-module $(f_!\mathfrak{F})[X] = \mathfrak{F}[Y]$ is flat for any flat contraherent cosheaf $\mathfrak{F}$ on X. For this purpose, consider a finite right resolution of the cosheaf $\mathfrak{F}$ by objects from $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ in $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$, and apply the functor $\Delta(Y,-)$ to it. It follows from Corollaries 5.2.2(c) and 3.4.4(c) that the sequence will remain exact. By Corollary 5.1.6(b), we obtain a finite right resolution of the module $\mathfrak{F}[Y]$ by flat $\mathcal{O}(X)$-modules, implying that the $\mathcal{O}(X)$-module $\mathfrak{F}[Y]$ is also flat. Part (c) follows from part (a) or (b) together with the adjunction (26). □

5.3. **Homology of locally cotorsion locally contraherent cosheaves.** Let X be a locally Noetherian scheme. Then the left derived functor of the functor of global cosections $\Delta(X,-)$ of locally cotorsion locally contraherent cosheaves on X is defined using left projective resolutions in the exact category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ (see Theorem 5.1.1(a)).

Notice that the derived functors $\mathbb{L}_*\Delta(X,-)$ computed in the exact category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ for a particular open covering $\mathbf{W}$ and in the whole category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ agree. The groups $\mathbb{L}_i\Delta(X,\mathfrak{C})$ are called the *homology groups* of a locally cotorsion locally contraherent cosheaf $\mathfrak{C}$ on the scheme X.

Let us show that $\mathbb{L}_{>0}\Delta(X,\mathfrak{F}) = 0$ for any coflasque locally cotorsion contraherent cosheaf $\mathfrak{F}$. By Corollary 5.1.3(b), any projective locally cotorsion contraherent cosheaf on X is coflasque. In view of Corollary 3.4.4(b), any resolution of an object of $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$ by objects of $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ in the category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ is exact with respect to the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$. By part (c) of the same Corollary, the functor $\Delta(X,-)$ preserves exactness of such sequences. Hence one can compute the derived functor $\mathbb{L}_*\Delta(X,-)$ using coflasque locally cotorsion contraherent resolutions.

Similarly, let X be a Noetherian scheme of finite Krull dimension. Then the left derived functor of the functor of global cosections $\Delta(X,-)$ of locally contraherent cosheaves on X is defined using left projective resolutions in the exact category $X\text{–}\mathsf{lcth}$ (see Corollary 5.2.4). The derived functors $\mathbb{L}_*\Delta(X,-)$ computed in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ for any prarticular open covering $\mathbf{W}$ and in the whole category $X\text{–}\mathsf{lcth}$ agree. Furthermore, one can compute the derived functor $\mathbb{L}_*\Delta(X,-)$ using coflasque left resolutions. The groups $\mathbb{L}_i\Delta(X,\mathfrak{C})$ are called the *homology groups* of a locally contraherent cosheaf $\mathfrak{C}$ on the scheme X.

It is clear from the above that the two definitions agree when they are both applicable; they also agree with the definitions given in Section 4.5 (cf. Lemma 4.2.1).

Using injective resolutions, one can similarly define the cohomology of quasi-coherent sheaves $\mathbb{R}^*\Gamma(X,-)$ on a locally Noetherian scheme X. Injective quasi-coherent sheaves on X being flasque, this definition agrees with the conventional sheaf-theoretical one. The full subcategory of flasque quasi-coherent sheaves in

X–qcoh is closed under extensions, cokernels of injective morphisms and infinite direct sums; we denote the induced exact category structure on it by X–$\mathsf{qcoh}^{\mathsf{fq}}$.

The following lemma is to be compared with Lemmas 4.6.1 and 4.6.2.

Lemma 5.3.1. (a) *Let X be a locally Noetherian scheme of Krull dimension D. Then the right homological dimension of any quasi-coherent sheaf on X with respect to the exact subcategory of flasque quasi-coherent sheaves X–$\mathsf{qcoh}^{\mathsf{fq}} \subset X$–$\mathsf{qcoh}$ does not exceed D.*

(b) *Let X be a locally Noetherian scheme of Krull dimension D. Then the left homological dimension of any locally cotorsion locally contraherent cosheaf on X with respect to the exact subcategory of coflasque locally cotorsion contraherent cosheaves X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \subset X$–$\mathsf{lcth}^{\mathsf{lct}}$ does not exceed D. Consequently, the same bound holds for the left homological dimension of any object of X–$\mathsf{lcth}^{\mathsf{lct}}$ with respect to the exact subcategory X–$\mathsf{ctrh}^{\mathsf{lct}}$.*

(c) *Let X be a Noetherian scheme of Krull dimension D. Then the left homological dimension of any locally contraherent cosheaf on X with respect to the exact subcategory of coflasque contraherent cosheaves X–$\mathsf{ctrh}_{\mathsf{cfq}} \subset X$–$\mathsf{lcth}$ does not exceed D. Consequently, the same bound holds for the left homological dimension of any object of X–lcth with respect to the exact subcategory X–ctrh.*

Proof. Follows from Lemma 3.4.7 and Corollary 3.4.2. ☐

Corollary 5.3.2. (a) *Let X be a locally Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, or abs the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fq}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ induced by the embedding of exact categories X–$\mathsf{qcoh}^{\mathsf{fq}} \longrightarrow X$–$\mathsf{qcoh}$ is an equivalence of triangulated categories.*

(b) *Let X be a locally Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ induced by the embedding of exact categories X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \longrightarrow X$–$\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is an equivalence of triangulated categories.*

For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ is an equivalence of categories.

(c) *Let X be a Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ induced by the embedding of exact categories X–$\mathsf{ctrh}_{\mathsf{cfq}} \longrightarrow X$–$\mathsf{lcth}_{\mathbf{W}}$ is an equivalence of triangulated categories.*

For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth})$ is an equivalence of categories.

Proof. Follows from Lemma 5.3.1 together with Proposition A.5.6. ☐

Corollary 5.3.3. (a) *Let X be a locally Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ induced by the embedding of exact categories X–$\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow X$–$\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is an equivalence of categories.*

For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *is an equivalence of categories.*

(b) *Let X be a Noetherian scheme of finite Krull dimension. Then for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *is an equivalence of triangulated categories.*

For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth})$ *is an equivalence of categories.*

Proof. Follows from Lemma 5.3.1(b-c) (see Corollaries 4.6.3–4.6.5 for comparison). □

The results below in this section purport to replace the above homological dimension estimates based on the Krull dimension with the ones based on the numbers of covering open affines.

Lemma 5.3.4. *Let X be a Noetherian scheme and $X = \bigcup_{\alpha=1}^N U_\alpha$ be its finite affine open covering. For each subset $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant N$ of indices $\{\alpha\}$, let $U_{\alpha_1} \cap \cdots \cap U_{\alpha_k} = \bigcup_{\beta=1}^n V_\beta$, where $n = n_{\alpha_1,\dots,\alpha_k}$, be a finite affine open covering of the intersection. Let M denote the supremum of the expressions $k - 1 + n_{\alpha_1,\dots,\alpha_k}$ taken over all the nonempty subsets of indices α_1, …, α_k. Then one has*

(a) $\mathbb{R}^{\geqslant M}\Gamma(X, \mathcal{E}) = 0$ *for any quasi-coherent sheaf $\mathcal{E}$ on X;*

(b) $\mathbb{L}_{\geqslant M}\Delta(X, \mathfrak{E}) = 0$ *for any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{E}$ on X, provided that the affine open covering $\{U_\alpha\}$ is subordinate to $\mathbf{W}$. Assuming additionally that the Krull dimension of X is finite, the same bound holds for any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{E}$ on X.*

Proof. We will prove part (b). The first assertion: let $\mathfrak{F}_\bullet \longrightarrow \mathfrak{E}$ be a left projective resolution of an object $\mathfrak{E} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. Consider the Čech resolution (27) for each cosheaf $\mathfrak{F}_i \in X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$. By Corollaries 5.1.3(a) and 5.1.6(b), this is a sequence of projective locally cotorsion contraherent cosheaves. Its restriction to each open subset $U_\alpha \subset X$ being naturally contractible, this finite sequence is exact in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$, and consequently also exact (i. e., even contractible) in $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$.

Denote the bicomplex of cosheaves we have obtained (without the rightmost term that is being resolved) by $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{F}_\bullet)$ and the corresponding bicomplex of the groups of global cosections by $C_\bullet(\{U_\alpha\}, \mathfrak{F}_\bullet)$. Now the total complex of the bicomplex $C_\bullet(\{U_\alpha\}, \mathfrak{F}_\bullet)$ is quasi-isomorphic (in fact, in this case even homotopy equivalent) to the complex $\Delta(X, \mathfrak{F}_\bullet)$ computing the homology groups $\mathbb{L}_*\Delta(X, \mathfrak{E})$.

On the other hand, for each $1 \leqslant k \leqslant N$, the complex $C_k(\{U_\alpha\}, \mathfrak{F}_\bullet)$ computes the direct sum of the homology of the cosheaves $j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{E}$ on $U_{\alpha_1} \cap \cdots \cap U_{\alpha_N}$ over all $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant N$. The schemes $U_{\alpha_1} \cap \cdots \cap U_{\alpha_N}$ being quasi-compact and separated, and the cosheaves $j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{E}$ being contraherent, the latter homology can be also computed by the Čech complexes $C_\bullet(\{V_\beta\}, j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{E})$ (see Section 4.5) and consequently vanish in the homological degrees $\geqslant n_1 + \cdots + n_k$.

To prove the second assertion, one uses a flat (or more generally, coflasque) left resolution of a contraherent cosheaf $\mathfrak{E}$ and argues as above using Corollary 5.2.10(b) (or Corollary 3.4.8(a), respectively). $\square$

Let us say that a locally cotorsion locally contraherent cosheaf $\mathfrak{P}$ on a locally Noetherian scheme X is *acyclic* if $\mathbb{L}_{>0}\Delta(X,\mathfrak{P})=0$. Acyclic locally cotorsion $\mathbf{W}$-locally contraherent cosheaves form a full subcategory in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ closed under extensions and kernels of admissible epimorphisms; when X is Noetherian (i. e., quasi-compact), this subcategory is also closed under infinite products. Hence it acquires the induced exact category structure, which we denote by $X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}$.

Similarly, we say that a locally contraherent cosheaf $\mathfrak{P}$ on a Noetherian scheme X of finite Krull dimension is *acyclic* if $\mathbb{L}_{>0}\Delta(X,\mathfrak{P})=0$. Acyclic $\mathbf{W}$-locally contraherent cosheaves form a full subcategory in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ closed under extensions, kernels of admissible epimorphisms, and infinite products. The induced exact category structure on this subcategory is denote by $X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}$. Clearly, one has $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}^{\mathsf{lct}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}$ (under the respective assumptions).

Finally, a quasi-coherent sheaf $\mathcal{P}$ on a locally Noetherian scheme X is said to be *acyclic* if $\mathbb{R}^{>0}\Gamma(X,\mathcal{P})=0$. Acyclic quasi-coherent sheaves form a full subcategory in $X\text{–}\mathsf{qcoh}$ closed under extensions and cokernels of injective morphisms; when X is Noetherian, this subcategory is also closed under infinite direct sums. The induced exact category structure on it is denoted by $X\text{–}\mathsf{qcoh}^{\mathsf{ac}}$.

Corollary 5.3.5. *Let X be a Noetherian scheme with an open covering $\mathbf{W}$ and M be the minimal possible value of the nonnegative integer defined in Lemma* 5.3.4 *(depending on an affine covering $X=\bigcup_\alpha X_\alpha$ subordinate to $\mathbf{W}$ and affine coverings of its intersections $X_{\alpha_1}\cap\dots\cap X_{\alpha_k}$). Then*

(a) *the right homological dimension of any quasi-coherent sheaf on X with respect to the exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{ac}}\subset X\text{–}\mathsf{qcoh}$ does not exceed $M-1$;*

(b) *the left homological dimension of any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ does not exceed $M-1$;*

(c) *assuming X has finite Krull dimension, the left homological dimension of any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ does not exceed $M-1$.*

Proof. Part (b): since there are enough projectives in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and these belong to $X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}$, the left homological dimension is well-defined. Now if $0\longrightarrow\mathfrak{Q}\longrightarrow\mathfrak{P}_{M-2}\longrightarrow\dotsb\longrightarrow\mathfrak{P}_0\longrightarrow\mathfrak{E}\longrightarrow 0$ is an exact sequence in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ with $\mathfrak{P}_i\in X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}$, then it is clear from Lemma 5.3.4(b) that $\mathfrak{Q}\in X\text{–}\mathsf{lcth}_{\mathbf{W},\mathsf{ac}}^{\mathsf{lct}}$. The proofs of parts (a) and (c) are similar. $\square$

Let $f\colon Y\longrightarrow X$ be a quasi-compact morphism of locally Noetherian schemes. By Corollary 5.2.8(b), the natural functor $\mathsf{Hot}^-(Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})\longrightarrow\mathsf{D}^-(Y\text{–}\mathsf{lcth}^{\mathsf{lct}})$ is an equivalence of triangulated categories. The derived functor $\mathbb{L}f_!\colon\mathsf{D}^-(Y\text{–}\mathsf{lcth}^{\mathsf{lct}})\longrightarrow\mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ is constructed by applying the functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}\longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ from

Corollary 5.1.6(a) termwise to bounded above complexes of projective locally cotorsion contraherent cosheaves. By Corollary 3.4.8(b), one can compute the derived functor $\mathbb{L}f_!$ using left resolutions of objects of $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$ by coflasque locally cotorsion contraherent cosheaves.

Similarly, if the scheme Y is Noetherian of finite Krull dimension, by Corollary 5.2.8(a), the natural functor $\mathsf{Hot}^-(Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{lcth})$ is an equivalence of triangulated categories. The derived functor $\mathbb{L}f_!\colon \mathsf{D}^-(Y\text{–}\mathsf{lcth}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth})$ is constructed by applying the functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{prj}} \longrightarrow X\text{–}\mathsf{ctrh}$ from Corollary 5.2.10(a) termwise to bounded above complexes of projective contraherent cosheaves. By Corollary 3.4.8(a), one can compute the derived functor $\mathbb{L}f_!$ using left resolutions by coflasque contraherent cosheaves.

Let $\mathbf{W}$ and $\mathbf{T}$ be open coverings of the schemes X and Y. We will call a locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y *acyclic with respect to f over* $\mathbf{W}$ (or $f/\mathbf{W}$*-acyclic*) if the object $\mathbb{L}f_!(\mathfrak{Q}) \in \mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ belongs to the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}} \subset \mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$. In other words, the complex $\mathbb{L}f_!(\mathfrak{Q})$ should have left homological dimension not exceeding 0 with respect to the exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ (in the sense of Section A.5).

Similarly, if the scheme Y is Noetherian of finite Krull dimension, we will call a $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y *acyclic with respect to f over* $\mathbf{W}$ (or $f/\mathbf{W}$*-acyclic* if the object $\mathbb{L}f_!(\mathfrak{Q}) \in \mathsf{D}^-(X\text{–}\mathsf{lcth})$ belongs to the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth} \subset \mathsf{D}^-(X\text{–}\mathsf{lcth})$. In other words, the complex $\mathbb{L}f_!(\mathfrak{Q})$ must have left homological dimension at most 0 with respect to the exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}$. According to Corollary A.5.3, an object of $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$ is $f/\mathbf{W}$-acyclic if and only if it is $f/\mathbf{W}$-acyclic as an object of $Y\text{–}\mathsf{lcth}_{\mathbf{T}}$. Any coflasque (locally cotorsion) contrarent cosheaf on Y is $f/\mathbf{W}$-acyclic.

It is easy to see that the full subcategory of $f/\mathbf{W}$-acyclic locally cotorsion $\mathbf{T}$-locally contraherent cosheaves in $Y\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ is closed under extensions, kernels of admissible epimorphisms, and infinite products. The full subcategory of $f/\mathbf{W}$-acyclic $\mathbf{T}$-locally contraherent cosheaves in $Y\text{–}\mathsf{lcth}_{\mathbf{W}}$ (defined under the appropriate assumptions above) has the same properties. We denote these subcategories with their induced exact category structures by $Y\text{–}\mathsf{lcth}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}}^{\mathsf{lct}}$ and $Y\text{–}\mathsf{lcth}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}}$, respectively.

Finally, the natural functor $\mathsf{Hot}^+(Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories by Corollary 4.11.2(a); and the derived functor $\mathbb{R}f_*\colon \mathsf{D}^+(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{qcoh})$ is constructed by applying the functor f_* termwise to bounded below complexes of injective quasi-coherent sheaves.

A quasi-coherent sheaf $\mathcal{Q}$ on Y is called *acyclic with respect to f* (or f*-acyclic*) if the object $\mathbb{R}f_*(\mathcal{Q}) \in \mathsf{D}^+(X\text{–}\mathsf{qcoh})$ belongs to the full subcategory $X\text{–}\mathsf{qcoh} \subset \mathsf{D}^+(X\text{–}\mathsf{qcoh})$. The full subcategory of f-acyclic quasi-coherent sheaves in $Y\text{–}\mathsf{qcoh}$ is closed under extensions, cokernels of injective morphisms, and infinite direct sums. We denote this exact subcategory by $Y\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{ac}}$.

Lemma 5.3.6. (a) *Let $\mathfrak{Q}$ be an $f/\mathbf{W}$-acyclic locally cotorsion $\mathbf{T}$-locally contraherent cosheaf on Y. Then the cosheaf of $\mathcal{O}_X$-modules $f_!\mathfrak{Q}$ is locally cotorsion $\mathbf{W}$-locally* *contraherent, and the object represented by it in the derived category* $\mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *is naturally isomorphic to* $\mathbb{L}f_!\mathfrak{Q}$.

(b) *Assuming that the scheme* Y *is Noetherian of finite Krull dimension, let* $\mathfrak{Q}$ *be an* $f/\mathbf{W}$*-acyclic* $\mathbf{T}$*-locally contraherent cosheaf on* Y*. Then the cosheaf of* $\mathcal{O}_X$*-modules* $f_!\mathfrak{Q}$ *is* $\mathbf{W}$*-locally contraherent, and the object represented by it in the derived category* $\mathsf{D}^-(X\text{–}\mathsf{lcth})$ *is naturally isomorphic to* $\mathbb{L}f_!\mathfrak{Q}$.

Proof. We will prove part (a), the proof of part (b) being similar. A complex $\cdots \longrightarrow \mathfrak{P}_2 \longrightarrow \mathfrak{P}_1 \longrightarrow \mathfrak{P}_0$ over $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ being isomorphic to an object $\mathfrak{P} \in X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ in $\mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ means that for each affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the complex of cotorsion $\mathcal{O}(U)$-modules $\cdots \longrightarrow \mathfrak{P}_2[U] \longrightarrow \mathfrak{P}_1[U] \longrightarrow \mathfrak{P}_0[U]$ is acyclic except at the rightmost term, its $\mathcal{O}(U)$-modules of cocycles are cotorsion, and the cokernel of the morphism $\mathfrak{P}_1 \longrightarrow \mathfrak{P}_0$ taken in the category of cosheaves of $\mathcal{O}_X$-modules, that is the cosheaf $U \longmapsto \operatorname{coker}(\mathfrak{P}_1[U] \to \mathfrak{P}_0[U])$, is identified with $\mathfrak{P}$.

Now let $\cdots \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{F}_1 \longrightarrow \mathfrak{F}_0$ be a left projective resolution of the object $\mathfrak{Q}$ in the exact category $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$; then the cosheaves $f_!\mathfrak{F}_i$ on X belong to $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$. It remains to notice that the functor $f_!$ preserves cokernels taken in the categories of cosheaves of $\mathcal{O}_Y$- and $\mathcal{O}_X$-modules. □

Lemma 5.3.7. *Let* $f\colon Y \longrightarrow X$ *be a morphism of Noetherian schemes with open coverings* $\mathbf{T}$ *and* $\mathbf{W}$*. Assume that either*

(a) $\mathbf{W}$ *is a finite affine open covering of* X*, or*

(b) *one of the schemes* X *or* Y *has finite Krull dimension.*

Then any locally cotorsion $\mathbf{T}$*-locally contraherent cosheaf on* Y *has finite left homological dimension with respect to the exact subcategory* $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}} \subset Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$.

Proof. Since $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}} \subset Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}}$, the left homological dimension with respect to the exact subcategory $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}} \subset Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ is well-defined for any quasi-compact morphism of locally Noetherian schemes $f\colon Y \longrightarrow X$. In view of Corollaries 5.1.6(a) and A.5.2, the left homological dimension of a cosheaf $\mathfrak{E} \in Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ with respect to $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}}$ does not exceed (in fact, is equal to) the left homological dimension of the complex $\mathbb{L}f_!(\mathfrak{E}) \in \mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$.

To prove part (a), denote by $M(Z,\mathbf{t})$ the minimal value of the nonnegative integer M defined in Lemma 5.3.4 and Corollary 5.3.5 for a given Noetherian scheme Z with an open covering $\mathbf{t}$. Let M be the maximal value of $M(f^{-1}(W),\mathbf{T}|_{f^{-1}(W)})$ taken over all affine open subschemes $W \in \mathbf{W}$. We will show that the left homological dimension of any object of $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ with respect to $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},\,f/\mathbf{W}\text{–}\mathsf{ac}}$ does not exceed $M-1$.

Set $Z = f^{-1}(W)$ and $\mathbf{t} = \mathbf{T}|_Z$. It suffices to check that for any Noetherian affine scheme W, a Noetherian scheme Z with an open covering $\mathbf{t}$, a morphism of schemes $g\colon Z \longrightarrow W$, and a cosheaf $\mathfrak{E} \in Z\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{t}}$, the complex of cotorsion $\mathcal{O}(W)$-modules $(\mathbb{L}g_!\mathfrak{E})[W]$ is isomorphic to a complex of cotorsion $\mathcal{O}(W)$-modules concentrated in the homological degrees $\leqslant M-1$ in the derived category $\mathcal{O}(W)\text{–}\mathsf{mod}^{\mathsf{cot}}$. The argument below follows the proof of Lemma 5.3.4.

Let $\mathfrak{F}_\bullet \longrightarrow \mathfrak{E}$ be a left projective resolution of the locally cotorsion locally contraherent cosheaf $\mathfrak{E}$ and U_α be a finite affine covering of the scheme Z subordinate to $\mathbf{t}$. Then the total complex of the bicomplex $C_\bullet(\{U_\alpha\}, \mathfrak{F}_\bullet)$ is homotopy equivalent to the complex $(g_!\mathfrak{F}_\bullet)[W]$ computing the object $(\mathbb{L}g_!\mathfrak{E})[W] \in \mathsf{D}^-(\mathcal{O}(W)\text{–}\mathsf{mod}^{\mathsf{cot}})$. On the other hand, for each $1 \leqslant k \leqslant N$ the complex $C_k(\{U_\alpha\}, \mathfrak{F}_\bullet)$ is the direct sum of the complexes $(g_{\alpha_1,\dots,\alpha_k!}j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{F}_\bullet)[W]$, where $g_{\alpha_1,\dots,\alpha_k}$ denotes the composition $g \circ j_{\alpha_1,\dots,\alpha_k} \colon U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \longrightarrow W$.

Let V_β be an affine open covering of the intersection $U_{\alpha_1} \cap \dots \cap U_{\alpha_k}$. Then the complex $(g_{\alpha_1,\dots,\alpha_k!}j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{F}_\bullet)[W]$ is homotopy equivalent to the total complex of the Čech bicomplex $C_\bullet(\{V_\beta\}, j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{F}_\bullet)$. For each $1 \leqslant l \leqslant n_{\alpha_1,\dots,\alpha_k}$, the complex $C_l(\{V_\beta\}, j^!_{\alpha_1,\dots,\alpha_k}\mathfrak{F}_\bullet)$ is the direct sum of the complexes $(h_{\beta_1,\dots,\beta_l!}e^!_{\beta_1,\dots,\beta_l}\mathfrak{F}_\bullet)[W]$, where $e_{\beta_1,\dots,\beta_l}$ are the embeddings $V_{\beta_1} \cap \dots \cap V_{\beta_l} \longrightarrow Z$ and $h_{\beta_1,\dots,\beta_l}$ are the compositions $g_{\alpha_1,\dots,\alpha_k} \circ e_{\beta_1,\dots,\beta_l} \colon V_{\beta_1} \cap \dots \cap V_{\beta_l} \longrightarrow W$.

Finally, the schemes $V_{\beta_1} \cap \dots \cap V_{\beta_l}$ being affine and $e^!_{\beta_1,\dots,\beta_l}\mathfrak{F}_\bullet$ being a projective resolution of a locally cotorsion contraherent cosheaf $e^!_{\beta_1,\dots,\beta_l}\mathfrak{E}$ on $V_{\beta_1} \cap \dots \cap V_{\beta_l}$, the complex of cotorsion $\mathcal{O}(W)$-modules $(h_{\beta_1,\dots,\beta_l!}e^!_{\beta_1,\dots,\beta_l}\mathfrak{F}_\bullet)[W]$ is isomorphic to the cotorsion $\mathcal{O}(W)$-module $\mathfrak{E}[V_{\beta_1} \cap \dots \cap V_{\beta_l}] \simeq (h_{\beta_1,\dots,\beta_l!}e^!_{\beta_1,\dots,\beta_l}\mathfrak{E})[W]$ in $\mathsf{D}^-(\mathcal{O}(W)\text{–}\mathsf{mod}^{\mathsf{cot}})$.

Part (a) is proven. Similarly one can show that the left homological dimension of any object of $Y\text{–}\mathsf{lcth}_{\mathbf{T}}$ with respect to $Y\text{–}\mathsf{lcth}_{\mathbf{T},f/\mathbf{W}\text{-ac}}$ does not exceed $M-1$ (assuming that the Krull dimension of Y is finite).

Now if the scheme X has finite Krull dimension D, pick a finite affine open covering $X = \bigcup_{\alpha=1}^N U_\alpha$ subordinate to $\mathbf{W}$. Then, by Lemma 5.3.1(b), the left homological dimension of any object of $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\{U_\alpha\}}$ with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\{U_\alpha\}}$ does not exceed D. It follows that the left homological dimension of any complex from $\mathsf{D}^-(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ does not exceed its left homological dimension with respect to $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\{U_\alpha\}}$ plus D. Hence the left homological dimension of any locally cotorsion locally contraherent cosheaf from $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ with respect to $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{-ac}}$ does not exceed its left homological dimension with respect to $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\{U_\alpha\}\text{-ac}}$ plus D. Here the former summand is finite by part (a).

If the scheme Y has finite Krull dimension D, then the left homological dimension in question does not exceed D by Lemma 5.3.1(a), since $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \subset Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{-ac}}$. ☐

Lemma 5.3.8. *Let $f\colon Y \longrightarrow X$ be a morphism of Noetherian schemes. Then any quasi-coherent sheaf on Y has finite right homological dimension with respect to the exact subcategory $Y\text{–}\mathsf{qcoh}^{f\text{-ac}} \subset Y\text{–}\mathsf{qcoh}$.*

Proof. Similar to (and simpler than) Lemma 5.3.7. ☐

Corollary 5.3.9. *Let $f\colon Y \longrightarrow X$ be a morphism of Noetherian schemes. Then*

(a) *for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, or abs, the triangulated functor $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{f\text{-ac}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh})$ induced by the embedding of exact categories $Y\text{–}\mathsf{qcoh}^{f\text{-ac}} \longrightarrow Y\text{–}\mathsf{qcoh}$ is an equivalence of categories;*

(b) *for any symbol* $\star = -$ *or* ctr*, the triangulated functor* $\mathsf{D}^\star(Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{–}\mathsf{ac}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}})$ *induced by the embedding of exact categories* $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{–}\mathsf{ac}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ *is an equivalence of categories;*

(c) *assuming that one of the conditions of Lemma* 5.3.7 *holds, for any symbol* $\star = \mathsf{b}$*,* $+$*,* $-$*,* $\varnothing$*,* $\mathsf{abs}+$*,* $\mathsf{abs}-$*,* ctr*, or* abs *the triangulated functor* $\mathsf{D}^\star(Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{–}\mathsf{ac}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}})$ *induced by the embedding of exact categories* $Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T},f/\mathbf{W}\text{–}\mathsf{ac}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}$ *is an equivalence of categories.*

Proof. Part (a) follows from Lemma 5.3.8 together with the dual version of Proposition A.5.6. Part (b) is provided by Proposition A.3.1. Part (c) follows from Lemma 5.3.7 together with Proposition A.5.6. □

5.4. **Background equivalences of triangulated categories.** The results of this section complement those of Sections 4.6 and 4.9–4.11.

Let X be a semi-separated Noetherian scheme. The *cotorsion dimension* of a quasi-coherent sheaf on X is defined as its right homological dimension with respect to the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset X\text{–}\mathsf{qcoh}$, i. e., the minimal length of a right resolution by cotorsion quasi-coherent sheaves. For the definition of the *very flat dimension* of a quasi-coherent sheaf on X, we refer to Section 4.9.

Lemma 5.4.1. *Let* $X = \bigcup_{\alpha=1}^N U_\alpha$ *be a finite affine open covering, and let* D *denote the Krull dimension of the scheme* X*. Then*

(a) *the very flat dimension of any flat quasi-coherent sheaf on* X *does not exceed* D*;*

(b) *the homological dimension of the exact category of flat quasi-coherent sheaves on* X *does not exceed* $N-1+D$*;*

(c) *the cotorsion dimension of any quasi-coherent sheaf on* X *does not exceed* $N-1+D$*;*

(d) *the right homological dimension of any flat quasi-coherent sheaf on* X *with respect to the exact subcategory of flat cotorsion quasi-coherent sheaves* $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ *does not exceed* $N-1+D$*;*

Proof. Part (a) follows from Theorem 1.5.6. Part (b): one proves that $\operatorname{Ext}_X^{>N-1+D}(\mathcal{F}, \mathcal{M}) = 0$ for any flat quasi-coherent sheaf $\mathcal{F}$ and any quasi-coherent $\mathcal{M}$ on X. One can use the Čech resolution (12) of the sheaf $\mathcal{M}$ and the natural isomorphisms $\operatorname{Ext}^*_X(\mathcal{F}, j_*\mathcal{G}) \simeq \operatorname{Ext}^*_U(j^*\mathcal{F}, \mathcal{G})$ for the embeddings of affine open subschemes $j\colon U \longrightarrow X$ and any quasi-coherent sheaves $\mathcal{F}$, $\mathcal{G}$ in order to reduce the question to the case of an affine scheme U. Then it remains to apply the same result from [58] discussed in Section 1.5. Taking into account Corollaries 4.1.9(c) and 4.1.11(b) (guaranteeing that the cotorsion dimension is well-defined), the Ext vanishing that we have proven implies part (c) as well. The right homological dimension in part (d) is well-defined due to Corollary 4.1.11(b), so (d) follows from (c) in view of the dual version of Corollary A.5.3 (cf. Lemma 4.6.9). □

Corollary 5.4.2. *Let* X *be a semi-separated Noetherian scheme of finite Krull dimension and* $d \geqslant 0$ *be any (finite) integer. Then the natural triangulated functors*

$\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ *and* $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ *are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ *are equivalences of categories.*

Proof. Follows from Lemma 5.4.1(b) together with the result of [52, Remark 2.1]. □

Corollary 5.4.3. *Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{co}$, *or* abs, *the triangulated functor* $\mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ *is an equivalence of triangulated categories.*

Proof. Follows from Lemma 5.4.1(a) together with Proposition A.5.6. □

Corollary 5.4.4. *Let X be a Noetherian scheme of finite Krull dimension. Then*

(a) *for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{ctr}$, *or* abs, *the triangulated functor* $\mathsf{D}^{\star}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{ctrh}_{\mathsf{cfq}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$ *is an equivalence of triangulated categories;*

(b) *for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{ctr}$, *or* abs, *the triangulated functor* $\mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *is an equivalence of triangulated categories;*

(c) *for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-$, *or* abs, *the triangulated functor* $\mathsf{D}^{\star}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{lcth})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}$ *is an equivalence of triangulated categories.*

Proof. Part (a) is provided by Lemma 5.2.5(b) together with the dual version of Proposition A.5.6. Parts (b-c) follow from part (a) and Corollary 5.3.2(b-c). Alternatively, in the case of a semi-separated Noetherian scheme X of finite Krull dimension, the assertions (b-c) can be obtained directly from Lemma 5.2.1(a). □

The following corollary is another restricted version of Theorem 4.6.6; it is to be compared with Corollary 4.6.10.

Corollary 5.4.5. *Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{co}$, *or* abs *there is a natural equivalence of triangulated categories* $\mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^{\star}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$.

Proof. Assuming $\star \neq \mathsf{co}$, by Lemma 5.4.1(d) together with the dual version of Proposition A.5.6 the triangulated functor $\mathsf{Hot}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is an equivalence of categories. In view of Corollary 5.4.2, the same assertion holds for $\star = \mathsf{co}$. Hence it remains to recall that the equivalence of categories from Lemma 4.6.7 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ (see the proof of Corollary 4.6.10(c)). □

Corollary 5.4.6. *Let $f\colon Y \longrightarrow X$ be a morphism of finite flat dimension between semi-separated Noetherian schemes. Then for any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{co}$, *or* abs *the equivalences of triangulated categories* $\mathsf{D}^{\star}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^{\star}(Y$

$\text{–ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ *and* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ *from Corollary* 5.4.5 *transform the right derived functor* $\mathbb{R}f_*$ (65) *into the left derived functor* $\mathbb{L}f_!$ (69).

Proof. Can be either deduced from Corollary 4.11.6(c), or proven directly in the similar way using Lemma 4.11.3(d). ☐

Let X be a locally Noetherian scheme with an open covering $\mathbf{W}$. As in Section 4.11, we denote by $X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}$ the full subcategory of objects of injective dimension $\leqslant d$ in $X\text{–}\mathsf{qcoh}$ and by $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}$ the full subcategory of objects of projective dimension $\leqslant d$ in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$. For a Noetherian scheme X of finite Krull dimension, let $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}$ denote the full subcategory of objects of projective dimension $\leqslant d$ in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. We set $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d} = X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\{X\},\mathsf{fpd}-d}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d} = X\text{–}\mathsf{lcth}_{\{X\},\mathsf{fpd}-d}$. Clearly, the projective dimension of an object of $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ or $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ does not change when the open covering $\mathbf{W}$ is replaced by its refinement.

The full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d} \subset X\text{–}\mathsf{qcoh}$ is closed under extensions, cokernels of admissible monomoprhisms, and infinite direct sums. The full subcategory $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is closed under extensions, kernels of admissible epimorphisms, and infinite products (see Corollary 5.1.5).

Corollary 5.4.7. (a) *Let X be a locally Noetherian scheme. Then the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$ *are equivalences of categories.*

(b) *Let X be a locally Noetherian scheme. Then the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

(c) *Let X be a Noetherian scheme of finite Krull dimension. Then the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

Proof. Part (a) follows from Corollary 4.11.1(a) and [52, Remark 2.1], while parts (b-c) follow from Proposition A.5.6 and the same Remark. ☐

A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{G}$ on a scheme X is said to have $\mathbf{W}$-*flat dimension not exceeding* d if the flat dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{G}[U]$ does not exceed d for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. The flat dimension of a cosheaf of $\mathcal{O}_X$-modules is defined as its $\{X\}$-flat dimension. $\mathbf{W}$-locally contraherent cosheaves of $\mathbf{W}$-flat dimension not exceeding d on a locally Noetherian scheme X form a full subcategory $X\text{–}\mathsf{lcth}^{\mathsf{ffd}-d}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ closed under extensions, kernels of admissible epimorphisms, and infinite products. We set $X\text{–}\mathsf{ctrh}^{\mathsf{ffd}-d} = X\text{–}\mathsf{lcth}^{\mathsf{ffd}-d}_{\{X\}}$.

The flat dimension of a contraherent cosheaf $\mathfrak{G}$ on an affine Noetherian scheme U is equal to the flat dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{G}[U]$ (see Section 3.7). Over a semi-separated Noetherian scheme X, a $\mathbf{W}$-locally contraherent cosheaf has $\mathbf{W}$-flat dimension $\leqslant d$ if and only if it admits a left resolution of length $\leqslant d$ by $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves (see Corollary 4.4.5(a)).

Hence it follows from Corollary 5.2.2(b) (applied to affine open subschemes $U \subset X$) that the $\mathbf{W}$-flat dimension of a $\mathbf{W}$-locally contraherent cosheaf on a locally Noetherian scheme X of finite Krull dimension does not change when the covering $\mathbf{W}$ is replaced by its refinement. According to part (a) of the same Corollary, on a semi-separated Noetherian scheme of finite Krull dimension the $\mathbf{W}$-flat dimension of a $\mathbf{W}$-locally contraherent cosheaf is equal to its colocally flat dimension; so $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d} = X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{clfd}-d}$. By Corollary 5.1.4, the $\mathbf{W}$-flat dimension of a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on a locally Noetherian scheme X coincides with its projective dimension in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ (and also does not depend on $\mathbf{W}$). So one has $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} = X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}^{\mathsf{lct}}$.

Lemma 5.4.8. *Let X be a Noetherian scheme of finite Krull dimension D. Then a $\mathbf{W}$-locally contraherent cosheaf on X has finite projective dimension in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ if and only if it has finite $\mathbf{W}$-flat dimension. More precisely, the inclusions of full subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-(d+D)}$ hold in the category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.*

Proof. The inclusion $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d}$ holds due to Corollary 5.2.4. Conversely, by the same Corollary any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{M}$ on X has a left resolution by flat contraherent cosheaves, so the $\mathbf{W}$-flat dimension of $\mathfrak{M}$ is equal to its left homological dimension with respect to $X\text{–}\mathsf{ctrh}^{\mathsf{fl}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see Corollary 5.2.2(b)). It remains to apply the last assertion of Corollary 5.2.6(b). ☐

Corollary 5.4.9. *For any Noetherian scheme X of finite Krull dimension and any (finite) integer $d \geqslant 0$, the natural triangulated functors $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d})$, and $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d})$ are equivalences of triangulated categories.*

Proof. It is clear from Lemma 5.4.8 that the homological dimension of the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}-d}$ is finite, so it remains to apply [52, Remark 2.1] (to obtain the equivalences between various derived categories of this exact category) and Proposition A.5.6 (to identify the absolute derived categories with the homotopy categories of projective objects). Alternatively, one can use [53, Theorem 3.6 and Remark 3.6]. ☐

The following theorem is the main result of this section.

Theorem 5.4.10. (a) *For any locally Noetherian scheme X, the natural functor $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories.*

(b) *For any locally Noetherian scheme X, the natural functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$ is an equivalence of triangulated categories.*

(c) *For any semi-separated Noetherian scheme X, the natural functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories.*

(d) *For any Noetherian scheme X of finite Krull dimension, the natural functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *are equivalences of triangulated categories.*

Proof. Part (a) is a standard result (see, e. g., [15, Lemma 1.7(b)]) which is a particular case of [53, Theorem 3.7 and Remark 3.7] and can be also obtained from the dual version of Proposition A.3.1(b). The key observation is that there are enough injectives in X–qcoh and the full subcategory X–$\mathsf{qcoh}^{\mathsf{inj}}$ they form is closed under infinite direct sums. Similarly, part (b) can be obtained either from Proposition A.3.1(b), or from the dual version of [53, Theorem 3.7 and Remark 3.7] (see also [53, Section 3.8]). In any case the argument is based on Theorem 5.1.1(a) and Corollary 5.1.5. Part (c) holds by Proposition A.3.1(b) together with Lemma 4.3.3, 4.4.1(a), or 4.4.3(a).

Finally, in part (d) the functors $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ are equivalences of categories by Corollary 5.4.9, and the functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of categories by Proposition A.3.1(b) together with Corollary 5.2.4. A direct proof of the equivalence $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is also possible; it proceeds along the following lines.

One has to use the more advanced features of the results of [53, Sections 3.7–3.8] involving the full generality of the conditions ($*$)–($**$). Alternatively, one can apply the more general Corollary A.6.2. Specifically, let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering; then it follows from Corollary 5.2.4(b) that an infinite product of projective contraherent cosheaves on X is a direct summand of a direct sum over α of the direct images of contraherent cosheaves on U_α correspoding to infinite products of very flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.

Infinite products of such modules may not be very flat, but they are certainly flat and contraadjusted. By the last assertion of Corollary 5.2.6(b), one can conclude that the projective dimensions of infinite products of projective objects in X–ctrh do not exceed the Krull dimension D of the scheme X. So the contraherent cosheaf analogue of the condition ($**$) holds for X–ctrh, or in other words, the assumption of Corollary A.6.2 is satisfied by the pair of exact categories $X\text{–}\mathsf{ctrh}_{\mathsf{prj}} \subset X\text{–}\mathsf{ctrh}$. ☐

The following corollary is to be compared with Corollaries 5.2.8(b) and 5.3.3(b).

Corollary 5.4.11. *For any locally Noetherian scheme X with an open covering $\mathbf{W}$, the triangulated functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ induced by the embedding of exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is an equivalence of triangulated categories.*

Proof. Follows from Theorem 5.4.10(b) applied to the coverings $\{X\}$ and $\mathbf{W}$ of the scheme X. Alternatively, one can apply directly Proposition A.3.1(b) together with Theorem 5.1.1(a). ☐

5.5. **Co-contra correspondence over a regular scheme.** Let X be a regular semi-separated Noetherian scheme of finite Krull dimension.

Theorem 5.5.1. (a) *The triangulated functor $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ induced by the embedding of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}$ is an equivalence of triangulated categories.*

(b) *The triangulated functor* $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}$ *is an equivalence of triangulated categories.*

(c) *There is a natural equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *provided by the derived functors* $\mathbb{R}\mathfrak{Hom}_X(\mathcal{O}_X, -)$ *and* $\mathcal{O}_X \odot^{\mathbb{L}}_X -$.

Proof. Part (a) actually holds for any symbol $\star \neq \mathsf{ctr}$ in the upper indices of the derived category signs, and is a particular case of Corollary 4.9.1(a). Indeed, one has $X\text{–}\mathsf{qcoh} = X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ provided that d is greater or equal to the Krull dimension of X. Similarly, part (b) actually holds for any symbol $\star \neq \mathsf{co}$ in the upper indices, and is a particular case of Corollary 4.9.1(c). Indeed, one has $X\text{–}\mathsf{lcth}_{\mathbf{W}} = X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ provided that d is greater or equal to the Krull dimension of X.

To prove part (c), notice that all the triangulated functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are equivalences of categories by Corollary 4.9.5 (since one also has $X\text{–}\mathsf{qcoh} = X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ provided that d is greater or equal to the Krull dimension of X). So it remains to apply Theorem 4.6.6. ☐

5.6. **Co-contra correspondence over a Gorenstein scheme.** Let X be a Gorenstein semi-separated Noetherian scheme of finite Krull dimension. We will use the following formulation of the Gorenstein condition: for any affine open subscheme $U \subset X$, the classes of $\mathcal{O}_X(U)$-modules of finite flat dimension, of finite projective dimension, and of finite injective dimension coincide.

Notice that neither of these dimensions can exceed the Krull dimension D of the scheme X. Accordingly, the class of $\mathcal{O}_X(U)$-modules defined by the above finite homological dimension conditions is closed under both infinite direct sums and infinite products. It is also closed under extensions and the passages to the cokernels of embeddings and the kernels of surjections.

Moreover, since the injectivity of a quasi-coherent sheaf on a Noetherian scheme is a local property, the full subcategories of quasi-coherent sheaves of finite flat dimension and of finite injective dimension coincide in $X\text{–}\mathsf{qcoh}$. Similarly, the full subcategories of locally contraherent cosheaves of finite flat dimension and of finite locally injective dimension coincide in $X\text{–}\mathsf{lcth}$. Neither of these dimensions can exceed D.

Theorem 5.6.1. (a) *The triangulated functors* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ *induced by the embeddings of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}} \longrightarrow X\text{–}\mathsf{qcoh}$ *are equivalences of triangulated categories.*

(b) *The triangulated functors* $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *induced by the embeddings of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{flid}} \longrightarrow X\text{–}\mathsf{ctrh}$ *are equivalences of triangulated categories.*

(c) *There is a natural equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{flid}})$ *provided by the derived functors* $\mathbb{R}\mathfrak{Hom}_X(\mathcal{O}_X, -)$ *and* $\mathcal{O}_X \odot^{\mathbb{L}}_X -$.

Proof. Parts (a-b): by Corollary 4.9.1(a,c), the functors $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ are equivalences of categories for any symbol $\star \ne \mathsf{ctr}$ and the functors $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ are equivalences of categories for any symbol $\star \ne \mathsf{co}$.

To prove that the functor $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is an equivalence of categories, notice that one has $X\text{–}\mathsf{qcoh}^{\mathsf{inj}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}D} = X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and the functor $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}D})$ is an equivalence of categories by Corollary 5.4.7(a), while the composition $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}D}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is an equivalence of categories by Theorem 5.4.10(a).

Similarly, to prove that the functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equialence of categories, notice that one has $X\text{–}\mathsf{ctrh}_{\mathsf{prj}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}D} = X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ and the functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}D})$ is an equivalence of categories by Corolary 5.4.9, while the composition $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}D}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of categories by Theorem 5.4.10(d).

To prove part (c), notice that the functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ are equivalences of categories by Corollary 5.4.2, while the functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ are equivalences of categories by Corollary 4.9.5(b).

Furthermore, consider the intersections $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$. As was explained in Section 4.10, the functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ is an equivalence of triangulated categories for any $\star \ne \mathsf{ctr}$, while the functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ is an equivalence of triangulated categories for any $\star \ne \mathsf{co}$.

Finally, it is clear from Lemma 4.10.2(a,d) (see also Lemma 4.11.3) that the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ of Lemma 4.6.7 identifies their full exact subcategories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}}$. So the induced equivalence of the derived categories $\mathsf{D}^{\mathsf{abs}}$ or D provides the desired equivalence of triangulated categories in part (c). ☐

5.7. **Co-contra correspondence over a scheme with a dualizing complex.** Let X be a semi-separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$ [30], which we will view as a finite complex of injective quasi-coherent sheaves on X. The following result complements the covariant Serre–Grothendieck duality theory as developed in the papers and the thesis [34, 49, 44, 15].

Theorem 5.7.1. *There are natural equivalences between the four triangulated categories* $\mathsf{D}^{\mathsf{abs}=\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$*,* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$*,* $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$*, and* $\mathsf{D}^{\mathsf{abs}=\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$*. (Here the notation* $\mathsf{abs} = \mathsf{co}$ *and* $\mathsf{abs} = \mathsf{ctr}$ *presumes the assertions that the corresponding derived categories of the second kind coincide for the exact category in question.) Among these, the equivalences* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *and* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ *do not require a dualizing complex and do not depend on it; all the remaining equivalences do and do.*

Proof. For any quasi-compact semi-separated scheme X with an open covering $\mathbf{W}$, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) = \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ by Corollary 4.9.5(b). For any Noetherian scheme X of finite Krull dimension, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ by Corollary 5.4.2.

For any semi-separated Noetherian scheme X, one has $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ by Theorem 5.4.10(a) and $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ by Corollary 4.6.8(b). Hence the desired equivalence $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$, which is provided by the derived functors

$$\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, -)\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$$

and

$$\mathcal{O}_X \odot^{\mathbb{L}}_X -\colon \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}).$$

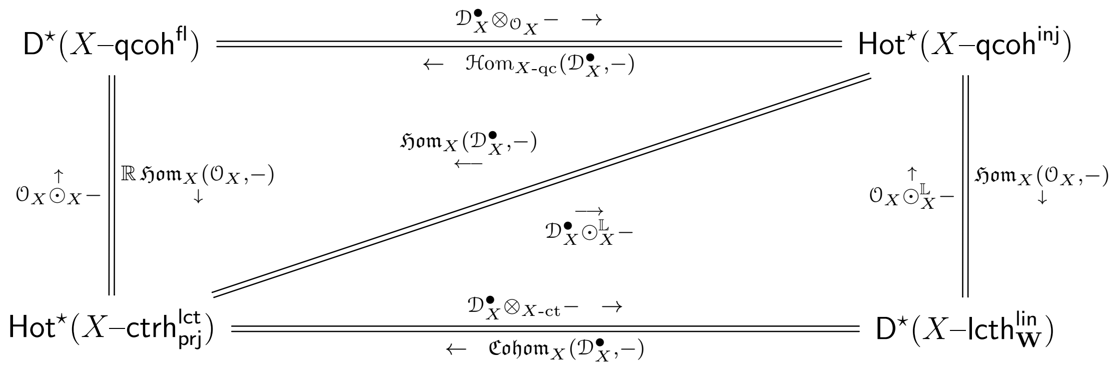
For any semi-separated Noetherian scheme X of finite Krull dimension, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ by Corollary 5.4.5, $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ by Theorem 5.4.10(b), and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ by Corollary 5.4.4(b). Alternatively, one can refer to the equivalence $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ holding by Corollary 5.4.3, $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ by Corollary 4.6.10(a), and $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ by Theorem 5.4.10(d). Either way, one gets the same desired equivalence $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$, which is provided by the derived functors

$$\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, -)\colon \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$$

and

$$\mathcal{O}_X \odot^{\mathbb{L}}_X -\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}).$$

Now we are going to construct a commutative diagram of equivalences of triangulated categories



for any symbol $\star = \mathsf{b}$, $\mathsf{abs}{+}$, $\mathsf{abs}{-}$, or abs.

The exterior vertical functors are constructed by applying the additive functors $\mathcal{O}_X \odot_X -$ and $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ to the given complexes termwise. The interior (derived) vertical functors have been defined in Corollaries 4.6.8(b) and 5.4.5. All the functors invoking the dualizing complex $\mathcal{D}^\bullet_X$ are constructed by applying the respective exact functors of two arguments to $\mathcal{D}^\bullet_X$ and the given unbounded complex termwise and totalizing the bicomplexes so obtained.

First of all, one notices that the functors in the interior upper triangle are right adjoint to the ones in the exterior. This follows from the adjunction (20) together with the adjunction of the tensor product of quasi-coherent sheaves and the quasi-coherent internal Hom.

The upper horizontal functors $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, -)$ are mutually inverse for the reasons explained in [44, Theorem 8.4 and Proposition 8.9] and [15, Theorem 2.5]. The argument in [15] is based on the observations that the morphism of finite complexes of flat quasi-coherent sheaves

$$\mathcal{F} \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$$

is a quasi-isomorphism for any sheaf $\mathcal{F} \in X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and the morphism of finite complexes of injective quasi-coherent sheaves

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$$

is a quasi-isomorphism for any sheaf $\mathcal{J} \in X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$.

Let us additionally point out that, according to Lemma 2.5.3(c) and [44, Lemma 8.7], the complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{J}^\bullet)$ is a complex of flat cotorsion quasi-coherent sheaves for any complex $\mathcal{J}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$. So the functor $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, -)$ actually lands in $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ (as does the functor $\mathcal{O}_X \odot_X -$ on the diagram, according to the proof of Corollary 5.4.5). The interior upper triangle is commutative due to the natural isomorphism (17). The exterior upper triangle is commutative due to the natural isomorphism (21).

In order to discuss the equivalence of categories in the lower horizontal line, we will need the following lemma. It is based on the definitions of the $\mathfrak{Cohom}$ functor in Section 3.6 and the contraherent tensor product functor $\otimes_{X\text{-ct}}$ in Section 3.7.

Lemma 5.7.2. *Let $\mathcal{J}$ be an injective quasi-coherent sheaf on a semi-separated Noetherian scheme X with an open covering $\mathbf{W}$. Then there are two well-defined exact functors*

$$\mathfrak{Cohom}_X(\mathcal{J}, -)\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$$

and

$$\mathcal{J} \otimes_{X\text{-ct}} -\colon X\text{–}\mathsf{ctrh}_{\mathsf{clf}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$$

between the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ of locally injective $\mathbf{W}$-locally contraherent cosheaves and colocally flat contraherent cosheaves on X. The functor $\mathcal{J} \otimes_{X\text{-ct}} -$ is left adjoint to the functor $\mathfrak{Cohom}_X(\mathcal{J}, -)$. Besides, the functor $\mathfrak{Cohom}_X(\mathcal{J}, -)$ takes values in the additive subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}} \subset X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$, while the functor $\mathcal{J} \otimes_{X\text{-ct}} -$ takes values in the additive subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$. For any quasi-coherent sheaf $\mathcal{M}$ and any colocally flat contraherent cosheaf $\mathfrak{F}$ on X there is a natural isomorphism

$$\mathcal{M} \odot_X (\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{F}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}) \odot_X \mathfrak{F} \tag{84}$$

of quasi-coherent sheaves on X.

Proof. Let us show that the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{J}, \mathfrak{K})$ is projective for any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{K}$ on X. Indeed, $\mathcal{J}$ is a direct summand of a finite direct sum of the direct images of injective quasi-coherent sheaves $\mathcal{I}$ from the embeddings of affine open subschemes $j\colon U \longrightarrow X$ subordinate to $\mathbf{W}$. So it suffices to consider the case of $\mathcal{J} = j_*\mathcal{I}$.

According to (40), there is a natural isomorphism of locally cotorsion ($\mathbf{W}$-locally) contraherent cosheaves $\mathfrak{Cohom}_X(j_*\mathcal{I}, \mathfrak{K}) \simeq j_! \, \mathfrak{Cohom}_U(\mathcal{I}, j^!\mathfrak{K})$ on X. The $\mathcal{O}(U)$-modules $\mathcal{I}(U)$ and $\mathfrak{K}[U]$ are injective, so $\operatorname{Hom}_{\mathcal{O}(U)}(\mathcal{I}(U), \mathfrak{K}[U])$ is a flat cotorsion $\mathcal{O}(U)$-module. In other words, the locally cotorsion contraherent cosheaf $\mathfrak{Cohom}_U(\mathcal{I}, j^!\mathfrak{K})$ is projective on U, and therefore its direct image with respect to j is projective on X (see Lemma 4.4.3(b) or Corollary 4.4.7(b)).

Now let $\mathfrak{F}$ be a colocally flat contraherent cosheaf on X. Then, in particular, $\mathfrak{F}$ is a flat contraherent cosheaf (Corollary 4.3.6), so the tensor product $\mathcal{J} \otimes_X \mathfrak{F}$ is a locally injective derived contrahereable cosheaf on X.

Moreover, by Corollary 4.3.4(c), $\mathfrak{F}$ is a direct summand of a finitely iterated extension of the direct images of flat contraherent cosheaves from affine open subschemes of X. It was explained in Section 3.5 that derived contrahereable cosheaves on affine schemes are contrahereable and the direct images of cosheaves with respect to affine morphisms preserve contrahereability. Besides, the full subcategory of contrahereable cosheaves on X is closed under extensions in the exact category of derived contrahereable cosheaves, and the functor $\mathcal{J} \otimes_X -$ takes short exact sequences of flat contraherent cosheaves to short exact sequences of derived contrahereable cosheaves on X (see Section 3.7).

So it follows from the isomorphism (48) that $\mathcal{J} \otimes_X \mathfrak{F}$ is a locally injective contrahereable cosheaf. Its contraherator $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{F} = \mathfrak{C}(\mathcal{J} \otimes_X \mathfrak{F})$ is consequently a locally injective contraherent cosheaf on X. Furthermore, according to Section 3.5 the (global) contraherator construction is an exact functor commuting with the direct images with respect to affine morphisms. Hence the contraherent cosheaf $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{F}$ is a direct summand of a finitely iterated extension of the direct images of (locally) injective contraherent cosheaves from affine open subschemes of X, i. e., $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{F}$ is a colocally projective locally injective contraherent cosheaf.

We have constructed the desired exact functors. A combination of the adjunction isomorphisms (35) and (30) makes them adjoint to each other. Finally, for any $\mathcal{M} \in X\text{–}\mathsf{qcoh}$ and $\mathfrak{F} \in X\text{–}\mathsf{ctrh}_{\mathsf{clf}}$ one has

$$\mathcal{M} \odot_X (\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{F}) = \mathcal{M} \odot_X \mathfrak{C}(\mathcal{J} \otimes_X \mathfrak{F}) \simeq \mathcal{M} \odot_X (\mathcal{J} \otimes_X \mathfrak{F}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}) \odot_X \mathfrak{F}$$

according to the isomorphisms (32) and (36). $\square$

Now we can return to the proof of Theorem 5.7.1. The functors in the interior lower triangle are left adjoint to the ones in the exterior, as it follows from the adjunction (20) and Lemma 5.7.2. Let us show that the lower horizontal functors are mutually inverse.

According to the proof of Corollary 4.6.8(b), the functor $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ induced by the embedding $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ is an equivalence of triangulated categories. Therefore, it suffices to show that for any cosheaf $\mathfrak{J} \in X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ the morphism of complexes of contraherent cosheaves

$$\mathcal{D}_X^\bullet \otimes_{X\text{-ct}} \mathfrak{Cohom}_X(\mathcal{D}_X^\bullet, \mathfrak{J}) \longrightarrow \mathfrak{J}$$

is a homotopy equivalence (or just a quasi-isomorphism in $X\text{–}\mathsf{ctrh}$), and for any cosheaf $\mathfrak{P} \in X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ the morphism of complexes of contraherent cosheaves

$$\mathfrak{P} \longrightarrow \mathfrak{Cohom}_X(\mathcal{D}^\bullet_X,\ \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{P})$$

is a homotopy equivalence (or just a quasi-isomorphism in $X\text{–}\mathsf{ctrh}$).

According to Corollaries 4.2.8 and 4.4.3, any object of $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ is a direct summand of a finite direct sum of direct images of objects in the similar categories on affine open subschemes of X. According to (43) and (48) together with the results of Section 3.5, both functors $\mathfrak{Cohom}_X(\mathcal{D}^\bullet_X, -)$ and $\mathcal{D}^\bullet_X \otimes_{X\text{-ct}} -$ commute with such direct images. So the question reduces to the case of an affine scheme U, for which the distinction between quasi-coherent sheaves and contraherent cosheaves mostly loses its significance, as both are identified with (appropriate classes of) $\mathcal{O}(U)$-modules. For this reason, the desired quasi-isomorphisms follow from the similar quasi-isomorphisms for quasi-coherent sheaves obtained in [15, proof of Theorem 2.5] (as quoted above).

Alternatively, one can argue in the way similar to the proof in [15]. Essentially, this means using an "inverse image localization" procedure in place of the "direct image localizaton" above. The argument proceeds as follows.

Let ${}'\mathcal{D}^\bullet_X \longrightarrow \mathcal{D}^\bullet_X$ be a quasi-isomorphism between a finite complex ${}'\mathcal{D}^\bullet_X$ of coherent sheaves over X and the complex of injective quasi-coherent sheaves $\mathcal{D}^\bullet_X$. Then the tensor product ${}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{F}$ is a finite complex of contraherent cosheaves for any flat contraherent cosheaf $\mathfrak{F}$ on X. Furthermore, the morphism ${}'\mathcal{D}^\bullet_X \longrightarrow \mathcal{D}^\bullet_X$ is a quasi-isomorphism of finite complexes over the exact category of coadjusted quasi-coherent cosheaves $X\text{–}\mathsf{qcoh}^{\mathsf{coa}}$ on X, hence the induced morphism ${}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{F} \longrightarrow \mathcal{D}^\bullet_X \otimes_X \mathfrak{F}$ is a quasi-isomorphism of finite complexes over the exact category of derived contrahereable cosheaves on X. It follows that the morphism ${}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{F} \simeq {}'\mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{F} \longrightarrow \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{F}$ is a quasi-isomorphism of finite complexes of contraherent cosheaves on X for any flat contraherent cosheaf $\mathfrak{F}$.

Let $\mathfrak{J}$ be a cosheaf from $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$. In order to show that the morphism of finite complexes $\mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{Cohom}_X(\mathcal{D}^\bullet_X, \mathfrak{J}) \longrightarrow \mathfrak{J}$ is a quasi-isomorphism over $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$, it suffices to check that the composition ${}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{Cohom}_X(\mathcal{D}^\bullet_X, \mathfrak{J}) \longrightarrow \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{Cohom}_X(\mathcal{D}^\bullet_X, \mathfrak{J}) \longrightarrow \mathfrak{J}$ is a quasi-isomorphism over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. The latter assertion can be checked locally, i. e., it simply means that for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the morphism ${}'\mathcal{D}^\bullet_X(U) \otimes_{\mathcal{O}_X(U)} \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{D}^\bullet_X(U), \mathfrak{J}[U]) \longrightarrow \mathfrak{J}[U]$ is a quasi-isomorphism of complexes of $\mathcal{O}_X(U)$-modules. This can be deduced from the condition that the morphism $\mathcal{O}_X(U) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X(U)}({}'\mathcal{D}^\bullet_X(U), \mathcal{D}^\bullet_X(U))$ is a quasi-isomorphism, as explained in the proof in [15].

Let $\mathfrak{F}$ be a flat contraherent cosheaf on X. Pick a bounded above complex of very flat quasi-coherent sheaves ${}''\mathcal{D}^\bullet_X$ over X together with a quasi-isomorphism ${}''\mathcal{D}^\bullet_X \longrightarrow {}'\mathcal{D}^\bullet_X$. Then the bounded below complex of contraherent cosheaves $\mathfrak{Cohom}_X({}''\mathcal{D}^\bullet_X,\ {}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{F})$ is well-defined. The morphisms of bounded below complexes $\mathfrak{Cohom}_X({}''\mathcal{D}^\bullet_X,\ {}'\mathcal{D}^\bullet_X \otimes_X \mathfrak{F}) \longrightarrow \mathfrak{Cohom}_X({}''\mathcal{D}^\bullet_X,\ \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{F})$ and $\mathfrak{Cohom}_X(\mathcal{D}^\bullet_X,\ \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{F}) \longrightarrow \mathfrak{Cohom}_X({}''\mathcal{D}^\bullet_X,\ \mathcal{D}^\bullet_X \otimes_{X\text{-ct}} \mathfrak{F})$ are quasi-isomorphisms over $X\text{–}\mathsf{ctrh}$.

Thus in order to show that the morphism $\mathfrak{F} \longrightarrow \mathfrak{Cohom}_X(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{X\text{-ct}} \mathfrak{F})$ is a quasi-isomorphism in X–$\mathsf{ctrh}^{\mathsf{fl}}$, it suffices to check that the morphism $\mathfrak{F} \longrightarrow \mathfrak{Cohom}_X({}''\mathcal{D}_X^\bullet, {}'\mathcal{D}_X^\bullet \otimes_X \mathfrak{F})$ is a quasi-isomorphism of bounded below complexes over X–ctrh.

The latter is again a local assertion, meaning simply that the morphism $\mathfrak{F}[U] \longrightarrow \mathrm{Hom}_{\mathcal{O}_X(U)}({}''\mathcal{D}_X^\bullet(U), {}'\mathcal{D}_X^\bullet(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U])$ is a quasi-isomorphism of complexes of $\mathcal{O}_X(U)$-modules for any affine open subscheme $U \subset X$. One proves it by replacing ${}''\mathcal{D}_X^\bullet(U)$ by a quasi-isomorphic bounded above complex ${}'''\mathcal{D}_X^\bullet(U)$ of finitely generated projective $\mathcal{O}_X(U)$-modules, and reducing again to the condition that the morphism $\mathcal{O}_X(U) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X(U)}({}'''\mathcal{D}_X^\bullet(U), {}'\mathcal{D}_X^\bullet(U))$ is a quasi-isomorphism (cf. [15]).

According to the proof of Corollary 4.6.8(b), the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ on the diagram actually lands in $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}})$ (as does the functor $\mathcal{D}_X^\bullet \otimes_{X\text{-ct}} -$, according to Lemma 5.7.2). The exterior upper triangle is commutative due to the natural isomorphism (19). The interior upper triangle is commutative due to the natural isomorphism (84).

The assertion that the two diagonal functors on the diagram are mutually inverse follows from the above. It can be also proven directly in the manner of the former of the above the proofs of the assertion that the two lower horizontal functors are mutually inverse. One needs to use the natural isomorphisms (45) and (46) for commutation with the direct images. ☐

5.8. **Co-contra correspondence over a non-semi-separated scheme.** The goal of this section is to obtain partial generalizations of Theorems 4.6.6 and 5.7.1 to the case of a non-semi-separated Noetherian scheme.

Theorem 5.8.1. *Let X be a Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$ there is a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$.*

Proof. According to Corollaries 5.3.3 and 5.4.4, one has $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{lcth}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ for any open covering $\mathbf{W}$ of the scheme X. We will construct an equivalence of triangulated categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ and then show that it takes the full subcategories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}(X\text{–}\mathsf{qcoh})$ into the full subcategories $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}) \subset \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ and back for all symbols $\star = \mathsf{b}$, $+$, or $-$.

By Lemma 3.4.7(a), the sheaf $\mathcal{O}_X$ has a finite right resolution by flasque quasi-coherent sheaves. We fix such a resolution $\mathcal{E}^\bullet$ for the time being. Given a complex $\mathcal{M}^\bullet$ over X–qcoh, we pick a complex $\mathcal{J}^\bullet$ over X–$\mathsf{qcoh}^{\mathsf{inj}}$ quasi-isomorphic to $\mathcal{M}^\bullet$ over X–qcoh (see Theorem 5.4.10(a), cf. Theorem 5.10.2 below) and assign to $\mathcal{M}^\bullet$ the total complex of the bicomplex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathcal{J}^\bullet)$ over X–$\mathsf{ctrh}^{\mathsf{lct}}$. Given a complex $\mathfrak{P}^\bullet$ over X–$\mathsf{lcth}^{\mathsf{lct}}$, we pick a complex $\mathfrak{F}^\bullet$ over X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ quasi-isomorphic to $\mathfrak{P}^\bullet$ over X–$\mathsf{lcth}^{\mathsf{lct}}$ (see Theorem 5.4.10(b), cf. Theorem 5.10.3(a) below) and assign to $\mathfrak{P}^\bullet$ the total complex of the bicomplex $\mathcal{E}^\bullet \odot_X \mathfrak{F}^\bullet$ over X–qcoh.

Let us first show that the complex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathcal{J}^\bullet)$ over X–$\mathsf{ctrh}^{\mathsf{lct}}$ is acyclic whenever a complex $\mathcal{J}^\bullet$ over X–$\mathsf{qcoh}^{\mathsf{inj}}$ is. For any scheme point $x \in X$, let $\mathfrak{m}_{x,X}$ denote the maximal ideal of the local ring $\mathcal{O}_{x,X}$. By [30, Proposition II.7.17], any injective quasi-coherent sheaf $\mathcal{J}$ on X can be presented as an infinite direct sum $\mathcal{J} = \bigoplus_{x\in X} \iota_{x*}\widetilde{I}_x$, where $\iota_x\colon \operatorname{Spec}\mathcal{O}_{x,X} \longrightarrow X$ are the natural morphisms and $\widetilde{I}_x$ are the quasi-coherent sheaves on $\operatorname{Spec}\mathcal{O}_{x,X}$ corresponding to infinite direct sums of copies of the injective envelopes of the $\mathcal{O}_{x,X}$-modules $\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}$. Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering. Set $S_\beta \subset X$ to be the set-theoretic complement to $\bigcup_{\alpha<\beta} U_\alpha$ in U_β, and consider the direct sum decomposition $\mathcal{J} = \bigoplus_{\alpha=1}^N \mathcal{J}_\alpha$ with $\mathcal{J}_\alpha = \bigoplus_{z\in S_\alpha} \iota_{z*}\widetilde{I}_z$.

The associated decreasing filtration $\mathcal{J}_{\geqslant\alpha} = \bigoplus_{\beta\geqslant\alpha}\mathcal{J}_\beta$ is preserved by all morphisms of injective quasi-coherent sheaves $\mathcal{J}$ on X (cf. Theorem 5.1.1 and Lemma 5.1.2). We obtain a termwise split filtration $\mathfrak{J}^\bullet_{\geqslant\alpha}$ on the complex $\mathfrak{J}^\bullet$ with the associated quotient complexes $\mathfrak{J}^\bullet_\alpha$ isomorphic to the direct images $j_{\alpha*}\mathcal{K}^\bullet_\alpha$ of complexes of injective quasi-coherent sheaves $\mathcal{K}^\bullet_\alpha$ from the open embeddings $j_\alpha\colon U_\alpha \longrightarrow X$. Moreover, for $\alpha = 1$ the complex of quasi-coherent sheaves $\mathcal{K}^\bullet_1 \simeq j_1^*\mathfrak{J}^\bullet$ is acyclic, since the complex $\mathfrak{J}^\bullet$ is; and the complex $j_{1*}\mathcal{K}^\bullet_1$ is acyclic by Corollary 3.4.9(a) or Lemma 3.4.7(a). It follows by induction that all the complexes $\mathcal{K}^\bullet_\alpha$ over U_α–$\mathsf{qcoh}^{\mathsf{inj}}$ are acyclic over U_α–qcoh.

Now one has $\mathfrak{Hom}_X(\mathcal{E}^\bullet, j_{\alpha*}\mathcal{K}^\bullet_\alpha) \simeq j_{\alpha!}\,\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$ by (45). The complex $\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$ over U_α–$\mathsf{ctrh}^{\mathsf{lct}}$ is quasi-isomorphic to $\mathfrak{Hom}_{U_\alpha}(\mathcal{O}_{U_\alpha}, \mathcal{K}^\bullet_\alpha)$, since $\mathcal{K}^\bullet_\alpha$ is a complex over U_α–$\mathsf{qcoh}^{\mathsf{inj}}$, while the complex $\mathfrak{Hom}_{U_\alpha}(\mathcal{O}_{U_\alpha}, \mathcal{K}^\bullet_\alpha)$ is acyclic over U_α–$\mathsf{ctrh}^{\mathsf{lct}}$, since the complex $\mathcal{K}^\bullet_\alpha$ is acyclic over U_α–$\mathsf{qcoh}^{\mathsf{cot}}$ (see Corollary 1.5.7 or Lemma 5.4.1(c)). So the complex $\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$ is acyclic over U_α–$\mathsf{ctrh}^{\mathsf{lct}}$; by Lemma 3.4.6(c), it is also a complex of coflasque contraherent cosheaves. By Corollary 3.4.9(c), or alternatively by Lemma 3.4.7(b) together with Corollary 3.4.8(b), it follows that the complex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, j_{\alpha*}\mathcal{K}^\bullet_\alpha)$ is acyclic over X–$\mathsf{ctrh}^{\mathsf{lct}}$.

Therefore, so is the complex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathfrak{J}^\bullet)$. Similarly one proves that the complex $\mathcal{E}^\bullet \odot_X \mathfrak{F}^\bullet$ is acyclic over X–qcoh whenever a complex $\mathfrak{F}^\bullet$ over X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ is acyclic over X–$\mathsf{ctrh}^{\mathsf{lct}}$. One has to use Theorem 5.1.1 and Lemma 5.1.2 (see the proof of Theorem 5.9.1(c) below), the isomorphism (47), and Lemma 3.4.6(d).

We have shown that the derived functors $\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{E}^\bullet, -)$ and $\mathcal{E}^\bullet \odot_X^{\mathbb{L}} -$ are well defined by the above rules $\mathcal{M}^\bullet \longmapsto \mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathfrak{J}^\bullet)$ and $\mathfrak{P}^\bullet \longmapsto \mathcal{E}^\bullet \odot_X \mathfrak{F}^\bullet$. It is a standard fact that the adjunction (20) makes such two triangulated functors adjoint to each other (cf. [52, Lemma 8.3]). Let us check that the adjunction morphism $\mathcal{E}^\bullet \odot_X^{\mathbb{L}} \mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathfrak{J}^\bullet) \longrightarrow \mathfrak{J}^\bullet$ is an isomorphism in $\mathsf{D}(X$–$\mathsf{qcoh})$ for any complex $\mathfrak{J}^\bullet$ over X–$\mathsf{qcoh}^{\mathsf{inj}}$. For the reasons explained above, one can assume $\mathfrak{J}^\bullet = j_{\alpha*}\mathcal{K}^\bullet_\alpha$ for some complex $\mathcal{K}^\bullet_\alpha$ over U_α–qcoh. Then $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathfrak{J}^\bullet) \simeq j_{\alpha!}\,\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$

Let $\mathfrak{G}^\bullet_\alpha$ be a complex over U_α–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ endowed with a quasi-isomorphism $\mathfrak{G}^\bullet_\alpha \longrightarrow \mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$ over U_α–$\mathsf{ctrh}^{\mathsf{lct}}$. Then $j_{\alpha!}\mathfrak{G}^\bullet_\alpha$ is a complex over X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$, and the morphism $j_{\alpha!}\mathfrak{G}^\bullet_\alpha \longrightarrow j_{\alpha!}\,\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}^\bullet_\alpha)$ is a quasi-isomorphism over X–$\mathsf{ctrh}^{\mathsf{lct}}$. So one has $\mathcal{E}^\bullet \odot_X^{\mathbb{L}} \mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathfrak{J}^\bullet) \simeq \mathcal{E}^\bullet \odot_X j_{\alpha!}\mathfrak{G}^\bullet_\alpha \simeq j_{\alpha*}(j_\alpha^*\mathcal{E}^\bullet \odot_{U_\alpha} \mathfrak{G}^\bullet_\alpha)$. Both $\mathcal{K}^\bullet_\alpha$ and $j_\alpha^*\mathcal{E}^\bullet \odot_{U_\alpha} \mathfrak{G}^\bullet_\alpha$ being complexes of flasque quasi-coherent sheaves on U_α, it remains to show that the natural morphism $j_\alpha^*\mathcal{E}^\bullet \odot_{U_\alpha} \mathfrak{G}^\bullet_\alpha \longrightarrow \mathcal{K}^\bullet_\alpha$ is a quasi-isomorphism over U_α–qcoh. Now the morphisms $\mathcal{O}_{U_\alpha} \odot_{U_\alpha} \mathfrak{G}^\bullet_\alpha \longrightarrow j_\alpha^*\mathcal{E}^\bullet \odot_{U_\alpha} \mathfrak{G}^\bullet_\alpha$ and $\mathcal{O}_{U_\alpha} \odot_{U_\alpha}$

$\mathfrak{G}_\alpha^\bullet \longrightarrow \mathcal{O}_{U_\alpha} \odot_{U_\alpha} \mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}_\alpha^\bullet) \longrightarrow \mathcal{O}_{U_\alpha} \odot_{U_\alpha} \mathfrak{Hom}_{U_\alpha}(\mathcal{O}_{U_\alpha}, \mathcal{K}_\alpha^\bullet) \longrightarrow \mathcal{K}_\alpha^\bullet$ are quasi-isomorphisms, and the desired assertion follows.

Similarly one shows that the adjunction morphism $\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{E}_X^\bullet,\, \mathcal{E}_X^\bullet \odot_X \mathfrak{F}^\bullet) \longrightarrow \mathfrak{F}^\bullet$ is an isomorphism in $\mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ for any complex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$. This finishes the construction of the equivalence of categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$. To show that it does not depend on the choice of a flasque resolution $\mathcal{E}^\bullet$ of the sheaf $\mathcal{O}_X$, consider an acyclic finite complex $\mathcal{L}^\bullet$ of flasque quasi-coherent sheaves on X. Then for any injective quasi-coherent sheaf $\mathcal{J}$ on X the complex $\mathfrak{Hom}_X(\mathcal{L}^\bullet, \mathcal{J})$ over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ is acyclic by construction. To show that the complex $\mathcal{L}^\bullet \odot_X \mathfrak{F}$ is acyclic over $X\text{–}\mathsf{qcoh}$ for any cosheaf $\mathfrak{F} \in X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$, one reduces the question to the case of an affine scheme X using Theorem 5.1.1(b) and Lemma 3.4.6(d).

Finally, it remains to show that the equivalence of categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ that we have constructed takes bounded above (resp., below) complexes to bounded above (resp., below) complexes and vice versa (up to quasi-isomorphism). If a complex $\mathcal{M}^\bullet$ over $X\text{–}\mathsf{qcoh}$ is bounded below, it has bounded below injective resolution $\mathcal{J}^\bullet$ and the complex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathcal{J}^\bullet)$ over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ is also bounded below. Now assume that a complex $\mathcal{J}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ has bounded above cohomology.

Arguing as above, consider its decreasing filtration $\mathcal{J}_{\geqslant\alpha}^\bullet$ with the associatived quotient complexes $\mathcal{J}_\alpha^\bullet \simeq j_{\alpha *}\mathcal{K}_\alpha^\bullet$. Using Lemma 3.4.7(a), one shows that the cohomology sheaves of the complexes $\mathcal{K}_\alpha^\bullet$ over $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ are also bounded above. By Corollary 1.5.7, the right homological dimension of $\mathcal{K}_\alpha^\bullet$ with respect to $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset U_\alpha^{-\mathsf{qcoh}}$ is finite, and it follows that the complex $\mathfrak{Hom}_{U_\alpha}(\mathcal{O}_X, \mathcal{K}_\alpha^\bullet)$ is quasi-isomorphic to a bounded above complex over $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}$. The complex $\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}_\alpha^\bullet)$ over $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ is quasi-isomorphic to $\mathfrak{Hom}_{U_\alpha}(\mathcal{O}_{U_\alpha}, \mathcal{K}_\alpha^\bullet)$. Finally, the complex $\mathfrak{Hom}_X(\mathcal{E}^\bullet, \mathcal{J}^\bullet) \simeq j_{\alpha!}\,\mathfrak{Hom}_{U_\alpha}(j_\alpha^*\mathcal{E}^\bullet, \mathcal{K}_\alpha^\bullet)$ is quasi-isomorphic to a bounded above complex over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ by Lemma 3.4.6(c) and the other results of Section 3.4.

Similarly one can show that for any complex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ quasi-isomorphic to a bounded below complex over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ the complex $\mathcal{E}^\bullet \odot_X \mathfrak{F}^\bullet$ over $X\text{–}\mathsf{qcoh}$ has bounded below cohomology sheaves. ☐

Now let X be a Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$ [30, Chapter 5]. As above, we will consider $\mathcal{D}_X^\bullet$ as a finite complex of injective quasi-coherent sheaves on X. The following partial version of the covariant Serre–Grothendieck duality holds without the semi-separatedness assumption on X.

Theorem 5.8.2. *The choice of a dualizing complex $\mathcal{D}_X^\bullet$ induces a natural equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$.*

Proof. According to Corollary 5.4.4(b), one has $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ for any open covering $\mathbf{W}$ of the scheme X. By Theorem 5.4.10(a-b), one has $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ and $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$. We will show that the functors $\mathfrak{Hom}_X(\mathcal{D}_X^\bullet, -)$ and $\mathcal{D}_X^\bullet \odot_X -$ induce an equivalence of the homotopy categories $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})$ for any symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$.

Let $\mathcal{J}$ be an injective quasi-coherent sheaf on X and $j\colon U\longrightarrow X$ be the embedding of an affine open subscheme. Then the results of Section 3.8 provide a natural isomorphism of contraherent cosheaves $\mathfrak{Hom}_X(\mathcal{J}, j_*\mathcal{J}) \simeq j_!\,\mathfrak{Hom}_U(j^*\mathcal{J}, \mathcal{J})$ on X for any injective quasi-coherent sheaf $\mathcal{J}$ on U and a natural isomorphism of quasi-coherent sheaves $\mathcal{J}\odot_X j_!\mathfrak{G} \simeq j_*(j^*\mathcal{J}\odot_X \mathfrak{G})$ on X for any flat cosheaf of $\mathcal{O}_U$-modules $\mathfrak{G}$ on U.

Notice that the functor j_* takes injective quasi-coherent sheaves to injective quasi-coherent sheaves and the functor $j_!$ takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves (Corollary 5.1.6(b)). Furthermore, let $X=\bigcup_\alpha U_\alpha$ be a finite affine open covering. It is clear from the classification theorems (see Theorem 5.1.1(b)) that any injective quasi-coherent sheaf or projective locally cotorsion contraherent cosheaf on X is a finite direct sum of the direct images of similar (co)sheaves from U_α.

It follows that the functors $\mathfrak{Hom}_X(\mathcal{J},-)$ and $\mathcal{J}\odot_X -$ take injective quasi-coherent sheaves to projective locally cotorsion contraherent cosheaves on X and back. By (20), these are two adjoint functors between the additive categories $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. Substituting $\mathcal{D}^\bullet_X$ in place of $\mathcal{J}$ and totalizing the finite complexes of complexes of (co)sheaves, we obtain two adjoint functors $\mathfrak{Hom}_X(\mathcal{D}^\bullet_X,-)$ and $\mathcal{D}^\bullet_X\odot_X -$ between the homotopy categories $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ and $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$.

In order to show that these are mutually inverse equivalences, it suffices to check that the adjunction morphisms $\mathcal{D}^\bullet_X\odot_X\mathfrak{Hom}_X(\mathcal{D}^\bullet_X,\mathcal{J})\longrightarrow\mathcal{J}$ and $\mathfrak{P}\longrightarrow\mathfrak{Hom}_X(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X\odot_X\mathfrak{P})$ are quasi-isomorphisms/homotopy equivalences of finite complexes for any $\mathcal{J}\in X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ and $\mathfrak{P}\in X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. Presenting $\mathcal{J}$ and $\mathfrak{P}$ as finite direct sums of the direct images of similar (co)sheaves from affine open subschemes of X and taking again into account the isomorphisms (45), (47) reduces the question to the case of an affine scheme, where the assertion is already known.

Alternatively, one can work directly in the greater generality of arbitrary (not necessarily locally cotorsion) and flat contraherent cosheaves. According to Theorem 5.4.10(d), one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})\simeq\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$. Let us show that the functors $\mathfrak{Hom}_X(\mathcal{D}^\bullet_X,-)$ and $\mathcal{D}^\bullet_X\odot_X -$ induce an equivalence of triangulated categories $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})\simeq\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ for any symbol $\star=\mathsf{b}$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs.

Given an injective quasi-coherent sheaf $\mathcal{J}$ on X, let us first check that the functor $\mathfrak{F}\longmapsto\mathcal{J}\odot_X\mathfrak{F}$ takes short exact sequences of flat contraherent cosheaves to short exact sequences of quasi-coherent sheaves on X. By the adjunction isomorphism (20), for any injective quasi-coherent sheaf $\mathcal{J}$ on X one has $\operatorname{Hom}_X(\mathcal{J}\odot_X\mathfrak{F},\mathcal{J})\simeq\operatorname{Hom}^X(\mathfrak{F},\mathfrak{Hom}_X(\mathcal{J},\mathcal{J}))$. The contraherent cosheaf $\mathfrak{Q}=\mathfrak{Hom}_X(\mathcal{J},\mathcal{J})$ being locally cotorsion, the functor $\mathfrak{F}\longmapsto\operatorname{Hom}^X(\mathfrak{F},\mathfrak{Q})$ is exact on $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ by Corollary 5.2.9(a).

Furthermore, by part (b) of the same Corollary any flat contraherent cosheaf $\mathfrak{F}$ on X is a direct summand of a finitely iterated extension of the direct images $j_!\mathfrak{G}$ of flat contraherent cosheaves $\mathfrak{G}$ on affine open subschemes $U\subset X$. Using the isomorphism (47), we conclude that the quasi-coherent sheaf $\mathcal{J}\odot_X\mathfrak{F}$ is injective. It follows, in particular, that the complex of quasi-coherent sheaves $\mathcal{J}\odot_X\mathfrak{F}^\bullet$ is contractible for any acyclic complex $\mathfrak{F}^\bullet$ over the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. Therefore, the same applies to the complex $\mathcal{D}^\bullet_X\odot_X\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$.

Finally, to prove that the map $\mathfrak{F} \longrightarrow \mathfrak{Hom}_X(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X \odot_X \mathfrak{F})$ is a quasi-isomorphism for any flat contraherent cosheaf $\mathfrak{F}$, it suffices again to consider the case $\mathfrak{F} = j_!\mathfrak{G}$, when the assertion follows from the isomorphisms (45), (47). Hence the morphism $\mathfrak{F}^\bullet \longrightarrow \mathfrak{Hom}_X(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X \odot_X \mathfrak{F}^\bullet)$ has a cone absolutely acyclic with respect to $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ for any compex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. □

5.9. **Compact generators.** Let D be a triangulated category where arbitrary infinite direct sums exist. We recall that object $C \in \mathsf{D}$ is called *compact* if the functor $\mathrm{Hom}_{\mathsf{D}}(C, -)$ takes infinite direct sums in D to infinite direct sums of abelian groups [47]. A set of compact objects $\mathsf{C} \subset \mathsf{D}$ is said to *generate* D if any object $X \in \mathsf{D}$ such that $\mathrm{Hom}_{\mathsf{D}}(C, X[*]) = 0$ for all $C \in \mathsf{C}$ vanishes in D.

Equivalently, this means that any full triangulated subcategory of D containing C and closed under infinite direct sums coincides with D. If C is a set of compact generators for D, then an object of D is compact if and only if it belongs to the minimal thick subcategory of D containing C.

Let $X\text{–}\mathsf{coh}$ denote the abelian category of coherent sheaves on a Noetherian scheme X.

Theorem 5.9.1. (a) *For any scheme X, the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ admits arbitrary infinite direct sums, while the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ admit infinite products.*

(b) *For any Noetherian scheme X, the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is compactly generated. The triangulated functor $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ induced by the embedding of abelian categories $X\text{–}\mathsf{coh} \longrightarrow X\text{–}\mathsf{qcoh}$ is fully faithful, and its image is the full subcategory of compact objects in $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$.*

(c) *For any Noetherian scheme X of finite Krull dimension, the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ admit arbitrary infinite direct sums and are compactly generated.*

Proof. Part (a) holds, because the abelian category $X\text{–}\mathsf{qcoh}$ admits arbitrary infinite direct sums and the full subcategory of coacyclic complexes in $\mathsf{Hot}(X\text{–}\mathsf{qcoh})$ is closed under infinite direct sums (see [48, Proposition 1.2.1 and Lemma 3.2.10]). Analogously, the exact categories $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ admit arbitrary infinite products and the full subcategories of contraacyclic complexes in $\mathsf{Hot}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are closed under infinite products.

In the assumption of part (b), the abelian category $X\text{–}\mathsf{qcoh}$ is a locally Noetherian Grothendieck category, so the assertions hold by Theorem 5.4.10(a) and [38, Proposition 2.3] (see also Lemma A.1.2). A more generally applicable assertion/argument can be found in [15, Proposition 1.5(d)] and/or [53, Section 3.11].

Part (c): notice first of all that all the categories $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are equivalent to each other by Corollaries 5.3.3 and 5.4.4(b). Furthermore, if the scheme X admits a dualizing complex, the assertion of part (c) follows from Theorem 5.8.2 and part (b). The following more complicated argument allows to prove the desired assertion in the stated generality.

By Theorem 5.4.10(b), the triangulated category in question is equivalent to the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$. Let us first consider the case when the scheme X is semi-separated. Then Corollary 5.4.5 identifies our triangulated category with $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$. It follows immediately that this triangulated category admits arbitrary infinite direct sums.

In the case of an affine Noetherian scheme U of finite Krull dimension, another application of Proposition A.5.6 allows to identify $\mathsf{D}^{\mathsf{co}}(U\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ with the homotopy category of complexes of projective $\mathcal{O}(U)$-modules, which is compactly generated by [35, Theorem 2.4]. More generally, the category $\mathsf{D}(U\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is equivalent to the homotopy category of projective $\mathcal{O}(U)$-modules for any affine scheme U by [49, Section 8] and is compactly generated for any affine Noetherian scheme U by [49, Proposition 7.14] (see also [50]). Besides, the homotopy category $\mathsf{Hot}(U\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ is equivalent to $\mathsf{D}(U\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ for any affine scheme U by [64, Corollary 5.8]. Finally, for any semi-separated Noetherian scheme X the triangulated category $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is compactly generated by [44, Theorem 4.10].

Now let us turn to the general case. First we have to show that the category $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ admits arbitrary infinite direct sums. Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering, and let $S_\beta \subset X$ denote the set-theoretic complement to $\bigcup_{\alpha<\beta} U_\alpha$ in U_β. Let $j_\alpha\colon U_\alpha \longrightarrow X$ denote the open embedding morphisms; then the direct image functor $j_{\alpha!}\colon \mathsf{Hot}(U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ is left adjoint to the inverse image functor $j_\alpha^!\colon \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{Hot}(U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ (see Corollaries 5.1.3(a) and 5.1.6(b), and the adjunction (26)). Hence the functor $j_{\alpha!}$ preserves infinite direct sums.

As explained in the proof of Theorem 5.1.1, any projective locally cotorsion contraherent cosheaf $\mathfrak{F}$ on X decomposes into a direct sum $\mathfrak{F} = \bigoplus_{\alpha=1}^N \mathfrak{F}_\alpha$, where each direct summand $\mathfrak{F}_\alpha$ is an infinite product over the points $z \in S_\alpha$ of the direct images of contraherent cosheaves on $\operatorname{Spec} \mathcal{O}_{z,X}$ corresponding to free contramodules over $\widehat{\mathcal{O}}_{z,X}$. According to Lemma 5.1.2, the associated increasing filtration $\mathfrak{F}_{\leqslant\alpha} = \bigoplus_{\beta\leqslant\alpha} \mathfrak{F}_\beta$ on $\mathfrak{F}$ is preserved by all morphisms of cosheaves $\mathfrak{F} \in X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$.

Given a family ${}^{(i)}\mathfrak{F}^\bullet$ of complexes over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$, we now see that every complex ${}^{(i)}\mathfrak{F}^\bullet$ is endowed with a finite termwise split filtration ${}^{(i)}\mathfrak{F}^\bullet_{\leqslant\alpha}$ such that the family of associated quotient complexes ${}^{(i)}\mathfrak{F}^\bullet_\alpha$ can be obtained by applying the direct image functor $j_{\alpha!}$ to a family of complexes over $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. It follows that the object $\bigoplus_i {}^{(i)}\mathfrak{F}^\bullet_\alpha$ exists in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$, and it remains to apply the following lemma (which is slightly stronger than [48, Proposition 1.2.1]).

Lemma 5.9.2. *Let $A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow A_i[1]$ be a family of distinguished triangles in a triangulated category D. Suppose that the infinite direct sums $\bigoplus_i A_i$ and $\bigoplus_i B_i$ exist in D. Then a cone C of the natural morphism $\bigoplus_i A_i \longrightarrow \bigoplus_i B_i$ is the infinite direct sum of the family of objects C_i in D.*

Proof. Set $A = \bigoplus_i A_i$ and $B = \bigoplus_i B_i$. By one of the triangulated category axioms, there exist morphisms of distinguished triangles $(A_i \to B_i \to C_i \to A_i[1]) \longrightarrow (A \to$

$B \to C \to A[1]$) whose components $A_i \longrightarrow A$ and $B_i \longrightarrow B$ are the natural embeddings. For any object $E \in \mathsf{D}$, apply the functor $\mathrm{Hom}_{\mathsf{D}}(-,E)$ to this family of morphisms of triangles and pass to the infinite product (of abelian groups) over i. The resulting morphism from the long exact sequence $\dots \longrightarrow \mathrm{Hom}_{\mathsf{D}}(A[1],E) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(C,E) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(B,E) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(A,E) \longrightarrow \dots$ to the long exact sequence $\dots \longrightarrow \prod_i \mathrm{Hom}_{\mathsf{D}}(A_i[1],E) \longrightarrow \prod_i \mathrm{Hom}_{\mathsf{D}}(C_i,E) \longrightarrow \prod_i \mathrm{Hom}_{\mathsf{D}}(B_i,E) \longrightarrow \prod_i \mathrm{Hom}_{\mathsf{D}}(A_i,E) \longrightarrow \dots$ is an isomorphism at the two thirds of all the terms, and consequently an isomorphism at the remaining terms, too. ☐

Denote temporarily the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ by $\mathsf{D}(X)$. To show that the category $\mathsf{D}(X)$ is compactly generated, we will use the result of [59, Theorem 5.15]. Let $Y \subset X$ be an open subscheme such that the category $\mathsf{D}(Y)$ is compactly generated (e. g., we already know this to hold when Y is semi-separated). Let $j\colon Y \longrightarrow X$ denote the open embedding morphism.

The composition $j^!j_!$ of the direct image and inverse image functors $j_!\colon \mathsf{D}(Y) \longrightarrow \mathsf{D}(X)$ and $j^!\colon \mathsf{D}(X) \longrightarrow \mathsf{D}(Y)$ is isomorphic to the identity endofunctor of $\mathsf{D}(Y)$, so the functor $j_!$ is fully faithful and the functor $j^!$ is a Verdier localization functor. Applying again Lemma 5.1.2, we conclude that the kernel of $j^!$ is the homotopy category of projective locally cotorsion contraherent cosheaves on X with vanishing restrictions to Y. Denote this homotopy category by $\mathsf{D}(Z,X)$, where $Z = X \setminus Y$, and its identity embedding functor by $i_!\colon \mathsf{D}(Z,X) \longrightarrow \mathsf{D}(X)$.

The functor $j_!$ is known to preserve infinite products, and the triangulated category $\mathsf{D}(Y)$ is assumed to be compactly generated; so it follows that there exists a triangulated functor $j^*\colon \mathsf{D}(X) \longrightarrow \mathsf{D}(Y)$ left adjoint to $j_!$ (see [48, Remark 6.4.5 and Theorem 8.6.1] and [38, Proposition 3.3(2)]). The existence of the functor $j_!$ left adjoint to $j^!$ implies existence of a functor $i^*\colon \mathsf{D}(X) \longrightarrow \mathsf{D}(Z,X)$ left adjoint to $i_!$; and the existence of the functor j^* left adjoint to $j_!$ implies existence of a functor $i_+\colon \mathsf{D}(Z,X) \longrightarrow \mathsf{D}(X)$ left adjoint to i^*.

The functors j^* and i_+ have double right adjoints (i. e., the right adjoints and the right adjoints to the right adjoints), hence they not only preserve infinite direct sums, but also take compact objects to compact objects. Furthermore, for any open subscheme $W \subset X$ with the embedding morphism $h\colon W \longrightarrow X$ one has the base change isomorphism $h^!j_! \simeq j'_!h'^!$, where j' and h' denote the open embeddings $W \cap Y \longrightarrow W$ and $W \cap Y \longrightarrow Y$. If the triangulated category $\mathsf{D}(W \cap Y)$ is compactly generated, one can pass to the left adjoint functors, obtaining an isomorphism of triangulated functors $j^*h_! \simeq h'_!j'^*$.

Let $X = \bigcup_\alpha U_\alpha$ be a finite affine (or, more generally, semi-separated) open covering, $Z_\alpha = X \setminus U_\alpha$ be the corresponding closed complements, and $i_{\alpha+}\colon \mathsf{D}(Z_\alpha, X) \longrightarrow \mathsf{D}(X)$ be the related fully faithful triangulated functors. It follows from the above that the images of the functors $i_{\alpha+}$ form a collection of Bousfield subcategories in $\mathsf{D}(X)$ pairwise intersecting properly in the sense of [59, Lemma 5.7(2)]. Furthermore, the category $\mathsf{D}(X)$ being generated by the images of the functors $j_{\alpha!}$, the intersection of the kernels of the functors $j_\alpha^*\colon \mathsf{D}(X) \longrightarrow \mathsf{D}(U_\alpha)$ is zero. These coincide with the images of the functors $i_{\alpha+}$. Thus the triangulated subcategories $i_{\alpha+}\mathsf{D}(Z_\alpha, X) \subset \mathsf{D}(X)$ form a *cocovering* (in the sense of [59]).

It remains to check that intersections of the images of $i_{\beta+}\mathsf{D}(Z_\beta, X)$ under the localization morphism $\mathsf{D}(X) \longrightarrow \mathsf{D}(U_\alpha)$ are compactly generated in $\mathsf{D}(U_\alpha)$. Let Y be a semi-separated Noetherian scheme of finite Krull dimension and $V \subset Y$ be an open subscheme with the closed complement $Z = Y \setminus V$. We will show that the image of the fully faithful triangulated functor $i_+ : \mathsf{D}(Z, Y) \longrightarrow \mathsf{D}(Y)$ is compactly generated in $\mathsf{D}(Y)$; this is clearly sufficient.

The result of Corollary 5.4.5 identifies $\mathsf{D}(Y) = \mathsf{Hot}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ with $\mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ and $\mathsf{D}(V) = \mathsf{Hot}(V\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ with $\mathsf{D}(V\text{–}\mathsf{qcoh}^{\mathsf{fl}})$. According to Corollary 5.4.6, this identification transforms the functor $j_! : \mathsf{D}(V) \longrightarrow \mathsf{D}(Y)$ into the derived functor $\mathbb{R}j_* : \mathsf{D}(V\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ constructed in (65). The latter functor is right adjoint to the functor $j^* : \mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(V\text{–}\mathsf{qcoh}^{\mathsf{fl}})$, which is therefore identified with the functor $j^* : \mathsf{D}(Y) \longrightarrow \mathsf{D}(V)$.

Finally, we refer to [44, Proposition 4.5 and Theorem 4.10] for the assertion that the kernel of the functor $j^* : \mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(V\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is compactly generated in $\mathsf{D}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})$. ☐

Theorem 5.9.3. (a) *For any scheme X, the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ admits infinite direct sums, while the derived categories $\mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ admit infinite products.*

(b) *For any quasi-compact semi-separated scheme X, the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ is compactly generated. The full triangulated subcategory of perfect complexes in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \subset \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \subset \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}(X\text{–}\mathsf{qcoh})$ is the full subcategory of compact objects in $\mathsf{D}(X\text{–}\mathsf{qcoh})$.*

(c) *For any Noetherian scheme X, the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ is compactly generated. The full triangulated subcategory of perfect complexes in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}(X\text{–}\mathsf{qcoh})$ is the full subcategory of compact objects in $\mathsf{D}(X\text{–}\mathsf{qcoh})$.*

(d) *For any quasi-compact semi-separated scheme X, the derived category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ admits infinite direct sums and is compactly generated.*

(e) *For any Noetherian scheme X of finite Krull dimension, the derived categories $\mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ admit infinite direct sums and are compactly generated.*

Proof. The proof of part (a) is similar to that of Theorem 5.9.1(a): the assersions hold, since the class of acyclic complexes over X–qcoh is closed under infinite direct sums, and the classes of acyclic complexes over $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ are closed under infinite products.

Parts (b) and (c) are particular cases of [59, Theorem 6.8], according to which the derived category $\mathsf{D}(X)$ of complexes of sheaves of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves is compactly generated for any quasi-compact quasi-separated scheme X. Here one needs to know that the natural functor $\mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X)$ is an equivalence of categories when X is either quasi-compact and semi-separated, or else Noetherian (cf. [65, Appendix B]). In the semi-separated case, this was proven in [8, Sections 5–6]. The proof in the Noetherian case is similar.

Alternatively, one can prove parts (b) and (c) directly in the way analogous to the argument in [59]. In either approach, one needs to know that the functor $\mathbb{R}j_*$ of derived direct image of complexes over Y–qcoh with respect to an open embedding $j\colon Y \longrightarrow X$ of schemes from the class under consideration is well-behaved. E. g., it needs to be local in the base, or form a commutative square with the derived functor of direct image of complexes of $\mathcal{O}_Y$-modules, etc. (cf. [43, Theorems 31 and 42]).

In the semi-separated case, one can establish such properties using contraadjusted resolutions and (the proof of) Corollary 4.1.13(a) (see the construction of the functor $\mathbb{R}f_*$ in Section 4.8 above). In the Noetherian case, one needs to use flasque resolutions and Corollary 3.4.9(a) (see the construction of the functor $\mathbb{R}f_*$ in Section 5.12 below).

Part (d) follows from part (b) together with Theorem 4.6.6. Part (e) follows from part (c) together with Theorem 5.8.1 and Corollary 5.4.4(b). □

5.10. **Homotopy projective complexes.** Let X be a scheme. A complex $\mathcal{J}^\bullet$ of quasi-coherent sheaves on X is called *homotopy injective* if the complex of abelian groups $\operatorname{Hom}_X(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$ is acyclic for any acyclic complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on X. The full subcategory of homotopy injective complexes in $\mathsf{Hot}(X\text{–}\mathsf{qcoh})$ is denoted by $\mathsf{Hot}(X\text{–}\mathsf{qcoh})^{\mathsf{inj}}$ and the full subcategory of complexes of injective quasi-coherent sheaves that are also homotopy injective is denoted by $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})^{\mathsf{inj}} \subset \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$.

Similarly, a complex $\mathfrak{P}^\bullet$ of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves is called *homotopy projective* if the complex of abelian groups $\operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{M}^\bullet)$ is acyclic for any acyclic complex $\mathfrak{M}^\bullet$ over the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. The full subcategory of homotopy projective complexes in $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$ is denoted by $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})_{\mathsf{prj}}$. We will see below in this section that the property of a complex of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on a Noetherian scheme X of finite Krull dimension to be homotopy projective does not change when the covering $\mathbf{W}$ is replaced by its refinement.

Finally, a complex $\mathfrak{F}^\bullet$ of $\mathbf{W}$-locally contraherent cosheaves is called *homotopy projective* if the complex of abelian groups $\operatorname{Hom}^X(\mathfrak{F}^\bullet, \mathfrak{M}^\bullet)$ is acyclic for any acyclic complex $\mathfrak{M}^\bullet$ over the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. The full subcategory of homotopy projective complexes in $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is denoted by $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})_{\mathsf{prj}}$. Let us issue a *warning* that our terminology is misleading: a homotopy projective complex of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves need not be homotopy projective as a complex of $\mathbf{W}$-locally contraherent cosheaves. It will be shown below that the property of a complex of $\mathbf{W}$-locally contraherent cosheaves on a Noetherian scheme X of finite Krull dimension to be homotopy projective does not change when the covering $\mathbf{W}$ is replaced by its refinement.

Lemma 5.10.1. (a) *Let X be a locally Noetherian scheme of finite Krull dimension with an open covering $\mathbf{W}$. Then a complex $\mathfrak{P}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ belongs to $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})_{\mathsf{prj}}$ if and only if the complex $\operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{E}^\bullet)$ is acyclic for any complex $\mathfrak{E}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}^{\mathsf{lct}}$.*

(b) *Let X be a Noetherian scheme of finite Krull dimension with an open covering $\mathbf{W}$. Then a complex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ belongs to $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})_{\mathsf{prj}}$ if and only if the complex $\mathrm{Hom}^X(\mathfrak{F}^\bullet,\mathfrak{E}^\bullet)$ is acyclic for any complex $\mathfrak{E}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$.*

Proof. We will prove part (a), part (b) being similar. The "only if" assertion holds by the definition. To check the "if", consider a complex $\mathfrak{M}^\bullet$ over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. By (the proof of) Theorem 5.4.10(b), there exists a complex $\mathfrak{E}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ together with a morphism of complexes of locally contraherent cosheaves $\mathfrak{E}^\bullet \longrightarrow \mathfrak{M}^\bullet$ with a cone contraacyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. Moreover, the complex Hom^X from any complex of projective locally cotorsion contraherent cosheaves to a contraacyclic complex over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ is acyclic. Hence the morphism $\mathrm{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{E}^\bullet) \longrightarrow \mathrm{Hom}^X(\mathfrak{P}^\bullet,\mathfrak{M}^\bullet)$ is a quasi-isomorphism. Finally, if the complex $\mathfrak{M}^\bullet$ is acyclic over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, then so is the complex $\mathfrak{E}^\bullet$, and by Lemma 5.3.1(b) it follows that the complex $\mathfrak{E}^\bullet$ is also acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}^{\mathsf{lct}}$. □

According to Lemma 5.10.1, the property of a complex over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ (respectively, over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$) to belong to $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})_{\mathsf{prj}}$ (resp., $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})_{\mathsf{prj}}$) does not depend on the covering $\mathbf{W}$ (in the assumptions of the respective part of the lemma). We will denote the full subcategory in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})$ (resp., $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$) consisting of the homotopy projective complexes by $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})_{\mathsf{prj}}$ (resp., $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})_{\mathsf{prj}}$). It is a standard fact that bounded above complexes of projectives are homotopy projective, $\mathsf{Hot}^-(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \subset \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})_{\mathsf{prj}}$ and $\mathsf{Hot}^-(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \subset \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})_{\mathsf{prj}}$.

The next result is essentially well-known.

Theorem 5.10.2. *Let X be a locally Noetherian scheme. Then the natural functors $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})^{\mathsf{inj}} \longrightarrow \mathsf{Hot}(X\text{–}\mathsf{qcoh})^{\mathsf{inj}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ are equivalences of triangulated categories.*

Proof. It is clear that both functors are fully faithful. The functor $\mathsf{Hot}(X\text{–}\mathsf{qcoh})^{\mathsf{inj}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ is an equivalence of categories by [1, Theorem 5.4]; this is applicable to any Grothendieck abelian category in place of $X\text{–}\mathsf{qcoh}$ (for an even more general statement, see [39, Theorem 6]).

To prove that any homotopy injective complex of quasi-coherent sheaves on a locally Noetherian scheme is homotopy equivalent to a homotopy injective complex of injective quasi-coherent sheaves, one can use (the proof of) Theorem 5.4.10(a). From any complex $\mathcal{M}^\bullet$ over $X\text{–}\mathsf{qcoh}$ there exists a closed morphism into a complex $\mathcal{J}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ with a coacyclic cone $\mathcal{E}^\bullet$. If the complex $\mathcal{M}^\bullet$ was homotopy injective, the morphism $\mathcal{E}^\bullet \longrightarrow \mathcal{M}^\bullet[1]$ is homotopic to zero, hence the complex $\mathcal{E}^\bullet$ is a direct summand of $\mathcal{J}^\bullet$ in $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$. Any morphism $\mathcal{E}^\bullet \longrightarrow \mathcal{J}^\bullet$ being also homotopic to zero, it follows that the complex $\mathcal{E}^\bullet$ is contractible. □

Theorem 5.10.3. *Let X be a Noetherian scheme of finite Krull dimension with an open covering $\mathbf{W}$. Then*

(a) *the natural functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})_{\mathsf{prj}} \longrightarrow \mathsf{Hot}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})_{\mathsf{prj}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ *are equivalences of triangulated categories;*

(b) *the natural functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})_{\mathsf{prj}} \longrightarrow \mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})_{\mathsf{prj}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *are equivalences of triangulated categories.*

Proof. We will prove part (b), part (a) being similar. Since both functors are clearly fully faithful, it suffices to show that the composition $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})_{\mathsf{prj}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of categories. This is equivalent to saying that the localization functor $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ has a left adjoint whose image is essentially contained in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$.

The functor in question factorizes into the composition of two localization functors $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$. The functor $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ has a left adjoint functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \hookrightarrow \mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ provided by Theorem 5.4.10(d); so it remains to show that the functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ has a left adjoint.

Since the latter functor preserves infinite products, the assertion follows from Theorem 5.9.1(c) and [38, Proposition 3.3(2)]. Here one also needs to know that the derived category $\mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \simeq \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ "exists" (i. e., morphisms between any given two objects form a set rather than a class). This is established by noticing that the classes of quasi-isomorphisms are locally small in $\mathsf{Hot}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ (see [66, Section 10.3.6 and Proposition 10.4.4]).

Similarly one can prove Theorem 5.10.2 for a Noetherian scheme X using Theorems 5.4.10(a) and 5.9.1(b); the only difference is that this time one needs a right adjoint functor, so [38, Proposition 3.3(1)] has to be applied. (Cf. [53, Section 5.5]). ☐

A complex $\mathfrak{F}^\bullet$ of $\mathbf{W}$-locally contraherent cosheaves on a scheme X is called *homotopy flat* if the complex of abelian groups $\operatorname{Hom}^X(\mathfrak{F}^\bullet, \mathfrak{M}^\bullet)$ is acyclic for any complex of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{M}^\bullet$ acyclic over the exact category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$. On a locally Noetherian scheme X of finite Krull dimension, the latter condition is equivalent to the acyclicity over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see Corollary 1.5.7). Notice that the property of a complex of $\mathbf{W}$-locally contraherent cosheaves to be homotopy flat may possibly change when the covering $\mathbf{W}$ is replaced by its refinement.

Lemma 5.10.4. *Let X be a Noetherian scheme of finite Krull dimension with an open covering $\mathbf{W}$. Then a complex of flat contraherent cosheaves $\mathfrak{F}^\bullet$ on X is homotopy flat if and only if the complex* $\operatorname{Hom}^X(\mathfrak{F}^\bullet, \mathfrak{E}^\bullet)$ *is acyclic for any complex $\mathfrak{E}^\bullet$ over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{cfq}}$.*

Proof. Similar to that of Lemma 5.10.1. The only difference is that one has to use Corollary 5.2.9(a) in order to show that the complex Hom^X from any complex of flat contraherent cosheaves on X to a contraacyclic complex over $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is acyclic. ☐

According to Lemma 5.10.4, the property of a complex over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ to belong to $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})^{\mathsf{fl}}$ does not depend on the covering $\mathbf{W}$ (on a Noetherian scheme X of finite Krull dimension). We denote the full subcategory in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})$ consisting of the homotopy flat complexes by $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})^{\mathsf{fl}}$. One can easily check that bounded above complexes of flat contraherent cosheaves are homotopy flat, $\mathsf{Hot}^-(X\text{–}\mathsf{ctrh}^{\mathsf{fl}}) \subset \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})^{\mathsf{fl}}$.

Theorem 5.10.5. *Let X be a Noetherian scheme of finite Krull dimension. Then the quotient category of the homotopy category of homotopy flat complexes of flat contraherent cosheaves $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})^{\mathsf{fl}}$ on X by its thick subcategory of acyclic complexes over the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ is equivalent to the derived category $\mathsf{D}(X\text{–}\mathsf{ctrh})$.*

Proof. By Corollary 5.2.6(b) and [52, Remark 2.1], any acyclic complex over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ is absolutely acyclic; and one can see from Corollary 5.2.9(a) that any absolutely acyclic complex over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ is homotopy flat. According to (the proof of) Theorem 5.10.3(b), there is a quasi-isomorphism into any complex over $X\text{–}\mathsf{ctrh}$ from a complex belonging to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})_{\mathsf{prj}} \subset \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{fl}})^{\mathsf{fl}}$. In view of [53, Lemma 1.6], it remains to show that any homotopy flat complex of flat contraherent cosheaves that is acyclic over $X\text{–}\mathsf{ctrh}$ is also acyclic over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$.

According again to Corollary 5.2.6(b) and the dual version of the proof of Proposition A.5.6, any complex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ admits a morphism into a complex $\mathfrak{P}^\bullet$ over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ with a cone absolutely acyclic with respect to $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. If the complex $\mathfrak{F}^\bullet$ was homotopy flat, it follows that the complex $\mathfrak{P}^\bullet$ is homotopy flat, too. This means that $\mathfrak{P}^\bullet$ is a homotopy projective complex of locally cotorsion contraherent cosheaves on X. If the complex $\mathfrak{F}^\bullet$ was also acyclic over $X\text{–}\mathsf{ctrh}$, so is the complex $\mathfrak{P}^\bullet$. It follows that $\mathfrak{P}^\bullet$ is acyclic over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$, and therefore contractible. We have proven that the complex $\mathfrak{F}^\bullet$ is absolutely acyclic over $X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$. ☐

5.11. **Special inverse image of contraherent cosheaves.** Recall that an affine morphism of schemes $f\colon Y \longrightarrow X$ is called *finite* if for any affine open subscheme $U \subset X$ the ring $\mathcal{O}_Y(f^{-1}(U))$ is a finitely generated module over the ring $\mathcal{O}_X(U)$. One can easily see that this condition on a morphism f is local in X.

Let $f\colon Y \longrightarrow X$ be a finite morphism of locally Noetherian schemes. Given a quasi-coherent sheaf $\mathcal{M}$ on X, one defines the quasi-coherent sheaf $f^!\mathcal{M}$ on Y by the rule

$$(f^!\mathcal{M})(V) = \mathcal{O}_Y(V) \otimes_{\mathcal{O}_Y(f^{-1}(U))} \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(f^{-1}(U)),\, \mathcal{M}(U))$$

for any affine open subschemes $V \subset Y$ and $U \subset X$ such that $f(V) \subset U$ [30, Section III.6]. The construction is well-defined, since for any pair of embedded affine open subschemes $U' \subset U \subset X$ one has

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X(U')}&(\mathcal{O}_Y(f^{-1}(U'),\, \mathcal{M}(U')) \\ &\simeq \operatorname{Hom}_{\mathcal{O}_X(U')}(\mathcal{O}_X(U') \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(f^{-1}(U)),\, \mathcal{O}_X(U') \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)) \\ &\simeq \mathcal{O}_X(U') \otimes_{\mathcal{O}_X(U)} \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(f^{-1}(U)),\, \mathcal{M}(U)).\end{aligned}$$

Indeed, one has $\operatorname{Hom}_R(L,\, F \otimes_R M) \simeq F \otimes_R \operatorname{Hom}_R(L, M)$ for any module M, finitely presented module L, and flat module F over a commutative ring R. (See Section 3.3 for a treatment of the non-semi-separatedness issue.)

The functor $f^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ is right adjoint to the exact functor $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$. Indeed, it suffices to define a morphism of quasi-coherent sheaves on Y on the modules of sections over the affine open subschemes $f^{-1}(U) \subset Y$. So given quasi-coherent sheaves $\mathcal{M}$ on X and $\mathcal{N}$ on Y, both groups of morphisms $\mathrm{Hom}_X(f_*\mathcal{N}, \mathcal{M})$ and $\mathrm{Hom}_Y(\mathcal{N}, f^!\mathcal{M})$ are identified with the group of all compatible collections of morphisms of $\mathcal{O}_X(U)$-modules $\mathcal{N}(f^{-1}(U)) \longrightarrow \mathcal{M}(U)$, or equivalently, compatible collections of morphisms of $\mathcal{O}_Y(f^{-1}(U))$-modules $\mathcal{N}(f^{-1}(U)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(f^{-1}(U)),\ \mathcal{M}(U))$.

Let $i\colon Z \longrightarrow X$ be a closed embedding of locally Noetherian schemes. Let $\mathbf{W}$ be an open covering of X and $\mathbf{T}$ be an open covering of Z such that i is a $(\mathbf{W},\mathbf{T})$-coaffine morphism. Given a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X, one defines a $\mathbf{T}$-flat $\mathbf{T}$-locally contraherent cosheaf $i^*\mathfrak{F}$ on Z by the rule

$$(i^*\mathfrak{F})[i^{-1}(U)] = \mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U]$$

for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Clearly, affine open subschemes of the form $i^{-1}(U)$ constitute a base of the topology of Z. The construction is well-defined, since for any pair of embedded affine open subschemes $U' \subset U \subset X$ subordinate to $\mathbf{W}$ one has

$$\begin{aligned}
&\mathcal{O}_Z(i^{-1}(U')) \otimes_{\mathcal{O}_X(U')} \mathfrak{F}[U'] \\
&\qquad\simeq (\mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U')) \otimes_{\mathcal{O}_X(U')} \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U'), \mathfrak{F}[U]) \\
&\qquad\qquad\simeq \mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U'), \mathfrak{F}[U]) \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_Z(i^{-1}(U))}(\mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U'),\ \mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U]) \\
&\qquad\qquad\qquad\simeq \mathrm{Hom}_{\mathcal{O}_Z(i^{-1}(U))}(\mathcal{O}_Z(i^{-1}(U')),\ \mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U]),
\end{aligned}$$

where the third isomorphism holds by Lemma 1.6.6(c). The $\mathcal{O}_Z(i^{-1}(U))$-module $\mathcal{O}_Z(i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U]$ is contraadjusted by Lemma 1.6.6(b).

Assuming that the morphism i is $(\mathbf{W},\mathbf{T})$-affine and $(\mathbf{W},\mathbf{T})$-coaffine, the exact functor $i^*\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}} \longrightarrow Z\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{fl}}$ is "partially left adjoint" to the exact functor $i_!\colon Z\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$. In other words, for any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X and any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Z there is a natural adjunction isomorphism

$$\mathrm{Hom}^X(\mathfrak{F}, i_!\mathfrak{Q}) \simeq \mathrm{Hom}^Z(i^*\mathfrak{F}, \mathfrak{Q}). \tag{85}$$

Indeed, it suffices to define a morphism of $\mathbf{T}$-locally contraherent cosheaves on Z on the modules of cosections over the open subschemes $i^{-1}(U) \subset Z$ for all affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. So both groups of morphisms in question are identified with the group of all compatible collections of morphisms of $\mathcal{O}_X(U)$-modules $\mathfrak{F}[U] \longrightarrow \mathfrak{Q}[i^{-1}(U)]$, or equivalently, compatible collections of morphisms of $\mathcal{O}_Z((i^{-1}(U))$-modules $\mathcal{O}_Z((i^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U] \longrightarrow \mathfrak{Q}[i^{-1}(U)]$.

Notice that for any open covering $\mathbf{T}$ of a closed subscheme $Z \subset X$ there exists an open covering $\mathbf{W}$ of the scheme X for which the embedding morphism i is $(\mathbf{W},\mathbf{T})$-affine and $(\mathbf{W},\mathbf{T})$-coaffine. For a locally Noetherian scheme X of finite Krull dimension and its closed subscheme Z, one has $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}} = X\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ and $Z\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{fl}} = Z\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ for any open coverings $\mathbf{W}$ and $\mathbf{T}$ (see Corollary 5.2.2(b)). In this case, the adjunction isomorphism (85) holds for any flat contraherent cosheaf $\mathfrak{F}$ on X and locally contraherent cosheaf $\mathfrak{Q}$ on Z. Most generally, the isomorphism

$$\operatorname{Hom}^{\mathcal{O}_X}(\mathfrak{F}, i_!\mathfrak{Q}) \simeq \operatorname{Hom}^{\mathcal{O}_Z}(i^*\mathfrak{F}, \mathfrak{Q})$$

holds for any $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on a locally Noetherian scheme X and any cosheaf of $\mathcal{O}_Z$-modules $\mathfrak{Q}$ on a closed subscheme $Z \subset X$.

Let $f\colon Y \longrightarrow X$ be a finite morphism of locally Noetherian schemes. Given a projective locally cotorsion locally contraherent cosheaf $\mathfrak{P}$ on X, one defines a projective locally contraherent cosheaf $f^*\mathfrak{P}$ on Y by the rule

$$(f^*\mathfrak{P})[V] = \operatorname{Hom}_{\mathcal{O}_Y(f^{-1})(U))}(\mathcal{O}_Y(V),\ \mathcal{O}_Y(f^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U])$$

for any affine open subschemes $V \subset Y$ and $U \subset X$ such that $f(V) \subset U$. The construction is well-defined, since for any affine open subschemes there is a natural isomorphism of $\mathcal{O}_Y(f^{-1}(U'))$-modules

$$\mathcal{O}_Y(f^{-1}(U')) \otimes_{\mathcal{O}_X(U')} \mathfrak{P}[U'] \simeq \operatorname{Hom}_{\mathcal{O}_Y(f^{-1}(U))}(\mathcal{O}_Y(f^{-1}(U')),\ \mathcal{O}_Y(f^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U])$$

obtained in the way similar to the above computation for the closed embedding case, except that Lemma 1.6.7(c) is being applied. The $\mathcal{O}_Y(f^{-1}(U))$-module $\mathcal{O}_Y(f^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U]$ is flat and cotorsion by Lemma 1.6.7(a); hence the $\mathcal{O}_Y(V)$-module $(f^*\mathfrak{P})[V]$ is flat and cotorsion by Corollary 1.6.5(a) and Lemma 1.3.5(a). A contraherent cosheaf that is locally flat and cotorsion on a locally Noetherian scheme Y belongs to $Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ by Corollary 5.1.4.

Assuming that the morphism f is $(\mathbf{W},\mathbf{T})$-affine, the functor $f^*\colon X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ is "partially left adjoint" to the exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$. In other words, for any $\mathbf{W}$-flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X and any contraherent cosheaf $\mathfrak{Q}$ on Y there is a natural adjunction isomorphism

$$\operatorname{Hom}^X(\mathfrak{P}, f_!\mathfrak{Q}) \simeq \operatorname{Hom}^Y(f^*\mathfrak{P}, \mathfrak{Q}). \tag{86}$$

Indeed, it suffices to define a morphism of $\mathbf{T}$-locally contraherent cosheaves on Y on the modules of cosections over the open subschemes $f^{-1}(U) \subset Y$ for all affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$, and the construction proceeds exactly in the same way as in the above case of a closed embedding i. Generally, the isomorphism

$$\operatorname{Hom}^{\mathcal{O}_X}(\mathfrak{P}, f_!\mathfrak{Q}) \simeq \operatorname{Hom}^{\mathcal{O}_Y}(f^*\mathfrak{P}, \mathfrak{Q})$$

holds for any projective locally cotorsion contraherent cosheaf $\mathfrak{P}$ on X and any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$ on Y.

The functor $f^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ for a finite morphism of locally Noetherian schemes $f\colon Y \longrightarrow X$ preserves infinite direct sums of quasi-coherent cosheaves. The functor $i^*\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}} \longrightarrow Z\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{fl}}$ for a $(\mathbf{W},\mathbf{T})$-affine closed embedding of locally Noetherian schemes $i\colon Z \longrightarrow X$ preserves infinite products of flat locally contraherent cosheaves. The functor $f^*\colon X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ for a finite morphism of locally Noetherian schemes $f\colon Y \longrightarrow X$ preserves infinite products of projective locally cotorsion contraherent cosheaves.

5.12. **Derived functors of direct and special inverse image.** For the rest of Section 5, the upper index $\star$ in the notation for derived and homotopy categories stands for one of the symbols b, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs.

Let X be a locally Noetherian scheme with an open covering $\mathbf{W}$. The following corollary is to be compared with Corollary 5.3.2.

Corollary 5.12.1. (a) *The triangulated functor* $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh^{fq}}) \longrightarrow \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ *is an equivalence of categories. The category* $\mathsf{D}(X\text{–}\mathsf{qcoh})$ *is equivalent to the quotient category of the homotopy category* $\mathsf{Hot}(X\text{–}\mathsf{qcoh^{fq}})$ *by the thick subcategory of complexes over* $X\text{–}\mathsf{qcoh^{fq}}$ *that are acyclic over* $X\text{–}\mathsf{qcoh}$.

(b) *The triangulated functor* $\mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh^{lct}_{cfq}}) \longrightarrow \mathsf{D^{ctr}}(X\text{–}\mathsf{lcth^{lct}_{\mathbf{W}}})$ *is an equivalence of categories. The category* $\mathsf{D}(X\text{–}\mathsf{lcth^{lct}_{\mathbf{W}}})$ *is equivalent to the quotient category of the homotopy category* $\mathsf{Hot}(X\text{–}\mathsf{ctrh^{lct}_{cfq}})$ *by the thick subcategory of complexes over* $X\text{–}\mathsf{ctrh^{lct}_{cfq}}$ *that are acyclic over* $X\text{–}\mathsf{lcth^{lct}_{\mathbf{W}}}$.

Proof. The first assertions in (a) and (b) hold, because there are enough injective quasi-coherent sheaves, which are flasque, and enough projective locally cotorsion contraherent cosheaves, which are coflasque. Since the class of flasque quasi-coherent sheaves is also closed under infinite direct sums, while the class of coflasque contraherent cosheaves is closed under infinite products (see Section 3.4), the desired assertions are provided by Proposition A.3.1(b) and its dual version. Now we know that there is a quasi-isomorphism from any complex over $X\text{–}\mathsf{qcoh}$ into a complex over $X\text{–}\mathsf{qcoh^{fq}}$ and onto any complex over $X\text{–}\mathsf{lcth^{lct}_{\mathbf{W}}}$ from a complex over $X\text{–}\mathsf{ctrh^{lct}_{cfq}}$, so the second assertions in (a) and (b) follow by [53, Lemma 1.6]. ☐

Let $f\colon Y \longrightarrow X$ be a quasi-compact morphism of locally Noetherian schemes. As it was mentioned in Section 5.3, the functor

$$\mathbb{R}f_*\colon \mathsf{D^+}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^+}(X\text{–}\mathsf{qcoh}) \tag{87}$$

can be constructed using injective or flasque resolutions. More generally, the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D^\star}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^\star}(X\text{–}\mathsf{qcoh}) \tag{88}$$

can be constructed for $\star = \mathsf{co}$ using injective (see Theorem 5.4.10(a)) or flasque (see Corollary 5.12.1(a)) resolutions, and for $\star = \varnothing$ using homotopy injective (see Theorem 5.10.2) or flasque (cf. Corollary 3.4.9(a)) resolutions.

When the scheme Y has finite Krull dimension, the functor $\mathbb{R}f_*$ can be constructed for any symbol $\star \neq \mathsf{ctr}$ using flasque resolutions (see Corollary 5.3.2(a)). Finally, when both schemes X and Y are Noetherian, the functor (88) can be constructed for any symbol $\star \neq \mathsf{ctr}$ using f-acyclic resolutions (see Corollary 5.3.9(a)).

The functor

$$\mathbb{L}f_!\colon \mathsf{D^-}(Y\text{–}\mathsf{lcth^{lct}}) \longrightarrow \mathsf{D^-}(X\text{–}\mathsf{lcth^{lct}}) \tag{89}$$

was constructed in Section 5.3 using projective or coflasque resolutions. More generally, the left derived functor

$$\mathbb{L}f_! \colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \tag{90}$$

can be constructed for $\star = \mathsf{ctr}$ using projective (see Theorem 5.4.10(b)) or coflasque (see Corollary 5.12.1(b)) resolutions; and for $\star = \varnothing$ using homotopy projective (when X is Noetherian of finite Krull dimension, see Theorem 5.10.3(a)) or coflasque (in the general case, cf. Corollary 3.4.9(c)) resolutions.

When the scheme Y has finite Krull dimension, the functor (90) can be constructed for any symbol $\star \neq \mathsf{co}$ using coflasque resolutions (see Corollary 5.3.2(b)). Finally, when both schemes X and Y are Noetherian and one of the conditions of Lemma 5.3.7 is satisfied, one can construct the left derived functor

$$\mathbb{L}f_! \colon \mathsf{D}^\star(Y\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{T}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$$

for any symbol $\star \neq \mathsf{co}$ using $f/\mathbf{W}$-acyclic resolutions (see Corollary 5.3.9(c)).

Now assume that the scheme Y is Noetherian of finite Krull dimension. The functor

$$\mathbb{L}f_! \colon \mathsf{D}^-(Y\text{–}\mathsf{lcth}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{lcth}) \tag{91}$$

was constructed in Section 5.3 using projective or coflasque resolutions. More generally, the left derived functor

$$\mathbb{L}f_! \colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \tag{92}$$

can be constructed for any symbol $\star \neq \mathsf{co}$ using coflasque resolutions (see Corollary 5.3.2(c); cf. Corollary 5.3.3).

For any quasi-compact morphism f of locally Noetherian schemes and any symbol $\star = \varnothing$ or co, the functor $\mathbb{R}f_*$ (88) preserves infinite direct sums, as it is clear from its construction in terms of complexes of flasque quasi-coherent sheaves (cf. [47, Lemma 1.4]). By Theorems 5.9.1(b), 5.9.3(c), and [38, Proposition 3.3(1)], it follows that whenever the scheme Y is Noetherian there exists a triangulated functor

$$f^! \colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \tag{93}$$

right adjoint to $\mathbb{R}f_*$.

For any symbol $\star = \varnothing$ or ctr, the functor $\mathbb{L}f_!$ (90) preserves infinite products, as it is clear from its construction in terms of complexes of coflasque locally cotorsion contraherent cosheaves. By Theorems 5.9.1(c), 5.9.3(e), and [38, Proposition 3.3(2)], it follows that whenever the scheme Y is Noetherian of finite Krull dimension there exists a triangulated functor

$$f^* \colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \tag{94}$$

left adjoint to $\mathbb{L}f_!$. Assuming again that the scheme Y is Noetherian of finite Krull dimension, the functor $\mathbb{L}f_!$ (92) preserves infinite products, and it follows that there exists a triangulated functor

$$f^* \colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}) \tag{95}$$

left adjoint to $\mathbb{L}f_!$.

Now let $f\colon Y \longrightarrow X$ be a finite morphism of locally Noetherian schemes. Notice that any finite morphism is affine, so the functor $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ is exact; hence the induced functor

$$f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \tag{96}$$

defined for any symbol $\star \neq \mathsf{ctr}$. The right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \tag{97}$$

of the special inverse image functor $f^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ from Section 5.11 is constructed for $\star = +$ or co in terms of injective resolutions (see Theorem 5.4.10(a)), and for $\star = \varnothing$ in terms of homotopy injective resolutions (see Theorem 5.10.2). The right derived functor $\mathbb{R}f^!$ (97) is right adjoint to the induced functor f_* (96).

Similarly, the functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ is well-defined and exact, as is the functor $f_!\colon Y\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{ctrh}$; hence the induced functors

$$f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \tag{98}$$

and

$$f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \tag{99}$$

defined for any symbol $\star \neq \mathsf{co}$.

The left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \tag{100}$$

of the special inverse image functor $f^*\colon X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}} \longrightarrow Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ from Section 5.11 is constructed for $\star = -$ or ctr in terms of projective (locally cotorsion) resolutions (see Theorem 5.4.10(b)). When the scheme X is Noetherian of finite Krull dimension, the functor (100) is constructed for $\star = \varnothing$ in terms of homotopy projective resolutions (see Theorem 5.10.3(a)). The left derived functor $\mathbb{L}f^*$ (100) is left adjoint to the induced functor $f_!$ (98).

Finally, let $i\colon Z \longrightarrow X$ be a closed embedding of Noetherian schemes of finite Krull dimension. Then the left derived functor

$$\mathbb{L}i^*\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(Z\text{–}\mathsf{ctrh}) \tag{101}$$

of the special inverse image functor $i^*\colon X\text{–}\mathsf{ctrh}^{\mathsf{fl}} \longrightarrow Z\text{–}\mathsf{ctrh}^{\mathsf{fl}}$ from Section 5.11 is constructed for $\star = -$ or ctr in terms of flat resolutions (see Theorem 5.4.10(d)), and for $\star = \varnothing$ in terms of homotopy projective (see Theorem 5.10.3(b)) or flat and homotopy flat (see Theorem 5.10.5) resolutions. The left derived functor $\mathbb{L}i^*$ (101) is left adjoint to the induced functor $i_!$ (99). Clearly, the constructions of the derived functors (100) and (101) agree wherever both a defined.

The following theorem generalizes Corollaries 4.11.6 and 5.4.6 to morphisms f of not necessarily finite flat dimension (between Noetherian schemes).

Theorem 5.12.2. (a) *Let $f\colon Y \longrightarrow X$ be a morphism of semi-separated Noetherian schemes. Then the equivalences of triangulated categories $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ and $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(Y\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ from Theorem 5.7.1 transform* *the triangulated functor* $f^!\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ (93) *into the triangulated functor* $f^!\colon \mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{ctrh^{lin}})$ (59).

(b) *Let* $f\colon Y \longrightarrow X$ *be a morphism of semi-separated Noetherian schemes of finite Krull dimension. Then the equivalences of triangulated categories* $\mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ *and* $\mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ *from Theorem* 5.7.1 *transform the triangulated functor* $f^*\colon \mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}})$ (57) *into the triangulated functor* $f^*\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ (95).

Proof. Part (a): let us show that our the equivalences of categories transform the functor $\mathbb{R}f_*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ (50, 88) into a functor left adjoint to the functor $f^!\colon \mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{ctrh^{lin}})$ (59). Let $\mathcal{J}^\bullet$ be a complex over $Y\text{–}\mathsf{qcoh^{inj}}$ and $\mathfrak{K}^\bullet$ be a complex over $X\text{–}\mathsf{ctrh^{lin}_{clp}}$. Pick an open covering $\mathbf{T}$ of the scheme Y such that f is a $(\{X\},\mathbf{T})$-coaffine morphism. We have to construct a natural isomorphism of abelian groups $\operatorname{Hom}_{\mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}})}(\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, f_*\mathcal{J}^\bullet), \mathfrak{K}^\bullet) \simeq \operatorname{Hom}_{\mathsf{D^{abs}}(Y\text{–}\mathsf{lcth^{lin}_{\mathbf{T}}})}(\mathfrak{Hom}_Y(\mathcal{O}_Y, \mathcal{J}^\bullet), f^!\mathfrak{K}^\bullet)$.

Let $f_*\mathcal{J}^\bullet \longrightarrow \mathcal{J}'^\bullet$ be a morphism from the complex $f_*\mathcal{J}^\bullet$ over $X\text{–}\mathsf{qcoh^{cta}}$ (see Corollary 4.1.13(a)) to a complex $\mathcal{J}'^\bullet$ over $X\text{–}\mathsf{qcoh^{inj}}$ with a cone coacyclic with respect to $X\text{–}\mathsf{qcoh}$; so $\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, f_*\mathcal{J}^\bullet) = \mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J}'^\bullet)$. Then both $\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J}'^\bullet)$ and $\mathcal{K}^\bullet$ are complexes over $X\text{–}\mathsf{ctrh^{lin}_{clp}}$, hence one has

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}})}(\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J}'^\bullet), \mathfrak{K}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh^{lin}_{clp}})}(\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J}'^\bullet), \mathfrak{K}^\bullet)\\ &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh^{inj}})}(\mathcal{J}'^\bullet,\ \mathcal{O}_X \odot_X \mathfrak{K}^\bullet)\\ \simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh^{cta}})}(f_*\mathcal{J}^\bullet,\ \mathcal{O}_X \odot_X \mathfrak{K}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh_{clp}})}(\mathfrak{Hom}_X(\mathcal{O}_X, f_*\mathcal{J}^\bullet), \mathfrak{K}^\bullet).\end{aligned}$$

Here the first isomorphism holds because $\mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh^{lin}_{clp}})$ (see the proof of Corollary 4.6.8(b)) and the second isomorphism is provided by the equivalence of categories $X\text{–}\mathsf{ctrh^{lin}_{clp}} \simeq X\text{–}\mathsf{qcoh^{inj}}$. The third isomorphism follows from the proof of Lemma A.1.3(a) and the fourth one comes from the equivalence of categories $X\text{–}\mathsf{qcoh^{cta}} \simeq X\text{–}\mathsf{ctrh_{clp}}$ (see Lemma 4.6.7).

Furthermore, by (44) and (25) one has

$$\begin{aligned}\operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh_{clp}})}(\mathfrak{Hom}_X(\mathcal{O}_X, f_*\mathcal{J}^\bullet), \mathfrak{K}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh_{clp}})}(f_!\,\mathfrak{Hom}_Y(\mathcal{O}_Y, \mathcal{J}^\bullet), \mathfrak{K}^\bullet)\\ \simeq \operatorname{Hom}_{\mathsf{Hot}(Y\text{–}\mathsf{lcth^{lin}_{\mathbf{T}}})}(\mathfrak{Hom}_Y(\mathcal{O}_Y, \mathcal{J}^\bullet), f^!\mathfrak{K}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{D^{abs}}(Y\text{–}\mathsf{lcth^{lin}_{\mathbf{T}}})}(\mathfrak{Hom}_Y(\mathcal{O}_Y, \mathcal{J}^\bullet), f^!\mathfrak{K}^\bullet),\end{aligned}$$

where the last isomorphism follows from the proof of Lemma A.1.3(b), as the objects of $Y\text{–}\mathsf{ctrh^{lin}_{clp}}$ are projective in the exact category $Y\text{–}\mathsf{lcth^{lin}_{\mathbf{T}}}$.

Part (b): we will show that the equivalences of categories $\mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh^{lct}})$ and $\mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh^{lct}})$ transform the functor $f^*\colon \mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}})$ (57) into the functor $f^*\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh^{lct}}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh^{lct}})$ (94). Let $\mathcal{L}^\bullet$ be a complex over $X\text{–}\mathsf{qcoh^{cot}} \cap X\text{–}\mathsf{qcoh^{fl}}$ and $\mathfrak{G}^\bullet$ be a complex over $Y\text{–}\mathsf{ctrh^{lct}_{prj}}$. We have to construct a natural isomorphism of abelian groups $\operatorname{Hom}_{\mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}})}(\mathcal{L}^\bullet,\ \mathcal{O}_X \odot_X^{\mathbb{L}} f_!\mathfrak{G}^\bullet) \simeq \operatorname{Hom}_{\mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}})}(f^*\mathcal{L}^\bullet,\ \mathcal{O}_Y \odot_Y \mathfrak{G}^\bullet)$.

Let $\mathfrak{F}^\bullet \longrightarrow f_!\mathfrak{G}^\bullet$ be a morphism into the complex $f_!\mathfrak{G}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ (see Corollary 4.5.3(b)) from a complex $\mathfrak{F}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ with a cone contraacyclic with respect to $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$; so $\mathcal{O}_X \otimes_X^{\mathbb{L}} f_!\mathfrak{G}^\bullet = \mathcal{O}_X \otimes_X \mathfrak{F}^\bullet$. Then both $\mathcal{L}^\bullet$ and $\mathcal{O}_X \odot_X \mathfrak{F}^\bullet$ are complexes over $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$, hence one has

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})}(\mathcal{L}^\bullet,\ \mathcal{O}_X \odot_X \mathfrak{F}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}})}(\mathcal{L}^\bullet,\ \mathcal{O}_X \odot_X \mathfrak{F}^\bullet)\\ &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})}(\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{L}^\bullet), \mathfrak{F}^\bullet)\\ \simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}})}(\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{L}^\bullet), f_!\mathfrak{G}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}})}(\mathcal{L}^\bullet,\ \mathcal{O}_X \odot_X f_!\mathfrak{G}^\bullet).\end{aligned}$$

Here the first isomorphism holds because $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ (see the proof of Corollary 5.4.5) and the second isomorphism is provided by the equivalence of categories $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ (see the proof of Corollary 4.6.10(c)). The third isomorphism follows from the proof of Lemma A.1.3(b) and the fourth one comes from the equivalence of categories $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \simeq X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ (see the proof of Corollary 4.6.8(a)).

Furthermore, one has

$$\begin{aligned}\operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}})}(\mathcal{L}^\bullet,\ \mathcal{O}_X \odot_X f_!\mathfrak{G}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{cot}})}(\mathcal{L}^\bullet,\ f_*(\mathcal{O}_Y \odot_Y \mathfrak{G}^\bullet))\\ \simeq \operatorname{Hom}_{\mathsf{Hot}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})}(f^*\mathcal{L}^\bullet,\ \mathcal{O}_Y \odot_Y \mathfrak{G}^\bullet) &\simeq \operatorname{Hom}_{\mathsf{D}^{\mathsf{abs}}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})}(f^*\mathcal{L}^\bullet,\ \mathcal{O}_Y \odot_Y \mathfrak{G}^\bullet).\end{aligned}$$

Here the first isomorphism holds according to the proof of Theorem 4.8.1 (since $\mathfrak{G}^\bullet$ is a complex of colocally projective contraherent cosheaves on Y) and the last one follows from the proof of Lemma A.1.3(a) (as the objects of $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ are injective in the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$). ☐

5.13. **Adjoint functors and bounded complexes.** The following theorem is a version of Theorem 4.8.1 for non-semi-separated Noetherian schemes. One would like to have it for an arbitrary morphism of Noetherian schemes of finite Krull dimension, but we are only able to present a proof in the case of a flat morphism.

Theorem 5.13.1. *Let $f\colon Y \longrightarrow X$ be a flat morphism of Noetherian schemes of finite Krull dimension. Then the equivalences of triangulated categories $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh})$ and $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh})$ from Theorem 5.8.1 transform the right derived functor $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ (88) into the left derived functor $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ (92).*

Proof. Clearly, for any morphism f of Noetherian schemes of finite Krull dimension and any symbol $\star \neq \mathsf{co}$ the equivalence of categories $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \simeq \mathsf{D}^\star(Y\text{–}\mathsf{ctrh})$ induced by the embedding of exact categories $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{ctrh}$ and the similar equivalence for X transform the left derived functor $\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ (90) into the functor (92). Hence it remains to check that the equivalences of triangulated categories $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ and $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ constructed in the proof of Theorem 5.8.1 transform the functor (88) into the functor (90).

Let $\mathcal{O}_X \longrightarrow \mathcal{E}^\bullet_X$ be a finite resolution of the sheaf $\mathcal{O}_X$ by flasque quasi-coherent sheaves on X. Then the morphism $\mathcal{O}_Y \longrightarrow f^*\mathcal{E}^\bullet_X$ is a quasi-isomorphism in Y–qcoh. Pick a finite resolution $f^*\mathcal{E}^\bullet_X \longrightarrow \mathcal{E}^\bullet_Y$ of the complex $f^*\mathcal{E}^\bullet_X$ by flasque quasi-coherent sheaves on Y. Then the composition $\mathcal{O}_Y \longrightarrow \mathcal{E}^\bullet_Y$ is also a quasi-isomorphism.

According to Section 3.8, for any complex $\mathcal{J}^\bullet$ over Y–$\mathsf{qcoh}^{\mathsf{inj}}$ there is a natural morphism $f_!\,\mathfrak{Hom}_Y(f^*\mathcal{E}_X,\mathcal{J}^\bullet) \longrightarrow \mathfrak{Hom}_X(\mathcal{E}^\bullet_X, f_*\mathcal{J}^\bullet)$ of complexes of cosheaves of $\mathcal{O}_X$-modules. Composing this morphism with the morphism $f_!\,\mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,\mathcal{J}^\bullet) \longrightarrow f_!\,\mathfrak{Hom}_Y(f^*\mathcal{E}^\bullet_X,\mathcal{J}^\bullet)$ induced by the quasi-isomorphism $f^*\mathcal{E}^\bullet_X \longrightarrow \mathcal{E}^\bullet_X$, we obtain a natural morphism of complexes of cosheaves of $\mathcal{O}_X$-modules $f_!\,\mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,\mathcal{J}^\bullet) \longrightarrow \mathfrak{Hom}_X(\mathcal{E}^\bullet_X, f_*\mathcal{J}^\bullet)$. Finally, pick a quasi-isomorphism $\mathfrak{F}^\bullet \longrightarrow \mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,\mathcal{J}^\bullet)$ of complexes over the exact category Y–$\mathsf{ctrh}^{\mathsf{lct}}$ acting from a complex $\mathfrak{F}^\bullet$ over Y–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{cfq}}$ to the complex $\mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,\mathcal{J}^\bullet)$. Applying the functor $f_!$ and composing again, we obtain a morphism $\mathbb{L}f_!\,\mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,\mathcal{J}^\bullet) = f_!\mathfrak{F}^\bullet \longrightarrow \mathfrak{Hom}_X(\mathcal{E}^\bullet_X, f_*\mathcal{J}^\bullet)$ of complexes over X–$\mathsf{ctrh}^{\mathsf{lct}}$. We have constructed a natural transformation

$$\mathbb{L}f_!\,\mathbb{R}\,\mathfrak{Hom}_Y(\mathcal{E}^\bullet_Y,-) \longrightarrow \mathbb{R}\,\mathfrak{Hom}_X(\mathcal{E}^\bullet_X,\mathbb{R}f_*(-))$$

of functors $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$. One can easily check that such natural transformations are compatible with the compositions of flat morphisms f.

Now one can cover the scheme X with affine open subschemes U_α and the scheme Y with affine open subschemes V_β so that for any β there exists α for which $f(V_\beta) \subset V_\alpha$. The triangulated category $\mathsf{D}(Y\text{–}\mathsf{qcoh})$ is generated by the derived direct images of objects from $\mathsf{D}(V_\alpha\text{–}\mathsf{qcoh})$. So it suffices to show that our natural transformation is an isomorphism whenever either both schemes X and Y are semi-separated, or the morphism f is an open embedding. The former case is covered by Theorem 4.8.1, while in the latter situation one can use the isomorphism (45) together with Lemma 3.4.6(c).

Alternatively, according to Section 3.8 for any complex $\mathfrak{P}^\bullet$ over Y–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ there is a natural morphism $\mathcal{E}^\bullet_X \odot_X f_!\mathfrak{P}^\bullet \longrightarrow f_*(f^*\mathcal{E}^\bullet_X \odot_Y \mathfrak{P}^\bullet)$ of complexes of quasi-coherent sheaves on X. Composing it with the morphism $f_*(f^*\mathcal{E}^\bullet_X \odot_Y \mathfrak{P}^\bullet) \longrightarrow f_*(\mathcal{E}^\bullet_Y \odot_Y \mathfrak{P}^\bullet)$ induced by the quasi-isomorphism $f^*\mathcal{E}^\bullet_X \longrightarrow \mathcal{E}^\bullet_X$, we obtain a natural morphism of complexes of quasi-coherent sheaves $\mathcal{E}^\bullet_X \odot_X f_!\mathfrak{P}^\bullet \longrightarrow f_*(\mathcal{E}^\bullet_Y \odot_Y \mathfrak{P}^\bullet)$ on X. Finally, pick a quasi-isomorphism $\mathcal{E}^\bullet_Y \odot_Y \mathfrak{P}^\bullet \longrightarrow \mathcal{K}^\bullet$ of complexes over Y–qcoh acting from the complex $\mathcal{E}^\bullet_Y \odot_Y \mathfrak{P}^\bullet$ to a complex $\mathcal{F}^\bullet$ over Y–$\mathsf{qcoh}^{\mathsf{fq}}$. Applying the functor f_* and composing again, we obtain a morphism $\mathcal{E}^\bullet_X \odot_X f_!\mathfrak{P}^\bullet \longrightarrow f_*\mathcal{F}^\bullet = \mathbb{R}f_*(\mathcal{E}^\bullet_Y \odot_Y \mathfrak{P}^\bullet)$ of complexes over X–qcoh. We have constructed a natural transformation

$$\mathcal{E}^\bullet_X \odot^{\mathbb{L}}_X \mathbb{L}f_!(-) \longrightarrow \mathbb{R}f_*(\mathcal{E}^\bullet_Y \odot^{\mathbb{L}}_Y -)$$

of functors $\mathsf{D}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$. To finish the proof, one continues to argue as above, using the isomorphism (47) and Lemma 3.4.6(d). ☐

Remark 5.13.2. Let $f\colon Y \longrightarrow X$ be a morphism of Noetherian schemes of finite Krull dimension. If the conclusion of Theorem 5.13.1 holds for f, it follows by adjunction that the equivalences of categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh})$ from Theorem 5.8.1 transform the functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ (93) into a functor right adjoint to $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ (92) and the functor $f^*\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ (95) into a functor left adjoint to $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ (88).

Of course, these are supposed to be the conventional derived functors of inverse image of quasi-coherent sheaves and contraherent cosheaves. One can notice, however, that the conventional constructions of such functors involve some difficulties when one is working outside of the situations covered by Theorems 4.8.1 and 5.13.1. The case of semi-separated schemes X and Y is covered by our exposition in Section 4 (see (55–56)), and in the case of a flat morphism f the underived inverse image would do (if one is working with locally cotorsion contraherent cosheaves).

In the general case, it is not clear if there exist enough flat quasi-coherent sheaves or locally injective contraherent cosheaves to make the derived functor constructions work. One can construct the derived functor $\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ using complexes of sheaves of $\mathcal{O}$-modules with quasi-coherent cohomology sheaves. We do *not* know how to define a derived functor $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ or $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ for an arbitrary morphism f of Noetherian schemes of finite Krull dimension.

Lemma 5.13.3. (a) *For any morphism $f\colon Y \longrightarrow X$ from a Noetherian scheme Y to a locally Noetherian scheme X such that either the scheme X is Noetherian or the scheme Y has finite Krull dimension, the triangulated functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ (93) takes $\mathsf{D}^+(X\text{–}\mathsf{qcoh})$ into $\mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ and induces a triangulated functor $f^!\colon \mathsf{D}^+(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ right adjoint to the right derived functor $\mathbb{R}f_*$ (87).*

(b) *For any morphism of Noetherian schemes of finite Krull dimension $f\colon Y \longrightarrow X$, the triangulated functor $f^*\colon \mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ (94) takes $\mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ into $\mathsf{D}^-(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ and induces a triangulated functor $f^*\colon \mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ left adjoint to the left derived functor $\mathbb{L}f_!$ (89).*

(c) *For any morphism of Noetherian schemes of finite Krull dimension $f\colon Y \longrightarrow X$, the triangulated functor $f^*\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ (95) takes $\mathsf{D}^-(X\text{–}\mathsf{ctrh})$ into $\mathsf{D}^-(Y\text{–}\mathsf{ctrh})$ and induces a triangulated functor $f^*\colon \mathsf{D}^-(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{ctrh})$ left adjoint to the left derived functor $\mathbb{L}f_!$ (91).*

Proof. In each part (a-c), the second assertion follows immediately from the first one (since the derived functors of direct image of bounded and unbounded complexes agree). In view of Corollary 5.4.4(b), part (c) is also equivalent to part (b).

To prove part (a), notice that a complex of quasi-coherent sheaves $\mathcal{N}^\bullet$ over Y has its cohomology sheaves concentrated in the cohomological degrees $\geqslant -N$ if and only if one has $\operatorname{Hom}_{\mathsf{D}(Y\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{N}^\bullet) = 0$ for any complex $\mathcal{L}^\bullet$ over $Y\text{–}\mathsf{qcoh}$ concentrated in the cohomological degrees $< -N$. This is true for any abelian category in place of $Y\text{–}\mathsf{qcoh}$. Now given a complex $\mathcal{M}^\bullet$ over $X\text{–}\mathsf{qcoh}$, one has $\operatorname{Hom}_{\mathsf{D}(Y\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, f^!\mathcal{M}^\bullet) \simeq \operatorname{Hom}_{\mathsf{D}(X\text{–}\mathsf{qcoh})}(\mathbb{R}f_*\mathcal{L}^\bullet, \mathcal{M}^\bullet) = 0$ whenever $\mathcal{M}^\bullet$ is concentrated in the cohomological degrees $\geqslant -N + M$, where M is a certain fixed constant. Indeed, the functor $\mathbb{R}f_*$ raises the cohomological degrees by at most the Krull dimension of the scheme Y (if it has finite Krull dimension), or by at most the constant from Lemmas 5.3.7–5.3.8 (if both schemes are Noetherian).

In order to deduce parts (b-c), we will use Theorem 5.13.1. The equivalence of triangulated categories $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh})$ from Theorem 5.8.1 identifies $\mathsf{D}^-(Y\text{–}\mathsf{qcoh})$ with $\mathsf{D}^-(Y\text{–}\mathsf{ctrh})$. Given a complex $\mathfrak{M}^\bullet$ from $\mathsf{D}^-(X\text{–}\mathsf{ctrh})$, we would like to show that the complex $\mathcal{N}^\bullet$ over $Y\text{–}\mathsf{qcoh}$ corresponding to the complex $\mathfrak{N}^\bullet = f^*\mathfrak{M}^\bullet$ over $Y\text{–}\mathsf{ctrh}$ belongs to $\mathsf{D}^-(Y\text{–}\mathsf{qcoh})$. This is equivalent to saying that the complex $j'^*\mathcal{N}^\bullet$ over $V\text{–}\mathsf{qcoh}$ belongs to $\mathsf{D}^-(V\text{–}\mathsf{qcoh})$ for the embedding of any small enough affine open subscheme $j'\colon V \longrightarrow Y$.

By Theorem 5.13.1 applied to the morphism j', the complex $j'^*\mathcal{N}^\bullet$ corresponds to the complex $j'^*\mathfrak{N}^\bullet$ over $V\text{–}\mathsf{ctrh}$. We can assume that the composition $f \circ j'\colon V \longrightarrow X$ factorizes through the embedding of an affine open subscheme $j\colon U \longrightarrow X$. Let f' denote the related morphism $V \longrightarrow U$. Then one has $j'^*\mathfrak{N}^\bullet \simeq f^*j^*\mathfrak{M}^\bullet$ in $\mathsf{D}(V\text{–}\mathsf{ctrh})$. Let $\mathcal{M}^\bullet$ denote the complex over $X\text{–}\mathsf{qcoh}$ corresponding to $\mathfrak{M}^\bullet$; by Theorem 5.8.1, one has $\mathcal{M}^\bullet \in \mathsf{D}^-(X\text{–}\mathsf{qcoh})$. Applying again Theorem 5.13.1 to the morphism j and Theorem 4.8.1 to the morphism f', we conclude that the complex $f^*j^*\mathfrak{M}$ over $\mathsf{D}(V\text{–}\mathsf{ctrh})$ corresponds to the complex $\mathbb{L}f^*j^*\mathcal{M}$ over $\mathsf{D}(V\text{–}\mathsf{qcoh})$. The latter is clearly bounded above, and the desired assertion is proven. □

Corollary 5.13.4. (a) *For any morphism $f\colon Y \longrightarrow X$ from a Noetherian scheme Y to a locally Noetherian scheme X such that either the scheme X is Noetherian or the scheme Y has finite Krull dimension, the triangulated functor $f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$* (93) *takes $\mathsf{D}^+(X\text{–}\mathsf{qcoh})$ into $\mathsf{D}^+(Y\text{–}\mathsf{qcoh})$, and the induced triangulated functor $f^!\colon \mathsf{D}^+(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ coincides with the one obtained in Lemma* 5.13.3.

(b) *For any morphism of Noetherian schemes of finite Krull dimension $f\colon Y \longrightarrow X$, the triangulated functor $f^*\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$* (94) *takes $\mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ into $\mathsf{D}^-(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$, and the induced triangulated functor $f^*\colon \mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ coincides with the one obtained in Lemma* 5.13.3.

(c) *For any morphism of Noetherian schemes of finite Krull dimension $f\colon Y \longrightarrow X$, the triangulated functor $f^*\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh})$* (95) *takes $\mathsf{D}^-(X\text{–}\mathsf{ctrh})$ into $\mathsf{D}^-(Y\text{–}\mathsf{ctrh})$, and the induced triangulated functor $f^*\colon \mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ coincides with the one obtained in Lemma* 5.13.3.

Proof. Notice first of all that there are natural fully faithful functors $\mathsf{D}^+(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, $\mathsf{D}^-(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$, $\mathsf{D}^-(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ (and similarly for Y) by Lemma A.1.2. Furthermore, one can prove the first assertion of part (a) in the way similar to the proof of Lemma 5.13.3(a).

A complex of quasi-coherent sheaves $\mathcal{N}^\bullet$ over Y is isomorphic in $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ to a complex whose terms are concentrated in the cohomological degrees $\geqslant -N$ if and only if one has $\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{N}^\bullet) = 0$ for any complex $\mathcal{L}^\bullet$ over $Y\text{–}\mathsf{qcoh}$ whose terms are concentrated in the cohomological degrees $< N$. This is true for any abelian category with exact functors of infinite direct sum in place of $Y\text{–}\mathsf{qcoh}$, since complexes with the terms concentrated in the degrees $\leqslant 0$ and $\geqslant 0$ form a t-structure on the coderived category [52, Remark 4.1]. For a more explicit argument, see [38, Lemma 2.2]. The rest of the proof of the first assertion is similar to that of Lemma 5.13.3(a); and both assertions can be proven in the way dual-analogous to the following proof of part (c).

It is clear from the constructions of the functors $\mathbb{L}f_!$ (92) in terms of coflasque resolutions that the functors $\mathbb{L}f_!\colon \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ and $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ form a commutative diagram with the Verdier localization functors $\mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$. According to (95) and (the proof of) Theorem 5.10.3(b), all the functors in this commutative square have left adjoints, which therefore also form a commutative square.

The functor $\mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ left adjoint to the localization functor can be constructed as the functor assigning to a complex over $X\text{–}\mathsf{ctrh}$ its homotopy projective resolution, viewed as an object of $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ (and similarly for Y). Since the homotopy projective resolution $\mathfrak{P}^\bullet$ of a bounded above complex $\mathfrak{M}^\bullet$ over $X\text{–}\mathsf{ctrh}$ can be chosen to be also bounded above, and the cone of the morphism $\mathfrak{P}^\bullet \longrightarrow \mathfrak{M}^\bullet$ is contraacyclic, part (c) follows from Lemma 5.13.3(c). □

Proposition 5.13.5. *Let $f\colon Y \longrightarrow X$ be a proper morphism (of finite type and) of finite flat dimension between semi-separated Noetherian schemes. Assume that the object $f^!\mathcal{O}_X$ is compact in $\mathsf{D}(Y\text{–}\mathsf{qcoh})$ (i. e., it is a perfect complex on Y). Then the functor $f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ (93) is naturally isomorphic to the functor $f^!\mathcal{O}_X \otimes^{\mathbb{L}'}_{\mathcal{O}_X} \mathbb{L}f^*$, where $f^!\mathcal{O}_X \in \mathsf{D}(Y\text{–}\mathsf{qcoh})$, while $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ is the functor constructed in (60) and $\otimes^{\mathbb{L}'}_{\mathcal{O}_X}$ is the tensor action functor (77).*

Proof. Instead of proving the desired assertion from scratch, we will deduce it step by step from the related results of [47, Section 5], using the results above in this section to brigde the gap between the derived and coderived categories.

Lemma 5.13.6. *Let $f\colon Y \longrightarrow X$ be a morphism of finite flat dimension between semi-separated Noetherian schemes. Then for any objects $\mathcal{L}^\bullet \in \mathsf{D}(Y\text{–}\mathsf{qcoh})$ and $\mathcal{M}^\bullet \in \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ there is a natural isomorphism $\mathbb{R}f_*\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_X} \mathcal{M}^\bullet \simeq \mathbb{R}f_*(\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{M}^\bullet)$ in $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, where $\mathbb{R}f_*$ denotes the derived direct image functors (50).*

Proof. For any objects $\mathcal{K}^\bullet \in \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathcal{M}^\bullet \in \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ one easily constructs a natural isomorphism $\mathbb{L}f^*(\mathcal{K}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_X} \mathcal{M}^\bullet) \simeq \mathbb{L}f^*\mathcal{K}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{M}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ (where $\mathbb{L}f^*$ denotes the derived functors (55, 60)). Substituting $\mathcal{K}^\bullet = \mathbb{R}f_*\mathcal{L}^\bullet$ with $\mathcal{L}^\bullet \in \mathsf{D}(Y\text{–}\mathsf{qcoh})$, one can consider the composition $\mathbb{L}f^*(\mathbb{R}f_*\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_X} \mathcal{M}^\bullet) \simeq \mathbb{L}f^*\mathbb{R}f_*\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{M}^\bullet \longrightarrow \mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{M}^\bullet$ of morphisms in $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$. By adjunction, we obtain the natural transformation $\mathbb{R}f_*\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_X} \mathcal{M}^\bullet \longrightarrow \mathbb{R}f_*(\mathcal{L}^\bullet \otimes^{\mathbb{L}'}_{\mathcal{O}_Y} \mathbb{L}f^*\mathcal{M}^\bullet)$ of functors $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \times \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$.

Since all the functor involved preserve infinite direct sums, it suffices to check that our morphism is an isomorphism for compact generators of the categories $\mathsf{D}(Y\text{–}\mathsf{qcoh})$ and $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, that is one can assume $\mathcal{L}^\bullet$ to be a perfect complex on X and $\mathcal{M}^\bullet$ to be a finite complex of coherent sheaves on Y (see Theorems 5.9.3(b) and 5.9.1(b)). In this case all the complexes involved are bounded below and, in view of Lemma A.1.2, the question reduces to the similar assertion for the conventional derived categories, which is known due to [47, Proposition 5.3]. □

In particular, for any object $\mathcal{M}^\bullet \in \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ we have a natural isomorphism $\mathbb{R}f_*f^!\mathcal{O}_X \otimes_{\mathcal{O}_X}^{\mathbb{L}'} \mathcal{M}^\bullet \simeq \mathbb{R}f_*(f^!\mathcal{O}_X \otimes_{\mathcal{O}_Y}^{\mathbb{L}'} \mathbb{L}f^*\mathcal{M}^\bullet)$. Composing it with the morphism induced by the adjunction morphism $\mathbb{R}f_*f^!\mathcal{O}_X \longrightarrow \mathcal{O}_X$, we obtain a natural morphism $\mathbb{R}f_*(f^!\mathcal{O}_X \otimes_{\mathcal{O}_Y}^{\mathbb{L}'} \mathbb{L}f^*\mathcal{M}^\bullet) \longrightarrow \mathcal{M}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$. We have constructed a natural transformation $f^!\mathcal{O}_X \otimes_{\mathcal{O}_Y}^{\mathbb{L}'} \mathbb{L}f^*\mathcal{M}^\bullet \longrightarrow f^!\mathcal{M}^\bullet$ of functors $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$.

Since the morphism f is proper, the functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ takes compact objects to compact objects [28, Théorème 3.2.1]. By [47, Theorem 5.1], it follows that the functor $f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ preserves infinite direct sums. So does the functor in the left-hand side of our morphism; therefore, it suffices to check that this morphism is an isomorphism when $\mathcal{M}^\bullet$ is a finite complex of coherent sheaves on X. Since we assume $f^!\mathcal{O}_X$ to be a perfect complex, in this case all the complexes involved are bounded below, and in view of Corollary 5.13.4(a), the question again reduces to the similar assertion for the conventional derived categories, which is provided by [47, Example 5.2 and Theorem 5.4]. ☐

Remark 5.13.7. The condition that $f^!\mathcal{O}_X$ is a perfect complex, which was not needed in [47, Section 5], cannot be dropped in the above Proposition, as one can see already in the case when X is the spectrum of a field and $Y \longrightarrow X$ is a finite morphism. The problem arises because of the difference between the left and right adjoint functors to the Verdier localization functor $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$.

To construct a specific counterexample, let k be a field and R be the quotient ring $R = k[x,y]/(x^2,xy,y^2)$, so that the images of the elements 1, x, and y form a basis in R over k. Set $X = \operatorname{Spec} k$ and $Y = \operatorname{Spec} R$. Then the injective coherent sheaf $\widetilde{R^*}$ on Y corresponding to the R-module $R^* = \operatorname{Hom}_k(R,k)$ represents the object $f^!\mathcal{O}_X \in \mathsf{D}(Y\text{–}\mathsf{qcoh})$. We have to show that the R-module R^* is not isomorphic in $\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod})$ to its left projective resolution.

Indeed, otherwise it would follow that the complex $\operatorname{Hom}_R(R^*, J^\bullet)$ is acyclic for any acyclic complex of injective R-modules $J^\bullet$. Since there is a short exact sequence of R-modules $0 \longrightarrow k^{\oplus 3} \longrightarrow R^{\oplus 2} \longrightarrow R^* \longrightarrow 0$, the complex $\operatorname{Hom}_R(k, J^\bullet)$ would then be also acyclic, making the complex of R-modules $J^\bullet$ contractible. This would mean that the coderived category of R-modules coincides with their derived category, which cannot be true, as their subcategories of compact objects are clearly different.

On the other hand, one can get rid of the semi-separability assumptions in Proposition 5.13.5 by constructing the functor $\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ in terms of complexes of sheaves of $\mathcal{O}$-modules with quasi-coherent cohomology sheaves (as it is done in [47]; cf. the proof of Theorem 5.9.3(b-c)) and obtaining the functor $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ from it using the techniques of [22].

5.14. **Compatibilities for a smooth morphism.** Let $f\colon Y \longrightarrow X$ be a smooth morphism of Noetherian schemes. Let $\mathcal{D}_X^\bullet$ be a dualizing complex for X; then $f^*\mathcal{D}_X^\bullet$ is a dualizing complex for Y [30, Theorem V.8.3]. The complex $f^*\mathcal{D}_X^\bullet$ being not necessarily a complex of injectives, let us pick a finite complex over $Y\text{–}\mathsf{qcoh}$ quasi-isomorphic to $f^*\mathcal{D}_X^\bullet$ and denote it temporarily by $\mathcal{D}_Y^\bullet$.

Corollary 5.14.1. *Assume the schemes X and Y to be semi-separated. Then*

(a) *the equivalences of triangulated categories* $\mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ *and* $\mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}}) \simeq \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ *from Theorem* 5.7.1 *related to the choice of the dualizing complexes* $\mathcal{D}_X^\bullet$ *and* $\mathcal{D}_Y^\bullet$ *on* X *and* Y *transform the inverse image functor* $f^*\colon \mathsf{D^{abs}}(X\text{–}\mathsf{qcoh^{fl}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}})$ (57) *into the (underived, as the morphism* f *is flat) inverse image functor* $f^*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ (60)*;*

(b) *the equivalences of triangulated categories* $\mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ *and* $\mathsf{D^{abs}}(Y\text{–}\mathsf{ctrh^{lin}}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ *from Theorem* 5.7.1 *related to the choice of the dualizing complexes* $\mathcal{D}_X^\bullet$ *and* $\mathcal{D}_Y^\bullet$ *on* X *and* Y *transform the inverse image functor* $f^!\colon \mathsf{D^{abs}}(X\text{–}\mathsf{ctrh^{lin}}) \longrightarrow \mathsf{D^{abs}}(Y\text{–}\mathsf{ctrh^{lin}})$ (59) *into the inverse image functor* $\mathbb{R}f^!\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ (62)*.*

Remark 5.14.2. It would follow from Conjecture 1.7.2 that the inverse image functor (62) in the formulation of part (b) of Corollary 5.14.1 is in fact an underived inverse image functor $f^!$, just as in part (a). In any event, the morphism f being at least flat, there is an underived inverse image functor $f^!\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh^{lct}}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh^{lct}})$ (64) (cf. the proof below).

Proof of Corollary 5.14.1. Part (a) (cf. [15, Section 3.8]): notice that for any finite complex $\mathcal{E}^\bullet$ of quasi-coherent sheaves on Y the functor $\mathsf{D^{abs}}(Y\text{–}\mathsf{qcoh^{fl}}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ constructed by tensoring complexes over $Y\text{–}\mathsf{qcoh^{fl}}$ with the complex $\mathcal{E}^\bullet$ over $\mathcal{O}_Y$ is well-defined, and replacing $\mathcal{E}^\bullet$ by a quasi-isomorphic complex leads to an isomorphic functor. So it remains to use the isomorphism $f^*(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \simeq f^*\mathcal{D}_X \otimes_{\mathcal{O}_X} f^*\mathcal{F}$ holding for any flat (or even arbitrary) quasi-coherent sheaf $\mathcal{F}$ on X.

Part (b): let $\mathbf{W}$ and $\mathbf{T}$ be open coverings of the schemes X and Y such that the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine. For any finite complex $\mathcal{E}^\bullet$ of quasi-coherent sheaves on Y, the functor $\mathsf{D^{abs}}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}})$ constructed by taking $\mathfrak{Cohom}_Y$ from $\mathcal{E}^\bullet$ to complexes over $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$ is well-defined, and replacing $\mathcal{E}^\bullet$ by a quasi-isomorphic complex leads to an isomorphic functor. So it remains to use the isomorphism (38) together with the fact that the equivalence of triangulated categories $\mathsf{D^{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \longrightarrow \mathsf{D^{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ induced by the embedding of exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and the similar equivalence for Y transform the functor (63) into the functor (61) for any morphism f of finite very flat dimension between quasi-compact semi-separated schemes. □

The following theorem is to be compared with Theorems 4.8.1 and 5.13.1, and Corollaries 4.11.6 and 5.4.6.

Theorem 5.14.3. *The equivalences of triangulated categories* $\mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ *and* $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ *from Theorem* 5.7.1 *or* 5.8.2 *related to the choice of the dualizing complexes* $\mathcal{D}_Y^\bullet$ *and* $\mathcal{D}_X^\bullet$ *on* Y *and* X *transform the right derived functor* $\mathbb{R}f_*\colon \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ (88) *into the left derived functor* $\mathbb{L}f_!\colon \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ (92)*.*

Proof. It suffices to show that the equivalences of triangulated categories $\mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh^{lct}})$ and $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh^{lct}})$ from the proof of Theorem 5.8.2 transform the functor (88) into the functor (90). The latter equivalences were constructed on the level of injective and projective resolutions as the equivalences of homotopy categories $\mathsf{Hot}(Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{Hot}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ induced by the functors $\mathfrak{Hom}_Y(\mathcal{D}^\bullet_Y, -)$ and $\mathcal{D}^\bullet_Y \odot_Y -$, and similarly for X. In particular, it was shown that these functors take complexes over $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ to complexes over $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ and back.

Furthermore, we notice that such complexes are adjusted to the derived functors $\mathbb{R}f_*$ and $\mathbb{L}f_!$ acting between the co/contraderived categories. The morphism f being flat, the direct image functors f_* and $f_!$ take $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ into $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ and $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ into $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ (see Corollary 5.1.6(b)).

According to Section 3.8, for any complex $\mathcal{J}^\bullet$ over $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ there is a natural morphism $f_!\,\mathfrak{Hom}_Y(f^*\mathcal{D}^\bullet_X, \mathcal{J}^\bullet) \longrightarrow \mathfrak{Hom}_X(\mathcal{D}^\bullet_X, f_*\mathcal{J}^\bullet)$ of complexes of cosheaves of $\mathcal{O}_X$-modules. Composing this morphism with the morphism $f_!\,\mathfrak{Hom}_Y(\mathcal{D}^\bullet_Y, \mathcal{J}^\bullet) \longrightarrow f_!\,\mathfrak{Hom}_Y(f^*\mathcal{D}^\bullet_X, \mathcal{J}^\bullet)$ induced by the quasi-isomorphism $f^*\mathcal{D}^\bullet_X \longrightarrow \mathcal{D}^\bullet_Y$, we obtain a natural morphism

$$f_!\,\mathfrak{Hom}_Y(\mathcal{D}^\bullet_Y, \mathcal{J}^\bullet) \longrightarrow \mathfrak{Hom}_X(\mathcal{D}^\bullet_X, f_*\mathcal{J}^\bullet) \tag{102}$$

of complexes over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. Similarly, according to Section 3.8 for any complex $\mathfrak{P}^\bullet$ over $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ there is a natural morphism $\mathcal{D}^\bullet_X \odot_X f_!\mathfrak{P}^\bullet \longrightarrow f_*(f^*\mathcal{D}^\bullet_X \odot_Y \mathfrak{P}^\bullet)$ of complexes over $X\text{–}\mathsf{qcoh}$. Composing it with the morphism $f_*(f^*\mathcal{D}^\bullet_X \odot_Y \mathfrak{P}^\bullet) \longrightarrow f_*(\mathcal{D}^\bullet_Y \odot_Y \mathfrak{P}^\bullet)$ induced by the quasi-isomorphism $f^*\mathcal{D}^\bullet_X \longrightarrow \mathcal{D}^\bullet_Y$, we obtain a natural morphism

$$\mathcal{D}^\bullet_X \odot_X f_!\mathfrak{P}^\bullet \longrightarrow f_*(\mathcal{D}^\bullet_Y \odot_Y \mathfrak{P}^\bullet) \tag{103}$$

of complexes over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$. The natural morphisms (102–103) are compatible with the adjunction (20) and with the compositions of the morphisms of schemes f.

It suffices to show that the morphism (102) is a homotopy equivalence (or just a quasi-isomorphism over $X\text{–}\mathsf{ctrh}$) for any one-term complex $\mathcal{J}^\bullet = \mathcal{J}$ over $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$, or equivalently, the morphism (103) is a homotopy equivalence (or just a quasi-isomorphism over $X\text{–}\mathsf{qcoh}$) for any one-term complex $\mathfrak{P}^\bullet = \mathfrak{P}$ over $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. According to (45), the morphism (102) is actually an isomorphism whenever both schemes X and Y are semi-separated, or the morphism f is affine, or it is an open embedding. According to (46–47), the morphism (103) is an isomorphism whenever the morphism f is affine, or it is an open embedding of an affine scheme.

We have already proven the desired assertion in the case of semi-separated Noetherian schemes X and Y. To handle the general case, cover the scheme X with semi-separated open subschemes U_α and the scheme Y with semi-separated open subschemes V_β so that for any β there exists α for which $f(V_\beta) \subset U_\alpha$. Decompose an object $\mathcal{J} \in Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ into a direct sum of the direct images of injective quasi-coherent sheaves from V_β. Since we know (102) to be an isomorphism for the morphisms $V_\beta \longrightarrow Y$, $U_\alpha \longrightarrow X$, and $V_\beta \longrightarrow U_\alpha$, it follows that this map is an isomorphism of complexes over $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ for the morphism f. □

5.15. **Compatibilities for finite and proper morphisms.** Let X be a Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet_X$, which we will view as a finite complex over

X–$\mathsf{qcoh}^{\mathsf{inj}}$. Let $f\colon Y \longrightarrow X$ be a finite morphism of schemes. Then $\mathcal{D}_Y^\bullet = f^!\mathcal{D}_X^\bullet$, where $f^!$ denotes the special inverse image functor from Section 5.11, is a dualizing complex for Y [30, Proposition V.2.4].

Theorem 5.15.1. *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ *and* $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ *from Theorem* 5.7.1 *or* 5.8.2 *related to the choice of the dualizing complexes* $\mathcal{D}_X^\bullet$ *and* $\mathcal{D}_Y^\bullet$ *on* X *and* Y *transform the right derived functor* $\mathbb{R}f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ (97) *into the left derived functor* $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ (100).

Proof. We will show that the equivalences of homotopy categories $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ and $\mathsf{Hot}(Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{Hot}(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$ from the proof of Theorem 5.8.2 transform the functor $f^!$ into the functor f^*. Notice that complexes over X–$\mathsf{qcoh}^{\mathsf{inj}}$ and X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ are adjusted to the derived functors $\mathbb{R}f^!$ and $\mathbb{L}f^*$ acting between the co/contraderived categories. The special inverse image functors $f^!$ and f^* take X–$\mathsf{qcoh}^{\mathsf{inj}}$ to Y–$\mathsf{qcoh}^{\mathsf{inj}}$ and X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ to Y–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$.

First of all, we will need the following base change lemma.

Lemma 5.15.2. *Let* $g\colon x \longrightarrow X$ *be a morphism of locally Noetherian schemes and* $f\colon Y \longrightarrow X$ *be a finite morphism. Set* $y = x \times_X Y$, *and denote the natural morphisms by* $f'\colon y \longrightarrow x$ *and* $g'\colon y \longrightarrow Y$. *Then*

(a) *for any quasi-coherent sheaf* $\mathcal{M}$ *on* X *there is a natural morphism of quasi-coherent sheaves* $g'^*f^!\mathcal{M} \longrightarrow f'^!g^*\mathcal{M}$ *on* y; *this map is an isomorphism for any* $\mathcal{M}$ *whenever the morphism* g *is flat;*

(b) *whenever the morphism* g *is an open embedding, for any projective locally cotorsion contraherent cosheaf* $\mathfrak{P}$ *on* X *there is a natural isomorphism of projective locally cotorsion contraherent cosheaves* $g'^!f^*\mathfrak{P} \simeq f'^*g^!\mathfrak{P}$ *on* y;

(c) *whenever the morphism* g *is quasi-compact, for any quasi-coherent sheaf* $\mathcal{m}$ *on* x *there is a natural isomorphism of quasi-coherent sheaves* $g'_*f'^!\mathcal{m} \simeq f^!g_*\mathcal{m}$ *on* Y;

(d) *whenever the morphism* g *is flat and quasi-compact, for any projective locally cotorsion contraherent cosheaf* $\mathfrak{p}$ *on* x *there is a natural isomorphism of projective locally cotorsion contraherent cosheaves* $g'_!f'^*\mathfrak{p} \simeq f^*g_!\mathfrak{p}$ *on* Y.

Proof. Notice that the functors being composed in parts (b) and (d) preserve the class of projective locally cotorsion contraherent cosheaves by Corollaries 5.1.3(a) and 5.1.6(b). Furthermore, parts (a) and (b) were essentially proven in Section 5.11. The assertion of part (c) is local in X, so one can assume X to be affine. Then if x is also affine, the assertion is obvious; and the general case of a Noetherian scheme x is handled by computing the global sections of m in terms of a finite affine covering u_α of x and finite affine coverings of the intersections $u_\alpha \cap u_\beta$. The proof of part (d) is similar to that of part (c). □

For an affine open subscheme $U \subset X$ with the open embedding morphism $j\colon U \longrightarrow X$, denote by V the full preimage $U \times_X Y \subset Y$ and by $j'\colon V \longrightarrow Y$ its open embedding morphism. Let $\mathcal{I}$ and $\mathcal{J}$ be injective quasi-coherent sheaves on X. Then the composition

$$\mathrm{Hom}_X(j_*j^*\mathcal{I},\mathcal{J}) \longrightarrow \mathrm{Hom}_Y(f^!j_*j^*\mathcal{I},f^!\mathcal{J}) \simeq \mathrm{Hom}_Y(j'_*j'^*f^!\mathcal{I},f^!\mathcal{J})$$

of the map induced by the functor $f^!$ with the isomorphism provided by Lemma 5.15.2(a,c) induces a morphism of $\mathcal{O}(V)$-modules

$$(f^*\mathfrak{Hom}_X(\mathcal{I},\mathcal{J}))[V] = \mathcal{O}(V)\otimes_{\mathcal{O}(U)}\mathrm{Hom}_X(j_*j^*\mathcal{I},\mathcal{J}) \\ \longrightarrow \mathrm{Hom}_Y(j'_*j'^*f^!\mathcal{I},f^!\mathcal{J}) = \mathfrak{Hom}_Y(f^!\mathcal{I},f^!\mathcal{J})[V],$$

defining a morphism of projective locally cotorsion contraherent cosheaves $f^*\mathfrak{Hom}_X(\mathcal{I},\mathcal{J}) \longrightarrow \mathfrak{Hom}_Y(f^!\mathcal{I},f^!\mathcal{J})$ on Y.

To show that this morphism is an isomorphism, decompose the sheaf $\mathcal{J}$ into a finite direct sum of the direct images of injective quasi-coherent sheaves $\mathcal{K}$ from embeddings $k\colon W\longrightarrow X$ of affine open subschemes $W\subset X$. Then the isomorphisms of Lemma 5.15.2(c-d) together with the isomorphism (45) reduce the question to the case of an affine scheme X. In the latter situation we have

$$\mathcal{O}(Y)\otimes_{\mathcal{O}(X)}\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{I}(X),\mathcal{J}(X)) \simeq \mathrm{Hom}_{\mathcal{O}(X)}(\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\mathcal{I}(X)),\mathcal{J}(X)) \\ \simeq \mathrm{Hom}_{\mathcal{O}(Y)}(\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\mathcal{I}(X)),\ \mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\mathcal{J}(X))),$$

since $\mathcal{O}(Y)$ is a finitely presented $\mathcal{O}(X)$-module and $\mathcal{I}(X)$, $\mathcal{J}(X)$ are injective $\mathcal{O}(X)$-modules.

Alternatively, let $\mathcal{M}$ be a quasi-coherent sheaf and $\mathfrak{P}$ be a projective locally cotorsion contraherent cosheaf on X. We keep the notation above for the open embeddings $j\colon U\longrightarrow X$ and $j'\colon V\longrightarrow Y$. Then the composition

$$j'_*j'^*f^!\mathcal{M}\otimes_{\mathcal{O}(V)}(f^*\mathfrak{P})[V] \simeq j'_*j'^*f^!\mathcal{M}\otimes_{\mathcal{O}(U)}\mathfrak{P}[U] \\ \simeq f^!j_*j^*\mathcal{M}\otimes_{\mathcal{O}(U)}\mathfrak{P}[U] \longrightarrow f^!(j_*j^*\mathcal{M}\otimes_{\mathcal{O}(U)}\mathfrak{P}[U])$$

provides a natural morphism $j'_*j'^*f^!\mathcal{M}\otimes_{\mathcal{O}(V)}(f^*\mathfrak{P})[V] \longrightarrow f^!(j_*j^*\mathcal{M}\otimes_{\mathcal{O}(U)}\mathfrak{P}[U])$ of quasi-coherent sheaves on Y. Passing to the inductive limit over U and noticing that the contratensor product $f^!\mathcal{M}\odot_Y f^*\mathfrak{P}$ can be computed on the diagram of affine open subschemes $V = U\times_X Y\subset Y$ (see Section 2.6), we obtain a morphism of quasi-coherent sheaves $f^!\mathcal{M}\odot_Y f^*\mathfrak{P}\longrightarrow f^!(\mathcal{M}\odot_X\mathfrak{P})$ on Y. When the quasi-coherent sheaf $\mathcal{M}=\mathcal{I}$ is injective, this is a morphism of injective quasi-coherent sheaves.

To show that this morphism is an isomorphism, decompose the cosheaf $\mathfrak{P}$ into a finite direct sum of the direct images of projective locally cotorsion contraherent cosheaves $\mathfrak{Q}$ from embeddings $k\colon W\longrightarrow X$ of affine open subschemes $W\subset X$. Then the isomorphisms of Lemma 5.15.2(c-d) together with the isomorphism (47) reduce the question to the case of an affine scheme X. In the latter situation we have

$$\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\mathcal{M}(X))\otimes_{\mathcal{O}(Y)}(\mathcal{O}(Y)\otimes_{\mathcal{O}(X)}\mathfrak{P}[X]) \\ \simeq \mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\mathcal{M}(X))\otimes_{\mathcal{O}(X)}\mathfrak{P}[X] \simeq \mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{O}(Y),\ \mathcal{M}(X)\otimes_{\mathcal{O}(X)}\mathfrak{P}[X]),$$

since the $\mathcal{O}(X)$-module $\mathcal{O}(Y)$ is finitely presented and the $\mathcal{O}(X)$-module $\mathfrak{P}[X]$ is flat. ☐

Now let X be a semi-separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$. Let $f\colon Y \longrightarrow X$ be a proper morphism (of finite type). Notice that, the morphism f being separated, the scheme Y is consequently semi-separated (and Noetherian), too. Set $\mathcal{D}_Y^\bullet$ to be a finite complex over $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ quasi-isomorphic to $f^!\mathcal{D}_X^\bullet$, where $f^!$ denotes the triangulated functor (93) right adjoint to the derived direct image functor $\mathbb{R}f_*$ (cf. Corollary 5.13.4(a) and [30, Remark before Proposition V.8.5]).

The following theorem is supposed to be applied together with Theorem 5.12.2 and/or Corollary 5.14.1.

Theorem 5.15.3. *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ *and* $\mathsf{D}^{\mathsf{abs}}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ *from Theorem* 5.7.1 *transform the triangulated functor* $f^*\colon \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ (57) *into the triangulated functor* $f^!\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ (93).

Proof. For any complex of flat quasi-coherent sheaves $\mathcal{F}^\bullet$ on X there is a natural isomorphism $f_*(\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet) \simeq f_*(\mathcal{D}_Y^\bullet) \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$ of complexes over $X\text{–}\mathsf{qcoh}$ (see (13)). Hence the adjunction morphism $f_*\mathcal{D}_Y^\bullet = \mathbb{R}f_*f^!\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet$ in $\mathsf{D}^+(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ induces a natural morphism $\mathbb{R}f_*(\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet) = f_*(\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet) \longrightarrow \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ (see (74)). We have constructed a natural transformation $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet \longrightarrow f^!(\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$ of functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$.

Since finite complexes of coherent sheaves $\mathcal{N}^\bullet$ on Y form a set of compact generators of $\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ (see Theorem 5.9.1(b)), in order to prove that our morphism is an isomorphism it suffices to show that the induced morphism of abelian groups $\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})}(\mathcal{N}^\bullet,\, \mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})}(\mathcal{N}^\bullet,\, f^!(\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet) \simeq \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})}(\mathbb{R}f_*\mathcal{N}^\bullet,\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet)$ is an isomorphism for any $\mathcal{N}^\bullet$. Both $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}^\bullet$ and $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$ being complexes of injective quasi-coherent sheaves, the Hom into them in the coderived categories coincides with the one taken in the homotopy categories of complexes of quasi-coherent sheaves.

The complexes $\mathcal{D}_Y^\bullet$, $\mathcal{D}_X^\bullet$ and $\mathcal{N}^\bullet$ being finite, the latter Hom only depends on a finite fragment of the complex $\mathcal{F}^\bullet$. This reduces the question to the case of a single coherent sheaf $\mathcal{N}$ on Y and a single flat quasi-coherent sheaf $\mathcal{F}$ on X; one has to show that the natural morphism of complexes of abelian groups $\operatorname{Hom}_Y(\mathcal{N},\, \mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{F}) \longrightarrow \operatorname{Hom}_X(\mathbb{R}f_*\mathcal{N},\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism. Notice that in the case $\mathcal{F} = \mathcal{O}_X$ this is the definition of $\mathcal{D}_Y^\bullet \simeq f^!\mathcal{D}_X^\bullet$.

In the case when there are enough vector bundles (locally free sheaves of finite rank) on X, one can pick a left resolution $\mathcal{L}^\bullet$ of the flat quasi-coherent sheaf $\mathcal{F}$ by infinite direct sums of vector bundles and argue as above, reducing the question further to the case when $\mathcal{F}$ is an infinite direct sum of vector bundles, when the assertion follows by compactness. In the general case, using the Čech resolution (12) of a flat quasi-coherent sheaf $\mathcal{F}$, one can assume $\mathcal{F}$ to be the direct image of a flat quasi-coherent sheaf $\mathcal{G}$ from an affine open subscheme $U \subset X$.

By the same projection formula (13) applied to the open embeddings $j\colon U \longrightarrow X$ and $j'\colon V = U \times_X Y \longrightarrow Y$ (which are affine morphisms), one has $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} j_*\mathcal{G} \simeq j_*(j^*\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_U} \mathcal{G})$ and $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*j_*\mathcal{G} \simeq j'_*(j'^*\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_V} f'^*\mathcal{G})$. Here we are also using the base change isomorphism $f^*j_*\mathcal{G} \simeq j'_*f'^*\mathcal{G}$, and denote by f' the morphism $V \longrightarrow U$. Using the adjunction of the direct image functors j_* and j^* together with the isomorphism $j^*\mathbb{R}f_*\mathcal{N} \simeq \mathbb{R}f'_*j'^*\mathcal{N}$ in $\mathsf{D^b}(U\text{–}\mathsf{coh})$, one can replace the morphism $f\colon Y \longrightarrow X$ by the morphism $f'\colon V \longrightarrow U$ into an affine scheme U in the desired assertion, where there are obviously enough vector bundles.

In the above argument one needs to know that the natural morphism $j'^*\mathcal{D}_Y^\bullet \simeq j'^*f^!\mathcal{D}_X^\bullet \longrightarrow f'^!j^*\mathcal{D}_X^\bullet$ is an isomorphism in $\mathsf{D}^+(V\text{–}\mathsf{qcoh})$. This is a result of Deligne [14]; see Theorem 5.16.1 below. □

5.16. **The extraordinary inverse image.** Let $f\colon Y \longrightarrow X$ be a separated morphism of finite type between Noetherian schemes. (What we call) Deligne's extaordinary inverse image functor $f^+\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ is a triangulated functor defined by the following rules:

(i) whenever f is an open embedding, $f^+ = f^*$ is the conventional inverse image functor (induced by the exact functor $f^*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ and left adjoint to the derived direct image functor $\mathbb{R}f_*\colon \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ (50, 88);

(ii) whenever f is a proper morphism, $f^+ = f^!$ is (what we call) Neeman's extraordinary inverse image functor (93), i. e., the functor right adjoint to the derived direct image functor $\mathbb{R}f_*$ (cf. Corollary 4.8.2);

(iii) given two composable separated morphisms of finite type $f\colon Y \longrightarrow X$ and $g\colon Z \longrightarrow Y$ between Noetherian schemes, one has a natural isomorphism of triangulated functors $(fg)^+ \simeq g^+f^+$.

Deligne constructs his extraordinary inverse image functor (which we denote by) $f^+\colon \mathsf{D}^+(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ in [14] (cf. [30, Theorem III.8.7]). Notice that by Lemma 5.13.3(a) and Corollary 5.13.4(a) the two versions of Neeman's extraordinary inverse image $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ and $f^!\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ have isomorphic restrictions to $\mathsf{D}^+(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}(X\text{–}\mathsf{qcoh}), \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$, both acting from $\mathsf{D}^+(X\text{–}\mathsf{qcoh})$ to $\mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ and being right adjoint to $\mathbb{R}f_*\colon \mathsf{D}^+(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{qcoh})$ (87); so there is no ambiguity here.

According to a counterexample of Neeman's [47, Example 6.5], there *cannot* exist a triangulated functor $f^+\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ defined for all, say, locally closed embeddings of affine schemes of finite type over a fixed field and satisfying (i) for open embeddings, (ii) for closed embeddings, and (iii) for compositions. Our aim is to show that there is *no* similar inconsistency in the rules (i-iii) to prevent existence of a functor f^+ acting between the coderived categories of (unbounded complexes of) quasi-coherent sheaves. Given that, and assuming compactifiability of separated morphisms of finite type between Noetherian schemes (a version of Nagata's theorem, see [14]), one would be able to proceed with the construction of the functor $f^+\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ (cf. [22, Sections 5 and 6]).

Theorem 5.16.1. *Let $g\colon Y \longrightarrow X$ and $g'\colon V \longrightarrow U$ be proper morphisms (of finite type) between Noetherian schemes, forming a commutative diagram with open* *embeddings* $h\colon U \longrightarrow X$ *and* $h'\colon V \longrightarrow Y$*. Then there is a natural isomorphism of triangulated functors* $h'^*g^! \simeq g'^!h^*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(V\text{–}\mathsf{qcoh})$*.*

Proof. We follow the approach in [14] (with occasional technical points borrowed from [22]). The argument is based on the results of [28, Chapitre 3].

Set $V' = U \times_X Y$. Then the natural morphism $V \longrightarrow V'$ is both an open embedding and a proper morphism, that is V' is a disconnected union of V and $V' \setminus V$. Hence one can assume that $V = V'$, i. e., the square is Cartesian.

The construction of the derived functor $\mathbb{R}f_*$ (88) being local in the base (since the flasqueness/injectivity properties of quasi-coherent sheaves are local), there is a natural isomorphism of triangulated functors $h^*\mathbb{R}g_* \simeq \mathbb{R}g'_*\,h'^*$. Given an object $\mathcal{M}^\bullet \in \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$, we now have a natural morphism $\mathbb{R}g'_*\,h'^*g^!\mathcal{M}^\bullet \simeq h^*\mathbb{R}g_*g^!\mathcal{M}^\bullet \longrightarrow h^*\mathcal{M}^\bullet$ in $\mathsf{D^{co}}(U\text{–}\mathsf{qcoh})$, inducing a natural morphism $h'^*g^!\mathcal{M}^\bullet \longrightarrow g'^!h^*\mathcal{M}^\bullet$ in $\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})$. We have constructed a morphism of functors $h'^*g^! \longrightarrow g'^!h^*$.

In order to prove that this morphism is an isomorphism in $\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})$, we will show that the induced morphism $\mathrm{Hom}_{\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet,\, h'^*g^!\mathcal{M}^\bullet) \longrightarrow \mathrm{Hom}_{\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet,\, g'^!h^*\mathcal{M}^\bullet)$ is an isomorphism of abelian groups for any finite complex of coherent sheaves $\mathcal{L}^\bullet$ on V. This would be sufficient in view of the fact that finite complexes of coherents form a set of compact generators of $\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})$ (see Theorem 5.9.1(b)).

For any category C, we denote by $\mathsf{Pro}\,\mathsf{C}$ the category of pro-objects in C (for our purposes, it suffices to consider projective systems indexed by the nonnegative integers). To any open embedding of Noetherian schemes $j\colon W \longrightarrow Z$ one assigns an exact functor (between abelian categories) $j_!\colon W\text{–}\mathsf{coh} \longrightarrow \mathsf{Pro}(Z\text{–}\mathsf{coh})$ defined by the following rule.

Let $\mathcal{I} \subset \mathcal{O}_Z$ denote the sheaf of ideals corresponding to some closed subscheme structure on the complement $Z \setminus W$. Given a coherent sheaf $\mathcal{F}$ on W, pick coherent sheaf $\mathcal{F}'$ on Z with $\mathcal{F}'|_W \simeq \mathcal{F}$. The functor $j_!$ takes the sheaf W to the projective system formed by the coherent sheaves $j_*(\mathcal{I}^n\mathcal{F}')$, $n \geqslant 0$ on Z (and their natural embeddings). One can check that this pro-object does not depend (up to a natural isomorphism) on the choice of a coherent extension $\mathcal{F}'$. In fact, one has $\mathrm{Hom}_W(\mathcal{F}, j^*\mathcal{G}) \simeq \mathrm{Hom}_Z(j_!\mathcal{F}, \mathcal{G})$ for any $\mathcal{G} \in Z\text{–}\mathsf{qcoh}$ [14, Proposition 4].

Passing to the bounded derived categories, one obtains a triangulated functor $j_!\colon \mathsf{D^b}(W\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^b}(\mathsf{Pro}\,Z\text{–}\mathsf{coh})$. Let $\mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh}) \subset \mathsf{Pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})$ denote the full subcategory formed by projective systems of complexes with uniformly bounded cohomology sheaves. The system of cohomology functors $\mathsf{pro}\,H^i\colon \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{Pro}(Z\text{–}\mathsf{coh})$ is conservative (a result applicable to any abelian category [14, Proposition 3]). Furthermore, there is a natural functor $\mathsf{D^b}(\mathsf{Pro}\,Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})$. Composing it with the above functor $j_!$ and passing to pro-objects in $W\text{–}\mathsf{coh}$ one constructs the functor $j_!\colon \mathsf{pro}\,\mathsf{D^b}(W\text{–}\mathsf{coh}) \longrightarrow \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})$.

On the other hand, the functor $h^*\colon \mathsf{D^b}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^b}(W\text{–}\mathsf{coh})$ induces a natural functor $\mathsf{pro}\,h^*\colon \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{pro}\,\mathsf{D^b}(W\text{–}\mathsf{coh})$.

Lemma 5.16.2. (a) *For any objects* $\mathcal{L}^\bullet \in \mathsf{pro}\,\mathsf{D^b}(W\text{–}\mathsf{coh})$ *and* $\mathcal{M}^\bullet \in \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *there is a natural isomorphism of abelian groups* $\mathrm{Hom}_{\mathsf{Pro}\,\mathsf{D^{co}}(W\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, h^*\mathcal{M}^\bullet) \simeq \mathrm{Hom}_{\mathsf{Pro}\,\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})}(h_!\mathcal{L}^\bullet, \mathcal{M}^\bullet)$*.*

(b) *For any objects* $\mathcal{F}^\bullet \in \mathsf{pro}\,\mathsf{D^b}(W\text{–}\mathsf{coh})$ *and* $\mathcal{G}^\bullet \in \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})$ *there is a natural isomorphism of abelian groups* $\mathrm{Hom}_{\mathsf{pro}\,\mathsf{D^b}(W\text{–}\mathsf{coh})}(\mathcal{F}^\bullet, h^*\mathcal{G}^\bullet) \simeq \mathrm{Hom}_{\mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})}(h_!\mathcal{F}^\bullet, \mathcal{G}^\bullet)$.

Proof. Notice that the assertion of part (a) is *not* true for the conventional unbounded derived categories. To prove it for the coderived categories, one assumes $\mathcal{M}^\bullet$ to be a complex over $Z\text{–}\mathsf{qcoh}^{\mathsf{inj}}$, so that Hom into $\mathcal{M}^\bullet$ in the coderived category is isomorphic to the similar Hom in the homotopy category of complexes. It is important here that $h^*\mathcal{M}^\bullet$ is a complex over $W\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ in these assumptions. The desired adjunction on the level of (pro)derived categories then follows from the above adjunction on the level of abelian categories. To deduce part (b), one can use the fact that the natural functor $\mathsf{D^b}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ is fully faithful (and similarly for W). ☐

Given a proper morphism $f\colon T \longrightarrow Z$ (of finite type) between Noetherian schemes, there is the direct image functor $\mathbb{R}f_*\colon \mathsf{D^b}(T\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^b}(Z\text{–}\mathsf{coh})$ [28, Théorème 3.2.1]. Passing to pro-objects, one obtains the induced functor $\mathsf{pro}\,\mathbb{R}f_*\colon\ \mathsf{pro}\,\mathsf{D^b}(T\text{–}\mathsf{coh}) \longrightarrow \mathsf{pro}\,\mathsf{D^b}(Z\text{–}\mathsf{coh})$.

Lemma 5.16.3. *Let* $V \longrightarrow U$, $Y \longrightarrow X$ *be a Cartesian square formed by proper morphisms of Noetherian schemes* $g\colon Y \longrightarrow X$ *and* $g'\colon V \longrightarrow U$, *and open embeddings* $h\colon U \longrightarrow X$ *and* $h'\colon V \longrightarrow Y$ *(as above). Then there is a natural isomorphism of functors* $\mathsf{pro}\,\mathbb{R}g_* \circ h'_! \simeq h_! \circ \mathsf{pro}\,\mathbb{R}g'_*\colon\ \mathsf{pro}\,\mathsf{D^b}(V\text{–}\mathsf{coh}) \longrightarrow \mathsf{pro}\,\mathsf{D^b}(X\text{–}\mathsf{coh})$.

Proof. Given an object $\mathcal{L}^\bullet \in \mathsf{pro}\,\mathsf{D}^b(V\text{–}\mathsf{coh})$, one has a natural morphism $\mathsf{pro}\,\mathbb{R}g'_*\mathcal{L}^\bullet \longrightarrow \mathsf{pro}\,\mathbb{R}g'_*\,\mathsf{pro}\,h'^*\,h'_!\mathcal{L}^\bullet \simeq \mathsf{pro}\,h^*\,\mathsf{pro}\,\mathbb{R}g_*\,h'_!\mathcal{L}^\bullet$ in $\mathsf{pro}\,\mathsf{D^b}(U\text{–}\mathsf{coh})$, inducing a natural morphism $h_!\,\mathsf{pro}\,\mathbb{R}g'_*\mathcal{L}^\bullet \longrightarrow \mathsf{pro}\,\mathbb{R}g_*\,h'_!\mathcal{L}^\bullet$ in $\mathsf{pro}\,\mathsf{D^b}(X\text{–}\mathsf{coh})$ by Lemma 5.16.2(b). We have constructed a morphism of functors $h_! \circ \mathsf{pro}\,\mathbb{R}g'_* \longrightarrow \mathsf{pro}\,\mathbb{R}g_* \circ h'_!$. In order to check that this morphism is an isomorphism, it suffices to consider the case of $\mathcal{L}^\bullet \in \mathsf{D^b}(V\text{–}\mathsf{coh})$ and show that the morphism in question becomes an isomorphism after applying the cohomology functors $\mathsf{pro}\,H^i$ taking values in $\mathsf{Pro}(X\text{–}\mathsf{coh})$.

Furthermore, one can restrict oneself to the case of a single coherent sheaf $\mathcal{L} \in V\text{–}\mathsf{coh}$. So we have to show that the morphisms $h_!\mathbb{R}^ig'_*(\mathcal{L}) \longrightarrow \mathsf{Pro}\,\mathbb{R}^ig_*(h'_!\mathcal{L})$ are isomorphisms in $\mathsf{Pro}(X\text{–}\mathsf{coh})$ for any coherent sheaf $\mathcal{L}$ on V and all $i \geqslant 0$. Finally, one can assume X to be an affine scheme (as all the constructions of the functors involved are local in the base, and the property of a morphism in $\mathsf{Pro}(X\text{–}\mathsf{coh})$ to be an isomorphism is local in X).

Let $\mathcal{L}'$ be a quasi-coherent sheaf on Y extending $\mathcal{L}$. Set $R = \mathcal{O}(X)$ and $I = \mathcal{O}_X(\mathcal{I})$, where $\mathcal{I}$ is a sheaf of ideals in $\mathcal{O}_X$ corresponding to a closed subscheme structure on $X \setminus U$. The question reduces to showing that the natural morphism between the pro-objects represented by the projective systems $I^nH^i(Y,\mathcal{L}')$ and $H^i(Y, I^n\mathcal{L}')$ is an isomorphism in $\mathsf{Pro}(R\text{–}\mathsf{mod})$. According to [28, Corollaire 3.3.2], the $(\bigoplus_{n=0}^\infty I^n)$-module $\bigoplus_{n=0}^\infty H^i(Y, I^n\mathcal{L}')$ is finitely generated; the desired assertion is deduced straightforwardly from this result. ☐

Now we can finish the proof of the theorem. Given a finite complex $\mathcal{L}^\bullet$ over $V\text{–}\mathsf{coh}$ and a complex $\mathcal{M}^\bullet$ over $X\text{–}\mathsf{qcoh}$, one has $\mathrm{Hom}_{\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet,\ h'^*g^!\mathcal{M}^\bullet) \simeq \mathrm{Hom}_{\mathsf{Pro}\,\mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})}(h_!\mathcal{L}^\bullet, g^!\mathcal{M}^\bullet) \simeq \mathrm{Hom}_{\mathsf{Pro}\,\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})}(\mathsf{pro}\,\mathbb{R}g_*\,h_!\mathcal{L}^\bullet, \mathcal{M}^\bullet)$ and $\mathrm{Hom}_{\mathsf{D^{co}}(V\text{–}\mathsf{qcoh})}$

$(\mathcal{L}^\bullet,\ g'^!h^*\mathcal{M}^\bullet) \simeq \mathrm{Hom}_{\mathsf{D^{co}}(U\text{–}\mathsf{qcoh})}(\mathbb{R}g'_*\mathcal{L}^\bullet,\ h^*\mathcal{M}^\bullet) \simeq \mathrm{Hom}_{\mathsf{Pro\,D^{co}}(X\text{–}\mathsf{qcoh})}(h_!\mathbb{R}g'_*\mathcal{L}^\bullet,\ \mathcal{M}^\bullet)$ by Lemma 5.16.2(a), so it remains to apply Lemma 5.16.3. ☐

The following theorem, which is the main result of this section, essentially follows from the several previous results. Let $f\colon Y \longrightarrow X$ be a morphism of finite type between Noetherian schemes. Let $\mathcal{D}_X^\bullet$ be a dualizing complex on X; set $\mathcal{D}_Y^\bullet$ to be a finite complex over $Y\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ quasi-isomorphic to $f^+\mathcal{D}_X^\bullet$ [30, Remark before Proposition V.8.5].

Theorem 5.16.4. *The equivalences of categories* $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ *and* $\mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ *from Theorem* 5.7.1 *or* 5.8.2 *related to the choice of the dualizing complexes* $\mathcal{D}_X^\bullet$ *and* $\mathcal{D}_Y^\bullet$ *on* X *and* Y *transform Deligne's extraordinary inverse image functor* $f^+\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ *into the functor* $f^*\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ (95) *left adjoint to the derived functor* $\mathbb{L}f_!$ (92)*, at least, in either of the following two situations:*

(a) *f is a separated morphism of semi-separated Noetherian schemes that can be factorized into an open embedding followed by a proper morphism;*
(b) *f is morphism of Noetherian schemes that can be factorized into a finite morphism followed by a smooth morphism.*

Proof. According to Theorem 5.15.1, desired assertion is true for finite morphisms of Noetherian schemes with dualizing complexes $f\colon Y \longrightarrow X$. Comparing Theorem 5.12.2(b) with Theorem 5.15.3, one comes to the same conclusion in the case of a proper morphism $f\colon Y \longrightarrow X$ (of finite type) between semi-separated Noetherian schemes with dualizing complexes.

On the other hand, let us consider the case of a smooth morphism f. Then it is essentially known ([30, Corollary VII.4.3], [47, Theorem 5.4], Proposition 5.13.5 above) that the functor $f^+\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ only differs from the conventional inverse image functor $f^*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ by a shift and a twist. Namely, for any complex $\mathcal{M}^\bullet$ over $X\text{–}\mathsf{qcoh}$ one has $f^!\mathcal{M}^\bullet \simeq \omega_{Y/X}[d]\otimes_{\mathcal{O}_Y} f^*\mathcal{M}^\bullet$, where d is the relative dimension and $\omega_{Y/X}$ is the line bundle of relative top forms on Y. In particular, in our present notation one would have $\mathcal{D}_Y^\bullet \simeq \omega_{Y/X}[n]\otimes_{\mathcal{O}_Y} f^*\mathcal{D}_X^\bullet$ (up to a quasi-isomorphism of finite complexes over $Y\text{–}\mathsf{qcoh}$).

By Theorem 5.14.3, the equivalences of triangulated categories $\mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ and $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ related to the choice of the dualizing complexes $f^*\mathcal{D}_X^\bullet$ and $\mathcal{D}_X^\bullet$ on Y and X transform the functor $\mathbb{R}f_*\colon \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ (88) into the functor $\mathbb{L}f_!\colon \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh})$ (92). Passing to the left adjoint functors, we conclude that the same equivalences transform the functor $f^*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Y\text{–}\mathsf{qcoh})$ (induced by the exact functor $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$, cf. (60)) into the functor $f^*\colon \mathsf{D^{ctr}}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D^{ctr}}(Y\text{–}\mathsf{ctrh})$ (95).

It remains to take the twist and the shift into account in order to deduce the desired assertion for the smooth morphism f. As a particular case, the above argument also covers the situation when f is an open embedding. ☐

Appendix A. Derived Categories of Exact Categories and Resolutions

In this appendix we recall and review some general results about the derived categories of the first and the second kind of abstract exact categories and their full subcategories, in presence of finite or infinite resolutions. There is nothing essentially new here. Two or three most difficult arguments are omitted or only briefly sketched with references to the author's previous works containing elaborated proofs of similar results in different (but more concrete) settings given in place of the details.

Note that the present one is still not the full generality for many results considered here. For most assertions concerning derived categories of the second kind, the full generality is that of exact DG-categories [53, Section 3.2 and Remark 3.5], which we feel is a bit too abstract to base our exposition on.

A.1. **Derived categories of the second kind.** Let E be an exact category. The homotopy categories of (finite, bounded above, bounded below, and unbounded) complexes over E will be denoted by $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{Hot}^{-}(\mathsf{E})$, $\mathsf{Hot}^{+}(\mathsf{E})$, and $\mathsf{Hot}(\mathsf{E})$, respectively. For the definitions of the conventional derived categories (of the first kind) $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{D}^{-}(\mathsf{E})$, $\mathsf{D}^{+}(\mathsf{E})$, and $\mathsf{D}(\mathsf{E})$ we refer to [46, 36] and [54, Section A.7]. Here are the definitions of the derived categories of the second kind [52, 53, 15].

An (unbounded) complex over E is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the all the total complexes of short exact sequences of complexes over E. Here a short exact sequence $0 \longrightarrow {}'K^\bullet \longrightarrow K^\bullet \longrightarrow {}''K^\bullet \longrightarrow 0$ of complexes over E is viewed as a bicomplex with three rows and totalized as such. The *absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ of the exact category E is defined as the quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of absolutely acyclic complexes.

Similarly, a bounded above (respectively, below) complex over E is called absolutely acyclic if it belongs to the minimal thick subcategory of $\mathsf{Hot}^{-}(\mathsf{E})$ (resp., $\mathsf{Hot}^{+}(\mathsf{E})$) containing all the total complexes of short exact sequences of bounded above (resp., below) complexes over E. We will see below that a bounded above (resp., below) complex over E is absolutely acyclic if and only if it is absolutely acyclic as an unbounded complex, so there is no ambiguity in our terminology. The bounded above (resp., below) absolute derived category of E is defined as the quotient category of $\mathsf{Hot}^{-}(\mathsf{E})$ (resp., $\mathsf{Hot}^{+}(\mathsf{E})$) by the thick subcategory of absolutely acyclic complexes and denoted by $\mathsf{D}^{\mathsf{abs}-}(\mathsf{E})$ (resp., $\mathsf{D}^{\mathsf{abs}+}(\mathsf{E})$).

We do not define the "absolute derived category of finite complexes over E", as it would not be any different from the conventional bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$. Indeed, any (bounded or unbounded) absolutely acyclic complex is acyclic; and any finite acyclic complex over an exact category is absolutely acyclic, since it is composed of short exact sequences. Moreover, any acyclic complex over an exact category of finite homological dimension is absolutely acyclic [52, Remark 2.1].

For comparison with the results below, we recall that for any exact category E the natural functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\pm}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$ are all fully faithful [54, Corollary A.11].

Lemma A.1.1. *For any exact category* E*, the functors* $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}-}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}+}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ *induced by the natural embeddings of the categories of bounded complexes into those of unbounded ones are all fully faithful.*

Proof. We will show that any morphism in $\mathsf{Hot}(\mathsf{E})$ (in an appropriate direction) between a complex bounded in a particular way and a complex absolutely acyclic with respect to the class of complexes unbounded in that particular way factorizes through a complex absolutely acyclic with respect to the class of correspondingly bounded complexes. For this purpose, it suffices to demonstrate that any absolutely acyclic complex can be presented as a termwise stabiling filtered inductive (or projective) limit of complexes absolutely acyclic with respect to the class of complexes bounded from a particular side.

Indeed, any short exact sequence of complexes over E is the inductive limit of short exact sequences of their subcomplexes of silly filtration, which are bounded below. One easily concludes that any absolutely acyclic complex is a termwise stabilizing inductive limit of complexes absolutely acyclic with respect to the class of complexes bounded below, and any absolutely acyclic complex bounded above is a termwise stabilizing inductive limit of finite acyclic complexes. This proves that the functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}-}(\mathsf{E})$ and $\mathsf{D}^{\mathsf{abs}+}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ are fully faithful.

On the other hand, any absolutely acyclic complex bounded below, being, by implication, an acyclic complex bounded below, is the inductive limit of its subcomplexes of canonical filtration, which are finite acyclic complexes. This shows that the functor $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}+}(\mathsf{E})$ is fully faithful, too. Finally, to prove that the functor $\mathsf{D}^{\mathsf{abs}-}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ is fully faithful, one presents any absolutely acyclic complex as a termwise stabilizing projective limit of complexes absolutely acyclic with respect to the class of complexes bounded above. □

Assume that infinite direct sums exist and are exact functors in the exact category E. Then a complex over E is called *coacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the total complexes of short exact sequences of complexes over E and closed under infinite direct sums. The *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$ of the exact category E is defined as the quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of coacyclic complexes.

Similarly, if the functors of infinite product are everywhere defined and exact in the exact category E, one calls a complex over E *contraacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the total complexes of short exact sequences of complexes over E and closed under infinite products. The *contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ of the exact category E is the quotient category of $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of contraacyclic complexes [52, Sections 2.1 and 4.1].

Lemma A.1.2. (a) *For any exact category* E *with exact functors of infinite direct sum, the full subcategory of bounded below complexes in* $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *is equivalent to* $\mathsf{D}^{+}(\mathsf{E})$*.*

(b) *For any exact category* E *with exact functors of infinite product, the full subcategory of bounded above complexes in* $\mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ *is equivalent to* $\mathsf{D}^{-}(\mathsf{E})$*.*

Proof. By [52, Lemmas 2.1 and 4.1], any bounded below acyclic complex over E is coacyclic and any bounded above acyclic complex over E is contraacyclic. Hence there are natural triangulated functors $\mathsf{D}^+(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ and $\mathsf{D}^-(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ (in the respective assumptions of parts (a) and (b)). It also follows that the subcomplexes and quotient complexes of canonical filtration of any co/contraacyclic complex remain co/contraacyclic. Hence any morphism in $\mathsf{Hot}(\mathsf{E})$ from a bounded above complex to a co/contraacyclic complex factorizes through a bounded above co/contraacyclic complex, and any morphism from a co/contraacyclic complex to a bounded below complex factorizes through a bounded below co/contraacyclic complex. Therefore, our triangulated functors are fully faithful [52, Remark 4.1]. □

Denote the full additive subcategory of injective objects in E by $\mathsf{E}^{\mathsf{inj}} \subset \mathsf{E}$ and the full additive subcategory of projective objects by $\mathsf{E}^{\mathsf{prj}} \subset \mathsf{E}$.

Lemma A.1.3. (a) *The triangulated functors* $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{Hot}^{\pm}(\mathsf{E}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(\mathsf{E})$, $\mathsf{Hot}(\mathsf{E}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$, *and* $\mathsf{Hot}(\mathsf{E}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *are fully faithful.*

(b) *The triangulated functors* $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E}^{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{Hot}^{\pm}(\mathsf{E}^{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(\mathsf{E})$, $\mathsf{Hot}(\mathsf{E}^{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$, *and* $\mathsf{Hot}(\mathsf{E}^{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ *are fully faithful.*

Proof. This is essentially a version of [53, Theorem 3.5] and a particular case of [53, Remark 3.5]. For any total complex $A^\bullet$ of a short exact sequence of complexes over E and any complex $J^\bullet$ over $\mathsf{E}^{\mathsf{inj}}$ the complex of abelian groups $\mathrm{Hom}_{\mathsf{E}}(A^\bullet, J^\bullet)$ is acyclic. Therefore, the same also holds for a complex $A^\bullet$ that can be obtained from such total complexes using the operations of cone and infinite direct sum (irrespectively even of such operations being everywhere defined or exact in E). Similarly, for any total complex $A^\bullet$ of a short exact sequence of complexes over E and any complex $P^\bullet$ over $\mathsf{E}^{\mathsf{prj}}$ the complex of abelian group is acyclic (hence the same also holds for any complexes that can be obtained from such total complexes using the operations of cone and infinite product). This semiorthogonality implies the assertions of Lemma. □

A.2. **Fully faithful functors.** Let E be an exact category and $\mathsf{F} \subset \mathsf{E}$ be a full subcategory closed under extensions and the passage to the kernels of admissible epimorphisms. We endow F with the induced exact category structure. Suppose that for any admissible epimorphism $E \longrightarrow F$ in E from an object $E \in \mathsf{E}$ to an object $F \in \mathsf{F}$ there exist an object $G \in \mathsf{F}$, an admissible epimorphism $G \longrightarrow F$ in F, and a morphism $G \longrightarrow E$ in E such that the triangle $G \longrightarrow E \longrightarrow F$ is commutative (cf. [36, Section 12]).

Proposition A.2.1. *For any symbol* $\star = \mathsf{b}$, $-$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star(\mathsf{F}) \longrightarrow \mathsf{D}^\star(\mathsf{E})$ *induced by the exact embedding functor* $\mathsf{F} \longrightarrow \mathsf{E}$ *is fully faithful. When* $\star = \mathsf{ctr}$, *it is presumed here that the functors of infinite product are everywhere defined and exact in the exact category* E *and preserve the full subcategory* $\mathsf{F} \subset \mathsf{E}$.

Proof. We use the notation $\mathsf{Hot}^\star(\mathsf{E})$ for the category $\mathsf{Hot}(\mathsf{E})$ if $\star = \varnothing$, co, ctr, or abs, the category $\mathsf{Hot}^-(\mathsf{E})$ if $\star = -$ or $\mathsf{abs}-$, the category $\mathsf{Hot}^+(\mathsf{E})$ if $\star = +$ or $\mathsf{abs}+$, and the category $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E})$ if $\star = \mathsf{b}$. Let us call an object of $\mathsf{Hot}^\star(\mathsf{E})$ *$\star$-acyclic* if it is annihilated by the localization functor $\mathsf{Hot}^\star(\mathsf{E}) \longrightarrow \mathsf{D}^\star(\mathsf{E})$.

In view of Lemma A.1.1, it suffices to consider the cases $\star = -$, abs, and ctr. In either case, we will show that any morphism in $\mathsf{Hot}^\star(\mathsf{E})$ from a complex $F^\bullet \in \mathsf{Hot}^\star(\mathsf{F})$ into a $\star$-acyclic complex $A^\bullet \in \mathsf{Hot}^\star(\mathsf{E})$ factorizes through a complex $G^\bullet \in \mathsf{Hot}^\star(\mathsf{F})$ that is $\star$-acyclic as a complex over F.

We start with the case $\star = -$. Assume for simplicity that both complexes $F^\bullet$ and $A^\bullet$ are concentrated in the nonpositive cohomological degrees; $F^\bullet$ is a complex over F and $A^\bullet$ is an exact complex over E. Notice that the morphism $A^{-1} \longrightarrow A^0$ must be an admissible epimorphism in this case. Set $G^0 = F^0$, and let $B^{-1} \in \mathsf{E}$ denote the fibered product of the objects F^0 and A^{-1} over A^0. Then there exists a unique morphism $A^{-2} \longrightarrow B^{-1}$ having a zero composition with the morphism $B^{-1} \longrightarrow F^0$ and forming a commutative diagram with the morphisms $A^{-2} \longrightarrow A^{-1}$ and $B^{-1} \longrightarrow A^{-1}$.

One easily checks that the complex $\cdots \longrightarrow A^{-3} \longrightarrow A^{-2} \longrightarrow B^{-1} \longrightarrow F^0 \longrightarrow 0$ is exact. Furthemore, there exists a unique morphism $F^{-1} \longrightarrow B^{-1}$ whose compositions with the morphisms $B^{-1} \longrightarrow F^0$ and $B^{-1} \longrightarrow A^{-1}$ are the differential $F^{-1} \longrightarrow F^0$ and the component $F^{-1} \longrightarrow A^{-1}$ of the morphism of complexes $F^\bullet \longrightarrow A^\bullet$. We have factorized the latter morphism of complexes through the above exact complex, whose degree-zero term belongs to F.

From this point on we proceed by induction in the homological degree. Suppose that our morphism of complexes $F^\bullet \longrightarrow A^\bullet$ has been factorized through an exact complex $\cdots \longrightarrow A^{-n-2} \longrightarrow A^{-n-1} \longrightarrow B^{-n} \longrightarrow G^{-n+1} \longrightarrow \cdots \longrightarrow G^0 \longrightarrow 0$, which coincides with the complex $A^\bullet$ in the degrees $-n-1$ and below, and whose terms belong to F in the degrees $-n+1$ and above. Since the full subcategory $\mathsf{F} \subset \mathsf{E}$ is assumed to be closed under extensions, the image Z^{-n} of the morphism $B^{-n} \longrightarrow G^{-n+1}$ belongs to F. Let $G^{-n} \longrightarrow Z^{-n}$ be an admissible epimorphism in F factorizable through the admissible epimorphism $B^{-n} \longrightarrow Z^{-n}$ in E.

Replacing, if necessary, the object G^{-n} by the direct sum $G^{-n} \oplus F^{-n}$, we can make the morphism $F^{-n} \longrightarrow B^{-n}$ factorizable through the morphism $G^{-n} \longrightarrow B^{-n}$. Let H^{-n-1} and Y^{-n-1} denote the kernels of the morphisms $G^{-n} \longrightarrow Z^{-n}$ and $B^{-n} \longrightarrow Z^{-n}$, respectively; then there is a natural morphism $H^{-n-1} \longrightarrow Y^{-n-1}$. Denote by B^{-n-1} the fibered product of the latter morphism with the morphism $A^{-n-1} \longrightarrow Y^{-n-1}$. There is a unique morphism $A^{-n-2} \longrightarrow B^{-n-1}$ having a zero composition with the morphism $B^{-n-1} \longrightarrow H^{-n-1}$ and forming a commutative diagram with the morphisms $A^{-n-2} \longrightarrow A^{-n-1}$ and $B^{-n-1} \longrightarrow A^{-n-1}$.

We have constructed an exact complex $\cdots \longrightarrow A^{-n-3} \longrightarrow A^{-n-2} \longrightarrow B^{-n-1} \longrightarrow G^{-n} \longrightarrow G^{-n+1} \longrightarrow \cdots \longrightarrow G^0 \longrightarrow 0$, coinciding with our previous complex in the degrees $-n+1$ and above and with the complex $A^\bullet$ in the degrees $-n-2$ and below, and having terms belonging to F in the degrees $-n$ and above. The morphism from the complex $F^\bullet$ into our previous intermediate complex factorizes through the new one (since the composition $F^{-n-1} \longrightarrow F^{-n} \longrightarrow G^{-n}$ factorizes uniquely through the morphism $H^{-n-1} \longrightarrow G^{-n}$, and then there exists a unique morphism $F^{-n-1} \longrightarrow B^{-n-1}$ whose compositions with the morphisms $B^{-n-1} \longrightarrow$

H^{-n-1} and $B^{-n-1} \longrightarrow A^{-n-1}$ are equal to the required ones). Continuing with this procedure ad infinitum provides the desired exact complex $G^\bullet$ over F.

The proof in the case $\star = \mathsf{abs}$ is similar to that of [15, Proposition 1.5], and the case $\star = \mathsf{ctr}$ is proven along the lines of [15, Remark 1.5] and [55, Theorem 4.2.1] (cf. the proofs of Proposition A.6.1 and Theorem B.5.3 below). Not to reiterate here the whole argument from [15, 55], let us restrict ourselves to a brief sketch.

One has to show that any morphism from a complex over F to a complex absolutely acyclic (contraacyclic) over E factorizes through a complex absolutely acyclic (contraacyclic) over F in the homotopy category $\mathsf{Hot}(\mathsf{E})$. This is checked by induction in the transformation rules using which one constructs arbitrary absolutely acyclic (contraacyclic) complexes over E from the total complexes of short exact sequences. Finally, the case of a morphism from a complex over F to the total compex of a short exact sequence over E is treated using the two lemmas below.

Lemma A.2.2. *Let $U^\bullet \longrightarrow V^\bullet \longrightarrow W^\bullet$ be a short exact sequence of complexes over an exact category E and $M^\bullet$ be its total complex. Then a morphism $N^\bullet \longrightarrow M^\bullet$ of complexes over E is homotopic to zero whenever its component $N^\bullet \longrightarrow W^\bullet[-1]$, which is a morphism of graded objects in E, can be lifted to a morphism of graded objects $N^\bullet \longrightarrow V^\bullet[-1]$.* □

Lemma A.2.3. *Let $U^\bullet \longrightarrow V^\bullet \longrightarrow W^\bullet$ be a short exact sequence of complexes over an exact category E and $M^\bullet$ be its total complex. Let Q be a graded object in E and $\widetilde{Q}^\bullet$ be the (contractible) complex over E freely generated by Q. Then a morphism of complexes $\tilde{q}\colon \widetilde{Q}^\bullet \longrightarrow M^\bullet$ has its component $\widetilde{Q}^\bullet \longrightarrow W^\bullet[-1]$ liftable to a morphism of graded objects $\widetilde{Q}^\bullet \longrightarrow V^\bullet[-1]$ whenever its restriction $q\colon Q \longrightarrow M^\bullet$ to the graded subobject $Q \subset \widetilde{Q}^\bullet$ has the same property.* □

In order to apply the lemmas, one needs to notice that, in our assumptions on E and F, for any admissible epimorphism $V \longrightarrow W$ in E, any object $F \in \mathsf{F}$, and any morphism $F \longrightarrow W$ in E there exist an object $Q \in \mathsf{F}$, an admissible epimorphism $Q \longrightarrow F$ in F, and a morphism $Q \longrightarrow V$ in E such that the square $Q \longrightarrow F$, $V \longrightarrow W$ is commutative. Otherwise, the argument is no different from the one in [15, 55]. □

A.3. **Infinite left resolutions.** Let E be an exact category and $\mathsf{F} \subset \mathsf{E}$ be a full subcategory closed under extensions and the passage to the kernels of admissible epimorphisms. Suppose further that every object of E is the image of an admissible epimorphism from an object belonging to F. We endow F with the induced structure of an exact category.

Proposition A.3.1. (a) *The triangulated functor $\mathsf{D}^-(\mathsf{F}) \longrightarrow \mathsf{D}^-(\mathsf{E})$ induced by the exact embedding functor $\mathsf{F} \longrightarrow \mathsf{E}$ is an equivalence of triangulated categories.*

(b) *If the infinite products are everywhere defined and exact in the exact category E, and preserve the full subcategory $\mathsf{F} \subset \mathsf{E}$, then the triangulated functor $\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ induced by the embedding $\mathsf{F} \longrightarrow \mathsf{E}$ is an equivalence of categories.*

Proof. The proof of part (a) is based on part (b) of the following lemma.

Lemma A.3.2. (a) *For any finite complex $E^{-d} \longrightarrow \cdots \longrightarrow E^0$ over E there exists a finite complex $F^{-d} \longrightarrow \cdots \longrightarrow F^0$ over F together with a morphism of complexes $F^\bullet \longrightarrow E^\bullet$ over E such that the morphisms $F^i \longrightarrow E^i$ are admissible epimorphisms in E and the cocone (or equivalently, the termwise kernel) of the morphism $F^\bullet \longrightarrow E^\bullet$ is quasi-isomorphic to an object of E placed in the cohomological degree $-d$.*

(b) *For any bounded above complex $\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0$ over E there exists a bounded above complex $\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0$ over F together with a quasi-isomorphism of complexes $F^\bullet \longrightarrow E^\bullet$ over E such that the morphisms $F^i \longrightarrow E^i$ are admissible epimorphisms in E.*

Proof. Pick an admissible epimorphism $F^0 \longrightarrow E^0$ with $F^0 \in \mathsf{F}$ and consider the fibered product G^{-1} of the objects E^{-1} and F^0 over E^0 in E. Then there exists a unique morphism $E^{-2} \longrightarrow G^{-1}$ having a zero composition with the morphism $G^{-1} \longrightarrow F^0$ and forming a commutative diagram with the morphisms $E^{-2} \longrightarrow E^{-1}$ and $G^{-1} \longrightarrow E^{-1}$. Continuing the construction, pick an admissible epimorphism $F^{-1} \longrightarrow G^{-1}$ with $F^{-1} \in \mathsf{F}$, consider the fibered product G^{-2} of E^{-2} and F^{-1} over G^{-1}, etc. In the case (a), proceed in this way until the object F^{-d} is constructed; in the case (b), proceed indefinitely. The desired assertions follow from the observation that natural morphism between the complexes $G^{-d} \longrightarrow F^{-d+1} \longrightarrow \cdots \longrightarrow F^0$ and $E^{-d} \longrightarrow E^{-d+1} \longrightarrow \cdots \longrightarrow E^0$ is a quasi-isomorphism for any $d \geqslant 1$. □

In view of [53, Lemma 1.6], in order to finish the proof of part (a) of Proposition it remains to show that any bounded above complex over F that is acyclic over E is also acyclic over F. This follows immediately from the condition that F is closed with respect to the passage to the kernels of admissible epimorphisms in E.

In the situation of part (b), the functor in question is fully faithful by Proposition A.2.1. A construction of a morphism with contraacyclic cone onto a given complex over E from an appropriately chosen complex over F is presented below.

Lemma A.3.3. (a) *For any complex $E^\bullet$ over E, there exists a complex $P^\bullet$ over F together with a morphism of complexes $P^\bullet \longrightarrow E^\bullet$ such that the morphism $P^i \longrightarrow E^i$ is an admissible epimorphism for each $i \in \mathbb{Z}$.*

(b) *For any complex $E^\bullet$ over E, there exists a bicomplex $P_\bullet^\bullet$ over F together with a morphism of bicomplexes $P_\bullet^\bullet \longrightarrow E^\bullet$ over E such that the complexes $P_j^\bullet$ vanish for all $j < 0$, while for each $i \in \mathbb{Z}$ the complex $\cdots \longrightarrow P_2^i \longrightarrow P_1^i \longrightarrow P_0^i \longrightarrow E^i \longrightarrow 0$ is acyclic with respect to the exact category E.*

Proof. To prove part (a), pick admissible epimorphisms $F^i \longrightarrow E^i$ onto all the objects $E^i \in \mathsf{E}$ from some objects $F^i \in \mathsf{F}$. Then the contractible complex $P^\bullet$ with the terms $P^i = F^i \oplus F^{i-1}$ (that is the complex freely generated by the graded object F^* over F) comes together with a natural morphism of complexes $P^\bullet \longrightarrow E^\bullet$ with the desired property. Part (b) is easily deduced from (a) by passing to the termwise kernel of the morphism of complexes $P_0^\bullet = P^\bullet \longrightarrow E^\bullet$ and iterating the construction. □

Lemma A.3.4. *Let A be an additive category with countable direct products. Let $\cdots \longrightarrow P^{\bullet\bullet}(2) \longrightarrow P^{\bullet\bullet}(1) \longrightarrow P^{\bullet\bullet}(0)$ be a projective system of bicomplexes over A.*

Suppose that for every pair of integers i, $j \in \mathbb{Z}$ the projective system $\cdots \longrightarrow P^{ij}(2) \longrightarrow P^{ij}(1) \longrightarrow P^{ij}(0)$ stabilizes, and let $P^{ij}(\infty)$ denote the corresponding limit. Then the total complex of the bicomplex $P^{\bullet\bullet}(\infty)$ constructed by taking infinite products along the diagonals is homotopy equivalent to a complex obtained from the total complexes of the bicomplexes $P^{\bullet\bullet}(n)$ (constructed in the same way) by iterated application of the operations of shift, cone, and countable product.

Proof. Denote by $T(n)$ and $T(\infty)$ the total complexes of, respectively, the bicomplexes $P^{\bullet\bullet}(n)$ and $P^{\bullet\bullet}(\infty)$. Then the short sequence of telescope construction

$$0 \longrightarrow T(\infty) \longrightarrow \prod_n T(n) \longrightarrow \prod_n T(n) \longrightarrow 0$$

is a termwise split short exact sequence of complexes over A. Indeed, at every term of the complexes the sequence decomposes into a countable product of sequences corresponding to the projective systems $P^{ij}(*)$ with fixed indices i, j and their limits $P^{ij}(\infty)$. It remains to notice that the telescope sequence of a stabilizing projective system is split exact, and a product of split exact sequences is split exact. □

Returning to part (b) of Proposition, given a complex $E^\bullet$ over E, one applies Lemma A.3.3(b) to obtain a bicomplex $P_\bullet^\bullet$ over F mapping onto $E^\bullet$. Let us show that the cone of the morphism onto $E^\bullet$ from the total complex $T^\bullet$ constructed by taking infinite products along the diagonals of the bicomplex $P_\bullet^\bullet$ is a contraacyclic complex over E. For this purpose, augment the bicomplex $P_\bullet^\bullet$ with the complex $E^\bullet$ and represent the resulting bicomplex as the termwise stabilizing projective limit of its quotient bicomplexes of canonical filtration with respect to the lower indices. The latter bicomplexes being finite exact sequences of complexes over E, the assertion follows from Lemma A.3.4. □

A.4. **Homotopy adjusted complexes.** The following simple construction (cf. [63]) will be useful for us when working with the conventional unbounded derived categories in Section 4 (see, specifically, Section 4.7).

Let E be an exact category. If the functors of infinite direct sum exist and are exact in E, we denote by $\mathsf{D}(\mathsf{E})^{\mathsf{lh}} \subset \mathsf{D}(\mathsf{E})$ the minimal full triangulated subcategory in $\mathsf{D}(\mathsf{E})$ containing the objects of E and closed under infinite direct sums. Similarly, if the functors of infinite product exist and are exact in E, we denote by $\mathsf{D}(\mathsf{E})^{\mathsf{rh}}$ the minimal full triangulated subcategory of $\mathsf{D}(\mathsf{E})$ containing the objects of E and closed under infinite products. It is not difficult to show (see the proof of Proposition A.4.3) that, in the assumptions of the respective definitions, $\mathsf{D}^-(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})^{\mathsf{lh}}$ and $\mathsf{D}^+(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})^{\mathsf{rh}}$.

We return temporarily to the assumptions of Section A.2 concerning a pair of exact categories $\mathsf{F} \subset \mathsf{E}$.

Lemma A.4.1. *Let $B^\bullet$ be a bounded above complex over F and $C^\bullet$ be an acyclic complex over E. Then any morphism of complexes $B^\bullet \longrightarrow C^\bullet$ over E factorizes through a bounded above acyclic complex $K^\bullet$ over F.*

Proof. The canonical truncation of exact complexes over the exact category E allows to assume the complex $C^\bullet$ to be bounded above. In this case, the assertion was established in the proof of Proposition A.2.1. (For a different argument leading to a slightly weaker conclusion, see [15, proof of Lemma 2.9].) ☐

Corollary A.4.2. *Let $B^\bullet$ be a bounded above complex over F and $C^\bullet$ be a complex over F that is acyclic as a complex over E. Then the group $\operatorname{Hom}_{\mathsf{D}(\mathsf{F})}(B^\bullet, C^\bullet)$ of morphisms in the derived category $\mathsf{D}(\mathsf{F})$ of the exact category F vanishes.*

Proof. Indeed, any morphism from $B^\bullet$ to $C^\bullet$ in $\mathsf{D}(\mathsf{F})$ can be represented as a fraction $B^\bullet \longrightarrow {}'C^\bullet \longleftarrow C^\bullet$, where $B^\bullet \longrightarrow {}'C^\bullet$ is a morphism of complexes over F and ${}'C^\bullet \longrightarrow C^\bullet$ is a quasi-isomorphism of such complexes. Then the complex ${}'C^\bullet$ is also acyclic over E, and it remains to apply Lemma A.4.1 to the morphism $B^\bullet \longrightarrow {}'C^\bullet$. ☐

From now on the assumptions of Section A.3 about a full subcategory F in an exact category E are enforced.

Proposition A.4.3. *Suppose that the exact category E is actually abelian, and that infinite direct sums are everywhere defined and exact in the category E and preserve the full exact subcategory $\mathsf{F} \subset \mathsf{E}$. Then the composition of natural triangulated functors $\mathsf{D}(\mathsf{F})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{E})$ is an equivalence of triangulated categories.*

Proof. We will show that any complex over E is the target of a quasi-isomorphism with the source belonging to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$. By [53, Lemma 1.6], it will follow, in particular, that $\mathsf{D}(\mathsf{E})$ is isomorphic to the localization of $\mathsf{D}(\mathsf{F})$ by the thick subcategory of complexes over F acyclic over E. By Corollary A.4.2, the latter subcategory is semiorthogonal to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$, so the same constuction of a quasi-isomorphism with respect to E will also imply that these two subcategories form a semiorthogonal decomposition of $\mathsf{D}(\mathsf{F})$. This would clearly suffice to prove the desired assertion.

Let $C^\bullet$ be a complex over E. Consider all of its subcomplexes of canonical truncation, pick a termwise surjective quasi-isomorphism onto each of them from a bounded above complex over F, and replace the latter with its finite subcomplex of silly filtration with, say, only two nonzero terms. Take the direct sum $B_0^\bullet$ of all the obtained complexes over F and consider the natural morphism of complexes $B_0^\bullet \longrightarrow C^\bullet$. This is a termwise surjective morphism of complexes which also acts surjectively on all the objects of coboundaries, cocycles, and cohomology. Next we apply the same construction to the kernel of this morphism of complexes, etc.

We have constructed an exact complex of complexes $\cdots \longrightarrow B_2^\bullet \longrightarrow B_1^\bullet \longrightarrow B_0^\bullet \longrightarrow C^\bullet \longrightarrow 0$ which remains exact after replacing all the complexes $B_i^\bullet$ and $C^\bullet$ with their cohomology objects (taken in the abelian category E). All the complexes $B_i^\bullet$ belong to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ by the construction. It remains to show that the totalization of the bicomplex $B_\bullet^\bullet$ obtained by taking infinite direct sums along the diagonals also belongs to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ and maps quasi-isomorphically onto $C^\bullet$.

The totalization of the bicomplex $B_\bullet^\bullet$ is a direct limit of the totalizations of its subbicomplexes of silly filtration in the lower indices $B_n^\bullet \longrightarrow \cdots \longrightarrow B_0^\bullet$. By the dual version of Lemma A.3.4, the former assertion follows. To prove the latter one, it suffices to apply the following result due to Eilenberg and Moore [16] to the bicomplex obtained by augmenting $B_\bullet^\bullet$ with $C^\bullet$. ☐

Lemma A.4.4. *Let* A *be an abelian category with exact functors of countable direct sum, and let* $D^\bullet_\bullet$ *be a bicomplex over* A *such that the complexes* $D^\bullet_j$ *vanish for all* $j < 0$*, while the complexes* $D^i_\bullet$ *are acyclic for all* $i \in \mathbb{Z}$*, as are the complexes* $H^i(D^\bullet_\bullet)$*. Then the total complex of the bicomplex* $D^\bullet_\bullet$ *obtained by taking infinite direct sums along the diagonals is acyclic.*

Proof. Denote by $S^\bullet(\infty)$ the totalization of the bicomplex $D^\bullet_\bullet$ and by $S^\bullet(n)$ the totalizations of its subbicomplexes of silly filtration $D^\bullet_n \longrightarrow \cdots \longrightarrow D^\bullet_0$. Consider the telescope short exact sequence

$$0 \longrightarrow \bigoplus\nolimits_n S^\bullet(n) \longrightarrow \bigoplus S^\bullet(n) \longrightarrow S^\bullet \longrightarrow 0$$

and pass to the long exact sequence of cohomology associated with this short exact sequence of complexes. The morphisms $\bigoplus_n H^i(S^\bullet(n)) \longrightarrow \bigoplus_n H^i(S^\bullet(n))$ in this long exact sequence are the differentials in the two-term complexes computing the derived functor of inductive limit $\varinjlim^*_n H^i(S^\bullet(n))$. It is clear from the conditions on the bicomplex $D^\bullet_\bullet$ that the morphisms of cohomology $H^i(S^\bullet(n-1)) \longrightarrow H^i(S^\bullet(n))$ induced by the embeddings of complexes $S^\bullet(n-1) \longrightarrow S^\bullet(n)$ vanish. Hence the morphisms $\bigoplus_n H^i(S^\bullet(n)) \longrightarrow \bigoplus_n H^i(S^\bullet(n))$ are isomorphisms and $H^*(S^\bullet) = 0$. $\square$

Remark A.4.5. In particular, it follows from Proposition A.4.3 that $\mathsf{D}(\mathsf{E}) = \mathsf{D}(\mathsf{E})^{\mathsf{lh}}$ for any abelian category E with exact functors of infinite direct sum. On the other hand, the following example is instructive. Let $\mathsf{E} = R\text{–}\mathsf{mod}$ be the abelian category of left modules over an associative ring R and $\mathsf{F} = R\text{–}\mathsf{mod}^{\mathsf{prj}}$ be the full additive subcategory of projective R-modules (with the induced trivial exact category structure). Then we have $\mathsf{D}(\mathsf{F}) = \mathsf{Hot}(\mathsf{F}) \neq \mathsf{D}(\mathsf{E})$, while $\mathsf{D}(\mathsf{F})^{\mathsf{lh}} \subsetneq \mathsf{D}(\mathsf{F})$ is the full subcategory of homotopy projective complexes in $\mathsf{Hot}(R\text{–}\mathsf{mod}^{\mathsf{prj}})$ [63]. Hence one can see (by considering $\mathsf{E} = \mathsf{F} = R\text{–}\mathsf{mod}^{\mathsf{prj}}$) that the assertion of Proposition A.4.3 is not generally true when the exact category E is not abelian.

Corollary A.4.6. *In the assumptions of Proposition* A.4.3*, the fully faithful functor* $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{F})$ *is left adjoint to the triangulated functor* $\mathsf{D}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{E})$ *induced by the embedding of exact categories* $\mathsf{F} \longrightarrow \mathsf{E}$*.*

Proof. Clear from the proof of Proposition A.4.3. $\square$

Keeping the assumptions of Proposition A.4.3, assume additionally that the exact category F has finite homological dimension. Then the natural functor $\mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{F})$ is an equivalence of triangulated categories [52, Remark 2.1]. Consider the composition of triangulated functors $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{F}) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$. The following result is a generalization of [15, Lemma 2.9].

Corollary A.4.7. *The functor* $\mathsf{D}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *so constructed is left adjoint to the Verdier localization functor* $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$*.*

Proof. One has to show that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathsf{E})}(B^\bullet, C^\bullet) = 0$ for any complex $B^\bullet \in \mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ and any acyclic complex $C^\bullet$ over E. This vanishing easily follows from Lemma A.4.1. $\square$

Finally, let $\mathsf{G} \subset \mathsf{F}$ be two full subcategories in an abelian category E, each satisfying the assumptions of Section A.3 and Proposition A.4.3. Assume that the exact category G has finite homological dimension. Consider the composition of triangulated functors $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{G})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{G}) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{G}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{F})$. The next corollary is a straightforward generalization of the previous one.

Corollary A.4.8. *The functor $\mathsf{D}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{F})$ constructed above is left adjoint to the composition of Verdier localization functors $\mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{E})$.* □

A.5. **Finite left resolutions.** We keep the assumptions of Section A.3. Assume additionally that the additive category E is, in the terminology of [46, 54], "semi-saturated" (i. e., it contains the kernels of its split epimorphisms, or equivalently, the cokernels of its split monomorphisms). Then the additive category F has the same property. The following results elaborate upon the ideas of [15, Remark 2.1].

We will say that an object of the derived category $\mathsf{D}^-(\mathsf{E})$ has *left F-homological dimension not exceeding m* if its isomorphism class can be represented by a bounded above complex $F^\bullet$ over F such that $F^i = 0$ for $i < -m$. By the definition, the full subcategory of objects of finite left F-homological dimension in $\mathsf{D}^-(\mathsf{E})$ is the image of the fully faithful triangulated functor $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^-(\mathsf{F}) \simeq \mathsf{D}^-(\mathsf{E})$.

Lemma A.5.1. *If a bounded above complex $G^\bullet$ over the exact subcategory F, viewed as an object of the derived category $\mathsf{D}^-(\mathsf{E})$, has left F-homological dimension not exceeding m, then the differential $G^{-m-1} \longrightarrow G^{-m}$ has a cokernel $'G^{-m}$ in the additive category F, and the complex $\cdots \longrightarrow G^{-m-1} \longrightarrow G^{-m} \longrightarrow {'G^{-m}} \longrightarrow 0$ over F is acyclic. Consequently, the complex $G^\bullet$ over F is quasi-isomorphic to the finite complex $0 \longrightarrow {'G^{-m}} \longrightarrow G^{-m+1} \longrightarrow G^{-m+2} \longrightarrow \cdots \longrightarrow 0$.*

Proof. In view of the equivalence of categories $\mathsf{D}^-(\mathsf{F}) \simeq \mathsf{D}^-(\mathsf{E})$, the assertion really depends on the exact subcategory F only. By the definition of the derived category, two complexes representing isomorphic objects in it are connected by a pair of quasi-isomorphisms. Thus it suffices to consider two cases when there is a quasi-isomorphism acting either in the direction $G^\bullet \longrightarrow F^\bullet$, or $F^\bullet \longrightarrow G^\bullet$ (where $F^\bullet$ is a bounded above complex over F such that $F^i = 0$ for $i < -m$).

In the former case, acyclicity of the cone of the morphism $G^\bullet \longrightarrow F^\bullet$ implies the existence of cokernels of its differentials and the acyclicity of canonical truncations, which provides the desired conclusion.

In the latter case, from acyclicity of the cone of the morphism $F^\bullet \longrightarrow G^\bullet$ one can similarly see that the morphism $G^{-m-2} \longrightarrow G^{-m-1} \oplus F^{-m}$ with the vanishing component $G^{-m-2} \longrightarrow F^{-m}$ has a cokernel, and it follows that the morphism $G^{-m-2} \longrightarrow G^{-m-1}$ also does. Denoting the cokernel of the latter morphism by $'G^{-m-1}$, one easily concludes that the complex $\cdots \longrightarrow G^{-m-2} \longrightarrow G^{-m-1} \longrightarrow {'G^{-m-1}} \longrightarrow 0$ is acyclic, and it remains to show that the morphism $'G^{-m-1} \longrightarrow G^{-m}$ is an admissible monomorphism in the exact category F. Indeed, the morphism $'G^{-m-1} \oplus F^{-m} \longrightarrow G^{-m} \oplus F^{-m+1}$ is; and hence so is its composition with the embedding of a direct summand $'G^{-m-1} \longrightarrow {'G^{-m-1}} \oplus F^{-m}$. □

Corollary A.5.2. *Let $G^\bullet$ be a finite complex over the exact category* E *such that $G^i = 0$ for $i < -m$ and $G^i \in \mathsf{F}$ for $i > -m$. Assume that the object represented by $G^\bullet$ in $\mathsf{D}^-(\mathsf{E})$ has left* F*-homological dimension not exceeding m. Then the object G^{-m} also belongs to* F.

Proof. Replace the object G^{-m} with its left resolution by objects from F and apply Lemma A.5.1. □

We say that an object $E \in \mathsf{E}$ has left F-homological dimension not exceeding m if the corresponding object of the derived category $\mathsf{D}^-(\mathsf{E})$ does. In other words, E must have a left resolution by objects of F of the length not exceeding m. Let us denote the left F-homological dimension of an object E by $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$.

Corollary A.5.3. *Let $\mathsf{E}' \subset \mathsf{E}$ be a (strictly) full semi-saturated additive subcategory with an induced exact category structure. Set $\mathsf{F}' = \mathsf{E}' \cap \mathsf{F}$, and assume that every object of E' is the image of an admissible epimorphism in the exact category E' acting from an object belonging to F'. Then for any object $E \in \mathsf{E}'$ one has $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E = \operatorname{ld}_{\mathsf{F}'/\mathsf{E}'} E$.*

Proof. Follows from Corollary A.5.2. □

Lemma A.5.4. *Let $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact triple in E. Then*
(a) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$;*
(b) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m+1$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m$;*
(c) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m-1$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m$.*

Proof. Let us prove part (a); the proofs of parts (b) and (c) are similar. The morphism $E''[-1] \longrightarrow E'$ in $\mathsf{D}^-(\mathsf{E}) \simeq \mathsf{D}^-(\mathsf{F})$ can be represented by a morphism of complexes $''F^\bullet \longrightarrow {}'F^\bullet$ in $\mathsf{Hot}^-(\mathsf{F})$. By Lemma A.5.1, both complexes can be replaced by their canonical truncations at the degree $-m$. Obviously, there is the induced morphism between the complexes truncated in this way, so we can simply assume that $''F^i = 0 = {}'F^i$ for $i < -m$. Moreover, the complex $''F^\bullet$ could be truncated even one step further, i. e., the morphism $''F^{-m} \longrightarrow {}''F^{-m+1}$ is an admissible monomorphism in the exact category F. From this one easily concludes that for the cone $G^\bullet$ of the morphism of complexes $''F^\bullet \longrightarrow {}'F^\bullet$ one has $G^i = 0$ for $i < -m-1$ and the morphism $G^{-m-1} \longrightarrow G^{-m}$ is an admissible monomorphism in F. □

Corollary A.5.5. (a) *Let $0 \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_0 \longrightarrow E \longrightarrow 0$ be an exact sequence in E. Then the left F-homological dimension $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$ does not exceed the supremum of the expressions $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E_i + i$ over $0 \leqslant i \leqslant n$ (where we set $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} 0 = -1$).*

(b) *Let $0 \longrightarrow E \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$ be an exact sequence in E. Then the left F-homological dimension $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$ does not exceed the supremum of the expressions $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E^i - i$ over $0 \leqslant i \leqslant n$.*

Proof. Part (a) follows by induction from Lemma A.5.4(c), and part (b) similarly follows from Lemma A.5.4(b). □

Proposition A.5.6. *Suppose that the left F-homological dimension of all objects $E \in \mathsf{E}$ does not exceed a fixed constant d. Then the triangulated functor $\mathsf{D}^\star(\mathsf{F}) \longrightarrow \mathsf{D}^\star(\mathsf{E})$*

induced by the exact embedding functor $\mathsf{F} \longrightarrow \mathsf{E}$ *is an equivalence of triangulated categories for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, *or* abs.

When $\star = \mathsf{co}$ *(respectively,* $\star = \mathsf{ctr}$*), it is presumed here that the functors of infinite direct sum (resp., infinite product) are everywhere defined and exact in the category* E *and preserve the full subcategory* $\mathsf{F} \subset \mathsf{E}$.

Proof. The cases $\star = -$ or ctr were considered in Proposition A.3.1 (and hold in its weaker assumptions). They can be also treated together with the other cases, as it is explained below.

In the cases $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$ one can argue as follows. Using the construction of Lemma A.3.3 and taking into account Corollary A.5.2, one produces for any $\star$-bounded complex $E^\bullet$ over E its finite left resolution $P_\bullet^\bullet$ of length d (in the lower indices) by $\star$-bounded (in the upper indices) complexes over F. The total complex of $P_\bullet^\bullet$ maps by a quasi-isomophism (in fact, a morphism with an absolutely acyclic cone) over the exact category E onto the complex $E^\bullet$.

By [53, Lemma 1.6], it remains to show that any complex $C^\bullet$ over F that is acyclic as a complex over E is also acyclic as a complex over F. For this purpose, we apply the same construction of the resolution $P_\bullet^\bullet$ to the complex $C^\bullet$. The complex $P_0^\bullet$ is acyclic over F and maps by a termwise admissible epimorphism onto the complex $C^\bullet$; it follows that the induced morphisms of the objects of cocycles are admissible epimorphisms, too. The passage to the cocycle objects of acyclic complexes also commutes with the passage to the kernels of termwise admissible epimorphisms.

We conclude that the cocycle objects of the acyclic complexes $P_i^\bullet$ form resolutions of the cocycle objects of the complex $C^\bullet$. Since the left F-homological dimension of objects of E does not exceed d and for $i < d$ the cocycle objects of $P_i^\bullet$ belong to F, so do the cocycle objects of the complex $P_d^\bullet$. Now the total complex of $P_\bullet^\bullet$ is acyclic over F and maps onto $C^\bullet$ with a cone acyclic over F.

The rather involved argument in the cases $\star = \mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs is similar to that in [15, Theorem 1.4] and goes back to the proof of [52, Theorem 7.2.2]. We do not reiterate the details here. ☐

A.6. **Finite homological dimension.** First we return to the assumptions of Section A.2. The following result is a partial generalization of Lemma A.1.3.

Proposition A.6.1. *Assume that infinite products are everywhere defined and exact in the exact category* E. *Suppose also that the exact category* F *has finite homological dimension. Then the triangulated functor* $\mathsf{D}^{\mathsf{abs}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ *induced by the exact embedding functor* $\mathsf{F} \longrightarrow \mathsf{E}$ *is fully faithful.*

Proof. A combination of the assertions of Proposition A.2.1 and [52, Remark 2.1] implies the assertion of Proposition in the case when F is preserved by the infinite products in E. The proof in the general case consists in a combination of the arguments proving the two mentioned results.

We will show that any morphism in $\mathsf{Hot}(\mathsf{E})$ from a complex over F to a complex contraacyclic over E factorizes through a complex absolutely acyclic over F. For this purpose, it suffices to check that the class of complexes over E having this property contains the total complexes of exact triples of complexes over E and is closed with respect to cones and infinite products.

The former two assersions are essentially proven in [15, Section 1.5] (see also Lemmas A.2.2–A.2.3 above). To prove the latter one, suppose that we are given a morphism of complexes $F^\bullet \longrightarrow \prod_\alpha A^\bullet_\alpha$ in $\mathsf{Hot}(\mathsf{E})$, where $F^\bullet$ is a complex over F. Suppose further that each component morphism $F^\bullet \longrightarrow A^\bullet_\alpha$ factorizes through a complex $G^\bullet_\alpha$ that is absolutely acyclic over F. Clearly, the morphism $F^\bullet \longrightarrow \prod_\alpha A^\bullet_\alpha$ factorizes through the complex $\prod_\alpha G^\bullet_\alpha$.

It follows from the proof in [52, Remark 2.1] (for a more generally applicable argument, see also [15, Section 1.6]) that there exists an integer n such that every complex absolutely acyclic over F can be obtained from totalizations of exact triples of complexes over F by applying the operation of the passage to a cone at most n times. Therefore, the product $\prod_\alpha G^\bullet_\alpha$ is an absolutely acyclic complex over E. Hence it follows from what we already know that the morphism $F^\bullet \longrightarrow \prod_\alpha G^\bullet_\alpha$ factorizes through a complex absolutely acyclic over F. □

Now the assumptions of Section A.5 are enforced. The assumption of the following corollary is essentially a generalization of the conditions $(*)$–$(**)$ from [53, Sections 3.7–3.8].

Corollary A.6.2. *In the situation of Proposition* A.6.1*, assume additionally that countable products of objects from* F *taken in the category* E *have finite left* F*-homological dimensions. Then the functor* $\mathsf{D}^{\mathsf{abs}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ *is an equivalence of triangulated categories.*

Proof. Clearly, it suffices to find for any complex over E a morphism into it from a complex over F with a cone contraacyclic over E. The following two-step construction procedure goes back to [53].

Given a complex $E^\bullet$ over E, we proceed as in the above proof of Proposition A.3.1(b), applying Lemma A.3.3(b) in order to obtain a bicomplex $P^\bullet_\bullet$ over F. The total complex $T^\bullet$ of the bicomplex $P^\bullet_\bullet$ constructed by taking infinite products along the diagonals maps onto $E^\bullet$ with a cone contraacyclic with respect to E.

By assumption, the left F-homological dimensions of the terms of the complex $T^\bullet$ are finite, and in fact bounded by a fixed constant d. Applying Lemma A.3.3(b) again together with Corollary A.5.2, we produce a finite complex of complexes $Q^\bullet_\bullet$ of length d (in the lower indices) over F mapping termwise quasi-isomorphically onto $T^\bullet$. Now the total complex of $Q^\bullet_\bullet$ maps onto $T^\bullet$ with a cone absolutely acyclic over E. □

The following generalization of the last result is straightforward. Let $\mathsf{G} \subset \mathsf{F}$ be two full subcategories of an exact category E, each satisfying the assumptions of Section A.3. Suppose that the pair of subcategories $\mathsf{F} \subset \mathsf{E}$ satisfies the assumptions of Proposition A.6.1, while countable products of objects from G taken in the category E have finite left F-homological dimension. Then the functor $\mathsf{D}^{\mathsf{abs}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ is an equivalence of triangulated categories.

Appendix B. Co-Contra Correspondence over a Flat Coring

The aim of this appendix is to extend the assertion of Theorem 5.7.1 from semi-separated Noetherian schemes to semi-separated Noetherian stacks. These are the stacks that can be represented by groupoids with affine schemes of vertices and arrows (see [2, Section 2.1]). We avoid explicit use of the stack language by working with what Kontsevich and Rosenberg call "finite covers" [37] instead. This naturally includes the noncommutative geometry situation. The corresponding algebraic language is that of flat corings over noncommutative rings; thus this appendix provides a bridge between the results of Section 5 and those of [52, Chapter 5].

B.1. **Contramodules over a flat coring.** Let A be an associative ring with unit. We refer to the memoir and monographs [17, 10, 52] for the definitions of a (coassociative) *coring* (with counit) $\mathcal{C}$ over A, a *left comodule* $\mathcal{M}$ over $\mathcal{C}$, and a *right comodule* $\mathcal{N}$ over $\mathcal{C}$. The definition of a *left contramodule* $\mathfrak{P}$ over $\mathcal{C}$ can be found in [17, Section III.5] or [52, Section 0.2.4 or 3.1.1] (see also [56, Section 2.5]). We denote the abelian groups of morphisms in the additive category of left $\mathcal{C}$-comodules by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{M})$ and the similar groups related to the additive category of left $\mathcal{C}$-contramodules by $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P},\mathfrak{Q})$.

The category $\mathcal{C}$–comod of left $\mathcal{C}$-comodules is abelian and the forgetful functor $\mathcal{C}$–$\mathsf{comod} \longrightarrow A$–$\mathsf{mod}$ is exact if and only if $\mathcal{C}$ is a flat right A-module [52, Section 1.1.2]. The category $\mathcal{C}$–contra of left $\mathcal{C}$-contramodules is abelian and the forgetful functor $\mathcal{C}$–$\mathsf{contra} \longrightarrow A$–$\mathsf{mod}$ is exact if and only if $\mathcal{C}$ is a projective left A-module [52, Section 3.1.2]. The following counterexample shows that the category $\mathcal{C}$–contra may be not abelian even though $\mathcal{C}$ is a flat left and right A-module.

Example B.1.1. Let us consider corings $\mathcal{C}$ of the following form. The coring $\mathcal{C}$ decomposes into a direct sum of A-A-bimodules $\mathcal{C} = \mathcal{C}_{11} \oplus \mathcal{C}_{12} \oplus \mathcal{C}_{22}$; the counit map $\mathcal{C} \longrightarrow A$ annihilates $\mathcal{C}_{12}$ and the comultiplication map takes $\mathcal{C}_{ik}$ into the direct sum $\bigoplus_j \mathcal{C}_{ij} \otimes_A \mathcal{C}_{jk}$. Assume further that the restrictions of the counit map to $\mathcal{C}_{11}$ and $\mathcal{C}_{22}$ are both isomorphisms $\mathcal{C}_{ii} \simeq A$. Notice that the data of an A-A-bimodule $\mathcal{C}_{12}$ determines the coring $\mathcal{C}$ in this case. A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is the same thing as a pair of left A-modules $\mathfrak{P}_1$ and $\mathfrak{P}_2$ endowed with an A-module morphism $\mathrm{Hom}_A(\mathcal{C}_{12},\mathfrak{P}_1) \longrightarrow \mathfrak{P}_2$.

The kernel of a morphism of left $\mathcal{C}$-contramodules $(f_1, f_2)\colon (\mathfrak{P}_1,\mathfrak{P}_2) \longrightarrow (\mathfrak{Q}_1,\mathfrak{Q}_2)$ (taken in the additive category $\mathcal{C}$–contra) is the $\mathcal{C}$-contramodule $(\ker f_1,\ \ker f_2)$. The cokernel of the morphism (f_1, f_2) can be computed as the $\mathcal{C}$-contramodule $(\operatorname{coker} f_1,\ \mathfrak{L})$, where $\mathfrak{L}$ is the cokernel of the morphism from $\mathfrak{P}_2$ to the fibered coproduct $\mathrm{Hom}_A(\mathcal{C}_{12},\ \operatorname{coker} f_1) \sqcup_{\mathrm{Hom}_A(\mathcal{C}_{12},\,\mathfrak{Q}_1)} \mathfrak{Q}_2$ (where $\ker f$ and $\operatorname{coker} f$ denote the kernel and cokernel of a morphism of A-modules f).

Now setting A be the ring of integers $\mathbb{Z}$ and $\mathcal{C}_{12}$ to be the (bi)module of rational numbers $\mathbb{Q}$ over $\mathbb{Z}$, one easily checks that the category $\mathcal{C}$–contra is not abelian. It suffices to consider $\mathfrak{P}_1 = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}$, $\mathfrak{P}_2 = 0$, and $\mathfrak{Q}_1 = \mathfrak{Q}_2 = \mathbb{Q}$, the structure morphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathfrak{Q}_1) \longrightarrow \mathfrak{Q}_2$ being the identity isomorphism. The morphism $f_1\colon \mathfrak{P}_1 \longrightarrow \mathfrak{Q}_1$ is chosen to be surjective, while of course $f_2 = 0$. Then the kernel of f is the $\mathcal{C}$-contramodule $(\ker f_1,\ 0)$ and the cokernel of the kernel of f is $(\mathbb{Q},0)$, while the cokernel of f is $(0,0)$ and kernel of the cokernel of f is $(\mathbb{Q},\mathbb{Q})$.

Similarly one can construct a nonflat coring $\mathcal{C}$ for which the category $\mathcal{C}\text{–}\mathsf{comod}$ is not abelian. For a coring $\mathcal{C} = \mathcal{C}_{11} \oplus \mathcal{C}_{12} \oplus \mathcal{C}_{22}$ with $\mathcal{C}_{ii} \simeq A$ as above, a left $\mathcal{C}$-comodule is the same thing as a pair of left A-modules $\mathcal{M}_1$ and $\mathcal{M}_2$ endowed with an A-module morphism $\mathcal{M}_1 \longrightarrow \mathcal{C}_{12} \otimes_A \mathcal{M}_2$. The cokernel and the kernel of an arbitrary morphism of left $\mathcal{C}$-comodules can be computed in the way dual-analogous to the above computation for $\mathcal{C}$-contramodules.

Setting $A = \mathbb{Z}$ and $\mathcal{C}_{12} = \mathbb{Z}/n$ with any $n \geqslant 2$, it suffices to consider the morphism of left $\mathcal{C}$-comodules $(g_1, g_2)\colon (\mathcal{L}_1, \mathcal{L}_2) \longrightarrow (\mathcal{M}_1, \mathcal{M}_2)$ with $\mathcal{L}_1 = \mathcal{L}_2 = \mathbb{Z}/n$, the structure morphism $\mathcal{L}_1 \longrightarrow \mathcal{L}_2/n\mathcal{L}_2$ being the identity map, $\mathcal{M}_1 = 0$, $\mathcal{M}_2 = \mathbb{Z}/n^2$, and an injective map $g_2\colon \mathcal{L}_2 \longrightarrow \mathcal{M}_2$. Then the cokernel of g is the $\mathcal{C}$-comodule $(0, \mathbb{Z}/n)$ and the kernel of the cokernel of g is $(0, \mathbb{Z}/n)$, while the kernel of g is $(0,0)$ and the cokernel of the kernel of g is $(\mathbb{Z}/n, \mathbb{Z}/n)$.

It is clear from the above example that the category of all left contramodules over a flat coring $\mathcal{C}$ does not have good homological properties in general—at least, unless one restricts the class of exact sequences under consideration to those whose exactness is preserves by the functor $\mathrm{Hom}_A(\mathcal{C}, -)$. Our preference is to restrict the class of contramodules instead (or rather, at the same time). So we will be interested in the category $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ of left $\mathcal{C}$-contramodules whose underlying left A-modules are cotorsion modules (see Section 1.3).

Assuming that $\mathcal{C}$ is a flat left A-module, this category has a natural exact category structure where a short sequence of contramodules is exact if and only if its underlying short sequence of A-modules is exact in the abelian category $A\text{–}\mathsf{mod}$. We denote the Ext groups computed in this exact category by $\mathrm{Ext}^{\mathcal{C},*}(\mathfrak{P}, \mathfrak{Q})$. Assuming that $\mathcal{C}$ is a flat right A-module, so the category $\mathcal{C}\text{–}\mathsf{comod}$ is abelian, we denote the Ext groups computed in this category by $\mathrm{Ext}^*_{\mathcal{C}}(\mathcal{L}, \mathcal{M})$.

Given a left A-module U, the left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A U$ is said to be *coinduced* from U. For any left $\mathcal{C}$-comodule $\mathcal{L}$, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\ \mathcal{C} \otimes_A U) \simeq \mathrm{Hom}_A(\mathcal{L}, U)$. Given a left A-module V, the left $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C}, V)$ is said to be *induced* from V. For any left $\mathcal{C}$-contramodule $\mathfrak{Q}$, there is a natural isomorphism $\mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_A(\mathcal{C}, V), \mathfrak{Q}) \simeq \mathrm{Hom}_A(V, \mathfrak{Q})$ [52, Sections 1.1.2 and 3.1.2]. By Lemma 1.3.3(a), the induced $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C}, V)$ is A-cotorsion whenever the coring $\mathcal{C}$ is a flat left A-module and the A-module V is cotorsion.

Notice that, assuming $\mathcal{C}$ to be a flat right A-module, the direct summands of $\mathcal{C}$-comodules coinduced from injective left A-modules are the injective objects of the abelian category $\mathcal{C}\text{–}\mathsf{comod}$, and there are enough of them. Similarly, assuming $\mathcal{C}$ to be a flat left A-module, the direct summands of $\mathcal{C}$-contramodules induced from flat cotorsion left A-modules are the projective objects of the exact category $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$, and there are enough of them (as one can see using Theorem 1.3.1(b)).

Recall that the *contratensor product* $\mathcal{N} \odot_{\mathcal{C}} \mathfrak{P}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is an abelian group constructed as the cokernel of the natural pair of maps $\mathcal{N} \otimes_A \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \rightrightarrows \mathcal{N} \otimes_A \mathfrak{P}$, one of which is induced by the contraaction map $\mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$, while the other one is the composition of the maps induced by the coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C}$ and the evaluation map $\mathcal{C} \otimes_A \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. For any right $\mathcal{C}$-comodule $\mathcal{N}$ and any left A-module V, there is a natural isomorphism $\mathcal{N} \odot_{\mathcal{C}} \mathrm{Hom}_A(\mathcal{C}, V) \simeq \mathcal{N} \otimes_A V$ [52, Sections 0.2.6 and 5.1.1–2].

Given two corings $\mathcal{C}$ and $\mathcal{E}$ over associative rings A and B, and a $\mathcal{C}$-$\mathcal{E}$-bicomodule $\mathcal{K}$, the rules $\mathfrak{P} \longmapsto \mathcal{K} \odot_{\mathcal{E}} \mathfrak{P}$ and $\mathcal{M} \longmapsto \mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M})$ define a pair of adjoint functors between the categories of left $\mathcal{E}$-contramodules and left $\mathcal{C}$-comodules. In the particular case of $\mathcal{C} = \mathcal{D} = \mathcal{K}$, the corresponding functors are denoted by $\Phi_{\mathcal{C}}\colon \mathcal{C}\text{–}\mathsf{contra} \longrightarrow \mathcal{C}\text{–}\mathsf{comod}$ and $\Psi_{\mathcal{C}}\colon \mathcal{C}\text{–}\mathsf{comod} \longrightarrow \mathcal{C}\text{–}\mathsf{contra}$, so $\Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}$ and $\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$. The functors $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ transform the induced left $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C}, U)$ into the coinduced left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A U$ and back, inducing an equivalence between the full subcategories of comodules and contramodules of this form in $\mathcal{C}\text{–}\mathsf{comod}$ and $\mathcal{C}\text{–}\mathsf{contra}$ [7, 52].

B.2. **Base rings of finite weak dimension.** Let $\mathcal{C}$ be a coring over an associative ring A. In this section we assume that $\mathcal{C}$ is a flat left and right A-module and, additionally, that A is a ring of finite weak dimension (i. e., the functor $\mathrm{Tor}^A(-,-)$ has finite homological dimension). Notice that the injective dimension of any cotorsion A-module is finite in this case.

The content of this section is very close to that of [52, Chapter 5, and, partly, Section 9.1], the main difference being that the left A-module projectivity assumption on the coring $\mathcal{C}$ used in [52] is weakened here to the flatness assumption. That is why the duality-analogy between comodules and contramodules is more obscure here than in *loc. cit.* Also, the form of the presentation below may be more in line with the main body of this paper than with [52].

Lemma B.2.1. (a) *Any $\mathcal{C}$-comodule can be presented as the quotient comodule of an A-flat $\mathcal{C}$-comodule by a finitely iterated extension of $\mathcal{C}$-comodules coinduced from cotorsion A-modules.*

(b) *Any A-cotorsion $\mathcal{C}$-contramodule has an admissible monomorphism into an A-injective $\mathcal{C}$-contramodule such that the cokernel is a finitely iterated extension of $\mathcal{C}$-contramodules induced from cotorsion A-modules.*

Proof. Part (a) is proven by the argument of [52, Lemma 1.1.3] used together with the result of Theorem 1.3.1(b). Part (b) is similar to [52, Lemma 3.1.3(b)]. ☐

A left $\mathcal{C}$-comodule $\mathcal{M}$ is said to be *cotorsion* if the functor $\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{M})$ takes short exact sequences of A-flat $\mathcal{C}$-comodules to short exact sequences of abelian groups. In particular, any $\mathcal{C}$-comodule coinduced from a cotorsion A-module is cotorsion.

An A-cotorsion left $\mathcal{C}$-contramodule $\mathfrak{P}$ is said to be *projective relative to A* (*$\mathcal{C}/A$-projective*) if the functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)$ takes short exact sequences of A-injective $\mathcal{C}$-contramodules to short exact sequences of abelian groups. In particular, any $\mathcal{C}$-contramodule induced from a cotorsion A-module is $\mathcal{C}/A$-projective.

Corollary B.2.2. (a) *A left $\mathcal{C}$-comodule $\mathcal{M}$ is cotorsion if and only if* $\mathrm{Ext}_{\mathcal{C}}^{>0}(\mathcal{L}, \mathcal{M}) = 0$ *for any A-flat left $\mathcal{C}$-comodule $\mathcal{L}$. In particular, the functor* $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, -)$ *takes* *short exact sequences of cotorsion left $\mathcal{C}$-comodules to short exact sequences of abelian groups. The class of cotorsion $\mathcal{C}$-comodules is closed under extensions and the passage to cokernels of injective morphisms.*

(b) *A left A-cotorsion $\mathcal{C}$-contramodule $\mathfrak{P}$ is $\mathcal{C}/A$-projective if and only if* $\operatorname{Ext}^{\mathcal{C},>0}(\mathfrak{P},\mathfrak{Q}) = 0$ *for any A-injective left $\mathcal{C}$-contramodule $\mathfrak{Q}$. In particular, the functor* $\operatorname{Hom}^{\mathcal{C}}(-,\mathfrak{Q})$ *takes short exact sequences of $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodules to short exact sequences of abelian groups. The class of $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodules is closed under extensions and the passage to kernels of admissible epimorphisms in* $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$.

Proof. Follows from there being enough A-flat $\mathcal{C}$-comodules in $\mathcal{C}\text{–}\mathsf{comod}$ and A-injective $\mathcal{C}$-contramodules in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$, i. e., weak forms of the assertions of Lemma B.2.1 (cf. [52, Lemma 5.3.1]). ☐

It follows, in particular, that the full subcategories of cotorsion $\mathcal{C}$-comodules and $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodules in $\mathcal{C}\text{–}\mathsf{comod}$ and $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ can be endowed with the induced exact category structures. We denote these exact categories by $\mathcal{C}\text{–}\mathsf{comod}^{\mathsf{cot}}$ and $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}_{\mathcal{C}/A\text{–}\mathsf{pr}}$, respectively.

Lemma B.2.3. (a) *Any $\mathcal{C}$-comodule admits an injective morphism into a finitely iterated extension of $\mathcal{C}$-comodules coinduced from cotorsion A-modules such that the quotient $\mathcal{C}$-comodule is A-flat.*

(b) *For any A-cotorsion $\mathcal{C}$-contramodule there exists an admissible epimorphism onto it from a finitely iterated extension of $\mathcal{C}$-contramodules induced from cotorsion A-modules such that the kernel is an A-injective $\mathcal{C}$-contramodule.*

Proof. The assertions follow from Lemma B.2.1 together with the existence of enough comodules coinduced from cotorsion modules in $\mathcal{C}\text{–}\mathsf{comod}$ and enough contramodules induced from cotorsion modules in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ by virtue of the argument from the second half of the proof of [18, Theorem 10] (cf. Lemmas 1.1.3, 4.1.3, 4.1.10, 4.2.4, 4.3.3, 5.2.7, etc.) An alternative argument (providing somewhat weaker assertions) can be found in [52, Lemma 9.1.2]. ☐

Corollary B.2.4. (a) *A $\mathcal{C}$-comodule is cotorsion if and only if it is a direct summand of a finitely iterated extension of $\mathcal{C}$-comodules coinduced from cotorsion A-modules.*

(b) *An A-cotorsion $\mathcal{C}$-contramodule is $\mathcal{C}/A$-projective if and only if it is a direct summand of a finitely iterated extension of $\mathcal{C}$-contramodules induced from cotorsion A-modules.*

Proof. Follows from Lemma B.2.3 and Corollary B.2.2. ☐

Corollary B.2.5. (a) *A $\mathcal{C}$-comodule is simultaneously cotorsion and A-flat if and only if it is a direct summand of a $\mathcal{C}$-comodule coinduced from a flat cotorsion A-module.*

(b) *An A-cotorsion $\mathcal{C}$-contramodule is simultaneously $\mathcal{C}/A$-projective and A-injective if and only if it is a direct summand of a $\mathcal{C}$-contramodule induced from an injective A-module.*

Proof. Both the "if" assertions are obvious. To prove the "only if" in part (a), consider an A-flat cotorsion $\mathcal{C}$-comodule $\mathcal{M}$. Using Theorem 1.3.1(a), pick an injective morphism $\mathcal{M} \longrightarrow P$ from $\mathcal{M}$ into a cotorsion A-module P such that the cokernel $P/\mathcal{M}$ is a flat A-module. Clearly, P is a flat cotorsion A-module. The cokernel of the composition of $\mathcal{C}$-comodule morphisms $\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{C} \otimes_A P$, being an extension of two A-flat $\mathcal{C}$-comodules, is also A-flat. According to Corollary B.2.2(a), it follows that the $\mathcal{C}$-comodule $\mathcal{M}$ is a direct summand of $\mathcal{C} \otimes_A P$.

To prove the "only if" in part (b), consider a $\mathcal{C}/A$-projective A-injective $\mathcal{C}$-contramodule $\mathfrak{P}$. The natural morphism of $\mathcal{C}$-contramodules $\mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ is surjective with an A-injective kernel. It remains to apply Corollary B.2.2(b) in order to conclude that the extension splits. □

Theorem B.2.6. (a) *For any cotorsion left $\mathcal{C}$-comodule $\mathcal{M}$, the left $\mathcal{C}$-contramodule $\Psi_{\mathcal{C}}(\mathcal{M})$ is a $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodule.*

(b) *For any $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodule $\mathfrak{P}$, the left $\mathcal{C}$-comodule $\Phi_{\mathcal{C}}(\mathfrak{P})$ is cotorsion.*

(c) *The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ restrict to mutually inverse equivalences between the exact subcategories $\mathcal{C}\text{–}\mathsf{comod}^{\mathsf{cot}} \subset \mathcal{C}\text{–}\mathsf{comod}$ and $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}_{\mathcal{C}/A\text{–}\mathsf{pr}} \subset \mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$.*

Proof. The functor $\Psi_{\mathcal{C}}$ takes cotorsion left $\mathcal{C}$-comodules $\mathcal{M}$ to A-cotorsion left $\mathcal{C}$-contramodules, since the functor $\mathrm{Hom}_A(F, \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})) \simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_A F,\ \mathcal{M})$ is exact on the exact category of flat left A-modules F. Furthermore, the functor $\Psi_{\mathcal{C}} \colon \mathcal{M} \longmapsto \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ takes short exact sequences of cotorsion $\mathcal{C}$-comodules to short exact sequences of A-cotorsion $\mathcal{C}$-contramodules, since $\mathcal{C}$ is a flat left A-module.

Similarly, the functor $\Phi_{\mathcal{C}}$ takes short exact sequences of $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodules to short exact sequences of $\mathcal{C}$-comodules, since $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}, \mathbb{Q}/\mathbb{Z}))$ and the left $\mathcal{C}$-contramodule $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}, \mathbb{Q}/\mathbb{Z})$ is A-injective ($\mathcal{C}$ being a flat right A-module).

In view of these observations, all the assertions follow from Corollary B.2.4. Alternatively, one could proceed along the lines of the proof of [52, Theorem 5.3], using the facts that any cotorsion $\mathcal{C}$-comodule has finite injective dimension in $\mathcal{C}\text{–}\mathsf{comod}$ and any $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodule has finite projective dimension in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ (see the proof of the next Theorem B.2.7). □

Theorem B.2.7. (a) *Assume additionally that countable direct sums of injective left A-modules have finite injective dimensions. Then the triangulated functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{C}\text{–}\mathsf{comod}^{\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ induced by the embedding of exact categories $\mathcal{C}\text{–}\mathsf{comod}^{\mathsf{cot}} \longrightarrow \mathcal{C}\text{–}\mathsf{comod}$ is an equivalence of triangulated categories.*

(b) *The triangulated functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}_{\mathcal{C}/A\text{–}\mathsf{pr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ induced by the embdding of exact categories $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}_{\mathcal{C}/A\text{–}\mathsf{pr}} \longrightarrow \mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ is an equivalence of triangulated categories.*

Proof. To prove part (a), we notice that the injective dimension of any cotorsion $\mathcal{C}$-comodule $\mathcal{M}$ (as an object of $\mathcal{C}\text{–}\mathsf{comod}$) does not exceed the weak homological dimension of the ring A. Indeed, one can compute the functor $\mathrm{Ext}_{\mathcal{C}}(-, \mathcal{M})$ using A-flat left resolutions of the first argument (which exist by Lemma B.2.1(a)). Given the description of injective $\mathcal{C}$-comodules in Section B.1, it follows from the assumptions of part (a) that countable direct sums of cotorsion $\mathcal{C}$-comodules have finite injective dimensions, too. It remains to apply the dual version of Corollary A.6.2.

Similarly, to prove part (b) one first notices that the projective dimension of any $\mathcal{C}/A$-projective A-cotorsion $\mathcal{C}$-contramodule does not exceed the supremum of the injective dimensions of cotorsion A-modules, i. e., the weak homological dimension of the ring A. Furthermore, it is clear from the description of projective A-cotorsion $\mathcal{C}$-contramodules in Section B.1 that infinite products of projective objects have finite projective dimensions in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$. Hence Corollary A.6.2 (or the subsequent remark) applies. Alternatively, one could argue in the way similar to the proof of [52, Theorem 5.4] (cf. Theorem B.3.1 below). □

Corollary B.2.8. *The coderived category of left $\mathcal{C}$-comodules $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ and the contraderived category of A-cotorsion left $\mathcal{C}$-contramodules $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ are naturally equivalent. The equivalence is provided by the derived functors of co-contra correspondence $\mathbb{R}\Psi_{\mathcal{C}}$ and $\mathbb{L}\Phi_{\mathcal{C}}$ constructed in terms of cotorsion right resolutions of comodules and relatively projective left resolutions of contramodules.*

Proof. Follows from Theorems B.2.6 and B.2.7. □

B.3. **Gorenstein base rings.** Let $\mathcal{C}$ be a coring over an associative ring A. We suppose the ring A to be left Gorenstein, in the sense that the classes of left A-modules of finite injective dimension and of finite flat dimension coincide. In this case, it is clear that both kinds of dimensions are uniformly bounded by a constant for those modules for which they are finite.

More generally, it will be sufficient to require that the classes of cotorsion left A-modules of finite injective dimension and of finite flat dimension coincide, countable direct sums of injective left A-modules have finite injective dimensions, and flat cotorsion A-modules have uniformly bounded injective dimensions. Then it follows that the flat and injective dimensions of cotorsion A-modules of finite flat/injective dimension are also uniformly bounded.

Theorem B.3.1. (a) *Assume that the coring $\mathcal{C}$ is a flat right A-module. Then the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ of the abelian category of left $\mathcal{C}$-comodules is equivalent to the quotient category of the homotopy category of the additive category of left $\mathcal{C}$-comodules coinduced from cotorsion A-modules of finite flat/injective dimension by its minimal thick subcategory containing the total complexes of short exact sequences of complexes of $\mathcal{C}$-comodules that at every term of the complexes are short exact sequences of $\mathcal{C}$-comodules coinduced from short exact sequences of cotorsion A-modules of finite flat/injective dimension.*

(b) *Assume that the coring $\mathcal{C}$ is a flat left A-module. Then the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ of the exact category of A-cotorsion left $\mathcal{C}$-contramodules is equivalent to the quotient category of the homotopy category of the additive category of left $\mathcal{C}$-contramodules induced from cotorsion A-modules of finite flat/injective dimension by its minimal thick subcategory cointaining the total complexes of short exact sequences of complexes of $\mathcal{C}$-contramodules that at every term of the complexes* *are short exact sequences of $\mathcal{C}$-contramodules induced from short exact sequences of cotorsion A-modules of finite flat/injective dimension.*

Proof. The argument proceeds along the lines of the proof of [52, Theorem 5.5] (see also [52, Question 5.4] and [53, Sections 3.9–3.10]).

Part (a): the relative cobar-resolution, totalized by taking infinite direct sums along the diagonals, provides a closed morphism with a coacyclic cone from any complex of left $\mathcal{C}$-comodules into a complex of coinduced $\mathcal{C}$-comodules. The construction from the proof of [52, Theorem 5.5] provides a closed morphism from any complex of coinduced left $\mathcal{C}$-comodules into a complex of $\mathcal{C}$-comodules termwise coinduced from injective A-modules such that this morphism is coinduced from an injective morphism of A-modules at every term of the complexes.

This allows to obtain a closed morphism with a coacyclic cone from any complex of coinduced left $\mathcal{C}$-comodules into a complex of $\mathcal{C}$-comodules termwise coinduced from injective A-modules (using the assumption that countable direct sums of injective A-modules have finite injective dimensions). The same construction from [52] can be also used to obtain a closed morphism from any complex of left $\mathcal{C}$-comodules with the terms coinduced from cotorsion A-modules of finite injective dimension into a complex of $\mathcal{C}$-comodules termwise coinduced from injective A-modules such that the cone is homotopy equivalent to a complex obtained from short exact sequences of complexes of $\mathcal{C}$-comodules termwise coinduced from short exact sequences of cotorsion A-modules of finite injective dimension using the operation of cone repeatedly.

The assertion of part (a) follows from these observations by a semi-orthogonal decomposition argument from the proofs of [52, Theorems 5.4–5.5].

Part (b): the contramodule relative bar-resolution, totalized by taking infinite products along the diagonals, provides a closed morphism with a contraacyclic cone onto any complex of A-cotorsion left $\mathcal{C}$-contramodules from a complex of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules. The construction dual to the one elaborated in the proof of [52, Theorem 5.5] provides a closed morphism onto any complex of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules from a complex of $\mathcal{C}$-contramodules termwise induced from flat cotorsion A-modules such that this morphism is induced from an admissible epimorphism of cotorsion A-modules at every term of the complexes.

This allows to obtain a closed morphism with a contraacyclic cone onto any complex of left $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules from a complex of $\mathcal{C}$-contramodules termwise induced from flat cotorsion A-modules (since an infinite product of flat cotorsion A-modules, being a cotorsion A-module of finite injective dimension, has finite flat dimension). The same construction can be also used to obtain a closed morphism onto any complex of left $\mathcal{C}$-contramodules with the terms induced from cotorsion A-modules of finite flat dimension from a complex of $\mathcal{C}$-contramodules termwise induced from flat cotorsion A-modules such that the cone is homotopy equivalent to a complex obtained from short exact sequences of complexes of $\mathcal{C}$-contramodules termwise induced from short exact sequences of cotorsion A-modules of finite flat dimension using the operation of cone repeatedly.

Now the dual version of the same semi-orthogonal decomposition argument from [52] implies part (b). ☐

Corollary B.3.2. *Assume that the coring $\mathcal{C}$ is a flat left and right A-module. Then the coderived category of left $\mathcal{C}$-comodules $\mathsf{D^{co}}(\mathcal{C}\text{–}\mathsf{comod})$ and the contraderived category of A-cotorsion left $\mathcal{C}$-contramodules $\mathsf{D^{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ are naturally equivalent. The equivalence is provided by the derived functors of co-contra correspondence $\mathbb{R}\Psi_{\mathcal{C}}$ and $\mathbb{R}\Phi_{\mathcal{C}}$ constructed in terms of right resolutions by complexes of $\mathcal{C}$-comodules termwise coinduced from cotorsion A-modules of finite injective/flat dimension and left resolutions by complexes of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules of finite flat/injective dimension.*

Proof. Follows from Theorem B.3.1 and the remarks in Section B.1. ☐

B.4. **Corings with dualizing complexes.** Let A and B be associative rings. We call a finite complex of A-B-bimodules $D^\bullet$ a *dualizing complex* for A and B [68, 12] if

- (i) $D^\bullet$ is simultaneously a complex of injective left A-modules and a complex of injective right B-modules (in the one-sided module structures obtained by forgetting the other module structure on the other side);
- (ii) as a complex of left A-modules, $D^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective A-modules, and similarly, as a complex of right B-modules, $D^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective B-modules;
- (iii) the "homothety" maps $A \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}(D^\bullet, D^\bullet)$ and $B \longrightarrow \mathrm{Hom}_A(D^\bullet, D^\bullet)$ are quasi-isomorphisms of complexes.

Lemma B.4.1. *Let $D^\bullet$ be a dualizing complex for associative rings A and B. Then*

(a) *if the ring A is left Noetherian and F is a flat left B-module, then the natural homomorphism of finite complexes of left B-modules $F \longrightarrow \mathrm{Hom}_A(D^\bullet,\ D^\bullet \otimes_B F)$ is a quasi-isomorphism;*

(b) *if the ring B is right coherent and J is an injective left A-module, then the natural homomorphism of finite complexes of left A-modules $D^\bullet \otimes_B \mathrm{Hom}_A(D^\bullet, J) \longrightarrow J$ is a quasi-isomorphism.*

Proof. Part (a): first of all, $D^\bullet \otimes_B F$ is a complex of injective left A-modules by Lemma 1.6.1(a). Let $'D^\bullet \longrightarrow D^\bullet$ be a quasi-isomorphism of complexes of left A-modules between a bounded above complex of finitely generated projective A-modules $'D^\bullet$ and the complex $D^\bullet$. Then it suffices to show that the induced morphism of complexes of abelian groups $F \longrightarrow \mathrm{Hom}_A({}'D^\bullet,\ D^\bullet \otimes_B F)$ is a quasi-isomorphism. Now the complex $\mathrm{Hom}_A({}'D^\bullet,\ D^\bullet \otimes_B F)$ is isomorphic to $\mathrm{Hom}_A({}'D^\bullet,\ D^\bullet) \otimes_B F$, and it remains to use the condition (iii) for the morphism $B \longrightarrow \mathrm{Hom}_A(D^\bullet, D^\bullet)$ together with the flatness condition on the B-module F.

Part (b): the complex $\mathrm{Hom}_A(D^\bullet, J)$ is a complex of flat left B-modules by Lemma 1.6.1(b). Let $''D^\bullet \longrightarrow D^\bullet$ be a quasi-isomorphism of complexes of right B-modules between a bounded above complex of finitely generated projective

B-modules $''D^\bullet$ and the complex $D^\bullet$. It suffices to show that the induced morphism of complexes of abelian groups $''D^\bullet \otimes_B \operatorname{Hom}_A(D^\bullet, J) \longrightarrow J$ is a quasi-isomorphism. The complex $''D^\bullet \otimes_B \operatorname{Hom}_A(D^\bullet, J)$ being isomorphic to $\operatorname{Hom}_A(\operatorname{Hom}_{B^{\mathrm{op}}}(''D^\bullet, D^\bullet), J)$, it remains to use the condition (iii) for the morphism $A \longrightarrow \operatorname{Hom}_{B^{\mathrm{op}}}(D^\bullet, D^\bullet)$ together with the injectivity condition on the A-module J. □

The following result is due to Christensen, Frankild, and Holm [12, Proposition 1.5].

Corollary B.4.2. *Let $D^\bullet$ be a dualizing complex for associative rings A and B. Assume that the ring A is left Noetherian. Then the projective dimension of any flat left B-module does not exceed the length of $D^\bullet$.*

Proof. Assume that the complex $D^\bullet$ is concentrated in the cohomological degrees from i to $i + d$. It suffices to show that $\operatorname{Ext}_B^{d+1}(F, G) = 0$ for any flat left B-modules F and G. Let $P_\bullet$ be a projective left resolution of the B-module F. By Lemma B.4.1(a), the natural map of complexes of abelian groups $\operatorname{Hom}_B(P_\bullet, G) \longrightarrow \operatorname{Hom}_B(P_\bullet,\ \operatorname{Hom}_A(D^\bullet,\ D^\bullet \otimes_B G))$ is a quasi-isomorphism. The right-hand side is isomorphic to the complex $\operatorname{Hom}_A(D^\bullet \otimes_B P_\bullet,\ D^\bullet \otimes_B G)$, which is quasi-isomorphic to $\operatorname{Hom}_A(D^\bullet \otimes_B F,\ D^\bullet \otimes_B G)$, since $D^\bullet \otimes_B G$ is a finite complex of injective A-modules. The corollary is proven. Notice that we have only used "a half of" the conditions (i-iii) imposed on a dualizing complex $D^\bullet$. □

Let $\mathcal{C}$ be a coring over an associative ring A and $\mathcal{E}$ be a coring over an associative ring B. We assume $\mathcal{C}$ to be a flat right A-module and $\mathcal{E}$ to be a flat left B-module. A *dualizing complex* for $\mathcal{C}$ and $\mathcal{E}$ is defined as a triple consisting of a finite complex of $\mathcal{C}$-$\mathcal{E}$-bicomodules $\mathcal{D}^\bullet$, a finite complex of A-B-bimodules $D^\bullet$, and a morphism of complexes of A-B-bimodules $\mathcal{D}^\bullet \longrightarrow D^\bullet$ with the following properties:

- (iv) $D^\bullet$ is a dualizing complex for the rings A and B;
- (v) $\mathcal{D}^\bullet$ is a complex of injective left $\mathcal{C}$-comodules (forgetting the right $\mathcal{E}$-comodule structure) and a complex of injective right $\mathcal{E}$-comodules (forgetting the left $\mathcal{C}$-comodule structure);
- (vi) the morphism of complexes of left $\mathcal{C}$-comodules $\mathcal{D}^\bullet \longrightarrow \mathcal{C} \otimes_A D^\bullet$ induced by the morphism of complexes of left A-modules $\mathcal{D}^\bullet \longrightarrow D^\bullet$ is a quasi-isomorphism;
- (vii) the morphism of complexes of right $\mathcal{E}$-comodules $\mathcal{D}^\bullet \longrightarrow D^\bullet \otimes_B \mathcal{E}$ induced by the morphism of complexes of right B-modules $\mathcal{D}^\bullet \longrightarrow D^\bullet$ is a quasi-isomorphism.

Lemma B.4.3. (a) *Suppose that the ring A is left Noetherian. Then for any $\mathcal{C}$-injective $\mathcal{C}$-$\mathcal{E}$-bicomodule $\mathcal{K}$ and any left $\mathcal{E}$-contramodule $\mathfrak{F}$ which is a projective object of $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$, the left $\mathcal{C}$-comodule $\mathcal{K} \odot_{\mathcal{E}} \mathfrak{F}$ is injective.*

(b) *Suppose that the ring B is right coherent. Then for any $\mathcal{E}$-injective $\mathcal{C}$-$\mathcal{E}$-bicomodule $\mathcal{K}$ and any injective left $\mathcal{C}$-comodule $\mathcal{J}$, the left $\mathcal{E}$-contramodule* $\operatorname{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{J})$ *is a projective object of $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$.*

Proof. Part (a): one can assume the left $\mathcal{E}$-contramodule $\mathfrak{F}$ to be induced from a flat (cotorsion) B-module F; then $\mathcal{K} \odot_{\mathcal{E}} \mathfrak{F} \simeq \mathcal{K} \otimes_B F$. Hence it suffices to check, e. g., that the class of injective left $\mathcal{C}$-comodules is preserved by filtered inductive limits. Let us show that $\mathcal{C}\text{–}\mathsf{comod}$ is a locally Noetherian Grothendieck abelian category.

The following is a standard argument (cf. [10]). Let $\mathcal{L}$ be a left $\mathcal{C}$-comodule. For any finitely generated A-submodule $U \subset \mathcal{L}$, the full preimage $\mathcal{L}_U$ of $\mathcal{C} \otimes_A U \subset \mathcal{C} \otimes_A \mathcal{L}$ with respect to the $\mathcal{C}$-coaction map $\mathcal{L} \longrightarrow \mathcal{C} \otimes_A \mathcal{L}$ is a $\mathcal{C}$-subcomodule in $\mathcal{L}$ contained in U. Since the left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A \mathcal{L}$ is a filtered inductive limit of its $\mathcal{C}$-subcomodules $\mathcal{C} \otimes_A U$, it follows that the $\mathcal{C}$-comodule $\mathcal{L}$ is a filtered inductive limit of its $\mathcal{C}$-subcomodules $\mathcal{L}_U$. Given that A is a left Noetherian ring, we can conclude that any left $\mathcal{C}$-comodule is the union of its A-finitely generated $\mathcal{C}$-subcomodules.

Part (b): one can assume the left $\mathcal{C}$-comodule $\mathcal{J}$ to be coinduced from a left A-module J, and the A-module J to have the form $\mathrm{Hom}_{\mathbb{Z}}(A, I)$ for a certain injective abelian group X. Then we have $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{J}) \simeq \mathrm{Hom}_A(\mathcal{K}, J) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, X)$. The $\mathcal{E}$-contramodule $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{K}, X)$ only depends on the right $\mathcal{E}$-comodule structure on $\mathcal{K}$, so one can assume $\mathcal{K}$ to be the right $\mathcal{E}$-comodule coinduced from an injective right B-module I. Now $\mathrm{Hom}_{\mathbb{Z}}(I \otimes_B \mathcal{E},\ X) \simeq \mathrm{Hom}_B(\mathcal{E}, \mathrm{Hom}_{\mathbb{Z}}(I, X))$ is the left $\mathcal{E}$-contramodule induced from the left B-module $\mathrm{Hom}_{\mathbb{Z}}(I, X)$. The latter is a flat cotorsion B-module by Lemmas 1.3.3(b) and 1.6.1(b). ☐

Lemma B.4.4. *Let $\mathcal{D}^\bullet \longrightarrow D^\bullet$ be a dualizing complex for corings $\mathcal{C}$ and $\mathcal{E}$. Then*

(a) *assuming that the ring A is left Noetherian, for any left $\mathcal{E}$-contramodule $\mathfrak{F}$ that is a projective object of $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ the adjunction morphism $\mathfrak{F} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet,\ \mathcal{D}^\bullet \odot_{\mathcal{E}} \mathfrak{F})$ is a quasi-isomorphism of finite complexes over the exact category $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$;*

(b) *assuming that the ring B is right coherent, for any injective left $\mathcal{C}$-comodule $\mathcal{J}$ the adjunction morphism $\mathcal{D}^\bullet \odot_{\mathcal{E}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$ is a quasi-isomorphism of finite complexes over the abelian category $\mathcal{C}\text{–}\mathsf{comod}$.*

Proof. Part (a): one can assume the $\mathcal{E}$-contramodule $\mathfrak{F}$ to be induced from a flat (cotorsion) left B-module F; then $\mathcal{D}^\bullet \odot_{\mathcal{E}} \mathfrak{F} \simeq \mathcal{D}^\bullet \otimes_B F$. It follows from (the proof of) Lemma B.4.3(a) that the morphism $\mathcal{D}^\bullet \otimes_B F \longrightarrow \mathcal{C} \otimes_A D^\bullet \otimes_B F$ induced by the quasi-isomorphism $\mathcal{D}^\bullet \longrightarrow \mathcal{C} \otimes_A D^\bullet$ from (vi) is a homotopy equivalence of complexes of injective left $\mathcal{C}$-comodules. Hence we have a quasi-isomorphism $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet,\ \mathcal{D}^\bullet \otimes_B F) \longrightarrow \mathrm{Hom}_A(\mathcal{D}^\bullet,\ D^\bullet \otimes_B F)$. Furthermore, by (vii) there is a quasi-isomorphism $\mathrm{Hom}_A(D^\bullet \otimes_B \mathcal{E},\ D^\bullet \otimes_B F) \longrightarrow \mathrm{Hom}_A(\mathcal{D}^\bullet,\ D^\bullet \otimes_B F)$ and a natural isomorphism $\mathrm{Hom}_A(D^\bullet \otimes_B \mathcal{E},\ D^\bullet \otimes_B F) \simeq \mathrm{Hom}_B(\mathcal{E},\ \mathrm{Hom}_A(D^\bullet,\ D^\bullet \otimes_B F))$. Finally, the natural morphism $\mathrm{Hom}_B(\mathcal{E}, F) \longrightarrow \mathrm{Hom}_B(\mathcal{E},\ \mathrm{Hom}_A(D^\bullet,\ D^\bullet \otimes_B F))$ is a quasi-isomorphism by Lemmas 1.3.3(b) and B.4.1(a).

Part (b): one can assume the $\mathcal{C}$-comodule $\mathcal{J}$ to be induced from an injective left A-module J; then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet, \mathcal{J}) \simeq \mathrm{Hom}_A(\mathcal{D}^\bullet, J)$. It follows from (the proof of) Lemma B.4.3(b) that the morphism $\mathrm{Hom}_A(D^\bullet \otimes_B \mathcal{E},\ J) \longrightarrow \mathrm{Hom}_A(\mathcal{D}^\bullet,\ J)$ induced by the quasi-isomorphism $\mathcal{D}^\bullet \longrightarrow D^\bullet \otimes_B \mathcal{E}$ from (vii) is a homotopy equivalence of complexes of projective objects in $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$. Hence we have a quasi-isomorphism $\mathcal{D}^\bullet \odot_{\mathcal{E}} \mathrm{Hom}_A(\mathcal{D}^\bullet, J) \longleftarrow \mathcal{D}^\bullet \odot_{\mathcal{E}} \mathrm{Hom}_A(D^\bullet \otimes_B \mathcal{E},\ J) \simeq \mathcal{D}^\bullet \odot_{\mathcal{E}} \mathrm{Hom}_B(\mathcal{E}, \mathrm{Hom}_A(D^\bullet, J)) \simeq$

$\mathcal{D}^\bullet \otimes_B \operatorname{Hom}_A(D^\bullet, J)$. Furthermore, by (vi) and Lemma 1.6.1(b) there is a quasi-isomorphism $\mathcal{D}^\bullet \otimes_B \operatorname{Hom}_A(D^\bullet, J) \longrightarrow \mathcal{C} \otimes_A D^\bullet \otimes_B \operatorname{Hom}_A(D^\bullet, J)$. Now it remains to apply Lemma B.4.1(b). □

The following theorem does not depend on the existence of any dualizing complexes.

Theorem B.4.5. (a) *Let $\mathcal{C}$ be a coring over an associative ring A; asssume that $\mathcal{C}$ is a flat right A-module and A is a left Noetherian ring. Then the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ of the abelian category of left $\mathcal{C}$-comodules is equivalent to the homotopy category of complexes of injective left $\mathcal{C}$-comodules.*

(b) *Let $\mathcal{E}$ be a coring over an associative ring B; assume that $\mathcal{E}$ is a flat left B-module and B is a right coherent ring. Then the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$ of the exact category of B-cotorsion left $\mathcal{E}$-contramodules is equivalent to the homotopy category of complexes of projective objects in $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$.*

Proof. Part (a) holds, since there are enough injective objects in $\mathcal{C}\text{–}\mathsf{comod}$ and the class of injectives is closed under infinite direct sums (the class of injective left A-modules being closed under infinite direct sums). Part (b) is true, because there are enough projective objects in $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ and the class of projectives is preserved by infinite products (the class of flat left B-modules being preserved by infinite products). See Proposition A.3.1(b) or [53, Sections 3.7 and 3.8] for further details. □

More generally, the assertion of part (a) holds if $\mathcal{C}$ is a flat right A-module and countable direct sums of injective left A-modules have finite injective dimensions. Similarly, the assertion of (b) is true if $\mathcal{E}$ is a flat left B-module and countable products of flat cotorsion left B-modules have finite flat dimensions (see Corollary A.6.2).

Corollary B.4.6. *Let $\mathcal{C}$ be a coring over an associative ring A and $\mathcal{E}$ be a coring over an associative ring B. Assume that $\mathcal{C}$ is a flat right A-module, $\mathcal{E}$ is a flat left B-module, the ring A is left Noetherian, and the ring B is right coherent. Then the data of a dualizing complex $\mathcal{D}^\bullet \longrightarrow D^\bullet$ for the corings $\mathcal{C}$ and $\mathcal{E}$ induces an equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$, which is provided by the derived functors $\mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet, -)$ and $\mathcal{D}^\bullet \odot_{\mathcal{E}}^{\mathbb{L}} -$.*

Proof. Follows from Lemmas B.4.3–B.4.4 and Theorem B.4.5. □

Now let us suppose that a coring $\mathcal{E}$ over an associative ring B is a projective left B-module. Then the category $\mathcal{E}\text{–}\mathsf{contra}$ is abelian with enough projectives; the latter are the direct summands of the left $\mathcal{E}$-contramodules induced from projective left B-modules.

Lemma B.4.7. *Assume that any left B-module has finite cotorsion dimension (or equivalently, any flat left B-module has finite projective dimension). Then*

(a) *any left $\mathcal{E}$-contramodule can be embedded into a B-cotorsion left $\mathcal{E}$-contramodule in such a way that the cokernel is a finitely iterated extension of $\mathcal{E}$-contramodules induced from flat left B-modules;*

(b) *any left $\mathcal{E}$-contramodule can be presented as the quotient contramodule of a finitely iterated extension of $\mathcal{E}$-contramodules induced from flat left B-modules by a B-cotorsion left $\mathcal{E}$-contramodule.*

Proof. The proof of part (a) is similar to that of [52, Lemma 3.1.3(b)] and uses Theorem 1.3.1(a). The proof of part (b) is similar to that of Lemma B.2.3 and based on part (a) and the fact that there are enough contramodules induced from flat (and even projective) B-modules in $\mathcal{E}\text{–}\mathsf{contra}$ (cf. Lemma 4.3.3). □

Let us call a left $\mathcal{E}$-contramodule $\mathfrak{F}$ *strongly contraflat* if the functor $\mathrm{Hom}^{\mathcal{E}}(\mathfrak{F},-)$ takes short exact sequences of B-cotorsion $\mathcal{E}$-contramodules to short exact sequences of abelian groups (cf. Section 4.3 and [52, Section 5.1.6 and Question 5.3]).

Assuming that any flat B-module has finite projective dimension, it follows from Lemma B.4.7(a) that the Ext groups computed in the exact category $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ and in the abelian category $\mathcal{E}\text{–}\mathsf{contra}$ agree. We denote these by $\mathrm{Ext}^{\mathcal{E},*}(-,-)$.

Corollary B.4.8. *Assume that any flat left B-module has finite projective dimension. Then*

(a) *A left $\mathcal{E}$-contramodule $\mathfrak{F}$ is strongly contraflat if and only if* $\mathrm{Ext}^{\mathcal{E},>0}(\mathfrak{F},\mathfrak{Q})=0$ *for any B-cotorsion left $\mathcal{E}$-contramodule $\mathfrak{Q}$. The class of strongly contraflat $\mathcal{E}$-contramodules is closed under extensions and the passage to kernels of surjective morphisms in $\mathcal{E}\text{–}\mathsf{contra}$.*

(b) *A left $\mathcal{E}$-contramodule is strongly contraflat if and only if it is a direct summand of a finitely iterated extension of $\mathcal{E}$-contramodules induced from flat B-modules.*

Proof. Part (a) follows from (a weak version of) Lemma B.4.7(a). Part (b) follows from Lemma B.4.7(b) and part (a). □

Theorem B.4.9. (a) *Assuming that any flat left B-module has finite projective dimension, for any symbol $\star=\mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs the triangulated functor $\mathsf{D}^\star(\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})\longrightarrow\mathsf{D}^\star(\mathcal{E}\text{–}\mathsf{contra})$ induced by the embedding of exact categories $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}\longrightarrow\mathcal{E}\text{–}\mathsf{contra}$ is an equivalence of triangulated categories.*

(b) *Assuming that countable products of projective left B-modules have finite projective dimensions (in particular, if the ring B is right coherent and flat left B-modules have finite projective dimensions), the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{E}\text{–}\mathsf{contra})$ of the abelian category of left $\mathcal{E}$-contramodules is equivalent to the homotopy category of complexes of projective left $\mathcal{E}$-contramodules.*

(c) *Assuming that countable products of flat left B-modules have finite projective dimensions, the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{E}\text{–}\mathsf{contra})$ is equivalent to the absolute derived category of the exact category of strongly contraflat left $\mathcal{E}$-contramodules.*

Proof. Part (a) follows from Lemma B.4.7(a) and the dual version of Proposition A.5.6. For part (b), see Corollary A.6.2 or [53, Section 3.8 and Remark 3.7]. Part (c) can be deduced from part (b) together with the fact that strongly contraflat left $\mathcal{E}$-contramodules have finite projective dimensions in $\mathcal{E}\text{–}\mathsf{contra}$ and Proposition A.5.6. Alternatively, notice that countable products of strongly contraflat $\mathcal{E}$-contramodules have finite projective dimensions by (the proof of) Corollary B.4.8(b), so Corollary A.6.2 applies. □

Corollary B.4.10. *In the situation of Corollary* B.4.6*, assume additionally that the coring $\mathcal{E}$ is a projective left B-module. Then the data of a dualizing complex*

$\mathcal{D}^\bullet \longrightarrow D^\bullet$ *for the corings* $\mathcal{C}$ *and* $\mathcal{E}$ *induces an equivalence of triangulated categories* $\mathsf{D^{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D^{ctr}}(\mathcal{E}\text{–}\mathsf{contra})$, *which is provided by the derived functors* $\mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{D}^\bullet, -)$ *and* $\mathcal{D}^\bullet \odot^{\mathbb{L}}_{\mathcal{E}} -$.

Proof. In addition to what has been said in Corollaries B.4.2, B.4.6 and Theorem B.4.9, we point out that in our present assumptions the assertions of Lemmas B.4.3(a) and B.4.4(a) apply to any strongly contraflat left $\mathcal{E}$-contramodule $\mathfrak{F}$. Indeed, in view of Corollary B.4.8(b) one only has to check that the functor $\mathcal{N} \odot_{\mathcal{E}} -$ takes short exact sequences of strongly contraflat left $\mathcal{E}$-contramodules to short exact sequences of abelian groups for any right $\mathcal{E}$-comodule $\mathcal{N}$. This follows from part (a) of the same Corollary, as $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{N} \odot_{\mathcal{E}} \mathfrak{F},\ \mathbb{Q}/\mathbb{Z}) \simeq \operatorname{Hom}^{\mathcal{E}}(\mathfrak{F}, \operatorname{Hom}_{\mathbb{Z}}(\mathcal{N}, \mathbb{Q}/\mathbb{Z}))$ and the left $\mathcal{E}$-contramodule $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{N}, \mathbb{Q}/\mathbb{Z})$ is B-cotorsion by Lemma 1.3.3(b). Therefore, one can construct the derived functor $\mathcal{D}^\bullet \odot^{\mathbb{L}}_{\mathcal{E}} -$ using strongly contraflat resolutions. ☐

B.5. **Base ring change.** Let $\mathcal{C}$ be a coring over an associative ring A. Given a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$, their *cotensor product* $\mathcal{N} \,\square_{\mathcal{C}}\, \mathcal{M}$ is an abelian group constructed as the kernel of the natural pair of maps $\mathcal{N} \otimes_A \mathcal{M} \rightrightarrows \mathcal{N} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$. Given a left $\mathcal{C}$-comodule $\mathcal{M}$ and a left $\mathcal{C}$-contramodule $\mathfrak{P}$, their group of *cohomomorphisms* $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M}, \mathfrak{P})$ is constructed as the cokernel of the natural pair of maps $\operatorname{Hom}_A(\mathcal{C} \otimes_A \mathcal{M},\ \mathfrak{P}) \rightrightarrows \operatorname{Hom}_A(\mathcal{M}, \mathfrak{P})$ [52, Sections 1.2.1 and 3.2.1].

Let $\mathcal{C}$ be a coring over an associative ring A and $\mathcal{E}$ be a coring over an associative ring B. A *map of corings* $\mathcal{C} \longrightarrow \mathcal{E}$ *compatible with a ring homomorphism* $A \longrightarrow B$ is an A-A-bimodule morphism such that the maps $A \longrightarrow B$, $\mathcal{C} \longrightarrow \mathcal{E}$, and the induced map $\mathcal{C} \otimes_A \mathcal{C} \longrightarrow \mathcal{E} \otimes_B \mathcal{E}$ form commutative diagrams with the comultiplication and counit maps in $\mathcal{C}$ and $\mathcal{E}$ [52, Section 7.1.1].

Let $\mathcal{C} \longrightarrow \mathcal{E}$ be a map of corings compatible with a ring map $A \longrightarrow B$. Given a left $\mathcal{C}$-comodule $\mathcal{M}$, one defines a left $\mathcal{E}$-comodule ${}_B\mathcal{M}$ by the rule ${}_B\mathcal{M} = B \otimes_A \mathcal{M}$, the coaction map being constructed as the composition $B \otimes_A \mathcal{M} \longrightarrow B \otimes_A \mathcal{C} \otimes_A \mathcal{M} \longrightarrow B \otimes_A \mathcal{E} \otimes_A \mathcal{M} \longrightarrow \mathcal{E} \otimes_A \mathcal{M} \simeq \mathcal{E} \otimes_B (B \otimes_A \mathcal{M})$. Similarly, given a right $\mathcal{C}$-comodule $\mathcal{K}$, there is a natural right $\mathcal{E}$-comodule structure on the tensor product $\mathcal{K}_B = \mathcal{K} \otimes_A B$. Given a left $\mathcal{C}$-contramodule $\mathfrak{P}$, one defines a left $\mathcal{E}$-contramodule ${}^B\mathfrak{P}$ by the rule ${}^B\mathfrak{P} = \operatorname{Hom}_A(B, \mathfrak{P})$, the contraaction map being constructed as the composition $\operatorname{Hom}_B(\mathcal{E}, \operatorname{Hom}_A(B, \mathfrak{P})) \simeq \operatorname{Hom}_A(\mathcal{E}, \mathfrak{P}) \longrightarrow \operatorname{Hom}_A(\mathcal{E} \otimes_A B,\ \mathfrak{P}) \longrightarrow \operatorname{Hom}_A(\mathcal{C} \otimes_A B,\ \mathfrak{P}) \simeq \operatorname{Hom}_A(B, \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P})) \longrightarrow \operatorname{Hom}_A(B, \mathfrak{P})$.

Assuming that $\mathcal{C}$ is a flat right A-module, the functor $\mathcal{M} \longmapsto {}_B\mathcal{M}\colon \mathcal{C}\text{–}\mathsf{comod} \longrightarrow \mathcal{E}\text{–}\mathsf{comod}$ has a right adjoint functor, which is denoted by $\mathcal{N} \longmapsto {}_{\mathcal{C}}\mathcal{N}\colon \mathcal{E}\text{–}\mathsf{comod} \longrightarrow \mathcal{C}\text{–}\mathsf{comod}$ and constructed by the rule ${}_{\mathcal{C}}\mathcal{N} = \mathcal{C}_B \square_{\mathcal{E}} \mathcal{M}$. In particular, the functor $\mathcal{N} \longmapsto {}_{\mathcal{C}}\mathcal{N}$ takes a coinduced left $\mathcal{E}$-comodule $\mathcal{E} \otimes_B U$ into the coinduced left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A U$ (where U is an arbitrary left B-module) [52, Section 7.1.2].

Given a coring $\mathcal{C}$ over an associative ring A and an associative ring homomorphism $A \longrightarrow B$, one can define a coring ${}_B\mathcal{C}_B$ over the ring B by the rule ${}_B\mathcal{C}_B = B \otimes_A \mathcal{C} \otimes_A B$. The counit in ${}_B\mathcal{C}_B$ is constructed as the composition $B \otimes_A \mathcal{C} \otimes_A B \longrightarrow B \otimes_A A \otimes_A B \longrightarrow B$, and the comultiplication is provided by the composition $B \otimes_A \mathcal{C} \otimes_A B \longrightarrow B \otimes_A \mathcal{C} \otimes_A \mathcal{C} \otimes_A B \simeq B \otimes_A \mathcal{C} \otimes_A A \otimes_A \mathcal{C} \otimes_A B \longrightarrow B \otimes_A \mathcal{C} \otimes_A B \otimes_A \mathcal{C} \otimes_A B \simeq$

$(B \otimes_A \mathcal{C} \otimes_A B) \otimes_B (B \otimes_A \mathcal{C} \otimes_A B)$. There is a natural map of corings $\mathcal{C} \longrightarrow {}_B\mathcal{C}_B$ compatible with the ring map $A \longrightarrow B$.

Assuming that $\mathcal{C}$ is a flat right A-module and B is a faithfully flat right A-module, the functors $\mathcal{M} \longrightarrow {}_B\mathcal{M}$ and $\mathcal{N} \longmapsto {}_\mathcal{C}\mathcal{N}$ are mutually inverse equivalences between the abelian categories of left $\mathcal{C}$-comodules and left ${}_B\mathcal{C}_B$-comodules. One proves this by applying the Barr–Beck monadicity theorem to the conservative exact functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow B\text{–}\mathsf{mod}$ taking a left $\mathcal{C}$-comodule $\mathcal{M}$ to the left B-module $B \otimes_A \mathcal{M}$ (see [37, Section 2.1.3] or [52, Section 7.4.1]).

Now let us assume that $\mathcal{C}$ is a flat left A-module, $\mathcal{E}$ is a flat left B-module, and B is a flat left A-module. Then the functor $\mathfrak{P} \longmapsto {}^B\mathfrak{P}$ takes A-cotorsion left $\mathcal{C}$-contramodules to B-cotorsion left $\mathcal{E}$-contramodules (by Lemma 1.3.3(a)) and induces an exact functor between the exact categories $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}} \longrightarrow \mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$. The left adjoint functor $\mathfrak{Q} \longmapsto {}^\mathcal{C}\mathfrak{Q}$ to this exact functor may not be everywhere defined, but one can easily see that it is defined on the full subcategory of left $\mathcal{E}$-contramodules induced from cotorsion left B-modules in the exact category $\mathcal{E}\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$, taking an induced contramodule $\mathfrak{Q} = \mathrm{Hom}_B(\mathcal{E}, V)$ over $\mathcal{E}$ to the induced contramodule ${}^\mathcal{C}\mathfrak{Q} = \mathrm{Hom}_A(\mathcal{C}, V)$ over $\mathcal{C}$ (where V is an arbitrary cotorsion left B-module; see Lemma 1.3.4(a)).

Finally, we assume that $\mathcal{C}$ is flat left A-module and B is a faithfully flat left A-module. It is clear from Example 3.2.1 that the functor $\mathfrak{P} \longmapsto {}^B\mathfrak{P}\colon \mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}} \longrightarrow {}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ is *not* an equivalence of exact categories in general (and not even in the case when $\mathcal{C} = A$). The aim of this section is to prove the following slightly weaker result (cf. Corollaries 4.6.3–4.6.5 and 5.3.3).

Theorem B.5.1. *Let $\mathcal{C}$ be a coassociative coring over an associative ring A and $A \longrightarrow B$ be an associative ring homomorphism. Assume that $\mathcal{C}$ is a flat left A-module and B is a faithfully flat left A-module. Then the triangulated functor $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$ induced by the exact functor $\mathfrak{P} \longmapsto {}^B\mathfrak{P}$ is an equivalence of triangulated categories.*

Corollary B.5.2. *In the assumptions of the previous Theorem, the exact functor $\mathfrak{P} \longmapsto {}^B\mathfrak{P}\colon \mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}} \longrightarrow {}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ is fully faithful and induces isomorphisms between the Ext groups computed in the exact categories $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ and ${}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$. Any short sequence (or, more generally, bounded above complex) in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ which this functor transforms into an exact sequence in ${}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ is exact in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$.*

Proof. See [52, Section 4.1]. □

The proof of Theorem B.5.1 is based on the following technical result, which provides adjusted resolutions for the construction of the left derived functor of the partial left adjoint functor $\mathfrak{Q} \longmapsto {}^\mathcal{C}\mathfrak{Q}$ to the exact functor $\mathfrak{P} \longmapsto {}^B\mathfrak{P}$. For completeness, we formulate three versions of the (co)induced resolution theorem; it is the third one that will be used in the argument below.

Theorem B.5.3. (a) *Let $\mathcal{C}$ be a coring over an associative ring A; assume that $\mathcal{C}$ is a flat right A-module. Then the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ of the abelian* *category of left $\mathcal{C}$-comodules is equivalent to the quotient category of the homotopy category of complexes of coinduced $\mathcal{C}$-comodules by its minimal triangulated subcategory containing the total complexes of the short exact sequences of complexes of $\mathcal{C}$-comodules termwise coinduced from short exact sequences of A-modules and closed under infinite direct sums.*

(b) *Let $\mathcal{C}$ be a coring over an associative ring A; assume that $\mathcal{C}$ is a projective left A-module. Then the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra})$ of the abelian category of left $\mathcal{C}$-contramodules is equivalent to the quotient category of the homotopy category of complexes of induced $\mathcal{C}$-contramodules by its minimal triangulated subcategory containing the total complexes of the short exact sequences of complexes of $\mathcal{C}$-contramodules termwise induced from short exact sequences of A-modules and closed under infinite products.*

(c) *Let $\mathcal{C}$ be a coring over an associative ring A; assume that $\mathcal{C}$ is a flat left A-module. Then the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ of the exact category of A-cotorsion left $\mathcal{C}$-contramodules is equivalent to the quotient category of the homotopy category of complexes of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules by its minimal triangulated subcategory containing the total complexes of the short exact sequences of complexes of $\mathcal{C}$-contramodules termwise induced from short exact sequences of cotorsion A-modules and closed under infinite products.*

Proof. This is yet another version of the results of [15, Proposition 1.5 and Remark 1.5], [55, Theorem 4.2.1], the above Propositions A.2.1 and A.3.1(b), etc. The difference is that the categories of coinduced comodules or induced contramodules are not even exact. This does not change much, however. Let us sketch a proof of part (c); the proofs of parts (a-b) are similar (but simpler).

The contramodule relative bar-resolution, totalized by taking infinite products along the diagonals, provides a closed morphism with a contraacyclic cone onto any complex of A-cotorsion left $\mathcal{C}$-contramodules from a complex of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules. This proves that the natural functor from the homotopy category of complexes of left $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules to the contraderived category of A-cotorsion left $\mathcal{C}$-contramodules is a Verdier localization functor [53, Lemma 1.6].

In order to prove that the natural functor from the quotient category of the homotopy category of termwise induced complexes in the formulation of the theorem to the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ is fully faithful, one shows that any morphism in $\mathsf{Hot}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ from a complex of contramodules termwise induced from cotorsion A-modules to a contraacyclic complex over $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ factorizes through an object of the minimal triangulated subcategory containing the total complexes of the short exact sequences of complexes of $\mathcal{C}$-contramodules termwise induced from short exact sequences of cotorsion A-modules and closed under infinite products.

Let us only explain the key step, as the argument is mostly similar to the ones spelled out in the references above. Let $\mathfrak{M}^\bullet$ be the total complex of a short exact sequence $\mathfrak{U}^\bullet \longrightarrow \mathfrak{V}^\bullet \longrightarrow \mathfrak{W}^\bullet$ of complexes of A-cotorsion $\mathcal{C}$-contramodules, and let $\mathfrak{E}^\bullet$ be a complex of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules.

The construction dual to the one explained in [52, Theorem 5.5] provides a closed morphism onto the complex $\mathfrak{E}^\bullet$ from a complex of $\mathcal{C}$-contramodules $\mathfrak{F}^\bullet$ termwise induced from flat cotorsion A-modules such that this morphism is induced from an admissible epimorphism of cotorsion A-modules at every term of the complexes (cf. the proof of Theorem B.3.1(b)).

Let $\mathfrak{K}^\bullet$ denote the kernel of the morphism of complexes $\mathfrak{F}^\bullet \longrightarrow \mathfrak{E}^\bullet$. Then the cone $\mathfrak{P}^\bullet$ of the admissible monomorphism $\mathfrak{K}^\bullet \longrightarrow \mathfrak{F}^\bullet$ maps naturally onto $\mathfrak{E}^\bullet$ with the cone isomorphic to the total complex of a short exact sequence of complexes of $\mathcal{C}$-contramodules termwise induced from a short exact sequence of cotorsion A-modules. As a morphism of graded $\mathcal{C}$-contramodules, the composition $\mathfrak{P}^\bullet \longrightarrow \mathfrak{E}^\bullet$ factorizes through the graded $\mathcal{C}$-contramodule $\mathfrak{F}^\bullet$. The latter being a projective graded object of $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$, it follows that the composition $\mathfrak{P}^\bullet \longrightarrow \mathfrak{E}^\bullet \longrightarrow \mathfrak{M}^\bullet$ is homotopic to zero by Lemma A.2.2. □

Proof of Theorem B.5.1. First of all, we notice that the functor $\mathfrak{Q} \longmapsto {}^{\mathcal{C}}\mathfrak{Q}$ acting between the full subcategories of left ${}_B\mathcal{C}_B$-contramodules induced from cotorsion B-modules and left $\mathcal{C}$-contramodules induced from cotorsion A-modules in ${}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$ and $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ is fully faithful. Indeed, one has

$$\operatorname{Hom}^{{}_B\mathcal{C}_B}(\operatorname{Hom}_B({}_B\mathcal{C}_B, U),\ \operatorname{Hom}_B({}_B\mathcal{C}_B, V)) \simeq \operatorname{Hom}_B(U, \operatorname{Hom}_B({}_B\mathcal{C}_B, V))$$
$$\simeq \operatorname{Hom}_B({}_B\mathcal{C}_B \otimes_B U,\ V) \simeq \operatorname{Hom}_A(\mathcal{C} \otimes_A U,\ V) \simeq \operatorname{Hom}^{\mathcal{C}}(\operatorname{Hom}_A(\mathcal{C}, U), \operatorname{Hom}_A(\mathcal{C}, V)).$$

Furthermore, the composition of functors $\mathfrak{Q} \longmapsto {}^{\mathcal{C}}\mathfrak{Q} \longmapsto {}^B({}^{\mathcal{C}}\mathfrak{Q})$ is naturally isomorphic to the identity endofunctor on the full subcategory of ${}_B\mathcal{C}_B$-contramodules induced from cotorsion B-modules in ${}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}$, as $\operatorname{Hom}_A(B, \operatorname{Hom}_A(\mathcal{C}, V)) \simeq \operatorname{Hom}_A(\mathcal{C} \otimes_A B,\ V) \simeq \operatorname{Hom}_B(B \otimes_A \mathcal{C} \otimes_A B,\ V)$.

Clearly, the functor $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}\mathfrak{Q}^\bullet$, acting between the homotopy categories of complexes of ${}_B\mathcal{C}_B$-contramodules termwise induced from complexes of cotorsion B-modules and complexes of $\mathcal{C}$-contramodules termwise induced from cotorsion A-modules, is "partially left adjoint" to the functor $\mathfrak{P}^\bullet \longmapsto {}^B\mathfrak{P}^\bullet\colon \mathsf{Hot}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}) \longrightarrow \mathsf{Hot}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$. By Theorem B.5.3(c), the former functor can be used to construct a left derived functor $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet\colon \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$. It is easy to see that the functor $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet$ is left adjoint to the functor $\mathfrak{P}^\bullet \longmapsto {}^B\mathfrak{P}^\bullet\colon \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$.

According to the above, the composition of the adjoint triangulated functors $\mathfrak{Q}^\bullet \longmapsto {}^B({}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet)\colon \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$ is isomorphic to the identity functor. Hence the functor $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet\colon \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ is fully faithful, while the functor $\mathfrak{P}^\bullet \longmapsto {}^B\mathfrak{P}^\bullet\colon \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}({}_B\mathcal{C}_B\text{–}\mathsf{contra}^{B\text{–}\mathsf{cot}})$ is a Verdier localization functor.

It remains to show that the triangulated functor $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet$ is essentially surjective. It is straightforward from the constructions that both functors $\mathfrak{P}^\bullet \longmapsto {}^B\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet \longmapsto {}^{\mathcal{C}}_{\mathbb{L}}\mathfrak{Q}^\bullet$ preserve infinite products. So it suffices to prove that the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}})$ coincides with its minimal triangulated subcategory containing all complexes of $\mathcal{C}$-contramodules termwise induced from left A-modules obtained by restriction of scalars from cotorsion left B-modules and closed under infinite products. In view of the argument of Lemmas A.3.3–A.3.4, we only need to check that there is an admissible epimorphism onto any object of $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ from an induced left $\mathcal{C}$-contramodule of the desired type.

The contraaction morphism $\mathrm{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ being an admissible epimorphism in $\mathcal{C}\text{–}\mathsf{contra}^{A\text{–}\mathsf{cot}}$ for any A-cotorsion left $\mathcal{C}$-contramodule $\mathfrak{P}$, the problem reduces to constructing an admissible epimorphism in $A\text{–}\mathsf{mod}^{\mathsf{cot}}$ onto any cotorsion left A-module from an A-module obtained by restriction of scalars from a cotorsion left B-module. Here we are finally using the assumption that B is a *faithfully* flat left A-module. According to [52, Section 7.4.1], the latter is equivalent to the ring homomorphism $A \longrightarrow B$ being injective and its cokernel being a flat left A-module. Assuming that, the morphism $\mathrm{Hom}_A(B,P) \longrightarrow \mathrm{Hom}_A(A,P) = P$ induced by the map $A \longrightarrow B$ is an admissible epimorphism in $A\text{–}\mathsf{mod}^{\mathsf{cot}}$ for any cotorsion left A-module P by Lemma 1.3.3(a). It remains to say that the left B-module $\mathrm{Hom}_A(B,P)$ is cotorsion by Lemma 1.3.5(a). ☐

Appendix C. Affine Noetherian Formal Schemes

This appendix is a continuation of [55, Appendix B]. Its goal is to construct the derived co-contra correspondence between quasi-coherent torsion sheaves and contraherent cosheaves of contramodules on an affine Noetherian formal scheme with a dualizing complex. The affineness condition allows to speak of (co)sheaves in terms of (contra)modules over a ring, simplifying the exposition.

C.1. **Torsion modules and contramodules.** Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Given an R-module E, we denote by ${}_{(n)}E \subset E$ its R-submodule consisting of all the elements annihilated by I^n. An R-module $\mathcal{M}$ is said to be *I-torsion* if $\mathcal{M} = \bigcup_{n\geqslant 1} {}_{(n)}\mathcal{M}$, i. e., for any element $m \in \mathcal{M}$ there exists an integer $n \geqslant 1$ such that $I^n m = 0$ in $\mathcal{M}$. An R-module $\mathfrak{P}$ is called an *(R,I)-contramodule* if $\mathrm{Ext}^*_R(R[s^{-1}],\mathfrak{P}) = 0$ for all $s \in I$, or equivalently, the system of equations $q_n = p_n + sq_{n+1}$, $n \geqslant 0$, is uniquely solvable in $q_n \in \mathfrak{P}$ for any fixed sequence $p_n \in \mathfrak{P}$ (cf. the beginning of Section 1.1).

Just as the category of I-torsion R-modules, the category of (R,I)-contramodules is abelian and depends only on the I-adic completion of the ring R [55, Theorem B.1.1] (cf. the ending part of Section 1.3 above). We denote the category of I-torsion R-modules by $(R,I)\text{–}\mathsf{tors}$ and the category of (R,I)-contramodules by $(R,I)\text{–}\mathsf{contra}$. Both categories are full subcategories in the abelian category of R-modules $R\text{–}\mathsf{mod}$, closed under the kernels and cokernels; the former subcategory is also closed under infinite direct sums, while the latter one is closed under infinite products.

Given a sequence of elements $r_n \in R$ converging to zero in the I-adic topology of R and a sequence of elements $p_n \in \mathfrak{P}$ of an (R,I)-contramodule $\mathfrak{P}$, the result of the "infinite summation operation" $\sum_n r_n p_n$ is well-defined as an element of $\mathfrak{P}$ [55, Section 1.2]. One can define the *contratensor product* $\mathfrak{P} \odot_{(R,I)} \mathcal{M}$ of an (R,I)-contramodule $\mathfrak{P}$ and an I-torsion R-module $\mathcal{M}$ as the quotient R-module of the tensor product $\mathfrak{P}\otimes_{\mathbb{Z}}\mathcal{M}$ by the relations $(\sum_n r_n p_n)\otimes m = \sum_n p_n\otimes r_n m$ for any sequence r_n converging to zero in R, any sequence of elements $p_n\in\mathfrak{P}$, and any element $m\in\mathcal{M}$. Here all but a finite number of summands in the right-hand side vanish due to the conditions imposed on the sequence r_n and the module $\mathcal{M}$.

In fact, however, for any (R,I)-contramodule $\mathfrak{P}$ and I-torsion R-module $\mathcal{M}$ there is a natural isomorphism $\mathfrak{P}\odot_{(R,I)}\mathcal{M}\simeq\mathfrak{P}\otimes_R\mathcal{M}$. This can be deduced from the fact that the functor $(R,I)\text{–}\mathsf{contra}\longrightarrow R\text{–}\mathsf{mod}$ is fully faithful, or alternatively, follows directly from the observation that any sequence of elements of $I^n\subset R$ converging to zero in the I-adic topology of R is a linear combination of a finite number of sequences converging to zero in R with the coefficients belonging to I^n (cf. [55, proof of Proposition B.9.1]).

Clearly, the tensor product $E\otimes_R\mathcal{M}$ of any R-module E and any I-torsion R-module $\mathcal{M}$ is an I-torsion R-module. Similarly, the R-module $\mathrm{Hom}_R(\mathcal{M},E)$ of R-linear maps from an I-torsion R-module $\mathcal{M}$ into any R-module E is an (R,I)-contramodule. Finally, the R-module $\mathrm{Hom}_R(E,\mathfrak{P})$ of homomorphisms from any R-module E into an (R,I)-contramodule $\mathfrak{P}$ is an (R,I)-contramodule [55, Sections 1.5 and B.2].

An (R,I)-contramodule $\mathfrak{F}$ is said to be (R,I)-*contraflat* if the functor $\mathcal{M}\longmapsto\mathfrak{F}\odot_{(R,I)}\mathcal{M}$ is exact on the abelian category of I-torsion R-modules. According to the above, this is equivalent to the functor $\mathcal{M}\longmapsto\mathfrak{F}\otimes_R\mathcal{M}$ being exact on the full abelian subcategory $(R,I)\text{–}\mathsf{tors}\subset R\text{–}\mathsf{mod}$.

By the Artin–Rees lemma, an object $\mathcal{K}\in(R,I)\text{–}\mathsf{tors}$ is injective if and only if it is injective in $R\text{–}\mathsf{mod}$, and if and only if its submodules ${}_{(n)}\mathcal{K}$ of elements annihilated by I^n are injective R/I^n-modules for all $n\geqslant 1$. By [55, Lemma B.9.2], an (R,I)-contramodule $\mathfrak{F}$ is (R,I)-contraflat if and only if it is a flat R-module, and if and only if its reductions $\mathfrak{F}/I^n\mathfrak{F}$ are flat R/I^n-modules for all $n\geqslant 1$. Furthermore, an object $\mathfrak{F}\in(R,I)\text{–}\mathsf{contra}$ is projective if and only if it is (R,I)-contraflat and its reduction $\mathfrak{F}/I\mathfrak{F}$ is a projective R/I-module, and if and only if all the reductions $\mathfrak{F}/I^n\mathfrak{F}$ are projective R/I^n-modules [55, Corollary B.8.2].

Theorem C.1.1. (a) *The coderived category* $\mathsf{D}^{\mathsf{co}}((R,I)\text{–}\mathsf{tors})$ *of the abelian category of I-torsion R-modules is equivalent to the homotopy category of complexes of injective I-torsion R-modules.*

(b) *The contraderived category* $\mathsf{D}^{\mathsf{ctr}}((R,I)\text{–}\mathsf{contra})$ *of the abelian category of (R,I)-contramodules is equivalent to the contraderived category of the exact category of R-flat (i. e., (R,I)-contraflat) (R,I)-contramodules.*

(c) *Assume that the Noetherian ring R/I has finite Krull dimension. Then the contraderived category* $\mathsf{D}^{\mathsf{ctr}}((R,I)\text{–}\mathsf{contra})$ *is equivalent to the absolute derived category of the exact category of R-flat (R,I)-contramodules.*

(d) *Assume that the Noetherian ring R/I has finite Krull dimension. Then the contraderived category* $\mathsf{D}^{\mathsf{ctr}}((R,I)\text{–}\mathsf{contra})$ *is equivalent to the homotopy category of complexes of projective (R,I)-contramodules.*

Proof. Part (a) holds, since there are enough injectives in the abelian category $(R,I)\text{–}\mathsf{tors}$ and the class of injective objects is closed under infinite direct sums. Part (b) is true, because there are enough (R,I)-contraflat (and even projective) objects in (R,I)–contra and the class of (R,I)-contraflat (R,I)-contramodules (or even flat R-modules) is closed under infinite products (see Proposition A.3.1(b)).

Parts (c-d) hold, since in their assumptions any (R,I)-contraflat (R,I)-contramodule $\mathfrak{F}$ has finite projective dimension in (R,I)–contra. Indeed, consider a projective resolution $\mathfrak{P}_\bullet$ of $\mathfrak{F}$ in (R,I)–contra and apply the functor $R/I\otimes_R -$ to it. Being a bounded above exact complex of flat R-modules, the complex $\mathfrak{P}_\bullet \longrightarrow \mathfrak{F}$ will remain exact after taking the tensor product. By Theorem 1.5.6, the R/I-modules of cycles in the complex $R/I\otimes_R\mathfrak{P}_\bullet$ are eventually projective, and it follows that the (R,I)-contramodules of cycles in $\mathfrak{P}_\bullet$ are eventually projective, too. Now it remains to apply Corollary A.6.2, Proposition A.5.6, and/or [52, Remark 2.1]. ☐

For noncommutative generalizations of the next two lemmas (as well as the theorem following them), see Sections C.5 and D.2.

Lemma C.1.2. (a) *For any R-module M, injective R-module J, and $n\geqslant 1$, there is a natural isomorphism of R/I^n-modules* $\operatorname{Hom}_R(M,J)/I^n\operatorname{Hom}_R(M,J)\simeq \operatorname{Hom}_{R/I^n}({}_{(n)}M,\,{}_{(n)}J)$.

(b) *For any R-module M, flat R-module F, and $n\geqslant 1$, there is a natural isomorphism of R/I^n-modules ${}_{(n)}(M\otimes_R F)\simeq {}_{(n)}M\otimes_{R/I^n}F/I^nF$.*

Proof. For any finitely generated R-module E there are natural isomorphisms $\operatorname{Hom}_R(\operatorname{Hom}_R(E,M),J)\simeq E\otimes_R\operatorname{Hom}_R(M,J)$ and $\operatorname{Hom}_R(E,M\otimes_R F)\simeq \operatorname{Hom}_R(E,M)\otimes_R F$. It remains to take $E=R/I^n$. ☐

A finite complex $\mathcal{D}^\bullet$ of injective I-torsion R-modules is said to be a *dualizing complex* for the pair (R,I) if for every integer $n\geqslant 1$ the complex of R/I^n-modules ${}_{(n)}\mathcal{D}^\bullet$ is a dualizing complex for the commutative Noetherian ring R/I^n. If a finite complex of injective R-modules $D^\bullet$ is a dualizing complex for the ring R, then its subcomplex $\mathcal{D}^\bullet=\bigcup_{n\geqslant 0}{}_{(n)}D^\bullet$ consisting of all the elements annihilated by some powers of $I\subset R$ is a dualizing complex for (R,I).

Lemma C.1.3. *Let $\mathcal{D}^\bullet$ be a finite complex of injective I-torsion R-modules such that the complex of R/I-modules ${}_{(1)}\mathcal{D}^\bullet$ is a dualizing complex for the ring R/I. Then $\mathcal{D}^\bullet$ is a dualizing complex for the pair (R,I).*

Proof. Clearly, ${}_{(n)}\mathcal{D}^\bullet$ is a finite complex of injective R/I^n-modules. Since any R/I^n-module K with finitely generated R/I-module ${}_{(1)}K$ is finitely generated itself, and the functors $\operatorname{Ext}^i_{R/I^n}(R/I,-)$ take finitely generated R/I^n-modules to finitely generated R/I-modules, one can check by induction that the R/I^n-modules of cohomology of the complex ${}_{(n)}\mathcal{D}^\bullet$ are finitely generated.

It remains to show that the natural map $R/I^n\longrightarrow\operatorname{Hom}_{R/I^n}({}_{(n)}\mathcal{D}^\bullet,\,{}_{(n)}\mathcal{D}^\bullet)$ is a quasi-isomorphism. Both the left-hand and the right-hand sides are finite complexes of flat (or, equivalently, projective) R/I^n-modules. By Lemma C.1.2(a), the functor $P\longmapsto P/IP$ transforms this morphism into the morphism $R/I\longrightarrow \operatorname{Hom}_{R/I}({}_{(1)}\mathcal{D}^\bullet,\,{}_{(1)}\mathcal{D}^\bullet)$, which is a quasi-isomorphism by assumption. Hence the desired assertion follows by Nakayama's lemma. ☐

Theorem C.1.4. *The data of a dualizing complex $\mathcal{D}^\bullet$ for a pair (R, I) with a Noetherian commutative ring R and an ideal $I \subset R$ induces an equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}((R, I)\text{–}\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{ctr}}((R, I)\text{–}\mathsf{contra})$, which is provided by the derived functors $\mathbb{R}\operatorname{Hom}_R(\mathcal{D}^\bullet, -)$ and $\mathcal{D}^\bullet \otimes_R^{\mathbb{L}} -$.*

Proof. The constructions of the derived functors are based on Theorem C.1.1(a,c). Applying the functor $\operatorname{Hom}_R(\mathcal{D}^\bullet, -)$ to a complex of injective I-torsion R-modules produces a complex of R-flat (R, I)-contramodules. Applying the functor $\mathcal{D}^\bullet \otimes_R -$ to a complex of R-flat (R, I)-contramodules produces a complex of injective I-torsion R-modules, and for any (absolutely) acyclic complex of flat R-modules $F^\bullet$, the complex of injective (I-torsion) R-modules $\mathcal{D}^\bullet \otimes_R F^\bullet$ is contractible.

It remains to show that the morphism of finite complexes $\mathcal{D}^\bullet \otimes_R \operatorname{Hom}_R(\mathcal{D}^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$ is a quasi-isomorphism for any I-torsion R-module $\mathcal{J}$, and the morphism of finite complexes $\mathfrak{F} \longrightarrow \operatorname{Hom}_R(\mathcal{D}^\bullet,\ \mathcal{D}^\bullet \otimes_R \mathfrak{F})$ is a quasi-isomorphism for any (R, I)-contramodule $\mathfrak{F}$. Both assertions follow from Lemma C.1.2, which implies that the morphisms become quasi-isomorphisms after applying the functors $\mathcal{K} \longmapsto {}_{(n)}\mathcal{K}$ and $\mathfrak{P} \longmapsto \mathfrak{P}/I^n\mathfrak{P}$. In the latter situation, one also has to use the assertion that the natural morphism $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/I^n\mathfrak{P}$ is an isomorphism for any R-flat (R, I)-contramodule $\mathfrak{P}$ (see [55, proof of Lemma B.9.2]). ☐

C.2. **Contraadjusted and cotorsion contramodules.** Let R be a Noetherian commutative ring and $s \in R$ be an element. In the spirit of the definitions from Sections 1.1 and C.1, let us say that an R-module P is *s-contraadjusted* if $\operatorname{Ext}^1_R(R[s^{-1}], P) = 0$, and that P is an *s-contramodule* if $\operatorname{Ext}^*_R(R[s^{-1}], P) = 0$. The property of an R-module P to be s-contraadjusted or an s-contramodule only depends on the abelian group P with the operator $s\colon P \longrightarrow P$.

An R-module P is an s-contramodule if and only if it is an $(R, (s))$-contramodule, where $(s) \subset R$ denotes the principal ideal generated by $s \in R$. More generally, given an ideal $I \subset R$, an R-module $\mathfrak{P}$ is an (R, I)-contramodule if and only if it is an s-contramodule for every $s \in I$, and it suffices to check this condition for any given set of generators of the ideal I [55, proof of Theorem B.1.1].

Recall that any quotient R-module of an s-contraadjusted R-module is s-contraadjusted. Given an R-module L, let us denote by $L(s) \subset L$ the image of the morphism $\operatorname{Hom}_R(R[s^{-1}], L) \longrightarrow L$ induced by the localization map $R \longrightarrow R[s^{-1}]$. Equivalently, the R-submodule $L(s) \subset L$ can be defined as the maximal s-divisible R-submodule in R, i. e., the sum of all R-submodules (or even all s-invariant abelian subgroups) in L in which the element s acts surjectively.

Therefore, if $P = L/L(s)$ denotes the corresponding quotient R-module, then one has $P(s) = 0$. Notice also that one has $\operatorname{Hom}_R(R[s^{-1}], P) = 0$ for any R-module P for which $P(s) = 0$. It follows that the R-quotient module $L/L(s)$ is an s-contramodule whenever an R-module L is s-contraadjusted.

Now let $s, t \in R$ be two elements; suppose that an R-module L is a t-contramodule. Then the R-module $\operatorname{Hom}_R(R[s^{-1}], L)$ is also a t-contramodule [55, Section B.2], as is the image $L(s)$ of the morphism of t-contramodules $\operatorname{Hom}_R(R[s^{-1}], L) \longrightarrow L$. Hence the quotient R-module $L/L(s)$ is also a t-contramodule. Assuming additionally that the R-module L was s-contraadjusted, the quotient module $L/L(s)$ is both a t- and an s-contramodule (i. e., it is an $(R,(t,s))$-contramodule).

Recall that an R-module L is said to be *contraadjusted* if it is s-contraadjusted for every element $s \in R$. Given a contraadjusted R-module L and an ideal $I \subset R$, one can apply the above construction of a quotient R-module successively for all the generators of the ideal I. Proceeding in this way, one in a finite number of steps (equal to the number of generators of I) obtains the unique maximal quotient R-module $\mathfrak{P}$ of the R-module L such that $\mathfrak{P}$ is an s-contramodule for every $s \in I$, i. e., $\mathfrak{P}$ is an (R,I)-contramodule. We denote this quotient module by $\mathfrak{P} = L/L(I)$.

More generally, all one needs in order to apply the above construction of the maximal quotient (R,I)-contramodule $\mathfrak{P} = L/L(I)$ to a given R-module L is that the ideal $I \subset R$ should have a set of generators $s_1, \dots, s_k$ such that L is an s_j-contramodule for every $j = 1, \dots, k$.

Lemma C.2.1. *Let R be a Noetherian commutative ring, $s \in R$ be an element, F be a flat R-module, and M be a finitely generated R-module. Then the natural R-module homomorphism* $\mathrm{Hom}_R(R[s^{-1}],\ M \otimes_R F) \longrightarrow M \otimes_R F$ *is injective. In other words, one has* $\mathrm{Hom}_R(R[s^{-1}]/R,\ M \otimes_R F) = 0$.

Proof. Consider the sequence of annihilator submodules of the elements $s^m \in R$, $m \geqslant 1$, in the R-module M. This is an increasing sequence of R-submodules in M. Let N be the maximal submodule in this sequence and n be a positive integer such that N is the annihilator of s^n in M. Then any element of the R-module $M \otimes_R F$ annihilated by s^{n+1} is also annihilated by s^n.

Indeed, the element s acts by an injective endomorphism of an R-module M/N; hence so does the element s^{n+1}. Since F is flat, it follows that s^{n+1} must act injectively in the tensor product $(M/N) \otimes_R F \simeq (M \otimes_R F)/(N \otimes_R F)$. Hence any element of $M \otimes_R F$ annihilated by s^{n+1} belongs to $N \otimes_R F$ and is therefore annihilated by s^n. ☐

Lemma C.2.2. *Let R be a Noetherian commutative ring, $s \in R$ be an element, and $I \subset R$ be an ideal. Then*

(a) *whenever an R-module L is s-contraadjusted, the R-modules $L(s)$ and $L/L(s)$ are also s-contraadjusted (and $L/L(s)$ is even an s-contramodule);*

(b) *whenever an R-module L is contraadjusted, the R-modules $L(s)$ and $L/L(s)$ are also contraadjusted;*

(c) *whenever an R-module L is contraadjusted, the R-modules $L(I)$ and $L/L(I)$ are also contraadjusted (and $L/L(I)$ is in addition an (R,I)-contramodule);*

(d) *whenever an R-module L is cotorsion and* $\mathrm{Hom}_R(R[s^{-1}]/R,\ L) = 0$*, the R-modules $L(s)$ and $L/L(s)$ are also cotorsion.*

Proof. The parenthesized assertions in (a) and (c) have been explained above. Since the class of cotorsion R-modules is closed under cokernes of injective morphisms, while the classes of contraadjusted and s-contraadjusted R-modules are even closed under quotients, it suffices to check the assertions related to the R-modules $L(s)$

and $L(I)$. Furthermore, there is a natural surjective morphism of R-modules $\mathrm{Hom}_R(R[s^{-1}], L) \longrightarrow L(s)$, which in the assumptions of (d) is an isomorphism.

Now part (d) is a particular case of Lemma 1.3.2(a), part (b) is provided by Lemma 1.2.1(b), and part (a) follows from the similar claim that the R-module $\mathrm{Hom}_R(R[s^{-1}], P)$ is s-contraadjusted for any s-contraadjusted R-module P. Finally, (b) implies (c) in view of the above recursive construction of the R-module $L(I)$ and the fact that the class of constraadjusted R-modules is closed under extensions. □

Corollary C.2.3. *Let R be a Noetherian commutative ring, $s \in R$ be an element, and $I \subset R$ be an ideal. In this setting*

(a) *if F is an s-contraadjusted flat R-module, then the R-modules $F(s)$ and $F/F(s)$ are also flat and s-contraadjusted (and $F/F(s)$ is even an s-contramodule);*

(b) *if F is a contraadjusted flat R-module, then $F(I)$ is a contraadjusted flat R-module and $F/F(I)$ is an R-flat R-contraadjusted (R, I)-contramodule;*

(c) *if F is a flat cotorsion R-module, then $F(I)$ is a flat cotorsion R-module and $F/F(I)$ is an R-flat R-cotorsion (R, I)-contramodule.*

Proof. Part (a): in order to prove that the R-module $F/F(s)$ is flat, let us check that the map $M \otimes_R F(s) \longrightarrow M \otimes_R F$ induced by the natural embedding $F(s) \longrightarrow F$ is injective for any finitely generated R-module M. By Lemma C.2.1, one has $F(s) \simeq \mathrm{Hom}_R(R[s^{-1}], F) \subset F$ and $(M \otimes_R F)(s) \simeq \mathrm{Hom}_R(R[s^{-1}],\ M \otimes_R F) \subset M \otimes_R F$. It remains to point out that the natural morphism $M \otimes_R \mathrm{Hom}_R(R[s^{-1}], F) \longrightarrow \mathrm{Hom}_R(R[s^{-1}],\ M \otimes_R F)$ is an isomorphism by Lemma 1.6.2.

Part (b): the above recursive construction of the (R, I)-contramodule $F/F(I)$ together with part (a) imply the assertion that $F/F(I)$ is a flat R-module. Now it remains to use Lemma C.2.2(c).

Part (c) can be proven in the way similar to part (b), using the fact that the class of cotorsion R-modules is closed under extensions and Lemma C.2.2(d). Alternatively, (c) can be deduced from Theorem 1.3.8. Indeed, if $F \simeq \prod_{\mathfrak{p}} \mathfrak{F}_{\mathfrak{p}}$, where $\mathfrak{p} \subset R$ are prime ideals and $\mathfrak{F}_{\mathfrak{p}}$ are $(R, \mathfrak{p})$-contramodules, then one easily checks that $F/F(I)$ is the product of $\mathfrak{F}_{\mathfrak{p}}$ over the prime ideals $\mathfrak{p}$ containing I, while $F(I)$ is the product of $\mathfrak{F}_{\mathfrak{q}}$ over all the other prime ideals $\mathfrak{q}$. □

Lemma C.2.4. (a) *Let $0 \longrightarrow P \longrightarrow K \longrightarrow F \longrightarrow 0$ be a short exact sequence of R-modules, where P is an s-contramodule, K is a contraadjusted R-module, and F is a flat R-module. Then $0 \longrightarrow P \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow 0$ is a short exact sequence of R-modules in which $K/K(s)$ is a contraadjusted R-module, $F/F(s)$ is a flat R-module, and all the three modules are s-contramodules.*

(b) *Let $0 \longrightarrow P \longrightarrow K \longrightarrow F \longrightarrow 0$ be a short exact sequence of R-modules, where P is an s-contramodule, K is a cotorsion R-module, and F is a flat R-module. Then $0 \longrightarrow P \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow 0$ is a short exact sequence of R-modules in which $K/K(s)$ is a cotorsion R-module, $F/F(s)$ is a flat R-module, and all the three modules are s-contramodules.*

Proof. Part (a): first of all let us show that the morphism of R-modules $K \longrightarrow F$ restricts to an isomorphism of their submodules $K(s) \longrightarrow F(s)$. Indeed, we have $\mathrm{Hom}_R(R[s^{-1}]/R,\ F) = 0$ by Lemma C.2.1 and $\mathrm{Hom}_R(R[s^{-1}]/R,\ P) \subset \mathrm{Hom}_R(R[s^{-1}], P) = 0$ by the definition of an s-contramodule, hence $\mathrm{Hom}_R(R[s^{-1}]/R,\ K) = 0$. Therefore, there are isomorphisms $K(s) \simeq \mathrm{Hom}_R(R[s^{-1}], K)$ and $F(s) \simeq \mathrm{Hom}_R(R[s^{-1}], F)$. Using the condition that P is an s-contramodule, that is $\mathrm{Ext}^*(R[s^{-1}], P) = 0$ again, we conclude that $K(s) \simeq F(s)$.

It follows that the sequence $0 \longrightarrow P \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow 0$ is exact. The R-module K being contraadjusted by assumption, its quotient R-module F is contraadjusted, too. Now the quotient module $K/K(s)$ is contraadjusted by Lemma C.2.2(b), the quotient module $F/F(s)$ is flat by Corollary C.2.3(a), and both are s-contramodules by Lemma C.2.2(a).

In part (b), it only remains to prove that $K/K(s)$ is a cotorsion R-module. This follows from the above argument and Lemma C.2.2(d). □

Corollary C.2.5. (a) *Let $0 \longrightarrow P \longrightarrow K \longrightarrow F \longrightarrow 0$ be a short exact sequence of R-modules, where P is an (R, I)-contramodule, K is a contraadjusted R-module, and F is a flat R-module. Then $0 \longrightarrow P \longrightarrow K/K(I) \longrightarrow F/F(I) \longrightarrow 0$ is a short exact sequence of (R, I)-contramodules in which $K/K(I)$ is a contraadjusted R-module and $F/F(I)$ is a flat R-module.*

(b) *Let $0 \longrightarrow P \longrightarrow K \longrightarrow F \longrightarrow 0$ be a short exact sequence of R-modules, where P is an (R, I)-contramodule, K is a cotorsion R-module, and F is a flat R-module. Then $0 \longrightarrow P \longrightarrow K/K(I) \longrightarrow F/F(I) \longrightarrow 0$ is a short exact sequence of (R, I)-contramodules in which $K/K(I)$ is a cotorsion R-module and $F/F(I)$ is a flat R-module.*

Proof. Follows by recursion from Lemma C.2.4. □

Lemma C.2.6. (a) *Let $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ be a short exact sequence of R-modules, where K is a contraadjusted R-module, F is a flat R-module, and P is an s-contramodule. Then $0 \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow P \longrightarrow 0$ is a short exact sequence of R-modules in which $K/K(s)$ is a contraadjusted R-module, $F/F(s)$ is a flat R-module, and all the three modules are s-contramodules.*

(b) *Let $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ be a short exact sequence of R-modules, where K is a cotorsion R-module, F is a flat R-module, and P is an s-contramodule. Then $0 \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow P \longrightarrow 0$ is a short exact sequence of R-modules in which $K/K(s)$ is a cotorsion R-module, $F/F(s)$ is a flat R-module, and all the three modules are s-contramodules.*

Proof. Part (a): first we prove that the morphism of R-modules $K \longrightarrow F$ restricts to an isomorphism of their submodules $K(s) \longrightarrow F(s)$. Indeed, $\mathrm{Hom}_R(R[s^{-1}]/R,\ K) \subset \mathrm{Hom}_R(R[s^{-1}]/R,\ F) = 0$ by Lemma C.2.1, hence $K(s) \simeq \mathrm{Hom}_R(R[s^{-1}], K)$ and $F(s) \simeq \mathrm{Hom}_R(R[s^{-1}], F)$. Since $\mathrm{Hom}_R(R[s^{-1}], P) = 0$ by assumption, we conclude that $K(s) \simeq F(s)$, and it follows that the sequence $0 \longrightarrow K/K(s) \longrightarrow F/F(s) \longrightarrow P \longrightarrow 0$ is exact.

The R-module F, being an extension of s-contraadjusted R-modules K and P, is s-contraadjusted, too. The rest of the argument coincides with the respective part of the proof of Lemma C.2.4, and so does the proof of part (b). □

Corollary C.2.7. (a) *Let* $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ *be a short exact sequence of* R*-modules, where* K *is a contraadjusted* R*-module,* F *is a flat* R*-module, and* P *is an* (R,I)*-contramodule. Then* $0 \longrightarrow K/K(I) \longrightarrow F/F(I) \longrightarrow P \longrightarrow 0$ *is a short exact sequence of* (R,I)*-contramodules in which* $K/K(I)$ *is a contraadjusted* R*-module and* $F/F(I)$ *is a flat* R*-module.*

(b) *Let* $0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$ *be a short exact sequence of* R*-modules, where* K *is a cotorsion* R*-module,* F *is a flat* R*-module, and* P *is an* (R,I)*-contramodule. Then* $0 \longrightarrow K/K(I) \longrightarrow F/F(I) \longrightarrow P \longrightarrow 0$ *is a short exact sequence of* (R,I)*-contramodules in which* $K/K(I)$ *is a cotorsion* R*-module and* $F/F(I)$ *is a flat* R*-module.*

Proof. Follows by recursion from Lemma C.2.6. □

Recall that, according to [55, Theorem B.8.1], the Ext groups/modules computed in the abelian categories R–mod and (R,I)–contra agree.

Corollary C.2.8. *Let* R *be a Noetherian commutative ring and* $I \subset R$ *be an ideal. Then*

(a) *any* (R,I)*-contramodule can be embedded into an* R*-cotorsion* (R,I)*-contramodule in such a way that the quotient* (R,I)*-contramodule is* R*-flat;*

(b) *any* (R,I)*-contramodule admits a surjective morphism onto it from an* R*-flat* (R,I)*-contramodule such that the kernel is an* R*-cotorsion* (R,I)*-contramodule;*

(c) *an* (R,I)*-contramodule* $\mathfrak{Q}$ *is* R*-cotorsion if and only if one has* $\mathrm{Ext}^1_R(\mathfrak{F},\mathfrak{Q}) = 0$ *for any* R*-flat* (R,I)*-contramodule* $\mathfrak{F}$*;*

(d) *an* (R,I)*-contramodule* $\mathfrak{F}$ *is* R*-flat if and only if one has* $\mathrm{Ext}^1_R(\mathfrak{F},\mathfrak{Q}) = 0$ *for any* R*-cotorsion* (R,I)*-contramodule* $\mathfrak{Q}$*.*

Proof. Parts (a-b) follow from Theorem 1.3.1 together with Corollaries C.2.5(b) and C.2.7(b). Part (c) is deduced from (a) and part (d) deduced from (b) easily. □

Let us denote by (R,I)–$\mathsf{contra}^{\mathsf{cta}}$ the full exact subcategory of R-contraadjusted (R,I)-contramodules and by (R,I)–$\mathsf{contra}^{\mathsf{cot}}$ the full exact subcategory of R-cotorsion (R,I)-contramodules in the abelian category (R,I)–contra.

Notice that the full subcategory (R,I)–$\mathsf{contra}^{\mathsf{cot}} \subset (R,I)$–$\mathsf{contra}$ depends only on the I-adic completion of the ring R, as one can see from Corollary C.2.8(c). The similar assertion for the full subcategory (R,I)–$\mathsf{contra}^{\mathsf{cot}} \subset (R,I)$–$\mathsf{contra}$ will follow from the results of Section C.3 below.

Theorem C.2.9. (a) *Let* R *be a Noetherian commutative ring and* $I \subset R$ *be an ideal. Then for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star((R,I)\text{–}\mathsf{contra}^{\mathsf{cta}}) \longrightarrow \mathsf{D}^\star((R,I)\text{–}\mathsf{contra})$ *induced by the embedding of exact categories* (R,I)–$\mathsf{contra}^{\mathsf{cta}} \longrightarrow (R,I)$–$\mathsf{contra}$ *is an equivalence of triangulated categories.*

(b) *Let* R *be a Noetherian commutative ring and* $I \subset R$ *be an ideal such that the quotient ring* R/I *has finite Krull dimension. Then for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star((R,I)\text{–}\mathsf{contra}^{\mathsf{cot}}) \longrightarrow$

$\mathsf{D}^\star((R,I)\text{–}\mathsf{contra})$ *induced by the embedding of exact categories* $(R,I)\text{–}\mathsf{contra}^{\mathsf{cot}} \longrightarrow (R,I)\text{–}\mathsf{contra}$ *is an equivalence of triangulated categories.*

Proof. Part (a) follows from Corollary C.2.5(a) or C.2.8(a) together with the opposite version of Proposition A.5.6. To prove part (b) in the similar way, one needs to know that in its assumptions any (R,I)-contramodule has finite right homological dimension with respect to the exact subcategory $(R,I)\text{–}\mathsf{contra}^{\mathsf{cot}} \subset (R,I)\text{–}\mathsf{contra}$. In view of Corollary C.2.8(c), this follows from the fact that any R-flat (R,I)-contramodule has finite projective dimension in $(R,I)\text{–}\mathsf{contra}$, which was established in the proof of Theorem C.1.1 (using [55, Corollary B.8.2]). □

Remark C.2.10. Let $X = \operatorname{Spec} R$ be the Noetherian affine scheme corresponding to the ring R and $U = X \setminus Z$ be the open complement to the closed subscheme $Z = \operatorname{Spec} R/I \subset X$. Denote by $j\colon U \longrightarrow X$ the natural open embedding morphism. Then the exact category $(R,I)\text{–}\mathsf{contra}^{\mathsf{cta}}$ is equivalent to the full exact subcategory in the exact category $X\text{–}\mathsf{ctrh}$ of contraherent cosheaves on X consisting of all the contraherent cosheaves $\mathfrak{P} \in X\text{–}\mathsf{ctrh}$ with vanishing restrictions $j^!\mathfrak{P} = \mathfrak{P}|_U$ to the open subscheme U. Similarly, the exact category $(R,I)\text{–}\mathsf{contra}^{\mathsf{cot}}$ is equivalent to the full exact subcategory in the exact category $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ consisting of all the (locally) cotorsion contraherent cosheaves $\mathfrak{P}$ on X for which $j^!\mathfrak{P} = 0$. Indeed, a contraadjusted R-module P is an s-contramodule if and only if the corresponding contraherent cosheaf $\mathfrak{P} = \check{P}$ on X vanishes in the restriction to $\operatorname{Spec} R[s^{-1}] \subset \operatorname{Spec} R$ (see Sections 2.2–2.4 for the definitions and notation). In particular, when the ring R/I is Artinian, the exact category $\ker(j^!\colon X\text{–}\mathsf{ctrh} \to U\text{–}\mathsf{ctrh}) = \ker(j^!\colon X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \to U\text{–}\mathsf{ctrh}^{\mathsf{lct}})$ is abelian and equivalent to the abelian category $(R,I)\text{–}\mathsf{contra} = (R,I)\text{–}\mathsf{contra}^{\mathsf{cta}} = (R,I)\text{–}\mathsf{contra}^{\mathsf{cot}}$.

C.3. **Very flat contramodules.** Unlike the flatness, cotorsion, and contraadjustness properties of (R,I)-contramodules, their very flatness property is *not* defined in terms of the similar property of R-modules. Instead, it is described in terms of the reductions modulo I^n and the very flatness properties of R/I^n-modules.

For a (straightforward) generalization of the next lemma, see Lemma D.1.9.

Lemma C.3.1. (a) *Let F be a flat R-module and $\mathfrak{Q}$ be an (R,I)-contramodule such that the R/I-module F/IF is very flat, the natural (R,I)-contramodule morphism $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/I^n\mathfrak{Q}$ is an isomorphism, and the R/I-module $\mathfrak{Q}/I\mathfrak{Q}$ is contraadjusted. Then one has $\operatorname{Ext}_R^{>0}(F,\mathfrak{Q}) = 0$.*

(b) *Let F be a flat R-module and $\mathfrak{Q}$ be an R-flat (R,I)-contramodule such that the R/I-module $\mathfrak{Q}/I\mathfrak{Q}$ is cotorsion. Then one has $\operatorname{Ext}_R^{>0}(F,\mathfrak{Q}) = 0$.*

Proof. Part (a): by (the proof of) Lemma 1.6.8(a), the R/I^n-modules $\mathfrak{Q}/I^n\mathfrak{Q}$ and the R/I-modules $I^{n-1}\mathfrak{Q}/I^n\mathfrak{Q}$ are contraadjusted, while by part (b) of the same lemma the R/I^n-modules F/I^nF are very flat. Now we follow the argument in the proof of [55, Proposition B.10.1]. The R-module F being flat by assumption, one has $\operatorname{Ext}_R^i(F,\ \mathfrak{Q}/I^n\mathfrak{Q}) \simeq \operatorname{Ext}_{R/I^n}^i(F/I^nF,\ \mathfrak{Q}/I^n\mathfrak{Q}) = 0$ for all $i > 0$ and any $n \geqslant 1$. The natural map $\operatorname{Hom}_R(F,\, \mathfrak{Q}/I^n\mathfrak{Q}) \longrightarrow \operatorname{Hom}_R(F,\, \mathfrak{Q}/I^{n-1}\mathfrak{Q})$ is surjective, since $\operatorname{Ext}^1_{R/I^n}(F/I^nF,\, I^{n-1}\mathfrak{Q}/I^n\mathfrak{Q}) = 0$. By [55, Lemma B.10.3], we have $\operatorname{Ext}^i_R(F,\, \varprojlim_n \mathfrak{Q}/I^n\mathfrak{Q}) \simeq \varprojlim^i_n \operatorname{Hom}_R(F,\, \mathfrak{Q}/I^n\mathfrak{Q}) = 0$ for all $i > 0$.

Part (b): by (the proof of) Lemma 1.6.8(c), the R/I^n-modules $\mathfrak{Q}/I^n\mathfrak{Q}$ and the R/I-modules $I^{n-1}\mathfrak{Q}/I^n\mathfrak{Q}$ are cotorsion, while by [55, proof of Lemma B.9.2] the natural map $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/I^n\mathfrak{Q}$ is an isomorphism. The argument continues exactly the same as in part (a). □

For generalizations of respective parts of the following corollary, see Lemma D.4.3 and Corollary D.5.5.

Corollary C.3.2. (a) *For any contraadjusted R-module Q, the R/I-module Q/IQ is contraadjusted. If the natural map $Q \longrightarrow \varprojlim_n Q/I^nQ$ is an isomorphism for an R-module Q and the R/I-module Q/IQ is contraadjusted, then the R-module Q is contraadjusted.*

(b) *A flat (R, I)-contramodule $\mathfrak{Q}$ is R-cotorsion if and only if the R/I-module $\mathfrak{Q}/I\mathfrak{Q}$ is cotorsion.*

Proof. Part (a): the first assertion is Lemma 1.6.6(b). Since the R/I-module F/IF is very flat for any very flat R-module F (see Lemma 1.2.2(b)), the second assertion follows from Lemma C.3.1(a). Part (b) is similarly a consequence of Lemma 1.6.7(a) and Lemma C.3.1(b). □

For a generalization of the next corollary, see Corollary D.4.8.

Corollary C.3.3. *Let F be a flat R-module for which the R/I-module F/IF is very flat. Then $\operatorname{Ext}^{>0}_R(F, \mathfrak{P}) = 0$ for any R-contraadjusted (R, I)-contramodule $\mathfrak{P}$.*

Proof. By Corollary C.2.7(a) or C.2.8(b), there exists a short exact sequence of (R, I)-contramodules $0 \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{P} \longrightarrow 0$, where the R-module $\mathfrak{Q}$ is contraadjusted and the R-module $\mathfrak{G}$ is flat. Then the R-module $\mathfrak{G}$ is also contraadjusted. By Corollary C.3.2(a), it follows that so are the R/I-modules $\mathfrak{Q}/I\mathfrak{Q}$ and $\mathfrak{G}/I\mathfrak{G}$. Furthermore, according to the proof of [55, Lemma B.9.2], one has $\mathfrak{G} \simeq \varprojlim_n \mathfrak{G}/I^n\mathfrak{G}$. The (R, I)-contramodule $\mathfrak{Q}$ being a subcontramodule of $\mathfrak{G}$, it follows that $\bigcap_n I^n\mathfrak{Q} = 0$, hence also $\mathfrak{Q} \simeq \varprojlim_n \mathfrak{Q}/I^n\mathfrak{Q}$. Now it remains to apply Lemma C.3.1(a) to (the R-module F and) the (R, I)-contramodules $\mathfrak{Q}$ and $\mathfrak{G}$. □

Lemma C.3.4. (a) *Any (R, I)-contramodule $\mathfrak{M}$ can be included in a short exact sequence of (R, I)-contramodules $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$, where the R/I^n-modules $\mathfrak{F}/I^n\mathfrak{F}$ are very flat, while the R-module $\mathfrak{P}$ is contraadjusted.*

(b) *Any (R, I)-contramodule $\mathfrak{M}$ can be included in a short exact sequence of (R, I)-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{M} \longrightarrow 0$, where the R/I^n-modules $\mathfrak{F}/I^n\mathfrak{F}$ are very flat, while the R-module $\mathfrak{P}$ is contraadjusted.*

Proof. First of all, let us show that any R-flat (R, I)-contramodule $\mathfrak{G}$ can be included in a short exact sequence of (R, I)-contramodules $0 \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow 0$, where the R/I^n-modules $\mathfrak{F}/I^n\mathfrak{F}$ are very flat, while the R-module $\mathfrak{E}$ is (flat and) contraadjusted. By Theorem 1.1.1(b), there exists a short exact sequence of R-modules

$0 \longrightarrow E \longrightarrow F \longrightarrow \mathfrak{G} \longrightarrow 0$ such that the R-module F is very flat, while the R-module E is contraadjusted. Then the R-module E is also flat.

The short sequence of R/I^n-modules $0 \longrightarrow E/I^nE \longrightarrow F/I^nF \longrightarrow \mathfrak{G}/I^n\mathfrak{G} \longrightarrow 0$ is exact for every $n \geqslant 1$. Set $\mathfrak{E} = \varprojlim_n E/I^nE$ and $\mathfrak{F} = \varprojlim_n F/I^nF$; recall that the natural map $\mathfrak{G} \longrightarrow \varprojlim_n \mathfrak{G}/I^n\mathfrak{G}$ is an isomorphism according to the proof of [55, Lemma B.9.2]. Passing to the projective limit, we obtain a short exact sequence of (R,I)-contramodules $0 \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow 0$. The R/I^n-modules $\mathfrak{F}/I^n\mathfrak{F} \simeq F/I^nF$ are very flat by Lemma 1.2.2(b), and the R-module $\mathfrak{E}$ is contraadjusted by Corollary C.3.2(a). Now it is easy to obtain part (a) from Corollary C.2.8(a) and part (b) from Corollary C.2.8(b).

Alternatively, in order to obtain the assertions of Lemma one can simply apply the constructions of Corollaries C.2.5(a) and C.2.7(a), taking F to be a very flat R-module and using Lemma 1.2.2(b) together with the observation that one has $(L/L(I))/I^n(L/L(I)) \simeq L/I^nL$ for every R-module L for which the ideal $I \subset R$ admits a set of generators s_j such that L is an s_j-contramodule for every j. □

Let us call an (R,I)-contramodule $\mathfrak{F}$ *very flat* if the functor $\mathrm{Hom}_R(\mathfrak{F},-)$ takes short exact sequences of R-contraadjusted (R,I)-contramodules to short exact sequences of abelian groups. The next corollary says that this definition is equivalent to the more familiar formulation is terms of the Ext^1 vanishing (cf. the definitions of very flat modules and contraadjusted sheaves in Sections 1.1 and 2.5).

Corollary C.3.5. *Let R be a Noetherian commutative ring and $I \subset R$ be an ideal. Then*

(a) *any (R,I)-contramodule can be embedded into an R-contraadjusted (R,I)-contramodule in such a way that the quotient (R,I)-contramodule is very flat;*

(b) *any (R,I)-contramodule admits a surjective morphism onto it from an very flat (R,I)-contramodule such that the kernel is an R-contraadjusted (R,I)-contramodule;*

(c) *an (R,I)-contramodule $\mathfrak{Q}$ is R-contraadjusted if and only if $\mathrm{Ext}^1_R(\mathfrak{F},\mathfrak{Q}) = 0$ for any very flat (R,I)-contramodule $\mathfrak{F}$;*

(d) *an (R,I)-contramodule $\mathfrak{F}$ is very flat if and only if $\mathrm{Ext}^1_R(\mathfrak{F},\mathfrak{Q}) = 0$ for any R-contraadjusted (R,I)-contramodule $\mathfrak{Q}$;*

(e) *an (R,I)-contramodule $\mathfrak{F}$ is very flat if and only if the R/I^n-module $\mathfrak{F}/I^n\mathfrak{F}$ is very flat for every $n \geqslant 1$.*

Proof. The "if" assertion in part (e) follows Corollary C.3.3 and [55, Lemma B.9.2]. Parts (a-b) are provided by the respective parts of Lemma C.3.4 together with the "if" assertion in (e). The "if" assertions in parts (c) and (d) are deduced from parts (a) and (b), respectively; and to prove the "only if", one only needs to know that any (R,I)-contramodule can be embedded into an R-contraadjusted one. Finally, the "only if" assertion in part (e) follows from the construction in Lemma C.3.4 on which the proof of part (b) is based. □

C.4. **Affine geometry of (R,I)-contramodules.** All rings in this section are presumed to be commutative and Noetherian. Let $f\colon R \longrightarrow S$ be a ring homomorphism,

I be an ideal in R, and J be an ideal in S. We denote by $Sf(I)$ the extension of the ideal $I \subset R$ in the ring S.

Lemma C.4.1. (a) *If $f(I) \subset J$, then any (S,J)-contramodule is also an (R,I)-contramodule in the R-module structure obtained by restriction of scalars via f.*

(b) *If $J \subset Sf(I)$, then an S-module is an (S,J)-contramodule whenever it is an (R,I)-contramodule in the R-module structure obtained by restriction of scalars via f.*

Proof. Part (a) holds, since an S-module Q is an (S,J)-contramodule if and only if the system of equations $q_n = p_n + tq_{n+1}$, $n \geqslant 0$, is uniquely solvable in $q_n \in Q$ for any fixed sequence $p_n \in Q$ and any $t \in J$. Part (b) is true, because it suffices to check the previous condition for the elements t belonging to any given set of generators of the ideal $J \subset S$ only [55, Sections B.1 and B.7]. □

Lemma C.4.2. *Assume that $J \subset Sf(I)$, and let $\mathfrak{P}$ be an (R,I)-contramodule. Then*

(a) *the S-module* $\operatorname{Hom}_R(S,\mathfrak{P})$ *is an (S,J)-contramodule;*

(b) *the S-module $S \otimes_R \mathfrak{P}$ is an (S,J)-contramodule whenever f is a finite morphism.*

Proof. Both assertions follow from Lemma C.4.1(b). Indeed, the R-module $\operatorname{Hom}_R(M,\mathfrak{P})$ is an (R,I)-contramodule for any R-module M, because the contramodule infinite summation operations can be defined on it (see the beginning of Section C.1), while the R-module $M \otimes_R \mathfrak{P}$ is an (R,I)-contramodule for any finitely generated R-module M, being the cokernel of a morphism between two finite direct sums of copies of the (R,I)-contramodule $\mathfrak{P}$. □

Denote by $\mathfrak{R}$ the I-adic completion of the ring R and by $\mathfrak{S}$ the J-adic completion of the ring S, both viewed as topological rings. By [55, Theorem B.1.1], the full subcategories $(R,I)\text{–}\mathsf{contra} \subset R\text{–}\mathsf{mod}$ and $(S,J)\text{–}\mathsf{contra} \subset S\text{–}\mathsf{mod}$ in the categories of R- and S-modules are equivalent to the categories $\mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{S}\text{–}\mathsf{contra}$ of contramodules over the topological rings $\mathfrak{R}$ and $\mathfrak{S}$.

Assume that $f(I) \subset J$; then the ring homomorphism $f\colon R \longrightarrow S$ induces a continuous homomorphism of topological rings $\phi\colon \mathfrak{R} \longrightarrow \mathfrak{S}$. According to [55, Section 1.8], there is a pair of adjoint functors of "contrarestriction" and "contraextension" of scalars $R^\phi\colon \mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ and $E^\phi\colon R\text{–}\mathsf{contra} \longrightarrow \mathfrak{S}\text{–}\mathsf{contra}$. While the functor R^ϕ is easily identified with the functor of restriction of scalars from Lemma C.4.1, the functor E^ϕ, which is left adjoint to R^ϕ, is defined by the rules that E^ϕ is right exact and takes the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]] = \varprojlim_n R/I^n[X]$ to the free $\mathfrak{S}$-contramodule $\mathfrak{S}[[X]] = \varprojlim_n S/J^n[X]$ for any set X.

When f is a finite morphism and $J = Sf(I)$, the functor E^ϕ is simply the functor of extension of scalars from Lemma C.4.2(b).

Lemma C.4.3. *For any R-flat (R,I)-contramodule $\mathfrak{F}$ there is a natural isomorphism of (S,J)-contramodules $E^\phi(\mathfrak{F}) \simeq \varprojlim_n (S \otimes_R \mathfrak{F})/J^n(S \otimes_R \mathfrak{F})$. In particular, the functor E^ϕ takes R-flat (R,I)-contramodules to S-flat (S,J)-contramodules and very flat (R,I)-contramodules to very flat (S,J)-contramodules.*

Proof. Since $F \longmapsto (S \otimes_R F)/J^n(S \otimes_R F) = (S \otimes_R F)/(J^n \otimes_R F) = S/J^n \otimes_R F$ is an exact functor on the category of flat R-modules F, so is the functor $F \longmapsto \varprojlim_n (S \otimes_R F)/J^n(S \otimes_R F)$. Hence it suffices to compute this functor for free $\mathfrak{R}$-contramodules, and indeed we have $S/J^n \otimes_R \mathfrak{R}[[X]] = S/J^n \otimes_{R/I^n} (R/I^n \otimes_R \mathfrak{R}[[X]]) = S/J^n \otimes_{R/I^n} R/I^n[X] = S/J^n[X]$ and $\varprojlim_n S/J^n \otimes_R \mathfrak{R}[[X]] = \varprojlim_n S/J^n[X] = \mathfrak{S}[[X]]$, as desired. The remaining assertions follow by the way of [55, Lemma B.9.2] and the above Corollary C.3.5(e) (see also Lemma D.1.3 below). □

Lemma C.4.4. (a) *If the map $\bar f\colon R/I \longrightarrow S/J$ is surjective, then the functor E^ϕ takes R-contraadjusted (R,I)-contramodules to S-contraadjusted (S,J)-contramodules.*

(b) *If the morphism $\bar f\colon R/I \longrightarrow S/J$ is finite, then the functor E^ϕ takes R-flat R-cotorsion (R,I)-contramodules to S-flat S-cotorsion (S,J)-contramodules.*

Proof. Part (a): by Corollary C.2.8(b) or Corollary C.3.5(b), any R-contraadjusted (R,I)-contramodule is a quotient (R,I)-contramodule of an R-flat R-contraadjusted (R,I)-contramodule. The functor E^ϕ being right exact and the class of contraadjusted S-modules being closed under quotents, it suffices to show that the S-module $E^\phi(\mathfrak{P})$ is contraadjusted for any R-flat R-contraadjusted (R,I)-contramodule $\mathfrak{P}$. Then, by Lemma C.4.3, one has $E^\phi(\mathfrak{P}) = \varprojlim_n E^\phi(\mathfrak{P})/J^nE^\phi(\mathfrak{P})$ and $E^\phi(\mathfrak{P})/JE^\phi(\mathfrak{P}) = S/J \otimes_R \mathfrak{P}$, so it remains to apply Corollary C.3.2(a) together with Lemma 1.6.6(b). Part (b) follows directly from Lemma C.4.3, Corollary C.3.2(b), and Lemma 1.6.7(a) in the similar way. □

We keep assuming that $f(I) \subset J$. Let $g\colon R \longrightarrow T$ be another ring homomorphism and $K = Tg(I) \subset T$ be the extension of the ideal $I \subset R$ in the ring T. Suppose that the commutative ring $H = S \otimes_R T$ is Noetherian, denote by $f'\colon T \longrightarrow H$ and $g'\colon S \longrightarrow H$ the related ring homomophisms, and set $L = Hg'(J) \subset H$.

Let $\mathfrak{T}$ and $\mathfrak{H}$ denote the adic completions of the rings T and H with respect to the ideals K and L, and let $\psi\colon \mathfrak{R} \longrightarrow \mathfrak{T}$, $\phi'\colon \mathfrak{T} \longrightarrow \mathfrak{H}$, and $\psi'\colon \mathfrak{S} \longrightarrow \mathfrak{H}$ be the induced homomorphisms of topological rings.

Lemma C.4.5. (a) *For any (T,K)-contramodule $\mathfrak{N}$ there is a natural isomorphism of (S,J)-contramodules $E^\phi R^\psi(\mathfrak{N}) \simeq R^{\psi'} E^{\phi'}(\mathfrak{N})$.*

(b) *Assume that the ring homomorphism $g\colon R \longrightarrow T$ induces an open embedding of affine schemes* $\operatorname{Spec} T \longrightarrow \operatorname{Spec} R$, *while the morphism $\bar f\colon R/I \longrightarrow S/J$ is finite. Then for any R-flat R-contraadjusted (R,I)-contramodule $\mathfrak{P}$ there is a natural isomorphism of (H,L)-contramodules $E^{\phi'}(\operatorname{Hom}_R(T,\mathfrak{P})) \simeq \operatorname{Hom}_S(H, E^\phi(\mathfrak{P}))$.*

Proof. Part (a): the functor of contraextension of scalars $E^\phi\colon (R,I)\text{–}\mathsf{contra} \longrightarrow (S,J)\text{–}\mathsf{contra}$ is left adjoint to the functor of (contra)restriction of scalars $R^\phi\colon (S,J)\text{–}\mathsf{contra} \longrightarrow (R,I)\text{–}\mathsf{contra}$, while the functor $R^\psi\colon (T,K)\text{–}\mathsf{contra} \longrightarrow (R,I)\text{–}\mathsf{contra}$ is left adjoint to the functor of coextension of scalars taking an (R,I)-contramodule $\mathfrak{P}$ to the (T,K)-contramodule $\operatorname{Hom}_R(T,\mathfrak{P})$. To obtain the desired isomorphism of functors, one can start with the obvious functorial isomorphism of (T,K)-contramodules

$\mathrm{Hom}_R(T, R^{\phi}(\mathfrak{Q})) \simeq R^{\phi'} \mathrm{Hom}_S(H, \mathfrak{Q})$ for any (S, J)-contramodule $\mathfrak{Q}$, and then pass to the left adjoint functors.

Part (b): the T-module $\mathrm{Hom}_R(T, \mathfrak{P})$ being flat by Corollary 1.6.5(a), one has $E^{\phi'}(\mathrm{Hom}_R(T, \mathfrak{P})) \simeq \varprojlim_n H/L^n \otimes_T \mathrm{Hom}_R(T, \mathfrak{P}) \simeq \varprojlim_n S/J^n \otimes_R \mathrm{Hom}_R(T, \mathfrak{P})$ and $\mathrm{Hom}_S(H, E^{\phi}(\mathfrak{P})) \simeq \mathrm{Hom}_R(T, E^{\phi}(\mathfrak{P})) \simeq \mathrm{Hom}_R(T, \varprojlim_n S/J^n \otimes_R \mathfrak{P}) \simeq \varprojlim_n \mathrm{Hom}_R(T, S/J^n \otimes_R \mathfrak{P})$. Finally, one has $S/J^n \otimes_R \mathrm{Hom}_R(T, \mathfrak{P}) \simeq \mathrm{Hom}_R(T, S/J^n \otimes_R \mathfrak{P})$ by Corollary 1.6.3(c), since the R-modules S/J^n are finitely generated. ☐

Lemma C.4.6. *Let $R \longrightarrow S_\alpha$ be a collection of morphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ is a finite open covering. Let I be an ideal in R and J_α be its extensions in S_α. Then a contraadjusted R-module P is an (R, I)-contramodule if and only if the S_α-modules $\mathrm{Hom}_R(S_\alpha, P)$ are (S_α, J_α)-contramodules for all α.*

Proof. The "only if" is a particular case of Lemma C.4.2(a), and to prove the "if" one can use the Čech sequence (2) together with Lemma C.4.1(a) and the facts that the class of (R, I)-contramodules is preserved by the functors Hom_R from any R-module as well as the passages to the cokernels of R-module morphisms (and, actually, kernels and extensions, too). ☐

C.5. **Noncommutative Noetherian rings.** The aim of this section is to generalize the main results of [55, Appendix B] and the above Section C.1 to noncommutative rings. Our exposition is somewhat sketchy with details of the arguments omitted when they are essentially the same as in the commutative case. For a further generalization, see Sections D.1–D.2 below.

Let R be a right Noetherian associative ring, and let $m \subset R$ be an ideal generated by central elements in R. Denote by $\mathfrak{R} = \varprojlim_n R/m^n$ the m-adic completion of the ring R, viewed as a topological ring in the projective limit ($= m$-adic) topology. We refer to [52, Remark A.3] and [55, Section 1.2] for the definitions of the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ of *left $\mathfrak{R}$-contramodules* and the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ (see also the beginning of Section D.1).

Theorem C.5.1. *The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful. If $s_1, \dots, s_k$ are some set of central generators of the ideal $m \subset R$, then the image of the forgetful functor consists precisely of those left R-modules P for which one has $\mathrm{Ext}^*_R(R[s_j^{-1}], P) = 0$ for all $1 \leqslant j \leqslant k$.*

In other words, extending the terminology of Section C.2 to the noncommutative situation, one can say that a left R-module is an $\mathfrak{R}$-contramodule if and only if it is an s_j-contramodule for every j.

Proof. The argument follows the proof of [55, Theorem B.1.1]. To show that $\mathrm{Ext}^*_R(R[s^{-1}], \mathfrak{P}) = 0$ for any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, one can simply (contra)restrict the scalars to the subring $K \subset R$ generated by s over $\mathbb{Z}$ in R, completed adically at the ideal $m \cap K$ or $(s) \subset K$, reducing to the commutative case, where the quoted theorem from [55, Appendix B] is directly applicable (cf. [55, Section B.2]).

To prove the fully faithfullness and the sufficiency of condition on an R-module P, consider the ring of formal power series $\mathfrak{T} = R[[t_1, \dots, t_k]]$ with central variables t_j and endow it with the adic topology of the ideal generated by $t_1 \dots, t_k$. There is a natural surjective open homomorphism of topological rings $\mathfrak{T} \longrightarrow \mathfrak{R}$ forming a commutative diagram with the ring homomorphisms $R \longrightarrow \mathfrak{T}$ and $R \longrightarrow \mathfrak{R}$ and taking t_j to s_j. Consider also the polynomial ring $T = R[t_1, \dots, t_k]$; there are natural ring homomorphisms $R \longrightarrow T \longrightarrow \mathfrak{T}$. The argument is based on two lemmas.

Lemma C.5.2. *The kernel $\mathfrak{J}$ of the ring homomorphism $\mathfrak{T} \longrightarrow \mathfrak{R}$ is generated by the central elements $t_j - s_j$ as an ideal in an abstract, nontopological ring $\mathfrak{T}$. Moreover, any family of elements in $\mathfrak{J}$ converging to zero in the topology of $\mathfrak{T}$ can be presented as a linear combination of k families of elements in $\mathfrak{T}$, converging to zero in the topology of $\mathfrak{T}$, with the coefficients $t_j - s_j$.*

Proof. The proof is similar to that in [55, Sections B.3–B.4]; the only difference is that one has to use the noncommutative versions of Hilbert basis theorem [24, Theorem 1.9 and Exercise 1ZA(c)] and Artin–Rees lemma [24, Theorem 13.3]. ☐

Lemma C.5.3. *The forgetful functor $\mathfrak{T}$–$\mathsf{contra} \longrightarrow T$–$\mathsf{mod}$ identifies the category of left contramodules over the topological ring $\mathfrak{T}$ with the full subcategory in the category of left T-modules consisting of all those modules which are t_j-contramodules for all j.*

Proof. This assertion is true for any associative ring R; the argument is the same as in [55, Sections B.5–B.6]. ☐

The proof of the theorem finishes similarly to [55, Section B.7]. The category $\mathfrak{R}$–contra is identified with the full subcategory in $\mathfrak{T}$–contra consisting of those left $\mathfrak{T}$-contramodules $\mathfrak{P}$ in which the elements $t_j - s_j$ act by zero. The latter category coincides with the category of T-modules in which the variables t_j act the same as the elements s_j and which are also t_j-contramodules for all j. ☐

Proposition C.5.4. *A left $\mathfrak{R}$-contramodule $\mathfrak{P}$ is a flat R-module if and only if the R/m^n-module $\mathfrak{P}/m^n\mathfrak{P}$ is flat for every $n \geqslant 1$. The natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/m^n\mathfrak{P}$ is an isomorphism in this case.*

Proof. The same as in [55, Lemma B.9.2]. One computes the functor $M \longmapsto M \otimes_R \mathfrak{P}$ on the category of finitely generated right R-modules M and uses (the noncommutative version of) the Artin–Rees lemma for such modules M. ☐

Proposition C.5.5. *For any flat R-module F such that the R/m-module F/mF is projective, and any R-contramodule $\mathfrak{Q}$, one has $\operatorname{Ext}_R^{>0}(F, \mathfrak{Q}) = 0$.*

Proof. This assertion, provable in the same way as [55, Proposition B.10.1] (see also Corollary D.1.10(b) below), does not depend on any Noetherianity assumptions. ☐

Corollary C.5.6. (a) *A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective if and only if it is a flat R-module and the R/m-module $\mathfrak{F}/m\mathfrak{F}$ is projective.*

(b) *The forgetful functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow R$–$\mathsf{mod}$ induces isomorphisms of all the Ext groups.*

Proof. To prove the "only if" assertion in part (a), it suffices to notice the isomorphism of R/m^n-modules $\mathfrak{R}[[X]]/m^n\mathfrak{R}[[X]] \simeq \mathfrak{R}/m^n[X]$, which holds for any set X and any $n \geqslant 1$, and use Proposition C.5.4. The "if" follows from the fully faithfulness assertion in Theorem C.5.1 and Proposition C.5.5, and part (b) follows from the same together with part (a). (Cf. [55, Section B.8] and Section D.1 below.) □

Corollary C.5.7. (a) *The contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ *of the abelian category of left* $\mathfrak{R}$*-contramodules is equivalent to the contraderived category of the exact category of* R*-flat left* $\mathfrak{R}$*-contramodules.*

(b) *Assume that all flat left* R/m*-modules have finite projective dimensions. Then the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ *is equivalent to the absolute derived category of the exact category of* R*-flat left* $\mathfrak{R}$*-contramodules.*

(c) *Assume that all flat left* R/m*-modules have finite projective dimensions. Then the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ *is equivalent to the homotopy category of complexes of projective left* $\mathfrak{R}$*-contramodules.*

Proof. Clearly, the class of flat left R-modules is closed under infinite products in our assumptions, so it remains to notice that projective left $\mathfrak{R}$-contramodules are R-flat and, in the assumption of parts (b-c), any R-flat left $\mathfrak{R}$-contramodule has finite projective dimension in $\mathfrak{R}\text{–}\mathsf{contra}$. The latter assertions follow from Corollary C.5.6(a). (Cf. Theorems C.1.1 and D.2.3.) □

Let K be a commutative ring, $I \subset K$ be a finitely generated ideal, and A and B be associative K-algebras. For any K-module L, we will denote by ${}_IL \subset L$ the submodule of elements annihilated by I in L.

Lemma C.5.8. (a) *For any left* A*-module* M *and injective left* A*-module* J *there is a natural isomorphism of* K/I*-modules* $\mathrm{Hom}_A(M,J)/I\,\mathrm{Hom}_A(M,J) \simeq \mathrm{Hom}_{A/IA}({}_IM, {}_IJ)$.

(b) *For any right* B*-module* N *and flat left* B*-module* F *there is a natural isomorphism of* K/I*-modules* ${}_I(N \otimes_B F) \simeq {}_IN \otimes_{B/IB} F/IF$.

Proof. For any finitely presented K-module E there are natural isomorphisms $\mathrm{Hom}_A(\mathrm{Hom}_K(E,M),J) \simeq E \otimes_K \mathrm{Hom}_A(M,J)$ and $\mathrm{Hom}_K(E,\, N \otimes_B F) \simeq \mathrm{Hom}_K(E,$ $N) \otimes_B F$. So it remains to take $E = K/I$. (Cf. Lemmas C.1.2 and D.2.5.) □

Let $D^\bullet$ be a finite complex of A-injective and B-injective A-B-bimodules over K (i. e., it is presumed that the left action of A and the right action of B restrict to one and the same action of K in $D^\bullet$). We are using the definition of a dualizing complex for a pair of noncommutative rings from Section B.4.

Lemma C.5.9. (a) *Assume that the ring* A *is left coherent and the ring* B *is right coherent. Then* ${}_ID^\bullet$ *is a dualizing complex for the rings* A/IA *and* B/IB *whenever* $D^\bullet$ *is dualizing complex the rings* A *and* B*.*

(b) *Assume that the ideal* I *is nilpotent, the ring* A *is left Noetherian, and the ring* B *is right Noetherian. Then* $D^\bullet$ *is a dualizing complex for the rings* A *and* B *whenever* ${}_ID^\bullet$ *is a dualizing complex for the rings* A/IA *and* B/IB*.*

Proof. Part (a): clearly, ${}_ID^\bullet$ is a complex of injective left A/IA-modules. The homothety map $B \longrightarrow \mathrm{Hom}_A(D^\bullet, D^\bullet)$ is a quasi-isomorphism of finite complexes of flat left B-modules (see Lemma 1.6.1(b)) and therefore remains a quasi-isomorphism after taking the tensor product with B/IB over B on the left, i. e., reducing modulo I. By Lemma C.5.8(a), it follows that the map $B/IB \longrightarrow \mathrm{Hom}_{A/IA}({}_ID^\bullet,\ {}_ID^\bullet)$ is a quasi-isomorphism.

A bounded above complex of left modules over a left coherent ring is quasi-isomorphic to a bounded above complex of finitely generated projective modules if and only if its cohomology modules are finitely presented. To show that the A/IA-modules of cohomology of the complex ${}_ID^\bullet$ are finitely presented, we notice that the complex of left A/IA-modules ${}_ID^\bullet$ is quasi-isomorphic to the complex $\mathrm{Hom}_{B^{\mathrm{op}}}(L^\bullet, D^\bullet)$ of right B-module homomorphisms from a left resolution $L^\bullet$ of the right B-module B/IB by finitely generated projective B-modules into the complex $D^\bullet$. Locally in the cohomological grading, the complex $\mathrm{Hom}_{B^{\mathrm{op}}}(L^\bullet, D^\bullet)$ is a finitely iterated cone of morphisms between shifts of copies of the complex $D^\bullet$, so its A-modules of cohomology are finitely presented. It remains to point out that an A/IA-module is finitely presented if and only if it is finitely presented as an A-module.

Part (b): whenever the ideal I is nilpotent, a morphism of finite complexes of flat B-modules is a quasi-isomorphism if and only if it becomes one after taking the tensor products with B/IB over B. This, together with the above argument, proves that the homothety map $B \longrightarrow \mathrm{Hom}_A(D^\bullet, D^\bullet)$ is a quasi-isomorphism. To show that the A-modules of cohomology of the complex $D^\bullet$ are finitely generated, one can consider the spectral sequence $E_2^{pq} = \mathrm{Ext}^p_{B^{\mathrm{op}}}(B/IB, H^qD^\bullet) \Longrightarrow H^{p+q}\,\mathrm{Hom}_{B^{\mathrm{op}}}(B/IB, D^\bullet)$ converging from the finite direct sums of copies of the A/I-modules ${}_IH^q(D^\bullet)$ (on the page E_1) to the cohomology A/IA-modules $H^n({}_ID^\bullet)$. Then one argues by increasing induction in q, using the fact that an A-module M is finitely generated provided that so is the A/IA-module ${}_IM$. (Cf. Lemmas C.1.3 and D.2.6.) ☐

Let K be a commutative Noetherian ring, $I \subset K$ be an ideal, and A and B be associative K-algebras such that the ring A is left Noetherian and the ring B is right Noetherian. A finite complex of A-B-bimodules $\mathcal{D}^\bullet$ over K is said to be a *dualizing complex for A and B over* (K, I) if

- (i) $\mathcal{D}^\bullet$ is a complex of injective left A-modules and a complex of injective right B-modules;
- (ii) any element in $\mathcal{D}^\bullet$ is annihilated by some power of the ideal I;
- (iii) for any (or, equivalently, some) $n \geqslant 1$, the complex ${}_{(n)}\mathcal{D}^\bullet = {}_{I^n}\mathcal{D}^\bullet$ is a dualizing complex for the rings A/I^nA and B/I^nB (cf. Lemma C.5.9).

A K-module (or A-module) is said to be *I-torsion* if every its element is annihilated by some power of the ideal $I \subset K$. A *left (B, I)-contramodule* is a left contramodule over the I-adic completion of the ring B, or, equivalently, a left B-module that is a (K, I)-contramodule in the K-module structure obtained by restriction of scalars (see Theorem C.5.1 and Section C.1).

We denote the abelian category of I-torsion left A-modules by (A, I)–tors and the abelian category of left (B, I)-contramodules by (B, I)–contra.

Theorem C.5.10. *The choice of a dualizing complex $\mathcal{D}^\bullet$ for the rings A and B over (K,I) induces an equivalence between the coderived category of I-torsion left A-modules $\mathsf{D}^{\mathsf{co}}((A,I)\text{–}\mathsf{tors})$ and the contraderived category of left (B,I)-contramodules $\mathsf{D}^{\mathsf{ctr}}((B,I)\text{–}\mathsf{contra})$. The equivalence is provided by the derived functors $\mathbb{R}\operatorname{Hom}_A(\mathcal{D}^\bullet,-)$ and $\mathcal{D}^\bullet\otimes_B^{\mathbb{L}}-$.*

Proof. Notice first of all that an object $\mathcal{J}\in(A,I)\text{–}\mathsf{tors}$ is injective if and only if the A/I^nA modules ${}_{(n)}\mathcal{J}={}_{I^n}\mathcal{J}$ are injective for all n (obviously), and if and only if it is an injective A-module (by the Artin–Rees lemma for centrally generated ideals in noncommutative Noetherian rings [24, Theorem 13.3]). The dual result for B-flat (B,I)-contramodules is provided by Proposition C.5.4.

Furthermore, we identify the coderived category $\mathsf{D}^{\mathsf{co}}((A,I)\text{–}\mathsf{tors})$ with the homotopy category of (complexes of) injective I-torsion A-modules (based on the facts that there are enough injectives and the class of injectives is closed with respect to infinite direct sums in $(A,I)\text{–}\mathsf{tors}$) and the contraderived category $\mathsf{D}^{\mathsf{ctr}}((B,I)\text{–}\mathsf{contra})$ with the absolute derived category of B-flat (B,I)-contramodules (based on Corollary C.5.7(b) and Lemma B.4.2). The rest of the argument is no different from the proof of Theorem C.1.4 (see also Lemma B.4.1) and based on Lemma C.5.8. ☐

Appendix D. Ind-Affine Ind-Schemes

The aim of this appendix is to lay some bits of preparatory groundwork for the definition of contraherent cosheaves of contramodules on ind-schemes. We construct enough very flat and contraadjusted contramodules on an ind-affine ind-scheme represented by a sequence of affine schemes and their closed embeddings with finitely generated defining ideals, and also enough cotorsion contramodules on an ind-Noetherian ind-affine ind-scheme of totally finite Krull dimension. A version of co-contra correspondence for ind-coherent ind-affine ind-schemes with dualizing complexes (and their noncommutative generalizations) is also worked out.

D.1. **Flat and projective contramodules.** Let $R_0\longleftarrow R_1\longleftarrow R_2\longleftarrow R_3\longleftarrow\cdots$ be a projective system of associative rings, indexed by the ordered set of positive integers, with surjective morphisms between them. Denote by $\mathfrak{R}$ the projective limit $\varprojlim_n R_n$, viewed as a topological ring in the topology of projective limit of discrete rings R_n. Clearly, the ring homomorphisms $\mathfrak{R}\longrightarrow R_n$ are surjective; let $\mathfrak{I}_n\subset\mathfrak{R}$ denote their kernels. Then the open ideals $\mathfrak{I}_n$ form a base of neighborhoods of zero in the topological ring $\mathfrak{R}$.

We are interested in left contramodules over the topological ring $\mathfrak{R}$ (see [52, Remark A.3] and [55, Section 1.2] or [56, Section 2.1] for the definition). They form an abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ having enough projective objects and endowed with an exact and faithful forgetful functor $\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow\mathfrak{R}\text{–}\mathsf{mod}$ preserving infinite products. Here $\mathfrak{R}\text{–}\mathsf{mod}$ denotes the abelian category of left modules over the ring $\mathfrak{R}$ (viewed as an abstract ring without any topology).

The projective $\mathfrak{R}$-contramodules are precisely the direct summands of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$. Here X is an arbitrary set of generators, and $\mathfrak{R}[[X]] = \varprojlim_n R_n[X]$ is the set of all maps $X \longrightarrow \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$.

For any left $\mathfrak{R}$-contramodule $\mathfrak{P}$ and any closed two-sided ideal $\mathfrak{J} \subset \mathfrak{R}$, we will denote by $\mathfrak{J}\times\mathfrak{P} \subset \mathfrak{P}$ the image of the contraaction map $\mathfrak{J}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. As usually, for any left module M over a ring R and any ideal $J \subset R$ the notation JM (or, as it may be sometimes convenient, $J\cdot M$) stands for the image of the action map $J \otimes_R M \longrightarrow M$. For an $\mathfrak{R}$-contramodule $\mathfrak{P}$, there is a (generally speaking, proper) inclusion $\mathfrak{J}\mathfrak{P} = \mathfrak{J}\cdot\mathfrak{P} \subset \mathfrak{J}\times\mathfrak{P}$. Of course, $\mathfrak{J}\times\mathfrak{P}$ is an $\mathfrak{R}$-subcontramodule in $\mathfrak{P}$, while $\mathfrak{J}\mathfrak{P}$ is in general only an $\mathfrak{R}$-submodule.

Lemma D.1.1. *For any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, the natural map to the projective limit $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/(\mathfrak{I}_n\times\mathfrak{P})$ is surjective.*

Proof. This assertion is true for any topological ring $\mathfrak{R}$ with a base of topology consisting of open right ideals, and any decreasing sequence of closed abelian subgroups $\mathfrak{R} \supset \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \mathfrak{I}_2 \supset \cdots$ converging to zero in the topology of $\mathfrak{R}$ (meaning that for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists $n \geqslant 0$ such that $\mathfrak{I}_n \subset \mathfrak{U}$). Indeed, it suffices to show that for any sequence of elements $p_n \in \mathfrak{I}_n\times\mathfrak{P}$, $n \geqslant 0$, there exists an element $p \in \mathfrak{P}$ such that $p - (p_0 + \cdots + p_n) \in \mathfrak{I}_{n+1}\times\mathfrak{P}$ for all $n \geqslant 0$.

Now each element p_n can be obtained as an infinite linear combination of elements of $\mathfrak{P}$ with the family of coefficients converging to zero in $\mathfrak{I}_n$. The countable sum of such expressions over all $n \geqslant 0$ is again an infinite linear combination of elements of $\mathfrak{P}$ with the coefficients converging to zero in $\mathfrak{R}$. The value of the latter, understood in the sense of the contramodule infinite summation operations in $\mathfrak{P}$, provides the desired element $p \in \mathfrak{P}$. Another (and perhaps more illuminating) argument can be found in [52, Lemma A.2.3] (while counterexamples showing that the map in question may not be injective are provided in [52, Section A.1]). ☐

The following result is a version of Nakayama's lemma for $\mathfrak{R}$-contramodules (see [55, Lemma 1.3.1] for a somewhat more familiar formulation).

Lemma D.1.2. *For any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, the equations $\mathfrak{I}_n\times\mathfrak{P} = \mathfrak{P}$ for all $n \geqslant 0$ imply $\mathfrak{P} = 0$.*

Proof. This assertion holds for any (complete and separated) topological ring $\mathfrak{R}$ with a base of topology consisting of open right ideals, and any sequence of closed abelian subgroups $\mathfrak{I}_0, \mathfrak{I}_1, \mathfrak{I}_2, \ldots \subset \mathfrak{R}$ with the property that the sequence of subgroups $\mathfrak{I}_0$, $\mathfrak{I}_0\mathfrak{I}_1$, $\mathfrak{I}_0\mathfrak{I}_1\mathfrak{I}_2, \ldots$ converges to zero in the topology of $\mathfrak{R}$ (i. e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists $n \geqslant 0$ such that $\mathfrak{I}_0 \cdots \mathfrak{I}_n \subset \mathfrak{U}$). Indeed, assume that the restrictions $\pi_n\colon \mathfrak{I}_n[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ are surjective for all $n \geqslant 0$. Let $p \in \mathfrak{P}$ be an element.

Notice that for any surjective map of sets $f\colon X \longrightarrow Y$ and any $n \geqslant 0$, the induced map $\mathfrak{I}_n[[f]]\colon \mathfrak{I}_n[[X]] \longrightarrow \mathfrak{I}_n[[Y]]$ is also surjective. Given a set X, define inductively the sets $\mathfrak{I}^{(-1)}[[X]] = X$ and $\mathfrak{I}^{(n)}[[X]] = \mathfrak{I}^{(n-1)}[[\mathfrak{I}_n[[X]]]]$ for all $n \geqslant 0$. Let $p_0 \in \mathfrak{I}_0[[\mathfrak{P}]]$ be a preimage of $p \in \mathfrak{P}$ under the map $\pi_0\colon \mathfrak{I}_0[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. Furthermore, let $p_n \in \mathfrak{I}^{(n)}[[\mathfrak{P}]]$ be a preimage of p_{n-1} under the map $\mathfrak{I}^{(n-1)}[[\pi_n]]\colon \mathfrak{I}^{(n)}[[\mathfrak{P}]] \longrightarrow \mathfrak{I}^{(n-1)}[[\mathfrak{P}]]$.

For any set X, the abelian group $\mathfrak{R}[[X]]$ is complete in its natural topology with the base of neighborhoods of zero formed by the subgroups $\mathfrak{U}[[X]]$, where $\mathfrak{U}\subset\mathfrak{R}$ are open right ideals. Besides, the "opening of parentheses"/monad multiplication map $\rho_X\colon\mathfrak{R}[[\mathfrak{R}[[X]]]]\longrightarrow\mathfrak{R}[[X]]$ is continuous, as is the map $\mathfrak{R}[[f]]\colon\mathfrak{R}[[X]]\longrightarrow\mathfrak{R}[[Y]]$ induced by any map of sets $f\colon X\longrightarrow Y$. For every $n\geqslant 0$, let $\rho_X^{(n)}\colon\mathfrak{I}^{(n)}[[X]]\longrightarrow\mathfrak{R}[[X]]$ denote (the restriction of) the iterated monad multiplication map.

Set $q_n=\rho^{(n-1)}_{\mathfrak{I}_n[[\mathfrak{P}]]}(p_n)\in\mathfrak{R}[[\mathfrak{I}_n[[\mathfrak{P}]]]]\subset\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]$ for all $n\geqslant 1$. Due to our convergence condition on the products of the subgroups $\mathfrak{I}_n\subset\mathfrak{R}$, the sum $\sum_{n=1}^\infty q_n$ converges in the topology of $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]$. Now we have $\mathfrak{R}[[\pi_n]](q_n)=\rho_{\mathfrak{P}}(q_{n-1})$ for all $n\geqslant 2$ and $\mathfrak{I}_0[[\pi_1]](q_1)=p_0$. Hence

$$\mathfrak{R}[[\pi]]\big(\textstyle\sum_{n=1}^\infty q_n\big)-\rho_{\mathfrak{P}}\big(\sum_{n=1}^\infty q_n\big)=p_0$$

in $\mathfrak{R}[[\mathfrak{P}]]$ and $p=\pi(\mathfrak{p}_0)=0$ by the contraassociativity equation. ☐

Lemma D.1.3. *Let $P_0\longleftarrow P_1\longleftarrow P_2\longleftarrow\cdots$ be a projective system of left R_n-modules in which the morphism $P_{n+1}\longrightarrow P_n$ identifies P_n with $R_n\otimes_{R_{n+1}}P_{n+1}$ for every $n\geqslant 0$. Let $\mathfrak{P}$ denote the $\mathfrak{R}$-contramodule $\varprojlim_n P_n$. Then the natural map $\mathfrak{P}\longrightarrow P_n$ identifies P_n with $\mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$. Conversely, for any left $\mathfrak{R}$-contramodule $\mathfrak{P}$ the projective system $P_n=\mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$ satisfies the above condition.*

Proof. Since P_n is an R_n-module and $\mathfrak{P}\longrightarrow P_n$ is a morphism of $\mathfrak{R}$-contramodules, the kernel of this morphism contains $\mathfrak{I}_n\times\mathfrak{P}$. To prove the inverse inclusion, consider an element $p\in\mathfrak{P}$ belonging to the kernel of the morphism $\mathfrak{P}\longrightarrow P_n$. The image of the element p in P_{n+1} belongs to $(\mathfrak{I}_n/\mathfrak{I}_{n+1})P_{n+1}$ (where $\mathfrak{I}_n/\mathfrak{I}_{n+1}$ is an ideal in R_{n+1}). Let us decompose this image accordingy into a finite linear combination of elements of P_{n+1} with coefficients from $\mathfrak{I}_n/\mathfrak{I}_{n+1}$, lift all the entering elements of P_{n+1} to $\mathfrak{P}$ and all the elements of $\mathfrak{I}_n/\mathfrak{I}_{n+1}$ to $\mathfrak{I}_n$, and subtract from p the corresponding finite linear combination of elements of $\mathfrak{P}$ with coefficients in $\mathfrak{I}_n$.

The image of the resulting element $p'\in\mathfrak{P}$ in P_{n+1} vanishes, so its image in P_{n+2} belongs to $(\mathfrak{I}_{n+1}/\mathfrak{I}_{n+2})P_{n+2}$. Continuing this process indefinitely, we obtain an expression of the original element p in the form of a countable linear combination of elements from $\mathfrak{P}$ with the coefficient sequence converging to zero in $\mathfrak{I}_n$. This proves the first assertion; the second one is straightforward. ☐

A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called *flat* if the map $\mathfrak{F}\longrightarrow\varprojlim_n\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ is an isomorphism and the R_n-modules $\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ are flat for all $n\geqslant 0$. We will see below in Corollary D.1.7 that the former condition, in fact, follows from the latter one.

Lemma D.1.4. *If $\mathfrak{G}\longrightarrow\mathfrak{F}$ is a surjective morphism of flat $\mathfrak{R}$-contramodules then its kernel $\mathfrak{H}$ is also a flat $\mathfrak{R}$-contramodule. Moreover, the sequences of R_n-modules $0\longrightarrow\mathfrak{H}/\mathfrak{I}_n\times\mathfrak{H}\longrightarrow\mathfrak{G}/\mathfrak{I}_n\times\mathfrak{G}\longrightarrow\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}\longrightarrow 0$ are exact.*

Proof. Clearly, for any short exact sequence of $\mathfrak{R}$-contramodules $0\longrightarrow\mathfrak{H}\longrightarrow\mathfrak{G}\longrightarrow\mathfrak{F}\longrightarrow 0$ there are short exact sequences of R_n-modules $0\longrightarrow\mathfrak{H}/\mathfrak{H}\cap(\mathfrak{I}_n\times\mathfrak{G})\longrightarrow\mathfrak{G}/\mathfrak{I}_n\times\mathfrak{G}\longrightarrow\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}\longrightarrow 0$ (because the map $\mathfrak{I}_n\times\mathfrak{G}\longrightarrow\mathfrak{I}_n\times\mathfrak{F}$ induced by a surjective map $\mathfrak{G}\longrightarrow\mathfrak{F}$ is surjective). If the R_n-module $\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ is flat, then the tensor product with R_{n-1} over R_n transforms this sequence into the similar sequence corresponding to the ideal $\mathfrak{I}_{n-1} \subset \mathfrak{R}$.

On the other hand, if the maps $\mathfrak{F} \longrightarrow \varprojlim_n \mathfrak{F}/\mathfrak{I}_n \times \mathfrak{F}$ and $\mathfrak{G} \longrightarrow \varprojlim_n \mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G}$ are isomorphisms, then the passage to the projective limit of the above short exact sequences allows to conclude that the map $\mathfrak{H} \longrightarrow \varprojlim_n \mathfrak{H}/\mathfrak{H} \cap (\mathfrak{I}_n \times \mathfrak{G})$ is an isomorphism. By Lemma D.1.3, it follows from these observations that $\mathfrak{H} \cap (\mathfrak{I}_n \times \mathfrak{G}) = \mathfrak{I}_n \times \mathfrak{H}$, so the sequences $0 \longrightarrow \mathfrak{H}/\mathfrak{I}_n \times \mathfrak{H} \longrightarrow \mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G} \longrightarrow \mathfrak{F}/\mathfrak{I}_n \times \mathfrak{F} \longrightarrow 0$ are exact. Finally, now if the R_n-modules $\mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G}$ are also flat, then so are the R_n-modules $\mathfrak{H}/\mathfrak{I}_n \times \mathfrak{H}$. ☐

Lemma D.1.5. *If $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$ is a short exact sequence of $\mathfrak{R}$-contramodules and the $\mathfrak{R}$-contramodules $\mathfrak{H}$ and $\mathfrak{F}$ are flat, then so is the $\mathfrak{R}$-contramodule $\mathfrak{G}$.*

Proof. In view of the proof of Lemma D.1.4, we only need to show that the map $\mathfrak{G} \longrightarrow \varprojlim_n \mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G}$ is an isomorphism. Choose a termwise surjective map onto the short exact sequence $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$ from a short exact sequence of free $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{W} \longrightarrow \mathfrak{V} \longrightarrow \mathfrak{U} \longrightarrow 0$ (e. g., $\mathfrak{W} = \mathfrak{R}[[\mathfrak{H}]]$, $\mathfrak{U} = \mathfrak{R}[[\mathfrak{F}]]$ or $\mathfrak{R}[[\mathfrak{G}]]$, and $\mathfrak{V} = \mathfrak{W} \oplus \mathfrak{U}$). Let $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{K} \longrightarrow 0$ be the corresponding short exact sequence of kernels.

Passing to the projective limit of short exact sequences $0 \longrightarrow \mathfrak{L}/\mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V}) \longrightarrow \mathfrak{V}/\mathfrak{I}_n \times \mathfrak{V} \longrightarrow \mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G} \longrightarrow 0$, we obtain a short exact sequence $0 \longrightarrow \varprojlim_n \mathfrak{L}/\mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V}) \longrightarrow \varprojlim_n \mathfrak{V}/\mathfrak{I}_n \times \mathfrak{V} \longrightarrow \varprojlim \mathfrak{G}/\mathfrak{I}_n \times \mathfrak{G} \longrightarrow 0$. The $\mathfrak{R}$-contramodule $\mathfrak{V}$ being free, the intersection $\bigcap_n \mathfrak{I}_n \times \mathfrak{V} \subset \mathfrak{V}$ vanishes; so both maps $\mathfrak{V} \longrightarrow \varprojlim_n \mathfrak{V}/\mathfrak{I}_n \times \mathfrak{V}$ and $\mathfrak{L} \longrightarrow \varprojlim_n \mathfrak{L}/\mathfrak{I}_n \times \mathfrak{L}$ are isomorphisms, while the map $\mathfrak{L} \longrightarrow \varprojlim_n \mathfrak{L}/\mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V})$ is, at least, injective. It remains to show that the latter map is surjective; equivalently, it means vanishing of the derived projective limit $\varprojlim^1_n \mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V})$.

Similarly, the assumptions that the maps $\mathfrak{H} \longrightarrow \varprojlim_n \mathfrak{H}/\mathfrak{I}_n \times \mathfrak{H}$ and $\mathfrak{F} \longrightarrow \varprojlim_n \mathfrak{F}/\mathfrak{I}_n \times \mathfrak{F}$ are isomorphisms are equivalently expressed as the vanishing of derived projective limits $\varprojlim^1_n \mathfrak{M} \cap (\mathfrak{I}_n \times \mathfrak{W})$ and $\varprojlim^1_n \mathfrak{K} \cap (\mathfrak{I}_n \times \mathfrak{U})$.

The short sequences $0 \longrightarrow \mathfrak{I}_n \times \mathfrak{W} \longrightarrow \mathfrak{I}_n \times \mathfrak{V} \longrightarrow \mathfrak{I}_n \times \mathfrak{U} \longrightarrow 0$ are exact, because the short sequence $0 \longrightarrow \mathfrak{W} \longrightarrow \mathfrak{V} \longrightarrow \mathfrak{U} \longrightarrow 0$ splits. Passing to the fibered product of two short exact sequences $0 \longrightarrow \mathfrak{I}_n \times \mathfrak{W} \longrightarrow \mathfrak{I}_n \times \mathfrak{V} \longrightarrow \mathfrak{I}_n \times \mathfrak{U} \longrightarrow 0$ and $0 \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{K} \longrightarrow 0$ over the short exact sequence $0 \longrightarrow \mathfrak{W} \longrightarrow \mathfrak{V} \longrightarrow \mathfrak{U} \longrightarrow 0$ (into which both of them are embedded), we obtain an exact sequence $0 \longrightarrow \mathfrak{M} \cap (\mathfrak{I}_n \times \mathfrak{W}) \longrightarrow \mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V}) \longrightarrow \mathfrak{K} \cap (\mathfrak{I}_n \times \mathfrak{U})$.

The $\mathfrak{R}$-contramodules $\mathfrak{U}$ and $\mathfrak{F}$ being flat, by Lemma D.1.4 we have $\mathfrak{K} \cap (\mathfrak{I}_n \times \mathfrak{U}) = \mathfrak{I}_n \times \mathfrak{K}$. It follows that the map $\mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V}) \longrightarrow \mathfrak{K} \cap (\mathfrak{I}_n \times \mathfrak{U})$ is surjective, so the whole sequence $0 \longrightarrow \mathfrak{M} \cap (\mathfrak{I}_n \times \mathfrak{W}) \longrightarrow \mathfrak{L} \cap (\mathfrak{I}_n \times \mathfrak{V}) \longrightarrow \mathfrak{K} \cap (\mathfrak{I}_n \times \mathfrak{U}) \longrightarrow 0$ is exact. Applying the right exact functor $\varinjlim^1_n$ (which is, in particular, exact in the middle), we obtain the desired vanishing. ☐

Lemma D.1.6. *Any projective $\mathfrak{R}$-contramodule is flat.*

Proof. It suffices to consider the case of a free left $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$. For any set X and any closed ideal $\mathfrak{J} \subset \mathfrak{R}$ one has $\mathfrak{J} \times (\mathfrak{R}[[X]]) = \mathfrak{J}[[X]]$ and $\mathfrak{R}[[X]]/\mathfrak{J}[[X]] \simeq$

$(\mathfrak{R}/\mathfrak{J})[[X]]$. So in particular $\mathfrak{R}[[X]]/(\mathfrak{I}_n\times\mathfrak{R}[[X]]) = R_n[X]$ is a flat (and even free and projective) left R_n-module and the natural $\mathfrak{R}$-contramodule morphism $\mathfrak{R}[[X]] \longrightarrow \varprojlim_n \mathfrak{R}[[X]]/(\mathfrak{I}_n\times\mathfrak{R}[[X]])$ is an isomorphism. □

Corollary D.1.7. *A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat if and only if the R_n-modules $\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ are flat for all $n \geqslant 0$.*

Proof. Set $\overline{\mathfrak{P}}_n = \mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$ for any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, and denote by $\mathfrak{P} \longmapsto \mathbb{L}_i\overline{\mathfrak{P}}_n$ the left derived functors of the right exact functor $\mathfrak{P} \longmapsto \overline{\mathfrak{P}}_n$ constructed in terms of projective left resolutions of left $\mathfrak{R}$-contramodules $\mathfrak{P}$. It follows from Lemmas D.1.4 and D.1.6 that flat $\mathfrak{R}$-contramodules are acyclic for the derived functors $\mathfrak{P} \longmapsto \mathbb{L}_*\overline{\mathfrak{P}}_n$, i. e., one has $\mathbb{L}_{>0}\overline{\mathfrak{F}}_n = 0$ for all flat left $\mathfrak{R}$-contramodules $\mathfrak{F}$.

Now let $\mathfrak{G}$ be a left $\mathfrak{R}$-contramodule such that all the R_n-modules $\overline{\mathfrak{G}}_n$ are flat. Set $\mathfrak{H} = \bigcap_n(\mathfrak{I}_n\times\mathfrak{G})$ and $\mathfrak{F} = \varprojlim_n \overline{\mathfrak{G}}_n = \mathfrak{G}/\mathfrak{H}$; according to Lemma D.1.3, one has $\overline{\mathfrak{G}}_n \simeq \overline{\mathfrak{F}}_n$. Since the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat, the long exact sequence of derived functors $\mathfrak{P} \longmapsto \mathbb{L}_*\overline{\mathfrak{P}}_n$ applied to the short exact sequence of left $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$ proves that the short sequence $0 \longrightarrow \overline{\mathfrak{H}}_n \longrightarrow \overline{\mathfrak{G}}_n \longrightarrow \overline{\mathfrak{F}}_n \longrightarrow 0$ is exact. We have shown that $\overline{\mathfrak{H}}_n = 0$ for all $n \geqslant 0$; by Lemma D.1.2, it follows that $\mathfrak{H} = 0$. (Cf. [55, Lemma B.9.2].) □

Given two $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, we will denote by $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q})$ and $\mathrm{Ext}^{\mathfrak{R},*}(\mathfrak{P},\mathfrak{Q})$ the Hom and Ext groups in the abelian category $\mathfrak{R}$–contra.

Lemma D.1.8. *Let $\mathfrak{F}$ be a flat left $\mathfrak{R}$-contramodule and $\mathfrak{P}$ be a left $\mathfrak{R}$-contramodule for which the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$ is an isomorphism. Set $F_n = \mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ and $P_n = \mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$, and assume that one has $\mathrm{Ext}^1_{R_0}(F_0, P_0) = 0 = \mathrm{Ext}^1_{R_{n+1}}(F_{n+1},\ \ker(P_{n+1} \to P_n))$ for all $n \geqslant 0$. Then $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{P}) = 0$.*

Proof. Let $\mathfrak{G} \longrightarrow \mathfrak{F}$ be a surjective morphism onto $\mathfrak{F}$ from a projective $\mathfrak{R}$-contramodule $\mathfrak{G}$ with the kernel $\mathfrak{H}$. Set $G_n = \mathfrak{G}/\mathfrak{I}_n\times\mathfrak{G}$ and similarly for H_n. By Lemmas D.1.6 and D.1.4, the $\mathfrak{R}$-contramodules $\mathfrak{G}$ and $\mathfrak{H}$ are flat, and the short sequences $0 \longrightarrow H_n \longrightarrow G_n \longrightarrow F_n \longrightarrow 0$ are exact. Let us show that any $\mathfrak{R}$-contramodule morphism $\mathfrak{H} \longrightarrow \mathfrak{P}$ can be extended to an $\mathfrak{R}$-contramodule morphism $\mathfrak{G} \longrightarrow \mathfrak{P}$.

The data of a morphism of $\mathfrak{R}$-contramodules $\mathfrak{H} \longrightarrow \mathfrak{P}$ is equivalent to that of a morphism of projective systems of R_n-modules $H_n \longrightarrow P_n$. Let us construct by induction an extension of this morphism to a morphism of projective systems $G_n \longrightarrow P_n$ for $n \geqslant 0$. Since $\mathrm{Ext}^1_{R_0}(F_0, P_0) = 0$, the case of $n = 0$ is clear. Assuming that the morphism $G_n \longrightarrow P_n$ has been obtained already, we will proceed to construct a compatible morphism $G_{n+1} \longrightarrow P_{n+1}$.

Since G_{n+1} is a projective R_{n+1}-module, the composition $G_{n+1} \longrightarrow G_n \longrightarrow P_n$ can be lifted to an R_{n+1}-module morphism $G_{n+1} \longrightarrow P_{n+1}$. (Notice that what is actually used here is the vanishing of $\mathrm{Ext}^1_{R_{n+1}}(G_{n+1},\ \ker(P_{n+1} \to P_n))$.) The composition of a morphism so obtained with the embedding $H_{n+1} \longrightarrow G_{n+1}$ differs from the given map $H_{n+1} \longrightarrow P_{n+1}$ by an R_{n+1}-module morphism $H_{n+1} \longrightarrow \ker(P_{n+1} \to P_n)$. Given that $\mathrm{Ext}^1_{R_{n+1}}(F_{n+1},\ \ker(P_{n+1} \to P_n)) = 0$, the latter map can be extended from H_{n+1} to G_{n+1} and added to the previously constructed map $G_{n+1} \longrightarrow P_{n+1}$. □

Suppose R is an associative ring endowed with a descending sequence of two-sided ideals $R \supset I_0 \supset I_1 \supset I_2 \supset \cdots$ such that the projective system of quotient rings R/I_n is isomorphic to our original projective system $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$. For example, one can always take $R = \mathfrak{R}$ and $I_n = \mathfrak{I}_n$; sometimes there may be other suitable choices of a ring R with the ideals I_n as well.

Then there is a natural ring homomorphism $R \longrightarrow \mathfrak{R}$ inducing the isomorphisms $R/I_n \simeq \mathfrak{R}/\mathfrak{I}_n$. The restriction of scalars provides a forgetful functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow R$–$\mathsf{mod}$, which is exact and faithful, and preserves infinite products. Given two R-modules L and M, we denote, as usually, by $\mathrm{Hom}_R(L, M)$ and $\mathrm{Ext}^*_R(L, M)$ the Hom and Ext groups in the abelian category of R-modules.

Lemma D.1.9. *Let F be a flat left R-module and $\mathfrak{P}$ be a left $\mathfrak{R}$-contramodule for which the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ is an isomorphism. Set $P_n = \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$, and assume that that one has $\mathrm{Ext}^1_{R_0}(F/I_0F,\, P_0) = 0 = \mathrm{Ext}^1_{R_{n+1}}(F/I_{n+1}F,\, \ker(P_{n+1} \to P_n))$ for all $n \geqslant 0$. Then $\mathrm{Ext}^1_R(F, \mathfrak{P}) = 0$.*

Proof. The proof is the same as in Lemma C.3.1. ☐

Corollary D.1.10. (a) *A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective if and only if the R_n-modules $\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ are projective for all $n \geqslant 0$.*

(b) *For any flat left R-module F such that the R_n-modules F/I_nF are projective for all $n \geqslant 0$, and any left $\mathfrak{R}$-contramodule $\mathfrak{Q}$, one has $\mathrm{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\, \mathfrak{Q}) \simeq \mathrm{Hom}_R(F, \mathfrak{Q})$ and $\mathrm{Ext}^{\mathfrak{R},>0}(\varprojlim_n F/I_nF,\, \mathfrak{Q}) = \mathrm{Ext}^{>0}_R(F, \mathfrak{Q}) = 0$.*

Proof. Part (a): the "only if" assertion holds by (the proof of) Lemma D.1.6. To prove the "if", notice first of all that the left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat by Corollary D.1.7. Consider a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$ with a projective $\mathfrak{R}$-contramodule $\mathfrak{G}$. The natural map $\mathfrak{H} \longrightarrow \varprojlim_n \mathfrak{H}/\mathfrak{I}_n{\times}\mathfrak{H}$ being an isomorphism because the map $\mathfrak{G} \longrightarrow \varprojlim_n \mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G}$ is (or by Lemma D.1.4), one has $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F}, \mathfrak{H}) = 0$ by Lemma D.1.8. Hence the short exact sequence splits and the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is a direct summand of the $\mathfrak{R}$-contramodule $\mathfrak{G}$.

Part (b): a natural map $\mathrm{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\, \mathfrak{Q}) \longrightarrow \mathrm{Hom}_R(F, \mathfrak{Q})$ for any R-module F and $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is induced by the R-module morphism $F \longrightarrow \varprojlim_n F/I_nF$. For an $\mathfrak{R}$-contramodule $\mathfrak{P}$ such that the map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ is an isomorphism, one has $\mathrm{Hom}_R(F, \mathfrak{P}) \simeq \varprojlim_n \mathrm{Hom}_{R_n}(F/I_nF,\, \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P})$, which is isomorphic to $\mathrm{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\, \mathfrak{P})$ by Lemma D.1.3.

When F is also a flat R-module with projective R_n-modules F/I_nF, one has $\mathrm{Ext}^{>0}_R(F, \mathfrak{P}) = 0$ by Lemma D.1.9 and $\mathrm{Ext}^{\mathfrak{R},>0}(\varprojlim_n F/I_nF,\, \mathfrak{P}) = 0$ by part (a). Now to prove the assertion of part (b) in the general case, it suffices to present an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ as the cokernel of an injective morphism of $\mathfrak{R}$-contramodules $\mathfrak{K} \longrightarrow \mathfrak{P}$ with $\mathfrak{P} = \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ (and, consequently, the same for $\mathfrak{K}$). ☐

When the ideals $\mathfrak{I}_n/\mathfrak{I}_{n+1} \subset R_{n+1}$ are nilpotent (i. e., for every $n \geqslant 0$ there exists $N_n \geqslant 1$ such that $(\mathfrak{I}_n/\mathfrak{I}_{n+1})^{N_n} = 0$), it follows from Corollary D.1.10(a) and [55, Lemma B.10.2] that a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective if and only if it is flat and the R_0-module $\mathfrak{F}/\mathfrak{I}_0\times\mathfrak{F}$ is projective. Similarly, it suffices to require that the R_0-module F/I_0F be projective in Corollary D.1.10(b) in this case.

D.2. **Co-contra correspondence.** In this section we consider a pair of projective systems $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ and $S_0 \longleftarrow S_1 \longleftarrow S_2 \longleftarrow \cdots$ of associative rings and surjective morphisms between them. Set $\mathfrak{R} = \varprojlim_n R_n$ and $\mathfrak{S} = \varprojlim_n S_n$, and denote by $\mathfrak{I}_n \subset \mathfrak{R}$ and $\mathfrak{J}_n \subset \mathfrak{S}$ the kernels of the natural surjective ring homomorphisms $\mathfrak{R} \longrightarrow R_n$ and $\mathfrak{S} \longrightarrow S_n$. We assume that the rings S_n are left coherent, the rings R_n are right coherent, the kernels $\mathfrak{J}_n/\mathfrak{J}_{n+1} \subset S_{n+1}$ of the ring homomorphisms $S_{n+1} \longrightarrow S_n$ are finitely generated as left ideals, and the kernels $\mathfrak{I}_n/\mathfrak{I}_{n+1} \subset R_{n+1}$ of the ring homomorphisms $R_{n+1} \longrightarrow R_n$ are finitely generated as right ideals.

Given a left $\mathfrak{S}$-module M, we denote by ${}_{S_n}M \subset M$ its submodule consisting of all the elements annihilated by $\mathfrak{J}_n$; so ${}_{S_n}M$ is the maximal left S_n-submodule in M. A left $\mathfrak{S}$-module $\mathcal{M}$ is said to be *discrete* if every its element is annihilated by the ideal $\mathfrak{J}_n$ for n large enough. In other words, a left $\mathfrak{S}$-module $\mathcal{M}$ is discrete if its increasing filtration ${}_{S_0}\mathcal{M} \subset {}_{S_1}\mathcal{M} \subset {}_{S_2}\mathcal{M} \subset \cdots$ is exhaustive, or equivalently, if the left action map $\mathfrak{S}\times\mathcal{M} \longrightarrow \mathcal{M}$ is continuous in (the projective limit topology of $\mathfrak{S}$ and) the discrete topology of $\mathcal{M}$. We denote the full abelian subcategory of discrete left $\mathfrak{S}$-modules by $\mathfrak{S}$–discr $\subset$ $\mathfrak{S}$–mod. Clearly, a discrete left $\mathfrak{S}$-module $\mathcal{J}$ is an injective object in $\mathfrak{S}$–discr if and only if all the left S_n-modules ${}_{S_n}\mathcal{J}$ are injective.

A discrete left $\mathfrak{S}$-module $\mathcal{M}$ is called *finitely presented* if there exists $n \geqslant 0$ such that $\mathcal{M} = {}_{S_n}\mathcal{M}$ and $\mathcal{M}$ is a finitely presented left S_n-module. It follows from the finite generatedness condition on the kernel of the ring homomorphism $S_{n+1} \longrightarrow S_n$ that any finitely presented left S_n-module is at the same time a finitely presented left S_{n+1}-module. A discrete left $\mathfrak{S}$-module $\mathcal{M}$ is finitely presented if and only if the functor of discrete $\mathfrak{S}$-module homomorphisms $\mathrm{Hom}_{\mathfrak{S}}(\mathcal{M}, -)$ from $\mathcal{M}$ commutes with filtered inductive limits in the abelian category of discrete left $\mathfrak{S}$-modules. The full subcategory of finitely presented discrete left $\mathfrak{S}$-modules is closed under kernels, cokernels, and extensions in $\mathfrak{S}$–discr; so it is an abelian category.

We refer to [57, Section 1] and [64, Section 6 and Appendix B], and the references therein, for discussions of fp-injective modules over coherent rings and fp-injective objects in locally coherent Grothendieck categories. A discrete left $\mathfrak{S}$-module $\mathcal{J}$ is called *fp-injective* if the functor $\mathrm{Hom}_{\mathfrak{S}}(-, \mathcal{J})$ takes short exact sequences of finitely presented discrete left $\mathfrak{S}$-modules to short exact sequences of abelian groups. Given two discrete left $\mathfrak{S}$-modules $\mathcal{L}$ and $\mathcal{M}$, we denote by $\mathrm{Ext}^*_{\mathfrak{S}}(\mathcal{L}, \mathcal{M})$ the Ext groups in the abelian category $\mathfrak{S}$–discr.

Lemma D.2.1. (a) *A discrete left $\mathfrak{S}$-module $\mathcal{J}$ is fp-injective if and only if the S_n-module ${}_{S_n}\mathcal{J}$ is fp-injective for every $n \geqslant 0$.*

(b) *A discrete left $\mathfrak{S}$-module $\mathcal{J}$ is fp-injective if and only if $\mathrm{Ext}^1_{\mathfrak{S}}(\mathcal{L}, \mathcal{J}) = 0$ for any finitely presented discrete left $\mathfrak{S}$-module $\mathcal{L}$, and if and only if $\mathrm{Ext}^{>0}_{\mathfrak{S}}(\mathcal{L}, \mathcal{J}) = 0$ for any such $\mathcal{L}$.*

(c) *The full subcategory of fp-injective discrete left $\mathfrak{S}$-modules is closed under extensions, the passages to the cokernels of injective morphisms, and infinite direct sums in* $\mathfrak{S}$–discr.

(d) *For any finitely presented discrete left $\mathfrak{S}$-module $\mathcal{L}$, the functor* $\mathrm{Hom}_{\mathfrak{S}}(\mathcal{L},-)$ *is exact on the exact category of fp-injective left $\mathfrak{S}$-modules.*

(e) *The functors $\mathcal{J} \longmapsto {}_{S_n}\mathcal{J}$ take short exact sequences of fp-injective discrete left $\mathfrak{S}$-modules to short exact sequences of fp-injective left S_n-modules.*

Proof. Part (a) is straightforward. To prove part (b), one notices that through any surjective morphism $\mathcal{E} \longrightarrow \mathcal{L}$ from a discrete left $\mathfrak{S}$-module $\mathcal{E}$ onto a finitely presented discrete left $\mathfrak{S}$-module $\mathcal{L}$ one can factorize a surjective morphism $\mathcal{K} \longrightarrow \mathcal{L}$ from a finitely presented discrete left $\mathfrak{S}$-module $\mathcal{K}$ onto $\mathcal{L}$. The first two assertions of part (c) follow from part (b), and the third one from part (a). Part (d) follows from part (b). Part (e) follows from part (d) applied to the discrete left $\mathfrak{S}$-modules $\mathcal{L} = S_n$. ☐

Lemma D.2.2. (a) *The supremum of injective dimensions of fp-injective discrete left $\mathfrak{S}$-modules (viewed as objects of the abelian category* $\mathfrak{S}$–discr*) is equal to the supremum of injective dimensions of fp-injective left S_n-modules (viewed as objects of the abelian category* S_n–mod*), taken over all $n \geqslant 0$.*

(b) *Assume that there is an integer $N \geqslant 0$ such that every left ideal in a ring S_n, for any $n \geqslant 0$, admits a set of generators of the carginality not exceeding $\aleph_N$. Then the injective dimension of any fp-injective discrete left $\mathfrak{S}$-module is not greater than $N+1$. In particular, if all the left ideals in the rings S_n admit at most countable sets of generators, then the injective dimension of any fp-injective discrete left $\mathfrak{S}$-module does not exceed* 1.

Proof. Part (a) follows from Lemma D.2.1(a,c,e). Part (b) is provided by [57, Proposition 2.3] together with part (a). ☐

Theorem D.2.3. (a) *The coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{S}\text{–}\mathsf{discr})$ *of the abelian category of discrete left $\mathfrak{S}$-modules is equivalent to the coderived category of the exact category of fp-injective left $\mathfrak{S}$-modules.*

(b) *If any fp-injective discrete left $\mathfrak{S}$-module has finite injective dimension, then the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{S}\text{–}\mathsf{discr})$ *is equivalent to the homotopy category of complexes of injective discrete left $\mathfrak{S}$-modules.*

(c) *The contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ *of the abelian category of left $\mathfrak{R}$-contramodules is equivalent to the contraderived category of the exact category of flat left $\mathfrak{R}$-contramodules.*

Proof. Parts (a-b) hold, because the category $\mathfrak{S}$–discr, being a Grothendieck abelian category, has enough injective objects; so it remains to use Lemma D.2.1(c) together with the dual versions of Propositions A.3.1(b) for part (a) and A.5.6 for part (b).

Since any left $\mathfrak{R}$-contramodule is a quotient contramodule of a flat (and even projective) one, and the class of flat left $\mathfrak{R}$-contramodules is closed under extensions and the passage to the kernels of surjective morphisms (see Lemmas D.1.4–D.1.6), in order to prove the assertion (b) it only remains to show that the class of flat left

$\mathfrak{R}$-contramodules is preserved by infinite products (see Proposition A.3.1(b)). The latter follows from the definition of flatness for left $\mathfrak{R}$-contramodules, the coherence condition on the rings R_n, and the next lemma. ☐

Lemma D.2.4. *For any family of left $\mathfrak{R}$-contramodules $\mathfrak{P}_\alpha$ and any $n \geqslant 0$, the two $\mathfrak{R}$-subcontramodules $\mathfrak{I}_n \times \prod_\alpha \mathfrak{P}_\alpha$ and $\prod_\alpha \mathfrak{I}_n \times \mathfrak{P}_\alpha$ coincide in $\prod_\alpha \mathfrak{P}_\alpha$.*

Proof. The former subcontramodule is obviously contained in the latter one; we have to prove the converse inclusion. For every $m \geqslant 0$, pick a finite set of generators $\bar r_m^\gamma$ of the right ideal $\mathfrak{I}_m/\mathfrak{I}_{m+1} \subset \mathfrak{R}/\mathfrak{I}_{m+1}$, and lift them to elements $r_m^\gamma \in \mathfrak{R}$. Then, for any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, any element of the subcontramodule $\mathfrak{I}_n \times \mathfrak{P} \subset \mathfrak{P}$ can be expressed in the form $\sum_{m \geqslant n}^\gamma r_m^\gamma p_m^\gamma$ with some $p_m^\gamma \in \mathfrak{P}$. In particular, any element p of the product $\prod_\alpha \mathfrak{I}_n \times \mathfrak{P}_\alpha$ can be presented in the form $p = \big(\sum_{m \geqslant n}^\gamma r_m^\gamma p_m^{\gamma,\alpha}\big)_\alpha$ with some elements $p_m^{\gamma,\alpha} \in \mathfrak{P}_\alpha$. Now one has $\big(\sum_{m \geqslant n}^\gamma r_m^\gamma p_m^{\gamma,\alpha}\big)_\alpha = \sum_{m \geqslant n}^\gamma r_m^\gamma((p_m^{\gamma,\alpha})_\alpha)$, which is an infinite sum of elements of $\prod_\alpha \mathfrak{P}_\alpha$ with the coefficient family still converging to zero in $\mathfrak{I}_n \subset \mathfrak{R}$, proving that p belongs to $\mathfrak{I}_n \times \prod_\alpha \mathfrak{P}_\alpha$. (Cf. [55, Lemmas 1.3.6–1.3.7].) ☐

A right $\mathfrak{R}$-module $\mathcal{N}$ is said to be discrete if every its element is annihilated by the ideal $\mathfrak{I}_n = \ker(\mathfrak{R} \to R_n)$ for $n \gg 0$, i. e., in the notation similar to the above, if $\mathcal{N} = \bigcup_n \mathcal{N}_{R_n}$. The abelian category of discrete right $\mathfrak{R}$-modules is denoted by $\mathsf{discr}\text{–}\mathfrak{R}$. For any associative ring S, any $\mathfrak{R}$-discrete S-$\mathfrak{R}$-bimodule $\mathcal{K}$, and any left S-module U, the abelian group $\operatorname{Hom}_S(\mathcal{N}, U)$ is naturally endowed with a left $\mathfrak{R}$-contramodule structure as the projective limit of the sequence of left R_n-modules $\operatorname{Hom}_S(\mathcal{N}, U) = \varprojlim_n \operatorname{Hom}_S(\mathcal{N}_{R_n}, U)$. For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, their *contratensor product* $\mathcal{N} \odot_\mathfrak{R} \mathfrak{P}$ is defined as the inductive limit of the sequence of abelian groups $\varinjlim_n \mathcal{N}_{R_n} \otimes_{R_n} \mathfrak{P}/(\mathfrak{I}_n \times \mathfrak{P})$.

To give a more fancy definition, the contratensor product $\mathcal{N} \odot_\mathfrak{R} \mathfrak{P}$ is the cokernel (of the difference) of the natural pair of abelian group homomorphisms $\mathcal{N} \otimes_\mathbb{Z} \mathfrak{R}[[\mathfrak{P}]] \rightrightarrows \mathcal{N} \otimes_\mathbb{Z} \mathfrak{P}$. Here one map is induced by the left contraaction map $\mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ and the other one is the composition $\mathcal{N} \otimes_\mathbb{Z} \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathcal{N}[\mathfrak{P}] \longrightarrow \mathcal{N} \otimes_\mathbb{Z} \mathfrak{P}$, where $\mathcal{N}[\mathfrak{P}]$ is the group of all finite formal linear combinations of elements of $\mathfrak{P}$ with the coefficients in $\mathcal{N}$, the former map to be composed is induced by the discrete right action map $\mathcal{N} \times \mathfrak{R} \longrightarrow \mathcal{N}$, and the latter map is just the obvious one. The contratensor product is a right exact functor of two arguments $\odot_\mathfrak{R} \colon \mathsf{discr}\text{–}\mathfrak{R} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathbb{Z}\text{–}\mathsf{mod}$.

For any $\mathfrak{R}$-discrete S-$\mathfrak{R}$-bimodule $\mathcal{K}$, any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, and any left S-module U, there are natural isomorphisms of abelian groups $\operatorname{Hom}_S(\mathcal{K} \odot_\mathfrak{R} \mathfrak{P}, U) \simeq \varprojlim_n \operatorname{Hom}_S(\mathcal{K}_{R_n} \otimes_{R_n} \mathfrak{P}/\mathfrak{I}_n \times \mathfrak{P}, U) \simeq \varprojlim_n \operatorname{Hom}_{R_n}(\mathfrak{P}/\mathfrak{I}_n \times \mathfrak{P}, \operatorname{Hom}_S(\mathcal{K}_{R_n}, U)) \simeq \operatorname{Hom}^\mathfrak{R}(\mathfrak{P}, \operatorname{Hom}_S(\mathcal{K}, U))$ (cf. [52, Sections 3.1.2 and 5.1.1]).

Given surjective ring homomorphisms $g \colon S \longrightarrow {}'S$ and $f \colon R \longrightarrow {}'R$, a left S-module M, and a right R-module N, we denote by ${}_{'S}M$ the submodule of all elements annihilated by the left action of $\ker(g)$ in M and by $N_{'R}$ the submodule of all elements annihilated by the right action of $\ker(f)$ in N. So ${}_{'S}M$ is the maximal left ${}'S$-submodule in M and $N_{'R}$ is the maximal right ${}'R$-submodule in N. Similarly, for any left R-module P we denote by ${}^{'R}P$ its maximal quotient left ${}'R$-module $P/\ker(f)P$.

Lemma D.2.5. *Assume that* $\ker(g)$ *is finitely generated as a left ideal in* S *and* $\ker(f)$ *is finitely generated as a right ideal in* R*. Let* K *be an* S*-*R*-bimodule such that its submodules* ${}_{'S}K$ *and* $K_{'R}$ *coincide; denote this* $'S$*-*$'R$*-subbimodule in* K *by* $'K$*. Then*

(a) *for any injective left* S*-module* J *there is a natural isomorphism of left* $'R$*-modules* ${}^{'R}\mathrm{Hom}_S(K,J) \simeq \mathrm{Hom}_{'S}({}'K, {}_{'S}J)$*;*

(b) *for any flat left* R*-module* F *there is a natural isomorphism of left* $'S$*-modules* ${}_{'S}(K \otimes_R F) \simeq {}'K \otimes_{'R} {}^{'R}F$*.*

Proof. Part (a): for any finitely presented right R-module E, there is a natural isomorphism $\mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(E,K),J) \simeq E \otimes_R \mathrm{Hom}_S(K,J)$. Taking $E = {}'R$, we get $\mathrm{Hom}_{'S}({}_{'S}K, {}_{'S}J) \simeq \mathrm{Hom}_S({}_{'S}K, J) = \mathrm{Hom}_S(K_{'R}, J) \simeq {}^{'R}\mathrm{Hom}_S(K,J)$.

Part (b): for any finitely presented left S-module E there is a natural isomorphism $\mathrm{Hom}_S(E,\ K \otimes_R F) \simeq \mathrm{Hom}_S(E,K) \otimes_R F$. Taking $E = {}'S$, we get $K_{'R} \otimes_{'R} {}^{'R}F \simeq K_{'R} \otimes_R F = {}_{'S}K \otimes_R F \simeq {}_{'S}(K \otimes_R F)$. ☐

The following definition of a dualizing complex for a pair of coherent noncommutative rings is a slight generalization of the one from [57, Section 4] (which is, in turn, a generalization of the one from the above Section B.4).

Let S be a left coherent ring and R be a right coherent ring. A complex of S-R-bimodules $D^\bullet$ is called a *dualizing complex* for the rings S and R if it satisfies the following conditions:

(i) the terms of the complex $D^\bullet$ are fp-injective left S-modules and fp-injective right R-modules;

(ii) the complex $D^\bullet$ is isomorphic, as an object of the coderived category of the exact category of S-fp-injective and R-fp-injective S-R-bimodules, to a finite complex of S-fp-injective and R-fp-injective S-R-bimodules;

(iii) the S-R-bimodules of cohomology $H^*(D^\bullet)$ of the complex $D^\bullet$ are finitely presented left S-modules and finitely presented right R-modules; and

(iv) the homothety maps $S \longrightarrow \mathrm{Hom}_{\mathsf{D}(\mathsf{mod}\text{-}R)}(D^\bullet, D^\bullet)$ and $R^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathsf{D}(S\text{-}\mathsf{mod})}(D^\bullet, D^\bullet[*])$ are isomorphisms of graded rings.

A dualizing complex in the sense of this definition satisfies the conditions of the definition of a dualizing complex in the sense of [57, Section 4] if and only if it is a finite complex of S-R-bimodules. Any dualizing complex in the sense of the above definition is isomorphic, as an object of the coderived category of S-fp-injective and R-fp-injective S-R-bimodules, to a dualizing complex in the sense of [57]. A dualizing complex in the sense of [57] satisfies the conditions of the definition of a dualizing complex of bimodules in the sense of our Appendix B if and only if it is a complex of S-injective and R-injective (rather than just fp-injective) bimodules. This is what is called a “strong dualizing complex” in [57, Section 3].

Lemma D.2.6. *Let* $D^\bullet$ *be a finite complex of* S*-fp-injective and* R*-fp-injective* S*-*R*-bimodules. Suppose that the subcomplexes* ${}_{'S}D^\bullet$ *and* $D^\bullet_{'R}$ *coincide in* $D^\bullet$*, and denote this complex of* $'S$*-*$'R$*-bimodules by* $'D^\bullet$*.*

(a) *Assume that the ring S is left coherent, the ring R is right coherent, all fp-injective left S-modules have finite injective dimensions, all fp-injective right R-modules have finite injective dimensions,* $\ker(g)$ *is finitely generated as a left ideal in S, and* $\ker(f)$ *is finitely generated as a right ideal in R. Then $'D^\bullet$ is a dualizing complex for the rings $'S$ and $'R$ whenever $D^\bullet$ is dualizing complex for the rings S and R.*

(b) *Assume that the ring S is left Noetherian, the ring R is right Noetherian, and the ideals* $\ker(g) \subset S$ *and* $\ker(f) \subset R$ *are nilpotent. Then $D^\bullet$ is a dualizing complex for the rings S and R whenever $'D^\bullet$ is a dualizing complex for the rings $'S$ and $'R$.*

Proof. Part (a): clearly, $'D^\bullet = {}_{'S}D^\bullet$ is a finite complex of fp-injective left $'S$-modules. Let $E^\bullet$ be a finite complex of injective left S-modules endowed with a quasi-isomorphism of complexes of S-modules $D^\bullet \longrightarrow E^\bullet$. The natural map $R \longrightarrow \operatorname{Hom}_S(D^\bullet, E^\bullet)$ is a quasi-isomorphism of finite complexes of flat left R-modules (see [57, Lemma 4.1(b)]) and therefore remains a quasi-isomorphism after taking the tensor product with $'R$ over R on the left, i. e., reducing modulo $\ker(f)$. By Lemma D.2.5, it follows that the map $'R \longrightarrow \operatorname{Hom}_{'S}('D^\bullet, {}_{'S}E^\bullet)$ is an quasi-isomorphism. It remains to notice that ${}_{'S}E^\bullet$ is a complex of injective left $'S$-modules and the induced map ${}_{'S}D^\bullet \longrightarrow {}_{'S}E^\bullet$ is a quasi-isomorphism of complexes of left $'S$-modules, as the functor $J \longmapsto {}_{'S}J^\bullet$ is exact on the exact category of fp-injective left S-modules.

To show that the left $'S$-modules of cohomology of the complex $'D^\bullet = D^\bullet_{'R}$ are finitely presented, we notice that it is quasi-isomorphic, as a complex of left $'S$-modules, to the complex $\operatorname{Hom}_{R^{\mathrm{op}}}(L^\bullet, D^\bullet)$ of right R-module homomorphisms from a left resolution $L^\bullet$ of the right R-module $'R$ by finitely generated projective R-modules into the complex $D^\bullet$. The argument finishes similarly to the proof of Lemma C.5.9(a). The proof of part (b) is also similar to that of Lemma C.5.9(b). $\square$

A complex of $\mathfrak{S}$-$\mathfrak{R}$-bimodules $\mathcal{D}^\bullet$ is called a *dualizing complex* for the projective systems of rings (S_n) and (R_n) if

(i) $\mathcal{D}^\bullet$ is a complex of fp-injective discrete left $\mathfrak{S}$-modules and a complex of fp-injective discrete right $\mathfrak{R}$-modules;
(ii) for every $n \geqslant 0$, the two subcomplexes ${}_{S_n}\mathcal{D}^\bullet$ and $\mathcal{D}^\bullet_{R_n}$ coincide in $\mathcal{D}^\bullet$; and
(iii) the latter subcomplex, denoted by ${}_{S_n}\mathcal{D}^\bullet = D^\bullet_n = \mathcal{D}^\bullet_{R_n} \subset \mathcal{D}^\bullet$, is a dualizing complex for the rings S_n and R_n.

The complex $\mathcal{D}^\bullet$ does not have to be bounded from either side, and neither do its subcomplexes $D^\bullet_n$; but the latter has to be isomorphic, as an object of the coderived category of S_n-fp-injective and R_n-fp-injective S_n-R_n-bimodules, to a finite (dualizing) complex of S_n-fp-injective and R_n-fp-injective S_n-R_n-bimodules.

Theorem D.2.7. *Assume that all fp-injective discrete left $\mathfrak{S}$-modules have finite injective dimensions. Then the choice of a dualizing complex $\mathcal{D}^\bullet$ for the projective systems of rings (S_n) and (R_n) induces an equivalence between the coderived category of discrete left $\mathfrak{S}$-modules* $\mathsf{D}^{\mathsf{co}}(\mathfrak{S}\text{–}\mathsf{discr})$ *and the contraderived category of left $\mathfrak{R}$-contramodules* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$. *The equivalence is provided by the derived functors*

$\mathbb{R}\operatorname{Hom}_{\mathfrak{S}}(\mathcal{D}^\bullet,-)$ *and* $\mathcal{D}^\bullet\odot^{\mathbb{L}}_{\mathfrak{R}}-$ *of the functor of discrete* $\mathfrak{S}$*-module homomorphisms from* $\mathcal{D}^\bullet$ *and the functor of contratensor product with* $\mathcal{D}^\bullet$ *over* $\mathfrak{R}$*.*

Proof. The constructions of the derived functors are based on Theorem D.2.3(b-c). By Lemmas D.2.5(a) and D.1.3 together with [57, Lemma 4.1(b)], applying the functor $\mathcal{J}^\bullet\longmapsto\operatorname{Hom}_{\mathfrak{S}}(\mathcal{D}^\bullet,\mathcal{J}^\bullet)\simeq\varprojlim_n\operatorname{Hom}_{S_n}({}_{S_n}\mathcal{D}^\bullet,{}_{S_n}\mathcal{J}^\bullet)$ to a complex of injective discrete left $\mathfrak{S}$-modules $\mathcal{J}^\bullet$ produces a complex of flat left $\mathfrak{R}$-contramodules.

By Lemma D.2.5(b) and [57, Lemma 4.1(a)], applying the functor $\mathfrak{F}^\bullet\longmapsto\mathcal{D}^\bullet\odot_{\mathfrak{R}}\mathfrak{F}^\bullet\simeq\varinjlim_n\mathcal{D}^\bullet_{R_n}\otimes_{R_n}\mathfrak{F}^\bullet/(\mathfrak{I}_n\times\mathfrak{F}^\bullet)$ to a complex of flat left $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ produces a complex of fp-injective discrete left $\mathfrak{S}$-modules. Let us show that the complex $\mathcal{D}^\bullet\odot_{\mathfrak{R}}\mathfrak{F}^\bullet$ is coacyclic whenever the complex $\mathfrak{F}^\bullet$ is contraacyclic.

Coacyclicity being preserved by inductive limits of sequences, it suffices to show that the complexes of left S_n-modules $D_n^\bullet\otimes_{R_n}\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet$ are coacyclic. The functor $\mathfrak{F}\longmapsto\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ preserves infinite products of left $\mathfrak{R}$-contramodules by Lemma D.2.4 and short exact sequences of flat left $\mathfrak{R}$-contramodules by Lemma D.1.4. Hence it takes contraacyclic complexes of flat left $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ to contraacyclic complexes of flat left R_n-modules $\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet$.

Now the exact category of flat left R_n-modules has finite homological dimension by Lemma D.2.2(a) and [57, Proposition 4.3], so any contraacyclic complex of flat left R_n-modules is absolutely acyclic by [52, Remark 2.1] (see also Section A.6), and the functor $D_n\otimes_{R_n}-$ clearly takes absolutely acyclic complexes of flat left R_n-modules to absolutely acyclic complexes of left S_n-modules. (Concerning the apparent ambiguity of our terminology, notice that the classes of complexes of flat objects contraacyclic or absolutely acyclic with respect to the abelian categories of arbitrary objects coincide with those of flat objects contraacyclic or absolutely acyclic with respect to the exact categories of flat objects by Proposition A.2.1.)

Furthermore, given a complex of injective discrete left $\mathfrak{S}$-modules $\mathcal{J}^\bullet$, in order to check that the adjunction morphism $\mathcal{D}^\bullet\odot_{\mathfrak{R}}\operatorname{Hom}_{\mathfrak{S}}(\mathcal{D}^\bullet,\mathcal{J}^\bullet)\longrightarrow\mathcal{J}^\bullet$ has a coacyclic cone, it suffices to show that so does the morphism $D_n^\bullet\otimes_{R_n}\operatorname{Hom}_{S_n}(D_n^\bullet,{}_{S_n}\mathcal{J}^\bullet)\longrightarrow{}_{S_n}\mathcal{J}^\bullet$ (in view of Lemma D.2.5, and because the class of coacyclic complexes is closed with respect to inductive limits of sequences). Given a complex of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$, choose a complex of injective discrete left $\mathfrak{S}$-modules $\mathcal{J}^\bullet$ endowed with a morphism of complexes of discrete left $\mathfrak{S}$-modules $\mathcal{D}^\bullet\odot_R\mathfrak{F}^\bullet\longrightarrow\mathcal{J}^\bullet$ with a coacyclic cone. This cone is a complex of fp-injective discrete left $\mathfrak{S}$-modules and, by Theorem D.2.3(a), it is also coacyclic with respect to the exact category of fp-injective discrete left $\mathfrak{S}$-modules. By Lemma D.2.1(e), it follows that the morphisms of complexes of left S_n-modules $D_n^\bullet\otimes_{R_n}\mathfrak{F}^\bullet/(\mathfrak{I}_n\times\mathfrak{F}^\bullet)\longrightarrow{}_{S_n}\mathcal{J}^\bullet$ have coacyclic cones, too. In order to check that the morphism of complexes of left $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet\longrightarrow\operatorname{Hom}_{\mathfrak{S}}(\mathcal{D}^\bullet,\mathcal{J}^\bullet)$ has a contraacyclic cone, it suffices to check that does the morphism $\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet\longrightarrow\operatorname{Hom}_{S_n}(D_n,{}_{S_n}\mathcal{J}^\bullet)$ (because the class of contraacyclic complexes is closed with respect to projective limits of sequences of surjective morphisms).

Finally, the pair of adjoint functors $\mathbb{R}\operatorname{Hom}_{S_n}(D_n^\bullet,-)$ and $D_n^\bullet\otimes^{\mathbb{L}}_{R_n}-$ between the coderived category of left S_n-modules and the contraderived category of left R_n-modules is connected by a chain of isomorphisms of pairs of adjoint functors with a similar pair of adjoint functors corresponding to a finite dualizing complex of S_n-R_n-bimodules. The latter pair of functors are mutually inverse equivalences by [57, Theorem 4.5]. $\square$

D.3. **Cotensor product of complexes of discrete modules.** Now we specialize from the setting of the previous section to the case when the two projective systems $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ and $S_0 \longleftarrow S_1 \longleftarrow S_2 \longleftarrow \cdots$ coincide and the rings $R_n = S_n$ are commutative. The morphisms $R_{n+1} \longrightarrow R_n$ are still assumed to be surjective, their kernel ideals to be finitely generated, and the rings R_n to be coherent. As above, we set $\mathfrak{R} = \varprojlim_n R_n$ and denote by $\mathfrak{I}_n \subset \mathfrak{R}$ the kernels of the surjective ring homomorphisms $\mathfrak{R} \longrightarrow R_n$.

A complex of $\mathfrak{R}$-modules $\mathcal{D}^\bullet$ is called a *dualizing complex* for the topological ring $\mathfrak{R}$ if it is a dualizing complex for the two coinciding projective systems of rings $(R_n) = (S_n)$ in the sense of the definition from the previous section, that is

- (i) $\mathcal{D}^\bullet$ is a complex of fp-injective discrete $\mathfrak{R}$-modules;
- (ii) for every $n \geqslant 0$, the maximal subcomplex of R_n-modules $D_n^\bullet = {}_{R_n}\mathcal{D}^\bullet$ in $\mathcal{D}^\bullet$ is isomorphic, as an object of the coderived category of the exact category of fp-injective R_n-modules, to a finite complex of fp-injective R_n-modules;
- (iii) the cohomology modules $H^*(D_n^\bullet)$ of the complex $D_n^\bullet$ are finitely presented R_n-modules; and
- (iv) the homothety map $R_n \longrightarrow \mathrm{Hom}_{\mathsf{D}(R_n\text{–}\mathsf{mod})}(D_n^\bullet, D_n^\bullet)$ is an isomorphism of graded rings.

One can conclude from Lemma D.2.6(a) that the property of a complex of $\mathfrak{R}$-modules to be a dualizing complex does not in fact depend on the choice of the projective system of discrete rings (R_n) satisfying the above-listed conditions and representing a given topological ring $\mathfrak{R}$.

The construction of the functor of *cotensor product* of complexes of discrete $\mathfrak{R}$-modules below in this section is to be compared with the constructions of the cotensor product of complexes of quasi-coherent sheaves on a Noetherian scheme with a dualizing complex in [15, Section B.2.5] and the semitensor product of complexes of modules in a relative situation in [57, Section 6].

The definition of the (*contramodule*) *tensor product* $\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q}$ of two $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ can be found in [55, Section 1.6]. The contramodule tensor product $\otimes^{\mathfrak{R}}$ is a right exact functor of two arguments defining an associative and commutative tensor category structure on the category $\mathfrak{R}$–contra with the unit object $\mathfrak{R}$. The functor $\otimes^{\mathfrak{R}}$ also commutes with infinite direct sums in $\mathfrak{R}$–contra. The tensor product of free $\mathfrak{R}$-contramodules is described by the rule $\mathfrak{R}[[X]] \otimes^{\mathfrak{R}} \mathfrak{R}[[Y]] \simeq \mathfrak{R}[[X \times Y]]$ for any sets X and Y.

Lemma D.3.1. *The tensor product of two flat $\mathfrak{R}$-contramodules $\mathfrak{F}$ and $\mathfrak{G}$ is a flat $\mathfrak{R}$-contramodule that can be constructed explicitly as $\mathfrak{F} \otimes^{\mathfrak{R}} \mathfrak{G} = \varprojlim_n (\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}) \otimes_{R_n} (\mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G})$. The contramodule tensor product functor is exact on the exact category of flat $\mathfrak{R}$-contramodules.*

Proof. The assertions of this Lemma hold true for any topological ring $\mathfrak{R}$ isomorphic to the topological projective limit of a sequence of surjective morphisms of commutative rings. Indeed, the functor defined by the above explicit formula takes flat $\mathfrak{R}$-contramodules to flat $\mathfrak{R}$-contramodules by Lemma D.1.3, and is exact on the exact category of flat $\mathfrak{R}$-contramodules by Lemma D.1.4 and because the projective limit of sequences of surjective maps is an exact functor. This functor is isomorphic to the functor $\otimes^{\mathfrak{R}}$ for flat $\mathfrak{R}$-contramodules $\mathfrak{F}$ and $\mathfrak{G}$, since both functors are right exact for flat $\mathfrak{R}$-contramodules and they are isomorphic for the free ones.

Moreover, there is a natural isomorphism of R_n-modules $(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q})/(\mathfrak{I}_n\times(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q})) \simeq (\mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}) \otimes_{R_n} (\mathfrak{Q}/\mathfrak{I}_n\times\mathfrak{Q})$ for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, because all the functors involved are right exact and there is such a natural isomorphism for all free or flat $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$. □

The tensor product of two complexes of $\mathfrak{R}$-contramodules $\mathfrak{P}^\bullet\otimes^{\mathfrak{R}}\mathfrak{Q}^\bullet$ is constructed by totalizing the bicomplex $\mathfrak{P}^i\otimes^{\mathfrak{R}}\mathfrak{Q}^j$ by taking infinite direct sums (in the category $\mathfrak{R}$–contra) along the diagonals. Since the reduction functor $\mathfrak{P}\longmapsto\mathfrak{I}_n\times\mathfrak{P}$, being left adjoint to the embedding functor R_n–mod $\longrightarrow$ $\mathfrak{R}$–contra, commutes with infinite direct sums, we know from Corollary D.1.7 that infinite direct sums of flat $\mathfrak{R}$-contramodules are flat and from Lemma D.3.1 that the tensor product of two complexes of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ can be constructed as the projective limit of the complexes of R_n-modules $(\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet)\otimes_{R_n}(\mathfrak{G}^\bullet/\mathfrak{I}_n\times\mathfrak{G}^\bullet)$.

Recall that all flat R_n-modules have finite projective dimensions provided that the ring R_n admits a dualizing complex and all fp-injective R_n-modules have finite injective dimensions [57, Proposition 4.3]; that all fp-injective R_n-modules have finite injective dimensions whenever all fp-injective discrete $\mathfrak{R}$-modules have finite injective dimensions; and that all fp-injective discrete $\mathfrak{R}$-modules have finite injective dimensions provided that, for all n, all ideals in the rings R_n are at most countably generated (Lemma D.2.2).

Lemma D.3.2. *Assume that all flat R_n-modules have finite projective dimensions. Then the tensor product $\mathfrak{F}^\bullet\otimes^{\mathfrak{R}}\mathfrak{G}^\bullet$ of two complexes of flat $\mathfrak{R}$-contramodules is contraacyclic whenever one of the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ is contraacyclic.*

Proof. First of all we recall that a complex of flat $\mathfrak{R}$-contramodules is contraacyclic as a complex in the abelian category of $\mathfrak{R}$-contramodules if and only if it is contraacyclic as a complex in the exact category of flat $\mathfrak{R}$-contramodules (Theorem D.2.3(c)). Furthermore, if the complex of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ is contraacyclic, then, by Lemmas D.1.4 and D.2.4, so is the complex of flat R_n-modules $\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet$. In view of the assumption of Lemma, by [52, Remark 2.1] it then follows that the latter complex of flat R_n-modules is absolutely acyclic. Hence the tensor product $(\mathfrak{F}^\bullet/\mathfrak{I}_n\times\mathfrak{F}^\bullet)\otimes_{R_n}(\mathfrak{G}^\bullet/\mathfrak{I}_n\times\mathfrak{G}^\bullet)$ is an absolutely acyclic complex of flat R_n-modules, too. Since the projective limit of sequences of surjective morphisms preserves contraacyclicity, we can conclude that the complex $\mathfrak{F}^\bullet\otimes^{\mathfrak{R}}\mathfrak{G}^\bullet$ is contraacyclic. □

The result of Lemma D.3.2 together with Theorem D.2.3(c) allows to define the left derived tensor product functor

$$\otimes^{\mathfrak{R},\mathbb{L}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$$

on the contraderived category of $\mathfrak{R}$-contramodules $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, providing it with the structure of an (associative and commutative) tensor triangulated category. In order to compute the contraderived category object $\mathfrak{P}^\bullet \otimes^{\mathfrak{R},\mathbb{L}} \mathfrak{Q}^\bullet$ for two complexes of $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, one picks two complexes of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ endowed with morphisms of complexes of contramodules with contraacyclic cones $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$, and puts $\mathfrak{P}^\bullet \otimes^{\mathfrak{R},\mathbb{L}} \mathfrak{Q}^\bullet = \mathfrak{F}^\bullet \otimes^{\mathfrak{R}} \mathfrak{G}^\bullet$. The one-term complex $\mathfrak{R}$ is the unit object of this tensor category structure.

Furthermore, there is a natural associativity isomorphism of discrete $\mathfrak{R}$-modules $\mathcal{M} \odot_{\mathfrak{R}} (\mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{Q}) \simeq (\mathcal{M} \odot_{\mathfrak{R}} \mathfrak{P}) \odot_{\mathfrak{R}} \mathfrak{Q}$ for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and any discrete $\mathfrak{R}$-module $\mathcal{M}$ (as one can see, e. g., from the second paragraph of the proof of Lemma D.3.1 together with the explicit construction of the contratensor product functor $\odot_{\mathfrak{R}}$ in Section D.2). The following lemma shows how to construct the left derived contratensor product functor

$$\odot_{\mathfrak{R}}^{\mathbb{L}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr})$$

providing the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr})$ with the structure of a triangulated module category over the triangulated tensor category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$.

Lemma D.3.3. *Assume that all flat R_n-modules have finite projective dimensions. Then the contratensor product $\mathcal{M}^\bullet \odot_{\mathfrak{R}} \mathfrak{F}^\bullet$ of a complex of discrete $\mathfrak{R}$-modules $\mathcal{M}^\bullet$ and a complex of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ is coacyclic provided that either the complex $\mathcal{M}^\bullet$ is coacyclic, or the complex $\mathfrak{F}^\bullet$ is contraacyclic.*

Proof. If the complex of $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ is contraacyclic then, according to the proof of Lemma D.3.2, the complexes of flat R_n-modules $\mathfrak{F}^\bullet/\mathfrak{I}_n{\times}\mathfrak{F}^\bullet$ are absolutely acyclic. It follows that the complexes of R_n-modules ${}_{R_n}\mathcal{M}^\bullet \otimes_{R_n} (\mathfrak{F}^\bullet/\mathfrak{I}_n{\times}\mathfrak{F}^\bullet)$ are absolutely acyclic, too; hence their inductive limit $\mathcal{M}^\bullet \odot_{\mathfrak{R}} \mathfrak{F}^\bullet$ is a coacyclic complex of discrete $\mathfrak{R}$-modules. The contratensor product of a complex of flat $\mathfrak{R}$-contramodules and a coacyclic complex of discrete $\mathfrak{R}$-modules is coacyclic since the functor of contratensor product with a complex of flat $\mathfrak{R}$-contramodules preserves infinite direct sums and exactness of short sequences of complexes of discrete $\mathfrak{R}$-modules. ☐

In order to compute the coderived category object $\mathcal{M}^\bullet \odot_{\mathfrak{R}}^{\mathbb{L}} \mathfrak{P}^\bullet$ for a complex of discrete $\mathfrak{R}$-modules $\mathcal{M}^\bullet$ and a complex of $\mathfrak{R}$-contramodules $\mathfrak{P}^\bullet$, one picks a complex of flat $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet$ endowed with a morphism of complexes of $\mathfrak{R}$-contramodules $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ with a contraacyclic cone, and puts $\mathcal{M}^\bullet \odot_{\mathfrak{R}}^{\mathbb{L}} \mathfrak{P}^\bullet = \mathcal{M}^\bullet \odot_{\mathfrak{R}} \mathfrak{F}^\bullet$.

Now suppose that we are given a dualizing complex $\mathcal{D}^\bullet$ for the topological ring $\mathfrak{R}$, and that all fp-injective discrete $\mathfrak{R}$-modules have finite injective dimensions. Then it is clear from the natural isomorphism $\mathcal{D}^\bullet \odot_{\mathfrak{R}} (\mathfrak{F}^\bullet \otimes^{\mathfrak{R}} \mathfrak{G}^\bullet) \simeq (\mathcal{D}^\bullet \odot_{\mathfrak{R}} \mathfrak{F}^\bullet) \odot_{\mathfrak{R}} \mathfrak{G}^\bullet$ that the equivalence of triangulated categories $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr})$ from Theorem D.2.7 transforms the derived contramodule tensor product functor $\otimes^{\mathfrak{R},\mathbb{L}}$ into the derived contratensor product functor $\odot_{\mathfrak{R}}^{\mathbb{L}}$.

The functor of *cotensor product* of complexes of discrete $\mathfrak{R}$-modules

$$\square_{\mathcal{D}^\bullet}\colon \mathsf{D^{co}}(\mathfrak{R}\text{–}\mathsf{discr}) \times \mathsf{D^{co}}(\mathfrak{R}\text{–}\mathsf{discr}) \longrightarrow \mathsf{D^{co}}(\mathfrak{R}\text{–}\mathsf{discr})$$

is obtained from the functor $\odot_{\mathfrak{R}}^{\mathbb{L}}$ or $\otimes^{\mathfrak{R},\mathbb{L}}$ by identifying one or both of the coderived categories $\mathsf{D^{co}}(R\text{–}\mathsf{discr})$ in the arguments of the functors with the contraderived category $\mathsf{D^{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ using the equivalence of Theorem D.2.7. Explicitly, this means that in order to compute the cotensor product of two object of the coderived category of discrete $\mathfrak{R}$-modules, one has to represent them by two complexes of injective discrete $\mathfrak{R}$-modules $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ and apply one of the formulas

$$\begin{aligned}\mathcal{I}^\bullet \square_{\mathcal{D}^\bullet} \mathcal{J}^\bullet &= \mathcal{D}^\bullet \odot_{\mathfrak{R}} (\operatorname{Hom}_{\mathfrak{R}}(\mathcal{D}^\bullet, \mathcal{I}^\bullet) \otimes^{\mathfrak{R}} \operatorname{Hom}_{\mathfrak{R}}(\mathcal{D}^\bullet, \mathcal{J}^\bullet)) \\ &\simeq \mathcal{I}^\bullet \odot_{\mathfrak{R}} \operatorname{Hom}_{\mathfrak{R}}(\mathcal{D}^\bullet, \mathcal{J}^\bullet) \simeq \mathcal{J}^\bullet \odot_{\mathfrak{R}} \operatorname{Hom}_{\mathfrak{R}}(\mathcal{D}^\bullet, \mathcal{I}^\bullet).\end{aligned}$$

Using the latter two formulas, it suffices that only one of the complexes of discrete $\mathfrak{R}$-modules $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ be a complex of injective discrete $\mathfrak{R}$-modules.

The functor of cotensor product of complexes of discrete $\mathfrak{R}$-modules $\square_{\mathcal{D}^\bullet}$ defines the structure of an (associative and commutative) tensor triangulated category on the coderived category $\mathsf{D^{co}}(\mathfrak{R}\text{–}\mathsf{discr})$. The dualizing complex $\mathcal{D}^\bullet$ is the unit object of this tensor category structure.

D.4. **Very flat and contraadjusted contramodules.** For the rest of the appendix we stick to a single projective system of associative rings and their surjective morphisms $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ with the projective limit $\mathfrak{R} = \varprojlim_n R_n$. In this section we assume that the rings R_n are commutative.

Then an $\mathfrak{R}$-contramodule $\mathfrak{F}$ is said to be *very flat* if the R_n-modules $\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ are very flat for all n (cf. Section C.3). It is clear from Corollary D.1.7 that any very flat $\mathfrak{R}$-contramodule is flat. In the case when the ideals $\mathfrak{I}_n/\mathfrak{I}_{n+1} \subset R_{n+1}$ are nilpotent and finitely generated, it follows from Lemma 1.6.8(b) that a flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ is very flat if and only if the R_0-module $\mathfrak{F}/\mathfrak{I}_0{\times}\mathfrak{F}$ is very flat.

Corollary D.4.1. *The class of very flat $\mathfrak{R}$-contramodules contains the projective $\mathfrak{R}$-contramodules and is closed under extensions and the passage to the kernels of surjective morphisms in $\mathfrak{R}$-contra. The projective dimension of any very flat $\mathfrak{R}$-contramodule (as an object of $\mathfrak{R}$–contra) does not exceed* 1.

Proof. Follows from Lemmas D.1.4–D.1.5 and Corollary D.1.10(a). □

An $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be *contraadjusted* if the functor $\operatorname{Hom}^{\mathfrak{R}}(-,\mathfrak{Q})$ takes short exact sequences of very flat $\mathfrak{R}$-contramodules to short exact sequences of abelian groups (or, equivalently, of $\mathfrak{R}$-modules, or of $\mathfrak{R}$-contramodules). It is clear from the first assertion of Corollary D.4.1 that an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is contraadjusted if and only if $\operatorname{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{Q}) = 0$ for any very flat $\mathfrak{R}$-contramodule $\mathfrak{F}$, and if and only if $\operatorname{Ext}^{\mathfrak{R},>0}(\mathfrak{F},\mathfrak{Q}) = 0$ for any very flat $\mathfrak{F}$.

It follows that the class of contraadjusted $\mathfrak{R}$-contramodules is closed under extensions and the passages to the cokernels of injective morphisms. Moreover, the second assertion of Corollary D.4.1 implies that any quotient $\mathfrak{R}$-contramodule of a contraadjusted $\mathfrak{R}$-contramodule is contraadjusted.

For the rest of this section we assume that (the rings R_n are commutative and) the ideals $\mathfrak{I}_n/\mathfrak{I}_{n+1} = \ker(R_{n+1} \to R_n) \subset R_{n+1}$ are finitely generated.

Lemma D.4.2. *Let R be a commutative ring and $I \subset R$ be a finitely generated ideal. Then the R-module IQ is contraadjusted for any contraadjusted R-module Q.*

Proof. The R-module IQ is a quotient module of a finite direct sum of copies of the R-module Q, and the class of contraadjusted R-modules is preserved by the passages to finite direct sums and quotients. ☐

Lemma D.4.3. *Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule for which the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ is an isomorphism. Then the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is contraadjusted whenever the R_n-modules $\mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ are contraadjusted for all n.*

Proof. Applying Lemma D.4.2 to the ring $R = R_{n+1}$ with the ideal $I = \mathfrak{I}_n/\mathfrak{I}_{n+1}$ and the R-module $Q = P_{n+1} = \mathfrak{P}/\mathfrak{I}_{n+1}{\times}\mathfrak{P}$, we conclude that the R_{n+1}-module $\ker(P_{n+1} \to P_n) = (\mathfrak{I}_n/\mathfrak{I}_{n+1})P_{n+1}$ is contraadjusted. Then it remains to make use of Lemma D.1.8. ☐

When the ideals $\mathfrak{I}_n/\mathfrak{I}_{n+1}$ are nilpotent, in view of Lemma 1.6.8(a) it suffices to require that the R_0-module $\mathfrak{P}/\mathfrak{I}_0{\times}\mathfrak{P}$ be contraadjusted in Lemma D.4.3.

Recall the above discussion of a ring R with ideals $I_n \subset R$ in Section D.1, and assume the ring R to be also commutative. The following construction plays a key role in our approach.

Lemma D.4.4. *Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule for which the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ is an isomorphism. Then the $\mathfrak{R}$-contramodule $\mathfrak{P}$ can be embedded into a contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{Q}$ in such a way that the quotient $\mathfrak{R}$-contramodule $\mathfrak{Q}/\mathfrak{P}$ is very flat.*

Proof. Let us consider $\mathfrak{P}$ as an R-module and embed it into a contraadjusted R-module K in such a way that the quotient R-module $F = K/\mathfrak{P}$ is very flat. Then, the R-module F being, in particular, flat, there are exact sequences of R_n-modules $0 \longrightarrow \mathfrak{P}/I_n\mathfrak{P} \longrightarrow K/I_nK \longrightarrow F/I_nF \longrightarrow 0$. Furthermore, we have surjective morphisms of R_n-modules $\mathfrak{P}/I_n\mathfrak{P} \longrightarrow \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ and the induced short exact sequences $0 \longrightarrow \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P} \longrightarrow K/(I_nK + \mathfrak{I}_n{\times}\mathfrak{P}) \longrightarrow F/I_nF \longrightarrow 0$.

Passing to the projective limits of these systems of short exact sequences, we obtain a natural morphism from the short exact sequence $0 \longrightarrow \varprojlim_n \mathfrak{P}/I_n\mathfrak{P} \longrightarrow \varprojlim_n K/I_nK \longrightarrow \varprojlim_n F/I_nF \longrightarrow 0$ to the short exact sequence $0 \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P} \longrightarrow \varprojlim_n K/(I_nK + \mathfrak{I}_n{\times}\mathfrak{P}) \longrightarrow \varprojlim_n F/I_nF \longrightarrow 0$. The map $\varprojlim_n \mathfrak{P}/I_n\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ is always surjective, since a surjective map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ (see Lemma D.1.1) factorizes through it. Hence the map $\varprojlim_n K/I_nK \longrightarrow \varprojlim_n K/(I_nK + \mathfrak{I}_n{\times}\mathfrak{P})$ is also surjective. Furthermore, the projective system $K/(I_nK + \mathfrak{I}_n{\times}\mathfrak{P})$ satisfies the condition of Lemma D.1.3, because $\mathfrak{I}_n{\times}\mathfrak{P} = I_n\mathfrak{P} + \mathfrak{I}_{n+1}{\times}\mathfrak{P}$.

Now put $\mathfrak{L} = \varprojlim_n K/I_nK$ and $\mathfrak{Q} = \varprojlim_n K/(I_nK + \mathfrak{I}_n{\times}\mathfrak{P})$, and also $\mathfrak{G} = \varprojlim_n F/I_nF$. By Lemma D.1.3, we have $\mathfrak{L}/\mathfrak{I}_n{\times}\mathfrak{L} = K/I_nK$ and $\mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G} = F/I_nF$.

Therefore, the R-contramodule $\mathfrak{G}$ is very flat. Besides, $\mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q} = K/(I_nK+\mathfrak{I}_n{\times}\mathfrak{P})$, and the natural map $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$ is an isomorphism. In addition, the R_n-modules K/I_nK are contraadjusted by Lemma 1.6.6(b). Finally, the R_n-module $\mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$ is a quotient module of a contraadjusted R_n-module $\mathfrak{L}/\mathfrak{I}_n{\times}\mathfrak{L} = K/I_nK$, and consequently is also contraadjusted. It remains to use Lemma D.4.3. □

Corollary D.4.5. *Any $\mathfrak{R}$-contramodule can be presented as the quotient contramodule of a very flat $\mathfrak{R}$-contramodule by a contraadjusted $\mathfrak{R}$-subcontramodule.*

Proof. Present an arbitrary $\mathfrak{R}$-contramodule $\mathfrak{P}$ as the quotient contramodule of a free $\mathfrak{R}$-contramodule $\mathfrak{H}$; apply Lemma D.4.4 to embed the kernel $\mathfrak{K}$ of the morphism $\mathfrak{H} \longrightarrow \mathfrak{P}$ into a contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{Q}$ so that the quotient contramodule $\mathfrak{F} = \mathfrak{Q}/\mathfrak{K}$ is very flat; consider the induced extensions $0 \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{P} \longrightarrow 0$ and $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$, and use Lemma D.1.5. It is important here that the map $\mathfrak{K} \longrightarrow \varprojlim_n \mathfrak{K}/\mathfrak{I}_n{\times}\mathfrak{K}$ is an isomorphism, since so is the map $\mathfrak{H} \longrightarrow \varprojlim_n \mathfrak{H}/\mathfrak{I}_n{\times}\mathfrak{H}$. (This is the same construction as in Lemma 1.1.3.) □

Corollary D.4.6. *Any $\mathfrak{R}$-contramodule can be embedded into a contraadjusted $\mathfrak{R}$-contramodule in such a way that the quotient contramodule is very flat.*

Proof. Use Corollary D.4.5 to present an arbitrary $\mathfrak{R}$-contramodule $\mathfrak{P}$ as the quotient $\mathfrak{R}$-contramodule of a very flat $\mathfrak{R}$-contramodule $\mathfrak{G}$ by a contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{K}$; apply Lemma D.4.4 to embed the $\mathfrak{R}$-contramodule $\mathfrak{G}$ into a contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{L}$ so that the quotient contramodule $\mathfrak{F} = \mathfrak{L}/\mathfrak{G}$ is very flat; and denote by $\mathfrak{N}$ the cokernel of the composition of injective morphisms of $\mathfrak{R}$-contramodules $\mathfrak{K} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{L}$. Now the $\mathfrak{R}$-contramodule $\mathfrak{N}$ is contraadjusted as the quotient contramodule of a contraadjusted $\mathfrak{R}$-contramodule (by a contraadjusted $\mathfrak{R}$-subcontramodule), and the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is embedded into $\mathfrak{N}$ with the cokernel $\mathfrak{F}$. (This construction comes from [52, proof of Lemma 9.1.2(a)].) □

Lemma D.4.7. *Let F be a flat R-module and $\mathfrak{Q}$ be an $\mathfrak{R}$-contramodule such that the R_n-modules F/I_nF are very flat, the map $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$ is an isomorphism, and the R_n-modules $\mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$ are contraadjusted. Then one has* $\operatorname{Ext}_R^{>0}(F,\mathfrak{Q}) = 0$.

Proof. This is a particular case of Lemma D.1.9. To show that its conditions are satisfied, one only has to recall that the ideals $\mathfrak{I}_n/\mathfrak{I}_{n+1} \subset R_{n+1}$ are finitely generated by our assumptions, and use Lemma D.4.2. □

Corollary D.4.8. *An $\mathfrak{R}$-contramodule is contraadjusted if and only if it is a contraadjusted R-module. In particular, the R_n-modules $\mathfrak{P}/I_n\mathfrak{P}$ and $\mathfrak{P}/\mathfrak{I}_n{\times}\mathfrak{P}$ are contraadjusted for any contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{P}$.*

Furthermore, one has $\operatorname{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\ \mathfrak{P}) \simeq \operatorname{Hom}_R(F,\mathfrak{P})$ *and* $\operatorname{Ext}^{\mathfrak{R},>0}(\varprojlim_n F/I_nF,\ \mathfrak{P}) = \operatorname{Ext}_R^{>0}(F,\mathfrak{P}) = 0$ *for any contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{P}$ and any flat R-module F for which the R_n-modules F/I_nF are very flat.*

Proof. According to the constructions of Lemma D.4.4 and Corollaries D.4.5–D.4.6, any contraadjusted $\mathfrak{R}$-contramodule $\mathfrak{P}$ can be obtained from the $\mathfrak{R}$-contramodules

$\mathfrak{Q}$ satisfying the conditions of Lemma D.4.7 using the operations of the passage to the cokernel of an injective morphism and the passage to a direct summand. This proves the equation $\mathrm{Ext}_R^{>0}(F,\mathfrak{Q})=0$, and consequently also the "only if" part of the first assertion; then the second assertion follows.

Conversely, suppose that an $\mathfrak{R}$-contramodule $\mathfrak{P}$ is a contraadjusted R-module. Let us present $\mathfrak{P}$ as the quotient contramodule of a (very) flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ by a contraadjusted $\mathfrak{R}$-subcontramodule $\mathfrak{Q}$. As we have proven, $\mathfrak{Q}$ is a contraadjusted R-module, and therefore so is the R-module $\mathfrak{F}$. Hence the R_n-modules $\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ are contraadjusted, and by Lemma D.4.3 it follows that $\mathfrak{F}$ is a contraadjusted $\mathfrak{R}$-contramodule. Now the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is contraadjusted as the quotient $\mathfrak{R}$-contramodule of a contraadjusted one.

The equation $\mathrm{Ext}^{\mathfrak{R},>0}(\varprojlim_n F/I_nF,\ \mathfrak{P})=0$ holds since the $\mathfrak{R}$-contramodule in the first argument is very flat, and to prove the isomorphism $\mathrm{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\ \mathfrak{P})\simeq \mathrm{Hom}_R(F,\mathfrak{P})$ one argues in the same way as in Corollary D.1.10(b). □

D.5. **Cotorsion contramodules.** A left $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is said to be *cotorsion* if the functor $\mathrm{Hom}^{\mathfrak{R}}(-,\mathfrak{Q})$ takes short exact sequences of flat left $\mathfrak{R}$-contramodules to short exact sequences of abelian groups. It follows from Lemmas D.1.4–D.1.6 that a left $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is cotorsion if and only if $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{Q})=0$ for any flat left $\mathfrak{R}$-contramodule $\mathfrak{F}$, and if and only if $\mathrm{Ext}^{\mathfrak{R},>0}(\mathfrak{F},\mathfrak{Q})=0$ for any flat $\mathfrak{F}$. Hence the class of cotorsion left $\mathfrak{R}$-contramodules is closed under extensions and the passages to the cokernels of injective morphisms and direct summands.

The *cotorsion dimension* of a left $\mathfrak{R}$-contramodule $\mathfrak{P}$ is defined as the supremum of the set of all integers d for which there exists a flat left $\mathfrak{R}$-contramodule $\mathfrak{F}$ such that $\mathrm{Ext}^{\mathfrak{R},d}(\mathfrak{F},\mathfrak{P})\neq 0$. Clearly, a nonzero $\mathfrak{R}$-contramodule is cotorsion if and only if its cotorsion dimension is equal to zero. Given a short exact sequence of $\mathfrak{R}$-contramodules $0\longrightarrow\mathfrak{K}\longrightarrow\mathfrak{L}\longrightarrow\mathfrak{M}\longrightarrow 0$ of the cotorsion dimensions k, l, and m, respectively, one has $l\leqslant\max(k,m)$, $k\leqslant\max(l,m+1)$, and $m\leqslant\max(l,k-1)$.

Lemma D.5.1. *Let $\mathfrak{Q}$ be a left $\mathfrak{R}$-contramodule for which the natural map $\mathfrak{Q}\longrightarrow\varprojlim_n\mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$ is an isomorphism. Set $Q_n=\mathfrak{Q}/\mathfrak{I}_n{\times}\mathfrak{Q}$, and assume that the left R_0-module Q_0 is cotorsion and the left R_n-modules $\ker(Q_{n+1}\to Q_n)$ are cotorsion for all $n\geqslant 0$. Then $\mathfrak{Q}$ is a cotorsion $\mathfrak{R}$-contramodule and a cotorsion left R-module.*

Proof. The equations $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{F},\mathfrak{Q})=0=\mathrm{Ext}^1_R(F,\mathfrak{Q})$ for any flat left $\mathfrak{R}$-contramodule $\mathfrak{F}$ and any flat left R-module F follow from Lemmas D.1.8 and D.1.9, respectively. □

For the rest of this section we assume that the rings R_n are commutative and Noetherian. The ring R is also presumed to be commutative.

Corollary D.5.2. *Let $\mathfrak{E}$ be a flat $\mathfrak{R}$-contramodule. Assume that the flat R_n-modules $E_n=\mathfrak{E}/\mathfrak{I}_n{\times}\mathfrak{E}$ are cotorsion for all $n\geqslant 0$. Then $\mathfrak{E}$ is a flat cotorsion $\mathfrak{R}$-contramodule and a cotorsion R-module.*

Proof. Since $\ker(E_{n+1}\to E_n)=(\mathfrak{I}_n/\mathfrak{I}_{n+1})E_{n+1}\simeq\mathfrak{I}_n/\mathfrak{I}_{n+1}\otimes_{R_{n+1}}E_{n+1}$, the assertions follow from Lemma D.5.1 in view of Lemma 1.6.4(a). □

Proposition D.5.3. *Any flat $\mathfrak{R}$-contramodule can be embedded into a flat cotorsion $\mathfrak{R}$-contramodule in such a way that the quotient $\mathfrak{R}$-contramodule is flat.*

Proof. In fact, we will even present a functorial construction of such an embedding of flat $\mathfrak{R}$-contramodules. It is based on the functorial construction of an embedding $G \longrightarrow \mathrm{FC}_S(G)$ of a flat module G over a commutative Noetherian ring S into a flat cotorsion module $\mathrm{FC}_S(G)$ with a flat quotient S-module $\mathrm{FC}_S(G)/G$, which was explained in Lemma 1.3.9. In addition, we will need the following lemma.

Lemma D.5.4. *Let T be a commutative Noetherian ring, $S = T/I$ be its quotient ring by an ideal $I \subset T$, and H be a flat T-module. Then there is a natural isomorphism of S-modules $\mathrm{FC}_T(H)/I\,\mathrm{FC}_T(H) \simeq \mathrm{FC}_S(H/IH)$ compatible with the natural morphisms $H \longrightarrow \mathrm{FC}_T(H)$ and $H/IH \longrightarrow \mathrm{FC}_S(H/IH)$.*

Proof. Since the ideal $I \subset T$ is finitely generated, the reduction functor $S \otimes_T -$ preserves infinite direct products. For any prime ideal $\mathfrak{q} \subset T$ that does not contain I and any $T_\mathfrak{q}$-module Q one has $Q/IQ = 0$; in particular, this applies to any $T_\mathfrak{q}$-contramodule. Now consider a prime ideal $\mathfrak{q} \supset I$ in T, and let $\mathfrak{p} = \mathfrak{q}/I$ be the related prime ideal in S. It remains to construct a natural isomorphism of S-modules $\widehat{H}_\mathfrak{q}/I\widehat{H}_q \simeq (\widehat{H/IH})_\mathfrak{p}$. Indeed, according to [55, proof of Lemma B.9.2] applied to the Noetherian ring T with the ideal $\mathfrak{q} \subset T$, the complete ring $\mathfrak{T} = \widehat{T}_\mathfrak{q}$, and the finitely generated T-module S, one has $S \otimes_T \widehat{H}_\mathfrak{q} \simeq \varprojlim_n S \otimes_T (H_\mathfrak{q}/\mathfrak{q}^n H_q) \simeq \varprojlim_n (S \otimes_T H)_\mathfrak{p}/\mathfrak{p}^n(S \otimes_T H)_\mathfrak{p}$, as desired. ☐

So let $\mathfrak{G}$ be a flat $\mathfrak{R}$-contramodule; set $G_n = \mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G}$. According to Lemma D.5.4, the flat cotorsion R_n-modules $\mathrm{FC}_{R_n}(G_n)$ naturally form a projective system satisfying the condition of Lemma D.1.3. Set $\mathfrak{C} = \varprojlim_n \mathrm{FC}_{R_n}(G_n)$; by Corollary D.5.2, the flat $\mathfrak{R}$-contramodule $\mathfrak{C}$ is cotorsion. The projective system of R_n-modules $\mathrm{FC}_{R_n}(G_n)/G_n$ also satisfies the condition of Lemma D.5.4, being the cokernel of a morphism of two projective systems that do; in view of the second assertion of Lemma 1.3.9, it follows that the $\mathfrak{R}$-contramodule $\mathfrak{F} = \varprojlim_n \mathrm{FC}_{R_n}(G_n)/G_n$ is flat. Finally, the short sequence $0 \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{F} \longrightarrow 0$ is exact, since $\varprojlim^1_n G_n = 0$. ☐

Corollary D.5.5. *A flat $\mathfrak{R}$-contramodule $\mathfrak{G}$ is cotorsion if and only if the R_n-modules $\mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G}$ are cotorsion for all $n \geqslant 0$. Any flat cotorsion $\mathfrak{R}$-contramodule is a cotorsion R-module.*

Proof. In view of Corollary D.5.2, we only have to show that the R_n-modules $\mathfrak{G}/\mathfrak{I}_n{\times}\mathfrak{G}$ are cotorsion for any flat cotorsion $\mathfrak{R}$-contramodule $\mathfrak{G}$. For this purpose, it suffices to apply the construction of Proposition D.5.3 to obtain a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{G} \longrightarrow 0$, where the $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat and the $\mathfrak{R}$-contramodule $\mathfrak{C}$, by construction, has the desired property. Now $\mathfrak{G}$ is an $\mathfrak{R}$-contramodule direct summand of $\mathfrak{C}$, since $\mathrm{Ext}^{\mathfrak{R},1}(\mathfrak{G}, \mathfrak{F}) = 0$ by assumption. ☐

From this point on and until the end of this section we assume that the Krull dimensions of the Noetherian commutative rings R_n are uniformly bounded by a constant D. Then it follows from the results of Section D.1 and Theorem 1.5.6 that the projective dimension of any flat $\mathfrak{R}$-contramodule does not exceed D. Therefore, the cotorsion dimension of any $\mathfrak{R}$-contramodule also cannot exceed D.

Lemma D.5.6. (a) *Any $\mathfrak{R}$-contramodule can be embedded into a cotorsion $\mathfrak{R}$-contramodule in such a way that the quotient $\mathfrak{R}$-contramodule is flat.*

(b *Any $\mathfrak{R}$-contramodule admits a surjective map onto it from a flat $\mathfrak{R}$-contramodule with the kernel being a cotorsion $\mathfrak{R}$-contramodule.*

Proof. Part (a): we will argue by decreasing induction in $0 \leqslant d \leqslant D$, showing that any $\mathfrak{R}$-contramodule $\mathfrak{P}$ can be embedded into an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ of cotorsion dimension $\leqslant d$ in such a way that the quotient $\mathfrak{R}$-contramodule $\mathfrak{Q}/\mathfrak{P}$ is flat.

Assume that we already know this for the cotorsion dimension $d+1$. Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule; pick a surjective morphism $\mathfrak{L} \longrightarrow \mathfrak{P}$ onto $\mathfrak{P}$ from a projective $\mathfrak{R}$-contramodule $\mathfrak{L}$. Denote the kernel of the morphism $\mathfrak{L} \longrightarrow \mathfrak{P}$ by $\mathfrak{K}$ and use the induction assumption to embed it into an $\mathfrak{R}$-contramodule $\mathfrak{N}$ of cotorsion dimension $\leqslant d+1$ so that the cokernel $\mathfrak{H} = \mathfrak{N}/\mathfrak{K}$ is flat.

Let $\mathfrak{G}$ be the fibered coproduct $\mathfrak{N} \sqcup_{\mathfrak{K}} \mathfrak{L}$; then the $\mathfrak{R}$-contramodule $\mathfrak{G}$ is flat as an extension of two flat $\mathfrak{R}$-contramodules $\mathfrak{H}$ and $\mathfrak{L}$. Now we use Proposition D.5.3 to embed $\mathfrak{G}$ into a flat cotorsion $\mathfrak{R}$-contramodule $\mathfrak{E}$ so that the cokernel $\mathfrak{F} = \mathfrak{E}/\mathfrak{G}$ is a flat $\mathfrak{R}$-contramodule. Then the cokernel $\mathfrak{Q}$ of the composition $\mathfrak{N} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{E}$ has cotorsion dimension $\leqslant d$. The $\mathfrak{R}$-contramodule $\mathfrak{P}$ embeds naturally into $\mathfrak{Q}$ with the cokernel $\mathfrak{F}$, so the desired assertion is proven.

We have also proven part (b) along the way (it suffices to notice that there is a natural surjective morphism $\mathfrak{G} \longrightarrow \mathfrak{P}$ with the kernel $\mathfrak{N}$); see also, e. g., the proof of Lemma 1.1.3. ☐

Corollary D.5.7. *Any cotorsion $\mathfrak{R}$-contramodule $\mathfrak{P}$ is also a cotorsion R-module. Besides, for any flat R-module F one has* $\operatorname{Hom}^{\mathfrak{R}}(\varprojlim_n F/I_nF,\ \mathfrak{P}) \simeq \operatorname{Hom}_R(F, \mathfrak{P})$ *and* $\operatorname{Ext}^{\mathfrak{R},>0}(\varprojlim_n F/I_nF,\ \mathfrak{P}) = \operatorname{Ext}_R^{>0}(F, \mathfrak{P}) = 0$.

Proof. Applying the construction of Lemma D.5.6(a) to the $\mathfrak{R}$-contramodule $\mathfrak{P}$, we obtain a short exact sequence of $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{F} \longrightarrow 0$ with a flat $\mathfrak{R}$-contramodule $\mathfrak{F}$ and an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ obtained from flat cotorsion $\mathfrak{R}$-contramodules using the operation of passage to the cokernel of an injective morphism at most D times. Then $\mathfrak{P}$ is a direct summand of $\mathfrak{Q}$, and since the class of cotorsion R-modules is preserved by the passages to the cokernels of injective morphisms and direct summands, the first assertion follows from Corollary D.5.5. The Ext-vanishing assertions now hold by the definitions, and the Hom isomorphism is obtained in the same way as in Corollaries D.1.10 and D.4.8. ☐

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