CONTRAMODULES

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Abstract. Contramodules are module-like algebraic structures endowed with infinite summation (or, occasionally, integration) operations satisfying natural axioms. Introduced originally by Eilenberg and Moore in 1965 in the case of coalgebras over commutative rings, contramodules experience a small renaissance now after being all but forgotten for three decades between 1970–2000. Here we present a review of various definitions and the most basic results related to contramodules (drawing mostly from our monographs and preprints [52, 54, 56, 57])—including contramodules over corings, topological associative rings, topological Lie algebras and topological groups, semicontramodules over semialgebras, and a "contra version" of the Bernstein–Gelfand–Gelfand category O. Several underived manifestations of the comodule-contramodule correspondence phenomenon are discussed.

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0. Introduction

0.0. Comodules over coalgebras or corings are familiar to many algebraists. Being asked about the natural ways to assign an abelian category to a coalgebra over a field, most people would probably mention the left comodules and the right comodules. This is indeed a good answer in the case of module categories over a *ring*, where considering the left modules or the right modules exhausts the basic possibilities. But the "left or right comodules" answer is strikingly *incomplete*, for in fact there are four such abelian categories. In addition to the left and right *comodules*, there are the left and right *contramodules*, which are no less basic, and very much analogous, or rather *dual-analogous* to, though different from, the comodules.

Contramodules were introduced, *on par with* comodules, in the classical 1965 AMS Memoir of Eilenberg and Moore [24], but little attention was paid. The 2003 monograph [14], which was supposed to contain state of the art on corings and comodules at the time, never mentioned contramodules. As it was noticed in the presentation [16], by the end of 2000's decade there still existed only *three* papers featuring contramodules that a MathSciNet search would bring: in addition to the Eilenberg and Moore's original memoir, there were the 1965 paper [63] by Vázquez García (in Spanish) and the rather remarkable 1970 paper of Barr [2]. The next mention of contramodules in any kind of mathematical literature that the present author is aware of comes in his own letters [51], written (in transliterated Russian) in 2000 and 2002.

The 2000–2002 letters were eventually noticed by two groups of authors [31, 15] and one of them got interested specifically in contramodules, so the number of relevant MathSciNet search hits grew a little by now (see, e. g., [17] and [66]). In the meantime, the present author's ideas about contramodules and the co-contra correspondence materialized in a sequence of long books, papers, and preprints [52, 54, 55, 56, 57]; there is also a presentation [58] and two later and shorter preprints [59, 60]. Still we feel that it may be difficult for a researcher or a student to navigate this corps of work without additional guidance. The present overview is intended to provide such guidance (the introductions to [57, 59, 60] may be also, hopefully, of some help).

0.1. A *coring* may be informally defined as a "coalgebra over a noncommutative ring" (or more precisely, a coalgebra object in the tensor category of bimodules over a ring). Eilenberg and Moore's original definition of contramodules [24] was formulated in the generality of coalgebras over commutative rings (i. e., coalgebra objects in the tensor category of modules), but the generalization to corings is straightforward. So a comodule over a coring can be described as "a comodule along the coalgebra variables in the coring and a module along the ring variables"; a contramodule over a coring is "a contramodule along the coalgebra and a module along the ring".

Another option is to consider "algebras over coalgebras" (or more precisely, algebra objects in the tensor categories of bicomodules); these are what we call *semialgebras*. The corresponding module objects are called the *semimodules* and the *semicontramodules*. Once again, a semimodule is "a module along the algebra variables in the semialgebra and a comodule along the coalgebra variables"; a semicontramodule is "a module along the algebra and a contramodule along the coalgebra".

In the maximal natural generality achieved in the monograph [52], one considers three-story towers of "algebras over coalgebras over rings", or *semialgebras over corings*. These still have four module categories attached to them, namely, the left and right semimodules and the left and right semicontramodules. That is the generality level in which the principal results of the main body of the book [52] are obtained.

There are many more "comodule-like" abelian categories in algebra than just comodules over corings or semimodules over semialgebras, though. Generally, just about every class of "discrete", "smooth", or "torsion" modules can be viewed as that of comodules "along a part of the variables" in one sense or another. Every such module category is typically accompanied by a much less familiar, but no less interesting, abelian category of contramodules. Hence one comes to the definitions of contramodules over topological rings and topological Lie algebras.

0.2. Generally, contramodules are modules with *infinite summation operations*, understood algebraically as operations of infinite arity subjected to natural axioms. Contramodules feel like being in some sense "complete", but carry no underlying topologies on them. Indeed, simple counterexamples [52, 61, 67] show that contramodule infinite summation operations *cannot* be interpreted as any kind of limit of finite partial sums (for all the finite partial sums of a particular series can vanish in a contramodule while the infinite sum does not).

Comodule categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives. Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives. The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projective objects are much less common than injective ones in "naturally appearing" abelian categories.

0.3. On the other hand, there is a remarkably simple case of contramodules over the adic completion of a Noetherian ring, where the forgetful functor from contramodules to modules is fully faithful, so the contramodule infinite summation operation can be *recovered* from the conventional module structure. In this setting, there is a different stream of literature, going back to the 1959 paper by Harrison [34], where contramodules were known and studied under different names (and neither the connection with the Eilenberg–Moore definition, *nor* the existence of the infinite summation operations were apparently ever noticed). The key modern term in this connection is the *MGM* (Matlis–Greenlees–May) *duality* [46, 23, 68, 59].

So (what we would call) *projective contramodules* over the ring of l-adic integers $\mathbb{Z}_l$ were studied in [34] in connection with the classification of (what Harrison called) *co-torsion abelian groups*. A definitive result in this direction was obtained by Enochs in [26], where (what are since known as) *flat cotorsion modules* over a Noetherian commutative ring were classified in terms of (what we call) projective contramodules over complete Noetherian local rings (see also [57, Theorem 1.3.8]). The argument in [26] was based on Matlis' classification of injective modules [45]. An equivalence between the categories of (what we would call) injective discrete modules and projective contramodules over $\mathbb{Z}_l$ was also noticed in [34].

As to the arbitrary (not necessarily projective) contramodules over $\mathbb{Z}_l$, these were studied under the name of *Ext-l-complete* abelian groups by Bousfield and Kan [12] and as *weakly l-complete* abelian groups by Jannsen [36]. Finally, contramodules over the adic completions of Noetherian (and certain other) rings became known as *cohomologically complete* modules in the papers of Yekutieli et al. [68, 69, 70]. All these names are derived from reflection over the basic fact that contramodules over $\mathbb{Z}_l$ and other adic completions are actually *always* adically complete, but *not* necessarily adically separated (as the above-mentioned counterexamples show).

0.4. In the author's own research, contramodules appear, first of all, in connection with the phenomenon of *comodule-contramodule correspondence* [58], which means *covariant* equivalences between appropriate categories of comodules and contramodules. The simplest example is the natural equivalence between the additive categories of injective left comodules and projective left contramodules over a coalgebra $\mathcal{C}$ over a field k. Attempting to extend this equivalence to *complexes* of left $\mathcal{C}$-comodules and left $\mathcal{C}$-contramodules using complexes of injective comodules and projective contramodules as resolutions, one discovers that unbounded acyclic complexes of contramodules are sometimes assigned to irreducible comodules and vice versa.

The same problem occurs in the more complicated situation of the correspondence between complexes of left semimodules and left semicontramodules over a semialgebra $\boldsymbol{\mathcal{S}}$ over $\mathcal{C}$ [27, 28, 64, 52]. Hence the *derived* co-contra correspondence is, generally speaking, an equivalence between *exotic*, rather than conventional, derived categories. The *coderived category* of $\mathcal{C}$-comodules is equivalent to the homotopy category of complexes of injective comodules, and similarly, the *contraderived category* of $\mathcal{C}$-contramodules is equivalent to the homotopy category of projective contramodules [54]. So the coderived category of left $\mathcal{C}$-comodules and the contraderived category of left $\mathcal{C}$-contamodules are naturally equivalent to each other, $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra})$ [52, Sections 0.2.6–7].

This phenomenon of equivalence between "derived categories of the second kind" is reproduced in a situation not involving comodules or contramodules in the papers [37, 42, 43, 60], where the homotopy categories of unbounded complexes of projective or injective modules over a ring are studied and an equivalence between them is sometimes obtained. An extension of this theory to quasi-coherent sheaves on nonaffine schemes was developed in the papers [49, 48, 55]; and an even more advanced version involving *contraherent cosheaves* was suggested in [57, Section 5.7].

0.5. In the relative situation of semimodules and semicontramodules over a semialgebra $\mathbf{S}$ over a coalgebra $\mathcal{C}$, the *derived semimodule-semicontramodule correspondence* is an equivalence between the *semiderived category* of left $\mathbf{S}$-semimodules and the *semi*(contra)*derived category* of left $\mathbf{S}$-semicontramodules,

$$\mathsf{D}^{\mathsf{si}}(\mathbf{S}\text{–}\mathsf{simod}) \simeq \mathsf{D}^{\mathsf{si}}(\mathbf{S}\text{–}\mathsf{sicntr}).$$

The former is a "mixture of the coderived category along the variables from $\mathcal{C}$ and the conventional derived category along the variables from $\mathbf{S}$ relative to $\mathcal{C}$" , while the latter is a "mixture of the contraderived category in the direction of $\mathcal{C}$ and the derived category in the direction of $\mathbf{S}$ relative to $\mathcal{C}$" [52, Corollary and Remark D.3.1]. A version of the derived semico-semicontra correspondence reproduced in a situation not involving contramodules can be found in [60, Section 5].

On the other hand, the coderived category of left comodules and the contraderived category of left contramodules over a *coring* $\mathcal{C}$ over a ring A are equivalent when the ring A has finite homological dimension (so the coderived and contraderived categories of A-modules are indistinguishable from their derived category). In other words, the coderived category of comodules and the contraderived category of contramodules are equivalent in the relative situation provided that the homological dimension "along the ring variables" is finite (when it is not, one needs a dualizing complex along the ring variables to be chosen).

Similarly, the conventional derived categories of comodules and contramodules may be equivalent in a relative situation mixing ring and coalgebra varables when the homological dimension "along the coalgebra variables" is finite. This includes, e. g., the case of quasi-compact semi-separated schemes, which are glued from the affine pieces by "a gluing procedure of finite homological dimension" (not exceeding the number of the pieces). The related version of derived co-contra correspondence for quasi-coherent sheaves and contraherent cosheaves was developed under the name of the "naïve co-contra correspondence" in [57, Chapter 4].

Furthermore, an affine Noetherian formal scheme is cut out from its ambient Noetherian scheme by a "cutting out procedure of finite homological dimension" [68, Corollaries 4.28 and 5.27]. This can be roughly explained by noticing that the formal completion of a scheme X along its closed subscheme Z consists in "subtracting from X the open complement U to Z in X", and U is a quasi-compact scheme whenever, say, X is affine and Z is defined by a finitely generated ideal. So the Matlis–Greenlees–May duality is, in fact, an *equivalence between the conventional derived categories of torsion modules and contramodules* over certain formal schemes [59].

0.6. We refrain from elaborating further upon the various derived categories and the derived co-contra correspondence in this paper, restricting ourselves mostly to the foregoing short discussion in the introduction. Indeed, it appears that the coderived and contraderived categories have attracted already some attention in the recent years, and a number of people have mastered the beginnings of the related techniques in one form or another. Besides, there is the presentation [58] discussing the philosophy of the derived co-contra correspondence.

Instead, we concentrate on the even more basic, and at the same time perhaps presently more counterintuitive, concepts of the abelian categories of contramodules. The simplest examples of the categories of contramodules over coalgebras over fields, the l-adic integers, the Virasoro algebra, and locally compact totally disconnected topological groups are discussed in Section 1. The key definitions of the categories of contramodules over topological rings, topological associative and Lie algebras, corings and semialgebras, and the category $\mathsf{O}^{\mathsf{ctr}}$ are introduced in Section 2. Tensor and Hom-like operations on the categories of contramodules and comodules and relations between various classes of objects adjusted to these operations (analogues and dual versions of the classes of flat, projective, and injective modules) are briefly considered in (the first half of) Section 3. Several *underived* co-contra correspondence constructions are discussed in the final Subsections 3.4–3.6.

Acknowledgement. I learned the definition of a contramodule from the hard copy of the Eilenberg–Moore 1965 AMS Memoir stored in the library of the Institute for Advanced Study, where I went in the Spring of 1999 to look for relevant literature after the discovery of (what are now known as) the coderived and contraderived categories shortly before. I was supported by an NSF grant at the time. Subsequently, almost all of my research on contramodules was done in Moscow, starting from the Summers of 2000 and 2002 and then in 2006–2014. I was partially supported by grants from EPDI, CRDF, INTAS, Pierre Deligne's Balzan prize, Simons Foundation, and several RFBR grants over the years. This paper was written when I was visiting the Technion in October 2014–March 2015, where I was supported in part by a fellowship from the Lady Davis Foundation. I am grateful to Dmitry Kaledin who suggested the idea of writing an overview on contramodules to me several years ago. I would like to thank Joseph Bernstein and Amnon Yekutieli for helpful discussions.

1. First Examples

1.1. **Contramodules over coalgebras over fields.** We start with recalling the largely familiar definitions. A coassociative *coalgebra* $\mathcal{C}$ with counit over a field k is a k-vector space endowed with a *comultiplication* map $\mu_\mathcal{C}\colon C \longrightarrow \mathcal{C}\otimes_k \mathcal{C}$ and a *counit* map $\varepsilon_\mathcal{C}\colon \mathcal{C}\longrightarrow k$ satisfying the equations dual to the equations on the multiplication and unit maps of an associative algebra with unit. Explicitly, the two compositions of the comultiplication map μ with the two maps $\mu\otimes\mathrm{id}_\mathcal{C}$ and $\mathrm{id}_\mathcal{C}\otimes\mu\colon \mathcal{C}\otimes_k\mathcal{C}\rightrightarrows\mathcal{C}\otimes_k\mathcal{C}\otimes_k\mathcal{C}$ induced by the comultiplication map

$$\mathcal{C}\longrightarrow\mathcal{C}\otimes_k\mathcal{C}\rightrightarrows\mathcal{C}\otimes_k\mathcal{C}\otimes_k\mathcal{C}$$

should be equal to each other, $(\mu\otimes\mathrm{id}_\mathcal{C})\circ\mu = (\mathrm{id}_\mathcal{C}\otimes\mu)\circ\mu$, and both the compositions of the comultiplication map with the two maps $\varepsilon\otimes\mathrm{id}_\mathcal{C}$ and $\mathrm{id}_\mathcal{C}\otimes\varepsilon\colon \mathcal{C}\otimes_k\mathcal{C}\rightrightarrows\mathcal{C}$

induced by the counit map ε

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_k \mathcal{C} \rightrightarrows \mathcal{C}$$

should be equal to the identity map, $(\varepsilon \otimes \mathrm{id}_{\mathcal{C}}) \circ \mu = \mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes \varepsilon) \circ \mu$.

A *left comodule* $\mathcal{M}$ over a coalgebra $\mathcal{C}$ is a k-vector space endowed with a *left coaction* map $\nu_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ satisfying the coassociativity and counitality equations. Explicitly, the two compositions of the coaction map ν with the two maps $\mu \otimes \mathrm{id}_{\mathcal{M}}$ and $\mathrm{id}_{\mathcal{C}} \otimes \nu \colon \mathcal{C} \otimes_k \mathcal{M} \rightrightarrows \mathcal{C} \otimes_k \mathcal{C} \otimes_k \mathcal{M}$ induced by the comultiplication and coaction maps

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M} \rightrightarrows \mathcal{C} \otimes_k \mathcal{C} \otimes_k \mathcal{M}$$

should be equal to each other, $(\mu \otimes \mathrm{id}_{\mathcal{M}}) \circ \nu = (\mathrm{id}_{\mathcal{C}} \otimes \nu) \circ \nu$, and the composition of the coaction map with the map $\varepsilon \otimes \mathrm{id}_{\mathcal{M}} \colon \mathcal{C} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ induced by the counit map $\varepsilon_{\mathcal{C}}$

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$$

should be equal to the identity map, $(\varepsilon \otimes \mathrm{id}_{\mathcal{M}}) \circ \nu = \mathrm{id}_{\mathcal{M}}$. A *right comodule* $\mathcal{N}$ over $\mathcal{C}$ is a k-vector space endowed with a right coaction map $\nu = \nu_{\mathcal{N}} \colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C}$ satisfying the similar equations, $(\nu \otimes \mathrm{id}_{\mathcal{C}}) \circ \nu = (\mathrm{id}_{\mathcal{N}} \otimes \mu) \circ \nu$

$$\mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C} \rightrightarrows \mathcal{N} \otimes_k \mathcal{C} \otimes_k \mathcal{C},$$

and $(\mathrm{id}_{\mathcal{N}} \otimes \varepsilon) \circ \nu = \mathrm{id}_{\mathcal{N}}$

$$\mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{C} \longrightarrow \mathcal{N}.$$

In order to arrive to the definition of a contramodule over $\mathcal{C}$, one only has to rewrite the most familiar definition of a module over an associative algebra in a slightly different form before quite formally dualizing it. Given an associative algebra A over k with the multiplication map $m \colon A \otimes_k A \longrightarrow A$ and the unit map $e \colon k \longrightarrow A$, one would usually define a left A-module M as a k-vector space endowed with a left action map $n \colon A \otimes_k M \longrightarrow M$ satisfying the associativity and unitality equations $n \circ (m \otimes \mathrm{id}_M) = n \circ (\mathrm{id}_A \otimes n)$

$$A \otimes_k A \otimes_k M \rightrightarrows A \otimes_k M \longrightarrow M$$

and $n \circ (e \otimes_k \mathrm{id}_M) = \mathrm{id}_M$

$$M \longrightarrow A \otimes_k M \longrightarrow M.$$

However, having a map n is the same thing as having a map

$$p \colon M \longrightarrow \mathrm{Hom}_k(A, M),$$

which then has to satisfy the associativity and unitality equations written in the form $\mathrm{Hom}(m, \mathrm{id}_M) \circ p = \mathrm{Hom}(\mathrm{id}_A, p) \circ p$

$$M \longrightarrow \mathrm{Hom}_k(A, M) \rightrightarrows \mathrm{Hom}_k(A \otimes_k A,\ M) \simeq \mathrm{Hom}_k(A, \mathrm{Hom}_k(A, M))$$

and $\mathrm{Hom}(e, \mathrm{id}_M) \circ p = \mathrm{id}_M$

$$M \longrightarrow \mathrm{Hom}_k(A, M) \longrightarrow M.$$

In this approach, the difference between the left and right modules lies in the way one identifies the the Hom from the tensor product $\mathrm{Hom}_k(A\otimes_k A,\, M)$ with the double Hom space $\mathrm{Hom}_k(A, \mathrm{Hom}_k(A, M))$: presuming the identification

$$\mathrm{Hom}_k(U \otimes_k V,\ W) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_k(U, W)) \tag{1}$$

leads to the definition of a *left* A-module, while identifying $\mathrm{Hom}_k(A \otimes_k A,\ N)$ with $\mathrm{Hom}_k(A, \mathrm{Hom}_k(A, N))$ by the rule

$$\mathrm{Hom}_k(U \otimes_k V,\ W) \simeq \mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \tag{2}$$

and writing the same equations produces the definition of a *right* A-module N.

Now we can formulate our main definition. A *left contramodule* $\mathfrak{P}$ over a coalgebra $\mathcal{C}$ is a k-vector space endowed with a *left contraaction* map

$$\pi_{\mathfrak{P}}\colon\ \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

satisfying the following *contraassociativity* and *contraunitality* equations. Firstly, the two compositions of the two maps $\mathrm{Hom}(\mu, \mathfrak{P})\colon\ \mathrm{Hom}_k(\mathcal{C}\otimes_k\mathcal{C}, \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$ and $\mathrm{Hom}(\mathcal{C}, \pi)\colon\ \mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$ induced by the comultiplication map $\mu = \mu_{\mathcal{C}}$ and the contraaction map $\pi = \pi_{\mathfrak{P}}$ with the contraaction map π

$$\mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})) \simeq \mathrm{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\ \mathfrak{P}) \rightrightarrows \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

should be equal to each other, $\pi \circ \mathrm{Hom}(\mu, \mathfrak{P}) = \pi \circ \mathrm{Hom}(\mathcal{C}, \pi)$, presuming the identification of $\mathrm{Hom}_k(\mathcal{C} \otimes_k \mathcal{C},\ \mathfrak{P}) \simeq \mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}))$ by the left rule (1). Secondly, the composition of the map $\mathrm{Hom}(\varepsilon, \mathfrak{P})\colon \mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P})$ induced by the counit map $\varepsilon = \varepsilon_{\mathcal{C}}$ with the contraaction map

$$\mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

should be equal to the identity map, $\pi \circ \mathrm{Hom}(\varepsilon, \mathfrak{P}) = \mathrm{id}_{\mathfrak{P}}$.

This definition can be found in [52, Section 0.2.4]; see also [54, Section 2.2] (the classical source is [24, Section III.5]). Using the identification by the right rule (2) instead of (1) produces the definition of a *right contramodule* over $\mathcal{C}$. The way to understand why (1) is the "left" rule and (2) is the "right" one lies in replacing a basic field k with a noncommutative ring; see Section 2.5 below.

1.2. **Basic properties of comodules and contramodules.** The simplest way to produce examples of contramodules is by applying the Hom functor to comodules in the first argument. Specifically, let $\mathcal{N}$ be a right comodule over a coalgebra $\mathcal{C}$ over k and V be a k-vector space. Then the vector space $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{N}, V)$ has a natural structure of left contramodule over $\mathcal{C}$. The left contraaction map $\pi_{\mathfrak{P}}$ is constructed by applying the functor $\mathrm{Hom}_k(-, V)$ to the right coaction map $\nu_{\mathcal{N}}$

$$\mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{N}, V)) \simeq \mathrm{Hom}_k(\mathcal{N} \otimes_k \mathcal{C},\ V) \longrightarrow \mathrm{Hom}_k(\mathcal{N}, V).$$

Let us denote by k–vect the category of k-vector spaces, by $\mathcal{C}$–comod the category of left $\mathcal{C}$-comodules, by comod–$\mathcal{C}$ the category of right $\mathcal{C}$-comodules, and by $\mathcal{C}$–contra the category of left $\mathcal{C}$-contramodules. The k-vector space of morphisms between left $\mathcal{C}$-comodules $\mathcal{L}$ and $\mathcal{M}$ will be denoted by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{M})$, and the vector space of morphisms between left $\mathcal{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ by $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{Q})$.

The category $\mathcal{C}\text{–}\mathsf{comod}$ is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow k\text{–}\mathsf{vect}$ is exact. To prove as much, one has to use the observation that the tensor product functor $\mathcal{C} \otimes_k -$ is exact, or more specifically, *left exact*. The forgetful functor also preserves inductive limits, so filtered inductive limits are exact functors in $\mathcal{C}\text{–}\mathsf{comod}$. The infinite products in $\mathcal{C}\text{–}\mathsf{comod}$ are not preserved by the forgetful functor (unless $\mathcal{C}$ is finite-dimensional) and are *not* exact in $\mathcal{C}\text{–}\mathsf{comod}$ in general.

In other words, the abelian category of $\mathcal{C}$-comodules satisfies Grothendieck's axioms Ab5 and Ab3*, but not in general Ab4* [33, N° 1.5]. It also admits a set of generators (for which one can take the finite-dimensional comodules), so it has enough injective objects [33, N° 1.10]. These can be explicitly described as follows.

A *cofree* left $\mathcal{C}$-comodule is a $\mathcal{C}$-comodule of the form $\mathcal{C} \otimes_k V$, where V is a k-vector space, with the left $\mathcal{C}$-coaction induced by the comultiplication in $\mathcal{C}$. For any left $\mathcal{C}$-comodule $\mathcal{L}$, there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\ \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{L}, V),$$

so cofree $\mathcal{C}$-comodules are injective. The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ embeds any left $\mathcal{C}$-comodule into a cofree one, so there are enough cofree $\mathcal{C}$-comodules. It follows that a $\mathcal{C}$-comodule is injective if and only if it is a direct summand of a cofree one [52, Sections 0.2.1, 1.1.2, and 5.1.5].

The category $\mathcal{C}\text{–}\mathsf{contra}$ is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow k\text{–}\mathsf{vect}$ is exact (here one has to observe that the functor $\mathrm{Hom}_k(\mathcal{C}, -)$ is exact, or more specifically, *right exact*). The forgetful functor also preserves infinite products, so infinite products are exact functors in $\mathcal{C}\text{–}\mathsf{contra}$. The infinite direct sums are *not* preserved by the forgetful functor (unless $\mathcal{C}$ is finite-dimensional) and are not exact in $\mathcal{C}\text{–}\mathsf{contra}$ in general. (However, *un*like the infinite products of $\mathcal{C}$-comodules, the infinite direct sums of $\mathcal{C}$-contramodules remain exact when the homological dimension of the category $\mathcal{C}\text{–}\mathsf{contra}$ does not exceed 1 [56, Remark 1.2.1].)

In other words, the abelian category of $\mathcal{C}$-contramodules satisfies Grothendieck's axioms Ab3 and Ab4*, but not in general Ab4 or Ab5*. It also has enough projective objects, which can be explicitly described as follows.

A *free* left $\mathcal{C}$-contramodule is a $\mathcal{C}$-contramodule of the form $\mathrm{Hom}_k(\mathcal{C}, V)$, where V is a k-vector space, with the left $\mathcal{C}$-contraaction constructed as explained in the beginning of this section. For any left $\mathcal{C}$-contramodule $\mathfrak{Q}$, there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_k(\mathcal{C}, V), \mathfrak{Q}) \simeq \mathrm{Hom}_k(V, \mathfrak{Q}),$$

so free $\mathcal{C}$-contramodules are projective. The contraaction map $\pi\colon \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ presents any $\mathcal{C}$-contramodule as the quotient contramodule of a free one, so there are enough free contramodules. It follows that a $\mathcal{C}$-contramodule is projective if and only if it is a direct summand of a free one [52, Sections 0.2.4, 3.1.2, and 5.1.5].

Notice that the class of injective $\mathcal{C}$-comodules is not only closed under infinite products in $\mathcal{C}\text{–}\mathsf{comod}$ (which holds in any abelian category), but also under infinite direct sums. Similarly, the class of projective $\mathcal{C}$-contramodules is not only closed under infinite direct sums in $\mathcal{C}\text{–}\mathsf{contra}$ (as in any abelian category), but also under infinite products. These observations are important for the theory of coderived and contraderived categories [54, Section 4.4, cf. Sections 3.7–3.8].

The correspondence assigning the free $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ to the cofree $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ is an equivalence between the additive categories of cofree left $\mathcal{C}$-comodules and free left $\mathcal{C}$-contramodules. Hence the additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules are equivalent, too [52, Sections 0.2.6 and 5.1.3] (see also [17] and [54, Sections 5.1–5.2]).

1.3. **Contramodules over the formal power series.** The linear duality functor identifies the category opposite to the category of conventional infinite-dimensional (otherwise known as discrete, or ind-finite-dimensional) vector spaces with the category of *linearly compact*, or pro-finite-dimensional, vector spaces. In particular, a coassociative coalgebra with counit is the same thing (up to inverting the arrows) as a linearly compact or pro-finite-dimensional topological associative algebra with unit. Notice that any coassociative coalgebra is the union of its finite-dimensional subcoalgebras [62, Section 2.2], so any topological associative algebra with a pro-finite-dimensional underlying topological vector space is a projective limit of finite-dimensional associative algebras.

In particular, one can idenfity coalgebras by the names of their dual linearly compact topological algebras. In this section we consider the simplest example of an infinite-dimensional coassociative coalgebra—the coalgebra $\mathcal{C}$ for which the dual topological algebra $\mathcal{C}^*$ is isomorphic to the algebra $k[[z]]$ of formal Taylor power series in one variable over a field k. Explicitly, $\mathcal{C}$ is the k-vector space with a countable basis consisting of the formal symbols 1^*, z^*, z^{2*}, …, z^{n*}, …, $n \in \mathbb{Z}_{\geqslant 0}$, with the comultiplication map given by the rule

$$\mu(z^{n*}) = \sum\nolimits_{i+j=n} z^{i*} \otimes z^{j*}$$

and the counit map $\varepsilon(1^*) = 1$, $\varepsilon(z^{n*}) = 0$ for $n > 0$.

Then a (left or right) $\mathcal{C}$-comodule $\mathcal{M}$ is the same thing as a k-vector space endowed with a *locally nilpotent* linear operator $z\colon \mathcal{M} \longrightarrow \mathcal{M}$. In other words, for any vector $m \in \mathcal{M}$ there must exist an integer $n \geqslant 1$ such that $z^n(m) = 0$ in $\mathcal{M}$. Indeed, given a linear operator z on $\mathcal{M}$ one would define the contraaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$ by the formula

$$\nu(m) = \sum\nolimits_{n=0}^{\infty} z^{n*} \otimes z^n(m),$$

and the local nilpotence condition is needed for the sum to be well-defined (i. e., finite) for every vector $m \in \mathcal{M}$.

A $\mathcal{C}$-contramodule structure on a k-vector space $\mathfrak{P}$ is, by the definition, the datum of a k-linear map $\pi\colon \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the contraassociativity and contraunitality axioms. Having such a map is the same thing as the following *infinite summation operation* being defined in $\mathfrak{P}$. For every sequence of vectors p_0, p_1, p_2 … $\in \mathfrak{P}$ there should be given a vector denoted figuratively by

$$\sum\nolimits_{n=0}^{\infty} z^n p_n \in \mathfrak{P}.$$

This infinitary operation in $\mathfrak{P}$ should satisfy the equations of linearity

$$\sum\nolimits_{n=0}^{\infty} z^n(ap_n + bq_n) = a\sum\nolimits_{n=0}^{\infty} z^n p_n + b\sum\nolimits_{n=0}^{\infty} z^n q_n,$$

contraassociativity

$$\sum\nolimits_{i=0}^{\infty} z^i\left(\sum\nolimits_{j=0}^{\infty} z^j p_{ij}\right) = \sum\nolimits_{n=0}^{\infty} z^n\left(\sum\nolimits_{i+j=n} p_{ij}\right),$$

and unitality

$$\sum\nolimits_{n=0}^{\infty} z^n p_n = p_0 \quad \text{when } p_1 = p_2 = p_3 = \dots = 0 \text{ in } \mathfrak{P}$$

for any p_n, q_n, $p_{ij} \in \mathfrak{P}$ and a, $b \in k$. Notice that in the main (middle) equation the first three summation signs denote the contramodule infinite summation operation, while the fourth one is the conventional finite sum of elements of a vector space [52, Section A.1.1].

As we will see below in Section 1.6, the $\mathcal{C}$-contramodule structure on a vector space $\mathfrak{P}$ is in fact determined by a single linear operator $z\colon \mathfrak{P} \longrightarrow \mathfrak{P}$,

$$z(p) = 1\cdot 0 + z\cdot p + z^2\cdot 0 + z^3\cdot 0 + \cdots$$

However, *un*like for the comodules, for contramodules over more complicated coalgebras the similar statement is, of course, no longer true.

1.4. **Contramodules over the l-adic integers.** A left module $\mathcal{M}$ over a topological ring $\mathfrak{R}$ is called *discrete* if the action map $\mathfrak{R}\times\mathcal{M} \longrightarrow \mathcal{M}$ is continuous in the discrete topology of $\mathcal{M}$ and the given topology of $\mathfrak{R}$. In other words, this means that the annihilator of every element of $\mathcal{M}$ must be an open left ideal in $\mathfrak{R}$. Discrete left $\mathfrak{R}$-modules form an abelian category, which we denote by $\mathfrak{R}$–discr.

The discussion of topological algebras dual to coalgebras in the previous section ignored one point which we now have to clarify. Given a coassociative coalgebra $\mathcal{C}$ over k, one can define the multiplication on the dual vector space $\mathcal{C}^*$ in two approximately equally natural ways which differ by the passage to the opposite algebra, i. e., switching the left and right arguments of the product map. Let us make the choice of defining the multiplication on $\mathcal{C}^*$ in in such a way that left $\mathcal{C}$-comodules acquire natural structures of left $\mathcal{C}^*$-modules and right $\mathcal{C}$-comodules become right $\mathcal{C}^*$-modules. Explicitly, this means applying the formula

$$\langle fg, c\rangle = \langle f, c_{(2)}\rangle\langle g, c_{(1)}\rangle$$

where $\langle\ ,\ \rangle$ denotes the natural pairing $\mathcal{C}^*\times\mathcal{C} \longrightarrow k$ and $c \longmapsto c_{(1)}\otimes c_{(2)}$ is Sweedler's symbolic notation for the comultiplication map μ [62, Section 1.2].

Then the category of left $\mathcal{C}$-comodules can be described as the full subcategory in the category of left $\mathcal{C}^*$-modules $\mathcal{C}^*$–mod consisting precisely of those $\mathcal{C}^*$-modules that are discrete with respect to the topology of $\mathcal{C}^*$. Similarly, a right $\mathcal{C}$-comodule is the same thing as a discrete right $\mathcal{C}^*$-module [62, Section 2.1].

Now the explicit description of contramodules over the coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[t]]$ given in the previous section raises the question about defining contramodules over topological rings other than pro-finite-dimensional algebras over fields. The most close analogues of the rings $k[[t]]$ being the rings of l-adic integers $\mathbb{Z}_l$, they are the natural starting point of the desired generalization (whose full development we postpone until Sections 2.1–2.3).

So let l be a prime number. Let us start with mentioning that a discrete module over the topological ring of l-adic integers $\mathbb{Z}_l$ is the same thing as an l-primary abelian group, i. e., an abelian group where the order of every element is a power of l. The category $\mathbb{Z}_l$–discr is abelian with exact functors of filtered inductive limits, which are also preserved by the embedding functor $\mathbb{Z}_l\text{–}\mathsf{discr} \longrightarrow \mathsf{Ab}$ into the category of abelian groups. The infinite products in $\mathbb{Z}_l$–discr are not preserved by the forgetful functor and *not* exact. In other words, the category $\mathbb{Z}_l$–discr satisfies Ab5 and Ab3*, but not Ab4*. It has enough injective objects, but no nonzero projectives. The injective discrete $\mathbb{Z}_l$-modules are precisely the direct sums of copies of the group $\mathbb{Q}_l/\mathbb{Z}_l$.

A $\mathbb{Z}_l$*-contramodule* $\mathfrak{P}$ is an abelian group endowed with the following infinite summation operation. For any sequence of elements $p_0, p_1, p_2 \ldots \in \mathfrak{P}$ an element denoted symbolically by

$$\sum\nolimits_{n=0}^{\infty} l^n p_n \in \mathfrak{P}$$

should be defined. This infinitary operation should satisfy the equations of additivity

$$\sum\nolimits_{n=0}^{\infty} l^n(p_n + q_n) = \sum\nolimits_{n=0}^{\infty} l^n p_n + \sum\nolimits_{n=0}^{\infty} l^n q_n,$$

contraassociativity

$$\sum\nolimits_{i=0}^{\infty} l^i \left(\sum\nolimits_{j=0}^{\infty} l^j p_{ij}\right) = \sum\nolimits_{n=0}^{\infty} l^n \left(\sum\nolimits_{i+j=n} p_{ij}\right),$$

and unitality + compatibility with the abelian group structure

$$\sum\nolimits_{n=0}^{\infty} l^n p_n = p_0 + p_1 + \cdots + p_1 \ (l \text{ summands } p_1) \ \text{ when } p_2 = p_2 = p_3 = \cdots = 0$$

for any elements p_n, q_n, and $p_{ij} \in \mathfrak{P}$.

For any l-primary abelian group $\mathcal{M}$ and abelian group V, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)$ has a natural $\mathbb{Z}_l$-contramodule structure provided by the rule

$$\left(\sum\nolimits_{n=0}^{\infty} l^n p_n\right)(m) = \sum\nolimits_{n=0}^{\infty} p_n(l^n m)$$

for any $p_n \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)$ and $m \in \mathcal{M}$. The category $\mathbb{Z}_l$–contra of $\mathbb{Z}_l$-contramodules is abelian and the forgetful functor $\mathbb{Z}_l\text{–}\mathsf{contra} \longrightarrow \mathsf{Ab}$ is exact. As we will see in Section 1.6, the forgetful functor is fully faithful. It preserves infinite products, but *not* infinite direct sums. Both the infinite direct sums and infinite products are exact functors in $\mathbb{Z}_l$–contra. In other words, the category $\mathbb{Z}_l$–discr satisfies Ab4 and Ab4* (but *not* Ab5 or Ab5*). It has enough projective objects, but no injectives.

The *free* $\mathbb{Z}_l$*-contramodule* generated by a set X is the set $\mathbb{Z}_l[[X]]$ of all infinite formal linear combinations $\sum_{x\in X} a_x x$ of elements of X with the coefficients $a_x \in \mathbb{Z}_l$ such that for every $n \geqslant 1$ all but a finite number of a_x are divisible by l^n in $\mathbb{Z}_l$. Notice that any formal linear combination satisfying this condition is, in fact, supported in an at most countable subset in X. As we will see below in Sections 2.1–2.2, for any $\mathbb{Z}_l$-contramodule $\mathfrak{P}$ the group of all $\mathbb{Z}_l$-contramodule morphisms $\mathbb{Z}_l[[X]] \longrightarrow \mathfrak{P}$ is isomorphic to the group $\mathfrak{P}^X$ of arbitrary maps of sets $X \longrightarrow \mathfrak{P}$. The classes of free and projective $\mathbb{Z}_l$-contramodules coincide.

The additive categories of injective discrete $\mathbb{Z}_l$-modules and projective $\mathbb{Z}_l$-contramodules are equivalent; the equivalence is provided by the functors $\mathcal{M} \longmapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, \mathcal{M})$ and $\mathfrak{P} \longmapsto \mathbb{Q}_l/\mathbb{Z}_l \otimes_{\mathbb{Z}} \mathfrak{P}$ [34, Proposition 2.1]. In particular, one has

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, \bigoplus\nolimits_X \mathbb{Q}_l/\mathbb{Z}_l) \simeq \mathbb{Z}_l[[X]] \quad \text{and} \quad \mathbb{Q}_l/\mathbb{Z}_l \otimes_Z \mathbb{Z}_l[[X]] \simeq \bigoplus\nolimits_X \mathbb{Q}_l/\mathbb{Z}_l.$$

1.5. **Counterexamples.** For any topological ring $\mathfrak{R}$, one can compute infinite products in the abelian category $\mathfrak{R}\text{–}\mathsf{discr}$ in the following way. Let $\mathcal{M}_\alpha$ be a family of discrete left $\mathfrak{R}$-modules; denote by M their product in the abelian category of arbitrary $\mathfrak{R}$-modules. Then the product of the family of objects $\mathcal{M}_\alpha$ in the category $\mathfrak{R}\text{–}\mathsf{discr}$ can be obtained as an $\mathfrak{R}$-submodule $\mathcal{M} \subset M$ consisting precisely of all the elements $m \in M$ whose annihilators in $\mathfrak{R}$ are open left ideals.

In particular, this provides a rule for computing infinite products in the abelian categories $\mathcal{C}\text{–}\mathsf{comod}$ of comodules over coalgebras over fields. Another way to formulate such a rule is as follows. In any abelian category, infinite products are left exact functors; in other words, they preserve kernels of morphisms. Since any $\mathcal{C}$-comodule can be presented as the kernel of a morphism of cofree $\mathcal{C}$-comodules, it suffices to know what the products of families of cofree $\mathcal{C}$-comodules are. The latter are easily seen to be given by the formula $\prod_\alpha \mathcal{C} \otimes_k V_\alpha = \mathcal{C} \otimes_k \prod_\alpha V_\alpha$.

Similarly, in order to compute the infinite direct sum of a family of objects in $\mathcal{C}\text{–}\mathsf{contra}$, one can present these as the cokernels of morphisms of free $\mathcal{C}$-contramodules. Since any $\mathcal{C}$-contramodule can be obtained as such a cokernel and the infinite direct sums preserve cokernels, it remains to use the formula $\bigoplus_\alpha \mathrm{Hom}_k(\mathcal{C}, V_\alpha) = \mathrm{Hom}_k(\mathcal{C}, \bigoplus_\alpha V_\alpha)$ for the direct sum of a family of free $\mathcal{C}$-contramodules.

Let us return to the example of the coalgebra $\mathcal{C}$ dual to the topological algebra of formal power series $k[[z]]$ considered in Section 1.3. Viewed as a discrete module over the algebra $\mathcal{C}^* = k[[z]]$, the coalgebra $\mathcal{C}$ can be identified with the quotient module $k((z))/k[[z]]$ of the $k[[z]]$-module of Laurent series $k((z))$ by its submodule $k[[z]]$. Consider the family of discrete $k[[z]]$-modules $z^{-n}k[[z]]/k[[z]]$, $n = 1, 2, \ldots$ They can be included into short exact sequences of discrete $k[[z]]$-modules

$$0 \longrightarrow z^{-n}k[[z]]/k[[z]] \longrightarrow k((z))/k[[z]] \longrightarrow k((z))/z^{-n}k[[z]] \longrightarrow 0.$$

Passing to the infinite product of these short exact sequences in the category $\mathcal{C}\text{–}\mathsf{comod}$ over all $n \geqslant 1$, one discovers that the map $k((z))/k[[z]] \otimes_k \prod_n k = \prod_n k((z))/k[[z]] \longrightarrow \prod_n k((z))/z^{-n}k[[z]] = k((z))/k[[z]] \otimes_k \prod_n kz^{-n}$ is not surjective, as, e. g., the vector $(z^{-n-1})_n \in \prod_n k((z))/z^{-n}k[[z]]$ does not belong to its image. One also computes the infinite product $\prod_n z^{-n}k[[z]]/k[[z]]$ in the category $\mathcal{C}\text{–}\mathsf{comod}$ as isomorphic to the inductive limit $\varinjlim_m \big(\prod_{n=1}^m z^{-n}k[[z]]/k[[z]] \times \prod_{n=m+1}^\infty z^{-m}k[[z]]/k[[z]]\big)$.

Now consider the family of $\mathcal{C}$-contramodules $k[[z]]/z^n k[[z]]$, $n = 1, 2 \ldots$ They can be viewed as parts of the short exact sequences of $\mathcal{C}$-contramodules

$$0 \longrightarrow z^n k[[z]] \longrightarrow k[[z]] \longrightarrow k[[z]/z^n k[[z]] \longrightarrow 0.$$

Passing to the infinite direct sum of these short exact sequences in the category $\mathcal{C}$–contra over all $n \geqslant 1$, one finds out that the map $\mathrm{Hom}_k(\mathcal{C}, \bigoplus_n kz^n) = \bigoplus_n z^n k[[z]] \longrightarrow \bigoplus_n k[[z]] = \mathrm{Hom}_k(\mathcal{C}, \bigoplus_n k)$ is injective. Its cokernel $\mathfrak{P} = \bigoplus_n k[[z]]/z^n k[[z]] \in \mathcal{C}$–$\mathsf{contra}$ is the $\mathcal{C}$-contramodule that we are interested in.

Let us start with introducing a more careful notation. Let $\mathfrak{E}$ denote the free $\mathcal{C}$-contramodule generated by a k-vector space E with a countable basis $e_1, e_2, e_3, \ldots$ Explicitly, $\mathfrak{E}$ is the set of all formal linear combinations $\sum_{n=1}^{\infty} a_n(z)e_n$, where the sequence of formal power series $a_n(z) \in k[[z]]$ converges to zero in the topology of $k[[z]]$. The $k[[z]]$-contramodule infinite summation operations on $\mathfrak{E}$ are defined in the obvious way. Let $\mathfrak{F}$ be the similar free $\mathcal{C}$-contramodule generated by a k-vector space F with a basis $f_1, f_2, f_3, \ldots$ Define a morphism of $\mathcal{C}$-contramodules $\mathfrak{F} \longrightarrow \mathfrak{E}$ by the rule $\sum_{n=1}^{\infty} b_n(z)f_n \longmapsto \sum_{n=1}^{\infty} z^n b_n(z)e_n$. Clearly, this morphism is injective; denote its cokernel by $\mathfrak{P} = \mathfrak{E}/\mathfrak{F}$.

Set $p_n \in \mathfrak{P}$ to be the images of the elements $e_n \in \mathfrak{E}$ under the surjective morphism $\mathfrak{E} \longrightarrow \mathfrak{P}$. Then the infinite sum $p = \sum_{n=1}^{\infty} z^n p_n$ is a nonzero vector in $\mathfrak{P}$, since the element $\sum_{n=1}^{\infty} z^n e_n$ does not belong to $\mathfrak{F} \subset \mathfrak{E}$ (there being no element $\sum_{n=1}^{\infty} f_n$ in $\mathfrak{F}$). On the other hand, every finite partial sum $zp_1 + z^2p_2 + \cdots + z^np_n = zp_1 + \cdots + z^np_n + z^{n+1} \cdot 0 + \cdots$ vanishes in $\mathfrak{P}$, the finite sum $ze_1 + \cdots + z^ne_n$ being the image of the vector $f_1 + \cdots + f_n \in \mathfrak{F}$ in $\mathfrak{E}$. It follows that our vector $p = z^n(p_n + zp_{n+1} + \cdots)$ belongs to $z^n\mathfrak{P}$ for every $n \geqslant 1$, so the z-adic topology on $\mathfrak{P}$ is not separated. This counterexample can be found in [52, Section A.1.1]; it also occured, under slightly different guises, in [61, Example 2.5] and [67, Example 3.20].

Among other things, $\mathfrak{P}$ provides an example of a $\mathcal{C}$-contramodule that does not have the form $\mathrm{Hom}_k(\mathcal{N}, V)$ for any $\mathcal{C}$-comodule $\mathcal{N}$. An example of a finite-dimensional contramodule not of this form (over a more complicated coalgebra $\mathcal{C}$) can be found in [52, Section A.1.2]. Concerning the above coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[z]]$, let us point out that the natural map $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/z^n\mathfrak{Q}$, though not necessarily injective, is always *surjective* for a $\mathcal{C}$-contramodule $\mathfrak{Q}$ [52, Lemma A.2.3]. Indeed, let $q_n \in \mathfrak{Q}$ be a sequence of vectors such that $q_{n+1} - q_n \in z^n\mathfrak{Q}$ for every $n = 1, 2, \ldots$ Suppose $q_{n+1} - q_n = z^n p_n$; then the infinite sum $q = q_0 + \sum_{n=1}^{\infty} z^n p_n$ provides an element $q \in \mathfrak{Q}$ for which $q - q_n \in z^n\mathfrak{Q}$ for every $n \geqslant 1$.

Similarly, consider the family of l-primary abelian groups $l^{-n}\mathbb{Z}/\mathbb{Z}$, $n = 1, 2, \ldots$ They can be included into short exact sequences of l-primary abelian groups

$$0 \longrightarrow l^{-n}\mathbb{Z}/\mathbb{Z} \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \mathbb{Q}_l/l^{-n}\mathbb{Z}_l \longrightarrow 0.$$

Passing to the infinite product of these short exact sequences in the category $\mathbb{Z}_l$–discr over all $n \geqslant 1$, one discovers that the map $\prod_n \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow \prod_n \mathbb{Q}_l/l^{-n}\mathbb{Z}_l$ is not surjective, as, e. g., the element $(l^{-n-1})_n \in \prod_n \mathbb{Q}_l/l^{-n}\mathbb{Z}_l$ does not belong to its image. One also computes the infinite product $\prod_n l^{-n}\mathbb{Z}/\mathbb{Z}$ in the category $\mathbb{Z}_l$–discr as isomorphic to the inductive limit $\varinjlim_m \bigl(\prod_{n=1}^{m} l^{-n}\mathbb{Z}/\mathbb{Z} \times \prod_{n=m+1}^{\infty} l^{-m}\mathbb{Z}/\mathbb{Z}\bigr)$.

Consider the family of $\mathbb{Z}_l$-contramodules $\mathbb{Z}/l^n\mathbb{Z}$, $n = 1, 2, \ldots$ They can be viewed as parts of the short exact sequences of $\mathbb{Z}_l$-contramodules

$$0 \longrightarrow l^n\mathbb{Z}_l \longrightarrow \mathbb{Z}_l \longrightarrow \mathbb{Z}/l^n\mathbb{Z} \longrightarrow 0.$$

Passing to the infinite direct sum of these short exact sequences in the category $\mathbb{Z}_l$–contra over all $n \geqslant 1$, one finds out that the map $\bigoplus_n l^n\mathbb{Z}_l \longrightarrow \bigoplus_n \mathbb{Z}_l$ is injective. Its cokernel $\mathfrak{P} = \bigoplus_n \mathbb{Z}/l^n\mathbb{Z} \in \mathcal{C}$–$\mathsf{contra}$ can be described as follows.

Let $\mathfrak{E}$ denote the free $\mathbb{Z}_l$-contramodule generated by a sequence of symbols e_1, e_2, e_3, … Explicitly, $\mathfrak{E}$ is the set of all formal linear combinations $\sum_{n=1}^\infty a_ne_n$, where the sequence of l-adic integers $a_n \in \mathbb{Z}_l$ converges to zero in the topology of $\mathbb{Z}_l$. Let $\mathfrak{F}$ be the similar $\mathbb{Z}_l$-contramodule generated by a sequence of symbols f_1, f_2, f_3, … Define a morphism of $\mathcal{C}$-contramodules $\mathfrak{F} \longrightarrow \mathfrak{E}$ by the rule $\sum_{n=1}^\infty b_nf_n \longmapsto \sum_{n=1}^\infty l^nb_ne_n$. Clearly, this morphism is injective; its cokernel $\mathfrak{E}/\mathfrak{F}$ is our $\mathbb{Z}_l$-contramodule $\mathfrak{P}$.

Set $p_n = e_n \bmod \mathfrak{F} \in \mathfrak{P}$. Then the infinite sum $p = \sum_{n=1}^\infty l^np_n$ is a nonzero element in $\mathfrak{P}$, since the element $\sum_{n=1}^\infty l^ne_n$ does not belong to $\mathfrak{F} \subset \mathfrak{E}$. On the other hand, every summand l^np_n vanishes in $\mathfrak{P}$, the element l^ne_n being the image of the element $f_n \in \mathfrak{F}$ in $\mathfrak{E}$. It follows that the element p belongs to $l^n\mathfrak{P}$ for every $n \geqslant 1$, so the l-adic topology on $\mathfrak{P}$ is not separated. Notice that the natural map $\mathfrak{Q} \longrightarrow \varprojlim_n \mathfrak{Q}/l^n\mathfrak{Q}$, though not necessarily injective, is always surjective for a $\mathbb{Z}_l$-contramodule $\mathfrak{P}$ [57, Lemma D.1.1]. The proof is similar to the above argument for $k[[z]]$-contramodules.

1.6. **Recovering the contramodule structure.** We have seen in the previous section that a $k[[z]]$-contramodule can contain infinitely z-divisible *vectors*, i. e., vectors $p \in \mathfrak{P}$ for which there exists a sequence of vectors $p_n \in \mathfrak{P}$ such that $p = z^np_n$ for every $n \geqslant 1$. Let us now show that *no* $k[[z]]$-contramodule can contain infinitely z-divisible $k[z]$*-submodules*. In other words, one can never choose the sequence of vectors $p_n \in \mathfrak{P}$ in a compatible way, i. e., any sequence of vectors $p_n \in \mathfrak{P}$ such that $p_n = zp_{n+1}$ for all $n \geqslant 0$ is the sequence of zero vectors.

Indeed, consider the expression $q = \sum_{n=0}^\infty z^np_n \in \mathfrak{P}$. By assumption, we have

$$\sum\nolimits_{n=0}^\infty z^np_n = \sum\nolimits_{n=0}^\infty z^n \cdot zp_{n+1} = \sum\nolimits_{n=0}^\infty z^{n+1}p_{n+1} = \sum\nolimits_{n=1}^\infty z^np_n,$$

that is $q = q - p_0$ and $p_0 = 0$. Here the last two equations conceal the use of the contraassociativity axiom from Section 1.3, which is being applied to the double sequence of vectors $p_{ij} = p_{i+1}$ when $j = 1$ and $p_{ij} = 0$ otherwise. The assertion we have proven is essentially a particular case of *Nakayama's lemma for contramodules* (see Section 2.1 below).

Now we are in the position to show that the forgetful functor $k[[z]]$–$\mathsf{contra} \longrightarrow k[z]$–$\mathsf{mod}$ (where we denote by $k[[z]]$–contra the category $\mathcal{C}$–contra of contramodules over the coalgebra $\mathcal{C}$ with $\mathcal{C}^* = k[[z]]$) is *fully faithful*, i. e., the $\mathcal{C}$-contramodule structure on a k-vector space $\mathfrak{P}$ can be uniquely recovered from the single linear operator $z\colon \mathfrak{P} \longrightarrow \mathfrak{P}$. Indeed, suppose that we want to "compute" the value of the infinite sum $\sum_{n=0}^\infty z^np_n$ in $\mathfrak{P}$. Consider the infinite system of linear equations

$$q_n = p_n + zq_{n+1}, \quad n = 0,\, 1,\, 2,\, \ldots \tag{3}$$

in the indeterminates $q_n \in \mathfrak{P}$. We have just shown that the related system of homogeneous linear equations $q_n = zq_{n+1}$ has no nonzero solutions in $\mathfrak{P}$. Hence a solution of the system (3) is unique if it exists. Given a $k[[z]]$-contramodule structure in $\mathfrak{P}$, one produces such a solution by setting

$$q_n = \sum\nolimits_{i=0}^{\infty} z^i p_{n+i}.$$

The value of $\sum_{n=0}^{\infty} z^n p_n$ can be recovered as the vector $q_0 \in \mathfrak{P}$.

We have essentially shown that a $k[z]$-module P admits an (always unique) $k[[z]]$-contramodule structure if and only if the system of nonhomogeneous linear equations (3) has a unique solution in q_n for every sequence of vectors $p_n \in P$. The latter condition is equivalent to the vanishing of the two Ext spaces $\mathrm{Ext}^*_{k[z]}(k[z,z^{-1}], P)$ (see [52, Remark A.1.1] and [56, Lemmas B.5.1 and B.7.1]).

Similarly, we have seen that a $\mathbb{Z}_l$-contramodule $\mathfrak{P}$ can contain infinitely l-divisible *elements*, i. e., there can be nonzero elements $p \in \mathfrak{P}$ for which there exists a sequence of elements $p_n \in \mathfrak{P}$ such that $p = l^n p_n$ for every $n \geqslant 1$. Let us show that no $\mathbb{Z}_l$-contramodule can contain infinitely l-divisible *subgroups.* In other words, one can never choose the sequence $p_n \in \mathfrak{P}$ in a compatible way, i. .e, any sequence of elements $p_n \in \mathfrak{P}$ such that $p_n = l p_{n+1}$ for all $n \geqslant 0$ is the zero sequence.

Indeed, consider the expression $\sum_{n=0}^{\infty} l^n p_n \in \mathfrak{P}$. By assumption, we have

$$\sum\nolimits_{n=0}^{\infty} l^n p_n = \sum\nolimits_{n=0}^{\infty} l^n \cdot l p_{n+1} = \sum\nolimits_{n=0}^{\infty} l^{n+1} p_{n+1} = \sum\nolimits_{n=1}^{\infty} l^n p_n,$$

that is $q = q - p_0$ and $p_0 = 0$. Here the first equation signifies the use of the "compatibility with the abelian group structure" axiom from Section 1.4, while the last two equations presume an application of the contraassociativity axiom.

Let us show that the forgetful functor $\mathbb{Z}_l\text{–}\mathsf{contra} \longrightarrow \mathsf{Ab}$ is fully faithful, i. e., a $\mathbb{Z}_l$-contramodule structure on an abelian group $\mathfrak{P}$ is uniquely determined by the abelian group structure. Suppose that we want to "compute" the value of the infinite sum $\sum_{n=0}^{\infty} l^n p_n$ in $\mathfrak{P}$. Consider the infinite system of linear equations

$$q_n = p_n + l q_{n+1}, \quad n = 0,\, 1,\, 2\, \ldots \tag{4}$$

in the indeterminates $q_n \in \mathfrak{P}$. We have just shown that the related system of homogeneous linear equations $q_n = l q_{n+1}$ has no nonzero solutions in $\mathfrak{P}$. Hence a solution of the system (4) is unique if it exists. Assuming a $\mathbb{Z}_l$-contramodule structure in $\mathfrak{P}$, one produces such a solution by setting

$$q_n = \sum\nolimits_{i=0}^{\infty} l^i p_{n+i}.$$

The value of $\sum_{n=0}^{\infty} l^n p_n$ can be recovered as the vector $q_0 \in \mathfrak{P}$.

We have essentially shown that an abelian group P admits an (always unique) $\mathbb{Z}_l$-contramodule structure if and only if the system of nonhomogeneous linear equations (4) has a unique solution in q_n for every sequence of elements $p_n \in P$. The latter condition is equivalent to the vanishing of the two Ext groups $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}[l^{-1}], P)$ (see [52, Remark A.3] and [56, Theorem B.1.1 and Lemma B.7.1]; cf. [12, Sections VI.3–4] and [36, Definition 4.6 and Remark 4.7]).

1.7. **Contramodules over the Virasoro algebra.** The definition of contramodules over the formal power series algebra in terms of infinite summation operations, as stated in Section 1.3, opens the door to generalizations of the notion of a contramodule to various topological algebraic structures, including not only associative rings, but also topological Lie algebras. In this section we demonstrate the possibility of such a definition in the simple example of the Virasoro Lie algebra.

The *punctured formal disk*, otherwise known as the *formal circle* over a field k is defined as a "space" such that the ring of functions on it is the ring of formal Laurent power series $k((z))$. The Lie algebra of vector fields on the formal circle $k[[z]]d/dz$ is the set of all expressions of the form $f(z)d/dz$ with $f(z) \in k((z))$, endowed with the obvious k-vector space structure and the Lie bracket $[f\,d/dz,\ g\,d/dz] = (f\,dg/dz - g\,df/dz)\,d/dz$. The vector fields $L_i = z^{i+1}\,d/dz$ form a topological basis in the vector space $k((z))d/dz$, in which the Lie bracket takes the form $[L_i, L_j] = (j-i)L_{i+j}$.

The *Virasoro algebra* $\mathbb{V}\mathrm{ir}$ is a central extension of the Lie algebra $k((z))d/dz$ with a one-dimensional kernel spanned by an element denoted by C. The k-vector space $\mathbb{V}\mathrm{ir} = k((z))d/dz \oplus kC$ has a topological basis formed by the vectors L_i, $i \in \mathbb{Z}$, and C, in which the Lie bracket is given by the rules

$$[L_i, C] = 0, \qquad [L_i, L_j] = (j-i)L_{i+j} + \delta_{i+j,0}\frac{i^3-i}{12}\,C,$$

where δ is the Kronecker symbol, for all i, $j \in \mathbb{Z}$ [28, 64, 38].

A *discrete module* $\mathcal{M}$ over the Virasoro algebra is a module over the Lie algebra $\mathbb{V}\mathrm{ir}$ for which the action map $\mathbb{V}\mathrm{ir} \times \mathcal{M} \longrightarrow \mathcal{M}$ is continuous in the z-adic topology of $\mathbb{V}\mathrm{ir}$ and the discrete topology of $\mathcal{M}$. In other words, $\mathcal{M}$ is a vector space endowed with linear operators L_i and $C\colon \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the above commutation relations and the discreteness condition, according to which for any vector $x \in \mathcal{M}$ there should exist an integer n such that $L_i x = 0$ for all $i > n$.

A *contramodule* $\mathfrak{P}$ over the Virasoro algebra $\mathbb{V}\mathrm{ir}$ is a k-vector space endowed with a linear operator $C\colon \mathfrak{P} \longrightarrow \mathfrak{P}$ and an infinite summation operation assigning to every sequence of vectors $p_{-n},\ p_{-n+1},\ p_{-n+2}\ \ldots \in \mathfrak{P}$, $n \in \mathbb{Z}$, a vector denoted symbolically by $\sum_{i=-n}^{\infty} L_i p_i \in \mathfrak{P}$. This infinitary operation, or rather, sequence of infinitary operations indexed by the integers n, should satisfy the equations of agreement

$$\sum\nolimits_{i=-n}^{\infty} L_i p_i = \sum\nolimits_{i=-m}^{\infty} L_i p_i \quad \text{when } -n < -m \text{ and } p_{-n} = \cdots = p_{-m-1} = 0,$$

linearity

$$\sum\nolimits_{i=-n}^{\infty} L_i(ap_i + bq_i) = a\sum\nolimits_{i=-n}^{\infty} L_i p_i + b\sum\nolimits_{i=-n}^{\infty} L_i q_i,$$

and the contra-Jacobi identity

$$\sum\nolimits_{i=-n}^{\infty} L_i(Cp_i) = C\sum\nolimits_{i=-n}^{\infty} L_i p_i$$

and

$$\sum\nolimits_{i=-n}^{\infty} L_i\left(\sum\nolimits_{j=-m}^{\infty} L_j p_{ij}\right) - \sum\nolimits_{j=-m}^{\infty} L_j\left(\sum\nolimits_{i=-n}^{\infty} L_i p_{ij}\right)$$
$$= \sum\nolimits_{h=-n-m}^{\infty} L_h\left(\sum\nolimits_{i+j=h}^{i\geqslant -n,\, j\geqslant -m}(j-i)p_{ij}\right) + C\sum\nolimits_{i+j=0}^{i\geqslant -n,\, j\geqslant -m}\frac{i^3-i}{12}p_{ij}.$$

for any p_i, q_i, $p_{ij} \in \mathfrak{P}$ and a, $b \in k$. This definition (for the Lie algebra $k((z))d/dz$ without the central extension) can be found in [52, Section D.2.7].

For any discrete module $\mathcal{M}$ over the Virasoro algebra and any k-vector space V, the vector space $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{M}, V)$ has a natural structure of $\mathbb{V}$ir-contramodule. The central element C acts in $\mathfrak{P}$ by the usual formula $(Cp)(x) = -p(Cx)$ for $p \in \mathfrak{P}$ and $x \in \mathcal{M}$, while the infinite summation operations are provided by the rule

$$\left(\sum\nolimits_{i=-n}^{\infty} L_i p_i\right)(x) = -\sum\nolimits_{i=-n}^{\infty} p_i(L_i x),$$

for $p_i \in \mathfrak{P}$ and $x \in \mathcal{M}$, where the second summation sign stands for the conventional sum of an eventually vanishing sequence of vectors in V.

The category $\mathbb{V}$ir–discr of discrete modules over the Virasoro algebra is abelian and the forgetful functor $\mathbb{V}$ir–discr $\longrightarrow$ k–vect is exact. Both the infinite direct sums and infinite products exist in $\mathbb{V}$ir–discr. The forgetful functor preserves infinite direct sums (but not infinite products), so filtered inductive limits are exact in $\mathbb{V}$ir–discr. In other words, the abelian category $\mathbb{V}$ir–discr satisfies the axioms Ab5 and Ab3*. It also admits a set of generators, so it has enough injectives.

The category $\mathbb{V}$ir–contra of contramodules over the Virasoro algebra is abelian and the forgetful functor $\mathbb{V}$ir–contra $\longrightarrow$ k–vect is exact. Both the infinite direct sums and the infinite products exist in $\mathbb{V}$ir–contra. The forgetful functor preserves infinite products, which are therefore exact functors in $\mathbb{V}$ir–contra; so this category satisfies Ab4 and Ab3*. There are also enough projective objects in $\mathbb{V}$ir–contra. We will explain their construction in Section 2.4 below.

1.8. **Contramodules over topological groups.** The aim of this section is to demonstrate the definition of contramodules over a locally compact, totally disconnected topological group. A typical example of such a group is the group $GL_n(\mathbb{Q}_l)$ of invertible square matrices over the rational l-adic numbers (endowed with the topology induced by the topology of $\mathbb{Q}_l$).

In this section, all the *topological spaces* are presumed to be Hausdorff, locally compact, and totally disconnected. Open-closed subsets in such a topological space X form a topology base [10, Corollaire II.4.4]. A *topological group* is a topological space with a group structure given by continuous multiplication and inverse element maps. Open subgroups in such a topological group G form a base of neighborhoods of zero [10, Corollaire III.4.6.1]. When G is compact, the same can be said about its open normal subgroups; so G is profinite.

A *discrete module* $\mathcal{M}$ over a topological group G is an abelian group endowed with an action of G provided by a continuous map $G \times \mathcal{M} \longrightarrow \mathcal{M}$ in the given topology of G and the discrete topology of $\mathcal{M}$. In other words, an action of G in

$\mathcal{M}$ is discrete if and only if the stabilizer of any element of $\mathcal{M}$ is an open subgroup in G. A discrete action can be also viewed as a map $\mathcal{M} \longrightarrow \mathcal{M}\{G\}$, where for any topological space X and abelian group A we denote by $A\{X\}$ the group of all locally constant A-valued functions $X \longrightarrow A$ on X. Denoting by G–mod the category of nontopological G-modules, i. e., abelian groups M endowed with an arbitrary action of G viewed as an abstract group, and by G–discr the category of discrete G-modules, there is a natural fully faithful functor G–$\mathsf{discr} \longrightarrow G$–$\mathsf{mod}$.

Let us introduce a bit more notation. Given a topological space X and an abelian group A, we denote by $A(X)$ the group of all locally constant compactly supported A-valued functions on X. For any topological spaces X and Y, there is a natural isomorphism $A(X \times Y) \simeq A(X)(Y)$. Furthermore, denote by $A[[X]]$ the abelian group of finitely additive compactly supported A-valued measures defined on the open-closed subsets of X. For any continuous map of topological spaces $X \longrightarrow Y$, the push-forward map $A[[X]] \longrightarrow A[[Y]]$ is defined [52, Section E.1.1].

For any topological spaces X, Y and an abelian group A, there is a natural map $A[[X\times Y]] \longrightarrow A[[X]][[Y]]$ assigning to an A-valued measure ν on $X\times Y$ the measure taking an open-closed subset $V \subset Y$ to the measure taking an open-closed subset $U \subset X$ to the element $\nu(U \times V) \in A$. This map is an isomorphism when *both* the spaces X and Y are discrete or *both* of them are compact, but not otherwise.

A *contramodule* over a topological group G is an abelian group $\mathfrak{P}$ endowed with a *G-contraaction* map $\pi\colon \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$, which can be viewed as an *integration operation* and denoted symbolically by

$$\pi(\mu) = \int_G g^{-1}(d\mu_g),$$

where $d\mu_g \in \mathfrak{P}$ denotes the value of a measure $\mu \in \mathfrak{P}[[G]]$ on a small piece of the group G containing an element $g \in G$, while $g^{-1}(d\mu_g) \in \mathfrak{P}$ is a small element in $\mathfrak{P}$ obtained by applying to $d\mu_g$ the presumed generalized action of $g^{-1} \in G$ in $\mathfrak{P}$.

The map π must satisfy the following *contraassociativity* and *contraunitality* equations. Firstly, the composition $\mathfrak{P}[[G \times G]] \longrightarrow \mathfrak{P}[[G]][[G]] \longrightarrow \mathfrak{P}[[G]]$ of the above-described map $\mathfrak{P}[[G \times G]] \longrightarrow \mathfrak{P}[[G]][[G]]$ with the iterated contraaction map $\mathfrak{P}[[G]][[G]] \longrightarrow \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$ should be equal to the composition $\mathfrak{P}[[G \times G]] \longrightarrow \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$ of the push-forward map $\mathfrak{P}[[G \times G]] \longrightarrow \mathfrak{P}[[G]]$ with respect to the multiplication map $G \times G \longrightarrow G$ with the contraaction map $\mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$,

$$\mathfrak{P}[[G \times G]] \rightrightarrows \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}.$$

Secondly, the point measure supported at the unit element $e \in G$ and taking a prescribed value $p \in \mathfrak{P}$ on the neighborhoods of e should be taken to the element p by the contraaction map,

$$\mathfrak{P} \longrightarrow \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}.$$

Given a point $g \in G$ and an element $p \in \mathfrak{P}$, denote by $g^{-1}(p) \in \mathfrak{P}$ the element one obtains by applying the contraaction map to the point measure supported at g and taking the value p on its neighborhoods. This rule defines a natural action of G (as an abstract, nontopological group) on any G-contramodule $\mathfrak{P}$, providing a forgetful functor $G\text{–}\mathsf{contra} \longrightarrow G\text{–}\mathsf{mod}$ [52, Section E.1.3].

For any discrete G-module $\mathcal{M}$ and an abelian group V, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)$ has a natural G-contramodule structure. The contraaction map $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)[[G]] \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)$ assigns to a measure μ the additive map taking an element $x \in \mathcal{M}$ to the value of the integral

$$\pi(\mu)(x) = \int_G d\mu_g(gm) \in V.$$

The $\mathcal{M}$-valued function $g \longmapsto gm$ being locally constant on G and the $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, V)$-valued measure μ being compactly supported in G, the integral is well-defined (cf. [52, Section E.1.4]).

The category of discrete G-modules is abelian and the forgetful functor $G\text{–}\mathsf{discr} \longrightarrow \mathsf{Ab}$ is exact. Filtered inductive limits are exact functors in $G\text{–}\mathsf{discr}$; they are also preserved by the forgetful functor. In other words, the category $G\text{–}\mathsf{discr}$ satisfies the axioms Ab5 and Ab3*. It also admits a set of generators, so it has enough injective objects. The category of G-contramodules is abelian and the forgetful functor $G\text{–}\mathsf{contra} \longrightarrow \mathsf{Ab}$ is exact. Infinite products are exact functors in $G\text{–}\mathsf{contra}$; they are also preserved by the forgetful functor. So the category $G\text{–}\mathsf{contra}$ satisfies the axioms Ab3 and Ab4*. It has enough projective objects.

The embedding functor $G\text{–}\mathsf{discr} \longrightarrow G\text{–}\mathsf{mod}$ and the forgetful functor $G\text{–}\mathsf{contra} \longrightarrow G\text{–}\mathsf{mod}$ have the similar properies, the forgetful functor $G\text{–}\mathsf{mod} \longrightarrow \mathsf{Ab}$ preserving the inductive and projective limits of any diagrams. We will see below in Section 2.6 how discrete G-modules and G-contramodules can be interpreted as *semimodules* and *semicontramodules* over a certain semialgebra $\boldsymbol{\mathcal{S}}$, opening the way to explicit constructions of injective and projective objects in $G\text{–}\mathsf{discr}$ and $G\text{–}\mathsf{contra}$.

2. Comodule and Contramodule Categories

2.1. **Contramodules over topological rings.** As we discussed in Sections 1.3–1.4, one would like to extend the definition of a contramodule from the topological algebras dual to coalgebras over fields to topological rings of more general nature. Before proceeding to present the desired definition, let us start with reintroducing the conventional modules over a ring.

Given a (nontopological) associative ring R with unit, one can define left R-modules in the following fancy way. For any set X, denote by $R[X]$ the set of formal linear combinations of elements of X with coefficients in R (i. e., the underlying set of the free R-module generated by X). The assignment $X \longmapsto R[X]$ is a covariant functor from the category of sets to itself. The key observation is that this functor has a natural structure of a *monad* [44, Chapter VI] on the category of sets.

In other words, for any set X there is a natural map of "opening the parentheses" $\phi_X \colon R[R[X]] \longrightarrow R[X]$, assigning a formal linear combination of elements of X to a formal linear combination of formal linear combinations. There is also a natural map $\epsilon_X \colon X \longrightarrow R[X]$ defined in terms of the zero and unit elements of the ring R.

The associativity and unitality axioms of a monad [44, Section VI.1] are satisfied by these two natural transformations.

Given the endofunctor $R[-]\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ endowed with the natural transformations ϕ and ϵ, one can define a left R-module as an algebra/module over this monad on the category of sets. In other words, a left $\mathfrak{R}$-module M is a set endowed with a map of sets $m\colon R[M] \longrightarrow M$ satisfying the associativity and unitality axioms from [44, Section VI.2]. Specifically, the two maps ϕ_M and $R[m]\colon R[R[M]] \longrightarrow R[M]$ should have equal compositions with the map m,

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M,$$

while the composition of the map $\epsilon_M\colon M \longrightarrow R[M]$ with the map m should be equal to the identity map id_M,

$$M \longrightarrow R[M] \longrightarrow M.$$

Now let $\mathfrak{R}$ be an associative topological ring with unit. We will have to assume that $\mathfrak{R}$ is complete and separated, and open right ideals form a base of neighborhoods of zero in $\mathfrak{R}$. In other words, the natural map $\mathfrak{R} \longrightarrow \varprojlim \mathfrak{R}/\mathfrak{J}$, where $\mathfrak{J}$ runs over all the open right ideals, must be a topological isomorphism. Notice that these are precisely the assumptions under which the discrete right $\mathfrak{R}$-modules are a good category to be assigned to $\mathfrak{R}$ (see the beginning of Section 1.4; cf. [9, Section 1.4]); otherwise, a discrete right $\mathfrak{R}$-module is the same thing as a discrete right module over the completion of $\mathfrak{R}$ in the new topology with a base of neighborhoods of zero consisting of the open right ideals in the original one.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_x r_x x$ of elements of X with the coefficients in $\mathfrak{R}$ for which the family of coefficients $r_x \in \mathfrak{R}$ converges to zero in the topology of $\mathfrak{R}$. The latter condition means that for any neighborhood of zero $U \subset \mathfrak{R}$ the set of all $x \in X$ for which $r_x \notin U$ must be finite. We will endow the functor $\mathfrak{R}[[-]]\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ with the structure of a monad on the category of sets by defining an "opening of infinite parentheses" map $\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ and a unit map $\epsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$.

In order to define the map ϕ_X, one essentially has to show that the infinite sums of products in $\mathfrak{R}$ that one obtains after opening the parentheses converge in the topology of $\mathfrak{R}$. That is where our assumptions about the topological ring $\mathfrak{R}$ have to be used. Indeed, one has $\mathfrak{R}[[X]] = \varprojlim_{\mathfrak{J}} \mathfrak{R}/\mathfrak{J}[X]$, where $\mathfrak{J}$ runs over all the open right ideals in $\mathfrak{R}$ (and the notation $A[X]$ for a set X and an abelian group A stands for the group of all finite formal linear combinations of the elements of X with coefficients in A).

Defining the "opening of parentheses" map $\mathfrak{R}/\mathfrak{J}[\mathfrak{R}[[X]]] \longrightarrow \mathfrak{R}/\mathfrak{J}[X]$ does not involve any actual infinite summation, since $\mathfrak{J} \subset \mathfrak{R}$ is an open right ideal. It remains to consider the composition $\mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}/\mathfrak{J}[\mathfrak{R}[[X]]] \longrightarrow \mathfrak{R}/\mathfrak{J}[X]$ and pass to the projective limit over $\mathfrak{J}$. The unit map ϵ_X is easy to define; one can say that it is the composition $X \longrightarrow \mathfrak{R}[X] \longrightarrow \mathfrak{R}[[X]]$. Checking the monad equations for the natural transformations ϕ and ϵ is straightforward.

A *left* $\mathfrak{R}$*-contramodule* $\mathfrak{P}$ is an algebra/module over this monad on the category of sets. In other words, it is a set endowed with an $\mathfrak{R}$*-contraaction* map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$

satisfying the (contra)associativity and unitality equations together with the natural transformations ϕ and ϵ. Specifically, the two maps $\phi_{\mathfrak{P}}$ and $\mathfrak{R}[[\pi]]\colon \mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ should have equal compositions with the contraaction map π,

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P},$$

while the composition of the map $\epsilon_{\mathfrak{P}}\colon \mathfrak{P} \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ with the contraaction map should be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$,

$$\mathfrak{P} \longrightarrow \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}.$$

This definition can be found in [52, Remark A.3] and [56, Section 1.2].

Notice that a systematic study of a class of monads on the category of sets, called the *algebraic* monads and viewed as "generalized rings", was undertaken by Durov in [22]. The definition above was in part inspired by Durov's work. However, the monad $X \longmapsto \mathfrak{R}[[X]]$ is *not* algebraic, as the functor $\mathfrak{R}[[-]]$ does not preserve filtered inductive limits of sets.

For any set X, the "opening of parentheses" map $\pi = \phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ provides the set $\mathfrak{R}[[X]]$ with a natural left $\mathfrak{R}$-contramodule structure. The $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$ are called the *free* $\mathfrak{R}$-contramodules. For the reasons common to all monads [44, Section VI.5], for any set X and any left $\mathfrak{R}$-contramodule $\mathfrak{Q}$ there is a natural bijection/isomorphism of abelian groups $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{Q}) \simeq \mathrm{Hom}_{\mathsf{Sets}}(X, \mathfrak{Q})$, where we denote by $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})$ the group of morphisms from an object $\mathfrak{P}$ to an object $\mathfrak{Q}$ in the category of $\mathfrak{R}$-contramodules.

Equivalently, an $\mathfrak{R}$-contramodule can be defined as a set endowed with the following *infinite summation operations*. For any set of indices $\{\alpha\}$, any family of elements $p_\alpha \in \mathfrak{P}_\alpha$, and any family of coefficients $r_\alpha \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$, the element denoted symbolically by $\sum_\alpha r_\alpha p_\alpha \in \mathfrak{P}$ must be defined. This series of infinitary operations in the set $\mathfrak{P}$ should satisfy the equations of contraassociativity

$$\sum\nolimits_\alpha r_\alpha \sum\nolimits_\beta r_{\alpha\beta} p_{\alpha\beta} = \sum\nolimits_{\alpha,\beta} (r_\alpha r_{\alpha\beta}) p_{\alpha\beta} \quad \text{if } r_\alpha \to 0 \text{ and } \forall \alpha\ r_{\alpha\beta} \to 0 \text{ in } \mathfrak{R},$$

unitality

$$\sum\nolimits_\alpha r_\alpha p_\alpha = p_{\alpha_0} \quad \text{if the set } \{\alpha\} \text{ consists of one element } \alpha_0 \text{ and } r_{\alpha_0} = 1,$$

and distributivity

$$\sum\nolimits_{\alpha,\beta} r_{\alpha\beta} p_\alpha = \sum\nolimits_\alpha \Big(\sum\nolimits_\beta r_{\alpha\beta}\Big) p_\alpha \quad \text{if } r_{\alpha\beta} \to 0 \text{ in } \mathfrak{R}.$$

Here the summation over double indices α, β presumes a set of pairs $\{(\alpha, \beta)\}$ mapping into another set $\{\alpha\}$ by a map denoted symbolically by $(\alpha, \beta) \longmapsto \alpha$ (i. e., the range of possible β's may depend on a chosen α). The summation sign in the parentheses in the third equation denotes the convergent sum in $\mathfrak{R}$, while all the other summation signs stand for the infinite summation operation in $\mathfrak{P}$. Our conditions on the topology of $\mathfrak{R}$ guarantee that the family $r_\alpha r_{\alpha\beta}$ converges to zero whenever both the family r_α does and the families $r_{\alpha\beta}$ do for every fixed β.

Restricting the summation operations to finite sets of indices $\{\alpha\}$, one discovers that every left $\mathfrak{R}$-contramodule has an underlying left $\mathfrak{R}$-module structure. Equivalently, one composes the contraaction map $\mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the natural embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ in order to endow the underlying set of an $\mathfrak{R}$-contramodule $\mathfrak{P}$ with the structure of an $\mathfrak{R}$-module. We have constructed the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ from the category of left $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$ to the category $\mathfrak{R}\text{–}\mathsf{mod}$ of left modules over the ring $\mathfrak{R}$ viewed as an abstract (nontopological) ring.

For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any abelian group V, the group of all additive maps $\mathrm{Hom}_{\mathsf{Ab}}(\mathcal{N}, V)$ has a natural left $\mathfrak{R}$-contramodule structure with the infinite summation operations defined by the rule

$$\left(\sum\nolimits_\alpha r_\alpha p_\alpha\right)(x) = \sum\nolimits_\alpha p_\alpha(xr_\alpha)$$

for any $p_\alpha \in \mathfrak{P}$, $x \in \mathcal{N}$ and a family of coefficients r_α converging to zero in the topology of $\mathfrak{R}$. Here the summation sign in the right-hand side denotes the sum of a family of elements in V all but a finite number of which vanish, as $xr_\alpha = 0$ for all but a finite subset of indices α.

For any topological ring $\mathfrak{R}$ the category $\mathfrak{R}\text{–}\mathsf{discr}$ of discrete left $\mathfrak{R}$-modules is abelian and the forgetful functor $\mathfrak{R}\text{–}\mathsf{discr} \longrightarrow \mathsf{Ab}$ is exact. Filtered inductive limits are exact functors in $\mathfrak{R}\text{–}\mathsf{discr}$; they are also preserved by the forgetful functor. In other words, the category $\mathfrak{R}\text{–}\mathsf{discr}$ satisfies the axioms Ab5 and Ab3* (but *not* in general Ab4*). It also admits a set of generators, so it has enough injectives.

For any complete and separated topological ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by the open right ideals the category of left $\mathfrak{R}$-contramodules is abelian and the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathsf{Ab}$ is exact. To convince oneself that this is so, one uses the definition of $\mathfrak{R}$-contramodules in terms of infinite summation operations in order to define the $\mathfrak{R}$-contramodule structures on the kernel and cokernel of any morphism of $\mathfrak{R}$-contramodules taken in the category of abelian groups. It helps to start from writing down the equations of compatibility of the contramodule infinite summation operations with the conventional finite operations in an abelian group or an $\mathfrak{R}$-module [56, Section 1.2].

Infinite products are exact functors in $\mathfrak{R}\text{–}\mathsf{contra}$; they are also preserved by the forgetful functor. There are enough projective objects in $\mathfrak{R}\text{–}\mathsf{contra}$; an $\mathfrak{R}$-contramodule is projective if and only if it is a direct summand of a free one. Infinite direct sums of free $\mathfrak{R}$-contramodules are computed by the rule $\bigoplus_\alpha \mathfrak{R}[[X_\alpha]] = \mathfrak{R}[[\coprod_\alpha X_\alpha]]$; to compute the infinite direct sum of a family of arbitrary $\mathfrak{R}$-contramodules, one can present them as the cokernels of morphisms of free contramodules and use the fact that infinite direct sums commute with cokernes. Hence the category $\mathfrak{R}\text{–}\mathsf{contra}$ satisfies the axioms Ab3 and Ab4* (but *not* in general Ab4).

The infinite products of discrete $\mathfrak{R}$-modules and the infinite direct sums of $\mathfrak{R}$-contramodules are not preserved by the respective forgetful functors in general. The embedding functor $\mathfrak{R}\text{–}\mathsf{discr} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ and the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ have the similar properties, the forgetful functor $\mathfrak{R}\text{–}\mathsf{mod} \longrightarrow \mathsf{Ab}$ preserving the inductive and projective limits of any diagrams.

The following version of *Nakayama's lemma for discrete modules and contramodules* over a topological ring is one of their most important properties.

Lemma. (a) *Let $\mathfrak{R}$ be a topological ring and $\mathfrak{m} \subset \mathfrak{R}$ be a topologically nilpotent ideal, i. e., for any neighborhood of zero $U \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset U$. Then for any nonzero discrete left $\mathfrak{R}$-module $\mathcal{M}$ the submodule ${}_{\mathfrak{m}}\mathcal{M} \subset \mathcal{M}$ of elements annihilated by $\mathfrak{m}$ is nonzero.*

(b) *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by the open right ideals, and let $\mathfrak{m} \subset \mathfrak{R}$ be a topologically nilpotent closed ideal. Then for any nonzero left $\mathfrak{R}$-contramodule $\mathfrak{P}$ the quotient contramodule $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ of $\mathfrak{P}$ by the image $\mathfrak{m}\mathfrak{P}$ of the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is nonzero.*

Here the map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is simply the restriction of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ to the subset $\mathfrak{m}[[\mathfrak{P}]] \subset \mathfrak{R}[[\mathfrak{P}]]$ of all formal linear combinations with (converging families of) coefficients in $\mathfrak{m}$. Notice that this version of Nakayama's lemma presumes *no* finite generatedness condition on either the discrete module *or* the contramodule; on the other hand, it requires a rather strong global topological nilpotency condition on the ideal $\mathfrak{m}$.

Proof. Part (a): let $x \in \mathcal{M}$ be a nonzero element and $U \subset \mathfrak{R}$ be its annihilator in $\mathfrak{R}$. The $\mathfrak{R}$-module $\mathcal{M}$ being discrete, U is an open neighborhood of zero in $\mathfrak{R}$; hence there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset U$, so $\mathfrak{m}^n x = 0$. It remains to consider the maximal integer $i \geqslant 0$ for which $\mathfrak{m}^i x \neq 0$; then $\mathfrak{m}^i x \subset {}_{\mathfrak{m}}\mathcal{M}$. The proof of part (b) is a bit more complicated; see [52, Lemma A.2.1] and [56, Lemma 1.3.1]. ☐

2.2. **Contramodules over the adic completions of Noetherian rings.** Let R be a right Noetherian associative ring, and let $m \subset R$ be an ideal generated by central elements in R. Denote by $\mathfrak{R} = \varprojlim_n R/m^n$ the m-adic completion of the ring R. In this section we explain how to describe the abelian category $\mathfrak{R}$–contra of left contramodules over the complete ring $\mathfrak{R}$ viewed as a topological ring in the projective limit topology (or, which is the same, the m-adic topology) in terms of conventional modules over the original ring R.

Theorem. *The composition of forgetful functors $\mathfrak{R}$–contra $\longrightarrow$ $\mathfrak{R}$–mod $\longrightarrow$ R–mod provides a fully faithful embedding of abelian categories $\mathfrak{R}$–contra $\longrightarrow$ R–mod. A left R-module P belongs to the full subcategory $\mathfrak{R}$–contra $\subset$ R–mod if and only if any of the following equivalent conditions holds:*

(a) *for any element $s \in m$ belonging to the center of the ring R and any $R[s^{-1}]$-module L one has $\operatorname{Ext}^i_R(L, P) = 0$ for all $i \geqslant 0$;*

(b) *for any element $s \in m$ belonging to the center of the ring R one has $\operatorname{Ext}^i_R(R[s^{-1}], P) = 0$ for all $i \geqslant 0$;*

(c) *for any element $s \in m$ one has $\operatorname{Ext}^*_{\mathbb{Z}[t]}(\mathbb{Z}[t, t^{-1}], P) = 0$, where $\mathbb{Z}[t]$ denotes the ring of polynomials in one variable with integral coefficients, $\mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials, and t acts in P by the multiplication with s;*

(d) *for any $j = 1, \ldots, n$ and any $i = 0$ or 1 one has $\operatorname{Ext}^i_R(R[s_j^{-1}], P) = 0$, where $s_1, \ldots, s_j$ is a fixed set of central generators of the ideal $m \subset R$;*

(e) *for any* $j = 1, \ldots, n$ *and any* $i = 0$ *or* 1 *one has* $\operatorname{Ext}^i_{\mathbb{Z}[t]}(\mathbb{Z}[t, t^{-1}], P) = 0$*, where* t *acts in* P *by the multiplication with* s_j*.*

In other words, an $\mathfrak{R}$-contramodule structure on a given left $\mathfrak{R}$-module is always *unique*, and the theorem lists equivalent conditions telling when it *exists*. Of course, for contramodules over topological rings more complicated then the adic completions no such description is in general possible.

In the case of a commutative ring R, the theorem essentially says that a contramodule over the m-adic completion of R is the same thing as a *cohomologically* m*-adically complete* R*-module* of Porta–Shaul–Yekutieli [68, 69, 70]. A very brief sketch of the proof of the above theorem is presented below; a detailed exposition can be found in [56, Appendix B] and [57, Section C.5] (see also [52, Remark A.1.1]).

Sketch of proof. First let us explain why any left R-module P admitting a left $\mathfrak{R}$-contramodule structure satisfies the conditions (a) and (c). The choice of an element $s \in m$ defines a continuous homomorphism of topological rings $\mathbb{Z}[[t]] \longrightarrow \mathfrak{R}$, thus endowing any left $\mathfrak{R}$-contramodule $\mathfrak{P}$ with a left $\mathbb{Z}[[t]]$-contramodule structure. One checks that this structure is inherited by the groups $\operatorname{Ext}^i_{\mathbb{Z}[t]}(L, \mathfrak{P})$ for any $\mathbb{Z}[t]$-module L and, when s is central in $\mathfrak{R}$, by the groups $\operatorname{Ext}^i_R(L, \mathfrak{P})$ for any R-module L. Now when t or s acts invertibly in L, the Ext groups in question turn out to be $\mathbb{Z}[[t]]$-contramodules with an invertible action of t, which have to vanish by the Nakayama Lemma 2.1(b) above (cf. [56, Section B.2]).

Furthermore, for any central element $s \in R$ and any left R-module P one has $\operatorname{Ext}^i_R(R[s^{-1}], P) \simeq \operatorname{Ext}^i_{\mathbb{Z}[t]}(\mathbb{Z}[t, t^{-1}], P)$ and, of course, both groups always vanish for $i > 1$ [56, Lemma B.7.1]. It remains to show that any left R-module P satisfying the condition (e) can be endowed with a left $\mathfrak{R}$-contramodule structure in a unique way. This is accomplished by the following sequence of lemmas.

Consider the topological ring of formal power series $\mathfrak{T} = R[[t_1, \ldots, t_n]]$ in the central variables $t_1, \ldots, t_n$ with coefficients in R; then there is a natural continuous ring homomorphism $\mathfrak{T} \longrightarrow \mathfrak{R}$ taking t_j to s_j. Consider also the ring of polynomials $T = R[t_1, \ldots, t_n]$ and the similar ring homomorphism $T \longrightarrow R$.

Lemma 1. *The ring homomorphism* $\mathfrak{T} \longrightarrow \mathfrak{R}$ *is surjective, and its kernel* $\mathfrak{J}$ *is generated by the central elements* $s_j - t_j$ *as an ideal in an abstract (nontopological) ring* $\mathfrak{T}$*. Moreover, any family of elements converging to zero in* $\mathfrak{R}$ *can be lifted to a family of elements converging to zero in* $\mathfrak{T}$*, and any family of elements in* $\mathfrak{J}$ *converging to zero in the topology of* $\mathfrak{T}$ *can be presented as a linear combination of* n *families of elements in* $\mathfrak{T}$*, each of them converging to zero, with the coefficients* $s_j - t_j$*.*

Proof. This is where the Noetherianness condition on the ring R is being used; see [56, Sections B.3–B.4] and [57, Lemma C.5.2]. □

Lemma 2. *The (contra)restriction of scalars functor* $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{T}\text{–}\mathsf{contra}$ *identifies the category of left* $\mathfrak{R}$*-contramodules with the full subcategory in* $\mathfrak{T}\text{–}\mathsf{contra}$ *consisting of those left* $\mathfrak{T}$*-contramodules in which the elements* $s_j - t_j \in \mathfrak{T}$ *act by zero.*

Proof. Follows from Lemma 1; see [56, Lemma B.4.1]. □

It is easy to interpret left $\mathfrak{T}$-contramodules as left R-modules endowed with infinite summation operations with the coefficients $t_1^{m_1}\cdots t_n^{m_n}$ (see [56, proof of Lemma B.5.1]; cf. Section 1.3 above). Hence it follows from Lemma 2 that the definition of contramodules over the l-adic integers given in Section 1.4 is equivalent to the general definition from Section 2.1 specialized to the case of $\mathfrak{R} = \mathbb{Z}_l$.

Lemma 3. *The forgetful functor* $\mathfrak{T}\text{–}\mathsf{contra} \longrightarrow T\text{–}\mathsf{mod}$ *identifies the category of left* $\mathfrak{T}$*-contramodules with the full subcategory in the category of left* T*-modules consisting of all those modules* Q *for which* $\mathrm{Ext}_T^*(T[t_j^{-1}], Q) = 0$ *for every* $j = 1, \ldots, n$.

Proof. The "unique recovering" argument here is just a more elaborated version of the reasoning from Section 1.6 above. See [56, Sections B.5–B.7] and [57, Lemma C.5.3] for the details. ☐

To finish the proof of the theorem, it remains to combine together the results of Lemmas 2 and 3. ☐

Denote by $\mathfrak{m} = \varprojlim_n m/m^n \subset \mathfrak{R}$ the extension of the ideal m in the ring $\mathfrak{R}$. The following result explains the term "cohomologically complete module" for left R-modules satisfying the equivalent conditions of Theorem.

Proposition 1. *For any left* R*-contramodule* $\mathfrak{P}$*, the natural map* $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{m}^n\mathfrak{P} = \varprojlim_n \mathfrak{P}/m^n\mathfrak{P}$ *is surjective.*

Proof. See [52, Lemma A.2.3] or [57, Lemma D.1.1]. ☐

Notice that the natural functor $\mathfrak{R}\text{–}\mathsf{discr} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful for any topological ring R with a base of neighborhoods of zero consisting of open left ideals J and its completion $\mathfrak{R} = \varprojlim_J R/J$. Moreover, when $\mathfrak{R} = \varprojlim_n R/m^n$ is the completion of the ring R in the adic topology of an ideal $m \subset R$ generated by a finite set of central elements s_j, an R-module M belongs to the full subcategory $\mathfrak{R}\text{–}\mathsf{discr} \subset R\text{–}\mathsf{mod}$ if and only if it is *m-torsion*, i. e., one has $R[s_j^{-1}] \otimes_R M = 0$ for all j or, which is the same, $\mathrm{Tor}_*^R(R[s_j^{-1}], M) = 0$ for all j.

Let us point out that the class of m-adically complete and separated left R-modules, i. e., left R-modules P for which the map $P \longrightarrow \varprojlim_n P/m^nP$ is an isomorphism, does *not* have good homological properties. Indeed, it is *not* preserved not only by the passages to the cokernels of injective morphisms (see Section 1.5), but *also* by extensions in $R\text{–}\mathsf{mod}$ or $\mathfrak{R}\text{–}\mathsf{contra}$ [61, Example 2.5]. The full subcategory of left $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra} \subset R\text{–}\mathsf{mod}$, on the other hand, not only contains all the m-adically complete and separated left R-modules, but is also closed under the kernels, cokernels, extensions, and projective limits in $R\text{–}\mathsf{mod}$.

In the respective assumptions on the topological ring $\mathfrak{R}$, let us denote by $\mathrm{Ext}_{\mathfrak{R}}^*(\mathcal{L}, \mathcal{M})$ the Ext groups in the abelian category $\mathfrak{R}\text{–}\mathsf{discr}$ and by $\mathrm{Ext}^{\mathfrak{R},*}(\mathfrak{P}, \mathfrak{Q})$ the Ext groups in the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$. The next proposition shows that the embeddings of abelian categories $\mathfrak{R}\text{–}\mathsf{discr} \longrightarrow R\text{–}\mathsf{mod}$ and $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ have good homological properties.

Proposition 2. (a) *Let R be a left Noetherian ring, $m \subset R$ be an ideal generated by central elements, and $\mathfrak{R} = \varprojlim_n R/m^n$ be the m-adic completion of R. Then the embedding functor $\mathfrak{R}$–$\mathsf{discr} \longrightarrow R$–$\mathsf{mod}$ induces isomorphisms on all the Ext groups,* $\mathrm{Ext}^i_{\mathfrak{R}}(\mathcal{L}, \mathcal{M}) \simeq \mathrm{Ext}^i_R(\mathcal{L}, \mathcal{M})$ *for all $\mathcal{L}$, $\mathcal{M} \in \mathfrak{R}$–$\mathsf{discr}$ and all $i \geqslant 0$.*

(b) *Let R be a right Noetherian ring, $m \subset R$ be an ideal generated by central elements, and $\mathfrak{R}$ be the m-adic completion of R. Then the embedding functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow R$–$\mathsf{mod}$ induces isomorphisms on all the Ext groups,* $\mathrm{Ext}^{\mathfrak{R},i}(\mathfrak{P}, \mathfrak{Q}) \simeq \mathrm{Ext}^i_R(\mathfrak{P}, \mathfrak{Q})$ *for all $\mathfrak{P}$, $\mathfrak{Q} \in \mathfrak{R}$–$\mathsf{contra}$ and all $i \geqslant 0$.*

Proof. Part (a): it follows from the Artin–Rees lemma (see [47, Theorem 8.5] and [32, Theorems 1.9 and 13.3]) that the functor $\mathfrak{R}$–$\mathsf{discr} \longrightarrow R$–$\mathsf{mod}$ preserves injectivity of objects (cf. [55, Section A.3]), which is clearly sufficient. Part (b): one shows that the functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow R$–$\mathsf{mod}$ takes free $\mathfrak{R}$-contramodules to flat R-modules and all $\mathfrak{R}$-contramodules to relatively cotorsion R-modules (see [56, Sections B.8–B.10] and [57, Propositions C.5.4–C.5.5]). ☐

2.3. **Contramodules over topological algebras over fields.** The "set-theoretical" definition of $\mathfrak{R}$-contramodules given in Section 2.1 is intended to incorporate "arithmetical" examples such as that of the ring $\mathfrak{R} = \mathbb{Z}_l$ of l-adic integers. In the case of a topological algebra $\mathfrak{R}$ over a field k, the definition can be simplified, facilitating the comparision with the notion of a contramodule over a coalgebra $\mathcal{C}$ over k.

A topological vector space V over a field k is said to have a *linear topology* if its open vector subspaces form a base of neighborhoods of zero in it. In the sequel, we presume all our topological vector spaces to have linear topologies and, unless otherwise mentioned, to be complete and separated. In other words, the natural map $V \longrightarrow \varprojlim_U V/U$, where U runs over all the open vector subspaces in V, should be a topological isomorphism (see [9] or [52, Section D.1]). Given a topological vector space V and an abstract (nontopological) vector space P over a field k, we denote by $V \otimes\hat{}\, P$ the projective limit $\varprojlim_U V/U \otimes_k P$ taken over all the open vector subspaces $U \subset V$, viewed as an abstract (nontopological) vector space.

For any associative algebra R over a field k, one can define left R-modules as modules over the monad $M \longmapsto R \otimes_k M$ on the category of k-vector spaces $M \in k$–vect. We would like to extend this definition of R-modules to the case of topological algebras over k. Let $\mathfrak{R}$ be a complete and separated topological associative algebra over a field k where open right ideal form a base of neighborhoods of zero. Then the functor $P \longmapsto \mathfrak{R} \otimes\hat{}\, P$ has a natural structure of a monad on the category of (nontopological) k-vector spaces $P \in k$–vect. Indeed, let us construct the natural transformations of multiplication and unit in this monad.

For any open right ideal $\mathfrak{J} \subset \mathfrak{R}$ the multiplication map $\mathfrak{R}/\mathfrak{J} \times \mathfrak{R} \longrightarrow \mathfrak{R}/\mathfrak{J}$, being continuous in the discrete topology of $\mathfrak{R}/\mathfrak{J}$ and the given topology of $\mathfrak{R}$, defines a structure of discrete right $\mathfrak{R}$-module on the quotient space $\mathfrak{R}/\mathfrak{J}$, so the annihilator of every element of $\mathfrak{R}/\mathfrak{J}$ is an open right ideal in $\mathfrak{R}$. Hence the multiplication map $\mathfrak{R}/\mathfrak{J} \otimes_k \mathfrak{R} \longrightarrow \mathfrak{R}/\mathfrak{J}$ induces a natural linear map $\mathfrak{R}/\mathfrak{J} \otimes_k (\mathfrak{R} \otimes\hat{}\, P) \longrightarrow \mathfrak{R}/\mathfrak{J} \otimes_k P$. Composing this map with the projection $\mathfrak{R} \otimes\hat{}\, (\mathfrak{R} \otimes\hat{}\, P) \longrightarrow \mathfrak{R}/\mathfrak{J} \otimes_k (\mathfrak{R} \otimes\hat{}\, P)$ and passing to the projective limit over open right ideals $\mathfrak{J}$, we obtain the desired monad multiplication map $\phi_P\colon \mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown P)\longrightarrow \mathfrak{R}\otimes^\frown P$.

The unit map $\epsilon_P\colon P\longrightarrow \mathfrak{R}\otimes^\frown P$ of our monad is obtained as the composition of the map $P\longrightarrow \mathfrak{R}\otimes_k P$ induced by the unit element in $\mathfrak{R}$ with the completion map $\mathfrak{R}\otimes_k P\longrightarrow \mathfrak{R}\otimes^\frown P$. Verifying the associativity and unitality axioms of a monad for the functor $\mathfrak{R}\otimes^\frown -\colon k\text{–}\mathsf{vect}\longrightarrow k\text{–}\mathsf{vect}$ endowed with the natural transformations ϕ and ϵ is straightforward.

A *left* $\mathfrak{R}$*-contramodule* $\mathfrak{P}$ is an algebra/module over this monad on the category of k-vector spaces. In other words, it is a k-vector space endowed with a *contraaction* map $\pi\colon \mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ satisfying the following *contraassociativity* and contraunitality equations. Firstly, the two maps $\phi_{\mathfrak{P}}$ and $\mathfrak{R}\otimes^\frown\pi\colon \mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown\mathfrak{P})\longrightarrow\mathfrak{R}\otimes^\frown\mathfrak{P}$ should have equal compositions with the contraaction map π,

$$\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown\mathfrak{P})\rightrightarrows\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}.$$

Secondly, the composition of the map $\epsilon_{\mathfrak{P}}\colon\mathfrak{P}\longrightarrow\mathfrak{R}\otimes^\frown\mathfrak{P}$ with the contraaction map should be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$,

$$\mathfrak{P}\longrightarrow\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}.$$

A *free* $\mathfrak{R}$*-contramodule* is an $\mathfrak{R}$-contramodule of the form $\mathfrak{P}=\mathfrak{R}\otimes^\frown P$, where P is a k-vector space, with the contraaction map $\pi=\phi_P\colon\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown P)\longrightarrow\mathfrak{R}\otimes^\frown P$. This definition of $\mathfrak{R}$-contramodules can be found in [52, Section D.5.2].

For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any (nontopological) k-vector space E, the vector space $\mathfrak{P}=\mathrm{Hom}_k(\mathcal{N},E)$ has a natural left $\mathfrak{R}$-contramodule structure provided by the contraaction map $\pi\colon\mathfrak{R}\otimes^\frown\mathrm{Hom}_k(\mathcal{N},E)\longrightarrow\mathrm{Hom}_k(\mathcal{N},E)$ defined symbolically by the formula

$$\pi(r\otimes^\frown f)(n)=f(nr),$$

where $r\in R$, $n\in\mathcal{N}$, $f\in\mathrm{Hom}_k(\mathcal{N},E)$, and the expression in the right-hand side makes sense, since the right action map $\mathcal{N}\otimes_k\mathfrak{R}\otimes_k\mathrm{Hom}_k(\mathcal{N},E)\longrightarrow\mathcal{N}\otimes_k\mathrm{Hom}_k(\mathcal{N},E)$ restricted to $n\otimes\mathfrak{R}\otimes\mathfrak{P}\subset\mathcal{N}\otimes_k\mathfrak{R}\otimes_k\mathfrak{P}$ factorizes through the surjection $n\otimes\mathfrak{R}\otimes\mathfrak{P}\longrightarrow n\otimes\mathfrak{R}/\mathfrak{J}\otimes\mathfrak{P}$ for a certain open right ideal $\mathfrak{J}\subset\mathfrak{R}$.

Let us show that our new definition of $\mathfrak{R}$-contramodules is equivalent to the one from Section 2.1 in the case of a topological algebra $\mathfrak{R}$ over a field k. The following argument can be found in [56, Section 1.10].

Recall that the category of k-vector spaces can be defined as the category of algebras/modules over the monad $X\longmapsto k[X]$ on the category of sets. Hence for any k-vector space P there is a natural action map $p\colon k[P]\longrightarrow P$. Moreover, this map is the coequalizer of the pair of maps $k[[P]]\rightrightarrows k[P]$, one of which is the "opening of parentheses" map ϕ_P, while the other one is the map $k[p]$ induced by the map p. Indeed, applying the forgetful functor $k\text{–}\mathsf{vect}\longrightarrow\mathsf{Sets}$ makes this even a split coequalizer with an explicit splitting defined in terms of the unit map of our monad [44, Sections VI.6–7]. Subtracting one of the maps in the pair from the other one, we obtain an exact sequence in the category $k\text{–}\mathsf{vect}$

$$k[k[P]]\longrightarrow k[P]\longrightarrow P\longrightarrow 0 \tag{5}$$

for any k-vector space P.

Notice the natural isomorphism $\mathfrak{R}\otimes^\frown k[X]\simeq\mathfrak{R}[[X]]$ for any set X. Furthermore, any additive functor on the category of k-vector spaces is exact. Thus applying the functor $\mathfrak{R}\otimes^\frown -$ to the exact sequence (5), we obtain an exact sequence

$$\mathfrak{R}[[k[P]]]\longrightarrow\mathfrak{R}[[P]]\longrightarrow\mathfrak{R}\otimes^\frown P\longrightarrow 0 \tag{6}$$

for any k-vector space P. In particular, we have obtained a natural surjective map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{R}\otimes^\frown\mathfrak{P}$ for any k-vector space $\mathfrak{P}$; composing it with the contraaction map $R\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ of a contramodule $\mathfrak{P}$ over the topological k-algebra $\mathfrak{R}$, we obtain a contraaction map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ defining the structure of a contramodule over the topological ring $\mathfrak{R}$ on the set $\mathfrak{P}$.

Conversely, starting from the contraaction map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ of a contramodule $\mathfrak{P}$ over the topological ring $\mathfrak{R}$, one can first of all compose it with the natural embedding $k[\mathfrak{P}]\longrightarrow\mathfrak{R}[[\mathfrak{P}]]$, defining a k-vector space structure $k[\mathfrak{P}]\longrightarrow\mathfrak{P}$ on the set $\mathfrak{P}$. Furthermore, restricting the contraassociativity equation $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]\rightrightarrows\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ to the subset $\mathfrak{R}[[k[\mathfrak{P}]]]\subset\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]$, one discovers that the two maps $\mathfrak{R}[[k[\mathfrak{P}]]]\rightrightarrows\mathfrak{R}[[\mathfrak{P}]]$ have equal compositions with the contraaction map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$. So we see from the exact sequence (6) that the contraaction map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ factorizes through the surjective map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{R}\otimes^\frown\mathfrak{P}$, providing $\mathfrak{P}$ with a contraaction map $\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ of a contramodule over the topological k-algebra $\mathfrak{R}$.

Of course, one still has to check that the map $\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ satisfies the contraassociativity and contraunitality equations if and only if the corresponding map $\mathfrak{R}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ does. Here it helps to notice that the natural map

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]]\longrightarrow\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown\mathfrak{P}) \tag{7}$$

is surjective, so any two maps $\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown\mathfrak{P})\longrightarrow\mathfrak{P}$ are equal to each other whenever their compositions with the map (7) are. We have also seen that the class of free $\mathfrak{R}$-contramodules as defined in this section coincides with the one introduced in Section 2.1 when $\mathfrak{R}$ is a topological algebra over a field.

Now we can finally compare our notion of a contramodule over a topological ring/topological algebra over a field k with the definition of a contramodule over a (coassociative) coalgebra $\mathcal{C}$ over k given in Section 1.1. Let $\mathfrak{R}=\mathcal{C}^*$ be the dual vector space to the coalgebra $\mathcal{C}$ endowed with its pro-finite-dimensional topological algebra structure (see Sections 1.3–1.4). Then there is a natural isomorphism $R\otimes^\frown P\simeq\operatorname{Hom}_k(\mathcal{C},P)$ for any k-vector space P, making an $\mathfrak{R}$-contraaction map $\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ the same thing as a $\mathcal{C}$-contraaction map $\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\longrightarrow\mathfrak{P}$.

The vector spaces $\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown\mathfrak{P})$ and $\operatorname{Hom}_k(\mathcal{C},\operatorname{Hom}_k(\mathcal{C},\mathfrak{P}))$ parametrizing the systems of contraassociativity equations on the two kinds of contraaction maps being also naturally isomorphic, one easily checks that a map $\mathfrak{R}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ defines a left $\mathfrak{R}$-contramodule structure on a k-vector space $\mathfrak{P}$ if and only if the corresponding map $\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\longrightarrow\mathfrak{P}$ defines a left $\mathcal{C}$-contramodule structure on $\mathfrak{P}$.

2.4. **Contramodules over topological Lie algebras.** The definition of the category of contramodules over the Virasoro algebra given in Section 1.7 calls for a generalization to a reasonably large class of topological Lie algebras.

There are some naïve approaches: for example, it is easy to define comodules and contramodules over Lie coalgebras in the way analogous to the definitions for coassociative coalgebras explained in Section 1.1. This provides the notion of a contramodule over a linearly compact topological Lie algebra. Notice that the class of all Lie coalgebras is in some sense not as narrow as that of coassociative coalgebras over fields: unlike in the coassociative case (see Section 1.3), a Lie coalgebra does not have to be the union of its finite-dimensional subcoalgebras.

Indeed, it suffices to consider the case of the Lie coalgebra $\mathcal{L}$ dual to the linearly compact Lie algebra $k[[z]]d/dz$ over a field k of zero characteristic, that is the Lie subalgebra topologically spanned by the generators L_{-1}, L_0, L_1, L_2 ... in the algebra $k((z))d/dz$. The Lie algebra $k[[z]]d/dz$ having no nonzero proper closed ideals, the Lie coalgebra $\mathcal{L}$ has no nonzero proper subcoalgebras at all. Nevertheless, the class of linearly compact Lie algebras does not even contain the Virasoro algebra.

So let us start with the class of locally linearly compact, or *Tate* Lie algebras. A topological vector space V is said to be *locally linearly compact*, or a *Tate vector space* if it has a linearly compact open subspace, or equivalently, if (linearly) compact open subspaces form a base of neighborhoods of zero in V. In other words, a topological vector space is a Tate vector space if it is topologically isomorphic to the direct sum of a compact vector space and a discrete vector space (see [9, Sections 1.1–1.2 and the references therein] and [52, Section D.1.1]).

A *Tate Lie algebra* $\mathfrak{g}$ is a Tate vector space endowed with a continuous Lie algebra structure, i. e., a Lie bracket $\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ that is continuous as a function of two variables. Any Tate Lie algebra has a base of neighborhoods of zero consisting of open Lie subalgebras (see footnotes in [7, Section 3.8.17] or [9, Section 1.4], or a paragraph in [52, Section D.1.8]). For example, the "Laurent totalization" $\mathfrak{g} = \bigoplus_{n<0}\mathfrak{g}_n\oplus\prod_{n\geqslant 0}\mathfrak{g}_n$ of any $\mathbb{Z}$-graded Lie algebra $\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ with finite-dimensional components $\mathfrak{g}_n$ is a Tate Lie algebra with compact open subalgebras $\prod_{i\geqslant n}\mathfrak{g}_i\subset\mathfrak{g}$, $n\geqslant 0$, forming a base of neighborhoods of zero. This includes such classical examples as the Virasoro and Kac–Moody Lie algebras.

A *contramodule* $\mathfrak{P}$ over a Tate Lie algebra $\mathfrak{g}$ over a field k is a k-vector space endowed with a *contraaction* map $\mathfrak{g}\otimes^{\frown}\mathfrak{P}\longrightarrow\mathfrak{P}$ satisfying the following (system of) contra-Jacobi equation(s). Given a compact vector space V, denote by $V^\vee$ the discrete vector space to dual to V, so that $V^{\vee *}$ is isomorphic to V as a topological vector space. For any abstract (nontopological) vector space P, there is then a natural isomorphism of (nontopological) vector spaces $V\otimes^{\frown}P\simeq\operatorname{Hom}_k(V^\vee,P)$. So in particular we have $\mathfrak{g}\otimes^{\frown}\mathfrak{P}\simeq\varinjlim_V\operatorname{Hom}_k(V^\vee,\mathfrak{P})$, where V runs over all the compact vector subspaces $V\subset\mathfrak{g}$.

Now let U, V, and $W\subset\mathfrak{g}$ be three compact vector subspaces for which $[U,V]\subset W$; then there is a natural cobracket map $W^\vee\longrightarrow V^\vee\otimes_k U^\vee$. It required that the composition

$$\mathrm{Hom}_k(V\otimes_k U,\ \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(W,\mathfrak{P}) \longrightarrow \mathfrak{g}\otimes^{\widehat{\ }}\mathfrak{P} \longrightarrow \mathfrak{P}$$

of the map induced by the cobracket map with the contraaction map should be equal to the difference of the interated contraaction maps

$$\mathrm{Hom}_k(V\otimes_k U,\ \mathfrak{P}) \simeq \mathrm{Hom}_k(U,\mathrm{Hom}_k(V,\mathfrak{P})) \longrightarrow \mathrm{Hom}_k(U,\mathfrak{P}) \longrightarrow \mathfrak{P}$$

and

$$\mathrm{Hom}_k(V\otimes_k U,\ \mathfrak{P}) \simeq \mathrm{Hom}_k(V,\mathrm{Hom}_k(U,\mathfrak{P})) \longrightarrow \mathrm{Hom}_k(V,\mathfrak{P}) \longrightarrow \mathfrak{P}.$$

This definition can be found in [52, Section D.2.7]. Contramodules over Tate Lie algebras serve as the coefficients for the theory of *semi-infinite cohomology* of Lie algebras (as opposed to the semi-infinite *homology* [29], [7, Section 3.8]); see [52, Section D.5.6] for the definition and Section 2.8 below for a brief overview.

In order to extend the definition of a $\mathfrak{g}$-contramodule to topological Lie algebras $\mathfrak{g}$ of more general nature, we need to introduce a bit more topological linear algebra background. The following three *topological tensor product* operations were defined in [9, Section 1.1] (see also [52, Section D.1.3]).

For any topological vector spaces V and W, the !-tensor product $V\otimes^! W$ is the completion of the tensor product $V\otimes_k W$ with respect to the topology with a base of neighborhoods of zero formed by the subspaces $V'\otimes W + V\otimes W' \subset V\otimes_k W$, where $V'\subset V$ and $W'\subset W$ are open vector subspaces. In other words, one has $V\otimes^! W = \varprojlim_{V',W'} V/V'\otimes_k W/W'$, with the projective limit topology.

Furthermore, the $*$-tensor product $V\otimes^* W$ is the completion of the tensor product $V\otimes_k W$ with respect to the topology in which a vector subspace $T\subset V\otimes_k W$ is open if and only if it satisfies the following three conditions:

(i) there exist open vector subspaces $V'\subset V,\ W'\subset W$ such that $V'\otimes_k W'\subset T$;
(ii) for any vector $v\in V$ there exists an open subspace $W''\subset W$ such that $v\otimes W''\subset T$;
(iii) for any vector $w\in W$ there exists an open subspace $V''\subset V$ such that $V''\otimes w\subset T$.

For any topological vector space U, a bilinear map $V\times W\longrightarrow U$ is continuous (as a function of two variables) if and only if it can be (always uniquely) extended to a continuous linear map $V\otimes^* W\longrightarrow U$.

Finally, the $\leftarrow$-tensor product $V\overset{\leftarrow}{\otimes}W$ is the completion of the tensor product $V\otimes_k W$ with respect to the topology in which a vector subspace $T\subset V\otimes_k W$ is open if and only if it satisfies the following two conditions:

(i) there exists an open vector subspace $V'\subset V$ such that $V'\otimes_k W\subset T$;
(ii) for any vector $v\in V$ there exists an open subspace $W''\subset W$ such that $v\otimes W''\subset T$.

The underlying abstract (nontopological) vector space of the topological tensor product $V\overset{\leftarrow}{\otimes}W$ does not depend on the topology on W and is naturally isomorphic to the completed tensor product $V\otimes^{\widehat{\ }} W$ introduced in Section 2.3. The multiplication map $\mathfrak{R}\times\mathfrak{R}\longrightarrow\mathfrak{R}$ of a (complete and separated) topological associative algebra $\mathfrak{R}$ can be (uniquely) extended to a continuous linear map $\mathfrak{R}\overleftarrow{\otimes}\mathfrak{R}\longrightarrow\mathfrak{R}$ if and only if open right ideals form a base of neighborhoods of zero in $\mathfrak{R}$ [9, Section 1.4].

Each of the three tensor product operations $\otimes^!$, $\otimes^*$, and $\overleftarrow{\otimes}$ defines an associative tensor/monoidal structure on the category of topological vector spaces; the former two tensor products are also commutative. In particular, given a topological associative algebra $\mathfrak{R}$ and a k-vector space P, there is a natural isomorphism of (nontopological) vector spaces $\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown P)\simeq(\mathfrak{R}\overleftarrow{\otimes}\mathfrak{R})\otimes^\frown P$; hence the monad multiplication map $\phi_P\colon\mathfrak{R}\otimes^\frown(\mathfrak{R}\otimes^\frown P)\longrightarrow\mathfrak{R}\otimes^\frown P$ from Section 2.3 defined whenever open right ideals form a base of neighborhoods of zero in $\mathfrak{R}$ [52, Section D.5.2].

For any topological vector space V, denote by $\bigwedge^{2,*}(V)$ the completion of the nontopological exterior square $\bigwedge^2(V)$ with respect to the topology in which a vector subspace $T\subset\bigwedge^2(V)$ is open if and only if there exists an open subspace $V'\subset V$ such that $\bigwedge^2(V')\subset T$ and for any vector $v\in V$ there exists an open subspace $V''\subset V$ such that $v\wedge V''\subset T$. For any topological vector space U, a skew-commutative bilinear map $V\times V\longrightarrow U$ is continuous if and only if it can be (uniquely) extended to a continuous linear map $\bigwedge^{2,*}(V)\longrightarrow U$. The topological vector space $\bigwedge^{2,*}(V)$ can be also viewed as a closed vector subspace in $V\otimes^* V$.

A *contramodule* $\mathfrak{P}$ over a topological Lie algebra $\mathfrak{g}$ over a field k is a k-vector space endowed with a contraaction map $\pi\colon\mathfrak{g}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$ satisfying the following *contra-Jacobi equation*. The composition

$$\textstyle\bigwedge^{2,*}(\mathfrak{g})\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{g}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}$$

of the map induced by the Lie bracket $\bigwedge^{2,*}(\mathfrak{g})\longrightarrow\mathfrak{g}$ of $\mathfrak{g}$ with the contraaction map should be equal to the composition of the map induced by the natural maps of topological vector spaces $\bigwedge^2(V)\longrightarrow V\otimes^* V\longrightarrow V\overleftarrow{\otimes}V$ considered in the case $V=\mathfrak{g}$ with the iterated contraaction map

$$\begin{aligned}\textstyle\bigwedge^{2,*}(V)\otimes^\frown\mathfrak{P}\longrightarrow(\mathfrak{g}\otimes^*\mathfrak{g})\otimes^\frown\mathfrak{P}\longrightarrow{}&(\mathfrak{g}\overleftarrow{\otimes}\mathfrak{g})\otimes^\frown\mathfrak{P}\\ &\simeq\mathfrak{g}\otimes^\frown(\mathfrak{g}\otimes^\frown\mathfrak{P})\longrightarrow\mathfrak{g}\otimes^\frown\mathfrak{P}\longrightarrow\mathfrak{P}.\end{aligned}$$

This definition can be found in [52, Section D.2.6].

A *discrete module* $\mathcal{M}$ over a topological Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module for which the action map $\mathfrak{g}\times\mathcal{M}\longrightarrow\mathcal{M}$ is continuous in the given topology of $\mathfrak{g}$ and the discrete topology of $\mathcal{M}$. In other words, this means that the annihilator of any element of $\mathcal{M}$ is an open subalgebra in $\mathfrak{g}$. So one can say that discrete $\mathfrak{g}$-modules are a good category to be assigned to $\mathfrak{g}$ when open subalgebras form a base of neighborhoods of zero in $\mathfrak{g}$ (cf. [9, Sections 1.4 and 2.4]); otherwise, a discrete $\mathfrak{g}$-module is the same thing as a discrete module over the completion of $\mathfrak{g}$ in the new topology with a base consisting of of the open subalgebras in the original one. For any discrete $\mathfrak{g}$-module $\mathcal{M}$ and any (nontopological) k-vector space E, the vector space $\mathfrak{P}=\operatorname{Hom}_k(\mathcal{M},E)$ has a natural $\mathfrak{g}$-contramodule structure provided by the contraaction map $\pi\colon\mathfrak{g}\otimes^\frown\operatorname{Hom}_k(\mathcal{M},E)$

defined symbolically by the formula

$$\pi(x \otimes^\frown f)(m) = -f(xm),$$

where $x \in \mathfrak{g}$, $m \in \mathcal{M}$, $f \in \operatorname{Hom}_k(\mathcal{M}, E)$ and the expression in the right-hand side makes sense due to the definition of the completed tensor product $\mathfrak{g} \otimes^\frown P$ and the discreteness condition on the $\mathfrak{g}$-module $\mathcal{M}$ [52, Section D.2.6].

The category $\mathfrak{g}$–discr of discrete $\mathfrak{g}$-modules is abelian and the embedding/forgetful functors $\mathfrak{g}$–$\mathsf{discr} \longrightarrow \mathfrak{g}$–$\mathsf{mod} \longrightarrow k$–$\mathsf{vect}$ from it to the categories of arbitrary $\mathfrak{g}$-modules and k-vector spaces are exact. Both infinite direct sums and infinite products exist in $\mathfrak{g}$–discr; the infinite direct sums are also preserved by the forgetful functors. It follows that filtered inductive limits are exact in $\mathfrak{g}$–discr. In other words, the category $\mathfrak{g}$–discr satisfies the axioms Ab5 and Ab3*, but not in general Ab4*. It also admits a set of generators, so it has enough injective objects.

Any $\mathfrak{g}$-contamodule $\mathfrak{P}$ has an underlying structure of a module over the Lie algebra $\mathfrak{g}$ viewed as an abstract (nontopological) Lie algebra; it is provided by the composition of maps $\mathfrak{g} \otimes \mathfrak{P} \longrightarrow \mathfrak{g} \otimes^\frown \mathfrak{P} \longrightarrow \mathfrak{P}$. The category $\mathfrak{g}$–contra is abelian and the forgetful functors $\mathfrak{g}$–$\mathsf{contra} \longrightarrow \mathfrak{g}$–$\mathsf{mod} \longrightarrow k$–$\mathsf{vect}$ are exact. Infinite products exist in the category $\mathfrak{g}$–contra and are preserved by the forgetful functor. The theorem below, when it is applicable, allows one to say more (cf. Sections 1.7 and 2.1).

The enveloping algebra $U(\mathfrak{g})$ of a topological Lie algebra $\mathfrak{g}$ can be endowed with a natural topology in two opposite ways. Let us denote by $U_l^\frown(\mathfrak{g})$ the completion of $U(\mathfrak{g})$ in the topology where the left ideals in $U(\mathfrak{g})$ generated by open subspaces in $\mathfrak{g}$ form a base of neighborhoods of zero, and by $U_r^\frown(\mathfrak{g})$ the completion of $U(\mathfrak{g})$ in the similar topology with a base formed by the right ideals generated by open subspaces in $\mathfrak{g}$. Using the assumption of continuity of the bracket in $\mathfrak{g}$, one can easily check that the multiplication in $U(\mathfrak{g})$ can be extended to continuous multiplications in $U_l^\frown(\mathfrak{g})$ and $U_r^\frown(\mathfrak{g})$. This construction was considered in [7, Section 3.8.17], [9, Section 2.4], and [52, Section D.5.1].

Theorem. (a) *For any topological Lie algebra $\mathfrak{g}$, the category of discrete $\mathfrak{g}$-modules is naturally isomorphic to the category of discrete left $U_l^\frown(\mathfrak{g})$-modules, $\mathfrak{g}$–$\mathsf{discr} \simeq U_l^\frown(\mathfrak{g})$–$\mathsf{discr}$. The datum of a discrete $\mathfrak{g}$-module structure on a vector space $\mathcal{M}$ is equivalent to the datum of a discrete left $U_l^\frown(\mathfrak{g})$-module structure on $\mathcal{M}$.*

(b) *For any topological Lie algebra $\mathfrak{g}$ admitting a countable base of neighborhoods of zero consisting of open Lie subalgebras in $\mathfrak{g}$, the category of $\mathfrak{g}$-contramodules is naturally isomorphic to the category of left $U_r^\frown(\mathfrak{g})$-contramodules, $\mathfrak{g}$–$\mathsf{contra} \simeq U_r^\frown(\mathfrak{g})$–$\mathsf{contra}$. The datum of a $\mathfrak{g}$-contramodule structure on a vector space $\mathfrak{P}$ is equivalent to the datum of a left $U_r^\frown(\mathfrak{g})$-contramodule structure on $\mathfrak{P}$.*

Proof. Part (a): any $\mathfrak{g}$-module can be viewed as an $U(\mathfrak{g})$-module and vice versa; it is obvious from the definitions that a $\mathfrak{g}$-module is discrete if and only if it its $U(\mathfrak{g})$-module structure extends to a structure of discrete left module over $U_l^\frown(\mathfrak{g})$. The proof of part (b) is more complicated; see [52, Section D.5.3]. ☐

Remark. General topology and topological algebra are known to be treacherous ground, and caution is advisable when working with topological vector spaces with linear topologies, as many assertions which appear to be natural at first glance turn out to be problematic at a closer look. In particular, the exposition in the paper [9], while correcting several mistakes or unfortunate definitions found in the previous book [7], may be still too optimistic on a few points.

For example, it is *not* clear how to prove that the quotient space V/U of a topological vector space V by a closed vector subspace U is *complete* in the quotient topology, or even if it is, that the induced map of topological tensor products $V \otimes^! W \longrightarrow (V/U) \otimes^! W$ or complete tensor products $V \otimes^{\frown} P \longrightarrow (V/U) \otimes^{\frown} P$ is always surjective. In both cases, the question is how to show that a map between projective limits of vector spaces is surjective. (Cf. [11, Exercice IV.4.10.b.α], where a related counterexample in the setting of topological vector spaces with nonlinear topologies compatible with the topology of the basic field of real numbers is considered.)

The problem does not arise for topological vector spaces with countable bases of neighborhoods of zero, as countable projective limits are better behaved, and indeed, any closed subspace U that has a countable base of neighborhoods of zero is a topological direct summand in V. However, the $*$-tensor product operation leads outside of the class of topological vector spaces with countable topologies.

In particular, we formulate our system of contra-Jacobi equations as being indexed by the complete tensor product $\bigwedge^{2,*}(\mathfrak{g}) \otimes^{\frown} \mathfrak{P}$, while the somewhat simpler alternative of having it indexed by the complete tensor product $(\mathfrak{g} \otimes^* \mathfrak{g}) \otimes^{\frown} \mathfrak{P}$ would work just as well when the characteristic of the field k is different from 2. Indeed, the natural map $(\mathfrak{g} \otimes^* \mathfrak{g}) \otimes^{\frown} \mathfrak{P} \longrightarrow \bigwedge^{2,*}(\mathfrak{g}) \otimes^{\frown} \mathfrak{P}$ is surjective in this case, the topological vector space $\bigwedge^{2,*}(\mathfrak{g})$ being a direct summand in $\mathfrak{g} \otimes^* \mathfrak{g}$. Then the contra-Jacobi equation could be written in the familiar form of the difference between the two compositions $(\mathfrak{g} \otimes^* \mathfrak{g}) \otimes^{\frown} \mathfrak{P} \rightrightarrows \mathfrak{g} \otimes^{\frown} (\mathfrak{g} \otimes^{\frown} \mathfrak{P}) \longrightarrow \mathfrak{g} \otimes^{\frown} \mathfrak{P} \longrightarrow \mathfrak{P}$ being equal to the composition $(\mathfrak{g} \otimes^* \mathfrak{g}) \otimes^{\frown} \mathfrak{P} \longrightarrow \mathfrak{g} \otimes^{\frown} \mathfrak{P} \longrightarrow \mathfrak{P}$. The desire to incorporate the characteristic 2 case leads to the somewhat more complicated definition above.

2.5. **Contramodules over corings.** The following scheme of categorical buildup is discussed in the introduction to the book [52]. Let K be a category endowed with an (associative, noncommutative) monoidal (tensor) category structure, M be a left module category over it, N be a right module category, and V be a category for which there is a pairing between the module categories N and M over K taking values in V.

This means that, in addition to the multiplication functor $\otimes\colon \mathsf{K} \times \mathsf{K} \longrightarrow \mathsf{K}$, there are also action functors

$$\otimes\colon \mathsf{N} \times \mathsf{K} \longrightarrow \mathsf{N} \quad \text{and} \quad \otimes\colon \mathsf{K} \times \mathsf{M} \longrightarrow \mathsf{M}$$

and a pairing functor $\otimes\colon \mathsf{N} \times \mathsf{M} \longrightarrow \mathsf{V}$. Furthermore, there are associativity constraints for the ternary multiplications

$$\mathsf{K} \times \mathsf{K} \times \mathsf{K} \longrightarrow \mathsf{K}, \quad \mathsf{N} \times \mathsf{K} \times \mathsf{K} \longrightarrow \mathsf{K}, \quad \mathsf{K} \times \mathsf{K} \times \mathsf{M} \longrightarrow \mathsf{M}, \quad \mathsf{N} \times \mathsf{K} \times \mathsf{M} \longrightarrow \mathsf{V}$$

satisfying the pentagonal diagram equations for products of four factors.

Let A be an associative ring object in K. Then one can consider the category ${}_A\mathsf{K}_A$ of A-A-bimodule objects in K, the category ${}_A\mathsf{M}$ of left A-module objects in M, and the category N_A of right A-module objects in N. When the categories K, M, N, and V are abelian (or additive categories with cokernels, or, at least, admit coequalizers), there are the functors $\otimes_A$ of tensor product over A, making ${}_A\mathsf{K}_A$ a tensor category, ${}_A\mathsf{M}$ a left module category over it, N_A a right module category, and providing a pairing $\mathsf{N}_A \times {}_A\mathsf{M} \longrightarrow \mathsf{V}$. The new tensor structures $\otimes_A$ are associative whenever the original tensor product functors $\otimes$ were right exact (preserved coequalizers).

Inverting the arrows in all the four categories, one comes to considering the situation of a *co*ring object $C \in \mathsf{K}$. Then there is the category ${}_C\mathsf{K}_C$ of C-C-bicomodule objects in K, the category ${}_C\mathsf{M}$ of left C-comodule objects in M, and the category N_C of right C-comodule objects in N. When the categories K, M, N, and V are abelian (or, at least, admit equalizers), there are the functors $\square_C$ of *cotensor product* over C, making ${}_C\mathsf{K}_C$ a tensor category, ${}_C\mathsf{M}$ a left module category over it, N_C a right module category, and providing a pairing $\mathsf{N}_C \times {}_C\mathsf{M} \longrightarrow \mathsf{V}$. The new tensor structures $\square_{\mathcal{C}}$ are associative whenever the functors $\otimes$ were left exact (preserved equalizers).

Now one may wish to iterate this construction, considering a coring object C in the category of A-A-bimodules ${}_A\mathsf{K}_A$, the categories of C-comodules in the categories of A-modules ${}_A\mathsf{M}$ and N_A, the category of C-C-bicomodules in ${}_A\mathsf{K}_A$, etc. Then one encounters the typical phenomenon of progressive relaxation/worsening of algebraic properties at every step of a buildup.

The functor $\otimes_A$ of tensor product over a ring object A is in most cases *not* left exact (being defined as a certain coequalizer, it does not preserve equalizers). Hence the cotensor product over a coring object $C \in {}_A\mathsf{K}_A$ will be only associative under certain (co)flatness conditions imposed on the objects involved. But the associativity is necessary to even *define* tensor products over ring objects. So when one makes the next step and considers a ring object S in the category of C-C-bicomodules in ${}_A\mathsf{K}_A$, one discovers that the functors of tensor product over S are only partially defined.

In this section, we consider coring objects $\mathcal{C}$ in the category of bicomodules over a conventional ring A (i. e., a ring object in the tensor category of abelian groups $\mathsf{K} = \mathsf{Ab}$). So let A be an associative ring (with unit).

A *coring* $\mathcal{C}$ over a ring A is an A-A-bimodule endowed with a *comultiplication* map $\mu\colon \mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C}$ and a *counit* map $\varepsilon\colon \mathcal{C} \longrightarrow A$ satisfying the following *linearity*, *coassociativity*, and *counitality* equations. First of all, both maps μ and ε must be *A-A-bimodule morphisms*. Secondly, the two compositions of the comultiplication map μ with the two maps $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mu\colon \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}$ induced by the comultiplication map should be equal to each other,

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C}.$$

Thirdly, both the compositions of the comultiplication map with the two maps $\varepsilon \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varepsilon\colon \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C}$ induced by the counit map ε should be equal to the identity map $\mathrm{id}_{\mathcal{C}}$,

$$\mathcal{C} \longrightarrow \mathcal{C} \otimes_A \mathcal{C} \rightrightarrows \mathcal{C}.$$

A *left comodule* $\mathcal{M}$ over a coring $\mathcal{C}$ over a ring A is a comodule object in the left module category of left A-modules over the coring object $\mathcal{C}$ in the tensor category of

A-A-bimodules. In other words, it is a left A-module endowed with a *left* $\mathcal{C}$-*coaction* map $\nu_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$ satisfying the following linearity, *coassociativity*, and *counitality* equations. First of all, the map $\nu = \nu_{\mathcal{M}}$ must be a *left A-module morphism.* Secondly, the compositions of the coaction map ν with the two maps $\mu \otimes \mathrm{id}$ and $\mathrm{id} \otimes \nu\colon \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}$ induced by the comultiplication and coaction maps should be equal to each other,

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \rightrightarrows \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{M}.$$

Thirdly, the composition of the coaction map with the map $\epsilon \otimes \mathrm{id}\colon \mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{M}$ induced by the counit map should be equal to the identity map $\mathrm{id}_{\mathcal{M}}$,

$$\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{M}.$$

Similarly, a *right comodule* $\mathcal{N}$ over $\mathcal{C}$ is a right A-module endowed with a right $\mathcal{C}$-coaction map $\nu_{\mathcal{N}}\colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C}$, which must be a right A-module morphism satisfying the coassociativity and counitality equations

$$\mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C} \rightrightarrows \mathcal{N} \otimes_A \mathcal{C} \otimes_A \mathcal{C},$$
$$\mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C} \longrightarrow \mathcal{N}.$$

These definitions can be found in [14, Sections 17.1 and 18.1] or [52, Section 1.1.1]. Corings and comodules also appear in noncommutative geometry, or more specifically, in connection with noncommutative semi-separated stacks [40, 41].

Before introducing $\mathcal{C}$-*contramodules*, let us discuss a bit more abstract nonsense. The conventional tensor calculus over a ring A includes, in addition to the tensor product functor $\otimes_A$, the functor Hom_A of homomorphisms of (say, left) A-modules. Applying the functor Hom_A to an A-A-bimodule E and a left A-module P produces a left A-module $\mathrm{Hom}_A(E,P)$. In fact, the functor Hom_A endows the category $A\text{–}\mathsf{mod}^{\mathsf{op}}$ *opposite to* the category of left A-modules with a *right* module category structure over the tensor category of A-A-bimodules $A\text{–}\mathsf{mod}\text{–}A$. Indeed, for any A-A-bimodules K and L and a left A-module P one has

$$\mathrm{Hom}_A(L, \mathrm{Hom}_A(K,P)) \simeq \mathrm{Hom}_A(K \otimes_A L,\ P),$$

or, denoting temporarily $P^{\mathsf{op}} *_A K = \mathrm{Hom}_A(K,P)^{\mathsf{op}}$,

$$(P^{\mathsf{op}} *_A K) *_A L \simeq P^{\mathsf{op}} *_A (K \otimes_A L).$$

(cf. the discussion of Hom space identification rules (1) and (2) in Section 1.1). In other words, one can say that the functor Hom_A makes the category of left A-modules a *left Hom category* over the tensor category of A-A-bimodules. The same functor $\mathrm{Hom}_A(-,-)$ provides a pairing between the left module category $A\text{–}\mathsf{mod}$ and the right module category $A\text{–}\mathsf{mod}^{\mathsf{op}}$ over the tensor category $A\text{–}\mathsf{mod}\text{–}A$ taking values in the category of abelian groups Ab.

A *left contramodule* $\mathfrak{P}$ over a coring $\mathcal{C}$ over a ring A is an object of the category opposite to the category of module objects in the right module category $A\text{–}\mathsf{mod}^{\mathsf{op}}$ over the coring object $\mathcal{C}$ in the tensor category $A\text{–}\mathsf{mod}\text{–}A$. In other words, it is a left A-module endowed with a *left* $\mathcal{C}$-*contraaction* map $\pi_{\mathfrak{P}}\colon \mathrm{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the following linearity, *contraassociativity*, and *contraunitality* equations. First of all, the map $\pi = \pi_{\mathfrak{P}}$ must be a left A-module morphism. Secondly, the compositions of the maps $\mathrm{Hom}(\mu, \mathfrak{P})\colon \mathrm{Hom}_A(\mathcal{C} \otimes_A \mathcal{C},\ \mathfrak{P}) \longrightarrow \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P})$ and $\mathrm{Hom}(\mathcal{C}, \pi)\colon \mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P})) \longrightarrow \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P})$ induced by the comultiplication and contraaction maps with the contraaction map should be equal to each other,

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P})) \simeq \mathrm{Hom}_A(\mathcal{C} \otimes_A \mathcal{C},\ \mathfrak{P}) \rightrightarrows \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}.$$

Thirdly, the composition of the map $\mathfrak{P} \longrightarrow \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P})$ induced by the counit map ε with the contraaction map should be equal to the identity map,

$$\mathfrak{P} \longrightarrow \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}.$$

This definition can be found in [52, Section 3.1.1]. In a slightly lesser generality of coassociative coalgebras over commutative rings, it was first given, together with the definition of a comodule, in the memoir [24, Section III.5].

For any right $\mathcal{C}$-comodule $\mathcal{N}$ endowed with a left action of a ring B by right $\mathcal{C}$-comodule endomorphisms and any left B-module V the abelian group $\mathrm{Hom}_B(\mathcal{N}, V)$ has a natural left $\mathcal{C}$-contramodule structure. Here the left action of A in $\mathrm{Hom}_B(\mathcal{N}, V)$ is induced by the right action of A in $\mathcal{N}$, and the left $\mathcal{C}$-contraaction morphism $\pi\colon \mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, V)) \longrightarrow \mathrm{Hom}_B(\mathcal{N}, V)$ is obtained by applying the contravariant functor $\mathrm{Hom}_B(-, V)$ to the right $\mathcal{C}$-coaction morphism $\nu\colon \mathcal{N} \longrightarrow \mathcal{N} \otimes_A \mathcal{C}$,

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_B(\mathcal{N}, V)) \simeq \mathrm{Hom}_B(\mathcal{N} \otimes_A \mathcal{C},\ V) \longrightarrow \mathrm{Hom}_B(\mathcal{N}, V).$$

The left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A V$, where V is a left A-module, is called the $\mathcal{C}$-comodule *coinduced* from an A-module V. For any left $\mathcal{C}$-comodule $\mathcal{L}$ there is a natural isomorphism of abelian groups

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\ \mathcal{C} \otimes_A V) \simeq \mathrm{Hom}_A(\mathcal{L}, V),$$

where $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{M})$ denotes the group of morphisms from a $\mathcal{C}$-comodule $\mathcal{L}$ to a $\mathcal{C}$-comodule $\mathcal{M}$ in the category $\mathcal{C}\text{–}\mathsf{comod}$ of left $\mathcal{C}$-comodules [52, Section 1.1.2].

The left $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C}, V)$, where V is a left A-module, is called the $\mathcal{C}$-contramodule *induced* from an A-module V. For any left $\mathcal{C}$-contramodule $\mathfrak{Q}$ there is a natural isomorphism of abelian groups

$$\mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_A(\mathcal{C}, V), \mathfrak{Q}) \simeq \mathrm{Hom}_A(V, \mathfrak{Q}),$$

where $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{Q})$ denotes the group of morphisms from a $\mathcal{C}$-contramodule $\mathfrak{P}$ to a $\mathcal{C}$-contramodule $\mathfrak{Q}$ in the category $\mathcal{C}\text{–}\mathsf{contra}$ of left $\mathcal{C}$-contramodules [52, Section 3.1.2].

Proposition. (a) *The following two conditions on a coring $\mathcal{C}$ are equivalent:*

- *the category of left $\mathcal{C}$-comodules is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow A\text{–}\mathsf{mod}$ is exact;*
- *the coring $\mathcal{C}$ is a flat right A-module.*

(b) *The following two conditions on a coring $\mathcal{C}$ are equivalent:*

- *the category of left $\mathcal{C}$-contramodules is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow A\text{–}\mathsf{mod}$ is exact;*
- *the coring $\mathcal{C}$ is a projective left A-module.*

Proof. One defines a $\mathcal{C}$-comodule or $\mathcal{C}$-contramodule structure on the kernel and cokernel of any morphism of left $\mathcal{C}$-comodules or left $\mathcal{C}$-contamodules computed in the category of abelian groups/left A-modules, assuming respectively that the functor $\mathcal{C} \otimes_A -\colon A\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ is exact (preserves kernels) or the functor $\operatorname{Hom}_A(\mathcal{C}, -)\colon A\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ is exact (preserves cokernels). This allows to show that the second condition implies the first one in either part (a) or (b).

To prove the converse implication in part (a), notice that the functor $\mathcal{C} \otimes_A -\colon A\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ is the composition of the coinduction functor $A\text{–}\mathsf{mod} \longrightarrow \mathcal{C}\text{–}\mathsf{comod}$ and the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow A\text{–}\mathsf{mod}$, the former of which is right adjoint to the latter one. Since any right adjoint functor between abelian categories is left exact, one concludes that the functor $\mathcal{C} \otimes_A -$ is left exact whenever the forgetful functor is exact. Similarly, in part (b) the functor $\operatorname{Hom}_A(\mathcal{C}, -)\colon A\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ is the composition of the induction functor $A\text{–}\mathsf{mod} \longrightarrow \mathcal{C}\text{–}\mathsf{contra}$ and the forgetful functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow A\text{–}\mathsf{mod}$, the former of which is left adjoint to the latter one. Since any left adjoint functor is right exact, the functor $\operatorname{Hom}_A(\mathcal{C}, -)$ is right exact whenever the forgetful functor is exact. □

Generally speaking, the cokernels of arbitrary morphisms exist in $\mathcal{C}\text{–}\mathsf{comod}$ and are preserved by the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow A\text{–}\mathsf{mod}$, but the kernels in $\mathcal{C}\text{–}\mathsf{comod}$ may be problematic when $\mathcal{C}$ is not a flat right A-module. Similarly, the kernels of arbitrary morphisms exist in $\mathcal{C}\text{–}\mathsf{contra}$ and are preserved by the forgeftul functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow A\text{–}\mathsf{mod}$, but the cokernels in $\mathcal{C}\text{–}\mathsf{contra}$ may be problematic when $\mathcal{C}$ is not a projective left A-module. Counterexamples showing that the categories $\mathcal{C}\text{–}\mathsf{comod}$ and $\mathcal{C}\text{–}\mathsf{contra}$ are *not* abelian in general can be found in [57, Example B.1.1].

Assume that the coring $\mathcal{C}$ is a flat right A-module; then, according to Proposition, the category $\mathcal{C}\text{–}\mathsf{comod}$ is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow A\text{–}\mathsf{mod}$ is exact. Both the infinite direct sums and infinite products exist in $\mathcal{C}\text{–}\mathsf{comod}$; the infinite direct sums are exact and are preserved by the forgetful functor. Filtered inductive limits are exact in the category of left $\mathcal{C}$-comodules; so it satisfies the axioms Ab5 and Ab3*, but *not* in general Ab4*. The category $\mathcal{C}\text{–}\mathsf{comod}$ also has a set of generators [14, Sections 3.13 and 18.14]; moreover, when A is a left Noetherian ring or $\mathcal{C}$ is a projective right A-module, every left $\mathcal{C}$-comodule is the union of its subcomodules that are finitely generated as A-modules [14, Sections 18.16 and 19.12].

The coaction map $\nu\colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$ embeds every left $\mathcal{C}$-comodule $\mathcal{M}$ as a subcomodule into the coinduced $\mathcal{C}$-comodule $\mathcal{C} \otimes_A \mathcal{M}$. Infinite products of coinduced $\mathcal{C}$-comodules are computed by the rule $\prod_\alpha \mathcal{C} \otimes_A V_\alpha = \mathcal{C} \otimes_A \prod_\alpha V_\alpha$ [14, Section 18.13]; to compute the product of an arbitrary family of left $\mathcal{C}$-comodules, one can present them as the kernels of morphisms of coinduced $\mathcal{C}$-comodules and use the fact that infinite products always commute with the kernels [52, Section 1.1.2]. There are enough injective objects in the category $\mathcal{C}\text{–}\mathsf{comod}$; a left $\mathcal{C}$-comodule is injective if and only if it is isomorphic to a direct summand of a $\mathcal{C}$-comodule $\mathcal{C} \otimes_A J$ coinduced from an injective left A-module J (see [14, Section 18.19] or [52, Section 5.1.5]).

Assume that the coring $\mathcal{C}$ is a projective left A-module; then, according to Proposition, the category $\mathcal{C}\text{–}\mathsf{contra}$ is abelian and the forgetful functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow A\text{–}\mathsf{mod}$

is exact. Both the infinite direct sums and infinite products exist in $\mathcal{C}$–contra; the infinite products are exact and are preserved by the forgetful functor. So the category of left $\mathcal{C}$-contramodules satisfies the axioms Ab3 and Ab4*, but *not* in general Ab4 or Ab5*.

The contraaction map $\pi\colon \mathrm{Hom}_A(\mathcal{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ presents every left $\mathcal{C}$-contramodule as a quotient contramodule of the induced $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C},\mathfrak{P})$. Infinite direct sums of induced $\mathcal{C}$-contramodules are computed by the rule $\bigoplus_\alpha \mathrm{Hom}_A(\mathcal{C},V_\alpha) = \mathrm{Hom}_A(\mathcal{C},\ \bigoplus_\alpha V_\alpha)$; to compute the direct sum of an arbitrary family of left $\mathcal{C}$-contramodules, one can present them as the cokernels of morphisms of induced $\mathcal{C}$-contramodules and use the fact that infinite direct sums always commute with the cokernels [52, Section 3.1.2]. There are enough projective objects in $\mathcal{C}$–contra; a left $\mathcal{C}$-contramodule is projective if and only if it is a direct summand of a $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C},F)$ induced from a projective left A-module F [52, Section 5.1.5].

The discussion in the beginning of this section suggests that one should consider, in addition to the categories of left $\mathcal{C}$-comodules, right $\mathcal{C}$-comodules, and left $\mathcal{C}$-contramodules, the pairing functors of *cotensor product* and *cohomomorphisms* acting from those categories to the category of abelian groups. Let us define these functors of two co/contramodule arguments now.

The *cotensor product* $\mathcal{N}\,\square_{\mathcal{C}}\,\mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ is an abelian group defined as the kernel of the difference of the pair of maps

$$\nu_{\mathcal{N}}\otimes\mathrm{id},\ \ \mathrm{id}\otimes\nu_{\mathcal{M}}\colon \mathcal{N}\otimes_A\mathcal{M} \rightrightarrows \mathcal{N}\otimes_A\mathcal{C}\otimes_A\mathcal{M}$$

one of which is induced by the $\mathcal{C}$-coaction in $\mathcal{N}$ and the other one by the $\mathcal{C}$-coaction in $\mathcal{M}$. For any right $\mathcal{C}$-comodule $\mathcal{N}$ and any left A-module V there is a natural isomorphism of abelian groups

$$\mathcal{N}\,\square_{\mathcal{C}}\,(\mathcal{C}\otimes_A V)\simeq \mathcal{N}\otimes_A V;$$

the similar formula holds for the cotensor product of a coinduced right $\mathcal{C}$-comodule and an arbitrary left $\mathcal{C}$-comodule. In particular, one has $\mathcal{N}\square_{\mathcal{C}}\mathcal{C}\simeq\mathcal{N}$ and $\mathcal{C}\square_{\mathcal{C}}\mathcal{M}\simeq\mathcal{M}$ (see [14, Section 21] or [52, Sections 0.2.1 and 1.2.1]).

The abelian group of *cohomomorphisms* $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{M},\mathfrak{P})$ from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is defined as the cokernel of (the difference of) the pair of maps

$$\begin{aligned}\mathrm{Hom}(\nu_{\mathcal{M}},\mathrm{id}),\ \ \mathrm{Hom}(\mathrm{id},\pi_{\mathfrak{P}})\colon\ &\mathrm{Hom}_A(\mathcal{C}\otimes_A\mathcal{M},\ \mathfrak{P})\\ &\simeq \mathrm{Hom}_A(\mathcal{M},\mathrm{Hom}_A(\mathcal{C},\mathfrak{P}))\rightrightarrows \mathrm{Hom}_A(\mathcal{M},\mathfrak{P}).\end{aligned}$$

For any left $\mathcal{C}$-comodule $\mathcal{M}$, and left $\mathcal{C}$-contramodule $\mathfrak{P}$, and any left A-module V, there are natural isomorphisms of abelian groups

$$\mathrm{Cohom}_{\mathcal{C}}(\mathcal{C}\otimes_A V,\ \mathfrak{P})\simeq\mathrm{Hom}_A(V,\mathfrak{P})$$
$$\mathrm{Cohom}_{\mathcal{C}}(\mathcal{M},\mathrm{Hom}_A(\mathcal{C},V))\simeq\mathrm{Hom}_A(\mathcal{M},V);$$

in particular, one has $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{C},\mathfrak{P})\simeq\mathfrak{P}$ [52, Sections 0.2.4 and 3.2.1].

Notice that the functor of cotensor product $\square_{\mathcal{C}}$ over a coring $\mathcal{C}$, being defined as the kernel of a morphism of cokernels, is *neither* left *nor* right exact in general. Similarly, the functor $\mathrm{Cohom}_{\mathcal{C}}$, being defined as the cokernel of a morphism of kernels, is neither left *nor* right exact (even when all the categories involved are abelian and all the forgetful functors are exact).

2.6. **Semicontramodules over semialgebras.** The notion of a *semialgebra* over a coalgebra over a field is dual to that of a coring in the same way as the notion of a coalgebra over a field is dual to that of an (associative) ring [1, 13, 51, 52]. In this section we present the related piece of theory, aiming to define semimodules and semicontramodules over semialgebras and interpret contramodules over topological groups as semicontramodules over certain semialgebras, as it was promised in Section 1.8.

Let $\mathcal{C}$ be a (coassociative) coalgebra (with counit) over a field k. In addition to the definitions of left $\mathcal{C}$-comodules, right $\mathcal{C}$-comodules and left $\mathcal{C}$-contramodules given in Section 1.1 and then repeated, in the greater generality of a coring $\mathcal{C}$, in the previous Section 2.5, we will also need the definition of a $\mathcal{C}$-$\mathcal{C}$-bicomodule.

Let $\mathcal{D}$ be another coalgebra over k. A $\mathcal{C}$-$\mathcal{D}$-*bicomodule* $\mathcal{K}$ is a k-vector space endowed with a left $\mathcal{C}$-comodule and a right $\mathcal{D}$-comodule structures $\nu'\colon \mathcal{K} \longrightarrow \mathcal{C}\otimes_k\mathcal{K}$ and $\nu''\colon \mathcal{K} \longrightarrow \mathcal{K}\otimes_k \mathcal{D}$ which commute with each other in the following sense. The composition of the left coaction map $\nu'\colon \mathcal{K} \longrightarrow \mathcal{C}\otimes_k \mathcal{K}$ with the map $\mathrm{id}\otimes\nu''$: $\mathcal{C}\otimes_k\mathcal{K} \longrightarrow \mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}$ induced by the right coaction map ν'' should be equal to the composition of the right coaction map $\nu''\colon \mathcal{K}\longrightarrow\mathcal{K}\otimes_k\mathcal{D}$ with the map $\nu'\otimes\mathrm{id}\colon \mathcal{K}\otimes_k \mathcal{D}\longrightarrow \mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}$ induced by the left coaction map ν'. Equivalently, the vector space $\mathcal{K}$ should be endowed with a $\mathcal{C}$-$\mathcal{D}$-*bicoaction* map $\nu\colon \mathcal{K}\longrightarrow \mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}$ satisfying the coassociativity and counitality equations $(\mu_{\mathcal{C}}\otimes\mathrm{id}_{\mathcal{K}}\otimes\mu_{\mathcal{D}})\circ\nu = (\mathrm{id}_{\mathcal{C}}\otimes\nu\otimes\mathrm{id}_{\mathcal{D}})\circ\nu$ and $(\varepsilon_{\mathcal{C}}\otimes\mathrm{id}_{\mathcal{K}}\otimes\varepsilon_{\mathcal{D}})\circ\nu = \mathrm{id}_{\mathcal{K}}$,

$$\mathcal{K}\longrightarrow\mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}\rightrightarrows\mathcal{C}\otimes_k\mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}\otimes_k\mathcal{D}$$
$$\mathcal{K}\longrightarrow\mathcal{C}\otimes_k\mathcal{K}\otimes_k\mathcal{D}\longrightarrow\mathcal{K}$$

(see [14, Sections 11.1 or 22.1] or [52, Sections 0.3.1 or 1.2.4]).

Recall from the end of the previous section that the *cotensor product* $\mathcal{N}\,\square_{\mathcal{C}}\,\mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ is the k-vector space constructed as the kernel of (the difference of) the pair of maps

$$\nu_{\mathcal{N}}\otimes\mathrm{id},\ \ \mathrm{id}\otimes\nu_{\mathcal{M}}\colon \mathcal{N}\otimes_k\mathcal{M}\rightrightarrows\mathcal{N}\otimes_k\mathcal{C}\otimes_k\mathcal{M}$$

induced by the $\mathcal{C}$-coactions maps in $\mathcal{N}$ and $\mathcal{M}$. Similarly, the k-vector space of *cohomomorphisms* from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is construced as the cokernel of the pair of maps

$$\mathrm{Hom}_k(\mathcal{C}\otimes_k\mathcal{M},\ \mathfrak{P})\simeq\mathrm{Hom}_k(\mathcal{M},\mathrm{Hom}_k(\mathcal{C},\mathfrak{P}))\rightrightarrows\mathrm{Hom}_k(\mathcal{M},\mathfrak{P})$$

one of which is induced by the $\mathcal{C}$-coaction in $\mathcal{M}$ and the other one by the $\mathcal{C}$-contraaction in $\mathfrak{P}$. The functor of cotensor product of comodules over a coalgebra $\mathcal{C}$ over a field k, being defined as the kernel of a morphism of exact functors, is left exact; while the functor of cohomomorphisms of comodules and contramodules over $\mathcal{C}$, defined as the cokernel of a morphism of exact functors, is right exact. For left $\mathcal{C}$-comodule $\mathcal{M}$, right $\mathcal{C}$-comodule $\mathcal{N}$, and k-vector space V, there is a natural isomorphism of k-vector spaces [52, Sections 0.2.4 and 3.2.2, and Proposition 3.2.3.1]

$$\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M}, \operatorname{Hom}_k(\mathcal{N}, V)) \simeq \operatorname{Hom}_k(\mathcal{N} \mathbin{\square}_{\mathcal{C}} \mathcal{M},\ V),$$

where the k-vector space $\operatorname{Hom}_k(\mathcal{N}, V)$ is endowed with a left $\mathcal{C}$-contramodule structure as explained in Section 1.2.

For any three coalgebras $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{N}$, and any $\mathcal{D}$-$\mathcal{E}$-bicomodule $\mathcal{M}$, the cotensor product $\mathcal{N} \mathbin{\square}_{\mathcal{D}} \mathcal{M}$ has a natural $\mathcal{C}$-$\mathcal{E}$-bicomodule structure. Furthermore, for any right $\mathcal{C}$-comodule $\mathcal{N}$, any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, and any left $\mathcal{C}$-comodule $\mathcal{M}$ there is a natural associativity isomorphism

$$(\mathcal{N} \mathbin{\square}_{\mathcal{C}} \mathcal{K}) \mathbin{\square}_{\mathcal{D}} \mathcal{M} \simeq \mathcal{N} \mathbin{\square}_{\mathcal{C}} (\mathcal{K} \mathbin{\square}_{\mathcal{D}} \mathcal{M}).$$

To put it simply, both the iterated cotensor products are identified with one and the same subspace in the vector space $\mathcal{N} \otimes_k \mathcal{K} \otimes_k \mathcal{M}$ (cf. the beginning of Section 2.5).

Similarly, for any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$ and any left $\mathcal{C}$-contramodule $\mathfrak{P}$, the space of cohomomorphisms $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}, \mathfrak{P})$ has a natural left $\mathcal{D}$-contramodule structure. One can define it by noticing that $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}, \mathfrak{P})$ is a quotient contramodule of the left $\mathcal{D}$-contramodule $\operatorname{Hom}_k(\mathcal{K}, \mathfrak{P})$, whose contramodule structure is induced by the right $\mathcal{D}$-comodule structure on $\mathcal{K}$ via the construction described in Section 1.2. For any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, any left $\mathcal{D}$-comodule $\mathcal{M}$, and any left $\mathcal{C}$-contramodule $\mathfrak{P}$, there is a natural associativity isomorphism

$$\operatorname{Cohom}_{\mathcal{C}}(\mathcal{K} \mathbin{\square}_{\mathcal{D}} \mathcal{M},\ \mathfrak{P}) \simeq \operatorname{Cohom}_{\mathcal{D}}(\mathcal{M},\ \operatorname{Cohom}_{\mathcal{C}}(\mathcal{K}, \mathfrak{P})).$$

Both the (iterated) Cohom spaces are identified with the quotient space of the vector space $\operatorname{Hom}_k(\mathcal{K} \otimes_k \mathcal{M},\ \mathfrak{P}) \simeq \operatorname{Hom}_k(\mathcal{M}, \operatorname{Hom}_k(\mathcal{K}, \mathfrak{P}))$ by one and the same vector subspace [52, Sections 0.3.4 or 3.2.4].

In particular, it follows from these associativity isomorphisms for a coalgebra $\mathcal{C} = \mathcal{D}$ that the category of $\mathcal{C}$-$\mathcal{C}$-bicomodules $\mathcal{C}\text{–}\mathsf{comod}\text{–}\mathcal{C}$ is an associative tensor category with respect to the cotensor product functor $\square_{\mathcal{C}}$, the category of left $\mathcal{C}$-comodules $\mathcal{C}\text{–}\mathsf{comod}$ is a left module category over $\mathcal{C}\text{–}\mathsf{comod}\text{–}\mathcal{C}$, and the category $\mathcal{C}\text{–}\mathsf{contra}^{\mathsf{op}}$ opposite to the category of left $\mathcal{C}$-contramodules is a right module category over $\mathcal{C}\text{–}\mathsf{comod}\text{–}\mathcal{C}$ with respect to the cohomomorphism functor $\operatorname{Cohom}_{\mathcal{C}}$.

A *semialgebra* $\boldsymbol{\mathcal{S}}$ over a coalgebra $\mathcal{C}$ over a field k is an associative ring object in the tensor category of $\mathcal{C}$-$\mathcal{C}$-bicomodules. In other words, it is a $\mathcal{C}$-$\mathcal{C}$-bicomodule endowed with a *semimultiplication* map $\mathbf{m}\colon \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$ and a *semiunit* map $\mathbf{e}\colon \mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$ satisfying the following *colinearity*, *semiassociativity* and *semiunitality* equations. First of all, the maps $\mathbf{m}$ and $\mathbf{e}$ must be $\mathcal{C}$-$\mathcal{C}$-bicomodule morphisms. Secondly, the compositions of the two maps $\mathbf{m} \mathbin{\square} \mathrm{id}_{\boldsymbol{\mathcal{S}}}$ and $\mathrm{id}_{\boldsymbol{\mathcal{S}}} \mathbin{\square} \mathbf{m}\colon \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \rightrightarrows \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}}$ induced by the semimultiplication map $\mathfrak{m}$ with the semimultiplication map

$$\boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \rightrightarrows \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$$

should be equal to each other, $\mathbf{m} \circ (\mathbf{m} \mathbin{\square} \mathrm{id}_{\boldsymbol{\mathcal{S}}}) = \mathbf{m} \circ (\mathrm{id}_{\boldsymbol{\mathcal{S}}} \mathbin{\square} \mathbf{m})$. Thirdly, both the compositions of the maps $\mathbf{e} \mathbin{\square} \mathrm{id}_{\boldsymbol{\mathcal{S}}}$ and $\mathrm{id}_{\boldsymbol{\mathcal{S}}} \mathbin{\square} \mathbf{e}\colon \boldsymbol{\mathcal{S}} \rightrightarrows \boldsymbol{\mathcal{S}} \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}}$ induced by the semiunit map $\mathbf{e}$ with the semimultiplication map $\mathbf{m}$

$$\mathfrak{S} \rightrightarrows \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{S} \longrightarrow \mathfrak{S}$$

should be equal to the identity map, $\mathbf{m} \circ (\mathbf{e} \,\square\, \mathrm{id}_{\mathfrak{S}}) = \mathrm{id}_{\mathfrak{S}} = \mathbf{m} \circ (\mathrm{id}_{\mathfrak{S}} \,\square\, \mathbf{e})$.

A *left semimodule* $\mathfrak{M}$ over a semialgebra $\mathfrak{S}$ over a coalgebra $\mathfrak{C}$ is a module object in the left module category of left $\mathfrak{C}$-comodules over the ring object $\mathfrak{S}$ in the tensor category of $\mathfrak{C}$-$\mathfrak{C}$-bicomodules. In other words, it is a left $\mathfrak{C}$-comodule endowed with a *left semiaction* map $\mathbf{n}\colon \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M} \longrightarrow \mathfrak{M}$ satisfying the following colinearity, *semiassociativity* and *semiunitality* equations. First of all, the map $\mathbf{n}$ must be a left $\mathfrak{C}$-comodule morphism. Secondly, the compositions of the two maps $\mathbf{m} \,\square\, \mathrm{id}_{\mathfrak{M}}$ and $\mathrm{id}_{\mathfrak{S}} \,\square\, \mathbf{n}\colon \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M} \rightrightarrows \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M}$ induced by the semimultiplication and semiaction maps with the semiaction map

$$\mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M} \rightrightarrows \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M} \longrightarrow \mathfrak{M}$$

should be equal to each other, $\mathbf{n} \circ (\mathbf{m} \,\square\, \mathrm{id}_{\mathfrak{M}}) = \mathbf{n} \circ (\mathrm{id}_{\mathfrak{S}} \,\square\, \mathbf{n})$. Thirdly, the composition of the map $\mathbf{e} \,\square_{\mathfrak{C}}\, \mathrm{id}_{\mathfrak{M}}\colon \mathfrak{M} \longrightarrow \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M}$ induced by the semiunit map $\mathbf{e}$ with the semiaction map $\mathbf{n}$

$$\mathfrak{M} \longrightarrow \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{M} \longrightarrow \mathfrak{M}$$

should be equal to the identity map, $\mathbf{n} \circ (\mathbf{e} \,\square_{\mathfrak{C}}\, \mathrm{id}_{\mathfrak{M}}) = \mathrm{id}_{\mathfrak{M}}$. A *right semimodule* $\mathfrak{N}$ over $\mathfrak{S}$ is a right $\mathfrak{C}$-comodule endowed with a right semiaction map $\mathbf{n}\colon \mathfrak{N} \,\square_{\mathfrak{C}}\, \mathfrak{S} \longrightarrow \mathfrak{N}$ satisfying the similar equations

$$\mathfrak{N} \,\square_{\mathfrak{C}}\, \mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{S} \rightrightarrows \mathfrak{N} \,\square_{\mathfrak{C}}\, \mathfrak{S} \longrightarrow \mathfrak{N},$$
$$\mathfrak{N} \longrightarrow \mathfrak{N} \,\square_{\mathfrak{C}}\, \mathfrak{S} \longrightarrow \mathfrak{N}.$$

These definitions can be found in [1, Sections 2.3 and 6.1], [13, Section 6], [15, Section 8], and [52, Sections 0.3.2 and 1.3.1]; see [52, Section 0.3.10] for some further references.

Before defining semicontramodules, let us recall from the discussion in Section 1.1 that there are two ways to define the conventional modules over associative algebras over k in tensor/polylinear algebra terms. In addition to the familiar definition of a left A-module M as a k-vector space endowed with a k-linear map $n\colon A \otimes_k M \longrightarrow M$ satisfying the associativity and unitality equations, one can also say that a left A-module structure on M is defined by a linear map $p\colon \mathrm{Hom}_k(A, M) \longrightarrow M$ satisfying the correspodingly rewritten equations.

A *left semicontramodule* $\mathfrak{P}$ over a semialgebra $\mathfrak{S}$ over a coalgebra $\mathfrak{C}$ is an object of the category opposite to the category of module objects in the right module category $\mathfrak{C}\text{–}\mathsf{contra}^{\mathsf{op}}$ over the ring object $\mathfrak{S}$ in the tensor category $\mathfrak{C}\text{–}\mathsf{comod}\text{–}\mathfrak{C}$. In other words, it is a left $\mathfrak{C}$-contramodule endowed with a *left semiaction* map $\mathbf{p}\colon \mathfrak{P} \longrightarrow \mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{S}, \mathfrak{P})$ satisfying the following *contralinearity*, *semicontraassociativity*, and *semicontraunitality* equations. First of all, the map $\mathbf{p}$ must be a left $\mathfrak{C}$-contramodule morphism. Secondly, the compositions of the semicontraaction map $\mathbf{p}$ with the two maps $\mathrm{Cohom}(\mathbf{m}, \mathfrak{P})\colon \mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{S}, \mathfrak{P}) \longrightarrow \mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{S} \,\square_{\mathfrak{C}}\, \mathfrak{S},\, \mathfrak{P})$

and $\mathrm{Cohom}(\mathcal{S},\mathbf{p})\colon \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P})\longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P}))$

$$\mathfrak{P}\longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P})\rightrightarrows \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S}\,\square_{\mathcal{C}}\,\mathcal{S},\ \mathfrak{P})\simeq \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P}))$$

should be equal to each other, $\mathrm{Cohom}(\mathbf{m},\mathfrak{P})\circ\mathbf{p} = \mathrm{Cohom}(\mathcal{S},\mathbf{p})\circ\mathbf{p}$, where the above identification $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{S}\,\square_{\mathcal{C}}\,\mathcal{S},\ \mathfrak{P})\simeq \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P}))$ is presumed. Thirdly, the composition of the semicontraaction map with the map $\mathrm{Cohom}_{\mathcal{C}}(\mathbf{e},\mathfrak{P})\colon \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P})\longrightarrow\mathfrak{P}$ induced by the semiunit map $\mathbf{e}$

$$\mathfrak{P}\longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathfrak{P})\longrightarrow\mathfrak{P}$$

should be equal to the identity map, $\mathrm{Cohom}_{\mathcal{C}}(\mathbf{e},\mathfrak{P})\circ\mathbf{p}=\mathrm{id}_{\mathfrak{P}}$. This definition can be found in [52, Sections 0.3.5 or 3.3.1].

For any right $\mathcal{S}$-semimodule $\mathcal{N}$ and any k-vector space V, the left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{N},V)$ has a natural left $\mathcal{S}$-semicontramodule structure. The left semicontraaction map $\mathbf{p}\colon \mathrm{Hom}_k(\mathcal{N},V)\longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathrm{Hom}_k(\mathcal{N},V))$ is constructed by applying the functor $\mathrm{Hom}_k(-,V)$ to the right semiaction map $\mathbf{n}$ of the $\mathcal{S}$-semimodule $\mathcal{N}$

$$\mathrm{Hom}_k(\mathcal{N},V)\longrightarrow \mathrm{Hom}_k(\mathcal{N}\,\square_{\mathcal{C}}\,\mathcal{S},\ V)\simeq \mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},\mathrm{Hom}_k(\mathcal{N},V)).$$

Generally speaking, the kernels of arbitrary morphisms exist in the category of left $\mathcal{S}$-semimodules $\mathcal{S}\text{–}\mathsf{simod}$ and are preserved by the forgetful functors $\mathcal{S}\text{–}\mathsf{simod}\longrightarrow \mathcal{C}\text{–}\mathsf{comod}\longrightarrow k\text{–}\mathsf{vect}$, but the cokernels in $\mathcal{S}\text{–}\mathsf{simod}$ may be problematic when $\mathcal{C}$ is not an injective right $\mathcal{C}$-comodule. Similarly, the cokernels of arbitrary morphisms exist in the category of left $\mathcal{S}$-semicontramodules $\mathcal{S}\text{–}\mathsf{sicntr}$ and are preserved by the forgetful functors $\mathcal{S}\text{–}\mathsf{sicntr}\longrightarrow\mathcal{C}\text{–}\mathsf{contra}\longrightarrow k\text{–}\mathsf{vect}$, but the kernels in $\mathcal{S}\text{–}\mathsf{sicntr}$ may be problematic when $\mathcal{C}$ is not an injective left $\mathcal{C}$-comodule.

Now let us assume that the semialgebra $\mathcal{S}$ is an injective right $\mathcal{C}$-comodule. Then the cotensor product functor $\mathcal{S}\,\square_{\mathcal{C}}\,-\colon \mathcal{C}\text{–}\mathsf{comod}\longrightarrow\mathcal{C}\text{–}\mathsf{comod}$ is exact, so the category $\mathcal{S}\text{–}\mathsf{simod}$ of left $\mathcal{S}$-semimodules is abelian and the forgetful functors $\mathcal{S}\text{–}\mathsf{simod}\longrightarrow \mathcal{C}\text{–}\mathsf{comod}\longrightarrow k\text{–}\mathsf{vect}$ are exact. Both the infinite direct sums and infinite products exist in $\mathcal{S}\text{–}\mathsf{simod}$ and both are preserved by the forgetful functor $\mathcal{S}\text{–}\mathsf{simod}\longrightarrow \mathcal{C}\text{–}\mathsf{comod}$, though only the infinite direct sums are preserved by the full forgetful functor $\mathcal{S}\text{–}\mathsf{simod}\longrightarrow k\text{–}\mathsf{vect}$.

Indeed, let $\mathcal{M}_\alpha$ be a family of left $\mathcal{S}$-semimodules and $\prod_\alpha\mathcal{M}_\alpha$ be their infinite product in the category of left $\mathcal{C}$-comodules $\mathcal{C}\text{–}\mathsf{comod}$; then one can easily construct a left semiaction map $\mathbf{m}\colon \mathcal{S}\square_{\mathcal{C}}\prod_\alpha\mathcal{M}_\alpha\longrightarrow\prod_\alpha\mathcal{M}_\alpha$ and show that the left $\mathcal{S}$-semimodule so obtained is the product of the family of objects $\mathcal{M}_\alpha$ in $\mathcal{S}\text{–}\mathsf{simod}$. So the category $\mathcal{S}\text{–}\mathsf{simod}$ satisfies the axioms Ab5 and Ab3*, but not in general Ab4*. It also has a set of generators, for which one can take the $\mathcal{S}$-semimodules $\mathcal{S}\,\square_{\mathcal{C}}\,\mathcal{L}$ induced from finite-dimensional left $\mathcal{C}$-comodules $\mathcal{L}$. Hence there are enough injective objects in $\mathcal{S}\text{–}\mathsf{simod}$; we will see in Section 3.5 below how one can construct them.

Assume that the semialgebra $\mathcal{S}$ is an injective left $\mathcal{C}$-comodule. Then the cohomomorphism functor $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{S},-)\colon \mathcal{C}\text{–}\mathsf{contra}\longrightarrow\mathcal{C}\text{–}\mathsf{contra}$ is exact, so the category $\mathcal{S}\text{–}\mathsf{sicntr}$ of left $\mathcal{S}$-semicontramodules is abelian and the forgetful functors $\mathcal{S}\text{–}\mathsf{sicntr}\longrightarrow \mathcal{C}\text{–}\mathsf{contra}\longrightarrow k\text{–}\mathsf{vect}$ are exact. Both the infinite direct sums and infinite products exist in the category $\mathcal{S}\text{–}\mathsf{sicntr}$ and both are preserved by the forgetful functor

$\mathbf{\mathfrak{S}}$–sicntr $\longrightarrow$ $\mathcal{C}$–contra, though only the infinite products are preserved by the full forgetful functor $\mathbf{\mathfrak{S}}$–$\mathsf{sicntr} \longrightarrow k$–$\mathsf{vect}$.

Indeed, let $\mathbf{\mathfrak{P}}_\alpha$ be a family of left $\mathbf{\mathfrak{S}}$-semicontramodules and $\bigoplus_\alpha \mathbf{\mathfrak{P}}_\alpha$ be their infinite direct sum in the category $\mathcal{C}$–contra. Then one can easily construct a left semicontraaction map $\mathbf{p} : \bigoplus_\alpha \mathbf{\mathfrak{P}}_\alpha \longrightarrow \mathrm{Cohom}_{\mathcal{C}}(\mathbf{\mathfrak{S}}, \bigoplus_\alpha \mathbf{\mathfrak{P}}_\alpha)$ and show that the left $\mathbf{\mathfrak{S}}$-semicontramodule so obtained is the direct sum of the family of objects $\mathbf{\mathfrak{P}}_\alpha$ in $\mathbf{\mathfrak{S}}$–sicntr. So the category $\mathbf{\mathfrak{S}}$–sicntr satisfies the axioms Ab3 and Ab4*, but not in general Ab4 or Ab5*. There are enough projective objects in $\mathbf{\mathfrak{S}}$–sicntr; we will see in Section 3.5 how to construct them.

Example. Let us explain the construction of the semialgebra $\mathbf{\mathfrak{S}}$ for which the category of $\mathbf{\mathfrak{S}}$-semicontramodules is equivalent to the category of contramodules over a (locally compact totally disconnected) topological group G, as it was promised in Section 1.8. In fact, we will see that for any given group G there is a whole family of such semialgebras $\mathbf{\mathfrak{S}}$ depending on the choice of a compact (i. e., profinite) open subgroup $H \subset G$. All of them are Morita equivalent to each other in the sense of [52, Section 8.4.5], i. e., the categories of (say, left) semimodules over all of them are equivalent, as are the categories of semicontramodules.

Given a commutative ring k, by a *discrete G-module over k* we mean a k-module $\mathcal{M}$ endowed with a k-linear discrete G-module structure $\mathcal{M} \longrightarrow \mathcal{M}\{G\}$; similarly, a *$G$-contramodule over k* is a k-module endowed with a k-linear G-contramodule structure $\mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$. In other words, a discrete G-module over k is a k-linear object in the additive category G–discr and a G-contramodule over k is a k-linear object in the additive category G–contra.

For the beginning, let k be a field. We will freely use the terminology and notation of Section 1.8; in particular, $k(X)$ denotes the vector space of locally constant compactly supported k-valued functions on a (locally compact totally disconnected) topological space X. Then for any topological spaces X and Y there is a natural isomorphism $k(X \times Y) \simeq k(X) \otimes_k k(Y)$. For any profinite group H, the inverse image map $k(H) \longrightarrow k(H \times H)$ with respect to the multiplication map $H \times H \longrightarrow H$, together with the map $k(H) \longrightarrow k$ of evaluation at the unit element $e \in H$, endow the vector space $k(H)$ with the structure of a coassociative coalgebra over k. For any k-vector space A one has $A\{H\} = A(H) \simeq k(H) \otimes_k A$ and $A[[H]] \simeq \mathrm{Hom}_k(k[[H]], A)$, so one can easily identify discrete G-modules over k with (left or right) $k(H)$-comodules and G-contramodules over k with $k(H)$-contramodules.

Let H be a compact open subgroup in a topological group G; then the left and right actions of H in G endow $k(G)$ with a natural structure of bicomodule over $k(H)$. Denote by $G \times^H G$ the quotient space of the Cartesian square $G \times G$ by the equivalence relation $(g'h, g'') \sim (g', hg'')$ for all g', $g'' \in G$ and $h \in H$. Being a disjoint union of G/H copies of G, this quotient is also a locally compact and totally disconnected topological space. The inverse image of functions with respect to the factorization map $G \times G \longrightarrow G \times^H G$ identifies the vector space $k(G \times^H G)$ with the cotensor product $k(G) \,\square_{k(H)}\, k(G) \subset k(G) \otimes_k k(G) = k(G \times G)$.

For any étale map (local homeomorphism) of topological spaces $p\colon X \longrightarrow Y$ and any abelian group A, the push-forward map $A(X) \longrightarrow A(Y)$, assigning to a function $f\colon X \longrightarrow A$ the function $p_*(f)\colon Y \longrightarrow A$,

$$(p_*f)(y) = \sum\nolimits_{p(x)=y} f(x), \qquad y \in Y, \quad x \in X,$$

is defined [52, Section E.1.1]. In particular, the push-forwards with respect to the multiplication map $G \times^H G \longrightarrow G$ and the embedding map $H \longrightarrow G$ provide natural left and right H-equivariant k-linear maps $k(G) \mathbin{\square}_{k(H)} k(G) \longrightarrow k(G)$ and $k(H) \longrightarrow k(G)$ endowing the vector space $\mathbf{S}_k(G,H) = k(G)$ with the structure of a semialgebra over the coalgebra $\mathcal{C}_k(H) = k(H)$ [52, Section E.1.2].

It is claimed that the category of (left or right) $\mathbf{S}_k(G,H)$-semimodules is isomorphic to the category of discrete G-modules over k; the datum of a $\mathbf{S}_k(G,H)$-semimodule structure on a k-vector space $\mathcal{M}$ is equivalent to the datum of a discrete G-module structure on $\mathcal{M}$. Indeed, according to [52, Sections E.1.3 and 10.2.2], the datum of a $\mathbf{S}_k(G,H)$-semimodule structure on $\mathcal{M}$ is equivalent to that of two structures of a G-module and a $\mathcal{C}_k(H)$-comodule satisfying two compatibility equations; essentially, this is the same as an action of G in $\mathcal{M}$ whose restriction to H comes from a $\mathcal{C}_k(H)$-coaction. It remains to notice that an action of G whose restriction to H is discrete is the same thing as a discrete action of G.

Similarly, the category of $\mathbf{S}_k(G,H)$-semicontramodules is isomorphic to the category of G-contramodules over k; the datum of a $\mathbf{S}_k(G,H)$-semicontramodule structure on a k-vector space $\mathfrak{P}$ is equivalent to the datum of a G-contramodule structure. Indeed, according to *loc. cit.* the datum of a $\mathbf{S}_k(G,H)$-semicontramodule structure on $\mathcal{M}$ is equivalent to that of two structures of a G-module and a $\mathcal{C}_k(H)$-contramodule satisfying two compatibility equations. Essentially, it is claimed that a contraassociative G-contraction map $\mathfrak{P}[[G]] \longrightarrow \mathfrak{P}$ can be uniquely recovered from its restriction to the point measures in G and the measures supported inside H, provided that the two compatibility equations are satisfied. One notices that there is an external product map $k[[G]] \otimes_k \mathfrak{P}[[G]] \longrightarrow \mathfrak{P}[[G \times G]]$ or $\mathfrak{P}[[G]] \otimes_k k[[G]] \longrightarrow \mathfrak{P}[[G \times G]]$ assiging to a k-valued and a $\mathfrak{P}$-valued measures on G a $\mathfrak{P}$-valued measure on $G \times G$. The contraassociativity equation in the definition of a G-contramodule, restricted to the external products of k-valued point measures in G and $\mathfrak{P}$-valued measures in H, taken in any fixed order, provides a prescription for the desired recovering of the G-contraaction map from its restrictions to the two specific kinds of measures.

Notice that the underlying k-vector space $k(G)$ of the semialgebras $\mathbf{S}_k(G,H)$ does not depend on the choice of a compact open subgroup $H \subset G$, but the semialgebra structure does, $\mathbf{S}_k(G,H)$ being a semialgebra over the coalgebra $\mathcal{C}_k(H) = k(H)$ depending on H. Still, the abelian categories of semimodules and semicontramodules over $\mathbf{S}_k(G,H)$ do not depend on H; but their semiderived categories and the functors of semi-infinite (co)homology and the derived semimodule-semicontramodule correspondence, whose construction is the aim of the book [52], *do* depend on H in a quite essential way [52, Section 8.4.6 and Remark E.3.2].

Now one would like to replace a field k with an arbitrary commutative ring, including first of all $k = \mathbb{Z}$. This was one of the motivating examples for developing the theory of semimodules and semicontramodules in the generality of semialgebras over corings over (generally speaking, noncommutative) rings rather than just over coalgebras over fields in the main body of the book [52]. One unpleasant technical complication that arises in this connection is the possible nonassociativity of cotensor product over a coring (see the discussion in the beginning of Section 2.5). Hence the importance of various sufficient conditions guaranteeing such associativity; see [14, Sections 11.6 and 22.5–6] and [52, Section 1.2.5].

In particular, the results of [14, 11.6(iv)] or [52, Proposition 1.2.5(f)] ensure that the notions of a semialgebra $\mathbf{S}_k(G, H)$ and arbitrary semimodules over it are unproblematic for any commutative ring k. To consider semicontramodules, one also needs associativity of the cohomomorphism functor, which holds in this case by the result of [52, Proposition 3.2.5(j)]. All the assertions mentioned above in this example still hold in this setting. The more advanced homological constructions and results of [52] in the application to the semialgebras $\mathbf{S}_k(G, H)$ depend on the assumption of the ring k having finite homological dimension, though.

2.7. **The category $\mathsf{O}^{\mathsf{ctr}}$.** The conventional concept of representations of an algebraic group G is that of comodules over the coalgebra of regular functions $\mathcal{C}(G)$ on G. Since every comodule over a coassociative coalgebra is the union of its finite-dimensional subcomodules [62, Section 2.1] (cf. the discussion in Sections 1.3–1.4), it means that infinite-dimensional representations of G are simply the unions of finite-dimensional representations, or the *ind-finite-dimensional* representations.

In particular, while every finite-dimensional representation of the Lie algebra $\mathfrak{g}$ of a simply connected semisimple complex algebraic group G can be integrated to a representation of G, this is *no* longer true for infinite-dimensional representations. Indeed, for any nonzero Lie algebra $\mathfrak{g}$ one can easily construct a module that is not a union of its finite-dimensional submodules (it suffices to consider the enveloping algebra $U(\mathfrak{g})$ with the action of $\mathfrak{g}$ by left multiplications).

The Lie correspondence takes a particularly simple form in the case of unipotent algebraic groups and nilpotent Lie algebras over a field of characteristic zero, which are two equivalent categories [20, Corollaire VI.2.4.5] (see [52, Section D.6.1] for further references and a discussion including the finite characteristic case). A module over a finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ comes from an (always unique) representation of the corresponding unipotent algebraic group G if and only if it is a union of finite-dimensional $\mathfrak{g}$-modules where all the vectors from $\mathfrak{g}$ act by nilpotent linear operators [50, Sections 3.3.5–7].

It is a classical idea to work with categories intermediate between those of representations of a Lie or algebraic group G and modules over its Lie algebra $\mathfrak{g}$. For this purpose, one starts from a Lie algebra $\mathfrak{g}$ with a chosen *subgroup* H, i. e., an algebraic group corresponding to a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then one considers $\mathfrak{g}$-modules $\mathcal{M}$ for which the restriction to $\mathfrak{h}$ of the action of $\mathfrak{g}$ in $\mathcal{M}$ can be/has been integrated to an algebraic action of H. As to the choice of the subgroup H, there are two basic approaches: given a complex (or real) semisimple Lie group G and its Lie algebra $\mathfrak{g}$, one can take H to be a maximal compact subgroup of G; or otherwise one can use a Borel (or maximal unipotent) subgroup of G in the role of H.

Modules over a semisimple Lie algebra $\mathfrak{g}$ integrable to representations of a maximal compact Lie subgroup H are classically known as *Harish-Chandra modules* [21, 65], while $\mathfrak{g}$-modules which can be integrated to an algebraic action of the Borel subgroup form what has been called the *BGG* (Bernstein–Gelfand–Gelfand) *category* O [4, 35]. Both can be united under an umbrella notion of *algebraic Harish-Chandra modules*, which means simply "modules over a pair consisting of a Lie algebra and an algebraic subgroup" (see a terminological discussion in [52, Remark D.2.5]).

An *algebraic Harish-Chandra pair* [5, Sections 1.8.2 and 3.3.2] is a set of data consisting of a Lie algebra $\mathfrak{g}$ over a field k, a finite-dimensional Lie subalgebra $\mathfrak{h} \subset H$, an algebraic group H over k whose Lie algebra is identified with $\mathfrak{h}$, and an action of H by automorphisms of the Lie algebra $\mathfrak{g}$. The following two compatibility conditions have to be satisfied. Firstly, the subalgebra $\mathfrak{h}$ must be an H-invariant subspace in $\mathfrak{g}$, and the restriction of the action of H in $\mathfrak{g}$ to $\mathfrak{h}$ should coincide with the adjoint action of H in $\mathfrak{h}$. Secondly, the action of $\mathfrak{h}$ in $\mathfrak{g}$ obtained by taking the derivative of the action of H in $\mathfrak{g}$ should coincide with the adjoint action of $\mathfrak{h}$ in $\mathfrak{g}$.

A *Harish-Chandra module* $\mathcal{M}$ over an algebraic Harish-Chandra pair $(\mathfrak{g}, H)$ is a k-vector space endowed with a $\mathfrak{g}$-module structure and an action of H satisfying the following compatibility conditions. Firstly, the restriction of the $\mathfrak{g}$-action in $\mathcal{M}$ to the Lie subalgebra $\mathfrak{h}$ should coincide with the derivative of the action of H in $\mathcal{M}$. Secondly, the $\mathfrak{g}$-action map $\mathfrak{g} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ should be H-equivariant.

In algebraic (rather than algebro-geometric) terms, an (affine) algebraic group G over a field k is described by the k-vector space $\mathcal{C}(G)$ of regular functions on G, endowed with a noncommutative convolution comultiplication $\mu\colon \mathcal{C}(G) \longrightarrow \mathcal{C}(G) \otimes_k \mathcal{C}(G)$ induced by the multiplication map $G \times G \longrightarrow G$ and a commutative pointwise multiplication $m\colon \mathcal{C}(G) \otimes_k \mathcal{C}(G) \longrightarrow \mathcal{C}(G)$. Together with the antipode map $s\colon \mathcal{C}(G) \longrightarrow \mathcal{C}(G)$ induced by the inverse element map $G \longrightarrow G$, these structures make $\mathcal{C}(G)$ a commutative Hopf algebra [62]. By a representation of G one conventionally means a $\mathcal{C}(G)$-comodule, while the multiplication on $\mathcal{C}(G)$ is being used in order to define a $\mathcal{C}(G)$-comodule structure on the tensor product $\mathcal{L} \otimes_k \mathcal{M}$ of any two k-vector spaces endowed with $\mathcal{C}(G)$-comodule structures. The antipode map s, being (always) an anti-automorphism for (both the multiplication and) the comultiplication of $\mathcal{C} = \mathcal{C}(G)$ (or any other Hopf algebra $\mathcal{C}$), allows to identify the categories of left and right $\mathcal{C}$-comodules, so the difference between them is inessential here.

Over a field k of characteristic 0, the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of an algebraic group G is interpreted as the algebra of left or right invariant differential operators on G or, simpler yet, the algebra of distributions ("delta functions") on G supported at the unit element $e \in G$. The k-vector space of distributions here is defined as the discrete dual vector space to the linearly compact k-vector space of functions on the formal completion of G at e. The noncommutative convolution multiplication $m\colon U(\mathfrak{g}) \otimes_k U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ in the Hopf algebra $U(\mathfrak{g})$ is induced by the multiplication map $G \times G \longrightarrow G$, while the commutative comultiplication $\mu\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes_k U(\mathfrak{g})$ is induced by the diagonal embedding $G \longrightarrow G \times G$ and the antipode map $s\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ simply multiplies every vector from $\mathfrak{g}$ by -1.

Evaluating a $\{e\}$-supported distribution at a regular function on G defines a natural pairing $\phi\colon \mathcal{C}(G) \otimes_k U(\mathfrak{g}) \longrightarrow k$. For example, the pairing with an element of $\mathfrak{g}$ assigns to a regular function on G the value of its derivative along the corresponding tangent vector at the origin $e \in G$. The pairing ϕ is compatible with the Hopf algebra structures on $\mathcal{C}(G)$ and $U(\mathfrak{g})$, transforming the comultiplication in the former into the multiplication in the latter and vice versa. In our left-right conventions (see Section 1.4 for a discussion), this compatibility is expressed by the equations

$$\phi(f, uv) = \phi(f_{(2)}, u)\phi(f_{(1)}, v)$$
$$\phi(fg, u) = \phi(f, u_{(2)})\phi(g, u_{(1)}),$$

for any f, $g \in \mathcal{C}(G)$ and u, $v \in U(\mathfrak{g})$, where $\mu(f) = f_{(1)} \otimes f_{(2)}$ and $\mu(u) = u_{(1)} \otimes u_{(2)}$ is Sweedler's symbolic notation for the comultiplication maps.

Given a $\mathcal{C}(G)$-comodule $\mathcal{M}$, one defines the "derivative" $U(\mathfrak{g})$-module structure $m\colon U(\mathfrak{g}) \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ on $\mathcal{M}$ as given by the composition $U(\mathfrak{g}) \otimes_k \mathcal{M} \longrightarrow U(\mathfrak{g}) \otimes_k \mathcal{C}(G) \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ of the maps induced by the coaction map and the pairing ϕ (with the arguments' positions inverted). Alternatively, one can obtain a left action map in the form $p\colon \mathcal{M} \longrightarrow \operatorname{Hom}_k(U(g), \mathcal{M})$ as the composition $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M} \longrightarrow \operatorname{Hom}_k(U(\mathfrak{g}), \mathcal{M})$ of the left coaction map and the map induced by the pairing ϕ. Furthermore, the adjoint action of G in itself preserves the origin, so, *unlike* the left and right actions of $\mathfrak{g}$ in the enveloping algebra $U(\mathfrak{g})$, the adjoint action of $\mathfrak{g}$ can be integrated to a representation of G in $U(\mathfrak{g})$ as well as in $\mathfrak{g}$. Hence both $\mathfrak{g}$ and $U(\mathfrak{g})$ are endowed with natural $\mathcal{C}(G)$-comodule structures.

Now an algebraic Harish-Chandra pair $(\mathfrak{g}, H)$ is described in pure algebraic terms as a set of data consisting of a Lie algebra $\mathfrak{g}$, a Hopf algebra $\mathcal{C}(H)$, a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a pairing $\phi\colon \mathcal{C}(H) \otimes_k U(\mathfrak{h}) \longrightarrow k$ compatible with the Hopf algebra structures, and a coaction of $\mathcal{C}(H)$ in $\mathfrak{g}$ satisfying the following compatibility conditions. Firstly, the coaction of $\mathcal{C}(H)$ in $\mathfrak{g}$ should be compatible with the Lie algebra structure on $\mathfrak{g}$; then there is also the induced coaction of $\mathcal{C}(H)$ in $U(\mathfrak{g})$. Secondly, the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ should be preserved by the $\mathcal{C}(H)$-coaction and the pairing ϕ should be compatible with the induced coaction of $\mathcal{C}(H)$ in $U(\mathfrak{h})$ and the adjoint coaction of $\mathcal{C}(H)$ in itself (equivalently, the restriction $\psi\colon \mathcal{C}(H) \times \mathfrak{h} \longrightarrow k$ of the pairing ϕ should be compatible with the adjoint $\mathcal{C}(H)$-coaction in $\mathcal{C}(H)$ and $\mathcal{C}(H)$-coaction in $\mathfrak{g}$ restricted to $\mathfrak{h}$). Thirdly, the adjoint action of $\mathfrak{h}$ in $\mathfrak{g}$ should coincide with the derivative $\mathfrak{h}$-action of the $\mathcal{C}(H)$-coaction, which is defined in terms of the given pairing ϕ.

Two generalizations of this setting, in two different directions, are discussed at length in the book [52]. A "quantum" version, with two noncommutative, noncocommutative Hopf algebras $\mathcal{C}$ and K in place of $\mathcal{C}(H)$ and $U(\mathfrak{h})$, an associative algebra R in the role of $U(\mathfrak{g})$, and "adjoint" coactions of $\mathcal{C}$ in K and R, is introduced in [52, Section C.1]. A "Tate" version, with the Hopf algebra $\mathcal{C}(H)$ of regular functions on an infinite-dimensional pro-affine pro-algebraic group H and a linearly compact open subalgebra $\mathfrak{h}$ in a Tate (locally linearly compact) Lie algebra $\mathfrak{g}$, can be found in [52, Section D.2.1] (see the overview in Section 2.8 below).

To repeat a previously given definition in the slightly new language, a Harish-Chandra module $\mathcal{M}$ over an algebraic Harish-Chandra pair $(\mathfrak{g}, H)$ is a k-vector space endowed with a $\mathfrak{g}$-module and a $\mathcal{C}(H)$-comodule structures satisfying two compatibility conditions. The restriction of the $\mathfrak{g}$-action in $\mathcal{M}$ to $\mathfrak{h}$ should coincide with the derivative $\mathfrak{h}$-action of the $\mathcal{C}(H)$-coaction, and the action map $\mathfrak{g} \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ should be a $\mathcal{C}(H)$-comodule morphism (where the coaction in the tensor product is defined in terms of the multiplication in the Hopf algebra $\mathcal{C}(H)$); equivalently, the action map $U(\mathfrak{g}) \otimes_k \mathcal{M} \longrightarrow \mathcal{M}$ should be a $\mathcal{C}(H)$-comodule morphism.

Before defining *Harish-Chandra contramodules*, let us say a few words about contramodules over the coalgebra $\mathcal{C}(G)$ of regular functions on an algebraic group G. Unfortunately, there does *not* seem to be any particular way to *interpret* a $\mathcal{C}(G)$-contramodule structure on a k-vector space $\mathfrak{P}$ in any terms more explicit than the general definition of a contramodule over a coalgebra over a field k. The only known exception is the case of a reductive algebraic group H over a field k of characteristic 0, when [52, Lemma A.2.2] or the last sentence of Section 1.2 apply and the semisimple abelian categories of $\mathcal{C}(H)$-comodules and $\mathcal{C}(H)$-contramodules are equivalent. Otherwise there is only the general intuition of contramodules as modules with infinite summation operations, supported by examples such as that of comodules and contramodules over the coalgebra $\mathcal{C}(H)$ of regular functions on the one-dimensional unipotent algebraic group $H = \mathbb{G}_a$ considered in Section 1.3.

However, the vectors spaces $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{M}, V)$ for all $\mathcal{C}(H)$-comodules $\mathcal{M}$ (i. e., vector spaces with the algebraic group H acting in them in the conventional sense) and all k-vector spaces V provide *examples* of $\mathcal{C}(H)$-contramodules for *any* algebraic group H (see Section 1.2). Moreover, for any commutative (for simplicity) Hopf algebra $\mathcal{C}$, a $\mathcal{C}$-comodule $\mathcal{M}$, and a $\mathcal{C}$-contramodule $\mathfrak{P}$, the k-vector space $\mathfrak{Q} = \mathrm{Hom}_k(\mathcal{M}, \mathfrak{P})$ has a natural $\mathcal{C}$-contramodule structure. To construct the desired left $\mathcal{C}$-contraaction map $\pi_{\mathfrak{Q}} \colon \mathrm{Hom}_k(\mathcal{C}, \mathrm{Hom}_k(\mathcal{M}, \mathfrak{P})) \longrightarrow \mathrm{Hom}_k(\mathcal{M}, \mathfrak{P})$, suppose that we are given a k-linear map $g \colon \mathcal{C} \longrightarrow \mathrm{Hom}_k(\mathcal{M}, \mathfrak{P})$ and a vector $m \in \mathcal{M}$. Consider the k-linear map $f \colon \mathcal{C} \longrightarrow \mathfrak{P}$ assigning to any element $c \in \mathcal{C}$ the vector

$$f(c) = g\big(s(m_{(-1)})c\big)(m_{(0)}) \in \mathfrak{P},$$

and set

$$\pi_{\mathfrak{Q}}(g) = \pi_{\mathfrak{P}}(f),$$

where $m \longmapsto m_{(-1)} \otimes m_{(0)}$ is the Sweedler notation for the left coaction map $\nu_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$, and $\pi_{\mathfrak{P}} \colon \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ is the left contraaction map of the original $\mathcal{C}$-contramodule $\mathfrak{P}$ [52, Section C.4.2].

Furthermore, given an algebraic group G and a $\mathcal{C}(G)$-contramodule $\mathfrak{P}$, one defines the "contraderivative" $U(\mathfrak{g})$-module structure $U(\mathfrak{g}) \otimes_k \mathfrak{P} \longrightarrow \mathfrak{P}$ on $\mathfrak{P}$ as given by the composition $U(\mathfrak{g}) \otimes_k \mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{C}(G), \mathfrak{P}) \longrightarrow \mathfrak{P}$ of the map

$$u \otimes p \longmapsto (c \mapsto \phi(c, u)p), \qquad u \in U(\mathfrak{g}), \;\; c \in \mathcal{C}(G), \;\; p \in \mathfrak{P}$$

induced by the pairing $\phi\colon \mathcal{C}(G) \otimes_k U(\mathfrak{g}) \longrightarrow k$ with the left contraaction map π (cf. [52, Sections 10.1.2]).

A *Harish-Chandra contramodule* $\mathfrak{P}$ over an algebraic Harish-Chandra pair $(\mathfrak{g}, H)$ is a k-vector space endowed with a $\mathfrak{g}$-module and a $\mathcal{C}(H)$-contramodule structures satisfying the following two compatibility conditions. Firstly, the restriction of the $\mathfrak{g}$-module structure on $\mathfrak{P}$ to the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ should coincide with the contraderivative $\mathfrak{h}$-module structure of the $\mathcal{C}(H)$-contramodule structure on $\mathfrak{P}$. Secondly, the $U(\mathfrak{g})$-action map in the form $\mathfrak{P} \longrightarrow \operatorname{Hom}_k(U(\mathfrak{g}), \mathfrak{P})$ should be a $\mathcal{C}(H)$-contramodule morphism, where the $\mathcal{C}(H)$-contramodule structure on $\operatorname{Hom}_k(U(\mathfrak{g}), \mathfrak{P})$ is obtained from the $\mathcal{C}(H)$-comodule structure on $U(\mathfrak{g})$ and the $\mathcal{C}(H)$-contramodule structure on $\mathfrak{P}$ as described above.

The latter condition is equivalent to the $\mathfrak{g}$-action map $\mathfrak{P} \longrightarrow \operatorname{Hom}_k(\mathfrak{g}, \mathfrak{P})$ being a $\mathcal{C}(H)$-contramodule morphism. To convince oneself that this is so, one can present the space $\operatorname{Hom}_k(U(\mathfrak{g}), \mathfrak{P})$ as the projective limit of the Hom spaces $\operatorname{Hom}_k(F_nU(\mathfrak{g}), \mathfrak{P})$, where F denotes the Poincaré–Birkhoff–Witt filtration of $U(\mathfrak{g})$, and further present every space $\operatorname{Hom}_k(F_nU(\mathfrak{g}), \mathfrak{P})$ as a subspace of the Hom space $\operatorname{Hom}_k(\bigoplus_{i=0}^n \mathfrak{g}^{\otimes n}, \mathfrak{P})$. Then it remains to use the fact that the $\mathcal{C}(H)$-comodule structure on $U(\mathfrak{g})$ is compatible with the associative algebra structure, i. e., the multiplication map $U(\mathfrak{g}) \otimes_k U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ is a $\mathcal{C}(H)$-comodule morphism, together with the assumption of associativity of the $U(\mathfrak{g})$-action in $\mathfrak{P}$.

Viewing the case of a semisimple Lie algebra $\mathfrak{g}$ with a Borel or maximal unipotent subgroup H as our main example, we denote by $\mathsf{O}(\mathfrak{g}, H)$ the category of Harish-Chandra modules over an algebraic Harish-Chandra pair $(\mathfrak{g}, H)$ and by $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, H)$ the category of Harish-Chandra contramodules. These are abelian categories with exact forgetful functors $\mathsf{O}(\mathfrak{g}, H) \longrightarrow k\text{–}\mathsf{vect}$ and $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, H) \longrightarrow k\text{–}\mathsf{vect}$, the former of which preserves infinite direct sums, while the latter preserves infinite products. The category $\mathsf{O}(\mathfrak{g}, H)$ satisfies the axioms Ab5 and Ab3* (but not Ab4*), while the category $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, H)$ satisfies the axioms Ab3 and Ab4* (but not Ab4 or Ab5*).

Now it is claimed that there is a semialgebra $\boldsymbol{\mathcal{S}}$ over the coalgebra $\mathcal{C} = \mathcal{C}(H)$ such that the categories $\mathsf{O}(\mathfrak{g}, H)$ and $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, H)$ are identified with the categories of semimodules and semicontramodules over $\boldsymbol{\mathcal{S}}$. More precisely, there are naturally *two* such semialgebras $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, H)$ and $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, H)$, differing by the left-right symmetry. The category of Harish-Chandra modules $\mathsf{O}(\mathfrak{g}, H)$ is isomorphic to the categories of left semimodules over $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, H)$ and right semimodules over $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, H)$, while the category of Harish-Chandra contramodules $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, H)$ is isomorphic to the category of left semicontramodules over $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, H)$.

The semialgebra $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, H)$ is constructed as the tensor product $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}(H)$, where the left $U(\mathfrak{h})$-module structure on $\mathcal{C}(H)$ is obtained by deriving the left coaction of $\mathcal{C}(H)$ in itself. The right $\mathcal{C}$-comodule structure on $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}(H)$ is induced by the right coaction of $\mathcal{C}$ in itself, while the left $\mathcal{C}$-comodule structure on this tensor product is defined in terms of the multiplication in $\mathcal{C}$, as the tensor product of the left coaction of $\mathcal{C}$ in itself and the adjoint coaction of $\mathcal{C}$ in $U(\mathfrak{g})$. The semiunit map $\mathbf{e}\colon \mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}^l = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}$ is induced by the embedding of algebras $U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$.

Finally, the semimultiplication map $\mathbf{m}\colon \boldsymbol{\mathcal{S}}^l\,\square_{\mathcal{C}}\,\boldsymbol{\mathcal{S}}^l \longrightarrow \boldsymbol{\mathcal{S}}^l$ is defined as the composition

$$\bigl(U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C}\bigr)\,\square_{\mathcal{C}}\,\bigl(U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C}\bigr) \simeq U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\bigl(\mathcal{C}\,\square_{\mathcal{C}}\,(U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C})\bigr) \\ \simeq U(\mathfrak{g})\otimes_{U(\mathfrak{h})}U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C} \longrightarrow U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C}$$

of the mutual associativity isomorphism of the tensor and cotensor products, whose natural existence in this case can be easily established from the fact that $U(\mathfrak{g})$ is a projective right $U(\mathfrak{h})$-module (by the Poincaré–Birkhoff–Witt theorem) [52, Section 1.2.3], and the map induced by the multiplication map $U(\mathfrak{g})\otimes_{U(\mathfrak{h})}U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$. This construction can be found in [52, Section 10.2.1].

Notice that for any left $\mathcal{C}$-comodule $\mathcal{M}$ there is a natural isomorphism

$$\boldsymbol{\mathcal{S}}^l\,\square_{\mathcal{C}}\,\mathcal{M} = \bigl(U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{C}\bigr)\,\square_{\mathcal{C}}\,\mathcal{M} \simeq U(\mathfrak{g})\otimes_{U(\mathfrak{h})}\mathcal{M},$$

where the $U(\mathfrak{h})$-module structure on $\mathcal{M}$ is obtained by deriving the $\mathcal{C}(H)$-comodule structure. Hence it follows that the datum of a left $\boldsymbol{\mathcal{S}}^l$-semimodule structure on a k-vector space $\boldsymbol{\mathcal{M}}$ is equivalent to that of a left $\mathcal{C}(H)$-comodule and a left $U(\mathfrak{g})$-module structures on $\boldsymbol{\mathcal{M}}$ satisfying two compatibility equations [52, Section 10.2.2]. These are easily seen to express the definition of a structure of Harish-Chandra module over $(\mathfrak{g},H)$ on the vector space $\boldsymbol{\mathcal{M}}$.

Similarly, the semialgebra $\boldsymbol{\mathcal{S}}^r = \boldsymbol{\mathcal{S}}^r(\mathfrak{g},H)$ is constructed as the tensor product $\mathcal{C}(H)\otimes_{U(\mathfrak{h})}U(\mathfrak{g})$, where the right $U(\mathfrak{h})$-module structure on $\mathcal{C}(H)$ is obtained by deriving the right coaction of $\mathcal{C}(H)$ in itself. The left $\mathcal{C}$-comodule structure on this tensor product is induced by the left coaction of $\mathcal{C}$ in itself, while the right $\mathcal{C}$-comodule structure is obtained by multiplying the right coaction of $\mathcal{C}$ in itself and the adjoint coaction of $\mathcal{C}$ in $U(\mathfrak{g})$. The semimultiplication and semiunit maps of the semialgebra $\boldsymbol{\mathcal{S}}^r$ are induced by the multiplication map $U(\mathfrak{g})\otimes_{U(\mathfrak{h})}U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ and the embedding $U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$, as above.

For any left $\mathcal{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism

$$\operatorname{Cohom}_{\mathcal{C}}(\boldsymbol{\mathcal{S}}^r,\mathfrak{P}) = \operatorname{Cohom}_{\mathcal{C}}(\mathcal{C}\otimes_{U(\mathfrak{h})}U(\mathfrak{g}),\,\mathfrak{P}) \simeq \operatorname{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}),\mathfrak{P}),$$

where the $U(\mathfrak{h})$-module structure on $\mathfrak{P}$ is obtained by contraderiving the $\mathcal{C}(H)$-contramodule structure. Hence one concludes that the datum of a left $\boldsymbol{\mathcal{S}}^r$-semicontramodule structure on a k-vector spaced $\boldsymbol{\mathfrak{P}}$ is equivalent to that of a left $\mathcal{C}$-contramodule and left $U(\mathfrak{g})$-module structures on $\boldsymbol{\mathfrak{P}}$ satisfying two compatibility equations (cf. Example in Section 2.6). These are easily seen to express the definition of a structure of Harish-Chandra contramodule over $(\mathfrak{g},H)$ on the vector space $\boldsymbol{\mathfrak{P}}$.

In addition to these descriptions of left $\boldsymbol{\mathcal{S}}^l$-semimodules and left $\boldsymbol{\mathcal{S}}^r$-semicontramodules, one would like to have an explicit interpretation of what it means to have a left $\boldsymbol{\mathcal{S}}^r$-semimodule structure on a k-vector space $\boldsymbol{\mathcal{M}}$. Such a description of left $\boldsymbol{\mathcal{S}}^r$-semimodules is indeed obtained in [52, Sections C.2 and C.4.3–4] under certain mild assumptions, which we will now discuss.

So far we used the notions of an "algebraic group H" and "a commutative Hopf algebra $\mathcal{C}(H)$" interchangeably, but in fact there are several differences between the two (cf. [19]). First of all, it is only *affine* algebraic groups that can be described by the algebras of global functions on them. As our aim is to consider linear representations of our algebraic groups, we can safely assume all of them to be affine.

Secondly, it is only *finitely generated* commutative algebras over a field that correspond to algebraic varieties; the spectrum of an arbitrary commutative algebra is, generally speaking, a *pro-affine pro-algebraic variety*. Considering Harish-Chandra pairs with pro-affine pro-algebraic groups H presumes also having an infinite-dimensional linearly compact Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Postponing this discussion to Section 2.8, we for now assume all our algebraic groups H to be finite-dimensional, or the commutative Hopf algebras $\mathcal{C}(H)$ to be finitely generated as algebras over k.

Thirdly and finally, *algebraic varieties* are generally assumed to be *reduced schemes*, i. e., to have no nilpotent elements in their structure sheaves (or, if they are affine, in the algebras of global functions on them). Now, over a field k of characteristic 0, every algebraic *group* scheme is a smooth variety, and over any field k every reduced algebraic group scheme is a smooth variety; but over a field k of finite characteristic there exist *nonreduced* algebraic group schemes. It suffices to consider the spectrum of the finite-dimensional algebra $\mathcal{C} = k[x]/(x^p)$ over a field k of characteristic p and notice that the rule $\mu(x) = 1 \otimes x + x \otimes 1$ describes a well-defined coassociative, counital comultiplication making $\mathcal{C}$ a Hopf algebra over k.

Now, assuming the algebraic group H to be a smooth finite-dimensional variety over a field k, there is the one-dimensional vector bundle of differential forms of the top degree on H. The group H acts in itself by the left and right multiplications, and there are the two induced actions in the vector space of global differential top forms $\mathcal{E} = \Omega^{top}(H)$. The subspace of left H-invariant top forms in $\mathcal{E}$ is always one-dimensional, as is the subspace of right H-invariant top forms, but these two subspaces may not coincide. A smooth algebraic group H admitting a nonzero *biinvariant* top form is said to be *unimodular*. An algebraic group H is unimodular if and only if all the operators of its adjoint representation $\mathrm{Ad}_H \colon H \longrightarrow \mathrm{GL}(\mathfrak{h})$ belong to the subgroup $\mathrm{SL}(\mathfrak{h}) \subset \mathrm{GL}(\mathfrak{h})$. All the reductive algebraic groups are unimodular, as are all the nilpotent ones; but many solvable groups are not.

Let us first assume the smooth algebraic group H to be unimodular. Then the semialgebras $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, H)$ and $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, H)$ are naturally isomorphic to each other; the isomorphism is provided by the maps given by the formulas

$$c \otimes_{U(\mathfrak{h})} u \longmapsto u_{[0]} \otimes_{U(\mathfrak{h})} cu_{[1]} \quad \text{and} \quad u \otimes_{U(\mathfrak{h})} c \longmapsto s(u_{[1]})c \otimes_{U(\mathfrak{h})} u_{[0]},$$

where $c \in \mathcal{C}(H)$, $u \in U(\mathfrak{g})$, the notation $c \otimes_{U(\mathfrak{h})} u$ and $u \otimes_{U(\mathfrak{h})} c$ stands for elements of $\boldsymbol{\mathcal{S}}^r = \mathcal{C}(H) \otimes_{U(\mathfrak{h})} U(\mathfrak{g})$ and $\boldsymbol{\mathcal{S}}^l = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}(H)$, respectively, and $u \longmapsto u_{[0]} \otimes u_{[1]}$ with $u_{[0]} \in U(\mathfrak{g})$ and $u_{[1]} \in \mathcal{C}(H)$ is the Sweedler notation for right coaction map defining the adjoint coaction of $\mathcal{C}(H)$ in $U(\mathfrak{g})$ [52, Section C.2.6]. Accordingly, the categories of left $\boldsymbol{\mathcal{S}}^r$-semimodules and left $\boldsymbol{\mathcal{S}}^l$-semimodules are naturally isomorphic.

In the general case of a nonunimodular smooth algebraic group H, the categories of left $\boldsymbol{\mathcal{S}}^r$-semimodules and left $\boldsymbol{\mathcal{S}}^l$-semimodules are still naturally equivalent, but the equivalence $\boldsymbol{\mathcal{S}}^r\text{–}\mathsf{simod} \simeq \boldsymbol{\mathcal{S}}^l\text{–}\mathsf{simod}$ does *not* commute with the forgetful functors

$\mathcal{S}^r$–simod $\longrightarrow$ $\mathcal{C}$–comod and $\mathcal{S}^l$–simod $\longrightarrow$ $\mathcal{C}$–comod. Instead, the two forgetful functors differ by a twist with the modular character $\det \mathrm{Ad}_H\colon H \longrightarrow GL(\mathfrak{h}) \longrightarrow k^*$ [52, Sections C.2.4–5].

When that the ground field k has characteristic 0 and the Harish-Chandra pair $(\mathfrak{g}, H)$ originates from a closed embedding of algebraic groups $H \subset G$ over k, choosing a biinvariant top form on H allows to interpret elements of the semialgebra $\mathcal{S}^l \simeq \mathcal{S}^r$ as distributions on the smooth variety G, supported in the smooth closed subvariety H and regular along H [52, Remark C.4.4]. In the nonunimodular case, the k-vector space of all distributions on G, supported in H and regular along H, has a natural structure of an $\mathcal{S}^l$-$\mathcal{S}^r$-*bisemimodule* providing the Morita equivalence between the semialgebras $\mathcal{S}^l$ and $\mathcal{S}^r$.

2.8. **Tate Harish-Chandra pairs.** It appears that one cannot integrate the Virasoro Lie algebra to a Lie group, but *a half* of it one easily can. Indeed, let k be a field of characteristic 0. Denote by $H(k)$ the set of all formal Taylor power series $a(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$ with a vanishing coefficient in degree 0 and a nonvanishing coefficient $a_1 \neq 0$ in degree 1. Then the composition multiplication $(a * b)(z) = a(b(z))$ defines a group structure on the set $H(k)$. This group is naturally the group of k-points of a certain pro-affine pro-algebraic group, which we denote by H. The Lie algebra $\mathfrak{h}$ of the pro-algebraic group H can be easily identified with the algebra of vector fields on the formal disk with a vanishing vector at the origin $zk[[z]]d/dz$, or with the closed subalgebra in the Virasoro Lie algebra topologically spanned by the basis vectors L_0, L_1, L_2, … (see Section 1.7).

Let us say a few words about Lie theory in the pro-algebraic group setting. For any subcoalgebra $\mathcal{D}$ in a commutative Hopf algebra $\mathcal{C}$, the subalgebra generated by $\mathcal{D} + s(\mathcal{D})$ in a Hopf subalgebra in $\mathcal{C}$. Since $\mathcal{C}$ is the union of its finite-dimensional subcoalgebras (see Section 1.3), it it also the union of its Hopf subalgebras that are finitely generated as commutative algebras. Thus there is no difference between the notions of a pro-affine pro-algebraic variety with a group structure and a pro-object in the category of affine algebraic groups. The Lie functor on the category of (pro-afffine) pro-algebraic groups can be simply obtained by passing to the pro-objects on both sides of the functor assigning a Lie algebra to an algebraic group; so the Lie algebra of a pro-algebraic group is a filtered projective limit of finite-dimensional Lie algebras. In particular, it follows that the topological Lie algebra $k[[z]]d/dz$, which has no closed ideals (see Section 2.4), *cannot* correspond to any pro-algebraic group, though its subalgebra $zk[[z]]d/dz$ does, as we have just seen.

We are interested in considering Harish-Chandra pairs with a Tate Lie algebra $\mathfrak{g}$ and pro-algebraic subgroup H corresponding to a open linearly compact subalgebra $\mathfrak{h} \subset \mathfrak{g}$. A precise definition presents a small technical difficulty in that one has to explain what it means for the coalgebra $\mathcal{C} = \mathcal{C}(H)$ to act in a Tate vector space $\mathfrak{g}$. Neither the notion of a $\mathcal{C}$-comodule *nor* that of a $\mathcal{C}$-contramodule are suitable for the task; rather, the compact vector space $\mathfrak{h}$, being dual to a $\mathcal{C}$-comodule, can be viewed as a $\mathcal{C}$-contramodule, while the quotient space $\mathfrak{g}/\mathfrak{h}$ has a $\mathcal{C}$-comodule structure.

However, an action of a algebraic group in a vector space is, of course, *not* determined by its restriction to an invariant subspace and the induced action in the quotient space. The authors of the manuscript [6], where (what we call) Tate Harish-Chandra pairs seem to have first appeared, solve the problem by working over the field of complex numbers $\mathbb{C}$ and considering an action of the *group of points* $H(\mathbb{C})$ of a pro-algebraic group H in a Tate Lie algebra $\mathfrak{g}$ [6, Section 3.1]. The approach taken in [52, Appendix D], which works over an arbitrary field, is to give the definition of a *continuous coaction* of a discrete coalgebra in a topological vector space.

Here we restrict ourselves to a brief sketch. Let $\mathcal{C}$ be a coassociative coalgebra and V be a topological vector space over k (see Sections 2.3–2.4). A *continuous right coaction* of $\mathcal{C}$ in V is a continuous linear map $V \longrightarrow V \otimes^! \mathcal{C}$, where $\mathcal{C}$ is considered as a discrete topological vector space, satisfying the coassociativity and counitality equations. Equivalently, a continuous right coaction can be defined as a continuous linear map $V \otimes^* \mathcal{C}^* \longrightarrow V$, where $\mathcal{C}^*$ is viewed as a linearly compact vector space, satisfying the associativity and unitality equations.

For any topological vector space V with a continuous coaction of a coassociative coalgebra $\mathcal{C}$, open subspaces of V invariant under the continuous coaction form a base of neighborhoods of zero in V. Given a topological vector space V endowed with a continuous coaction of a coalgebra $\mathcal{C}$ and a topological vector space W endowed with a continuous coaction of a coalgebra $\mathcal{D}$, all the three topological tensor products $V \otimes^! W$, $V \otimes^* W$, and $V \overleftarrow{\otimes} W$ are naturally endowed with a continuous coaction of the coalgebra $\mathcal{C} \otimes_k \mathcal{D}$. In particular, when two topological vector spaces V and W are endowed with continuous coactions of a Hopf algebra $\mathcal{C}$, the three topological tensor products acquire the tensor product coactions of $\mathcal{C}$ [52, Sections D.1.3–4].

A continuous coaction of a commutative Hopf algebra $\mathcal{C}$ in a topological Lie algebra $\mathfrak{g}$ is said to be *compatible* with the Lie algebra structure if the bracket map $[\,,\,]\colon \mathfrak{g} \otimes^* \mathfrak{g} \longrightarrow \mathfrak{g}$ commutes with the continuous coactions of $\mathcal{C}$. Similarly one can speak about compatibility of topological associative algebra structures, continuous actions, pairings, etc. with continuous coactions of $\mathcal{C}$ [52, Section D.1.5]. A Tate Lie algebra $\mathfrak{g}$ with a continuous coaction of a commutative Hopf algebra $\mathcal{C}$ compatible with the Lie algebra structure has a base of neighborhoods of zero consisting of $\mathcal{C}$-invariant compact open Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ [52, Section D.1.8].

A *Tate Harish-Chandra pair* $(\mathfrak{g}, \mathcal{C})$ is a set of data consisting of a Tate Lie algebra $\mathfrak{g}$, a commutative Hopf algebra $\mathcal{C}$, a continuous coaction of $\mathcal{C}$ in $\mathfrak{g}$ compatible with the Lie algebra structure, a $\mathcal{C}$-invariant compact open subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a continuous pairing $\psi\colon \mathcal{C} \times \mathfrak{h} \longrightarrow k$, where $\mathcal{C}$ is endowed with the discrete topology. These data should satisfy four compatibility equations: the pairing ψ should be compatible with the comultiplication in $\mathcal{C}$ and the Lie bracket in $\mathfrak{g}$, and also with the multiplication in $\mathcal{C}$; the pairing ψ should be compatible with the restriction to $\mathfrak{h}$ of the continuous coaction of $\mathfrak{h}$ in $\mathcal{C}$ and the adjoint coaction of $\mathcal{C}$ in itself; the action of $\mathfrak{h}$ in $\mathfrak{g}$ obtained by deriving the continuous coaction of $\mathcal{C}$ in $\mathfrak{g}$ using the paring ψ should coincide with the adjoint action of $\mathfrak{h}$ in $\mathfrak{g}$. We refer to [52, Section D.2.1] for the details.

In particular, let $\mathfrak{g}$ be a Tate Lie algebra with a pro-nilpotent compact open subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i. e., $\mathfrak{h}$ is the projective limit of a filtered projective system of finite-dimensional nilpotent Lie algebras. Assume further that the discrete $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$ is nilpotent or "ind-nilpotent", i. e., it is the union of finite-dimensional nilpotent modules over finite-dimensional quotient Lie algebras of $\mathfrak{h}$ by its open ideals. Then, over a field of characteristic 0, one can integrate the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ to a Tate Harish-Chandra pair with the Hopf algebra $\mathcal{C} = \mathcal{C}(H)$ of the pro-unipotent pro-algebraic group H corresponding to the Lie algebra $\mathfrak{h}$. A version of this construction is applicable over fields of arbitrary characteristic [52, Section D.6]. In particular, let $\mathfrak{g} = \bigoplus_{n<0} \mathfrak{g}_n \oplus \prod_{n\geqslant 0} \mathfrak{g}_n$ be the Laurent totalization a $\mathbb{Z}$-graded Lie algebra with finite-dimensional components (see Section 2.4); then for any integer $m \geqslant 1$ the Lie subalgebra $\mathfrak{h} = \prod_{n\geqslant m} \mathfrak{g}_m \subset \mathfrak{g}$ satisfies the above nilpotency conditions, so there is the corresponding Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$.

A *Harish-Chandra module* $\mathcal{M}$ over a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ is a k-vector space endowed with a discrete $\mathfrak{g}$-module and a $\mathcal{C}$-comodule structures satisfying the following two compatibility equations. Firstly, the derivative $\mathfrak{h}$-action of the $\mathcal{C}$-coaction in $\mathcal{M}$, which is always a discrete action due to the continuity/discreteness condition imposed on the pairing ψ, should coincide with the restriction of the $\mathfrak{g}$-action in $\mathcal{M}$ to $\mathfrak{h}$. Secondly, the action map $\mathfrak{g}\otimes^*\mathcal{M} \longrightarrow \mathcal{M}$ should be compatible with the continuous coactions of $\mathcal{C}$; equivalently, the action map $\mathfrak{g}/U \otimes_k \mathcal{L} \longrightarrow \mathcal{M}$ should be a morphism of $\mathcal{C}$-comodules for any finite-dimensional $\mathcal{C}$-subcomodule $\mathcal{L} \subset \mathcal{M}$ and any $\mathcal{C}$-invariant compact open subspace $U \subset \mathfrak{g}$ annihilating $\mathcal{L}$ [52, Section D.2.5].

The pairing $\psi\colon \mathcal{C} \times \mathfrak{h} \longrightarrow k$ can be viewed as a Lie algebra morphism $\mathfrak{h} \longrightarrow \mathcal{C}^*$ (where the Lie algebra structure on $\mathcal{C}^*$ is defined in terms of its associative algebra structure), and as such, can be uniquely extended to an associative algebra morphism $U(\mathfrak{h}) \longrightarrow \mathcal{C}^*$, providing a pairing $\phi\colon \mathcal{C} \otimes_k U(\mathfrak{h}) \longrightarrow k$ compatible with the Hopf algebra structures on both sides. When the pairing ϕ is nondegenerate in $\mathcal{C}$ [52, condition D.2.2 (iv)], the derivative action functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow \mathfrak{h}\text{–}\mathsf{discr}$ is fully faithful [52, Section 10.1.4], which allows to simplify the definition of a Harish-Chandra module over $(\mathfrak{g}, \mathcal{C})$. Namely, the second one of the above two compatibility equations holds automatically in this case and can be dropped, so Harish-Chandra modules over $(\mathfrak{g}, \mathcal{C})$ can be simply defined as discrete $\mathfrak{g}$-modules whose discrete $\mathfrak{h}$-module structure comes from a $\mathcal{C}$-comodule structure (cf. [52, Section D.2.2]). In particular, this nondegeneracy condition holds in the above example of a Tate Harish-Chandra pair associated with a Tate Lie algebra $\mathfrak{h}$ with a pro-nilpotent compact open subalgebra $\mathfrak{g}$ acting ind-nilpotently in $\mathfrak{g}/\mathfrak{h}$ (over any field k).

A *Harish-Chandra contramodule* $\mathfrak{P}$ over a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ is a k-vector space endowed with a $\mathfrak{g}$-contramodule and a $\mathcal{C}$-contramodule structures satisfying the following two compatibility equations. Firstly, the contraderivative $\mathfrak{h}$-contraaction of the $\mathcal{C}$-contraaction in $\mathfrak{P}$, which is defined as the composition

$$\mathfrak{h} \otimes^{\frown} \mathfrak{P} \simeq \operatorname{Hom}_k(\mathfrak{h}^\vee, \mathfrak{P}) \longrightarrow \operatorname{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$$

of the map induced by the pairing ψ and the $\mathcal{C}$-contraaction map, should coincide with the restriction of the $\mathfrak{g}$-contraaction in $\mathfrak{P}$ to the subalgebra $\mathfrak{h}$ (see the definition of a contramodule over a Tate Lie algebra $\mathfrak{g}$ in Section 2.4). Secondly, for any $\mathcal{C}$-invariant compact open subspace $U \subset \mathfrak{g}$ the $\mathfrak{g}$-contraaction map $\operatorname{Hom}_k(U^\vee, \mathfrak{P}) \longrightarrow \mathfrak{P}$ should be a morphism of $\mathcal{C}$-contramodules (where the $\mathcal{C}$-contramodule structure on the Hom space from a $\mathcal{C}$-comodule $U^\vee$ into a $\mathcal{C}$-contramodule $\mathfrak{P}$ is provided by the construction from Section 2.7). This definition can be found in [52, Sections D.2.7–8].

For any Harish-Chandra module $\mathcal{M}$ over $(\mathfrak{g}, \mathcal{C})$ and any k-vector space E, the vector space $\mathfrak{P} = \operatorname{Hom}_k(\mathcal{M}, E)$ has a natural Harish-Chandra contramodule structure with the $\mathcal{C}$-contraaction in $\mathfrak{P}$ provided by the construction from Section 1.2 and the $\mathfrak{g}$-contraaction in $\mathfrak{P}$ defined according to the construction from Section 2.4.

The categories $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ and $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C})$ of Harish-Chandra modules and contramodules over a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ are abelian, and the forgetful functors $\mathsf{O}(\mathfrak{g}, \mathcal{C}) \longrightarrow k\text{–}\mathsf{vect}$ and $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C}) \longrightarrow k\text{–}\mathsf{vect}$ are exact. Both the infinite direct sums and infinite products exist in the cagories $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ and $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C})$, but only the direct sums are preserved by the forgetful functor $\mathsf{O}(\mathfrak{g}, \mathcal{C}) \longrightarrow k\text{–}\mathsf{vect}$ and only the products are preserved by the functor $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C}) \longrightarrow k\text{–}\mathsf{vect}$. The category $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ satisfies the axioms Ab5 and Ab3* (but not Ab4*), while the category $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ satisfies the axioms Ab3 and Ab4* (but not Ab4 or Ab5*).

There are enough injective objects in the category $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ and enough projective objects in the category $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C})$. The identification of these categories with categories of semimodules and semicontramodules over certain semialgebras, which we will now briefly discuss, will make the explicit constructions of such injective and projective objects explained below in Section 3.5 applicable in this case.

As in Section 2.7, the semialgebras $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, \mathcal{C})$ and $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, \mathcal{C})$ are defined as the tensor products

$$\boldsymbol{\mathcal{S}}^l = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C} \quad \text{and} \quad \boldsymbol{\mathcal{S}}^r = \mathcal{C} \otimes_{U(\mathfrak{h})} U(\mathfrak{g}),$$

where, as above, $U(\mathfrak{g})$ and $U(\mathfrak{h})$ denote the universal enveloping algebras of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ considered as abstract Lie algebras without any topologies. The left and right $U(\mathfrak{h})$-module structures on $\mathcal{C}$ are obtained by deriving the left and right coactions of $\mathcal{C}$ in itself using the pairing ϕ. The right coaction of $\mathcal{C}$ in $\boldsymbol{\mathcal{S}}^l$ and the left coaction in $\boldsymbol{\mathcal{S}}^r$ are induced by the right and left coactions of $\mathcal{C}$ in itself. The construction of the left coaction of $\mathcal{C}$ in $\boldsymbol{\mathcal{S}}^l$ and the right coaction in $\boldsymbol{\mathcal{S}}^r$ is rather delicate [52, Section D.2.3]. Once the $\mathcal{C}$-$\mathcal{C}$-bicomodule structures on $\boldsymbol{\mathcal{S}}^l$ and $\boldsymbol{\mathcal{S}}^r$ has been defined, the constructions of the semimultiplication and semiunit maps are similar to those in Section 2.7 (see [52, Section 10.2.1]), though one still has to check that the maps $\boldsymbol{\mathcal{S}}^l \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}}^l \longrightarrow \boldsymbol{\mathcal{S}}^l$ and $\boldsymbol{\mathcal{S}}^r \mathbin{\square}_{\mathcal{C}} \boldsymbol{\mathcal{S}}^r \longrightarrow \boldsymbol{\mathcal{S}}^r$ so obtained are morphisms of $\mathcal{C}$-$\mathcal{C}$-bicomodules (the nontrivial part is to show that the former is a morphism of left $\mathcal{C}$-comodules and the latter a morphism of right ones).

Now the category $\mathsf{O}(\mathfrak{g}, \mathcal{C})$ of Harish-Chandra modules over a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ is isomorphic to the category of left semimodules over the semialgebra $\boldsymbol{\mathcal{S}}^l(\mathfrak{g}, \mathcal{C})$; the datum of a left $\boldsymbol{\mathcal{S}}^l$-semimodule structure on a k-vector space $\boldsymbol{\mathcal{M}}$ is equivalent to that of a Harish-Chandra module structure on $\boldsymbol{\mathcal{M}}$. Similarly, the category $\mathsf{O}^{\mathsf{ctr}}(\mathfrak{g}, \mathcal{C})$ of Harish-Chandra contramodules over $(\mathfrak{g}, \mathcal{C})$ is isomorphic to the category of left semicontramodules over the semialgebra $\boldsymbol{\mathcal{S}}^r(\mathfrak{g}, \mathcal{C})$; the datum of a left

$\mathbf{S}^r$-semicontramodule structure on a k-vector space $\mathfrak{P}$ is equivalent to that a Harish-Chandra contramodule structure on $\mathfrak{P}$ [52, Sections 10.2.2, D.2.5, and D.2.8].

As in Section 2.7, one would like to have also an explicit description of what it means to have a left $\mathbf{S}^r$-semimodule structure on a k-vector space $\mathfrak{M}$. In the infinite-dimensional situation, one cannot hope to have a Morita equivalence between the semialgebras $\mathbf{S}^l$ and $\mathbf{S}^r$. Rather, the determinantal anomaly moves one step higher in the cohomological degree when one passes to Tate vector spaces, and what used to be a twist with the modular character in the finite-dimensional case becomes *a shift of the central charge* in the Tate Harish-Chandra situation. Before formulating the precise assertion, let us give the definition of a *central extension* $\varkappa\colon (\mathfrak{g}', \mathcal{C}) \longrightarrow (\mathfrak{g}, \mathcal{C})$ of a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$.

A *morphism of Tate Harish-Chandra pairs* $(\mathfrak{g}', \mathcal{C}') \longrightarrow (\mathfrak{g}, \mathcal{C})$ can be defined as a set of data consisting of a continuous morphism of Tate Lie algebras $\mathfrak{g}' \longrightarrow \mathfrak{g}$ and a morphism of Hopf algebras $\mathcal{C} \longrightarrow \mathcal{C}'$ such that the map $\mathfrak{g}' \longrightarrow \mathfrak{g}$ commutes with the continuous $\mathcal{C}'$-coactions and takes the subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ into the subalgebra $\mathfrak{h} \subset \mathfrak{g}$, while the maps $\mathcal{C} \longrightarrow \mathcal{C}'$ and $\mathfrak{h}' \longrightarrow \mathfrak{h}$ are compatible with the pairings ψ and ψ'. A *central extension of Tate Harish-Chandra pairs with the kernel k* is a morphism $(\mathfrak{g}', \mathcal{C}') \longrightarrow (\mathfrak{g}, \mathcal{C})$ such that $\mathcal{C}' = \mathcal{C}$ is the same Hopf algebra and $\mathcal{C} \longrightarrow \mathcal{C}'$ the identity map, $\mathfrak{g}' \longrightarrow \mathfrak{g}$ is a quotient map of topological vector spaces with a one-dimensional kernel $k \subset \mathfrak{g}'$ in which a fixed basis vector $1_{\mathfrak{g}'} \in k \subset \mathfrak{g}'$ is chosen, the map of Lie subalgebras $\mathfrak{h}' \longrightarrow \mathfrak{h}$ is an isomorphism, the kernel $k = k \cdot 1_{\mathfrak{g}'}$ lies in the center of the Lie algebra $\mathfrak{g}'$, and the coaction of the Hopf algebra $\mathcal{C}$ in the subspace $k \subset \mathfrak{g}'$ is trivial [52, Section D.2.2]. As it is usual for extensions with a fixed kernel, the set of all (isomorphism classes of) central extensions of a given Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ with the kernel k has a natural structure of abelian group (and in fact, even of a k-vector space) with respect to the Baer sum operation [52, Section D.3.1].

The Lie algebra $\mathfrak{gl}(V)$ of continuous endomorphisms of a Tate vector space V has a canonical central extension $\mathfrak{gl}(V)^\sim$ with the kernel k defined in terms of the trace functional on the space of all continuous linear operators $V \longrightarrow V$ of finite rank [7, Sections 2.7.8 and 3.8.17–18] (see also [52, Sections D.1.6–8]; a historical discussion can be found in [7, the beginning of Section 2.7]). The central extension $\gamma_0\colon\ \mathfrak{gl}(V)^\sim \longrightarrow \mathfrak{gl}(V)$ can be characterized by the property that there is a well-defined linear action of the Lie algebra $\mathfrak{gl}(V)^\sim$ in the space $\bigwedge^{\infty/2}(V)$ of semi-infinite exterior forms over V lifting the projective action of the Lie algebra $\mathfrak{gl}(V)$ (see [29], [40, Lecture 4], and [8, Sections 4.2.13 and 7.13.16]). One chooses the canonical basis element $1_{\mathfrak{gl}^\sim} \in k \subset \mathfrak{gl}(V)^\sim$ so that it acts by the identity operator in the space of semi-infinite forms; abusing the terminology, we say that $\bigwedge^{\infty/2}(V)$ is a "$\mathfrak{gl}(V)$-module with the central charge γ_0".

Pulling back the central extension $\mathfrak{gl}(\mathfrak{g})^\sim \longrightarrow \mathfrak{gl}(\mathfrak{g})$ with respect to the adjoint representation $\mathfrak{ad}_{\mathfrak{g}}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, one obtains the canonical central extension $\varkappa_0\colon \mathfrak{g}^\sim \longrightarrow \mathfrak{g}$ of an arbitrary Tate Lie algebra $\mathfrak{g}$. The above convention for the choice of the canonical basis element $1_{\mathfrak{gl}^\sim}$ and the consequent choice of the basis element $1_{\mathfrak{g}^\sim} \in k \subset \mathfrak{g}$

provide it that for any discrete module M over the Lie algebra $\mathfrak{g}^\sim$ where the element $1_{\mathfrak{g}^\sim}$ acts by minus the identity operator ("a discrete $\mathfrak{g}$-module with the central charge $-\varkappa_0$") there is a well-defined discrete action of the original Lie algebra $\mathfrak{g}$ in the tensor product $\bigwedge^{\infty/2}(V)\otimes_k M$, allowing to define the *semi-infinite homology* of $\mathfrak{g}$ with coefficients in M [7, Sections 3.8.19–22] as the homology of a natural differential on $\bigwedge^{\infty/2}(V)\otimes_k M$ (see [7, the beginning of Section 3.8] for a historical discussion). In addition, in [52, Sections D.5.5–6], the semi-infinite *cohomology* of a Tate Lie algebra $\mathfrak{g}$ with coefficients in any $\mathfrak{g}^\sim$*-contramodule* $\mathfrak{P}$ where $1_{\mathfrak{g}^\sim}$ acts by the identity ("a $\mathfrak{g}$-contramodule with the central charge $\varkappa_0$") is also defined as the cohomology of a natural differential on the space $\mathrm{Hom}_k(\bigwedge^{\infty/2}(V),\,\mathfrak{P})$.

In particular, for the Lie algebra $\mathfrak{g}=k((z))d/dz$ of vector fields on the formal circle, the canonical central extension $\mathfrak{g}^\sim$ is the Virasoro algebra $\mathbb{V}\mathrm{ir}$ (see Section 1.7). The canonical basis element $1\in k\subset\mathbb{V}\mathrm{ir}$ is $-C/26$, i. e., the central element $C\in\mathbb{V}\mathrm{ir}$ acts by the scalar -26 in the space of semi-infinite forms $\bigwedge^{\infty/2}\big(k((z))d/dz\big)$. So the semi-infinite homology and cohomology is defined for any discrete $\mathbb{V}\mathrm{ir}$-module (or, as we will say, "$k((z))d/dz$-module") with the central charge $C=26$ and any $\mathbb{V}\mathrm{ir}$-contramodule ("$k((z))d/dz$-contramodule") with the central charge -26.

When a Tate vector space V is endowed with a continuous coaction of a commutative Hopf algebra $\mathcal{C}$, the topological Lie algebra $\mathfrak{gl}(V)^\sim$ acquires the induced continuous coaction of $\mathcal{C}$ [52, Sections D.1.6–7]. Hence a continuous coaction of a commutative Hopf algebra $\mathcal{C}$ in a Tate Lie algebra $\mathfrak{g}$ always lifts naturally to a continuous coaction of $\mathcal{C}$ in the canonical central extension $\mathfrak{g}^\sim$ of $\mathfrak{g}$. Furthermore, the canonical central extension $\mathfrak{g}^\sim\longrightarrow\mathfrak{g}$ splits naturally over any compact open Lie subalgebra $\mathfrak{h}\subset\mathfrak{g}$ [7, Section 3.8.19], [52, Section D.1.8] (warning: over a pair of embedded compact open Lie subalgebras $\mathfrak{h}'\subset\mathfrak{h}''\subset\mathfrak{g}$, these splittings do *not* agree, but rather differ by a relative adjoint trace character). For any Tate Harish-Chandra pair $(\mathfrak{g},\mathcal{C})$, this allows to extend the canonical central extension $\varkappa_0\colon\mathfrak{g}^\sim\longrightarrow\mathfrak{g}$ of the Tate Lie algebra $\mathfrak{g}$ to a canonical central extension of Tate Harish-Chandra pairs $(\mathfrak{g}^\sim,\mathcal{C})\longrightarrow(\mathfrak{g},\mathcal{C})$, which we will denote also by $\varkappa_0$.

Let $\varkappa\colon(\mathfrak{g}',\mathcal{C}')\longrightarrow(\mathfrak{g},\mathcal{C})$ be a central extension of Tate Harish-Chandra pairs with the kernel $k=k\cdot1_{\mathfrak{g}'}$. By a *Harish-Chandra module* $\mathcal{M}$ *over* $(\mathfrak{g},\mathcal{C})$ *with the central charge* $\varkappa$ we mean a Harish-Chandra module over the Tate Harish-Chandra pair $(\mathfrak{g}',\mathcal{C})$ such that the central element $1_{\mathfrak{g}'}\in\mathfrak{g}'$ acts by the identity operator in $\mathcal{M}$. Similarly, a *Harish-Chandra contramodule* $\mathfrak{P}$ over a Tate Harish-Chandra pair $(\mathfrak{g},\mathcal{C})$ *with the central charge* $\varkappa$ is a Harish-Chandra contramodule over $(\mathfrak{g}',\mathcal{C})$ such that the central element $1_{\mathfrak{g}'}$ contraacts by the identity operator in $\mathfrak{P}$, that is the composition

$$\mathfrak{P}\simeq k\otimes^{\frown}\mathfrak{P}\longrightarrow\mathfrak{g}'\otimes^{\frown}\mathfrak{P}\longrightarrow\mathfrak{P}$$

of the map induced by the choice of the element $1_{\mathfrak{g}'}\in\mathfrak{g}'$ with the $\mathfrak{g}'$-contraaction map is equal to the identity map $\mathfrak{P}\longrightarrow\mathfrak{P}$.

The category $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ of Harish-Chandra modules over $(\mathfrak{g},\mathcal{C})$ with the central charge $\varkappa$ is abelian; the fully faithful embedding functor $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})\longrightarrow\mathsf{O}(\mathfrak{g}',\mathcal{C})$ is exact and preserves both the infinite direct sums and products. The category $\mathsf{O}^{\mathsf{ctr}}_\varkappa(\mathfrak{g},\mathcal{C})$ of

Harish-Chandra contramodules over $(\mathfrak{g}, \mathcal{C})$ with the central charge $\varkappa$ is also abelian; the fully faithful embedding functor $\mathsf{O}_\varkappa(\mathfrak{g}, \mathcal{C}) \longrightarrow \mathsf{O}(\mathfrak{g}', \mathcal{C})$ is exact and preserves the infinite direct sums and products. There are enough injective objects in the category $\mathsf{O}_\varkappa(\mathfrak{g}', \mathcal{C})$ and enough projective objects in the category $\mathsf{O}^{\mathsf{ctr}}_\varkappa(\mathfrak{g}', \mathcal{C})$; we will see below in Section 3.5 how one can construct them.

Our next goal is to define semialgebras $\boldsymbol{\mathcal{S}}^l = \boldsymbol{\mathcal{S}}^l_\varkappa(\mathfrak{g}, \mathcal{C})$ and $\boldsymbol{\mathcal{S}}^r = \boldsymbol{\mathcal{S}}^r_\varkappa(\mathfrak{g}, \mathcal{C})$ such that the categories of Harish-Chandra modules and contramodules over a Tate Harish-Chandra pair $(\mathfrak{g}, \mathcal{C})$ with the central charge $\varkappa$ could be identified with the categories of semimodules and semicontramodules over $\boldsymbol{\mathcal{S}}^l$ and $\boldsymbol{\mathcal{S}}^r$. Let us start with the related elementary construction of the modified universal enveloping algebra corresponding to a central extension of (nontopological) Lie algebras.

Given a central extension $\varkappa\colon \mathfrak{g}' \longrightarrow \mathfrak{g}$ with the kernel $k = k \cdot 1_{\mathfrak{g}'}$, the algebra $U_\varkappa(\mathfrak{g})$ is constructed as the quotient algebra $U(\mathfrak{g}')/(1_{U(\mathfrak{g}')} - 1_{\mathfrak{g}'})$ of the enveloping algebra $U(\mathfrak{g}')$ by the ideal generated by the difference between the unit element $1_{U(\mathfrak{g}')}$ of the associative algebra $U(\mathfrak{g}')$ and the fixed basis vector $1_{\mathfrak{g}'}$ in the kernel of the central extension. Then the category of left $U_\varkappa(\mathfrak{g})$-modules is isomorphic to the category of "$\mathfrak{g}$-modules with the central charge $\varkappa$", i. e., $\mathfrak{g}'$-modules where the element $1_{\mathfrak{g}'}$ acts by the identity operator, while the category of right $U_\varkappa(\mathfrak{g})$-modules is isomorphic to the category of $\mathfrak{g}$-modules with the central charge $-\varkappa$.

Now the semialgebras $\boldsymbol{\mathcal{S}}^l_\varkappa(\mathfrak{g}, \mathcal{C})$ and $\boldsymbol{\mathcal{S}}^r_\varkappa(\mathfrak{g}, \mathcal{C})$ are constructed as the tensor products

$$\boldsymbol{\mathcal{S}}^l = U_\varkappa(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C} \quad \text{and} \quad \boldsymbol{\mathcal{S}}^r = \mathcal{C} \otimes_{U(\mathfrak{h})} U_\varkappa(\mathfrak{g}).$$

The datum of a left $\boldsymbol{\mathcal{S}}^l$-semimodule structure on a given k-vector space $\boldsymbol{\mathcal{M}}$ is equivalent to that of a Harish-Chandra module structure over $(\mathfrak{g}, \mathcal{C})$ with the central charge $\varkappa$, while the datum of a left $\boldsymbol{\mathcal{S}}^r$-semicontramodule structure on a given k-vector space $\mathfrak{P}$ is equivalent to that of a Harish-Chandra contramodule structure over $(\mathfrak{g}, \mathcal{C})$ with the central charge $\varkappa$ [52, Sections D.2.2, D.2.5 and D.2.8]. The following theorem, when its condition is satisfied, allows to describe left $\boldsymbol{\mathcal{S}}^r$-semimodules.

Theorem. *Assume that the pairing $\phi\colon \mathcal{C} \otimes_k U(\mathfrak{h}) \longrightarrow k$ is nondegenerate in $\mathcal{C}$. Then there is a natural isomorphism of semialgebras*

$$\boldsymbol{\mathcal{S}}^r_{\varkappa+\varkappa_0}(\mathfrak{g}, \mathcal{C}) \simeq \boldsymbol{\mathcal{S}}^l_\varkappa(\mathfrak{g}, \mathcal{C})$$

over the coalgebra $\mathcal{C}$.

Proof. This is one of the most difficult, and at the same time a singularly least well-understood result of the book [52]. The precise formulation, where the desired isomorphism is uniquely characterized by a certain list of conditions, can be found in [52, Section D.3.1]. The lengthy proof, which is based on the relative nonhomogeneous quadratic duality theory developed in [52, Chapter 11] (see [52, Section 0.4] for an introduction), occupies the rest of [52, Section D.3]. ☐

3. Tensor Operations and Adjusted Objects

3.1. **Comodules and contramodules over coalgebras over fields.** Let $\mathcal{C}$ be a coassociative coalgebra over a field k. The constructions of the *cotensor product* $\mathcal{N}\,\square_{\mathcal{C}}\,\mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ and the vector space of *cohomomorphisms* $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M},\mathfrak{P})$ from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{C}$-contramodule $\mathfrak{P}$ were already presented at the end Section 2.5 and repeated at the beginning of Section 2.6. The main function of these two tensor operations on comodules and contramodules in the theories developed in the book [52] is to provide the tensor and module category structures in whose terms the notion of a semialgebra and the categories of semimodules and semicontramodules are subsequently defined.

Let us now introduce the definition of the *contratensor product* of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-contramodule $\mathfrak{P}$, which plays a key role in the comodule-contramodule correspondence constructions. The contratensor product $\mathcal{N}\odot_{\mathcal{C}}\mathfrak{P}$ is a k-vector space defined as the cokernel of (the difference of) the pair of maps

$$(\mathrm{id}\otimes\mathrm{ev}_{\mathcal{C}})\circ(\nu_N\otimes\mathrm{id}),\ \ \mathrm{id}\otimes\pi_{\mathfrak{P}}\colon \mathcal{N}\otimes_k\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\rightrightarrows\mathcal{N}\otimes_k\mathfrak{P},$$

one of which is the composition $\mathcal{N}\otimes_k\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\longrightarrow\mathcal{N}\otimes_k\mathcal{C}\otimes_k\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\longrightarrow\mathcal{N}\otimes_k\mathfrak{P}$ of the map induced by the $\mathcal{C}$-coaction in $\mathcal{N}$ and the map induced by the evaluation map $\mathrm{ev}_{\mathcal{C}}\colon\mathcal{C}\otimes_k\operatorname{Hom}_k(\mathcal{C},\mathfrak{P})\longrightarrow\mathfrak{P}$, while the other one is induced by the $\mathcal{C}$-contraaction in $\mathfrak{P}$. The functor of contratensor product of comodules and contramodules over a coalgebra $\mathcal{C}$ is right exact. For any right $\mathcal{C}$-comodule $\mathcal{N}$ and k-vector space V there is a natural isomorphism of k-vector spaces

$$\mathcal{N}\odot_{\mathcal{C}}\operatorname{Hom}_k(\mathcal{C},V)\simeq\mathcal{N}\otimes_k V.$$

Furthermore, for any right $\mathcal{C}$-comodule $\mathcal{N}$, left $\mathcal{C}$-contramodule $\mathfrak{P}$, and k-vector space V there is a natural isomorphism of k-vector spaces

$$\operatorname{Hom}_k(\mathcal{N}\odot_{\mathcal{C}}\mathfrak{P},\ V)\simeq\operatorname{Hom}^{\mathcal{C}}(\mathfrak{P},\operatorname{Hom}_k(\mathcal{N},V)),$$

where the vector space $\operatorname{Hom}_k(\mathcal{N},V)$ is endowed with a left $\mathcal{C}$-contramodule structure as explained in Section 1.2 [52, Sections 0.2.6 and 5.1.1].

For any two coalgebras $\mathcal{C}$ and $\mathcal{D}$ over k, any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, and any left $\mathcal{C}$-comodule $\mathcal{M}$, the vector space of $\mathcal{C}$-comodule homomorphisms $\operatorname{Hom}_{\mathcal{C}}(\mathcal{K},\mathcal{M})$ has a natural left $\mathcal{D}$-contramodule structure. One can define it by noticing that $\operatorname{Hom}_{\mathcal{C}}(\mathcal{K},\mathcal{M})$ is a subcontramodule in the left $\mathcal{D}$-contramodule $\operatorname{Hom}_k(\mathcal{K},\mathcal{M})$, whose contramodule structure is induced by the right $\mathcal{D}$-comodule structure on $\mathcal{K}$. Similarly, for any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$ and left $\mathcal{D}$-contramodule $\mathfrak{P}$ the vector space $\mathcal{K}\odot_{\mathcal{D}}\mathfrak{P}$ has a natural left $\mathcal{C}$-comodule structure.

For any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, left $\mathcal{C}$-comodule $\mathcal{M}$, and left $\mathcal{D}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of k-vector spaces

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{K}\odot_{\mathcal{D}}\mathfrak{P},\ \mathcal{M})\simeq\operatorname{Hom}^{\mathcal{D}}(\mathfrak{P},\operatorname{Hom}_{\mathcal{C}}(\mathcal{K},\mathcal{M})).$$

In other words, the functor $\mathcal{K}\odot_{\mathcal{D}}-\colon\mathcal{D}\text{–}\mathsf{contra}\longrightarrow\mathcal{C}\text{–}\mathsf{comod}$ is left adjoint to the functor $\operatorname{Hom}_{\mathcal{C}}(\mathcal{K},-)\colon\mathcal{C}\text{–}\mathsf{comod}\longrightarrow\mathcal{D}\text{–}\mathsf{contra}$ [52, Section 5.1.2]. The adjoint functors $\operatorname{Hom}_{\mathcal{C}}(\mathcal{C},-)$ and $\mathcal{C}\odot_{\mathcal{C}}-$ of comodule homomorphisms from and contratensor product with the $\mathcal{C}$-$\mathcal{C}$-bicomodule $\mathcal{K} = \mathcal{C}$ restricted to the additive subcategories of injective $\mathcal{C}$-comodules and projective $\mathcal{C}$-contramodules provide the equivalence of additive categories $\mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}} \simeq \mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ described at the end of Section 1.2.

Relations between (or rather, in the case of a coalgebra $\mathcal{C}$ over a field k, the coincidences of) the following classes of adjusted objects, together with the classes of injective and projective objects, in the comodule and contramodule categories play an important role in the co/contramodule and semico/semicontramodule theory. A discussion of these coincidences is the aim of the remaining part of this section. We will see in the proof of Proposition 3.5 below how these results are being used.

A left $\mathcal{C}$-comodule $\mathcal{M}$ is called *coflat* if the functor $-\mathbin{\square_{\mathcal{C}}}\mathcal{M}\colon \mathsf{comod}\text{–}\mathcal{C} \longrightarrow k\text{–}\mathsf{vect}$ of cotensor product with $\mathcal{M}$ is exact on the abelian category of right $\mathcal{C}$-comodules. A left $\mathcal{C}$-comodule $\mathcal{M}$ is called *coprojective* if the functor $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M},-)\colon \mathcal{C}\text{–}\mathsf{contra} \longrightarrow k\text{–}\mathsf{vect}$ is exact on the abelian category of left $\mathcal{C}$-contramodules.

Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is called *contraflat* if the functor $-\odot_{\mathcal{C}}\mathfrak{P}\colon \mathsf{comod}\text{–}\mathcal{C} \longrightarrow k\text{–}\mathsf{vect}$ of contratensor product with $\mathfrak{P}$ is exact on the abelian category of right $\mathcal{C}$-comodules. A left $\mathcal{C}$-contramodule $\mathfrak{P}$ is called *coinjective* if the functor $\operatorname{Cohom}_{\mathcal{C}}(-,\mathfrak{P})\colon \mathcal{C}\text{–}\mathsf{comod}^{\mathsf{op}} \longrightarrow k\text{–}\mathsf{vect}$ is exact on the abelian category of left $\mathcal{C}$-comodules [52, Section 0.2.9].

Lemma. *Let $\mathcal{C}$ be a coassociative coalgebra over a field k. Then*

(a) *a $\mathcal{C}$-comodule is coflat if and only if it is coprojective and if and only if it is injective;*

(b) *a $\mathcal{C}$-contramodule is contraflat if and only if it is coinjective and if and only if it is projective.*

Proof. Part (a): it is clear from the natural isomorphism $\operatorname{Hom}_k(\mathcal{N}\mathbin{\square_{\mathcal{C}}}\mathcal{M},\, V) \simeq \operatorname{Cohom}_{\mathcal{C}}(\mathcal{M}, \operatorname{Hom}_k(\mathcal{N}, V))$ for any right $\mathcal{C}$-comodule $\mathcal{N}$, left $\mathcal{C}$-comodule $\mathcal{M}$, and k-vector space V (see Section 2.6) that any coprojective left $\mathcal{C}$-comodule $\mathcal{M}$ is coflat, and from the natural isomorphism $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{C}\otimes_k V,\ \mathfrak{P}) \simeq \operatorname{Hom}_k(V,\mathfrak{P})$ for any k-vector space V and left $\mathcal{C}$-contramodule $\mathfrak{P}$ (see Section 2.5) that any injective left $\mathcal{C}$-comodule $\mathcal{M}$ is coprojective.

Conversely, by a comodule version of Baer's criterion, a left $\mathcal{C}$-comodule $\mathcal{M}$ is injective whenever the functor $\operatorname{Hom}_{\mathcal{C}}(-,\mathcal{M})$ is exact on the category of *finite-dimensional* left $\mathcal{C}$-comodules. Indeed, a $\mathcal{C}$-comodule morphism into $\mathcal{M}$ from a subcomodule of any $\mathcal{C}$-comodule can be successively extended to larger and larger subcomodules in the way of a transfinite induction or Zorn's lemma, and one only has to deal with subcomodules of finite-dimensional $\mathcal{C}$-comodules in the process. It remains to notice the natural right $\mathcal{C}$-comodule structure on the dual vector space $\mathcal{L}^*$ to any finite-dimensional left $\mathcal{C}$-comodule $\mathcal{L}$, and the natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{L},\mathcal{M}) \simeq \mathcal{L}^*\mathbin{\square_{\mathcal{C}}}\mathcal{M}$$

for any finite-dimesional left $\mathcal{C}$-comodule $\mathcal{L}$ and any left $\mathcal{C}$-comodule $\mathcal{M}$, in order to conclude that any coflat left $\mathcal{C}$-comodule $\mathcal{M}$ is injective.

Part (b): since any $\mathcal{C}$-comodule is a union of its finite-dimensional subcomodules, and the functor of contratensor product $-\odot_{\mathcal{C}}\mathfrak{P}$ preserves inductive limits (in its comodule argument), a left $\mathcal{C}$-contramodule $\mathfrak{P}$ is contraflat whenever the functor $-\odot_{\mathcal{C}}\mathfrak{P}$ is exact on the category of finite-dimensional right $\mathcal{C}$-comodules. It remains to notice the natural isomorphism

$$\operatorname{Cohom}_{\mathcal{C}}(\mathcal{L},\mathfrak{P})\simeq\mathcal{L}^*\odot_{\mathcal{C}}\mathfrak{P}$$

for any finite-dimensional left $\mathcal{C}$-comodule $\mathcal{L}$ and any left $\mathcal{C}$-comodule $\mathfrak{P}$ in order to conclude that any coinjective left $\mathcal{C}$-comodule is coflat. The assertion that every projective $\mathcal{C}$-contramodule is coinjective follows immediately from the natural isomorphism $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M},\operatorname{Hom}_k(\mathcal{C},V))\simeq\operatorname{Hom}_k(\mathcal{M},V)$ for any left $\mathcal{C}$-comodule $\mathcal{M}$ and k-vector space V (see Section 2.5).

Showing that every contraflat $\mathcal{C}$-contramodule is projective is much more difficult. This assertion was formulated as a conjecture in [51] and eventually proven in [52, Section A.3]. The result in question is a generalization of the classical theorem that flat modules over a finite-dimensional associative algebra are projective [3, Theorem P and Examples I.3(1)]. The argument is [3] is based on the structure theory of Artinian rings. Similarly, the proof of projectivity of contraflat contramodules in [52] is based on the structure theory of coalgebras over fields [62, Sections 9.0–1] and the structure theory of contramodules over them developed in [52, Section A.2], and first of all, on the contramodule Nakayama lemma (see Section 2.1).

In the exposition below, we restrict ourselves to proving that any coinjective $\mathcal{C}$-contramodule is projective. This is the result that has an important application to semicontramodules that will be considered in Section 3.5. The argument that we describe here also has an advantage of being generalizable to comodules and contramodules over corings over arbitrary noncommutative rings [52, Lemma 5.2].

The assertion comes out as an unexpected consequence of one of the results establishing mutual associativity of the cotensor product/cohomomorphism operations $\square_{\mathcal{C}}$ or $\operatorname{Cohom}_{\mathcal{C}}$ with the operations of contratensor product, comodule homomorphisms, or contramodule homomorphisms $\odot_{\mathcal{C}}$, $\operatorname{Hom}_{\mathcal{C}}$, or $\operatorname{Hom}^{\mathcal{C}}$. The operations from the first and the second list are exact on different sides, so they are only mutually associative under certain adjustness conditions on the objects involved [52, Section 5.2] (cf. [51], where these mutual associativity assertions were formulated in a less general form insufficient for deducing the corollary under discussion).

Let $\mathcal{C}$ and $\mathcal{D}$ be two coalgebras over a field k.

Proposition 1. *Let $\mathcal{N}$ be a right $\mathcal{D}$-comodule, $\mathcal{K}$ be a $\mathcal{D}$-$\mathcal{C}$-bicomodule, and $\mathfrak{P}$ be a left $\mathcal{C}$-contramodule. Then there is a natural map of k-vector spaces*

$$(\mathcal{N}\,\square_{\mathcal{D}}\,\mathcal{K})\odot_{\mathcal{C}}\mathfrak{P}\longrightarrow\mathcal{N}\,\square_{\mathcal{D}}\,(\mathcal{K}\odot_{\mathcal{C}}\mathfrak{P}),$$

which is an isomorphism whenever $\mathfrak{P}$ is a contraflat left $\mathcal{C}$-comodule or $\mathcal{N}$ is a coflat right $\mathcal{C}$-comodule.

Proposition 2. *Let $\mathcal{L}$ be a left $\mathcal{D}$-comodule, $\mathcal{K}$ be a $\mathcal{C}$-$\mathcal{D}$-bicomodule, and $\mathcal{M}$ be a left $\mathcal{C}$-comodule. Then there is a natural map of k-vector spaces*

$$\mathrm{Cohom}_{\mathcal{D}}(\mathcal{L}, \mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M})) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{K} \mathbin{\square}_{\mathcal{D}} \mathcal{L},\ \mathcal{M}),$$

which is an isomorphism whenever $\mathcal{M}$ is an injective left $\mathcal{C}$-comodule or $\mathcal{L}$ is a coprojective left $\mathcal{D}$-comodule.

Proposition 3. *Let $\mathfrak{P}$ be a left $\mathcal{C}$-contramodule, $\mathcal{K}$ be a $\mathcal{D}$-$\mathcal{C}$-bicomodule, and $\mathfrak{Q}$ be a left $\mathcal{D}$-contramodule. Then there is a natural map of k-vector spaces*

$$\mathrm{Cohom}_{\mathcal{D}}(\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathrm{Cohom}_{\mathcal{D}}(\mathcal{K}, \mathfrak{Q})),$$

which is an isomorphism whenever $\mathfrak{P}$ is a projective left $\mathcal{C}$-contramodule or $\mathfrak{Q}$ is a coinjective left $\mathcal{D}$-contramodule.

Proof. To construct the natural map in Proposition 1, one considers the composition

$$(\mathcal{N} \mathbin{\square}_{\mathcal{D}} \mathcal{K}) \otimes_k \mathfrak{P} \longrightarrow \mathcal{N} \otimes_k \mathcal{K} \otimes_k \mathfrak{P} \longrightarrow \mathcal{N} \otimes_k (\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P})$$

and observes that it is has equal compositions with the two maps $(\mathcal{N} \mathbin{\square}_{\mathcal{D}} \mathcal{K}) \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \rightrightarrows (\mathcal{N} \mathbin{\square}_{\mathcal{D}} \mathcal{K}) \otimes_k \mathfrak{P}$, as well as with the two maps $\mathcal{N} \otimes_k (\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P}) \rightrightarrows \mathcal{N} \otimes_k \mathcal{D} \otimes_k (\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P})$. This map is an isomorphism when the $\mathcal{C}$-contramodule $\mathfrak{P}$ is contraflat, since the exact sequence

$$0 \longrightarrow \mathcal{N} \mathbin{\square}_{\mathcal{D}} \mathcal{K} \longrightarrow \mathcal{N} \otimes_k \mathcal{K} \longrightarrow \mathcal{N} \otimes_k \mathcal{D} \otimes_k \mathcal{K}$$

remains exact after applying the functor $- \odot_{\mathcal{C}} \mathfrak{P}$, and when the $\mathcal{C}$-comodule $\mathcal{N}$ is coflat, since the exact sequence

$$\mathcal{K} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathcal{K} \otimes_k \mathfrak{P} \longrightarrow \mathcal{K} \odot_{\mathcal{C}} \mathfrak{P} \longrightarrow 0$$

remains exact after applying the functor $\mathcal{N} \mathbin{\square}_{\mathcal{D}} -$.

To construct the natural map in Proposition 3, one considers the composition

$$\mathrm{Hom}_k(\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{Q}) \longrightarrow \mathrm{Hom}_k(\mathcal{K} \otimes_k \mathfrak{P},\ \mathfrak{Q}) \simeq \mathrm{Hom}_k(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{K}, \mathfrak{Q})) \longrightarrow \mathrm{Hom}_k(\mathfrak{P}, \mathrm{Cohom}_{\mathcal{D}}(\mathcal{K}, \mathfrak{Q}))$$

and observes that it has equal compositions with the two maps $\mathrm{Hom}_k(\mathcal{D} \otimes_k (\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P}), \mathfrak{Q}) \rightrightarrows \mathrm{Hom}_k(\mathcal{K} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{Q})$, as well as with the two maps $\mathrm{Hom}_k(\mathfrak{P}, \mathrm{Cohom}_{\mathcal{D}}(\mathcal{K}, \mathfrak{Q})) \rightrightarrows \mathrm{Hom}_k(\mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}), \mathrm{Cohom}_{\mathcal{D}}(\mathcal{K}, \mathfrak{Q}))$. This map is an isomorphism when the $\mathcal{C}$-contramodule $\mathfrak{P}$ is projective, since the exact sequence

$$\mathrm{Hom}_k(\mathcal{D} \otimes_k \mathcal{K},\ \mathfrak{Q}) \longrightarrow \mathrm{Hom}_k(\mathcal{K}, \mathfrak{Q}) \longrightarrow \mathrm{Cohom}_{\mathcal{D}}(\mathcal{K}, \mathfrak{Q}) \longrightarrow 0$$

remains exact after applying the functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)$, and when the $\mathcal{D}$-contramodule $\mathfrak{Q}$ is coinjective, since the exact sequence

$$\mathcal{K} \otimes_k \mathrm{Hom}_k(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathcal{K} \otimes_k \mathfrak{P} \longrightarrow \mathcal{K} \odot_{\mathcal{C}} \mathfrak{P} \longrightarrow 0$$

remains exact after applying the functor $\mathrm{Cohom}_{\mathcal{D}}(-, \mathfrak{Q})$. The proof of Proposition 2 is similar. ☐

Now we can prove that any coinjective left $\mathcal{C}$-contramodule $\mathfrak{P}$ is projective. Let $l\colon \mathfrak{E} \longrightarrow \mathfrak{P}$ be a surjective $\mathcal{C}$-contramodule morphism. According to the second assertion of Proposition 3 applied to the coalgebras $\mathcal{C} = \mathcal{D}$, the bicomodule $\mathcal{K} = \mathcal{C}$, and the contramodules $\mathfrak{Q} = \mathfrak{P}$, there is a commutative diagram of maps of k-vector spaces with a lower horizontal isomorphism

$$\begin{array}{ccc}
\operatorname{Cohom}_{\mathcal{C}}(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{E}) & \longrightarrow & \operatorname{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{E}) \\
{\scriptstyle \operatorname{Cohom}_{\mathcal{C}}(\mathcal{C}\odot_{\mathcal{C}}\mathfrak{P},\, l)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}^{\mathcal{C}}(\mathfrak{P},\, l)} \\
\operatorname{Cohom}_{\mathcal{C}}(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{P}) & = & \operatorname{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{P})
\end{array}$$

The leftmost vertical map, being a quotient of the surjective map

$$\operatorname{Hom}_k(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ l)\colon\ \operatorname{Hom}_k(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{E}) \longrightarrow \operatorname{Hom}_k(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{P},\ \mathfrak{P}),$$

is surjective; so the rightmost vertical map is surjective, too. It follows that the morphism $\mathrm{id}\colon \mathfrak{P} \longrightarrow \mathfrak{P}$ can be lifted to a $\mathcal{C}$-contramodule morphism $\mathfrak{P} \longrightarrow \mathfrak{E}$, i. e., our surjective morphism of $\mathcal{C}$-contramodules $l\colon \mathfrak{E} \longrightarrow \mathfrak{P}$ splits. □

3.2. **Contramodules over pro-Artinian local rings.** A *pro-Artinian commutative ring* $\mathfrak{R}$ is the projective limit of a filtered projective system of Artinian commutative rings and surjective morphisms between them. Equivalently, $\mathfrak{R}$ is a complete and separated topological commutative ring where open ideals form a base of neighborhoods of zero and all the discrete quotient rings are Artinian [56, Section A.2].

A *pro-Artinian local ring* is the projective limit of a filtered projective system of Artinian commutative local rings and surjective morphisms between them. E. g., any complete Noetherian local ring can be viewed as a pro-Artinian local ring.

Contramodules over pro-Artinian local rings are suggested in [56] for use in the role of coefficients in homological theories involving tensor operations, infinite direct sums and products, and reductions to the residue field. The point is that the use of contramodules with their well-behaved reductions (satisfying Nakayama's lemma irrespectively of any finite generatedness assumptions) allows to extend to pro-Artinian local rings many results originally provable over a field only.

Let $\mathfrak{R}$ be a pro-Artinian local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. The category of $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$ and the reduction functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{m}\mathfrak{P}$ acting from it to the category of k-vector spaces have the following formal properties:

- (i) $\mathfrak{R}\text{–}\mathsf{contra}$ is an abelian category with infinite direct sums and products; the infinite product functors in $\mathfrak{R}\text{–}\mathsf{contra}$ are exact and preserved by the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$;
- (ii) $\mathfrak{R}\text{–}\mathsf{contra}$ is a tensor category with a right exact tensor product functor $\otimes^{\mathfrak{R}}$ and an internal Hom functor $\operatorname{Hom}_{\mathfrak{R}}$; the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ takes the unit object of the tensor structure $\mathfrak{R} \in \mathfrak{R}\text{–}\mathsf{contra}$ to the unit object $\mathfrak{R} \in \mathfrak{R}\text{–}\mathsf{mod}$ and commutes with the internal Hom functors;

(iii) there are enough projective objects in $\mathfrak{R}$–contra; these are called the *free $\mathfrak{R}$-contramodules*, as they are precisely the direct sums of copies of the object $\mathfrak{R}$; the class of free $\mathfrak{R}$-contramodules is preserved by infinite direct sums, infinite products, and the operations $\otimes^{\mathfrak{R}}$ and $\mathrm{Hom}_{\mathfrak{R}}$;
(iv) the reduction functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow k$–$\mathsf{vect}$ preserves infinite direct sums, infinite products, commutes with the tensor products, and commutes with the internal Hom from free $\mathfrak{R}$-contramodules;
(v) the reduction functor does not annihilate any objects: $\mathfrak{P}/\mathfrak{m}\mathfrak{P} = 0$ implies $\mathfrak{P} = 0$ for any $\mathfrak{P} \in \mathfrak{R}$–$\mathsf{contra}$; in other words, a morphism of free $\mathfrak{R}$-contramodules $\mathfrak{F} \longrightarrow \mathfrak{G}$ is an isomorphism whenever its reduction $\mathfrak{F}/\mathfrak{m}\mathfrak{F} \longrightarrow \mathfrak{G}/\mathfrak{m}\mathfrak{G}$ is an isomorphism of k-vector spaces.

The constructions of these structures and the proofs of the listed assertions can be found in [56, Sections 1.1–1.6]. Let us point out two caveats, which are in fact two ways to formulate one and the same observation. Firstly, the infinite direct sums of $\mathfrak{R}$-contramodules are *not* always exact functors. Secondly, the functors of tensor product $\mathfrak{F} \otimes^{\mathfrak{R}} -$ with a free $\mathfrak{R}$-contramodule $\mathfrak{F}$ is *not* always exact in $\mathfrak{R}$–contra.

Both problems do not occur in the homological dimension 1 case. E. g., in the case of the ring of l-adic integers $\mathfrak{R} = \mathbb{Z}_l$ or the ring of formal power series in one variable $\mathfrak{R} = k[[z]]$ the infinite direct sums in $\mathfrak{R}$–contra are still exact, as are the tensor products with free $\mathfrak{R}$-contramodules. Of course, even in these cases the infinite direct sums and tensor products of $\mathfrak{R}$-contramodules are not preserved by the forgetful functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow \mathfrak{R}$–$\mathsf{mod}$ (and one would not expect them to be).

The definition of a left contramodule over a complete and separated topological ring $\mathfrak{R}$ with open right ideals forming a base of neighborhoods of zero was explained in Section 2.1. Let us now define the operation of tensor product $\otimes^{\mathfrak{R}}$ of contramodules over a commutative topological ring $\mathfrak{R}$ with open ideals forming a base of neighborhoods of zero.

The following definition of a contrabilinear map for conramodules over a commutative topological ring was suggested to the author by Deligne. Let $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{K}$ be three contramodules over $\mathfrak{R}$. A map $b\colon \mathfrak{P}\times\mathfrak{Q} \longrightarrow \mathfrak{K}$ is called *contrabilinear* over $\mathfrak{R}$ if for any two families of coefficients r_α and $s_\beta \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$ and any two families of elements $p_\alpha \in \mathfrak{P}$ and $q_\beta \in \mathfrak{Q}$ the equation

$$b\Big(\sum\nolimits_\alpha r_\alpha p_\alpha,\ \sum\nolimits_\beta s_\beta q_\beta\Big) = \sum\nolimits_{\alpha,\beta}(r_\alpha s_\beta)\, b(p_\alpha, q_\beta)$$

holds in $\mathfrak{K}$. An $\mathfrak{R}$-contramodule $\mathfrak{L}$ endowed with an $\mathfrak{R}$-contrabilinear map $\mathfrak{P}\times\mathfrak{Q} \longrightarrow \mathfrak{L}$ is called the *contramodule tensor product* of the $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ if for any $\mathfrak{R}$-contramodule $\mathfrak{K}$ and any $\mathfrak{R}$-contrabilinear map $\mathfrak{P}\times\mathfrak{Q} \longrightarrow \mathfrak{K}$ there exists a unique $\mathfrak{R}$-contramodule morphism $\mathfrak{L} \longrightarrow \mathfrak{K}$ making the triangle diagram $\mathfrak{P}\times\mathfrak{Q} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{K}$ commutative [56, Section 1.6].

One has to check that for any $\mathfrak{R}$-subcontramodules $\mathfrak{P}' \subset \mathfrak{P}$ and $\mathfrak{Q}' \subset \mathfrak{Q}$ any $\mathfrak{R}$-contrabilinear map $\mathfrak{P} \times \mathfrak{Q} \longrightarrow \mathfrak{K}$ annihilating $\mathfrak{P}' \times \mathfrak{Q}$ and $\mathfrak{P} \times \mathfrak{Q}'$ factorizes uniquely through a contrabilinear map $\mathfrak{P}/\mathfrak{P}' \times \mathfrak{Q}/\mathfrak{Q}' \longrightarrow \mathfrak{K}$. Then it follows that the contramodule tensor product $\mathfrak{P}/\mathfrak{P}' \otimes^{\mathfrak{R}} \mathfrak{Q}/\mathfrak{Q}'$ can be obtained as the cokernel of the natural morphism of contramodule tensor products

$$\mathfrak{P}' \otimes^{\mathfrak{R}} \mathfrak{Q} \oplus \mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{Q}' \longrightarrow \mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{Q},$$

so in order to produce the contramodule tensor products of arbitrary pairs of $\mathfrak{R}$-contramodules it remains to explain what the tensor products of free $\mathfrak{R}$-contramodules are. Here one notices that setting $\mathfrak{R}[[X]] \otimes^{\mathfrak{R}} \mathfrak{R}[[Y]] \simeq \mathfrak{R}[[X \times Y]]$ for any two sets X and Y does the job. Moreover, one has $\mathfrak{R}[[X]] \otimes^{\mathfrak{R}} \mathfrak{P} \simeq \bigoplus_X \mathfrak{P}$ for any set X and any $\mathfrak{R}$-contramodule $\mathfrak{P}$, and generally the contramodule tensor product is a right exact functor preserving infinite direct sums in the category $\mathfrak{R}$–contra.

The set $\mathrm{Hom}_R(\mathfrak{P}, \mathfrak{Q})$ of all $\mathfrak{R}$-contramodule morphisms between two $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ is endowed with the $\mathfrak{R}$-contramodule structure provided by the pointwise infinite summation operations

$$\Big(\sum\nolimits_\alpha r_\alpha f_\alpha\Big)(p) = \sum\nolimits_\alpha r_\alpha f_\alpha(p)$$

for any family of morphisms $f_\alpha \colon \mathfrak{P} \longrightarrow \mathfrak{Q}$, any element $p \in \mathfrak{P}$, and any family of coefficients $r_\alpha \in \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$ [56, Section 1.5]. One easily checks from the definitions that

$$\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{Q},\ \mathfrak{K}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{P}, \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{Q}, \mathfrak{K}))$$

for any $\mathfrak{R}$-contramodules $\mathfrak{P}$, $\mathfrak{Q}$, and $\mathfrak{K}$, as is required of the internal Hom functor in a tensor category.

Given an $\mathfrak{R}$-contramodule $\mathfrak{P}$, one denotes by $\mathfrak{m}\mathfrak{P} \subset \mathfrak{P}$ the image of the contraaction map $\mathfrak{R}[[\mathfrak{P}]] \supset \mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. The passage to the quotient k-vector space $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ provides the construction of the reduction functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{m}\mathfrak{P}$.

We have explained the constructions of all the structures mentioned in (i-v). The proof of the assertion that the class of projective $\mathfrak{R}$-contramodules is closed under the operations of infinite product and the internal Hom functor depends on the assumption that the ring $\mathfrak{R}$ is pro-Artinian [56, Lemma 1.3.7], as does the proof of the fact that the reduction functor preserves infinite products [56, Lemma 1.3.6]. The proof of the Nakayama lemma (v) is based on the assumption of topological nilpotence of the ideal $\mathfrak{m}$ implied by the definition of a pro-Artinial local ring (see Section 2.1), and the assertion that all projective $\mathfrak{R}$-contramodules are free follows from the Nakayama lemma [56, Lemma 1.3.2].

In addition to $\mathfrak{R}$-contramodules, the coefficient formalism developed in [56, Section 1] also includes *discrete* $\mathfrak{R}$*-modules* or $\mathfrak{R}$*-comodules*; see Section 3.6 for a short discussion and [56, Sections 1.4 and 1.9] for the details.

3.3. **Flat contramodules over topological rings.** The aim of this section is to describe a certain class of adjusted contramodules over topological associative rings with a countable base of neighborhoods of zero formed by open two-sided ideals that plays a crucial role in [57, Appendix D] and will probably prove to be important and useful in other contexts as well. Before passing to the general setting, let us briefly return to the discussion of contramodules over the adic completions of Noetherian rings by centrally generated ideals from Section 2.2.

Let m be an ideal generated by central elements in a right Noetherian ring R; denote by $\mathfrak{R} = \varprojlim_n R/m^n$ the m-adic completion of the ring R, viewed as a complete and separated topological ring in its natural topology. Let $\mathfrak{m} = \varprojlim_n m/m^n$ denote the ideal generated by the image of $m \subset R$ in the ring $\mathfrak{R}$.

The following result is [57, Proposition C.5.4]; in the commutative case, its proof can be found in [56, Lemma B.9.2].

Lemma 1. *A left $\mathfrak{R}$ contramodule $\mathfrak{P}$ is a flat R-module if and only if the R/m^n-module $\mathfrak{P}/\mathfrak{m}^n\mathfrak{P}$ is flat for every $n \geqslant 1$. The natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{m}^n\mathfrak{P}$ is an isomorphism if this is the case.* □

In other words, the nonseparatedness phenomenon demonstrated by the counterexamples from [61, 67, 52] described in Section 1.5 above does not occur for $\mathfrak{R}$-contramodules satisfying either of the two equivalent flatness conditions from the first sentence of Lemma. In the more general setting below, what was an assertion of a lemma becomes a well-behaved definition.

In the rest of the section we follow [57, Section D.1]. Let $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow R_3 \longleftarrow \cdots$ be a projective system of associative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{R} = \varprojlim_n R_n$ and endow it with the topology of projective limit of discrete rings R_n. Let $\mathfrak{I}_n \subset \mathfrak{R}$ denote the kernels of the natural surjective morphisms of rings $\mathfrak{R} \longrightarrow R_n$; then the open ideals $\mathfrak{I}_n$ form a base of neighborhoods of zero in the topological ring $\mathfrak{R}$. To keep our notation in line with that of [57, Appendix D], we switch a bit away from our previous notation pattern and denote by $\mathfrak{J}\times\mathfrak{P} \subset \mathfrak{P}$ the image of the contraaction map $\mathfrak{J}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ for any closed ideal $\mathfrak{J} \subset \mathfrak{R}$ and any $\mathfrak{R}$-contramodule $\mathfrak{P}$.

We recall from Section 2.1 that the category of left $\mathfrak{R}$-contramodules $\mathfrak{R}$–contra is an abelian category with infinite direct sums, exact infinite products, and enough projective objects. The next result is [57, Lemma D.1.1]; see also [52, Lemma A.2.3].

Lemma 2. *For any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \mathfrak{P}/\mathfrak{I}_n\times\mathfrak{P}$ is surjective.* □

Here is the promised definition. A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called *flat* if the map $\mathfrak{F} \longrightarrow \varprojlim_n \mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ is surjective and the R_n-module $\mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F}$ is flat for every n. The following proposition is the main related result.

Proposition 1. *The class of flat left $\mathfrak{R}$-contramodules is closed under extensions and the passages to the kernels of surjective morphisms. For any short exact sequence of flat $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{F} \longrightarrow 0$, the short sequences of R_n-modules $0 \longrightarrow \mathfrak{H}/\mathfrak{I}_n\times\mathfrak{H} \longrightarrow \mathfrak{G}/\mathfrak{I}_n\times\mathfrak{G} \longrightarrow \mathfrak{F}/\mathfrak{I}_n\times\mathfrak{F} \longrightarrow 0$ are exact.*

Proof. This is [57, Lemmas D.1.3 and D.1.4]. □

The next result provides a characterization of projective $\mathfrak{R}$-contramodules, generalizing the results of [67, Corollary 4.5] and [56, Corollary B.8.2] (see also [69, Theorem 1.10] and [57, Corollary C.5.6]).

Proposition 2. *A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is a projective object in $\mathfrak{R}$–contra if and only if it is flat and the left $\mathfrak{R}_n$-modules $\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ are projective for all $n \geqslant 0$. In other words, a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective if and only if the map $\mathfrak{F} \longrightarrow \varprojlim_n \mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ is an isomorphism and the R_n-modules $\mathfrak{F}/\mathfrak{I}_n{\times}\mathfrak{F}$ are projective.*

Proof. This is [57, Corollary D.1.8(a)]. □

3.4. **Underived co-contra correspondence over corings.** It was already mentioned in the end of Section 1.2 that the categories of *injective* left *comodules* and *projective* left *contramodules* over a coalgebra $\mathcal{C}$ over a field k are naturally equivalent. Similarly, at the end of Section 1.4 we pointed out the equivalence between the categories of injective discrete modules and projective contramodules over the ring of l-adic integers $\mathbb{Z}_l$. These are the simplest instances of a very general homological phenomenon called the *comodule-contramodule correspondence*, which has many manifestations in algebra [46, 23, 68, 37, 42, 43, 54, 56, 59, 60], algebraic geometry [49, 48, 55, 57], and representation theory [27, 28, 64, 52]. In the remaining three sections we restrict ourselves to an overview of those versions of the co-contra correspondence that can be readily formulated on the level of *additive* or *exact* categories, while referring to the presentation [58] and the introduction to the paper [59] for a discussion of the derived comodule-contramodule correspondence.

Let $\mathcal{C}$ be a coassociative coring over an associative ring A (see Section 2.5). Then the assignment of the left $\mathcal{C}$-contramodule $\mathrm{Hom}_A(\mathcal{C}, V)$ to the left $\mathcal{C}$-comodule $\mathcal{C} \otimes_A V$ and vice versa establishes an equivalence between the full additive subcategories of coinduced $\mathcal{C}$-comodules in $\mathcal{C}$–comod and induced $\mathcal{C}$-contramodules in $\mathcal{C}$–contra. Indeed, one has

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_A U,\ \mathcal{C} \otimes_A V) \simeq \mathrm{Hom}_A(\mathcal{C} \otimes_A U,\ V) \\ \simeq \mathrm{Hom}_A(U, \mathrm{Hom}_A(\mathcal{C}, V)) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathrm{Hom}_A(\mathcal{C}, U), \mathrm{Hom}_A(\mathcal{C}, V))$$

for any left A-modules U and V [52, Section 0.2.6]. This is a particular case of the isomorphism of *Kleisli categories* for a pair (left adjoint comonad, right adjoint monad) in any base category [39, 17].

Adjoining the direct summands to the full subcategories of coinduced $\mathcal{C}$-comodules and induced $\mathcal{C}$-contramodules, one obtains what are called the full subcategories of relatively injective (or $\mathcal{C}/A$-injective) $\mathcal{C}$-comodules and relatively projective $\mathcal{C}$-contramodules in the book [14, Sections 18.17–18] and the paper [17, Sections 2.7–8]. This may be the standard terminology; still in the monograph [52, Section 5.1.3] we chose to call these *quite relatively injective* (quite $\mathcal{C}/A$-injective) $\mathcal{C}$-comodules and *quite relatively projective* (quite $\mathcal{C}/A$-projective) $\mathcal{C}$-contramodules, while preserving the shorter terms with the single word "relatively" for the wider and more important classes of comodules and contramodules discussed below.

A left $\mathcal{C}$-comodule $\mathcal{J}$ is said to be *quite* $\mathcal{C}/A$*-injective* if the short sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{J}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{J}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{J}) \longrightarrow 0 \tag{8}$$

is exact for every short exact sequence of left $\mathcal{C}$-comodules $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0$ that *splits* as a short exact sequence of left *A-modules.* We recall that the category of left $\mathcal{C}$-comodules is not abelian in general, there being a problem with the kernels of morphisms (see Section 2.5). However, any A-split surjection of $\mathcal{C}$-comodules has a kernel preserved by the forgetful functor $\mathcal{C}\text{–}\mathsf{comod} \longrightarrow A\text{–}\mathsf{mod}$.

Moreover, the category of left $\mathcal{C}$-comodules with the class of all A-split short exact sequences is an exact category (see [18] for the definition, discussion, and references); the quite $\mathcal{C}/A$-injective $\mathcal{C}$-comodules are simply the injective objects of this exact category structure. In particular, the coaction morphism $\mathcal{M} \longrightarrow \mathcal{C} \otimes_A \mathcal{M}$ of any left $\mathcal{C}$-comodule $\mathcal{M}$ is split by the A-module map $\mathcal{C} \otimes_A \mathcal{M} \longrightarrow \mathcal{M}$ induced by the counit ε of the coring $\mathcal{C}$. Considering the corresponding A-split short exact sequence of $\mathcal{C}$-comodules, one easily concludes that a $\mathcal{C}$-comodule is quite $\mathcal{C}/A$-injective if and only if it is a direct summand of a coinduced $\mathcal{C}$-comodule.

Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{F}$ is said to be *quite* $\mathcal{C}/A$*-projective* if the short sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}^{\mathcal{C}}(\mathfrak{F}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathcal{C}}(\mathfrak{F}, \mathfrak{Q}) \longrightarrow \operatorname{Hom}^{\mathcal{C}}(\mathfrak{F}, \mathfrak{R}) \longrightarrow 0 \tag{9}$$

is exact for every short exact sequence of left $\mathcal{C}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ that *splits* as a short exact sequence of left A-modules. Recall that the category of left $\mathcal{C}$-contramodules is not abelian in general, there being a problem with the cokernels of morphisms. However, any A-split embedding of $\mathcal{C}$-contramodules has a cokernel preserved by the forgetful functor $\mathcal{C}\text{–}\mathsf{contra} \longrightarrow A\text{–}\mathsf{mod}$.

Moreover, the category of left $\mathcal{C}$-contramodules with the class of all A-split short exact sequences is an exact category; the quite $\mathcal{C}/A$-projective $\mathcal{C}$-contramodules are simply the projective objects of this exact category structure. In particular, the contraaction morphism $\operatorname{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ of any $\mathcal{C}$-contramodule $\mathfrak{P}$ is split by the A-module map $\mathfrak{P} \longrightarrow \operatorname{Hom}_A(\mathcal{C}, \mathfrak{P})$ induced by the counit of the coring $\mathcal{C}$. Considering the corresponding A-split short exact sequence of $\mathcal{C}$-contramodules, one easily concludes that a $\mathcal{C}$-contramodule is quite $\mathcal{C}/A$-injective if and only if it is a direct summand of an induced $\mathcal{C}$-contramodule.

So the full additive subcategories of quite $\mathcal{C}/A$-injective $\mathcal{C}$-comodules in $\mathcal{C}\text{–}\mathsf{comod}$ and quite $\mathcal{C}/A$-projective $\mathcal{C}$-contramodules in $\mathcal{C}\text{–}\mathsf{contra}$ are equivalent for any coassociative coring $\mathcal{C}$. When the coring $\mathcal{C}$ coincides with the ring A (i. e., the counit map $\mathcal{C} \longrightarrow A$ is bijective), this reduces to the identity equivalence of the category of left A-modules with itself. However, the category of A-modules is abelian and not only additive; viewing it just as an additive category is rather unsatisfactory from our point of view. Still the categories of quite relatively injective comodules and quite relatively projective contramodules do not seem to carry any homological structures beyond those of additive categories; in particular, they do *not* have any nontrivial exact category structures. The following definitions [52, Sections 5.1.4 and 5.3] are purported to overcome this drawback (cf. [53, Sections 4.1 and 4.3], where similarly defined relatively adjusted modules are called, more in line with the traditional terminology, "weakly relatively projective" and "weakly relatively injective").

Assume that the coring $\mathcal{C}$ is a projective left and a flat right A-module; then the categories of left $\mathcal{C}$-comodules and $\mathcal{C}$-contramodules are abelian. A left $\mathcal{C}$-comodule $\mathcal{J}$ is called *injective relative to* A ($\mathcal{C}/A$-injective) if the short sequence of Hom groups (8) is exact for any short exact sequence of left $\mathcal{C}$-comodules $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0$ that are *projective as left A-modules*. Similarly, a left $\mathcal{C}$-contramodule $\mathfrak{F}$ is called *projective relative to* A ($\mathcal{C}/A$-projective) if the short sequence of Hom groups (9) is exact for any short exact sequence of left $\mathcal{C}$-contramodules $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ that are *injective as left A-modules*.

Assume further that the ring A has finite left homological dimension (i. e., the category of left A-modules has finite homological dimension; cf. the last sentence of Section 2.6). Then the full subcategory $\mathcal{C}\text{–}\mathsf{comod}_{\mathcal{C}/A\text{–}\mathsf{inj}}$ of $\mathcal{C}/A$-injective $\mathcal{C}$-comodules is closed under extensions and the passages to the cokernels of injective morphisms in $\mathcal{C}\text{–}\mathsf{comod}$. Similarly, the full subcategory $\mathcal{C}\text{–}\mathsf{contra}_{\mathcal{C}/A\text{–}\mathsf{proj}}$ of $\mathcal{C}/A$-projective $\mathcal{C}$-contramodules is closed under extensions and the passages to the kernels of subjective morphisms in $\mathcal{C}\text{–}\mathsf{contra}$ [52, Lemma 5.3.1]. Being closed under extensions, the full subcategories $\mathcal{C}\text{–}\mathsf{comod}_{\mathcal{C}/A\text{–}\mathsf{inj}}$ and $\mathcal{C}\text{–}\mathsf{contra}_{\mathcal{C}/A\text{–}\mathsf{proj}}$ inherit the exact category structures of the abelian categories $\mathcal{C}\text{–}\mathsf{comod}$ and $\mathcal{C}\text{–}\mathsf{contra}$. Moreover, the following strong converse assertions hold.

Lemma. (a) *The subcategory of* $\mathcal{C}/A$*-injective* $\mathcal{C}$*-comodules* $\mathcal{C}\text{–}\mathsf{comod}_{\mathcal{C}/A\text{–}\mathsf{inj}} \subset \mathcal{C}\text{–}\mathsf{comod}$ *is the minimal full subcategory of the abelian category* $\mathcal{C}\text{–}\mathsf{comod}$ *containing the coinduced* $\mathcal{C}$*-comodules and closed under extensions and direct summands.*

(b) *The subcategory of* $\mathcal{C}/A$*-projective* $\mathcal{C}$*-contramodules* $\mathcal{C}\text{–}\mathsf{contra}_{\mathcal{C}/A\text{–}\mathsf{proj}} \subset \mathcal{C}\text{–}\mathsf{contra}$ *is the minimal full subcategory of the abelian category* $\mathcal{C}\text{–}\mathsf{contra}$ *containing the induced* $\mathcal{C}$*-contramodules and closed under extensions and direct summands.*

Proof. This is the strengthening of the result of [52, Remark 9.1] that one obtains by replacing the resolution technique of [52, Lemma 9.1.2] with that of [25, second half of the proof of Theorem 10]; see also [57, Corollary B.2.4].

To be more specific, let us prove part (b). Let $\mathfrak{P}$ be a left $\mathcal{C}$-contramodule. The $\mathcal{C}$-contraaction map $\mathfrak{G} = \mathrm{Hom}_A(\mathcal{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ is a surjective morphism of $\mathcal{C}$-contramodules; let us denote its kernel by $\mathfrak{K}$. According to [52, Lemma 3.1.3(b)], there exists an injective $\mathcal{C}$-contramodule morphism $\mathfrak{K} \longrightarrow \mathfrak{E}$ from $\mathfrak{K}$ into an A-injective left $\mathcal{C}$-contramodule $\mathfrak{E}$ such that the quotient contramodule $\mathfrak{E}/\mathfrak{K}$ is a finitely iterated extension of induced $\mathcal{C}$-contramodules. Denote by $\mathfrak{F}$ the fibered coproduct $(\mathfrak{E} \oplus \mathfrak{G})/\mathfrak{K}$ of the $\mathcal{C}$-contramodules $\mathfrak{E}$ and $\mathfrak{G}$ over $\mathfrak{K}$; then $\mathfrak{F}$ is an extension of the $\mathcal{C}$-contramodules $\mathfrak{E}/\mathfrak{K}$ and $\mathfrak{G}$, and there is an surjective morphism $\mathfrak{F} \longrightarrow \mathfrak{P}$ with the kernel $\mathfrak{E}$.

Now suppose that the $\mathcal{C}$-contramodule $\mathfrak{P}$ is $\mathcal{C}/A$-projective. Then the Ext group $\mathrm{Ext}^{\mathcal{C},1}(\mathfrak{P}, \mathfrak{E})$ in the abelian category $\mathcal{C}\text{–}\mathsf{contra}$ vanishes by [52, Lemma 5.3.1(b)], hence the $\mathcal{C}$-contramodule $\mathfrak{P}$ is a direct summand of a finitely iterated extension of induced $\mathcal{C}$-contramodules $\mathfrak{F}$. Notice that the length of the iterated extension in this construction is bounded by the left homological dimension of the ring A. $\square$

According to [52, Theorem 5.3], the exact categories of $\mathcal{C}/A$-injective left $\mathcal{C}$-comodules and $\mathcal{C}/A$-projective left $\mathcal{C}$-contramodules are naturally equivalent

$$\mathcal{C}\text{–comod}_{\mathcal{C}/A\text{–inj}} \simeq \mathcal{C}\text{–contra}_{\mathcal{C}/A\text{–proj}}.$$

The equivalence is provided by the functor $\Psi_{\mathcal{C}}\colon \mathcal{M} \longmapsto \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ of $\mathcal{C}$-comodule homomorphisms from the left $\mathcal{C}$-comodule $\mathcal{C}$ and the functor $\Phi_{\mathcal{C}}\colon \mathfrak{P} \longmapsto \mathcal{C} \odot_{\mathcal{C}} \mathfrak{P}$ of *contratensor product* of left $\mathcal{C}$-contramodules with the right $\mathcal{C}$-comodule $\mathcal{C}$ [52, Sections 0.2.6–7 and 5.1.1] (cf. Section 3.1; see also [17, Section 5]).

3.5. **Underived semico-semicontra correspondence.** Let $\mathcal{S}$ be a semialgebra over a coalgebra $\mathcal{C}$ over a field k (see Section 2.6). Assume that $\mathcal{S}$ is an injective left $\mathcal{C}$-comodule and an injective right $\mathcal{C}$-comodule, so the categories of left $\mathcal{S}$-semimodules and left $\mathcal{S}$-semicontramodules $\mathcal{S}\text{–simod}$ and $\mathcal{S}\text{–sicntr}$ are abelian. The full subcategory $\mathcal{S}\text{–simod}_{\mathcal{C}\text{–inj}}$ of left $\mathcal{S}$-semimodules that are *injective as left* $\mathcal{C}$*-comodules* is obviously closed under extensions (and cokernels of injective morphisms) in $\mathcal{S}\text{–simod}$, while the full subcategory $\mathcal{S}\text{–sicntr}_{\mathcal{C}\text{–proj}}$ of left $\mathcal{S}$-semicontramodules that are *projective as left* $\mathcal{C}$*-contramodules* is closed under extensions (and kernels of surjective morphisms) in $\mathcal{S}\text{–sicntr}$. Hence the full subcategories $\mathcal{S}\text{–simod}_{\mathcal{C}\text{–inj}}$ and $\mathcal{S}\text{–sicntr}_{\mathcal{C}\text{–proj}}$ inherit the exact category structures of the abelian categories $\mathcal{S}\text{–simod}$ and $\mathcal{S}\text{–sicntr}$.

According to [52, Sections 0.3.7 and 6.2] (see also [51]), the exact categories of $\mathcal{C}$-injective left $\mathcal{S}$-semimodules and $\mathcal{C}$-projective left $\mathcal{S}$-semicontramodules are naturally equivalent. The equivalence is provided by the functor $\Psi_{\mathcal{S}}\colon \boldsymbol{\mathcal{M}} \longmapsto \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}, \boldsymbol{\mathcal{M}})$ of $\mathcal{S}$-semimodule homomorphisms from the left $\mathcal{S}$-semimodule $\mathcal{S}$ and the functor $\Phi_{\mathcal{S}}\colon \boldsymbol{\mathfrak{P}} \longmapsto \mathcal{S} \circledcirc_{\mathcal{S}} \boldsymbol{\mathfrak{P}}$ of (*semi*)*contratensor product* of left $\mathcal{S}$-semicontramodules with the right $\mathcal{S}$-semimodule $\mathcal{S}$ (cf. the definition of the contratensor product over a coalgebra in Section 3.1). Furthermore, the equivalence of exact categories $\mathcal{S}\text{–simod}_{\mathcal{C}\text{–inj}} \simeq \mathcal{S}\text{–sicntr}_{\mathcal{C}\text{–proj}}$ forms a commutative diagram of functors with the equivalence $\mathcal{C}\text{–comod}_{\text{inj}} \simeq \mathcal{C}\text{–contra}_{\text{proj}}$ between the additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules, and the forgetful functors $\mathcal{S}\text{–simod}_{\mathcal{C}\text{–inj}} \longrightarrow \mathcal{C}\text{–comod}_{\text{inj}}$ and $\mathcal{S}\text{–sicntr}_{\mathcal{C}\text{–proj}} \longrightarrow \mathcal{C}\text{–contra}_{\text{proj}}$,

$$\begin{array}{ccc} \mathcal{S}\text{–simod}_{\mathcal{C}\text{–inj}} & \underset{\leftarrow\ \Phi_{\mathcal{S}}}{\overset{\Psi_{\mathcal{S}}\ \rightarrow}{=\!=\!=\!=}} & \mathcal{S}\text{–sicntr}_{\mathcal{C}\text{–proj}} \\ \downarrow & & \downarrow \\ \mathcal{C}\text{–comod}_{\text{inj}} & \underset{\leftarrow\ \Phi_{\mathcal{C}}}{\overset{\Psi_{\mathcal{C}}\ \rightarrow}{=\!=\!=\!=}} & \mathcal{C}\text{–contra}_{\text{proj}}. \end{array} \tag{10}$$

The particular case considered in Section 2.8, with the semialgebra

$$\mathcal{S}^r_{\varkappa+\varkappa_0}(\mathfrak{g}, \mathcal{C}) \simeq \mathcal{S} = \mathcal{S}^l_{\varkappa}(\mathfrak{g}, \mathcal{C})$$

corresponding to a central extension $\varkappa\colon (\mathfrak{g}', \mathcal{C}) \longrightarrow (\mathfrak{g}, \mathcal{C})$ of Tate Harish-Chandra pairs satisfying the condition of Theorem 2.8 is especially notable. In this situation we obtain the commutative diagram of equivalences of exact/additive categories and forgetful functors

$$
(11)\qquad
\begin{array}{ccc}
\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})_{\mathcal{C}\text{–}\mathsf{inj}} & \underset{\leftarrow\ \Phi_{\mathcal{S}}}{\overset{\Psi_{\mathcal{S}}\ \rightarrow}{=\!=\!=\!=}} & \mathsf{O}^{\mathsf{ctr}}_{\varkappa+\varkappa_0}(\mathfrak{g},\mathcal{C})_{\mathcal{C}\text{–}\mathsf{proj}} \\
\downarrow & & \downarrow \\
\mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}} & \underset{\leftarrow\ \Phi_{\mathcal{C}}}{\overset{\Psi_{\mathcal{C}}\ \rightarrow}{=\!=\!=\!=}} & \mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}},
\end{array}
$$

where $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})_{\mathcal{C}\text{–}\mathsf{inj}} \subset \mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ and $\mathsf{O}^{\mathsf{ctr}}_\varkappa(\mathfrak{g},\mathcal{C})_{\mathcal{C}\text{–}\mathsf{proj}} \subset \mathsf{O}^{\mathsf{ctr}}_\varkappa(\mathfrak{g},\mathcal{C})$ denote the full exact subcategories of $\mathcal{C}$-injective Harish-Chandra modules and $\mathcal{C}$-projective Harish-Chandra contramodules with the central charge $\varkappa$.

It was pointed out by Feigin–Fuchs [27], [28, Remark 2.4] and Meurman–Frenkel–Rocha-Caridi–Wallach [64] back in the first half of 1980's that the categories of Verma modules over the Virasoro algebra on any pair of complementary levels $C = c$ and $C = 26 - c$ are anti-isomorphic. The above result extends this classical observation to the whole exact subcategories of $\mathcal{C}$-adjusted objects in the abelian categories O and $\mathsf{O}^{\mathsf{ctr}}$ over any Tate Harish-Chandra pair satisfying the nondegeneracy condition.

Indeed, consider the Tate Harish-Chandra pair $(\mathfrak{g},\mathcal{C}) = \big(k((z))d/dz, \mathcal{C}(H)\big)$ over a field k of characteristic 0 with the pro-algebraic subgroup H corresponding to the compact open Lie subalgebra $\mathfrak{h} = zk[[z]]d/dz \subset k((z))d/dz$ as described in the beginning of Section 2.8. The group H acts in the Lie algebra $\mathfrak{g}$ by changing the independent variable in the vector fields, $a^{-1}\big(f(z)d/dz\big) = f(a(z))\,d/da(z) = f(a(z))/a'(z)\,d/dz$ for all $a \in H(k)$, $f \in k((z))$.

A *Verma module* over $\mathbb{V}\mathrm{ir}$ is an $U(\mathbb{V}\mathrm{ir})$-module induced from a one-dimensional module kv_0 over the compact open subalgebra $\mathfrak{h} \oplus kC \subset \mathbb{V}\mathrm{ir}$. The subalgebra $z^2k[[z]]d/dz \subset \mathfrak{h}$ topologically spanned by the basis vectors L_n with $n \geqslant 1$ acts by zero in kv_0, while the generators C and L_0 act by certain scalars c and $h_0 \in k$. These modules belong to the categories $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ with the respective central charges $\varkappa = c$, but, as such, do *not* play any noticeable role in our theory. Indeed, they do *not* belong to the subcategories $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})_{\mathcal{C}\text{–}\mathsf{inj}} \subset \mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$, being freely generated as modules over a Lie subalgebra *complementary* to $\mathfrak{h}$ in $\mathbb{V}\mathrm{ir}$ and having no particular adjustness properties as comodules over $\mathcal{C} = \mathcal{C}(H)$.

The *contragredient* Verma modules are relevant for us instead. Notice first of all that the pro-algebraic group H contains a subgroup whose group of points consists of the coordinate changes $z \longmapsto a_1 z$ multiplying the coordinate z with a scalar factor $a_1 \in k$. The category of comodules (as well as contramodules) over (the coalgebra of) this algebraic group, which is isomorphic to the multiplicative group $\mathbb{G}_m$, is equivalent to the category of graded k-vector spaces. In particular, the Verma modules $\mathcal{M}(c, h_0)$ over $\mathbb{V}\mathrm{ir}$ carry the grading by the weights of the semisimple operator L_0.

Furthermore, the discrete Lie subalgebra $\bigoplus_n kL_n \oplus C \subset \mathbb{V}\mathrm{ir}$ spanned *nontopologically* by the generators C and $L_n \in \mathbb{V}\mathrm{ir}$ has an involutive automorphism σ given by the rules $\sigma(L_n) = -L_{-n}$ and $\sigma(C) = -C$. The *contragredient Verma module*

$\mathcal{M}(c,h_0)^\vee$ is the graded dual vector space to $\mathcal{M}(c,h_0)$ endowed by the induced action of the Lie subalgebra $\bigoplus_n kL_n \oplus C$, twisted by the automorphism σ and then extended to the whole Lie algebra $\mathbb{V}\mathrm{ir}$ by continuity. Both the passage to the dual module and the involution σ change the sign of the central charge; hence $\mathcal{M}(c,h_0)^\vee$ is again a $\mathbb{V}\mathrm{ir}$-module with the central charge c. So the full subcategory of Verma modules on the level $\varkappa = c$ (with a varying parameter $h_0 \in k$) in $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ is anti-isomorphic to the full subcategory of contragredient Verma modules in the same category $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$.

Denoting by $H_+ \subset H$ the pro-unipotent pro-algebraic subgroup whose points are the power series $a(z) = 1 + a_2z^2 + a_3z^3 + \cdots$ with $a_n \in k$ for $n \geqslant 2$, the contragredient Verma modules can be described as precisely those objects of the category $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ whose structures of $\mathcal{C}(H_+)$-comodules are those of cofree comodules with one cogenerator. Alternatively, the contragredient Verma modules are distinguished among all the objects of $\mathsf{O}_\varkappa(\mathfrak{g},\mathcal{C})$ by the property that their underlying $\mathcal{C}$-comodules are the "minimal possible", i. e., *indecomposable* injective comodules.

Finally, consider the category of Harish-Chandra *contramodules* over $(\mathfrak{g},\mathcal{C})$ with the central charge $\varkappa + \varkappa_0 = -26 + c$ whose $\mathcal{C}(H_+)$-contramodule structures are those of free contramodules with one generator, or equivalently, whose underlying $\mathcal{C}$-contramodules are indecomposable projective contramodules. This full subcategory in $\mathsf{O}^{\mathsf{ctr}}_{\varkappa+\varkappa_0}(\mathfrak{g},\mathcal{C})$ is anti-equivalent to the category of contragredient Verma modules with the central charge $26 - c$ via the linear duality functor $\mathfrak{P} = \mathcal{N}^* = \mathrm{Hom}_k(\mathcal{N},k)$. It is also *equivalent* to the category of Verma modules with the central charge $26 - c$ via the functor assigning to a Verma module $\mathcal{M}$ the *infinite product* $\mathfrak{P} = \prod_{n\in\mathbb{Z}} \mathcal{M}_n$ of its grading components, endowed with the σ-twisted action of the Virasoro Lie algebra and a natural $\mathcal{C}$-contramodule structure (cf. the discussion of graded contramodules in [52, Section 11.1.1] and [54, Remark 2.2]).

So the classical duality between the categories of Verma modules on the complementary levels appears in our setting, after the naïve twisting and linear duality are taken into account, as the restriction of the equivalence of exact categories (11) to the full subcategories of objects "of the minimal possible size". (Cf. [52, Corollary and Remark D.3.1], where a discussion of these results in the *derived* comodule-contramodule correspondence context can be found; see also [52, Sections 0.2.6–7] for relevant counterexamples demonstrating how exotic derived categories appear in the derived co-contra correspondence.)

Now let us explain, as it was promised in Sections 2.6 and 2.8, how to use the equivalence of exact categories (10) in order to construct injective objects in the category $\mathbf{S}$–simod and projective objects in the category $\mathbf{S}$–sicntr. As above, we assume that the semialgebra $\mathbf{S}$ is an injective left and right $\mathcal{C}$-comodule.

Proposition. (a) *There are enough injective objects in the abelian category of left $\mathbf{S}$-semimodules. A left $\mathbf{S}$-semimodule is injective if and only if it is a direct summand of an $\mathbf{S}$-semimodule of the form $\Phi_{\mathbf{S}}(\mathrm{Hom}_k(\mathbf{S},V))$, where V is a k-vector space.*

(b) *There are enough projective objects in the abelian category of left $\mathbf{S}$-semicontramodules. A left $\mathbf{S}$-semicontramodule is projective if and only if it is a direct summand of an $\mathbf{S}$-semicontramodule of the form $\Psi_{\mathbf{S}}(\mathbf{S}\otimes_k V)$, where V is a k-vector space.*

Proof. A left semimodule $\mathbf{M}$ over a semialgebra $\mathbf{S}$ over a coalgebra $\mathcal{C}$ over a field k is called *semiprojective* if it is a direct summand of the $\mathbf{S}$-semimodule $\mathbf{S}\otimes_k V$ for some k-vector space V. Similarly, a left semicontramodule $\mathbf{P}$ over $\mathbf{S}$ is called *semiinjective* if it is a direct summand of the $\mathbf{S}$-semicontramodule $\mathrm{Hom}_k(\mathbf{S},V)$ for some vector space V [52, Sections 3.4.3 and 9.2].

The semiprojective semimodules are projective objects in the exact category of $\mathcal{C}$-injective left $\mathbf{S}$-semimodules, as the functor

$$\mathbf{M}\longmapsto \mathrm{Hom}_{\mathbf{S}}(\mathbf{S}\otimes_k V,\ \mathbf{M})\simeq \mathrm{Hom}_k(V,\Psi_{\mathbf{S}}(\mathbf{M}))\\ \simeq \mathrm{Hom}_k(V,\Psi_{\mathcal{C}}(\mathbf{M}))=\mathrm{Hom}_k(V,\mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\mathbf{M}))\simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}\otimes_k V,\ \mathbf{M})$$

is exact on $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–}\mathsf{inj}}$. For any $\mathcal{C}$-injective left $\mathbf{S}$-semimodule $\mathbf{M}$, the semiaction morphism $\mathbf{S}\,\square_{\mathcal{C}}\,\mathbf{M}\longrightarrow\mathbf{M}$ is an admissible epimorphism in $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–}\mathsf{inj}}$ from a semiprojective left $\mathbf{S}$-semimodule $\mathbf{S}\,\square_{\mathcal{C}}\,\mathbf{M}$ onto $\mathbf{M}$; so the projective objects of the category $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–}\mathsf{inj}}$ are precisely the semiprojective semimodules.

Similarly, the semiinjective semicontramodules are injective objects in the exact category of $\mathcal{C}$-projective left $\mathbf{S}$-semicontramodules, as the functor

$$\mathbf{P}\longmapsto \mathrm{Hom}^{\mathbf{S}}(\mathbf{P},\ \mathrm{Hom}_k(\mathbf{S},V))\simeq \mathrm{Hom}_k(\Phi_{\mathbf{S}}(\mathbf{P}),V)\\ \simeq \mathrm{Hom}_k(\Phi_{\mathcal{C}}(\mathbf{P}),V)\simeq \mathrm{Hom}^{\mathcal{C}}(\mathbf{P},\ \mathrm{Hom}_k(\mathcal{C},V))$$

is exact on $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–}\mathsf{proj}}$. For any $\mathcal{C}$-projective left $\mathbf{S}$-semicontramodule $\mathbf{P}$, the semicontraaction morphism $\mathbf{P}\longrightarrow\mathrm{Cohom}_{\mathcal{C}}(\mathbf{S},\mathbf{P})$ is an admissible monomorphism in $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–}\mathsf{proj}}$ from $\mathbf{P}$ into a semiinjective left $\mathbf{S}$-semicontramodule $\mathrm{Cohom}_{\mathcal{C}}(\mathbf{S},\mathbf{P})$. So the injective objects of the category $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–}\mathsf{proj}}$ are precisely the semiinjective semicontramodules.

The functors $\Psi_{\mathbf{S}}$ and $\Phi_{\mathbf{S}}$ being mutually inverse equivalences between the exact categories $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–}\mathsf{inj}}$ and $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–}\mathsf{proj}}$, it follows that the $\mathbf{S}$-semimodules $\Phi_{\mathbf{S}}(\mathrm{Hom}_k(\mathbf{S},V))$ and their direct summands are the injective objects of the exact category $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–}\mathsf{inj}}$, while the $\mathbf{S}$-semicontramodules $\Psi_{\mathbf{S}}(\mathbf{S}\otimes_k V)$ and their direct summands are the projective objects of the exact category $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–}\mathsf{proj}}$. Furthermore, any left $\mathbf{S}$-semimodule can be embedded into a $\mathcal{C}$-injective $\mathbf{S}$-semimodule. This assertion is provided by the combination of the construction of [52, Lemma 1.3.3] (see also [51]) with the result of Lemma 3.1(a) above.

Similarly, any left $\mathbf{S}$-semicontramodule is the quotient contramodule of a $\mathcal{C}$-projective $\mathbf{S}$-semicontramodule. To prove this fact, one has to combine the construction of [52, Lemma 3.3.3] (which was present already in [51]) with the assertion of [52, Lemma 5.2 or 5.3.2] whose proof we reproduced, in our generality of coalgebras over fields, in Lemma 3.1(b) above. Therefore, any left $\mathbf{S}$-semimodule can be embedded into an $\mathbf{S}$-semimodule of the form $\Phi_{\mathbf{S}}(\mathrm{Hom}_k(\mathbf{S},V))$, and any left $\mathbf{S}$-semicontramodule is the quotient contramodule of an $\mathbf{S}$-semicontramodule of the form $\Psi_{\mathbf{S}}(\mathbf{S}\otimes_k V)$.

Finally, in order to show that any injective object $\mathfrak{J}$ of the exact category $\mathbf{S}\text{–}\mathsf{simod}_{\mathcal{C}\text{–inj}}$ is also an injective object in the abelian category $\mathbf{S}\text{–}\mathsf{simod}$, suppose that we are given an injective morphism $\mathfrak{J} \longrightarrow \mathfrak{L}$ from $\mathfrak{J}$ into a left $\mathbf{S}$-semimodule $\mathfrak{L}$. Let $\mathfrak{L} \longrightarrow \mathfrak{M}$ be an injective morphism from $\mathfrak{L}$ into a $\mathcal{C}$-injective left $\mathbf{S}$-semimodule $\mathfrak{M}$. Then $\mathfrak{J}$ is a direct summand in $\mathfrak{M}$, so there is a projection $\mathfrak{M} \longrightarrow \mathfrak{J}$ splitting the embedding $\mathfrak{J} \longrightarrow \mathfrak{M}$. Restricting this projection to the subsemimodule $\mathfrak{L} \subset \mathfrak{M}$, we see that the embedding $\mathfrak{J} \longrightarrow \mathfrak{L}$ also splits.

Similarly, in order to prove that any projective object $\mathfrak{F}$ of the exact category $\mathbf{S}\text{–}\mathsf{sicntr}_{\mathcal{C}\text{–proj}}$ is also a projective object in the abelian category $\mathbf{S}\text{–}\mathsf{sicntr}$, suppose that we are given a surjective morphism $\mathfrak{Q} \longrightarrow \mathfrak{F}$ onto $\mathfrak{F}$ from a left $\mathbf{S}$-semicontramodule $\mathfrak{Q}$. Let $\mathfrak{P} \longrightarrow \mathfrak{Q}$ be a surjective morphism onto $\mathfrak{Q}$ from a $\mathcal{C}$-projective left $\mathbf{S}$-semicontramodule $\mathfrak{P}$. Then $\mathfrak{F}$ is a direct summand in $\mathfrak{P}$, so there is a section $\mathfrak{F} \longrightarrow \mathfrak{P}$ splitting the surjection $\mathfrak{P} \longrightarrow \mathfrak{F}$. Composing this section with the projection $\mathfrak{P} \longrightarrow \mathfrak{Q}$, we obtain a section $\mathfrak{F} \longrightarrow \mathfrak{Q}$ showing that the surjection $\mathfrak{Q} \longrightarrow \mathfrak{F}$ also splits (cf. [52, proof of Lemma 9.2.1]). ☐

3.6. **Co-contra correspondence over topological rings.** In this last section we discuss generalizations of the equivalence between the additive categories of injective comodules and projective contramodules over a coalgebra over a field to topological rings $\mathfrak{R}$ more complicated than the linearly compact topological algebras (which are dual to coalgebras over fields). For examples of *derived* co-contra correspondence over topological rings the reader is referred to [57, Sections C.1, C.5, and D.2].

First let us suppose that $\mathfrak{R}$ is a pro-Artinian commutative ring (see Section 3.2). By the definition, an $\mathfrak{R}$-*comodule* is an ind-object in the abelian category opposite to the category of discrete $\mathfrak{R}$-modules of finite length [56, Section 1.4]. There is a natural contravariant functor $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$ assigning to every $\mathfrak{R}$-comodule a pro-object in the category of discrete $\mathfrak{R}$-modules of finite length. Furthermore, there is a distinguished object $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ in the category $\mathfrak{R}\text{–}\mathsf{comod}$ of $\mathfrak{R}$-comodules such that $\mathcal{C}^{\mathrm{op}} = \mathfrak{R}$; the functor $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$, viewed as a contravariant functor from $\mathfrak{R}\text{–}\mathsf{comod}$ to the category of abelian groups, is represented by $\mathcal{C}(\mathfrak{R})$. A *cofree* $\mathfrak{R}$-comodule is a direct sum of copies of the $\mathfrak{R}$-comodule $\mathcal{C}$; the cofree $\mathfrak{R}$-comodules are injective, and any $\mathfrak{R}$-comodule can be embedded into a cofree one.

According to Matlis' duality (see [45, Corollary 4.3] or [47, Theorem 18.6]), choosing an injective hull of the irreducible module over an Artinian commutative local ring R fixes an anti-equivalence of the category of R-modules of finite length with itself. Passing to the inductive limit of such auto-anti-equivalences over all the discrete quotient rings of a pro-Artinian commutative ring $\mathfrak{R}$, one obtains an auto-anti-equivalence of the category of discrete $\mathfrak{R}$-modules of finite length depending on the choice of a "minimal injective cogenerator" of the abelian category $\mathfrak{R}\text{–}\mathsf{discr}$, i. e., an injective hull of the direct sum of the irreducible discrete $\mathfrak{R}$-modules. Hence choosing such an injective object $\mathcal{E} \in \mathfrak{R}\text{–}\mathsf{discr}$ identifies the category of discrete $\mathfrak{R}$-modules with the category of $\mathfrak{R}$-comodules; the equivalence of categories $\mathfrak{R}\text{–}\mathsf{comod} \simeq \mathfrak{R}\text{–}\mathsf{discr}$ takes the object $\mathcal{C} \in \mathfrak{R}\text{–}\mathsf{comod}$ to the object $\mathcal{E}$ [56, Section 1.9].

For any pro-Artinian commutative ring $\mathfrak{R}$, the categories of injective $\mathfrak{R}$-comodules and projective $\mathfrak{R}$-contramodules are naturally equivalent, $\mathfrak{R}\text{–}\mathsf{comod}_{\mathsf{inj}} \simeq \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$. The equivalence is provided by the functor $\Psi_{\mathfrak{R}}\colon \mathcal{M} \longmapsto \operatorname{Hom}_{\mathfrak{R}}(\mathcal{C}, \mathcal{M})$ of $\mathfrak{R}$-comodule homomorphisms from the $\mathfrak{R}$-comodule $\mathcal{C}(\mathfrak{R})$ and the functor $\Phi_{\mathfrak{R}}\colon \mathfrak{P} \longmapsto \mathcal{C} \odot_{\mathfrak{R}} \mathfrak{P}$ of *contratensor product* of $\mathfrak{R}$-contramodules with the $\mathfrak{R}$-comodule $\mathcal{C}(\mathfrak{R})$. It assigns the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ to the cofree $\mathfrak{R}$-comodule $\bigoplus_X \mathcal{C}(\mathfrak{R})$ for any set X. This result can be found in [52, Section 1.5].

More generally, let $\mathfrak{R}$ be a *right pseudo-compact* topological ring, i. e., a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals $\mathfrak{J}$ for which the right $\mathfrak{R}$-modules $\mathfrak{R}/\mathfrak{J}$ have finite length [30, § IV.3]. We define *left* $\mathfrak{R}$*-comodules* as the ind-objects in the abelian category opposite to the category of discrete right $\mathfrak{R}$-modules of finite length. The category of left $\mathfrak{R}$-comodules $\mathfrak{R}\text{–}\mathsf{comod}$ is anti-equivalent to the category of *pseudo-compact right* $\mathfrak{R}$*-modules*, i. e., pro-objects in the category of discrete right $\mathfrak{R}$-modules of finite length or, which is the same, compact and separated topological right $\mathfrak{R}$-modules with a base of neighborhoods of zero formed by open submodules with discrete quotient modules of finite length. As above, we denote this anti-equivalence by $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$.

There is a distinguished left $\mathfrak{R}$-comodule $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ for which $\mathcal{C}^{\mathrm{op}} = \mathfrak{R}$; the functor $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$, viewed as a contravariant functor from $\mathfrak{R}\text{–}\mathsf{comod}$ to the category of abelian groups, is represented by $\mathcal{C}(\mathfrak{R})$. A *cofree* left $\mathfrak{R}$-comodule is a direct sum of copies of the $\mathfrak{R}$-comodule $\mathcal{C}$; the cofree $\mathfrak{R}$-comodules are injective, and any left $\mathfrak{R}$-comodule can be embedded into a cofree one. The abelian category $\mathfrak{R}\text{–}\mathsf{comod}$ is locally finite [30, § II.4]; and the choice of an injective cogenerator $\mathcal{E}$ in any locally finite abelian category A fixes an equivalence between A and the category of left comodules over the topological ring $\mathfrak{R} = \operatorname{End}_{\mathsf{A}}(\mathcal{E})^{\mathrm{op}}$ opposite to the ring of endomorphisms of the object $\mathcal{E} \in \mathsf{A}$. The topology on the ring $\mathfrak{R}$ is defined to have a base of neighborhoods of zero consisting of (the right ideals opposite to) the annihilators of submodules of finite length $\mathcal{M} \subset \mathcal{E}$. The equivalence of categories $\mathsf{A} \simeq \mathfrak{R}\text{–}\mathsf{comod}$ assigns the object $\mathcal{C} \in \mathfrak{R}\text{–}\mathsf{comod}$ to the object $\mathcal{E}$ [30, Corollaire VI.4.1].

For any right pseudo-compact topological ring $\mathfrak{R}$, the categories of injective left $\mathfrak{R}$-comodules and projective left $\mathfrak{R}$-contramodules are naturally equivalent,

$$\mathfrak{R}\text{–}\mathsf{comod}_{\mathsf{inj}} \simeq \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}.$$

Indeed, the injective left $\mathfrak{R}$-comodules are the direct summands of the cofree $\mathfrak{R}$-comodules $\bigoplus_X \mathcal{C}(R)$, and the projective left $\mathfrak{R}$-contramodules are the direct summands of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$, where X denotes arbitrary sets. It remains to compute the groups of morphisms

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{R}}\left(\textstyle\bigoplus_X \mathcal{C},\, \bigoplus_Y \mathcal{C}\right) &\simeq \textstyle\prod_X \operatorname{Hom}_{\mathfrak{R}}\left(\mathcal{C},\, \bigoplus_Y \mathcal{C}\right) \simeq \prod_X \varprojlim_{\mathcal{M}\subset\mathcal{C}} \operatorname{Hom}_{\mathfrak{R}}\left(\mathcal{M},\, \bigoplus_Y \mathcal{C}\right) \\ \simeq \textstyle\prod_X \varprojlim_{\mathcal{M}\subset\mathcal{C}} \bigoplus_Y \operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{C}) &\simeq \textstyle\prod_X \varprojlim_{\mathfrak{J}\subset\mathfrak{R}} \bigoplus_Y \mathfrak{R}/\mathfrak{J} = \prod_X \varprojlim_{\mathfrak{J}\subset\mathfrak{R}} \mathfrak{R}/\mathfrak{J}[Y] \\ &\simeq \textstyle\prod_X \mathfrak{R}[[Y]] \simeq \prod_X \operatorname{Hom}^{\mathfrak{R}}\left(\mathfrak{R}, \mathfrak{R}[[Y]]\right) \simeq \operatorname{Hom}^{\mathfrak{R}}\left(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]\right) \end{aligned}$$

between the cofree $\mathfrak{R}$-comodules and the free $\mathfrak{R}$-contramodules in terms of projective limits over the subcomodules of finite length $\mathcal{M} \subset \mathcal{C}$ and the open right ideals $\mathfrak{J} \subset \mathfrak{R}$ in order obtain an isomorphism of the categories they form. In other words, the additive categories of projective left $\mathfrak{R}$-contramodules and projective pseudo-compact right $\mathfrak{R}$-modules are *anti-equivalent* (cf. [52, the end of Remark A.3]).

One would like to generalize this equivalence from locally finite to locally Noetherian abelian categories, i. e., abelian categories satisfying the axiom Ab5 and admitting a set of generators consisting of Noetherian objects, or equivalently, Ab5-categories where every object is the union of its Noetherian subobjects and isomorphism classes of Noetherian objects form a set [30, § II.4] (cf. [42]).

A remark at the end of [30, § IV.3] suggests considering topological rings $\mathfrak{R}$ with a base of neighborhoods of zero formed by (say, right) ideals $\mathfrak{J}$ such that the quotient modules $\mathfrak{R}/\mathfrak{J}$ are Artinian, and topological right $\mathfrak{R}$-modules with a base of neighborhoods of zero formed by open $\mathfrak{R}$-submodules with Artinian quotient modules. Then the opposite abelian category $\mathsf{E}(\mathfrak{R})$ to such category of $\mathfrak{R}$-modules is locally Noetherian, and the object $\mathcal{C}$ opposite to the right $\mathfrak{R}$-module $\mathfrak{R}$ is injective in it.

However, the direct summands of direct sums of copies of the object $\mathcal{C}$ do *not* exhaust the class of injective objects in $\mathsf{E}(\mathfrak{R})$, as one can see already in the example of the discrete ring $\mathfrak{R} = \mathbb{Z}$ with the Artinian right $\mathfrak{R}$-module $\mathbb{Q}_l/\mathbb{Z}_l$, which admits no surjective continuous morphisms from topological products of copies of the right $\mathfrak{R}$-module $\mathfrak{R}$. Furthermore, given a locally Noetherian abelian category A, choosing an injective object $\mathcal{E} \in \mathsf{A}$ such that all the injectives in A are direct summands of the direct sums of copies of $\mathcal{E}$ and leads to a topological ring $\mathfrak{R} = \operatorname{End}_{\mathsf{A}}(\mathcal{E})^{\mathrm{op}}$ which does *not* satisfy the above Artinianness condition in general.

Indeed, it suffices to take $\mathsf{A} = \mathsf{Ab}$ and $\mathcal{E} = \mathbb{R}/\mathbb{Z}$; then the Noetherian object $\mathbb{Z} \in \mathsf{A}$ is embeddable into $\mathcal{E}$ by means of any irrational number in $\mathbb{R}/\mathbb{Z}$, hence the discrete right $\mathfrak{R}$-module $\operatorname{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}$ is the quotient module of $\mathfrak{R} = \operatorname{End}_{\mathsf{Ab}}(\mathbb{R}/\mathbb{Z})^{\mathrm{op}}$ by a certain open right ideal, and it is *not* an Artinian module. *Nor* does the functor assigning the $\mathfrak{R}$-module $\operatorname{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ to the abelian group $\mathbb{Z}$ appear anywhere close to being an anti-equivalence of abelian categories.

So we choose a slightly different path and start from an arbitrary locally Noetherian abelian category A. Recall that, being a Grothendieck abelian category (an Ab5-category with a set of generators), any locally Noetherian category has enough injectives [33, N° 1.10].

Theorem. *For any locally Noetherian abelian category* A *there exists a unique abelian category* B *with enough projectives such that the full additive subcategories of injective objects in* A *and projective objects in* B *are (covariantly) equivalent,*

$$\mathsf{A}_{\mathsf{inj}} \simeq \mathsf{B}_{\mathsf{proj}}.$$

All the infinite direct sums and products exist in the abelian categories A *and* B*, and both the subcategories of injective objects in* A *and projective objects in* B *are closed under both the infinite direct sums and the infinite products.*

Proof. To prove uniqueness, we notice that an abelian category with enough projectives is determined by its full additive subcategory of projective objects. Indeed, given an abelian category B' with enough projectives and an abelian category B'', any additive functor $\mathsf{B}'_{\mathsf{proj}} \longrightarrow \mathsf{B}''$ can be uniquely extended to a right exact functor $\mathsf{B}' \longrightarrow \mathsf{B}''$. In particular, B' and B'' be two abelian categories with equivalent full subcategories of projectives $\mathsf{B}'_{\mathsf{proj}} \simeq \mathsf{B}''_{\mathsf{proj}}$. Then the embedding functor $\mathsf{B}'_{\mathsf{proj}} \longrightarrow \mathsf{B}''$ extends uniquely to a right exact functor $\mathsf{B}' \longrightarrow \mathsf{B}''$, while the embedding functor $\mathsf{B}''_{\mathsf{proj}} \longrightarrow \mathsf{B}'$ extends uniquely to a right exact functor $\mathsf{B}'' \longrightarrow \mathsf{B}'$. The compositions $\mathsf{B}' \longrightarrow \mathsf{B}'' \longrightarrow \mathsf{B}'$ and $\mathsf{B}'' \longrightarrow \mathsf{B}' \longrightarrow \mathsf{B}''$ being right exact functors isomorphic to the identity functors on the full subcategories of projective objects, they are also naturally isomorphic to identity functors on the whole abelian categories B' and B''.

Now, given a locally Noetherian abelian category A, choose an injective object $\mathcal{E} \in \mathsf{A}$ such that all the injectives in A are directs summands of the direct sums of copies of $\mathcal{E}$. E. g., one can take $\mathcal{E}$ to be the direct sum of injective envelopes of all the quotient objects of Noetherian generators of A. Consider the topological ring $\mathfrak{R} = \mathrm{End}_{\mathsf{A}}(\mathcal{E})^{\mathrm{op}}$ with a base of neighborhoods of zero formed by the right ideals opposite to the annihilators of Noetherian submodules $\mathcal{M} \subset \mathcal{E}$ in $\mathrm{End}_{\mathsf{A}}(\mathcal{E})$. Set $\mathsf{B} = \mathfrak{R}\text{–}\mathsf{contra}$ to be the abelian category of left $\mathfrak{R}$-contramodules.

To identify the full subcategory of direct sums of copies of the object $\mathcal{E}$ in A with the full subcategory of free $\mathfrak{R}$-contramodules in B, one computes the Hom groups

$$\begin{aligned}\mathrm{Hom}_{\mathsf{A}}\big(\textstyle\bigoplus_X \mathcal{E},\, \bigoplus_Y \mathcal{E}\big) &\simeq \textstyle\prod_X \mathrm{Hom}_{\mathsf{A}}\big(\mathcal{E},\, \bigoplus_Y \mathcal{E}\big) \simeq \prod_X \varprojlim_{\mathcal{M}\subset\mathcal{E}} \mathrm{Hom}_{\mathsf{A}}\big(\mathcal{M},\, \bigoplus_Y \mathcal{E}\big) \\ &\simeq \textstyle\prod_X \varprojlim_{\mathcal{M}\subset\mathcal{E}} \bigoplus_Y \mathrm{Hom}_{\mathsf{A}}(\mathcal{M},\mathcal{E}) \simeq \prod_X \varprojlim_{\mathfrak{J}\subset\mathfrak{R}} \bigoplus_Y \mathfrak{R}/\mathfrak{J} \\ &\simeq \textstyle\prod_X \mathfrak{R}[[Y]] \simeq \prod_X \mathrm{Hom}^{\mathfrak{R}}\big(\mathfrak{R},\mathfrak{R}[[Y]]\big) \simeq \mathrm{Hom}^{\mathfrak{R}}\big(\mathfrak{R}[[X]],\mathfrak{R}[[Y]]\big)\end{aligned}$$

in both subcategories in terms of projective limits over the Noetherian subobjects $\mathcal{M} \subset \mathcal{E}$ and the open right ideals $\mathfrak{J} \subset \mathfrak{R}$. Adjoining the direct summands (the images of idempotent endomorphisms) to both the full subcategories, one obtains the desired equivalence between the full subcategories of injective objects in A and projective objects in B.

Any abelian category with infinite direct sums and a set of generators has infinite products by Freyd's Special Adjoint Functor existence Theorem [44, Corollary V.8]; and the category of left contramodules over a topological ring $\mathfrak{R}$ with a base of neighborhoods of zero consisting of open right ideals always has both the infinite direct sums and products, as we have seen in Section 2.1. In any abelian category the infinite direct sums of projective objects are projective and the infinite products of injective objects are injective.

The infinite direct sums of injective objects in a locally Noetherian abelian category A are injective [30, Corollaire II.4.1 and Proposition IV.2.6]. Finally, to prove that the infinite products of projective objects in our category B are projective, we notice that any family of objects of the full subcategory $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$ has an infinite product in $\mathsf{B}_{\mathsf{proj}}$ (since any family of objects of $\mathsf{A}_{\mathsf{inj}}$ has an infinite product in $\mathsf{A}_{\mathsf{inj}}$). It is claimed that whenever an abelian category B has enough projectives and the full subcategory of projectives $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$ has infinite products, these are also the infinite products of objects of $\mathsf{B}_{\mathsf{proj}}$ in the whole category B, i. e., the embedding functor $\mathsf{B}_{\mathsf{proj}} \longrightarrow \mathsf{B}$ preserves infinite products.

Indeed, let an object $X \in \mathsf{B}$ be presented as the cokernel of a morphism of projective objects $Q \longrightarrow P$, so that there is an initial fragment of projective resolution $Q \longrightarrow P \longrightarrow X \longrightarrow 0$ in B. Let F_α be a family of objects in $\mathsf{B}_{\mathsf{proj}}$ and F be their product in $\mathsf{B}_{\mathsf{proj}}$. Then one computes the group $\mathrm{Hom}_{\mathsf{B}}(X, F)$ as the kernel of the map of abelian groups $\mathrm{Hom}_{\mathsf{B}}(P, F) \longrightarrow \mathrm{Hom}_{\mathsf{B}}(Q, F)$, which is isomorphic to the product of the kernels of the maps $\mathrm{Hom}_{\mathsf{B}}(P, F_\alpha) \longrightarrow \mathrm{Hom}_{\mathsf{B}}(Q, F_\alpha)$, that is the product of the groups $\mathrm{Hom}_{\mathsf{B}}(X, F_\alpha)$ over the indices α. □

References

[1] M. Aguiar. Internal categories and quantum groups. Cornell Univ. Ph.D. Thesis, 1997. Available from `http://www.math.cornell.edu/~maguiar/`.

[2] M. Barr. Coequalizers and free triples. *Math. Zeitschrift* **116**, #4, p. 307–322, 1970.

[3] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.* **95**, p. 466–488, 1960.

[4] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand. On a category of $\mathfrak{g}$-modules. *Functional Analysis and its Appl.* **10**, #2, p. 87–92, 1976.

[5] A. Beilinson, J. Bernstein. A proof of Jantzen conjectures. *Advances in Soviet Math.* **16**, #1, p. 1–50, 1993.

[6] A. Beilinson, B. Feigin, B. Mazur. Notes on conformal field theory (incomplete), 1991. Available from `http://www.math.sunysb.edu/~kirillov/manuscripts.html`.

[7] A. Beilinson, V. Drinfeld. Chiral algebras. AMS Colloquium Publications, 51. American Math. Society, Providence, RI, 2004.

[8] A. Beilinson, V. Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. February 2000. Available from `http://www.math.utexas.edu/~benzvi/Langlands.html`.

[9] A. Beilinson. Remarks on topological algebras. *Moscow Math. Journ.* **8**, #1, p. 1–20, 2008. `arXiv:0711.2527 [math.QA]`

[10] N. Bourbaki. Topologie générale. Chapitres 1 à 4. Springer-Verlag Berlin–Heidelberg–New York, 2007. Réimpression inchangée de l'édition originale, Hermann, Paris, 1971.

[11] N. Bourbaki. Espaces vectoriels topologiques. Chapitres 1 à 5. Springer, 2007. Réimpression inchangée de l'édition originale, Masson, Paris, 1981.

[12] A. K. Bousfield, D. M. Kan. Homotopy limits, completions and localizations. *Lecture Notes in Math.* **304**, Springer, 1972–1987.

[13] T. Brzeziński. The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties. *Algebras and Representation Theory* **5**, #4, p. 389–410, 2002. `arXiv:math.RA/0002105`

[14] T. Brzezinski, R. Wisbauer. Corings and comodules. London Mathematical Society Lecture Note Series, 309. Cambridge University Press, Cambridge, 2003.

[15] T. Brzeziński. Flat connections and (co)modules. New Techniques in Hopf Algebras and Graded Ring Theory, Universa Press, Wetteren, 2007, pp. 35–52. `arXiv:math.QA/0608170`

[16] T. Brzeziński. Contramodules. Slides of the presentation at the conference on "Categories in Geometry", Split, September 2007. Available from `http://www.irb.hr/korisnici/zskoda/BrzezinskiSplitSlides.pdf`.

[17] G. Böhm, T. Brzeziński, R. Wisbauer. Monads and comonads in module categories. *Journ. of Algebra* **233**, #5, p. 1719–1747, 2009. `arXiv:0804.1460 [math.RA]`

[18] T. Bühler. Exact categories. *Expositiones Math.* **28**, #1, p. 1–69, 2010. `arXiv:0811.1480 [math.HO]`

[19] P. Deligne, J. S. Milne. Tannakian categories. *Lecture Notes Math.* **900**, Springer-Verlag, Berlin–Heidelberg–New York, 1982, p. 101–228.

[20] M. Demazure, P. Gabriel. Groupes algébriques, Tome I. North-Holland, Amsterdam, 1970.

[21] J. Dixmier. Enveloping Algebras. Translated from French. North-Holland, Amsterdam, 1977. Or: Graduate Studies in Math. 11, AMS, Providence, RI, 1996.

[22] N. Durov. New approach to Arakelov geometry. Doctoral Dissertation, University of Bonn, 2007. `arXiv:0704.2030 [math.AG]`

[23] W. G. Dwyer, J. P. C. Greenlees. Complete modules and torsion modules. *American Journ. of Math.* **124**, #1, p. 199–220, 2002.

[24] S. Eilenberg, J. C. Moore. Foundations of relative homological algebra. *Memoirs of the American Math. Society* **55**, 1965.

[25] P. C. Eklof, J. Trlifaj. How to make Ext vanish. *Bulletin of the London Math. Society* **33**, #1, p. 41–51, 2001.

[26] E. Enochs. Flat covers and flat cotorsion modules. *Proceedings of the American Math. Society* **92**, #2, p. 179–184, 1984.

[27] B. L. Feigin, D. B. Fuchs. Verma modules over the Virasoro algebra. *Functional Analysis and its Appl.* **17**, #3, p. 241–242, 1983.

[28] B. L. Feigin, D. B. Fuchs. Verma modules over the Virasoro algebra. Topology (Leningrad, 1982), p. 230–245, *Lecture Notes in Math.* **1060**, Springer-Verlag, Berlin, 1984.

[29] B. L. Feigin. The semi-infinite homology of Kac-Moody and Virasoro Lie algebras. *Russian Math. Surveys* **39**, #2, p. 155–156, 1984.

[30] P. Gabriel. Des catégories abéliennes. *Bulletin de la Soc. Math. de France* **90**, p. 323–448, 1962.

[31] D. Gaitsgory, D. Kazhdan. Algebraic groups over a 2-dimensional local field: some further constructions. Studies in Lie theory, p. 97–130, *Progress in Math.* **243**, Birkhäuser Boston, Boston, MA, 2006. `arXiv:math.RT/0406282`

[32] K. Goodearl, R. Warfield. An introduction to noncommutative Noetherian rings. London Mathematical Society Student Texts, 16. Cambridge University Press, 1989–2004.

[33] A. Grothendieck. Sur quelques points d'algèbre homologique. *Tohoku Math. Journ. (2)* **9**, #2, p. 119–221, 1957.

[34] D. K. Harrison. Infinite abelian groups and homological methods. *Annals of Math.* **69**, #2, p. 366–391, 1959.

[35] J. E. Humphreys. Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$. Graduate Studies in Math. 94, AMS, Providence, RI, 2008.

[36] U. Jannsen. Continuous étale cohomology. *Mathematische Annalen* **280**, #2, p. 207–245, 1988.

[37] P. Jørgensen. The homotopy category of complexes of projective modules. *Advances in Math.* **193**, #1, p. 223–232, 2005. `arXiv:math.RA/0312088`

[38] V. G. Kac, A. K. Raina. Bombay lectures on highest weight representations of infinite-dimensional Lie algebras. Advanced Series in Math. Physics, vol. 2, World Scientific, Singapore–New Jersey–Hong Kong, 1987.

[39] M. Kleiner. Adjoint monads and an isomorphism of the Kleisli categories. *Journ. of Algebra* **133**, #1, p. 79–82, 1990.

[40] M. Kontsevich, A. Rosenberg. Noncommutative smooth spaces. *The Gelfand Mathematical Seminars 1996–1999*, p. 85–108, Birkhäuser Boston, Boston, MA, 2000. `arXiv:math.AG/9812158`

[41] M. Kontsevich, A. Rosenberg. Noncommutative spaces and flat descent. Max-Planck-Institut für Mathematik (Bonn) preprint MPIM2004-36.

[42] H. Krause. The stable derived category of a Noetherian scheme. *Compositio Math.* **141**, #5, p. 1128–1162, 2005. `arXiv:math.AG/0403526`

[43] S. Iyengar, H. Krause. Acyclicity versus total acyclicity for complexes over noetherian rings. *Documenta Math.* **11**, p. 207–240, 2006.
[44] S. MacLane. Categories for the working mathematician. Graduate Texts in Mathematics, 5. Springer-Verlag, New York–Berlin, 1971–1998.
[45] E. Matlis. Injective modules over Noetherian rings. *Pacific Journ. of Math.* **8**, #3, p. 511–528, 1958.
[46] E. Matlis. The higher properties of R-sequences. *Journ. of Algebra* **50**, #1, p. 77–112, 1978.
[47] H. Matsumura. Commutative ring theory. Translated by M. Reid. Cambidge Studies in Advanced Mathematics, 8. Cambridge University Press, 1986–2006.
[48] D. Murfet. The mock homotopy category of projectives and Grothendieck duality. Ph. D. Thesis, Australian National University, September 2007. Available from `http://www.therisingsea.org/thesis.pdf`.
[49] A. Neeman. The homotopy category of flat modules, and Grothendieck duality. *Inventiones Math.* **174**, p. 225–308, 2008.
[50] A. L. Onishchik, E. B. Vinberg. Lie groups and algebraic groups. Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990.
[51] L. Positselski. Seriya pisem pro polubeskonechnye (ko)gomologii associativnyh algebr ("A series of letters about the semi-infinite (co)homology of associative algebras", transliterated Russian). 2000, 2002. Available from `http://positselski.livejournal.com/314.html` or `http://posic.livejournal.com/413.html`.
[52] L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. `arXiv:0708.3398 [math.CT]`
[53] R. Bezrukavnikov, L. Positselski. On semi-infinite cohomology of finite-dimensional graded algebras. *Compositio Math.* **146**, #2, pp. 480–496, 2010. `arXiv:0803.3252 [math.QA]`
[54] L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs of the American Math. Society* **212**, #996, 2011. vi+133 pp. `arXiv:0905.2621 [math.CT]`
[55] A. I. Efimov, L. Positselski. Coherent analogues of matrix factorizations and relative singularity categories. Electronic preprint `arXiv:1102.0261 [math.CT]`.
[56] L. Positselski. Weakly curved A_∞-algebras over a topological local ring. Electronic preprint `arXiv:1202.2697 [math.CT]`.
[57] L. Positselski. Contraherent cosheaves. Electronic preprint `arXiv:1209.2995 [math.CT]`.
[58] L. Positselski. Comodule-contramodule correspondence. Slides of the presentation at the meeting in Třešt', Czech Republic, April 2014. Expanded version. Available from `http://positselski.narod.ru/co-contra.pdf`.
[59] L. Positselski. Dedualizing complexes and MGM duality. Electronic preprint `arXiv:1503.05523 [math.CT]`.
[60] L. Positselski. Coherent rings, fp-injective modules, dualizing complexes, and covariant Serre–Grothendieck duality. Electronic preprint `arXiv:1504.00700 [math.CT]`.
[61] A.-M. Simon. Approximations of complete modules by complete big Cohen–Macaulay modules over a Cohan–Macaulay local ring. *Algebras and Representation Theory* **12**, #2–5, p. 385–400, 2009.
[62] M. E. Sweedler. Hopf algebras. Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.
[63] R. Vázquez García. The category of the triples in a category. (Spanish) *Anales del Instituto de Matemáticas, Universidad Nacional Autónoma de México* **5**, p. 21–34, 1965.
[64] A. Rocha-Caridi, N. Wallach. Characters of irreducible representations of the Virasoro algebra. *Math. Zeitschrift* **185**, #1, p. 1–21, 1984.

[65] N. Wallach. Real reductive groups I, II. Pure and Applied Mathematics, 132. Academic Press, 1988, 1992.

[66] R. Wisbauer. Comodules and contramodules. *Glasgow Math. Journ.* **52**, #A, pp. 151–162, 2010.

[67] A. Yekutieli. On flatness and completion for infinitely generated modules over noetherian rings. *Communications in Algebra* **39**, #11, p. 4221–4245, 2010. `arXiv:0902.4378 [math.AC]`

[68] M. Porta, L. Shaul, A. Yekutieli. On the homology of completion and torsion. *Algebras and Representation Theory* **17**, #1, p. 31–67, 2014. `arXiv:1010.4386 [math.AC]`

[69] M. Porta, L. Shaul, A. Yekutieli. Cohomologically cofinite complexes. *Communications in Algebra* **43**, #2, p. 597–615, 2015. `arXiv:1208.4064 [math.AC]`

[70] A. Yekutieli. A separated cohomologically complete module is complete. *Communications in Algebra* **43**, #2, p. 616–622, 2015. `arXiv:1312.2714 [math.AC]`

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