WEAKLY CURVED A_∞-ALGEBRAS OVER A TOPOLOGICAL LOCAL RING

LEONID POSITSELSKI

Abstract. We define and study the derived categories of the first kind for curved DG- and A_∞-algebras complete over a pro-Artinian local ring with the curvature elements divisible by the maximal ideal of the local ring. We develop the Koszul duality theory in this setting and deduce the generalizations of the conventional results about A_∞-modules to the weakly curved case. The formalism of contramodules and comodules over pro-Artinian topological rings is used throughout the paper. Our motivation comes from the Floer–Fukaya theory.

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0. Introduction

0.1. The conventional definition of the derived category involves localizing the homotopy category (or alternatively, the category of complexes and closed morphisms between them) by the class of quasi-isomorphisms. In fact, one needs more than just such a definition in order to do homological algebra in a derived category. Constructing derived functors and computing the Ext groups requires having appropriate classes of resulutions.

The *classical homological algebra* can be roughly described as the study of derived categories that can be equivalently defined as the localizations of the homotopy categories by quasi-isomorphisms and the full subcategories in the homotopy categories formed by complexes of projective or injective objects, or DG-modules that are projective/injective as graded modules.

This is true, e. g., for appropriately bounded complexes over an abelian or exact category with enough projectives or injectives, or for bounded DG-modules over a DG-algebra with nonpositive cohomological grading, or over a connected simply connected DG-algebra with nonnegative cohomological grading.

0.2. It is known since the pioneering work of Spaltenstein [33] that one can work with the unbounded derived categories of modules and sheaves using resolutions satisfying stronger conditions than termwise projectivity, flatness or injectivity. These classes of resolutions, now known as *homotopy projective*, *homotopy flat*, etc., complexes, are defined by conditions imposed on the complex as a whole and depending on the differential in the complex, rather than only on its terms. This approach was extended to unbounded DG-modules over unbounded DG-rings by Keller [20] and Bernstein–Lunts [3].

Another point of view, first introduced in Hinich's paper [17] in the case of cocommutative DG-coalgebras, involves strenghtening the conditions imposed on quasi-isomophisms rather than the conditions on resolutions. Extended to DG-comodules by Lefèvre-Hasegawa [23] and others, this theory took its fully developed form in the present author's monograph [28] and memoir [29], where the general definitions of the *derived categories of the second kind* were given.

The terminology of *two kinds of derived categories* goes back to the classical paper of Husemoller, Moore, and Stasheff [18], where the distinction between *differential derived functors of the first* and *the second kind* was introduced. The conventional unbounded derived category, studied by Spaltenstein, Keller, et al., is called *the derived category of the first kind*. The definition of the derived category of the second kind has several versions, called the *absolute derived*, *complete derived*, *coderived*, and *contraderived category* [29, 30]. Here the "coderived category" terminology comes from Keller's brief exposition [21] of the related results from Lefèvre-Hasegawa's thesis.

0.3. As the latter terminological system suggests, the coderived categories are more suitable for comodules, and similarly the contraderived categories more suitable for contramodules [9], than for modules. The philosophy of "taking the derived categories of the first kind for modules, and the derived categories of the second kind for comodules or contramodules" was used in the author's monograph on semi-infinite homological algebra [28]. It works well in Koszul duality, too [29], although the derived categories of the second kind for DG-modules also have their uses in the case of DG-algebras whose underlying graded algebras have finite homological dimension.

Unlike the conventional quasi-isomorphism, the equivalence relations used to define the derived categories of the second kind are not reflected by forgetful functors; so one cannot tell whether, say, a DG-comodule is trivial in the coderived category (*coacyclic*) or not just by looking on its underlying complex of vector spaces. Thus the forgetful functors between derived categories of the second kind are generally *not* conservative (which stands in the way of possible application of the presently popular techniques of the ∞-categorical Barr–Beck theorem [24]).

On the other hand, derived categories of the second kind make perfect sense for curved DG-modules or DG-comodules [29], to which the conventional definition of the derived category (of the first kind) is not applicable, as curved structures have no cohomology groups. This sometimes forces one to consider derived categories of the second kind for modules, including modules over rings of infinite homological dimension, inspite of all the arising technical complications [30, 31].

The aim of this paper is to show how the derived category of the first kind can be defined for curved DG-modules and curved A_∞-modules, if only in a rather special situation of algebras over a complete local ring with the curvature element divisible by the maximal ideal of the local ring.

0.4. Let us explain the distinction between the derived categories of the first and the second kind in some more detail (see also [29, Sections 0.1–0.3] and [28, Preface and Section 0.2.9]). The classical situation, when there is no difference between the two kinds of derived categories, is special in that no infinite summation occurs when one totalizes a resolution of a complex or a DG-module. When the need to use the infinite summation arises, however, one is forced to choose between taking direct sums or direct products.

Informally, this means specifying the direction along the diagonals of a bicomplex (or a similar double-indexed family of groups with several differentials) in which the terms "increase" or "decrease" in the order of magnitude. Using the appropriate kind of completion is presumed, in which one takes infinite direct sums in the "increasing" direction and infinite products in the "decreasing" one. The components of the total differential of the bicomplex-like structure are accordingly ordered; and the spectral sequence in which one first passes to the cohomology with respect to the dominating component of the differential converges (at least in the weak sense of [8]) to the cohomology of the related totalization.

In the A_∞-algebra situation, the conventional theory of the first kind presumes that the operations m_i, $i \geqslant 1$ are ordered so that $m_1 \gg m_2 \gg m_3 \gg \cdots$ in the order of magnitude, i. e., the component m_1 dominates. So, in particular, if the differential $d = m_1$ is acyclic on a given A_∞-algebra or A_∞-module, then such an algebra or module vanishes in the homotopy category and the higher operations m_2, m_3, etc. on it do not matter from the point of view of the theory of the first kind.

On the other hand, considering the derived category of the second kind, e. g., for CDG-modules over a CDG-algebra, means choosing the multiplication as the dominating component, i. e., setting $m_2 \gg m_1 \gg m_0$. Hence the importance of the underlying graded algebras or modules of CDG-algebras or CDG-modules, with the differentials and the curvature elements in them forgotten, in the study of the coderived, contraderived, and absolute derived categories.

Of course, the above vague wording should be taken with a grain of salt, and the notation is symbolic: it is not any particular maps m_i (some of which may well happen to vanish for some particular algebras) but the whole vector spaces of such maps that are ordered in the "order of magnitude".

0.5. A theory of the second kind for curved A_∞ structures can also be developed. Essentially, it would mean that no "divergent" infinite sequences of the higher operations should be allowed to occur. As usually, it is technically easier to do that for coalgebras, where the "convergence condition" on the higher comultilpications μ_i appears naturally. This theory is reasonably well-behaved [29, Sections 7.4–7.6].

It may be possible to have a theory of the second kind for curved A_∞-algebras, too, e. g., by restricting oneself to those curved A_∞-algebras and A_∞-modules in each of which there is a finite number of nonvanishing higher operations m_i only [29, final sentences of Remark 7.3]. However, proving theorems about such A_∞-modules would involve working with topological coalgebras, which seems to be technically quite unpleasant (infinite operations on modules would be problematic, etc.)

A more delicate approach might involve imposing the convergence condition according to which the operations m_i eventually vanish in the restriction to the tensor powers of every fixed finite-dimensional subspace of the curved A_∞-algebra, and similarly for the higher components f_i of the curved A_∞-morphisms, modules, etc. The bar-construction of such an A_∞-algebra would be defined as a DG-coalgebra object in the tensor category of ind-pro-finite-dimensional vector spaces.

Perhaps the most reasonable way to deal with the forementioned problem would require replacing the ground category of vector spaces with that of pro-vector spaces, with the approximate effect of interchanging the roles of algebras and coalgebras [28, Remark 2.7]. This is also what one may wish to do should the need to develop a theory of the first kind for A_∞-coalgebras arise (cf. [29, Remark 7.6]).

0.6. On the other hand, having a theory of the first kind for curved A_∞-algebras would essentially mean setting $m_0 \gg m_1 \gg m_2 \gg \cdots$, i. e., designating the curvature element m_0 as the dominant term. The problem is that m_0, being just an element of a curved A_∞-algebra, is too silly a structure to be allowed to dominate unrestrictedly. When nondegenerate enough (and it is all to easy for an element of a vector space to be nondegenerate enough) and made dominating, it would kill all the other structure of such a curved A_∞-algebra or A_∞-module.

That is why every curved A_∞-algebra over a field which is either considered as nonunital and has a nonzero curvature element, or has a curvature element not proportional to the unit, is A_∞-isomorphic to a curved A_∞-algebra with $m_i = 0$ for all $i \geqslant 1$ [29, Remark 7.3]. Moreover, every A_∞-module over a (unital or not) curved A_∞-algebra with a nonzero curvature element over a field is contractible.

0.7. Hence the alternative of developing a theory of the first kind for curved A_∞-algebras over a local ring $\mathfrak{R}$, with the curvature element being required to be divisible by the maximal ideal $\mathfrak{m} \subset \mathfrak{R}$. Let us first discuss this idea in the simplest case of the ring of formal power series $\mathfrak{R} = k[[\epsilon]]$, where k is a field.

In terms of the above ordering metaphor, this means having two scales of orders of magnitude at the same time. On the one hand, the ϵ-adic topology is presumed,

i. e., $1 \gg \epsilon \gg \epsilon^2 \gg \cdots$ On the other hand, it is a theory of the first kind, so $m_0 \gg m_1 \gg m_2 \gg \cdots$ Given that m_0 is assumed to be divisible by ϵ and m_1 isn't, the question which of the two scales has the higher priority arises immediately.

If we want to make our theory as far from trivial as possible, the natural answer is to designate the ϵ-adic scale as the more important one. In the theory developed in this paper, this is achieved by having the topology of a complete local ring $\mathfrak{R}$ built into the tensor categories of $\mathfrak{R}$-modules in which our A_∞-algebras and A_∞-modules live. That is where $\mathfrak{R}$-contramodules (and also $\mathfrak{R}$-comodules) come into play.

0.8. In the conventional setting of (uncurved) DG- and A_∞-algebras over a field, the notion of A_∞-morphisms can be used to define the derived category of DG-modules. Indeed, the homotopy category of A_∞-modules over an A_∞-algebra coincides with their derived category, and the derived category of A_∞-modules over a DG-algebra is equivalent to the derived category of DG-modules. So the complex of A_∞-morphisms between two DG-modules over a DG-algebra computes the Hom between them in the derived category of DG-modules.

A similar definition of the derived category of curved DG-modules over a curved DG-algebra was suggested in [26]. Then it was shown in the subsequent paper [22] that the "derived category of CDG-modules" defined in this way vanishes entirely whenever the curvature element of the CDG-algebra is nonzero and one is working over a field (as it follows from the above discussion).

One of the results of this paper is the demonstration of a setting in which this kind of definition of the derived category of curved DG-modules is nontrivial and reasonably well-behaved.

0.9. Before we start explaining what $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules are, let us have a look on the situation from another angle.

The passage from uncurved to curved algebras is supposed not only to expand the class of algebras being considered, but also enlarge the sets of morphisms between them. In fact, the natural functor from DG-algebras to CDG-algebras is faithful, but not fully faithful [27]. Together with the curvature elements in algebras, *change-of-connection* elements in morphisms between algebras are naturally supposed to come.

One of the consequences of the existence of the change-of-connection morphisms in the category of CDG-algebras is the impossibility of extending to CDG-modules the conventional definition of the derived category (of the first kind) of DG-modules over a DG-algebra. Quite simply, CDG-isomorphic DG-algebras may have entirely different derived categories of DG-modules. Moreover, the derived category of DG-modules over a DG-algebra is invariant under quasi-isomorphisms of DG-algebras; and this already is incompatible with the functoriality with respect to change-of-connection morphisms. Indeed, *any* two DG-algebras over a field can be connected by a chain of transformations, some of which are quasi-isomorphisms, while the other ones are CDG-isomorphisms of DG-algebras [29, Examples 9.4].

0.10. So another problem with the naïve attempt to develop a theory of the first kind for curved A_∞-algebras over a field is that one cannot have change-of-connection morphisms in it. One can say that the A_∞-morphisms between such A_∞-algebras are too numerous, in that all the operations m_1, m_2, … can be killed by A_∞-isomorphisms if only the curvature element m_0 is not proportional to the unit, and still they are too few, in that morphisms with nonvanishing change-of-connection components f_0 cannot be considered.

To be more precise, recall that an A_∞-*morphism* $f\colon A \longrightarrow B$ is defined a sequence of maps $f_i\colon A^{\otimes i} \longrightarrow B$, where $i \geqslant 1$ [23]. For curved A_∞-algebras, one would like to define curved A_∞-morphisms as similar sequences of maps f_i starting with $i = 0$, the component $f_0 \in B^1$ being the change-of-connection element. The problem is that the compatibility equations on the maps f_i in terms of the A_∞-operations $m_i^A\colon A^{\otimes i} \longrightarrow A$ and $m_i^B\colon B^{\otimes i} \longrightarrow B$ contain a meaningless infinite summation when $f_0 \neq 0$ and one is working, e. g., over a field of coefficients (unless $m_i^B = 0$ for $i \gg 0$).

The explanation is that while the maps m_i^A are interpreted as the components of a coderivation of the tensor coalgebra cogenerated by the graded vector space A, the maps f_i are the components of a morphism between such tensor coalgebras. And while coderivations may not preserve coaugmentations of conilpotent coalgebras over fields, coalgebra morphisms always do.

0.11. The latter problem can be solved by having f_0 divisible by the maximal ideal $\mathfrak{m}$ of a complete local ring $\mathfrak{R}$ and the components of the graded $\mathfrak{R}$-modules A and B complete in the $\mathfrak{m}$-adic topology, to make the relevant infinite sums convergent. One also wants the components of one's A_∞-algebras over $\mathfrak{R}$ to be free (complete) $\mathfrak{R}$-modules, so that their completed tensor product over $\mathfrak{R}$ is an exact functor.

This is a good definition of the category of curved A_∞-algebras to work with; but when dealing with A_∞-modules, it is useful to have an abelian category to which their components may belong. And the category of (infinitely generated) $\mathfrak{m}$-adically complete $\mathfrak{R}$-modules is *not* an abelian already for $\mathfrak{R} = k[[\epsilon]]$. The natural abelian category into which complete $\mathfrak{R}$-modules are embedded is that of $\mathfrak{R}$-*contramodules*.

In particular, when $\mathfrak{R} = \mathbb{Z}_l$ is the ring of l-adic integers, the abelian category of $\mathfrak{R}$-contramodules is that of the *weakly l-complete abelian groups* of Jannsen [19], known also as the *Ext-p-complete abelian groups* of Bousfield–Kan [4] (where $p = l$). Contramodules over $k[[\epsilon]]$ are very similar [28, Remarks A.1.1 and A.3].

0.12. Generally, contramodules are modules with infinite summation operations. Among other things, they provide a way of having an abelian category of nontopological modules with some completeness properties over a coring or a topological ring. Defined originally by Eilenberg and Moore [9] as natural counterparts of comodules over coalgebras over commutative rings, contramodules were studied and used in the present author's monograph [28] for the purposes of the semi-infinite cohomology theory and the comodule-contramodule correspondence.

In particular, $k[[\epsilon]]$-contramodules form a full subcategory of the category of $k[[\epsilon]]$-modules (and even a full subcategory of the category of $k[\epsilon]$-modules). This subcategory contains all the $k[[\epsilon]]$-modules M such that $M \simeq \varprojlim_n M/\epsilon^n M$, and also some other $k[[\epsilon]]$-modules (hence the "weakly complete" terminology). The natural map $\mathfrak{M} \longrightarrow \varprojlim_n \mathfrak{M}/\epsilon^n\mathfrak{M}$ is surjective for every $k[[\epsilon]]$-contramodule $\mathfrak{M}$, but it may not be injective [28, Section A.1.1 and Lemma A.2.3].

The more familiar *comodules*, on the other hand, are basically discrete or torsion modules. For a pro-Artinian topological ring, they are defined as the opposite category to that of Gabriel's *pseudo-compact modules* [14]. Notice that our $\mathfrak{R}$-comodules are *not* literally discrete modules over a topological ring $\mathfrak{R}$, although any choice of an injective hull of the irreducible discrete module over a pro-Artinian commutative local ring $\mathfrak{R}$ provides an equivalence between these two abelian categories (and there is even a *natural* equivalence when $\mathfrak{R}$ is a profinite-dimensional algebra over a field or a profinite ring). So the $k[[\epsilon]]$-comodules are just $k[[\epsilon]]$-modules with a locally nilpotent action of ϵ.

The conventional formalism of tensor operations (i. e., the tensor product and Hom) on modules or bimodules over rings can be extended to comodules and contramodules over noncocommutative corings, where the natural operations are in much greater abundance and variety (there are five of them to be found in [28, 29]). For contramodules and comodules over a pro-Artinian commutative ring, we define the total of *seven* operations in this paper.

This allows to consider, in particular, curved A_∞-modules over $\mathfrak{R}$-free $\mathfrak{R}$-complete curved A_∞-algebras with, alternatively, either $\mathfrak{R}$-contramodule ("weakly complete") or $\mathfrak{R}$-comodule ("torsion", "discrete") coefficients. By another instance of the derived comodule-contramodule correspondence, the corresponding two homotopy categories are naturally equivalent.

0.13. Yet another reason to work with complete modules or contramodules rather than just conventional modules over a local ring is the need to use Nakayama's lemma as the basic technical tool. The point is, the conventional version of Nakayama's lemma for modules over local rings only holds for finitely generated modules. Of course, one does not want to restrict oneself to finite-dimensional vector spaces or finitely generated modules over the coefficient ring when doing the homological algebra of A_∞-algebras and A_∞-modules.

So we want to have an abelian category of modules with infinite direct sums and products where Nakayama's lemma holds. Notice that Nakayama's lemma holds for infinitely generated modules over an Artinian local ring. As a natural generalization of this obvious observation, the appropriate version of Nakayama's lemma for infinitely generated contramodules over a topological ring with a topologically nilpotent maximal ideal was obtained in [28, Section A.2 and Remark A.3].

For comodules over a pro-Artinian local ring, we use the dual version of Nakayama's lemma, which is clearly true.

0.14. We call curved DG-algebras in the tensor category of free $\mathfrak{R}$-contramodules with the curvature element divisible by the maximal ideal $\mathfrak{m} \subset \mathfrak{R}$ *weakly curved*

DG-algebras, or *wcDG-algebras* over $\mathfrak{R}$. Morphisms of wcDG-algebras are CDG-algebra morphisms with the change-of-connection elements divisible by $\mathfrak{m}$.

CDG-modules over a wcDG-algebra are referred to as wcDG-modules. The similar terminology is used for A_∞-algebras: a *weakly curved* A_∞*-algebra*, or a *wc* A_∞*-algebra*, is a curved A_∞-algebra in the tensor category of free $\mathfrak{R}$-contramodules with the curvature element divisible by $\mathfrak{m}$.

So we can summarize much of the preceding discussion by saying that *theories* (i. e., derived categories and derived functors) *of the first kind make sense in the weakly curved, but not in the strongly curved case.*

0.15. Let us explain how we define the equivalence relation on wcDG-modules and wc A_∞-modules. First assume that the underlying graded $\mathfrak{R}$-module of our weakly curved module $\mathfrak{M}$ over $\mathfrak{A}$ is a free graded $\mathfrak{R}$-contramodule.

In this case we simply apply the functor of reduction modulo $\mathfrak{m}$ to obtain an uncurved DG- or A_∞-module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over an uncurved algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. The weakly curved module $\mathfrak{M}$ is viewed as a trivial object of our triangulated category of modules if the complex $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is acyclic. In particular, when the curvature element of $\mathfrak{A}$ in fact vanishes, this condition means that the complex of free $\mathfrak{R}$-contramodules $\mathfrak{M}$ should be contractible (rather than just acyclic).

So the triangulated category of wcDG- or wc A_∞-modules that we construct is not actually their derived category of the first kind, but rather a mixed, or *semiderived category* [28]. It behaves as the derived category of the first kind in the direction of $\mathfrak{A}$ relative to $\mathfrak{R}$, and the derived category of the second kind, or more precisely the contraderived category, along the variables from $\mathfrak{R}$.

For this reason we call the weakly curved modules $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is an acyclic complex of vector spaces *semiacyclic*. The prefix "semi" here means roughly "halfway between acyclic and contraacyclic" or even "halfway between acyclic and contractible"; so it should be thought of as a condition *stronger* than the acyclicity.

As to wcDG- or wc A_∞-modules $\mathfrak{N}$ over $\mathfrak{A}$ whose underlying graded $\mathfrak{R}$-modules are $\mathfrak{R}$-contramodules that are not necessarily free, we replace them with $\mathfrak{R}$-free wc $\mathfrak{A}$-modules $\mathfrak{M}$ isomorphic to $\mathfrak{N}$ in the contraderived category of weakly curved $\mathfrak{A}$-modules before reducing modulo $\mathfrak{m}$ to check for semiacyclicity. So our semiderived category of arbitrary $\mathfrak{R}$-contramodule wcDG- or wc A_∞-modules over $\mathfrak{A}$ is the quotient category of their contraderived category by the kernel of the derived reduction functor. To construct such an $\mathfrak{R}$-free resolution $\mathfrak{M}$ of a given weakly curved $\mathfrak{A}$-module $\mathfrak{N}$, it suffices to find an $\mathfrak{R}$-free left resolution for $\mathfrak{N}$ in the abelian category of $\mathfrak{R}$-contramodule weakly curved $\mathfrak{A}$-modules and totalize it by taking infinite products along the diagonals.

The definition of the equivalence relation on wcDG-modules or wc A_∞-modules over $\mathfrak{A}$ whose underlying graded $\mathfrak{R}$-modules are cofree $\mathfrak{R}$-comodules is similar, except that functor $\mathcal{M} \longmapsto {}_{\mathfrak{m}}\mathcal{M}$ of passage to the maximal submodule annihilated by $\mathfrak{m}$ is used to obtain a complex of vector spaces from a curved $\mathfrak{A}$-module in this case. And for arbitrary $\mathfrak{R}$-comodule weakly curved $\mathfrak{A}$-modules, the semiderived category is constructed as the quotient category of the coderived category of such modules by the kernel of the derived $\mathfrak{m}$-annihilated submodule functor.

0.16. In particular, when the categories of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules have finite homological dimensions (e. g., $\mathfrak{R}$ is a regular complete Noetherian local ring), any acyclic complex of free $\mathfrak{R}$-contramodules or cofree $\mathfrak{R}$-modules is contractible. In this case, our semiderived category of wcDG- or wc A_∞-modules can be viewed as a true derived category of the first kind and called simply the *derived category.*

On the other hand, it is instructive to consider the case of the ring of dual numbers $R = k[\epsilon]/\epsilon^2$. In this case, there is no difference between R-contramodules and R-comodules, which are both just R-modules; and accordingly no difference between R-contramodule and R-comodule curved A-modules.

Still, their semiderived categories are different, in the sense that the two equivalence relations on weakly curved A-modules (in other words, the two classes of semiacyclic curved modules) are different in the case of weakly curved modules that are not (co)free over R. Indeed, they are different already for $A = R$, as the classes of coacyclic and contraacyclic complexes of R-modules are different [29, Examples 3.3].

The two semiderived categories of weakly curved A-modules are equivalent (as they are for any weakly curved algebra $\mathfrak{A}$ over any pro-Artinian local ring $\mathfrak{R}$), but the equivalence is a nontrivial construction when applied to modules that are not R-(co)free.

0.17. The most striking aspect of the theory of (semi)derived categories of weakly curved modules developed in this paper is just how nontrivial these are. On the one hand, there is a general tendency of the curvature to trivialize the categories of modules, well-known to the specialists now (see, e. g., [22]).

One reason for this is that apparently no curved modules over a given curved algebra can be pointed out *a priori* that would not be known to vanish in the homotopy category already. In particular, a curved algebra has *no* natural structure of a curved module over itself. Some explicit constructions of CDG-modules are used in the proofs of the general theorems about them in [29, 30, 31], but these always produce contractible CDG-modules. There are lots of examples of nontrivial CDG-modules, but these are CDG-modules over CDG-algebras of some special types (CDG-bimodules [30], Koszul CDG-algebras [27], change-of-connection transformations of DG-algebras, etc.)

In particular, an example from [22] shows that our (semi)derived category of weakly curved modules over a wcDG- or wc A_∞-algebra $\mathfrak{A}$ may vanish entirely even when the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ has a nonzero cohomology algebra. This can already happen over $\mathfrak{R} = k[[\epsilon]]$, or indeed, over $\mathfrak{R} = k[\epsilon]/\epsilon^2$.

Specifically, let $\mathfrak{A} = \mathfrak{R}[x, x^{-1}]$ be the graded algebra over $\mathfrak{R}$ generated by an element x of degree 2 and its inverse element x^{-1}, with the only relation saying that these two elements should be inverse to each other, endowed with the zero differential, vanishing higher operations $m_i = 0$ for $i \geqslant 3$, and the curvature element $h = \epsilon x$. Then the homotopy categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree wcDG-modules over

$\mathfrak{A}$ already vanish, as consequently do the (semi)derived categories of wcDG- and wc A_∞-modules over $\mathfrak{A}$ (see Example 5.3.5).

For a similar wcDG-algebra $\mathfrak{A}' = \mathfrak{R}[x]$ with $\deg x = 2$, $d = 0$ and $h = \epsilon x$, one obtains a nonvanishing (semi)derived category of wcDG- or wc A_∞-modules in which all the $\mathfrak{R}$-contramodules of morphisms are annihilated by ϵ (see Example 6.6.1).

0.18. On the other hand, the obvious expectation that the $\mathfrak{R}$-(contra)modules of morphisms in the triangulated category of weakly curved $\mathfrak{A}$-modules are always torsion modules is most emphatically *not true*. The reason for the obvious expectation is, of course, that the homotopy category of curved A_∞-modules is trivial over a field.

The explanation is that one cannot quite localize contramodules. The functor assigning to an $\mathfrak{R}$-contramodule the tensor product of its underlying $\mathfrak{R}$-module with the field of quotients of $\mathfrak{R}$ does *not* preserve either the tensor product or the internal Hom of contramodules; nor, indeed, does it preserve even infinite direct sums.

0.19. Our computations of the $\mathfrak{R}$-contramodules Hom in certain (semi)derived categories of wcDG- and wc A_∞-modules are based on the Koszul duality theorems generalizing those in [29]. The semiderived category of wcDG-modules over a wcDG-algebra $\mathfrak{A}$ is equivalent to the coderived category of CDG-comodules and the contraderived category of CDG-contramodules over the CDG-coalgebra $\mathfrak{C} = \mathrm{Bar}(\mathfrak{A})$ obtained by applying the bar construction to $\mathfrak{A}$. Similarly, the co/contraderived category of CDG-co/contramodules over an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ that is conilpotent modulo $\mathfrak{m}$ is equivalent to the semiderived category of wcDG-modules over the cobar construction $\mathrm{Cob}(\mathfrak{C})$.

In particular, it follows that the semiderived category of wcDG-modules over a wcDG-algebra $\mathfrak{A}$ is equivalent to the semiderived category of wc A_∞-modules over $\mathfrak{A}$ considered as a wc A_∞-algebra, so our lumping together of the wcDG- and wc A_∞-modules in the preceding discussion is justified. On the other hand, the semiderived category of wc A_∞-modules over a wc A_∞-algebra $\mathfrak{A}$ is equivalent to the semiderived category of CDG-modules over the enveloping wcDG-algebra of $\mathfrak{A}$.

0.20. To obtain a specific example of a nontrivial Hom computation in a (semi)derived category of wcDG- or wc A_∞-modules, one can start with an ungraded $\mathfrak{R}$-free coalgebra $\mathfrak{C}$ considered as a CDG-coalgebra concentrated in degree 0 with a zero differential and a zero curvature function. Then the co/contraderived category of, say, $\mathfrak{R}$-free CDG-co/contramodules over $\mathfrak{C}$ is just the co/contraderived category of the exact category of $\mathfrak{R}$-free $\mathfrak{C}$-co/contramodules. In particular, the exact category of $\mathfrak{R}$-free comodules over $\mathfrak{C}$ embeds into its coderived category (and similarly for contramodules).

So considering $\mathfrak{C}$ as a comodule over itself we obtain an example of an object in the coderived category whose endomorphism ring is a nonvanishing *free* $\mathfrak{R}$-contramodule. On the other hand, whenever the k-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent, by the Koszul duality theorems mentioned above the coderived category of $\mathfrak{R}$-free comodules over $\mathfrak{C}$ is equivalent to the semiderived category of wcDG- or wc A_∞-modules over $\mathrm{Cob}(\mathfrak{C})$. It remains to pick an $\mathfrak{R}$-free coalgebra $\mathfrak{C}$ that is conilpotent modulo $\mathfrak{m}$ but has no coaugmentation over $\mathfrak{R}$ in order produce an example of a semiderived category of wcDG- or wc A_∞-modules with a nonzero $\mathfrak{R}$-free Hom contramodule.

In the simplest case of the coalgebra dual to the $\mathfrak{R}$-free algebra of finite rank $\mathfrak{R}[y]/(y^2=\epsilon)$ (with $\mathfrak{R}=k[[\epsilon]]$ or $k[\epsilon]/\epsilon^2$, as above) we obtain the wcDG-algebra $\mathfrak{A}=\mathrm{Cob}(\mathfrak{C})$ that is freely generated over $\mathfrak{R}$ by an element x of degree 1, with the zero differential and the curvature element $h=\epsilon x^2$. The cobar construction assigns to the cofree comodule $\mathfrak{C}$ over $\mathfrak{C}$ a wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$ with an underlying graded $\mathfrak{A}$-module freely generated by two elements. The algebra of endomorphisms of $\mathfrak{M}$ in the (semi)derived category of wcDG- or wc A_∞-modules over $\mathfrak{A}$ is isomorphic to $\mathfrak{R}[y]/(y^2=\epsilon)$, so it is a free $\mathfrak{R}$-contramodule of rank two (see Example 6.6.2).

0.21. One of our most important results in this paper is that the semiderived category wcDG- or wc A_∞-modules over a wcDG- or wc A_∞-algebra $\mathfrak{A}$ over $\mathfrak{R}$ is compactly generated. So are the coderived category of CDG-comodules and the contraderived category of CDG-contramodules over an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$.

In the case case of wcDG- or wc A_∞-modules we present a *single*, if not quite explicit, compact generator. In order to construct this generator, one has to consider $\mathfrak{R}$-comodule coefficients. In the semiderived category of $\mathfrak{R}$-comodule wcDG- or wc A_∞-modules over $\mathfrak{A}$, the weakly curved module $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is the desired compactly generating object. As we have already mentioned, the semiderived categories of $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule weakly curved modules over $\mathfrak{A}$ are equivalent, but the equivalence is a somewhat complicated construction.

In the case of the coderived category of CDG-comodules over $\mathfrak{C}$, we also consider $\mathfrak{R}$-comodule coefficients, and have CDG-comodules whose underlying graded $\mathfrak{R}$-comodules have finite length form a triangulated subcategory of compact generators. For the contraderived category of CDG-contramodules over $\mathfrak{C}$, there is, once again, no explicit construction: one just has to identify this category with the coderived category of CDG-comodules.

While mainly interested in the curved modules and co/contramodules in the tensor category of $\mathfrak{R}$-contramodules, it is chiefly for the purposes of these constructions of compact generators that we pay so much attention to the $\mathfrak{R}$-comodule coefficients and the $\mathfrak{R}$-comodule-contramodule correspondence in our exposition.

0.22. Mixing contramodules with comodules is a tricky business, though. We have already discussed $\mathfrak{R}$-contramodule weakly curved $\mathfrak{A}$-modules and $\mathfrak{R}$-comodule weakly curved $\mathfrak{A}$-module in this introduction; and now we have just mentioned $\mathfrak{R}$-comodule curved $\mathfrak{C}$-comodules. So let us use the occasion to *warn* the reader that, apparently, it makes little sense to consider arbitrary $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules or $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules, as such categories of graded modules are not even abelian, the relevant functors on the categories of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules not having the required exactness properties.

The (exotic derived) categories of $\mathfrak{R}$-free curved $\mathfrak{C}$-comodules and $\mathfrak{R}$-cofree curved $\mathfrak{C}$-contramodules make perfect sense and are well-behaved; and so are the categories of arbitrary $\mathfrak{R}$-contramodule (or just $\mathfrak{R}$-free) curved $\mathfrak{C}$-contramodules and arbitrary

$\mathfrak{R}$-comodule (or just $\mathfrak{R}$-cofree) curved $\mathfrak{C}$-comodules. The (exotic derived) categories of $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule (weakly) curved $\mathfrak{A}$-modules are also well-behaved, as are the categories of $\mathfrak{R}$-free or $\mathfrak{R}$-cofree (weakly) curved $\mathfrak{A}$-modules. But arbitrary (other than just $\mathfrak{R}$-free or $\mathfrak{R}$-cofree) $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules or $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules are generally problematic.

0.23. To end, let us briefly discuss the motivation and possible applications. We are not in the position to suggest here any specific ways in which the techniques we are developing could be applied in Fukaya's Lagrangian Floer theory. In fact, the Novikov ring, which is the coefficient ring of the Floer–Fukaya theory, is *not* pro-Artinian (*nor* is it a topological local ring in our definition); so our results do not seem to be at present directly applicable.

Thus we restrict ourselves to stating that curved A_∞-algebras *do* seem to appear in the Floer–Fukaya business, and that their curvature (and change-of-connection) elements *do* seem to be, by the definition, divisible by appropriate maximal ideal(s) of the coefficient ring(s) [12, 13, 6]. Our study *does* imply that, generally speaking, quite nontrivial derived categories of modules can be associated with curved algebras of this kind. Working over a complete local ring, rather than over a field, is the price one has to pay for being able to obtain these derived categories of modules.

Furthermore, the semiderived categories of weakly curved DG- and A_∞-modules have all the usual properties of the derived categories of DG- and A_∞-modules over algebras over fields. The only caveat is that the nontriviality is *not guaranteed*: the triangulated categories of weakly curved modules may sometimes vanish when, on the basis of the experience with uncurved modules over algebras over fields, one would not expect them to (as it was noticed in [22]).

0.24. Another possible application has to do with the deformation theory of DG-algebras. As pointed out in [22], if one presumes that the deformations of DG-algebras should be controlled by their Hochschild cohomology complexes, one discovers that deformations in the class of CDG-algebras are to be considered on par with the conventional DG-algebra deformations.

A curved infinitesimal or formal deformation of a DG-algebra A over a field k is a wcDG-algebra $\mathfrak{A}$ over the ring $R = k[\epsilon]/\epsilon^2$ or $\mathfrak{R} = k[[\epsilon]]$, respectively. The problem of constructing deformations of the derived categories of DG-modules corresponding to curved deformations of DG-algebras was discussed in [22] (cf. the recent paper [7]).

Without delving into the implications of the deformation theory viewpoint, let us simply state that what seems to be a reasonable definition of the conventional derived category of wcDG-modules in the case of a pro-Artinian topological local ring $\mathfrak{R}$ of finite homological dimension is developed in this paper. So, at least, the case of a formal deformation may be (in some way) covered by our theory.

0.25. I am grateful to all the people who have been telling me about the curved A_∞-algebras appearing in the Floer–Fukaya theory throughout the recent years, and particularly Tony Pantev, Maxim Kontsevich, Alexander Kuznetsov, Dmitri Orlov, Anton Kapustin, and Andrei Losev. It would have never occured to me to consider curved algebras over local rings without their influence. I also wish to thank Bernhard Keller, Pedro Nicolás, and particularly Wendy Lowen for an interesting discussion of infinitesimal curved deformations of DG-algebras. Finally, I would like to thank Sergey Arkhipov for suggesting that tensor products of contramodules should be defined, B. Keller for directing my attention to Gabliel's pseudo-compact modules, Alexander Efimov for answering my numerous questions about the Fukaya theory, and Ed Segal for a conversation about curved A_∞-algebras over a field. The author was partially supported by the Simons Foundation grant and RFBR grants while working on this paper.

1. $\mathfrak{R}$-Contramodules and $\mathfrak{R}$-Comodules

1.1. **Topological rings.** Unless specified otherwise, all *rings* in this paper are assumed to be associative, commutative and unital. Some of the most basic of our results will be equally applicable to noncommutative rings.

Let $\mathfrak{R}$ be a topological ring in which open ideals form a base of neighborhoods of zero. We will always assume that $\mathfrak{R}$ is separated and complete; in other words, the natural map $\mathfrak{R} \longrightarrow \varprojlim_{\mathfrak{I}} \mathfrak{R}/\mathfrak{I}$, where the projective limit is taken over all open ideals $\mathfrak{I} \subset \mathfrak{R}$, is an isomorphism.

A *topological local ring* $\mathfrak{R}$ is a topological ring with a topologically nilpotent open ideal $\mathfrak{m}$ such that the quotient ring $\mathfrak{R}/\mathfrak{m}$ is a field. Here "topologically nilpotent" means that for any open ideal $\mathfrak{I} \subset \mathfrak{m}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{I}$. Clearly, a topological local ring is also local as an abstract ring. For example, the completion of any (discrete) ring by the powers of any of its maximal ideals is a topological local ring in the adic topology.

A topological ring is called *pro-Artinian* if its discrete quotient rings are Artinian rings. The projective limit of any filtered diagram of Artinian rings and surjective morphisms between them is a pro-Artinian ring (see Corollary A.2.1). For example, a complete Noetherian local ring $\mathfrak{R}$ with the maximal ideal $\mathfrak{m}$ is a pro-Artinian topological ring in the $\mathfrak{m}$-adic topology (cf. Appendix B). The dual vector space to any coassociative, cocommutative and counital coalgebra over a field is a pro-Artinian topological ring in its linearly compact topology.

1.2. **$\mathfrak{R}$-contramodules.** Given an (abstract) ring R, one can define R-modules in the following fancy way. For any set X, let $R[X]$ denote the set of all finite formal linear combinations of the elements of X with coefficients in R. The embedding $X \longrightarrow R[X]$ defined in terms of the zero and unit elements of R and the "opening of parentheses" map $R[R[X]] \longrightarrow R[X]$ make the functor $X \longmapsto R[X]$ a monad on the category of sets. The R-modules are the algebras/modules over this monad.

Now let $\mathfrak{R}$ be a topological ring. For any set X, let $\mathfrak{R}[[X]]$ be the set of all (infinite) formal linear combinations $\sum_{x\in X} r_x x$ of the elements of X with coefficients in $\mathfrak{R}$ such that the family of coefficients $r_x \in \mathfrak{R}$ converges to zero as "x goes to infinity". This means that for any open ideal $\mathfrak{I} \subset \mathfrak{R}$ the set of all $x \in X$ such that $r_x \notin \mathfrak{I}$ is finite.

There is the obvious embedding $\varepsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$. The "opening of parentheses" map $\rho_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ is defined by the rule

$$\textstyle\sum_{y\in\mathfrak{R}[[X]]} r_y y \longmapsto \sum_{x\in X}(\sum_y r_y r_{yx})x, \quad \text{where } y=\sum_{x\in\mathfrak{R}} r_{yx}x.$$

Here the infinite sum $\sum_{y\in\mathfrak{R}[[X]]} r_y r_{yx}$ converges in $\mathfrak{R}$, since the family r_y converges to zero and $\mathfrak{R}$ is complete. In fact, this construction is applicable to any noncommutative topological ring $\mathfrak{R}$ where open right ideals form a base of neighborhoods of zero [28, Remark A.3].

The above natural transformations $\varepsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$ and $\rho_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ define the structure of an (associative and unital) monad on the functor $X \longmapsto \mathfrak{R}[[X]]$. An $\mathfrak{R}$-*contramodule* is a module over this monad. In other words, an $\mathfrak{R}$-contramodule $\mathfrak{P}$ is a set endowed with a map of sets $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ satisfying the conventional associativity and unitality axioms $\pi_{\mathfrak{P}}\circ\mathfrak{R}[[\pi_{\mathfrak{P}}]] = \pi_{\mathfrak{P}}\circ\rho_{\mathfrak{P}}$ and $\pi_P\circ\varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$ for an algebra/module over a monad $X \longmapsto \mathfrak{R}[[X]]$. We call the map $\pi_{\mathfrak{P}}$ the *contraaction map* and the above associativity equation the *contraassociativity equation*. The category of $\mathfrak{R}$-contramodules is denoted by $\mathfrak{R}\text{–}\mathsf{contra}$.

The morphism of monads $\mathfrak{R}[X] \longrightarrow \mathfrak{R}[[X]]$ defines the (nontopological) $\mathfrak{R}$-module structure on the underlying set of every $\mathfrak{R}$-contramodule, so we have the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ from the category of $\mathfrak{R}$-contramodules to the abelian category of $\mathfrak{R}$-modules.

Equivalently, an $\mathfrak{R}$-contramodule can be defined as a set endowed with the following "infinite summation" operations. For any family of elements r_α converging to zero in $\mathfrak{R}$ and any family of elements $p_\alpha\in\mathfrak{P}$ (where α runs over some index set) there is a well-defined element $\sum_\alpha r_\alpha p_\alpha\in\mathfrak{P}$. These operations must satisfy the idenitities of unitality:

$$\textstyle\sum_\alpha r_\alpha p_\alpha = p_{\alpha_0} \quad \text{if the set } \{\alpha\} \text{ consists of one element } \alpha_0 \text{ and } r_{\alpha_0}=1,$$

associativity:

$$\textstyle\sum_\alpha r_\alpha\sum_\beta r_{\alpha\beta}p_{\alpha\beta} = \sum_{\alpha,\beta}(r_\alpha r_{\alpha\beta})p_{\alpha\beta} \quad \text{if } r_\alpha\to 0 \text{ and } \forall\alpha\ r_{\alpha\beta}\to 0 \text{ in } \mathfrak{R},$$

and distributivity:

$$\textstyle\sum_{\alpha,\beta} r_{\alpha\beta}p_\alpha = \sum_\alpha(\sum_\beta r_{\alpha\beta})p_\alpha \quad \text{if } r_{\alpha\beta}\to 0 \text{ in } \mathfrak{R}.$$

Here the summation over α,β presumes a set of pairs $\{(\alpha,\beta)\}$ mapping into another set by $(\alpha,\beta)\longmapsto\alpha$ (i. e., the range of possible β's may depend on a chosen α).

The finite and infinite operations are compatible in the sense of the equations

$$\textstyle\sum_\alpha r_\alpha(p_\alpha+q_\alpha) = \sum_\alpha r_\alpha p_\alpha + \sum_\alpha r_\alpha q_\alpha, \quad \sum_\alpha(r'_\alpha+r''_\alpha)p_\alpha = \sum_\alpha r'_\alpha p_\alpha + \sum_\alpha r''_\alpha p_\alpha,$$

and

$$\textstyle\sum_\alpha r_\alpha(s_\alpha p_\alpha) = \sum_\alpha(r_\alpha s_\alpha)p_\alpha, \quad \sum_\alpha(sr_\alpha)p_\alpha = s\sum_\alpha r_\alpha p_\alpha$$

for p_α, $q_\alpha\in\mathfrak{P}$ and r_α, r'_α, r''_α, s, $s_\alpha\in\mathfrak{R}$ with r_α, r'_α, r''_α converging to zero. Using these identities, one can define the $\mathfrak{R}$-contramodule structures on the kernel and cokernel of an $\mathfrak{R}$-contramodule morphism $\mathfrak{P}\longrightarrow\mathfrak{Q}$ taken in the category of $\mathfrak{R}$-modules.

Hence $\mathfrak{R}$–contra is an abelian category and $\mathfrak{R}$–$\mathsf{contra} \longrightarrow \mathfrak{R}$–$\mathsf{mod}$ is an exact functor. One also easily checks that infinite products exist in $\mathfrak{R}$–contra and the forgetful functor to $\mathfrak{R}$–mod preserves them.

For the reasons common to all monads, for any set X the set $\mathfrak{R}[[X]]$ has a natural $\mathfrak{R}$-contramodule structure. The functor taking a set X to the $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ is left adjoint to the forgetful functor from $\mathfrak{R}$–contra to the category of sets. We call contramodules of the form $\mathfrak{R}[[X]]$ *free $\mathfrak{R}$-contramodules*. Free contramodules are projective objects in the abelian category of $\mathfrak{R}$-contramodules, there are enough of them, and hence every projective $\mathfrak{R}$-contramodule is a direct summand of a free $\mathfrak{R}$-contramodule.

For any collection of sets X_α, the free contramodule $\mathfrak{R}[[\coprod_\alpha X_\alpha]]$ generated by the disjoint union of X_α is the direct sum of the free contramodules $\mathfrak{R}[[X_\alpha]]$ in the category $\mathfrak{R}$–contra. This allows to compute, at least in principle, the direct sum of every collection of $\mathfrak{R}$-contramodules, by presenting them as cokernels of morphisms of free contramodules and using the fact that infinite direct sums commute with cokernels. So infinite direct sums exist in $\mathfrak{R}$–contra.

Remark 1.2.1. The functors of infinite direct sum are not exact in $\mathfrak{R}$–contra in general; in fact, one can check that they are not exact already for the ring $\mathfrak{R} = k[[z,t]]$ of formal power series in two variables over a field k. However, when the abelian category of $\mathfrak{R}$-contramodules has homological dimension not exceeding 1, the infinite direct sums in $\mathfrak{R}$–contra *are* exact.

Indeed, the derived functor of infinite direct sum in $\mathfrak{R}$–contra can be defined and computed using left projective resolutions. Now if every subcontramodule of a projective contramodule is projective, it remains to check that the direct sum of a family of injective morphisms of projective contramodules is an injective morphism. This follows from the fact that the natural map from the direct sum of a family of projective contramodules to their direct product is injective.

The latter assertion holds for contramodules over any topological ring $\mathfrak{R}$. It suffices to prove it for free contramodules, for which it can be checked in terms of the explicit constructions of all the objects involved.

1.3. **Nakayama's lemma.** Topological rings without units and contramodules over them are defined in the way similar to the above definitions, and so is the property of *topological nilpotence* of a topological ring without unit.

The following result is a generalization of [19, Lemma 4.11] (cf. Appendix B).

Lemma 1.3.1. *Let $\mathfrak{m}$ be a topologically nilpotent topological ring without unit, and $\mathfrak{P}$ be a nonzero $\mathfrak{m}$-contramodule. Then the image of the contraaction map $\pi_\mathfrak{P}\colon \mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ differs from $\mathfrak{P}$.*

Proof. The following proof does not depend on the commutativity assumption on $\mathfrak{m}$ (cf. [28, Lemma A.2.1 and Remark A.3]). Assume that the map $\pi = \pi_\mathfrak{P}$ is surjective; let $p \in \mathfrak{P}$ be an element. Notice that for any surjective map of sets $f\colon X \longrightarrow Y$, the induced map $\mathfrak{m}[[f]]\colon \mathfrak{m}[[X]] \longrightarrow \mathfrak{m}[[Y]]$ is surjective.

Define inductively $\mathfrak{m}^{(0)}[[X]] = X$ and $\mathfrak{m}^{(n)}[[X]] = \mathfrak{m}[[\mathfrak{m}^{(n-1)}[[X]]]]$ for $n \geqslant 1$. Let $p_1 \in \mathfrak{m}[[\mathfrak{P}]]$ be a preimage of p under the map π. Furthermore, let $p_n \in \mathfrak{m}^{(n)}[[\mathfrak{P}]]$ be a preimage of p_{n-1} under the map $\mathfrak{m}^{(n-1)}[[\pi]]$.

For any set X, let $\rho_X^{(n)} \colon \mathfrak{m}^{(n)}[[X]] \longrightarrow \mathfrak{m}[[X]]$ denote the iterated monad multiplication/contraaction map. The abelian group $\mathfrak{m}[[X]]$ is complete in its natural topology with the base of neighborhoods of zero formed by the subgroups $I[[X]]$, where $I \subset \mathfrak{m}$ are open ideals. Besides, the map $\rho_X \colon \mathfrak{m}[[\mathfrak{m}[[X]]]] \longrightarrow \mathfrak{m}[[X]]$ is continuous.

Set $q_n = \rho_{\mathfrak{m}[[\mathfrak{P}]]}^{(n-1)}(p_n) \in \mathfrak{m}^{(2)}[[\mathfrak{P}]]$ for all $n \geqslant 2$. Since $\mathfrak{m}$ is topologically nilpotent, the sum $\sum_n q_n$ converges in the topology of $\mathfrak{m}[[\mathfrak{m}[[\mathfrak{P}]]]]$. Now we have $\mathfrak{m}[[\pi]](q_n) = \rho_{\mathfrak{P}}(q_{n-1})$ for all $n \geqslant 3$ and $\mathfrak{m}[[\pi]](q_2) = p_1$. Hence

$$\mathfrak{m}[[\pi]](\textstyle\sum_{n=2}^{\infty} q_n) - \rho_{\mathfrak{P}}(\textstyle\sum_{n=2}^{\infty} q_n) = p_1$$

and $p = \pi_{\mathfrak{P}}(p_1) = 0$ by the contraassociativity equation. □

Let $\mathfrak{R}$ be a topological local ring with the maximal ideal k. Denote by $k = \mathfrak{R}/\mathfrak{m}$ the residue field. Assign to any $\mathfrak{R}$-contramodule $\mathfrak{P}$ its quotient group $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ by the image $\mathfrak{m}\mathfrak{P}$ of the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. Then $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ is a vector space over k. In particular, for a free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ we obtain the vector space $\mathfrak{R}[[X]]/\mathfrak{m}(\mathfrak{R}[[X]]) = k[X]$ with the basis X over k.

Lemma 1.3.2. *Over a topological local ring $\mathfrak{R}$, the classes of free and projective contramodules coincide.*

Proof. Let $\mathfrak{P}$ be a projective $\mathfrak{R}$-contramodule. Picking a basis X in the vector space $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$, we obtain an $\mathfrak{R}$-contramodule morphism $\mathfrak{R}[[X]] \longrightarrow \mathfrak{P}/\mathfrak{m}\mathfrak{P}$, where $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ is endowed with an $\mathfrak{R}$-contramodule structure induced by its k-vector space structure. Since the $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ is projective, this morphism can be lifted to an $\mathfrak{R}$-contramodule morphism $f \colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{P}$ (it suffices to choose preimages in $\mathfrak{P}$ for all elements of X). By Lemma 1.3.1, the cokernel of the morphism f is a zero contramodule, so f is surjective. Since $\mathfrak{P}$ is projective, it follows that f is a projection onto a direct summand in the abelian category of $\mathfrak{R}$-contramodules, so $\mathfrak{R}[[X]] = \mathfrak{P} \oplus \mathfrak{Q}$. Then $\mathfrak{Q}/\mathfrak{m}\mathfrak{Q} = 0$ and it remains to apply Lemma 1.3.1 again in order to conclude that $\mathfrak{Q} = 0$. □

Lemma 1.3.3. *Let $\mathfrak{K}^\bullet$ be a (possibly unbounded) complex of free contramodules over a topological local ring $\mathfrak{K}$ with the maximal ideal $\mathfrak{m}$. Then the complex $\mathfrak{K}^\bullet$ is contractible if and only if the complex of vector spaces $\mathfrak{K}^\bullet/\mathfrak{m}\mathfrak{K}^\bullet$ is contractible (i. e., acyclic).*

Proof. Choose a contracting homotopy for $\mathfrak{K}^\bullet/\mathfrak{m}\mathfrak{K}^\bullet$ and lift it to a homotopy map h on $\mathfrak{K}^\bullet$ using projectivity of the contramodules $\mathfrak{K}^i$. Then the endomorphism $dh + hd$ of the component $\mathfrak{K}^i$ is an identity map modulo $\mathfrak{m}\mathfrak{K}^i$. It remains to show that any morphism f of free $\mathfrak{R}$-contramodules such that the induced map $f/\mathfrak{m}f$ is an isomorphism of vectors spaces is an isomorphism of $\mathfrak{R}$-contramodules. The proof of this assertion repeats the above proof of Lemma 1.3.2. □

Remark 1.3.4. The results below in this section are applicable to any noncommutative right pro-Artinian topological ring (i. e., a filtered projective limit of noncommutative rings with surjective morphisms between them). However, we will only prove them for a pro-Artinian topological local ring here, using the above version of Nakayama's lemma as the main technical tool. The general case of a pro-Artinian commutative ring can be deduced by decomposing such a ring $\mathfrak{R}$ as an infinite product of topological local rings $\mathfrak{R}_\alpha$ and any contramodule over $\mathfrak{R}$ as the infinite product of contramodules over $\mathfrak{R}_\alpha$ (see [28, Lemma A.1.2]). In the noncommutative case, one proceeds by lifting the primitive idempotents of the maximal prosemisimple quotient ring of $\mathfrak{R}$ to a converging family of idempotents in $\mathfrak{R}$, etc. We do not go into that here, because only local rings are important for our purposes.

Let $\mathfrak{R}$ be a pro-Artinian (topological local) ring and $\mathfrak{J} \subset \mathfrak{R}$ be a closed ideal. By Corollary A.2, the topological ring $\mathfrak{R}/\mathfrak{J}$ is complete. Moreover, for any set X the natural map $\mathfrak{R}[[X]] \longrightarrow \mathfrak{R}/\mathfrak{J}[[X]]$ is surjective, since it is a morphism of $\mathfrak{R}$-contramodules that is surjective modulo $\mathfrak{m}$. Hence we have the exact sequence $0 \longrightarrow \mathfrak{J}[[X]] \longrightarrow \mathfrak{R}[[X]] \longrightarrow \mathfrak{R}/\mathfrak{J}[[X]] \longrightarrow 0$.

To any $\mathfrak{R}$-contramodule $\mathfrak{P}$, one can assign its quotient group $\mathfrak{P}/\mathfrak{J}\mathfrak{P}$ by the image $\mathfrak{J}\mathfrak{P}$ of the contraaction map $\mathfrak{J}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. Then $\mathfrak{P}/\mathfrak{J}\mathfrak{P}$ is a contramodule over the quotient ring $\mathfrak{R}/\mathfrak{J}$; it is the maximal quotient contramodule of $\mathfrak{P}$ that is a contramodule over $\mathfrak{R}/\mathfrak{J}$. In other words, the functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{J}\mathfrak{P}$ is left adjoint to the functor of contrarestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ (sending an $\mathfrak{R}/\mathfrak{J}$-contramodule $\mathfrak{Q}$ to the same set $\mathfrak{Q}$ considered as an $\mathfrak{R}$-contramodule).

Hence the functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{J}\mathfrak{P}$ is right exact and commutes with infinite direct sums. One easily checks that it takes the free contramodule $\mathfrak{R}[[X]]$ to the free contramodule $\mathfrak{R}/\mathfrak{J}[[X]]$.

In particular, for an open ideal $\mathfrak{I} \subset \mathfrak{R}$ the functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{I}\mathfrak{P}$ takes values in the category of $\mathfrak{R}/\mathfrak{I}$-modules. This functor is well-defined for any topological ring $\mathfrak{R}$.

Lemma 1.3.5. *Let $\mathfrak{R}$ be a topological local ring. Then an $\mathfrak{R}$-contramodule $\mathfrak{P}$ is projective if and only if the $\mathfrak{R}/\mathfrak{I}$-module $\mathfrak{P}/\mathfrak{I}\mathfrak{P}$ is projective for every open ideal $\mathfrak{I} \subset \mathfrak{R}$.*

Proof. The "only if" assertion is obvious; let us prove the "if". Pick a basis X in the k-vector space $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ and consider the free $\mathfrak{R}$-contramodule $\mathfrak{F} = \mathfrak{R}[[X]]$. Lifting the elements of X into $\mathfrak{P}$, we construct a morphism of $\mathfrak{R}$-contramodules $\mathfrak{F} \longrightarrow \mathfrak{P}$ such that the induced morphism of k-vector spaces $\mathfrak{F}/\mathfrak{m}\mathfrak{F} \longrightarrow \mathfrak{P}/\mathfrak{m}\mathfrak{P}$ is an isomorphism.

By Lemma 1.3.1, the morphism $\mathfrak{F} \longrightarrow \mathfrak{P}$ is surjective. For any open ideal $\mathfrak{I} \subset \mathfrak{R}$, consider the morphism of projective/free $\mathfrak{R}/\mathfrak{I}$-modules $\mathfrak{F}/\mathfrak{I}\mathfrak{F} \longrightarrow \mathfrak{P}/\mathfrak{m}\mathfrak{P}$. This morphism is split, so for its kernel K we have $K/(\mathfrak{m}/\mathfrak{I})K = 0$, hence $K = 0$.

Now consider the kernel $\mathfrak{K}$ of the morphism $\mathfrak{F} \longrightarrow \mathfrak{P}$. We have shown that $\mathfrak{K}$ is contained in $\mathfrak{I}\mathfrak{F}$ as a subcontramodule of $\mathfrak{F}$, for any open ideal $\mathfrak{I} \subset \mathfrak{R}$. The $\mathfrak{R}$-contramodule $\mathfrak{F}$ being free and the intersection of all open ideals $\mathfrak{I}$ in $\mathfrak{R}$ being zero, the intersection of all subcontramodules $\mathfrak{I}\mathfrak{F}$ in $\mathfrak{F}$ is also zero; thus $\mathfrak{K} = 0$. (Cf. [28, Lemma A.3 and Remark A.3].) $\square$

Lemma 1.3.6. *For any closed ideal $\mathfrak{J}$ in a pro-Artinian ring $\mathfrak{R}$, the functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{J}\mathfrak{P}$ from the category of $\mathfrak{R}$-contramodules to the category of $\mathfrak{R}/\mathfrak{J}$-contramodules preserves infinite products.*

Proof. As mentioned above (see Remark 1.3.4), we will assume that $\mathfrak{R}$ is a topological local ring. Presenting arbitrary contramodules $\mathfrak{P}_\alpha$ as the cokernels of morphisms of free $\mathfrak{R}$-contramodules, we reduce the question to the case of the product of free $\mathfrak{R}$-contramodules $\mathfrak{P}_\alpha = \mathfrak{R}[[X_\alpha]]$. Set $\mathfrak{Q} = \prod_\alpha \mathfrak{R}[[X_\alpha]]$.

Pick a basis X in the product of vector spaces $\prod_\alpha k[X_\alpha]$. The morphism of $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]] \longrightarrow k[X] \longrightarrow \prod k[[X_\alpha]]$ can be lifted to a morphism of $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]] \longrightarrow \prod_\alpha \mathfrak{R}[[X_\alpha]]$. For any open ideal $\mathfrak{I} \subset \mathfrak{R}$, the composition $\mathfrak{R}[[X]] \longrightarrow \prod_\alpha \mathfrak{R}[[X_\alpha]] \longrightarrow \prod_\alpha \mathfrak{R}/\mathfrak{I}[X_\alpha]$ factorizes through $\mathfrak{R}/\mathfrak{I}[X]$. The map $\mathfrak{R}[[X]] \longrightarrow \prod_\alpha \mathfrak{R}[[X_\alpha]]$ is the projective limit of the maps $\mathfrak{R}/\mathfrak{I}[X] \longrightarrow \prod_a \mathfrak{R}/\mathfrak{I}[X_\alpha]$ taken over the open ideals $\mathfrak{I}$.

For any Artinian local ring R with the maximal ideal m and any collection of R-modules P_α one has $(\prod_\alpha P_\alpha)/m(\prod_\alpha P_\alpha) \simeq \prod_\alpha P_\alpha/mP_\alpha$, since the ideal m is finitely generated. Moreover, the product of projective modules over a right Artinian ring is projective [2, 5]. Hence the morphism of free $\mathfrak{R}/\mathfrak{I}$-modules $\mathfrak{R}/\mathfrak{I}[X] \longrightarrow \prod_a \mathfrak{R}/\mathfrak{I}[X_\alpha]$, being an isomorphism modulo m, is itself an isomorphism.

Passing to the projective limit over all open ideals $\mathfrak{I} \subset \mathfrak{R}$, we conclude that the map $\mathfrak{R}[[X]] \longrightarrow \prod_\alpha \mathfrak{R}[[X_\alpha]]$ is an isomorphism. Similarly, passing to the projective limit over the open ideals $\mathfrak{I}$ containing $\mathfrak{J}$, we have the isomorphism $\mathfrak{R}/\mathfrak{J}[[X]] \simeq \prod_\alpha \mathfrak{R}/\mathfrak{J}[[X_\alpha]]$. The two isomorphisms form a commutative square with the natural surjections. It remains to recall that $\mathfrak{R}[[X]]/\mathfrak{J}(\mathfrak{R}[[X]]) = \mathfrak{R}/\mathfrak{J}[[X]]$. ☐

Along the way, we have also proven the next lemma.

Lemma 1.3.7. *The class of projective contramodules over a pro-Artinian ring $\mathfrak{R}$ is closed under infinite products.* ☐

Notice that any projective $\mathfrak{R}$-contramodule is a direct summand of an infinite product of copies of the $\mathfrak{R}$-contramodule $\mathfrak{R}$, as one can immediately see from the above proof.

Remark 1.3.8. The functor of contrarestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ does *not* preserve infinite direct sums in general. It suffices to consider the case when $\mathfrak{m}^2 = 0$ and $\mathfrak{m} = \mathfrak{J}$ is an infinite dimensional (linearly compact) k-vector space. However, when $\mathfrak{R}$ is a Noetherian local ring with the $\mathfrak{m}$-adic topology, the functor of contrarestriction of scalars from $\mathfrak{R}/\mathfrak{J}$ to $\mathfrak{R}$ does preserve infinite direct sums, and moreover, has a right adjoint functor, constructed as follows. Given an $\mathfrak{R}$-contramodule $\mathfrak{P}$, one can consider the subset ${}_\mathfrak{J}\mathfrak{P} = \mathrm{Hom}^\mathfrak{R}(\mathfrak{R}/\mathfrak{J}, \mathfrak{P}) \subset \mathfrak{P}$ of all elements annihilated by $\mathfrak{J}$ acting in $\mathfrak{P}$ viewed as an $\mathfrak{R}$-module. This is clearly an $\mathfrak{R}$-subcontramodule of $\mathfrak{P}$; in the above Noetherian assumption, one can check that the contraaction map $\mathfrak{J}[[{}_\mathfrak{J}\mathfrak{P}]] \longrightarrow {}_\mathfrak{J}\mathfrak{P}$ vanishes.

1.4. **$\mathfrak{R}$-comodules.** Let $\mathfrak{R}$ be a pro-Artinian topological ring. A *comodule* over $\mathfrak{R}$ is an ind-object of the abelian category opposite to the category of discrete $\mathfrak{R}$-modules of finite length. By the definition, $\mathfrak{R}$-comodules form a locally Noetherian (and even locally finite) Grothendieck abelian category.

Comodules over pro-Artinian rings have no underlying groups or sets; however, there is a natural contravariant functor $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$ assigning a pro-object in the category of discrete $\mathfrak{R}$-modules of finite length to every $\mathfrak{R}$-comodule. An exact, conservative functor of projective limit acts from the latter category to the category of abelian groups (see Corollary A.3.1). In fact, for any $\mathfrak{R}$-comodule $\mathcal{M}$ the abelian group $\mathcal{M}^{\mathrm{op}}$ has a natural $\mathfrak{R}$-contramodule structure.

We denote the category of $\mathfrak{R}$-comodules by $\mathfrak{R}\text{–}\mathsf{comod}$. There is a distinguished object $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ in this category such that $\mathcal{C}^{\mathrm{op}} = \mathfrak{R}$; for any $\mathfrak{R}$-comodule $\mathcal{M}$, the $\mathfrak{R}$-comodule morphisms $\mathcal{M} \longrightarrow \mathcal{C}$ correspond bijectively to the elements of $\mathcal{M}^{\mathrm{op}}$. Hence $\mathcal{C}(\mathfrak{R})$ is an injective cogenerator of the category $\mathfrak{R}\text{–}\mathsf{comod}$. Since this category is locally Noetherian, direct sums of copies of the object $\mathcal{C}$ are also injective. We will call these *cofree* $\mathfrak{R}$-comodules. One can easily see that there are enough of them, so any injective $\mathfrak{R}$-comodule is a direct summand of a cofree one.

As any Grothendieck abelian category, the category $\mathfrak{R}\text{–}\mathsf{comod}$ has arbitrary infinite products. One can describe them in the following way.

Given a closed ideal $\mathfrak{J} \subset \mathfrak{R}$, we assign to any $\mathfrak{R}$-comodule $\mathcal{M}$ its maximal subcomodule ${}_{\mathfrak{J}}\mathcal{M}$ that is a comodule over $\mathfrak{R}/\mathfrak{J}$. In other words, the functor $\mathcal{M} \longmapsto {}_{\mathfrak{J}}\mathcal{M}$ is right adjoint to the functor of corestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$ (induced on the ind-objects by the functor sending a discrete $\mathfrak{R}/\mathfrak{J}$-module of finite length to the same abelian group considered as an $\mathfrak{R}$-module).

Hence the functor $\mathcal{M} \longmapsto {}_{\mathfrak{J}}\mathcal{M}$ is left exact and preserves infinite products. It follows that $\prod_\alpha \mathcal{M}_\alpha = \varinjlim_{\mathfrak{J}} \prod_\alpha {}_{\mathfrak{J}}\mathcal{M}_\alpha$ for any family of objects $\mathcal{M}_\alpha \in \mathfrak{R}\text{–}\mathsf{comod}$, where the inductive limit is taken over all open ideals $\mathfrak{J} \subset \mathfrak{R}$. As to the functors of infinite product in the abelian category of comodules over an Artinian ring $R = \mathfrak{R}/\mathfrak{J}$, these are exact, since the category $R\text{–}\mathsf{comod}$ has enough projectives (because the category of R-modules of finite length has enough injectives).

Clearly, the functor $\mathcal{M} \longmapsto {}_{\mathfrak{J}}\mathcal{M}$ also commutes with all filtered inductive limits. For any closed ideal $\mathfrak{J} \subset \mathfrak{R}$, one has ${}_{\mathfrak{J}}\mathcal{C}(\mathfrak{R}) = \mathcal{C}(\mathfrak{R}/\mathfrak{J})$. Conversely, an $\mathfrak{R}$-comodule $\mathcal{J}$ is injective if and only if the $\mathfrak{R}/\mathfrak{J}$-comodule ${}_{\mathfrak{J}}\mathcal{J}$ is injective for every open ideal $\mathfrak{J} \subset \mathfrak{R}$. We use the conventional duality on the category of finite-dimensional k-vector spaces in order to identify comodules over the field k (considered as a discrete topological ring) with (possibly infinite-dimensional) k-vector spaces.

Lemma 1.4.1. *Let $\mathfrak{R}$ be a pro-Artinian topological local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k = \mathfrak{R}/\mathfrak{m}$, and let $\mathcal{M}$ be a nonzero $\mathfrak{R}$-comodule. Then the k-vector space ${}_{\mathfrak{m}}\mathcal{M}$ is nonzero.*

Proof. One can present the ind-object $\mathcal{M}$ as the inductive limit of a filtered diagram of $\mathfrak{R}$-comodules $\mathcal{M}_\alpha$ of finite length and injective morphisms $\mathcal{M}_\alpha \longrightarrow \mathcal{M}_\beta$ between them. Then the maps of k-vector spaces ${}_{\mathfrak{m}}(\mathcal{M}_\alpha) \longrightarrow {}_{\mathfrak{m}}(\mathcal{M}_\beta)$ are also injective, hence so are the maps ${}_{\mathfrak{m}}(\mathcal{M}_\alpha) \longrightarrow {}_{\mathfrak{m}}\mathcal{M}$. It remains to use the fact that for any nonzero module (of finite length) M over an Artinian local ring R with the maximal ideal m, the quotient module/vector space M/mM is nonzero. $\square$

Lemma 1.4.2. *Over a pro-Artinian topological local ring $\mathfrak{R}$, the classes of cofree and injective comodules coincide.*

Lemma 1.4.3. *Let $\mathcal{K}^\bullet$ be a (possibly unbounded) complex of cofree comodules over a pro-Artinian local ring $\mathfrak{R}$ with the maximal ideal $\mathfrak{m}$ and the residue field $k = \mathfrak{R}/\mathfrak{m}$. Then the complex $\mathcal{K}^\bullet$ is contractible if and only if the complex of k-vector spaces ${}_{\mathfrak{m}}\mathcal{K}^\bullet$ is contractible (i. e., acyclic).*

Proofs of Lemmas 1.4.2 *and* 1.4.3. Completely analogous to the proofs of Lemmas 1.3.2 and 1.3.3, up to the duality and with the use of Lemma 1.3.1 replaced by that of Lemma 1.4.1. □

Remark 1.4.4. The functor of corestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$ has also a left adjoint functor $\mathcal{M} \longmapsto \mathcal{M}/\mathfrak{J}\mathcal{M}$. The latter is defined on comodules of finite length by the rule $(\mathcal{M}/\mathfrak{J}\mathcal{M})^{\mathrm{op}} = {}_{\mathfrak{J}}(\mathcal{M}^{\mathrm{op}})$, where ${}_{\mathfrak{J}}M$ denotes the maximal submodule annihilated by $\mathfrak{J}$ in a discrete $\mathfrak{R}$-module M, and extended to arbitrary $\mathfrak{R}$-comodules as a functor preserving inductive limits. It follows that the functor of corestriction of scalars from $\mathfrak{R}/\mathfrak{J}$ to $\mathfrak{R}$ preserves infinite products.

1.5. **Hom and contratensor product.** For any topological ring $\mathfrak{R}$ and any two $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, the set $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})$ of all $\mathfrak{R}$-contamodule homomorphisms $\mathfrak{P} \longrightarrow \mathfrak{Q}$ has a natural $\mathfrak{R}$-contramodule structure with the infinite summation operation $(\sum_\alpha r_\alpha f_\alpha)(x) = \sum_\alpha r_\alpha f_\alpha(x)$, where $f_\alpha \in \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})$ are any contramodule homomorphisms, $r_\alpha \in \mathfrak{R}$ is a family of elements converging to zero, the infinite sum in the left hand side belongs to $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})$, and the infinite sum in the right hand side is taken in $\mathfrak{Q}$.

In particular, for any set X and any $\mathfrak{R}$-contramodule $\mathfrak{Q}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{Q}) \simeq \prod_{x\in X} \mathfrak{Q}$. Hence it follows from Lemma 1.3.7 that the functor $\mathrm{Hom}^{\mathfrak{R}}(-,-)$ preserves the class of projective contramodules over a pro-Artinian topological ring $\mathfrak{R}$. Clearly, the functor $\mathrm{Hom}^{\mathfrak{R}}$ is left exact and takes infinite direct sums of contramodules in the first argument and infinite products in the second argument to infinite products; it also becomes exact when a projective contramodule is substituted as the first argument.

For any closed ideal $\mathfrak{J} \subset \mathfrak{R}$ and any two $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, there is a natural morphism of $\mathfrak{R}/\mathfrak{J}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})/\mathfrak{J}\,\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}/\mathfrak{J}}(\mathfrak{P}/\mathfrak{J}\mathfrak{P}, \mathfrak{Q}/\mathfrak{J}\mathfrak{Q})$. By Lemma 1.3.6, this morphism is an isomorphism whenever the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is projective. Besides, for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}/\mathfrak{J}$-contramodule $\mathfrak{Q}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q}) \simeq \mathrm{Hom}^{\mathfrak{R}/\mathfrak{J}}(\mathfrak{P}/\mathfrak{J}\mathfrak{P}, \mathfrak{Q})$, where the $\mathfrak{R}$-contramodule structures on $\mathfrak{Q}$ and the Hom contramodule in the right hand side is obtained from the $\mathfrak{R}/\mathfrak{J}$-contramodule structures using the functor of contrarestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$.

For any pro-Artinian topological ring $\mathfrak{R}$ and any $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$, the set $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N})$ of all $\mathfrak{R}$-comodule morphisms $\mathcal{M} \longrightarrow \mathcal{N}$ has a natural $\mathfrak{R}$-contramodule structure, which is constructed as follows. If $\mathcal{M} = \varinjlim_\alpha \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ are $\mathfrak{R}$-comodules of finite length, then $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N}) \simeq \varprojlim_\alpha \mathrm{Hom}_{\mathfrak{R}}(\mathcal{M}_\alpha, \mathcal{N})$ and the

$\mathfrak{R}$-contramodule structure on the left hand side of this isomorphism is defined as the projective limit of the contramodule structures on the Hom groups in the right hand side (recall that the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ preserves projective limits). Now $\mathcal{M}_\alpha$ is a module of finite length over a discrete Artinian quotient ring $R_\alpha = \mathfrak{R}/\mathfrak{I}_\alpha$ of the topological ring $\mathfrak{R}$, and $\operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}_\alpha, \mathcal{N}) \simeq \operatorname{Hom}_{R_\alpha}(\mathcal{M}_\alpha, {}_{\mathfrak{I}_\alpha}\mathcal{N})$ is an R_α-module, hence also an $\mathfrak{R}$-contramodule.

Clearly, the functor $\operatorname{Hom}_{\mathfrak{R}}$ is left exact and assigns infinite products of contramodules to infinite direct sums of comodules in the first argument and infinite products of comodules in the second argument. Substituting $\mathcal{N} = \mathcal{C}(\mathfrak{R})$, we obtain the contravariant functor $\mathfrak{R}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ taking $\mathcal{M}$ to $\mathcal{M}^{\mathrm{op}}$ that was introduced in Section 1.4.

For any closed ideal $\mathfrak{J} \subset \mathfrak{R}$ and two $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$, there is a natural morphism of $\mathfrak{R}/\mathfrak{J}$-contramodules $\operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N})/\mathfrak{J} \operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N}) \longrightarrow \operatorname{Hom}_{\mathfrak{R}}({}_{\mathfrak{J}}\mathcal{M}, {}_{\mathfrak{J}}\mathcal{N})$. This morphism is an isomorphism whenever the $\mathfrak{R}$-comodule $\mathcal{N}$ is injective. Indeed, the case $\mathcal{M} = \mathcal{N} = \mathcal{C}(\mathfrak{R})$ is obvious; when both $\mathcal{M}$ and $\mathcal{N}$ are cofree $\mathfrak{R}$-comodules, the assertion follows from Lemma 1.3.6. Finally, since the functor $\mathcal{N} \longmapsto {}_{\mathfrak{J}}\mathcal{N}$ preserves injectivity, the assertion for an injective comodule $\mathcal{N}$ and an arbitrary comodule $\mathcal{M}$ can be deduced by presenting $\mathcal{M}$ as the kernel of a morphism of injective comodules. Besides, for any $\mathfrak{R}/\mathfrak{J}$-comodule $\mathcal{M}$ and $\mathfrak{R}$-comodule $\mathcal{N}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N}) \simeq \operatorname{Hom}_{\mathfrak{R}/\mathfrak{J}}(\mathcal{M}, {}_{\mathfrak{J}}\mathcal{N})$, where the $\mathfrak{R}$-comodule structure on $\mathcal{M}$ and the $\mathfrak{R}$-contramodule structure on the Hom contramodule in the right hand side are defined using the functors of contra- and corestriction of scalars.

For any pro-Artinian topological ring $\mathfrak{R}$, an $\mathfrak{R}$-contramodule $\mathfrak{P}$, and an $\mathfrak{R}$-comodule $\mathcal{M}$, the *contratensor product* $\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M}$ is an $\mathfrak{R}$-comodule defined by the rules

(i) $\mathfrak{P} \odot_{\mathfrak{R}} \varinjlim_\alpha \mathcal{M}_\alpha = \varinjlim_\alpha \mathfrak{P}/\mathfrak{I}_\alpha\mathfrak{P} \odot_{R_\alpha} \mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ is a comodule over a discrete Artinian quotient ring $R_\alpha = \mathfrak{R}/\mathfrak{I}_\alpha$ of the topological ring $\mathfrak{R}$;

(ii) for any Artinian ring R, the functor $\odot_R$ commutes with inductive limits in both arguments; and

(iii) $(Q \odot_R \mathcal{N})^{\mathrm{op}} = \operatorname{Hom}_R(Q, \mathcal{N}^{\mathrm{op}})$ for any module Q and any module of finite length $\mathcal{N}^{\mathrm{op}}$ over an Artinian ring R.

Clearly, the functor $\odot_{\mathfrak{R}}$ is right exact and commutes with infinite direct sums in both arguments. It is also obvious that there is a natural isomorphism of $\mathfrak{R}$-comodules $\mathfrak{R} \odot_{\mathfrak{R}} \mathcal{M} \simeq \mathcal{M}$ for any $\mathfrak{R}$-comodule $\mathcal{M}$. A natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M}, \mathcal{N}) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, \operatorname{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N}))$ can easily be constructed for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$. In particular, there is a natural isomorphism of $\mathfrak{R}$-contramodules $(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M})^{\mathrm{op}} \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathcal{M}^{\mathrm{op}})$ for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$.

Given a closed ideal $\mathfrak{J} \subset \mathfrak{R}$, for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$ there is a natural morphism of $\mathfrak{R}/\mathfrak{J}$-comodules $\mathfrak{P}/\mathfrak{J}\mathfrak{P} \odot_{\mathfrak{R}/\mathfrak{J}} {}_{\mathfrak{J}}\mathcal{M} \simeq \mathfrak{P} \odot_{\mathfrak{R}} {}_{\mathfrak{J}}\mathcal{M} \longrightarrow {}_{\mathfrak{J}}(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M})$, which is an isomorphism when $\mathfrak{P}$ is a projective $\mathfrak{R}$-contramodule. Here the contratensor product over $\mathfrak{R}$ in the middle term is identified with the $\mathfrak{R}/\mathfrak{J}$-comodule from which its $\mathfrak{R}$-comodule structure is obtained by the corestriction of scalars.

Let $\mathcal{N}$ be an $\mathfrak{R}$-comodule. It follows from the above assertion that the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$ defined by the rule $\mathfrak{P} \longmapsto \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{P}$ is left adjoint to the functor $\mathfrak{R}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ defined by the rule $\mathcal{M} \longmapsto \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{M})$. For any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$, set $\Phi_{\mathfrak{R}}(\mathfrak{P}) = \mathcal{C}(\mathfrak{R}) \odot_{\mathfrak{R}} \mathfrak{P}$ and $\Psi_{\mathfrak{R}}(\mathcal{M}) = \mathrm{Hom}^{\mathfrak{R}}(\mathcal{C}(\mathfrak{R}), \mathcal{M})$.

The following result is a generalization of [16, Proposition 2.1] and [4, VI.4.5].

Proposition 1.5.1. *The functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ restrict to mutually inverse equivalences between the additive categories of free $\mathfrak{R}$-contramodules and cofree $\mathfrak{R}$-comodules.*

Proof. Clearly, the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ take the contramodule $\mathfrak{R}$ to the comodule $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ and back; the functor $\Phi_{\mathfrak{R}}$ preserves infinite direct sums and the functor $\Psi_{\mathfrak{R}}$ preserves infinite products. It follows that the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ take projective contramodules to injective comodules and back. It remains to check, e. g., that the functor $\Psi_{\mathfrak{R}}$ preserves infinite direct sums of injective comodules.

Indeed, a morphism of projective contramodules $\mathfrak{P} \longrightarrow \mathfrak{Q}$ is an isomorphism whenever so are all the morphisms $\mathfrak{P}/\mathfrak{I}\mathfrak{P} \longrightarrow \mathfrak{Q}/\mathfrak{I}\mathfrak{Q}$ for open ideals $\mathfrak{I} \subset \mathfrak{R}$. Now $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{C}, \mathcal{M})/\mathfrak{I}\,\mathrm{Hom}_{\mathfrak{R}}(\mathcal{C}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathfrak{R}/\mathfrak{I}}(\mathcal{C}(\mathfrak{R}/\mathfrak{I}), {}_{\mathfrak{I}}\mathcal{M})$ for an injective $\mathfrak{R}$-comodule $\mathcal{M}$ and the functor $\mathrm{Hom}_R(\mathcal{N}, -)$ preserves infinite direct sums whenever $\mathcal{N}$ is a comodule of finite length over an Artinian ring R. $\square$

1.6. **Contramodule tensor product and Ctrhom.** For any topological ring $\mathfrak{R}$, the tensor product operation $\otimes^{\mathfrak{R}}$ on the category of contramodules over $\mathfrak{R}$ is defined in the following way.

For free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$ and $\mathfrak{R}[[Y]]$, set $\mathfrak{R}[[X]] \otimes^{\mathfrak{R}} \mathfrak{R}[[Y]] = \mathfrak{R}[[X \times Y]]$. There is a natural map $\mathfrak{R}[[X]] \otimes_{\mathfrak{R}} \mathfrak{R}[[Y]] \longrightarrow \mathfrak{R}[[X]] \otimes^{\mathfrak{R}} \mathfrak{R}[[Y]]$, where $\otimes_{\mathfrak{R}}$ denotes the tensor product in the category of $\mathfrak{R}$-modules $\mathfrak{R}\text{–}\mathsf{mod}$, taking the tensor $\sum_{x\in X} r_x x \otimes_{\mathfrak{R}} \sum_{y\in Y} r_y y$ to the formal sum $\sum_{(x,y)\in X\times Y} r_x r_y (x,y)$, where r_x, $r_y \in \mathfrak{R}$ are two families of elements converging to zero.

Let $f\colon \mathfrak{R}[[X']] \longrightarrow \mathfrak{R}[[X'']]$ and $g\colon \mathfrak{R}[[Y']] \longrightarrow \mathfrak{R}[[Y'']]$ be two homomorphisms of free $\mathfrak{R}$-contramodules. The data of the morphisms f and g is equivalent to the data of two families of elements $f(x') \in \mathfrak{R}[[X'']]$ and $g(y') \in \mathfrak{R}[[Y'']]$, where $x' \in X'$ and $y' \in Y'$. Define the homomorphism $(f \otimes g)\colon \mathfrak{R}[[X' \times Y']] \longrightarrow \mathfrak{R}[[X'' \times Y'']]$ by the rule $(f \otimes g)(x' \otimes y') = f(x') \otimes_{\mathfrak{R}} g(y') \in \mathfrak{R}[[X'' \times Y'']]$.

One readily checks that we have constructed a biadditive tensor product functor $\otimes^{\mathfrak{R}}$ on the category of free $\mathfrak{R}$-contramodules. Since there are enough free contramodules in $\mathfrak{R}\text{–}\mathsf{contra}$, this functor extends in a unique way to a right exact functor $\otimes^{\mathfrak{R}}\colon \mathfrak{R}\text{–}\mathsf{contra} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$.

The functor of tensor product of free contramodules is naturally associative, commutative, and unital with the unit object $\mathfrak{R}$; hence so is the functor of tensor product of arbitrary $\mathfrak{R}$-contramodules. The functor of tensor product of contramodules also preserves infinite direct sums, since the functor of tensor product of free contramodules does. In particular, the functor $\mathfrak{R}[[X]] \otimes^{\mathfrak{R}} -$ assigns to any $\mathfrak{R}$-contramodule $\mathfrak{P}$ the direct sum of X copies of $\mathfrak{P}$.

There is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{Q},\mathfrak{N}))\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q},\ \mathfrak{N})$ for any three $\mathfrak{R}$-contramodules $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{N}$; so $\operatorname{Hom}^{\mathfrak{R}}$ is the internal Hom functor for the tensor product functor $\otimes^{\mathfrak{R}}$. As above, it suffices to construct this isomorphism for free $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, which is easy.

Assuming that $\mathfrak{R}$ is a pro-Artinian topological ring, one easily checks that the reduction functor $\mathfrak{P}\longmapsto\mathfrak{P}/\mathfrak{J}\mathfrak{P}$ takes tensor products of $\mathfrak{R}$-contramodules to tensor products of $\mathfrak{R}/\mathfrak{J}$-contramodules for any closed ideal $\mathfrak{J}\subset\mathfrak{R}$. Besides, whenever the functor of contrarestriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{contra}\longrightarrow\mathfrak{R}\text{–}\mathsf{contra}$ preserves infinite direct sums (see Remark 1.3.8), there is a natural isomorphism of $\mathfrak{R}$-contramodules $\mathfrak{R}/\mathfrak{J}\otimes^{\mathfrak{R}}\mathfrak{P}\simeq\mathfrak{P}/\mathfrak{J}\mathfrak{P}$ for any $\mathfrak{R}$-contramodule $\mathfrak{P}$, where the $\mathfrak{R}$-contramodule structures on $\mathfrak{R}/\mathfrak{J}$ and $\mathfrak{P}/\mathfrak{J}\mathfrak{P}$ are defined in terms of the functor of contrarestriction of scalars. Indeed, both sides are right exact functors of the argument $\mathfrak{P}$ commuting with infinite direct sums, and the isomorphism holds for $\mathfrak{P}=\mathfrak{R}$.

Finally, for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and $\mathfrak{R}$-comodule $\mathcal{M}$ there is a natural isomorphism of $\mathfrak{R}$-comodules $(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q})\odot_{\mathfrak{R}}\mathcal{M}\simeq\mathfrak{P}\odot_{\mathfrak{R}}(\mathfrak{Q}\odot_{\mathfrak{R}}\mathcal{M})$. In other words, the functor $\odot_{\mathfrak{R}}$ makes $\mathfrak{R}\text{–}\mathsf{comod}$ a module category over the tensor category $\mathfrak{R}\text{–}\mathsf{contra}$.

For any pro-Artinian topological ring $\mathfrak{R}$, an $\mathfrak{R}$-contramodule $\mathfrak{P}$, and an $\mathfrak{R}$-comodule $\mathcal{M}$, the $\mathfrak{R}$-comodule $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M})$ of *contrahomomorphisms* from $\mathfrak{P}$ to $\mathcal{M}$ is defined by the rules

(i) $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M})=\varinjlim_{\mathfrak{J}}\operatorname{Ctrhom}_{\mathfrak{R}/\mathfrak{J}}(\mathfrak{P}/\mathfrak{J}\mathfrak{P},\ {}_{\mathfrak{J}}\mathcal{M})$, where the inductive limit is taken over all open ideals $\mathfrak{J}\subset\mathfrak{R}$;

(ii) for any Artinian ring R, the functor Ctrhom_R takes inductive limits in the first argument to projective limits;

(iii) for any module of finite length Q over an Artinian ring R, the functor $\operatorname{Ctrhom}_R(Q,-)$ preserves filtered inductive limits; and

(iv) $\operatorname{Ctrhom}_R(Q,\mathcal{N})^{\mathrm{op}}=Q\otimes_R\mathcal{N}^{\mathrm{op}}$ for any modules of finite length Q and $\mathcal{N}^{\mathrm{op}}$ over an Artinian ring R.

Clearly, the functor $\operatorname{Ctrhom}_{\mathfrak{R}}$ is left exact in both arguments. The functor $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{R}[[X]],-)$ assigns to any $\mathfrak{R}$-comodule $\mathcal{M}$ the direct product of X copies of $\mathcal{M}$. Presenting an arbitrary $\mathfrak{R}$-contramodule as the cokernel of a morphism of free $\mathfrak{R}$-contramodules, one can conclude that the functor $\operatorname{Ctrhom}_{\mathfrak{R}}$ transforms infinite directs sums of $\mathfrak{R}$-contramodules in the first argument and infinite products of $\mathfrak{R}$-comodules in the second argument into infinite products of $\mathfrak{R}$-comodules.

In particular, there is a natural isomorphism $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{R},\mathcal{M})\simeq\mathcal{M}$; and one can construct a natural isomorphism of $\mathfrak{R}$-comodules $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q},\ \mathcal{M})\simeq\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{Q},\mathcal{M}))$ for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and $\mathfrak{R}$-comodule $\mathcal{M}$ by reducing the question to the case of free $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ first. In other words, the functor $\operatorname{Ctrhom}_{\mathfrak{R}}$ makes the category opposite to $\mathfrak{R}\text{–}\mathsf{comod}$ a module category over $\mathfrak{R}\text{–}\mathsf{contra}$.

Finally, for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{P}\odot_{\mathfrak{R}}\mathcal{M},\ \mathcal{N})\simeq\operatorname{Hom}_{\mathfrak{R}}(\mathcal{M},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{N}))$. Given a closed ideal $\mathfrak{J}\subset\mathfrak{R}$, for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$ there is a natural isomorphism of $\mathfrak{R}/\mathfrak{J}$-comodules ${}_{\mathfrak{J}}\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M})\simeq\operatorname{Ctrhom}_{\mathfrak{R}/\mathfrak{J}}(\mathfrak{P}/\mathfrak{J}\mathfrak{P},\ {}_{\mathfrak{J}}\mathcal{M})$.

Besides, there is a natural isomorphism of $\mathfrak{R}$-comodules $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{R}/\mathfrak{J},\mathcal{M}) \simeq {}_{\mathfrak{J}}\mathcal{M}$, where the $\mathfrak{R}$-comodule structure on the right hand side is defined by means of the functor of restriction of scalars $\mathfrak{R}/\mathfrak{J}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$.

Lemma 1.6.1. *Let $\mathfrak{A}$ be an (associative, noncommutative, unital) algebra in the tensor category of free contramodules over a pro-Artinian topological ring $\mathfrak{R}$, and let $\mathfrak{J} \subset \mathfrak{R}$ be a closed ideal. Then any idempotent element in $\mathfrak{A}/\mathfrak{J}\mathfrak{A}$ can be lifted to an idempotent element in $\mathfrak{A}$.*

Proof. Notice first of all that $\mathfrak{A}/\mathfrak{J}\mathfrak{A}$ is an algebra in the tensor category of free contramodules over $\mathfrak{R}/\mathfrak{J}$, the underlying abelian groups to $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{J}\mathfrak{A}$ have (noncommutative) ring structures, and the natural projection $\mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{J}\mathfrak{A}$ is a morphism of rings, so the question about lifting idempotents makes sense. As usually, we will assume for simplicity that $\mathfrak{R}$ is local.

To prove the assertion, we will apply Zorn's lemma. Let $e_{\mathfrak{J}} \in \mathfrak{A}/\mathfrak{J}\mathfrak{A}$ be our idempotent element. Consider the set of all pairs $(\mathfrak{K}, e_{\mathfrak{K}})$, where $\mathfrak{K} \subset \mathfrak{R}$ is a closed ideal contained in $\mathfrak{J}$ and $e_{\mathfrak{K}} \in \mathfrak{A}/\mathfrak{K}\mathfrak{A}$ is an idempotent element lifting $e_{\mathfrak{J}}$. Endow this set with the obvious partial order relation.

Given a linearly ordered subset of elements $(\mathfrak{K}_\alpha, e_{\mathfrak{K}_\alpha})$, consider the intersection $\mathfrak{K}$ of all the ideals $\mathfrak{K}_\alpha$. By Corollary A.4.1, any open ideal $\mathfrak{I} \subset \mathfrak{R}$ containing $\mathfrak{K}$ contains also one of the ideals $\mathfrak{K}_\alpha$. Hence the ring $\mathfrak{A}/\mathfrak{K}\mathfrak{A} = \varprojlim_{\mathfrak{I}\supset\mathfrak{K}} \mathfrak{A}/\mathfrak{I}\mathfrak{A}$ (where $\mathfrak{I} \subset \mathfrak{R}$ are open ideals) is the projective limit of the rings $\mathfrak{A}/\mathfrak{K}_\alpha\mathfrak{A}$. So we can construct an idempotent element $e_{\mathfrak{K}} \in \mathfrak{A}/\mathfrak{K}\mathfrak{A}$ such that the pair $(\mathfrak{K}, e_{\mathfrak{K}})$ majorates all the pairs $(\mathfrak{K}_\alpha, e_{\mathfrak{K}_\alpha})$.

Let $(\mathfrak{K}, e_{\mathfrak{K}})$ be an element of our poset such that $\mathfrak{K} \neq 0$. Pick an open ideal $\mathfrak{I} \subset \mathfrak{R}$ not containing $\mathfrak{K}$. Then the ring $\mathfrak{A}/\mathfrak{K}\mathfrak{A}$ is the quotient ring of $\mathfrak{A}/(\mathfrak{I}\cap\mathfrak{K})\mathfrak{A}$ by the ideal $\mathfrak{K}\mathfrak{A}/(\mathfrak{I}\cap\mathfrak{K})\mathfrak{A}$. Since $\mathfrak{K}/(\mathfrak{I}\cap\mathfrak{K}) \simeq (\mathfrak{K}+\mathfrak{I})/\mathfrak{I}$ is a discrete and nilpotent closed ideal in $\mathfrak{R}/(\mathfrak{I}\cap\mathfrak{K})$, the ideal $\mathfrak{K}\mathfrak{A}/(\mathfrak{I}\cap\mathfrak{K})\mathfrak{A} \subset \mathfrak{A}/(\mathfrak{I}\cap\mathfrak{K})\mathfrak{A}$ is also nilpotent. Thus it remains to use the well-known lemma on lifting idempotents modulo a nil ideal (see, e. g., [11, Lemma I.12.1]) in order to lift $e_{\mathfrak{K}}$ to an idempotent element $e_{\mathfrak{I}\cap\mathfrak{K}} \in \mathfrak{A}/(\mathfrak{I}\cap\mathfrak{K})\mathfrak{A}$, completing the Zorn lemma argument and the whole proof.

Alternatively, one can use [11, Lemmas I.12.1–2] to the effect that for any fixed, not necessarily idempotent lifting $a \in \mathfrak{A}$ of the element $e_{\mathfrak{J}}$ and any open ideal $\mathfrak{I} \subset \mathfrak{R}$ there exists a unique idempotent $e_{\mathfrak{I}} \in \mathfrak{A}/\mathfrak{I}\mathfrak{A}$ expressible as a polynomial in $a \bmod \mathfrak{I}\mathfrak{A}$ with integer coefficients and lifting the idempotent element $e_{\mathfrak{J}} \bmod (\mathfrak{I}+\mathfrak{J})\mathfrak{A}/\mathfrak{J}\mathfrak{A} \in \mathfrak{A}/(\mathfrak{I}+\mathfrak{J})\mathfrak{A}$. The compatible system of idempotents $e_{\mathfrak{I}}$ defines the desired idempotent element of $\mathfrak{A} = \varprojlim_{\mathfrak{I}} \mathfrak{A}/\mathfrak{I}\mathfrak{A}$ lifting $e_{\mathfrak{J}}$. ☐

1.7. **Cotensor product and Cohom.** The following two operations on $\mathfrak{R}$-comodules and $\mathfrak{R}$-contramodules are not as important for our purposes in this paper as the previous five. We consider them largely for the sake of completeness (however, cf. the proof of Corollary 1.9.4 below, and the constructions of functors $\otimes_{\mathfrak{B}}$ and $\operatorname{Hom}_{\mathfrak{B}}$ in Section 4.1 and functors $\square_{\mathfrak{C}}$ and $\operatorname{Cohom}_{\mathfrak{C}}$ in Section 4.4).

Let $\mathfrak{R}$ be a pro-Artinian topological ring. The cotensor product operation $(\mathcal{M},\mathcal{N}) \longmapsto \mathcal{M} \square_{\mathfrak{R}} \mathcal{N}$ on the category of $\mathfrak{R}$-comodules is obtained by passing to the category of ind-objects and the opposite category from the operation of the tensor product over $\mathfrak{R}$ of discrete $\mathfrak{R}$-modules of finite length. Clearly, $\square_{\mathfrak{R}}$ is an associative and commutative tensor category structure on $\mathfrak{R}\text{–}\mathsf{comod}$ with the unit object $\mathcal{C}(\mathfrak{R})$.

The functor $\square_{\mathfrak{R}}$ is left exact and commutes with infinite direct sums. For any closed ideal $\mathfrak{J} \subset \mathfrak{R}$ and any $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$ there are natural isomorphisms of $\mathfrak{R}/\mathfrak{J}$-comodules $\mathcal{C}(\mathfrak{R}/\mathfrak{J}) \square_{\mathfrak{R}} \mathcal{M} \simeq {}_{\mathfrak{J}}\mathcal{M}$ and ${}_{\mathfrak{J}}(\mathcal{M} \square_{\mathfrak{R}} \mathcal{N}) \simeq {}_{\mathfrak{J}}\mathcal{M} \square_{\mathfrak{R}} \mathcal{N} \simeq {}_{\mathfrak{J}}\mathcal{M} \square_{\mathfrak{R}/\mathfrak{J}} {}_{\mathfrak{J}}\mathcal{N}$, where the two cotensor products over $\mathfrak{R}$ are identified with the $\mathfrak{R}/\mathfrak{J}$-comodules from which their $\mathfrak{R}$-comodule structures are obtained by the restriction of scalars.

For any $\mathfrak{R}$-comodule $\mathcal{M}$ and $\mathfrak{R}$-contramodule $\mathfrak{P}$, the $\mathfrak{R}$-contramodule $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P})$ of *cohomomorphisms* from $\mathcal{M}$ to $\mathfrak{P}$ is defined as follows. For a cofree $\mathfrak{R}$-comodule $\mathcal{M} = \bigoplus_{x \in X} \mathcal{C}(\mathfrak{R})$ and a projective $\mathfrak{R}$-contramodule $\mathfrak{P}$, one sets $\operatorname{Cohom}_{\mathfrak{R}}(\bigoplus_X \mathcal{C}(\mathfrak{R}),\ \mathfrak{P}) = \prod_X \mathfrak{P}$. It follows that $\operatorname{Cohom}_{\mathfrak{R}}(\bigoplus_X \mathcal{C}(\mathfrak{R}),\ \mathfrak{P}) \simeq \varprojlim_{\mathfrak{J}} \operatorname{Cohom}_{\mathfrak{R}/\mathfrak{J}}({}_{\mathfrak{J}} \bigoplus_X \mathcal{C}(\mathfrak{R}),\ \mathfrak{P}/\mathfrak{J}\mathfrak{P})$, where the projective limit is taken over all open ideals $\mathfrak{J} \subset \mathfrak{R}$. To define the (contravariant) action of $\operatorname{Cohom}_{\mathfrak{R}}(-, \mathfrak{P})$ on morphisms of cofree $\mathfrak{R}$-comodules $\mathcal{M}$, one can first do so for an Artinian ring R, which is easy; then pass to the projective limit over $\mathfrak{J}$. Alternatively, set $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P}) = \operatorname{Hom}^{\mathfrak{R}}(\Psi_{\mathfrak{R}}(\mathcal{M}), \mathfrak{P})$ (see Section 1.5) for any cofree $\mathfrak{R}$-comodule $\mathcal{M}$ and any $\mathfrak{R}$-contramodule $\mathfrak{P}$.

Since there are enough cofree comodules and projective contramodules, one can extend the above-defined functor in a unique way to a right exact functor $\operatorname{Cohom}^{\mathfrak{R}}\colon \mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{op}} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$. The functor $\operatorname{Cohom}_{\mathfrak{R}}$ transforms infinite direct sums of $\mathfrak{R}$-comodules in the first argument and infinite products of $\mathfrak{R}$-contramodules in the second argument to infinite products of $\mathfrak{R}$-contramodules.

For any $\mathfrak{R}$-comodules $\mathcal{M}$ and $\mathcal{N}$ and any $\mathfrak{R}$-contramodule $\mathfrak{P}$, there are natural isomorphisms of $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{C}(\mathfrak{R}), \mathfrak{P}) \simeq \mathfrak{P}$ and $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M} \square_{\mathfrak{R}} \mathcal{N}, \mathfrak{P}) \simeq \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{N}, \mathfrak{P}))$. So the functor $\operatorname{Cohom}_{\mathfrak{R}}$ makes the category opposite to $\mathfrak{R}\text{–}\mathsf{contra}$ a module category over the tensor category $\mathfrak{R}\text{–}\mathsf{comod}$.

Given a closed ideal $\mathfrak{J} \subset \mathfrak{R}$, for any $\mathfrak{R}$-comodule $\mathcal{M}$ and $\mathfrak{R}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of $\mathfrak{R}/\mathfrak{J}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P})/\mathfrak{J} \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P}) \simeq \operatorname{Cohom}_{\mathfrak{R}/\mathfrak{J}}({}_{\mathfrak{J}}\mathcal{M},\ \mathfrak{P}/\mathfrak{J}\mathfrak{P})$ (since this is so for a cofree $\mathfrak{R}$-comodule $\mathcal{M}$ and a projective $\mathfrak{R}$-contramodule $\mathfrak{P}$, and all the functors are exact "from the same side").

For any $\mathfrak{R}$-comodule $\mathcal{M}$, there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{R}) \simeq \mathcal{M}^{\mathrm{op}}$. It follows that the functor $\operatorname{Cohom}_{\mathfrak{R}}(-, \mathfrak{P})$ into a projective $\mathfrak{R}$-contramodule $\mathfrak{P}$ is exact, since any projective $\mathfrak{R}$-contramodule is a direct summand of a product of copies of $\mathfrak{R}$. More generally, for any $\mathfrak{R}$-comodules $\mathfrak{N}$ and $\mathfrak{M}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{N}^{\mathrm{op}}) \simeq (\mathcal{N} \square_{\mathfrak{R}} \mathcal{M})^{\mathrm{op}}$.

For any $\mathfrak{R}$-comodule $\mathcal{N}$ of finite length, there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{N}, \mathfrak{P}) \simeq \mathcal{N}^{\mathrm{op}} \otimes_{\mathfrak{R}} \mathfrak{P} \simeq \mathcal{N}^{\mathrm{op}} \otimes^{\mathfrak{R}} \mathfrak{P}$, where the $\mathfrak{R}$-contramodule structure on the middle term is obtained by restriction of scalars from the R-module structure, R being a discrete Artinian quotient ring of $\mathfrak{R}$ such that $\mathcal{N}$ is a comodule over R. Indeed, the second isomorphism follows from the results of Section 1.6; and to prove the first one, it suffices to notice that both functors of the second argument $\mathfrak{P}$ are right exact and preserve infinite products, and they produce the same $\mathfrak{R}$-contramodule for $\mathfrak{P} = \mathfrak{R}$.

Given a closed ideal $\mathfrak{J} \subset \mathfrak{R}$, for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{C}(\mathfrak{R}/\mathfrak{J}), \mathfrak{P}) \simeq \mathfrak{P}/\mathfrak{J}\mathfrak{P}$, where the $\mathfrak{R}$-contramodule structure on the right hand side is obtained by the restriction of scalars from the $\mathfrak{R}/\mathfrak{J}$-contramodule structure. One proves this in the way similar to the above: both functors of the argument $\mathfrak{P}$ are right exact and preserve infinite products, and the isomorphism holds for $\mathfrak{P} = \mathfrak{R}$.

Lemma 1.7.1. (a) *For any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodules $\mathcal{N}$ and $\mathcal{M}$, there is a natural morphism of $\mathfrak{R}$-comodules $\mathfrak{P} \odot_{\mathfrak{R}} (\mathcal{N} \mathbin{\square}_{\mathfrak{R}} \mathcal{M}) \longrightarrow (\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{N}) \mathbin{\square}_{\mathfrak{R}} \mathcal{M}$. This morphism is an isomorphism whenever either the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is projective, or the $\mathfrak{R}$-comodule $\mathcal{M}$ is injective.*

(b) *For any $\mathfrak{R}$-comodules $\mathcal{L}$, $\mathcal{K}$ and $\mathcal{M}$, there is a natural morphism of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{L}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}, \mathcal{M})) \longrightarrow \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L} \mathbin{\square}_{\mathfrak{R}} \mathcal{K},\ \mathcal{M})$. This morphism is an isomorphism whenever one of the $\mathfrak{R}$-comodules $\mathcal{L}$ and $\mathcal{M}$ is injective.*

(c) *For any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and $\mathfrak{R}$-comodule $\mathcal{M}$, there is a natural morphism of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{R}}(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M}, \mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{Q}))$. This morphism is an isomorphism whenever one of the $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ is projective.*

(d) *For any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ and $\mathfrak{R}$-comodule $\mathcal{M}$, there is a natural morphism of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathfrak{Q})) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{Q}))$. This morphism is an isomorphism whenever either the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is projective, or the $\mathfrak{R}$-comodule $\mathcal{M}$ is injective.*

Proof. Part (a): presenting $\mathcal{M}$ and $\mathcal{N}$ as filtered inductive limits of comodules of finite length, we reduce the problem of constructing the desired morphism to the case of a discrete Artinian ring R and modules P, $\mathcal{M}^{\mathrm{op}}$, $\mathcal{N}^{\mathrm{op}}$ over it. Presenting the $\mathfrak{R}$-module P as a filtered inductive limit of finitely generated modules, we can assume all the three modules to have finite length. Then we have $(P \odot_R (\mathcal{N} \square_{\mathfrak{R}} \mathcal{M}))^{\mathrm{op}} = \mathrm{Hom}_R(P, \mathcal{N}^{\mathrm{op}} \otimes_R \mathcal{M}^{\mathrm{op}})$ and $((P \odot_R \mathcal{N}) \mathbin{\square}_{\mathfrak{R}} \mathcal{M})^{\mathrm{op}} = \mathrm{Hom}_R(P, \mathcal{N}^{\mathrm{op}}) \otimes_R \mathcal{M}^{\mathrm{op}}$, and there is a natural morphism $\mathrm{Hom}_R(P, N) \otimes_R M \longrightarrow \mathrm{Hom}_R(P, N \otimes_R M)$ for any R-modules P, M, N. To check the second assertion, notice that both sides commute with infinite direct sums in all the three arguments, so it remains to consider the cases $P = \mathfrak{R}$ or $\mathcal{M} = \mathcal{C}(\mathfrak{R})$, which are straightforward.

Part (b): it suffices to constructed the desired functorial morphism in the case when the $\mathfrak{R}$-comodule $\mathcal{L}$ is injective, because the left hand side takes kernels in the argument $\mathcal{L}$ to cokernels. So we can assume $\mathcal{L} = \Phi_{\mathfrak{R}}(\mathfrak{F})$ with $\mathfrak{F}$ projective; then $\mathcal{L} \mathbin{\square}_{\mathfrak{R}} \mathcal{K} = \mathfrak{F} \odot_{\mathfrak{R}} \mathcal{K}$ by part (a). The functorial isomorphisms $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{L}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}, \mathcal{M})) \simeq \mathrm{Hom}^{\mathfrak{R}}(\Psi_{\mathfrak{R}}(\mathcal{L}), \mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}, \mathcal{M})) \simeq \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{F}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}, \mathcal{M})) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathfrak{F} \odot_{\mathfrak{R}} \mathcal{K},\ \mathcal{M}) \simeq \mathrm{Hom}_{\mathfrak{R}}(cL \mathbin{\square}_{\mathfrak{R}} \mathcal{K},\ \mathcal{M})$. We have already proven the second assertion in the case when $\mathfrak{L}$ is injective; to deal with the remaining case, notice that both sides commute with infinite products in the argument $\mathcal{M}$, so it remains to consider the case $\mathcal{M} = \mathcal{C}(\mathfrak{R})$. In this case we have $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{L}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}, \mathcal{C}(\mathfrak{R})) \simeq \mathrm{Cohom}_{\mathfrak{R}}(\mathcal{L}, \mathcal{K}^{\mathrm{op}}) \simeq (\mathcal{L} \mathbin{\square}_{\mathfrak{R}} \mathcal{K})^{\mathrm{op}} \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L} \mathbin{\square}_{\mathfrak{R}} \mathcal{K},\ \mathcal{C}(\mathfrak{R}))$.

Part (c): it suffices to construct the desired functorial morphism in the case when the $\mathfrak{R}$-contramodule $\mathfrak{P}$ is projective, because the right hand side takes cokernels in the argument $\mathfrak{P}$ to kernels. Assuming $\mathfrak{P}$ is projective, the left hand side takes kernels in the argument $\mathcal{M}$ to cokernels, so it suffices to consider the case when $\mathcal{M}$ is injective. Then we can assume $\mathcal{M} = \Phi_R(\mathfrak{F})$ with $\mathfrak{F}$ projective; hence $\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M} = \Phi_{\mathfrak{R}}(\mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{F})$. The functorial isomorphisms $\operatorname{Cohom}_{\mathfrak{R}}(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M},\ \mathfrak{Q}) \simeq \operatorname{Hom}^{\mathfrak{R}}(\Psi_{\mathfrak{R}}(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M}),\ \mathfrak{Q}) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P} \otimes^{\mathfrak{R}} \mathfrak{F},\ \mathfrak{Q}) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{F}, \mathfrak{Q})) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{Q}))$ provide the desired morphism. To prove the second assertion, notice that both sides take infinite direct sums in the argument $\mathfrak{P}$ and infinite products in the argument $\mathfrak{Q}$ to infinite products. The case $\mathfrak{P} = \mathfrak{R}$ is easy, and in the case $\mathfrak{Q} = \mathfrak{R}$ it remains to recall that $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{R}) = \mathcal{M}^{\mathrm{op}}$ and $(\mathfrak{P} \odot_{\mathfrak{R}} \mathcal{M})^{\mathrm{op}} = \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, \mathcal{M}^{\mathrm{op}})$.

The proof of the first assertion of part (d) is similar to that of the first assertions of parts (b-c); and the second assertion of part (d) is easy. ☐

1.8. **Change of ring.** Let $\eta\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ be a continuous homomorphism of topological rings. Then for any set X there is a natural map of sets $\mathfrak{R}[[X]] \longrightarrow \mathfrak{S}[[X]]$ taking $\sum_x r_x x$ to $\sum_x \eta(r_x)x$. One easily checks that this is a morphism of monads, so any $\mathfrak{S}$-contramodule $\mathfrak{Q}$ acquires an induced $\mathfrak{R}$-contramodule structure. We denote the $\mathfrak{R}$-contramodule so obtained by $R^\eta(\mathfrak{Q})$.

The functor of contrarestriction of scalars $R^\eta\colon \mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ is exact and preserves infinite products. It has a left adjoint functor of contraextension of scalars $E^\eta\colon \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{S}\text{–}\mathsf{contra}$, which can be defined on free contramodules by the rule $E^\eta(\mathfrak{R}[[X]]) = \mathfrak{S}[[X]]$ and extended to arbitrary contramodules as a right exact functor. The functor E^η preserves infinite direct sums and tensor products of contramodules. For any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{S}$-contramodule $\mathfrak{Q}$, there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P}, R^\eta\mathfrak{Q}) \simeq R^\eta \operatorname{Hom}^{\mathfrak{S}}(E^\eta\mathfrak{P}, \mathfrak{Q})$.

Now assume that η is a profinite morphism of pro-Artinian topological rings, i. e., the discrete quotient rings of $\mathfrak{S}$ are finite algebras over the discrete quotient rings of $\mathfrak{R}$. Then any discrete module of finite length over $\mathfrak{S}$ is also a discrete module of finite length over $\mathfrak{R}$ in the induced structure. Passing to the opposite categories and the ind-objects, we obtain the functor of corestriction of scalars $R_\eta\colon \mathfrak{S}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$. The functor R_η is exact and preserves infinite direct sums.

The functor R_η has a right adjoint functor of coextension of scalars $E_\eta\colon \mathfrak{R}\text{–}\mathsf{comod} \longrightarrow \mathfrak{S}\text{–}\mathsf{comod}$, which can be defined on injective cogenerators by the rule $E_\eta(\prod_{x\in X} \mathcal{C}(\mathfrak{R})) = \prod_{x\in X} \mathcal{C}(\mathfrak{S})$ and extended to arbitrary comodules as a left exact functor. As a right adjoint to a functor taking Noetherian objects to Noetherian objects, the functor E_η preserves both infinite direct sums and infinite products. For any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$, there is a natural isomorphism of $\mathfrak{S}$-comodules $\operatorname{Ctrhom}_{\mathfrak{S}}(E^\eta\mathfrak{P}, E_\eta\mathcal{M}) \simeq E_\eta \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{P}, \mathcal{M})$. For any $\mathfrak{R}$-comodule $\mathcal{M}$ and $\mathfrak{S}$-comodule $\mathcal{N}$, there is a natural isomorphism of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{R}}(R_\eta\mathcal{M}, \mathcal{N}) \simeq R^\eta \operatorname{Hom}_{\mathfrak{S}}(\mathcal{M}, E_\eta\mathcal{N})$.

Using Lemma 1.3.6, one can check that the functor E^η also preserves infinite products when η is a profinite morphism of pro-Artinian topological rings. For any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{K}$, there is a natural morphism of $\mathfrak{S}$-contramodules

$E^\eta \operatorname{Hom}^{\mathfrak R}(\mathfrak P,\mathfrak K) \longrightarrow \operatorname{Hom}^{\mathfrak S}(E^\eta\mathfrak P, E^\eta\mathfrak K)$, which is an isomorphism whenever $\mathfrak P$ is a projective $\mathfrak R$-contramodule. For any $\mathfrak R$-comodules $\mathcal L$ and $\mathcal M$, there is a natural morphism of $\mathfrak S$-contramodules $E^\eta \operatorname{Hom}_{\mathfrak R}(\mathcal L,\mathcal M) \longrightarrow \operatorname{Hom}_{\mathfrak S}(E_\eta\mathcal L, E_\eta\mathcal M)$, which is an isomorphism whenever $\mathcal M$ is an injective $\mathfrak R$-comodule (see the argument for the case $\mathfrak S = \mathfrak R/\mathfrak J$ in Section 1.5). For any $\mathfrak R$-contramodule $\mathfrak P$ and $\mathfrak S$-comodule $\mathcal N$, there is a natural isomorphism of $\mathfrak R$-comodules $\mathfrak P \odot_{\mathfrak R} R^\eta\mathcal N \simeq R_\eta(E^\eta\mathfrak P \odot_{\mathfrak S} \mathcal N)$. For any $\mathfrak R$-contramodule $\mathfrak P$ and $\mathfrak R$-comodule $\mathcal M$, there is a natural morphism of $\mathfrak S$-comodules $E^\eta(\mathfrak P) \odot_{\mathfrak S} E^\eta(\mathcal M) \longrightarrow E_\eta(\mathfrak P \odot_{\mathfrak R} \mathcal M)$, which is an isomorphism whenever $\mathfrak P$ is a projective $\mathfrak R$-contramodule.

For any $\mathfrak R$-contramodule $\mathfrak P$ and $\mathfrak S$-contramodule $\mathfrak Q$, there is a natural morphism of $\mathfrak R$-contramodules $\mathfrak P \otimes^{\mathfrak R} R^\eta\mathfrak Q \longrightarrow R^\eta(E^\eta\mathfrak P \otimes^{\mathfrak R} \mathfrak Q)$. For any $\mathfrak R$-contramodule $\mathfrak P$ and $\mathfrak S$-comodule $\mathcal N$, there is a natural morphism of $\mathfrak R$-comodules $R_\eta \operatorname{Ctrhom}_{\mathfrak S}(E^\eta\mathfrak P,\mathcal N) \longrightarrow \operatorname{Ctrhom}_{\mathfrak R}(\mathfrak P, R_\eta\mathcal N)$. For any $\mathfrak S$-contramodule $\mathfrak Q$ and $\mathfrak R$-comodule $\mathcal M$, there is a natural morphism of $\mathfrak R$-comodules $R_\eta \operatorname{Ctrhom}_{\mathfrak S}(\mathfrak Q, E_\eta\mathcal M) \longrightarrow \operatorname{Ctrhom}_{\mathfrak R}(R^\eta\mathfrak Q,\mathcal M)$.

For any $\mathfrak R$-comodule $\mathcal M$ and $\mathfrak S$-comodule $\mathcal N$, there is a natural isomorphism of $\mathfrak R$-comodules $\mathcal M \,\square_{\mathfrak R}\, R_\eta\mathcal N \simeq R_\eta(E_\eta\mathcal M \,\square_{\mathfrak S}\, \mathcal N)$. For any $\mathfrak R$-comodules $\mathcal L$ and $\mathcal M$, there is a natural isomorphism of $\mathfrak S$-comodules $E_\eta(\mathcal L \,\square_{\mathfrak R}\, \mathcal M) \simeq E_\eta(\mathcal L) \,\square_{\mathfrak S}\, E_\eta(\mathcal M)$. For any $\mathfrak R$-comodule $\mathcal M$ and $\mathfrak S$-contramodule $\mathfrak Q$, there is a natural isomorphism of $\mathfrak R$-contramodules $\operatorname{Cohom}_{\mathfrak R}(\mathcal M, R^\eta\mathfrak Q) \simeq R^\eta \operatorname{Cohom}_{\mathfrak S}(E_\eta\mathcal M,\mathfrak Q)$. For any $\mathfrak S$-comodule $\mathcal N$ and $\mathfrak R$-contramodule $\mathfrak P$, there is a natural isomorphism of $\mathfrak R$-contramodules $\operatorname{Cohom}_{\mathfrak R}(R_\eta\mathcal N,\mathfrak P) \simeq R^\eta \operatorname{Cohom}_{\mathfrak S}(\mathcal N, E^\eta\mathfrak P)$. For any $\mathfrak R$-comodule $\mathcal M$ and $\mathfrak R$-contramodule $\mathfrak P$, there is a natural isomorphism of $\mathfrak S$-contramodules $E^\eta \operatorname{Cohom}_{\mathfrak R}(\mathcal M,\mathfrak P) \simeq \operatorname{Cohom}_{\mathfrak S}(E_\eta\mathcal M, E^\eta\mathfrak P)$.

Let $\mathfrak R\text{–}\mathsf{contra}^{\mathsf{free}}$ and $\mathfrak R\text{–}\mathsf{comod}^{\mathsf{cofr}}$ denote the additive categories of, respectively, free $\mathfrak R$-contramodules and cofree $\mathfrak R$-comodules; the similar notation applies to $\mathfrak S$.

Proposition 1.8.1. *The equivalences of categories $\Phi_{\mathfrak R} = \Psi_{\mathfrak R}^{-1}\colon \mathfrak R\text{–}\mathsf{contra}^{\mathsf{free}} \simeq \mathfrak R\text{–}\mathsf{comod}^{\mathsf{cofr}}$ and $\Phi_{\mathfrak S} = \Psi_{\mathfrak S}^{-1}\colon \mathfrak S\text{–}\mathsf{contra}^{\mathsf{free}} \simeq \mathfrak S\text{–}\mathsf{comod}^{\mathsf{cofr}}$ from Proposition* 1.5.1 *transform the contraextension-of-scalars functor $E^\eta\colon \mathfrak R\text{–}\mathsf{contra}^{\mathsf{free}} \longrightarrow \mathfrak S\text{–}\mathsf{contra}^{\mathsf{free}}$ into the coextension-of-scalars functor $E_\eta\colon \mathfrak R\text{–}\mathsf{comod}^{\mathsf{cofr}} \longrightarrow \mathfrak S\text{–}\mathsf{comod}^{\mathsf{cofr}}$ and back.*

Proof. It follows from the above that for any $\mathfrak R$-contramodule $\mathfrak P$ there is a natural morphism of $\mathfrak S$-comodules $\Phi_{\mathfrak S}E^\eta(\mathfrak P) \longrightarrow E_\eta\Phi_{\mathfrak R}(\mathfrak P)$, which is an isomorphism whenever $\mathfrak P$ is projective. Similarly, for any $\mathfrak R$-comodule $\mathcal M$ there is a natural morphism of $\mathfrak S$-contramodules $E^\eta\Psi_{\mathfrak R}(\mathcal M) \longrightarrow \Psi_{\mathfrak S}E_\eta(\mathcal M)$, which is an isomorphism whenever $\mathcal M$ is injective. □

1.9. **Discrete $\mathfrak R$-modules.** Let $\mathfrak R$ be a pro-Artinian topological ring. Denote by $\mathfrak R\text{–}\mathsf{discr}$ the abelian category of discrete $\mathfrak R$-modules. Clearly, it is a locally Noetherian (and even locally finite) Grothendieck abelian category.

For each irredicible discrete $\mathfrak R$-module, pick its injective envelope in $\mathfrak R\text{–}\mathsf{discr}$, and denote by C the direct sum of all the injective objects obtained in this way. For any discrete $\mathfrak R$-module M and any closed ideal $\mathfrak J \subset \mathfrak R$, denote by ${}_{\mathfrak J}M$ the maximal submodule of M annihilated by $\mathfrak J$. Then ${}_{\mathfrak J}C$ is a direct sum of injective hulls of all the irreducible objects in $\mathfrak R/\mathfrak J\text{–}\mathsf{discr}$.

Applying this assertion in the case of an open ideal $\mathfrak{I} \subset \mathfrak{R}$ and using the standard results in commutative algebra of Artinian rings [25, Theorem 18.6], one concludes that the functor $M \longmapsto \mathrm{Hom}_{\mathfrak{R}\text{–}\mathsf{discr}}(M, C)$ is an exact auto-anti-equivalence of the abelian category of discrete $\mathfrak{R}$-modules of finite length. Passing to the ind-objects, we obtain an equivalence of abelian categories $\mathfrak{R}\text{–}\mathsf{discr} \simeq \mathfrak{R}\text{–}\mathsf{comod}$ identifying the injective object $C \in \mathfrak{R}\text{–}\mathsf{discr}$ with the canonical injective object $\mathcal{C} \in \mathfrak{R}\text{–}\mathsf{comod}$. For discrete $\mathfrak{R}$-modules and $\mathfrak{R}$-comodules M and $\mathcal{M}$ of finite length, the functor $M \longmapsto \mathrm{Hom}_{\mathfrak{R}\text{–}\mathsf{discr}}(M, C)$ is then identified with the functor $\mathcal{M} \longmapsto \mathcal{M}^{\mathrm{op}}$ (both functors taking values in the category of discrete $\mathfrak{R}$-modules of finite length).

Using this description of $\mathfrak{R}$-comodules, the constructions of the operations of contratensor product and contrahomomorphisms in Sections 1.5–1.6 can be presented in the following much simpler form.

For any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and discrete $\mathfrak{R}$-module M, the contratensor product $\mathfrak{P} \odot_{\mathfrak{R}} M$ is a discrete $\mathfrak{R}$-module defined by the rule $\mathfrak{P} \odot_{\mathfrak{R}} \varinjlim_\alpha M_\alpha = \varinjlim_\alpha \mathfrak{P}/\mathfrak{I}_\alpha\mathfrak{P} \otimes_{R_\alpha} M_\alpha$, where M_α is a module over a discrete Artinian quotient ring $R_\alpha = \mathfrak{R}/\mathfrak{I}_\alpha$ of the topological ring $\mathfrak{R}$ and the tensor product in the right hand side is taken in the category of R_α-modules. The discrete $\mathfrak{R}$-module of contrahomomorphisms $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P}, M)$ is defined as the inductive limit $\varinjlim_{\mathfrak{I}} \mathrm{Hom}_{\mathfrak{R}/\mathfrak{I}}(\mathfrak{P}/\mathfrak{I}\mathfrak{P},\ {}_{\mathfrak{I}}M)$ taken over all the open ideals $\mathfrak{I} \subset \mathfrak{R}$, where $\mathrm{Hom}_{\mathfrak{R}/\mathfrak{I}}$ denotes the internal Hom in the abelian tensor category of $\mathfrak{R}/\mathfrak{I}$-modules.

The cotensor product of discrete $\mathfrak{R}$-modules can be constructed by the rule $\varinjlim_\alpha N_\beta \,\square_{\mathfrak{R}} \varinjlim_\alpha M_\alpha = \varinjlim_{\alpha,\beta} \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Hom}_{\mathfrak{R}}(N_\beta, C) \otimes_{\mathfrak{R}} \mathrm{Hom}_{\mathfrak{R}}(M_\alpha, C),\ C)$, where M_α and N_β are discrete $\mathfrak{R}$-modules of finite length. In particular, the cotensor product with a discrete $\mathfrak{R}$-module N of finite length is described as $N \,\square_{\mathfrak{R}} M = \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Hom}_{\mathfrak{R}}(N, C),\ M)$.

Lemma 1.9.1. *A contramodule $\mathfrak{P}$ over a pro-Artinian topological ring $\mathfrak{R}$ is projective if and only if either of the following equivalent conditions holds:*

(a) *the functor $\mathcal{N} \longmapsto \mathfrak{P} \odot_{\mathfrak{R}} \mathcal{N}$ is exact on the abelian category of $\mathfrak{R}$-comodules;*
(b) *the contravariant functor $\mathcal{M} \longmapsto \mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P})$ from the abelian category of $\mathfrak{R}$-comodules to the abelian category of $\mathfrak{R}$-contramodules is exact.*

Proof. In both cases, the "only if" assertions have been already proven in Sections 1.5 and 1.7; so it remains to prove the "if". In both cases, restricting oneself to $\mathfrak{R}$-contramodules $\mathfrak{Q}$ or $\mathfrak{R}$-comodules $\mathcal{N}$ obtained by restriction of scalars from contramodules or comodules over $\mathfrak{R}/\mathfrak{I}$, where $\mathfrak{I} \subset \mathfrak{R}$ are open ideals, and using Lemma 1.3.5, one reduces the question to the case of modules over a discrete Artinian ring R. In the latter situation, identifying R-comodules with R-modules as explained above in this section, we see that the functor in part (a) is simply the functor $\otimes_R$, so the assertion follows from the fact that a flat module over an Artinian ring is projective [2]. Finally, restricting ourselves to R-comodules $\mathcal{M}$ of finite length in part (d), we also identify the functor under consideration with the functor of tensor product over R (see Section 1.7). (Cf. [28, Section 0.2.9].) $\square$

Lemma 1.9.2. *A comodule $\mathcal{M}$ over a pro-Artinian topological ring $\mathfrak{R}$ is injective if and only if either of the following equivalent conditions holds:*

(a) *the functor $\mathcal{N} \longmapsto \mathcal{N} \,\square_{\mathfrak{R}}\, \mathcal{M}$ is exact on the abelian category of $\mathfrak{R}$-comodules;*

(b) *the functor $\mathcal{M} \longmapsto \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M}, \mathfrak{P})$ is exact on the abelian category of $\mathfrak{R}$-contramodules.*

Proof. In both cases, the "only if" assertions have been already proven in Section 1.7; it remains to prove the "if". Restricting ourselves to $\mathfrak{R}$-comodules $\mathcal{N}$ or $\mathfrak{R}$-contramodules $\mathfrak{P}$ obtained by restriction of scalars from contramodules or comodules over $\mathfrak{R}/\mathfrak{I}$, where $\mathfrak{I} \subset \mathfrak{R}$ are open ideals, and using the relevant assertion from Section 1.4, we can reduce the question to the case of comodules over a discrete Artinian ring R. Restricting to $\mathfrak{R}$-contramodules $\mathfrak{P}$ of the form $\mathfrak{P} = \mathcal{N}^{\mathrm{op}}$, we conclude that it suffices to prove part (a). Finally, it remains to restrict to $\mathfrak{R}$-comodules $\mathcal{N}$ of finite length and use the above description of $\square_{\mathfrak{R}}$ in terms of discrete $\mathfrak{R}$-modules. $\square$

Remark 1.9.3. The assertions of Lemma 1.9.1 do not apply to the functors $\otimes^{\mathfrak{R}}$ and $\operatorname{Ctrhom}_{\mathfrak{R}}$, because the substitution of a projective contramodule at the first argument does not make these functors exact. The reason is that infinite direct sums of $\mathfrak{R}$-contramodules are not exact functors, and neither are infinite products of $\mathfrak{R}$-comodules.

Similarly, the assertions of Lemma 1.9.2 do not apply to the functors $\odot_{\mathfrak{R}}$ and $\operatorname{Ctrhom}_{\mathfrak{R}}$, because the substitution of an injective comodule at the second argument does not make these functors exact. In the case of the functor $\odot_{\mathfrak{R}}$, it suffices to consider a discrete Artinian ring $\mathfrak{R}$ (essentially, one would like to put projective comodules in the second argument of $\odot_{\mathfrak{R}}$, but these do not exist in general). A counterexample for the functor $\operatorname{Ctrhom}_{\mathfrak{R}}$ with $\mathfrak{R} = k[[\epsilon]]$ can be obtained from the examples of $k[[\epsilon]]$-contramodules with divisible elements [28, Section A.1.1].

Corollary 1.9.4. *For any pro-Artinian topological ring $\mathfrak{R}$, the homological dimensions of the abelian categories of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules coincide.*

Proof. The right derived functor

$$\operatorname{Coext}_{\mathfrak{R}}^{-i}\colon \mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{op}} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}, \quad i = 0,\ 1,\ 2,\ \ldots$$

of the functor $\operatorname{Cohom}_{\mathfrak{R}}$ is constructed by either replacing the first argument of $\operatorname{Cohom}_{\mathfrak{R}}$ with its right injective resolution, or replacing the second argument with its left projective resolution. The same derived functor is obtained in both ways, since $\operatorname{Cohom}_{\mathfrak{R}}$ from an injective $\mathfrak{R}$-comodule or into a projective $\mathfrak{R}$-contramodule is an exact functor. It follows from Lemmas 1.9.1(b) and 1.9.2(b) that the homological dimensions of both abelian categories $\mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{R}\text{–}\mathsf{comod}$ are equal to the homological dimension of the derived functor $\operatorname{Coext}_{\mathfrak{R}}$ (i. e., the maximal integer i for which $\operatorname{Coext}_{\mathfrak{R}}^{-i}$ is a nonzero functor). $\square$

The common value of the homological dimensions of the abelian categories $\mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{R}\text{–}\mathsf{comod}$ is called the *homological dimension* of a pro-Artinian topological ring $\mathfrak{R}$. Notice that this dimension is also equal to the homological dimension of the abelian category of discrete $\mathfrak{R}$-modules of finite length.

In particular, when $\mathfrak{R}$ is a complete Noetherian local ring, the category of discrete $\mathfrak{R}$-modules of finite length is contained in the category of finitely generated $\mathfrak{R}$-modules, and the latter is contained in the category of $\mathfrak{R}$-contramodules. It follows that the homological dimension of $\mathfrak{R}$ as a pro-Artinian topological ring is equal to the homological dimension of $\mathfrak{R}$ as an abstract commutative ring, i. e., the Krull dimension of $\mathfrak{R}$ when $\mathfrak{R}$ is regular and infinity otherwise.

Let $\mathfrak{T} \longrightarrow \mathfrak{R}$ be a profinite morphism of pro-Artinian topological rings (see Section 1.8). Let $C_\mathfrak{T}$ be a direct sum of injective hulls of the irreducible objects of $\mathfrak{T}$–discr. Then the set $C_\mathfrak{R}$ of all continuous $\mathfrak{T}$-module homomorphisms $\mathfrak{R} \longrightarrow C_\mathfrak{T}$ (with the discrete topology on $C_\mathfrak{T}$) endowed with its natural $\mathfrak{R}$-module structure is a direct sum of injective hulls of all the irreducible objects in $\mathfrak{R}$–discr. This construction agrees with the compositions of profinite morphisms $\mathfrak{T} \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{S}$.

Assuming such morphisms to be given and the modules $C_\mathfrak{R}$ and $C_\mathfrak{S}$ being defined in terms of $C_\mathfrak{T}$, we have the equivalences of categories $\mathfrak{R}$–$\mathsf{discr} \simeq \mathfrak{R}$–$\mathsf{comod}$ and $\mathfrak{S}$–$\mathsf{discr} \simeq \mathfrak{S}$–$\mathsf{comod}$ corresponding to $C_\mathfrak{R}$ and $C_\mathfrak{S}$. These identifications being presumed, the functor of corestriction of scalars R_η for the morphism $\eta\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ assigns to a discrete module N over $\mathfrak{S}$ the same abelian group N considered as a discrete module over $\mathfrak{R}$ in the module structure induced by η. The functor of coextension of scalars E_η assigns to a discrete $\mathfrak{R}$-module M the set of all continuous $\mathfrak{R}$-module homomorphisms $\mathfrak{S} \longrightarrow M$ endowed with its natural discrete $\mathfrak{S}$-module structure.

1.10. **Special classes of rings.** The above construction provides a natural choice of an object $C_\mathfrak{R}$ for some special classes of pro-Artinian topological rings $\mathfrak{R}$, such as the profinite-dimensional algebras over a field k (for which one can use $\mathfrak{T} = k = C_\mathfrak{T}$) or the profinite rings (for which one can take $\mathfrak{T} = \widehat{\mathbb{Z}} = \prod_l \mathbb{Z}_l$ to be the product of all the rings of l-adic integers and $C_\mathfrak{T} = \mathbb{Q}/\mathbb{Z}$).

Let us discuss the case of a topological algebra $\mathfrak{R}$ over a field k in some more detail. First of all, in this case there is the following simplified definition of $\mathfrak{R}$-contramodules.

Given a vector space V over k, set $\mathfrak{R}\otimes\widehat{\ }V = \varprojlim_\mathfrak{I} \mathfrak{R}/\mathfrak{I}\otimes_k V$, where the projective limit is taken over all the open ideals $\mathfrak{I} \subset \mathfrak{R}$. The unit element of $\mathfrak{R}$ induces the natural map $V \longrightarrow \mathfrak{R}\otimes\widehat{\ }V$. The natural map $\mathfrak{R}\otimes\widehat{\ }(\mathfrak{R}\otimes\widehat{\ }V) \longrightarrow \mathfrak{R}\otimes\widehat{\ }V$ is obtained by passing to the projective limit of the natural maps $\mathfrak{R}/\mathfrak{I}\otimes_k(\mathfrak{R}\otimes\widehat{\ }V) \longrightarrow \mathfrak{R}/\mathfrak{I}\otimes_k\mathfrak{R}/\mathfrak{I}\otimes_k V \longrightarrow V$. This construction can be extended to any noncommutative topological algebra $\mathfrak{R}$ over k where open right ideals from a base of neighborhoods of zero [28, Section D.5.2].

The above two natural transformations define a monad structure on the endofunctor $V \longmapsto \mathfrak{R}\otimes\widehat{\ }V$ of the category of k-vector spaces. The category of algebras/modules over this monad is equivalent to the category of $\mathfrak{R}$-contramodules as defined in Section 1.2. Indeed, for any k-vector space V there is a natural morphism of k-vector spaces $\mathfrak{R}[[V]] \longrightarrow \mathfrak{R}\otimes\widehat{\ }V$ obtained by passing to the projective limit of the maps $\mathfrak{R}/\mathfrak{I}[V] \longrightarrow \mathfrak{R}/\mathfrak{I}\otimes_k V$. Choosing a basis X in the vector space V, we see that the composition $\mathfrak{R}[[X]] \longrightarrow \mathfrak{R}[[V]] \longrightarrow \mathfrak{R}\otimes\widehat{\ }V$ is an isomorphism, so the map $\mathfrak{R}[[V]] \longrightarrow \mathfrak{R}\otimes\widehat{\ }V$ is surjective.

It essentially remains to check that for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ the contraaction map $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ factorizes through $\mathfrak{R}\otimes\hat{}\,\mathfrak{P}$. Equivalently, the image of an element of $\mathfrak{R}[[\mathfrak{P}]]$ under $\pi_{\mathfrak{P}}$ must be equal to the image of the corresponding element of $\mathfrak{R}[[X]]$ under the same map, where X is a basis of $\mathfrak{P}$ and $\mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{R}[[X]]$ is the projection onto a direct summand constructed above. This follows straightforwardly from the contraassociativity axiom: one has $\pi_{\mathfrak{P}}(\sum_{p\in\mathfrak{P}} r_p p) = \pi_{\mathfrak{P}}(\sum_{p\in\mathfrak{P}} r_p \sum_{x\in X} a_{p,x}x) = \pi_{\mathfrak{P}}(\sum_{x\in X}(\sum_{p\in\mathfrak{P}} a_{p,x}r_p)x)$, where $r_p \in \mathfrak{R}$ is a family of elements converging to zero and $p = \sum_{x\in X} a_{p,x}x$ is the presentation of an element $p \in \mathfrak{P}$ as a finite linear combination of the basis elements $x \in X$ with the coefficients $a_{p,x} \in k \subset \mathfrak{R}$.

The free $\mathfrak{R}$-contramodules are exactly the $\mathfrak{R}$-contramodules of the form $\mathfrak{R}\otimes\hat{}\,V$, where V is a k-vector space. The infinite direct sums and the tensor product of free $\mathfrak{R}$-contramodules are described by the rules $\bigoplus_\alpha \mathfrak{R}\otimes\hat{}\,V_\alpha = \mathfrak{R}\otimes\hat{}\bigoplus_\alpha V_\alpha$ and $(\mathfrak{R}\otimes\hat{}\,U)\otimes^{\mathfrak{R}}(\mathfrak{R}\otimes\hat{}\,V) = \mathfrak{R}\otimes\hat{}\,(U\otimes_k V)$, where V_α, U, V are k-vector spaces. Given a morphism $\eta\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ of topological algebras over k, the functor of contraextension of scalars E^η takes $\mathfrak{R}\otimes\hat{}\,V$ to $\mathfrak{S}\otimes\hat{}\,V$.

Now assume that $\mathfrak{R}$ is a profinite-dimensional algebra over k; in other words, $\mathfrak{R} = \mathcal{C}^*$ is the dual vector space to a coassociative, cocommutative, and counital coalgebra $\mathcal{C}$ over k. Then one has $\mathfrak{R}\otimes\hat{}\,V \simeq \operatorname{Hom}_k(\mathcal{C}, V)$, so our definition of $\mathfrak{R}$-contramodules is equivalent to the classical definition of $\mathcal{C}$-contramodules in [9]. As explained above, the category of $\mathfrak{R}$-comodules is naturally equivalent to the category of discrete $\mathfrak{R}$-modules, which are clearly the same as $\mathcal{C}$-comodules in the conventional sense. In fact, these assertions do not depend on the assumption that $\mathfrak{R}$ is commutative and $\mathcal{C}$ is cocommutative.

The distinguished $\mathfrak{R}$-comodule $\mathcal{C}(\mathfrak{R})$ is identified with the comodule $\mathcal{C}$ over the coalgebra $\mathcal{C}$, and the cofree $\mathfrak{R}$-comodules are the same as the cofree $\mathcal{C}$-comodules, i. e., those of the form $\mathcal{C}\otimes_k V$, where V is a k-vector space. The infinite products of cofree $\mathfrak{R}$-comodules and free $\mathfrak{R}$-contramodules are described by the rules $\prod_\alpha \mathcal{C}\otimes_k V_\alpha = \mathcal{C}\otimes_k \prod_\alpha V_\alpha$ and $\prod_\alpha \operatorname{Hom}_k(\mathcal{C}, V_\alpha) = \operatorname{Hom}_k(\mathcal{C}, \prod_\alpha V_\alpha)$; there are similar formulas for the infinite direct sums.

Finally, the operations of comodule Hom (taking values in contramodules), contratensor product $\odot_{\mathfrak{R}}$, cotensor product $\square_{\mathfrak{R}}$, and cohomomorphisms $\operatorname{Cohom}_{\mathfrak{R}}$, as defined in Sections 1.5 and 1.7, correspond to the operations of comodule Hom, contratensor product $\odot_{\mathcal{C}}$, cotensor product $\square_{\mathcal{C}}$, and cohomomorphisms $\operatorname{Cohom}_{\mathcal{C}}$, as defined in [28, Section 0.2], under the equivalences of categories $\mathfrak{R}\text{–}\mathsf{contra} \simeq \mathcal{C}\text{–}\mathsf{contra}$ and $\mathfrak{R}\text{–}\mathsf{comod} \simeq \mathcal{C}\text{–}\mathsf{comod}$. Given a morphism of profinite-dimensional algebras $\mathfrak{R} \longrightarrow \mathfrak{S}$ dual to a morphism of coalgebras $\mathcal{D} \longrightarrow \mathcal{C}$, where $\mathfrak{R} = \mathcal{C}^*$ and $\mathfrak{S} = \mathcal{D}^*$, the functors of contra/coextension of scalars defined in Section 1.8 correspond under our equivalences of categories to the similar functors defined in [28, Section 7.1.2] (in the case of coalgebras over a field).

2. $\mathfrak{R}$-Free and $\mathfrak{R}$-Cofree wcDG-Modules

For the rest of the paper (with the exception of the appendices), unless otherwise specified, $\mathfrak{R}$ denotes a pro-Artinian commutative topological local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k = \mathfrak{R}/\mathfrak{m}$.

We fix once and for all a grading group datum $(\Gamma, \sigma, 1)$ [30, Section 1.1]; unless a grading by the integers is specifically mentioned, all our gradings (on algebras, modules, complexes, etc.) are presumed to be Γ-gradings. The conventions and notation of *loc. cit.* are being used in connection with the Γ-gradings.

2.1. **$\mathfrak{R}$-free graded algebras and modules.** A *graded $\mathfrak{R}$-contramodule* is a family of $\mathfrak{R}$-contramodules indexed by elements of the grading group Γ. A *free graded $\mathfrak{R}$-contramodule* is a graded $\mathfrak{R}$-contramodule whose grading components are free $\mathfrak{R}$-contramodules.

The operations of tensor product $\otimes^{\mathfrak{R}}$ and internal homomorphisms $\mathrm{Hom}^{\mathfrak{R}}$ are extended to graded $\mathfrak{R}$-contramodules in the conventional way. Specifically, the components of $\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q}$ are the contramodule direct sums of the tensor products of the appropriate components of $\mathfrak{P}$ and $\mathfrak{Q}$, that is $(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q})^n = \bigoplus_{i+j=n}\mathfrak{P}^i\otimes^{\mathfrak{R}}\mathfrak{Q}^j$, while the components of $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q})$ are the direct products of the Hom contramodules between the components of $\mathfrak{P}$ and $\mathfrak{Q}$, i. e., $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q})^n = \prod_{j-i=n}\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{P}^i,\mathfrak{Q}^j)$. Notice that the operations $\otimes^{\mathfrak{R}}$ and $\mathrm{Hom}^{\mathfrak{R}}$ preserve the class of free (graded) $\mathfrak{R}$-contramodules (see Sections 1.2–1.3).

An *$\mathfrak{R}$-free graded algebra* $\mathfrak{B}$ is, by the definition, a graded algebra object in the tensor category of free $\mathfrak{R}$-contramodules; in other words, it is a free graded $\mathfrak{R}$-contramodule endowed with an (associative, noncommutative) homogeneous multiplication map $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{B}\longrightarrow\mathfrak{B}$ and a homogeneous unit map $\mathfrak{R}\longrightarrow\mathfrak{B}$ (or, equivalently, a unit element $1\in\mathfrak{B}^0$).

An *$\mathfrak{R}$-free graded left module* $\mathfrak{M}$ over $\mathfrak{B}$ is a graded left module over $\mathfrak{B}$ in the tensor category of free $\mathfrak{R}$-contramodules, i. e., a free graded $\mathfrak{R}$-contramodule endowed with an (associative and unital) homogeneous $\mathfrak{B}$-action map $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{M}\longrightarrow\mathfrak{M}$. *$\mathfrak{R}$-free graded right modules* $\mathfrak{N}$ over $\mathfrak{B}$ are defined in the similar way. Alternatively, one can define $\mathfrak{B}$-module structures on free graded $\mathfrak{R}$-contramodules in terms of the action maps $\mathfrak{M}\longrightarrow\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{M})$, and similarly for $\mathfrak{N}$. The conventional super sign rule has to be used: e. g., the two ways to represent a left action are related by the rule $f_x(b) = (-1)^{|x||b}bx$, while for a right action it is $f_y(b) = yb$, etc.

In fact, the category of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules is enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules, so the abelian group of morphisms between two $\mathfrak{R}$-free graded $\mathfrak{B}$-modules $\mathfrak{L}$ and $\mathfrak{M}$ is the underlying abelian group of the degree-zero component of a certain naturally defined (not necessarily free) graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M})$. This $\mathfrak{R}$-contramodule is constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{L},\mathfrak{M})\rightrightarrows\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{L},\mathfrak{M})$ induced by the actions of $\mathfrak{B}$ in $\mathfrak{L}$ and $\mathfrak{M}$. There is a nonobvious sign rule here, which will be needed when working with differential modules, and which is different for the left and the right graded modules; see [29, Section 1.1].

The *tensor product* $\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M}$ of an $\mathfrak{R}$-free graded right $\mathfrak{B}$-module $\mathfrak{N}$ and an $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{M}$ is a (not necessarily free) graded $\mathfrak{R}$-contramodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{M}\rightrightarrows\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{M}$.

The additive category of $\mathfrak{R}$-free graded (left or right) $\mathfrak{B}$-modules has a natural exact category structure: a short sequence of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules is said to be exact if it is (split) exact as a short sequence of free graded $\mathfrak{R}$-contramodules. The exact category of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules admits infinite direct sums and products, which are preserved by the forgetful functors to the category of free graded $\mathfrak{R}$-contramodules and preserve exact sequences.

If a morphism of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules has an $\mathfrak{R}$-free kernel (resp., cokernel) in the category of graded $\mathfrak{R}$-contramodules, then this kernel (resp., cokernel) is naturally endowed with an $\mathfrak{R}$-free graded $\mathfrak{B}$-module structure, making it the kernel (resp., cokernel) of the same morphism in the additive category of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules.

There are enough projective objects in the exact category of $\mathfrak{R}$-free graded left $\mathfrak{B}$-modules; these are the direct summands of the graded $\mathfrak{B}$-modules $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U}$ freely generated by free graded $\mathfrak{R}$-contramodules $\mathfrak{U}$. Similarly, there are enough injectives in this exact category, and these are the direct summands of the graded $\mathfrak{B}$-modules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{V})$ cofreely cogenerated by free graded $\mathfrak{R}$-contramodules $\mathfrak{V}$ (for the sign rule, see [29, Section 1.5]).

Lemma 2.1.1. (a) *Let $\mathfrak{P}$ be a projective $\mathfrak{R}$-free graded left $\mathfrak{B}$-module and $\mathfrak{M}$ be an arbitrary $\mathfrak{R}$-free graded left $\mathfrak{B}$-module. Then the graded $\mathfrak{R}$-contramodule* $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{M})$ *is free, and the functor of reduction modulo* $\mathfrak{m}$ *induces an isomorphism of graded k-vector spaces*

$$\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{M})/\mathfrak{m}\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{M})\simeq\operatorname{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M}).$$

(b) *Let $\mathfrak{L}$ be an arbitrary $\mathfrak{R}$-free graded left $\mathfrak{B}$-module and $\mathfrak{J}$ be an injective $\mathfrak{R}$-free graded left $\mathfrak{B}$-module. Then the graded $\mathfrak{R}$-contramodule* $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{J})$ *is free, and the functor of reduction modulo* $\mathfrak{m}$ *induces an isomorphism of graded k-vector spaces*

$$\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{J})/\mathfrak{m}\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{J})\simeq\operatorname{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{J}/\mathfrak{m}\mathfrak{J}).$$

(c) *For any $\mathfrak{R}$-free graded right $\mathfrak{B}$-module $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{B}$-module, there is a natural isomorphism of graded k-vector spaces*

$$(\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M})/\mathfrak{m}(\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M})\simeq(\mathfrak{N}/\mathfrak{m}\mathfrak{N})\otimes_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{M}/\mathfrak{m}\mathfrak{M}).$$

For any $\mathfrak{R}$-free graded right $\mathfrak{B}$-module $\mathfrak{N}$ and any projective $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{P}$, the graded $\mathfrak{R}$-contramodule $\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{P}$ is free.

Proof. Clearly, $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ is a graded k-algebra, and $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is a graded left $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$-module for any $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{M}$, since the functor of reduction modulo $\mathfrak{m}$ preserves tensor products of $\mathfrak{R}$-contramodules. For any $\mathfrak{R}$-free left $\mathfrak{B}$-modules $\mathfrak{L}$ and $\mathfrak{M}$, this functor induces a natural map of graded abelian groups

$$\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M})\longrightarrow\operatorname{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M}).$$

One easily checks that this map is a morphism of graded $\mathfrak{R}$-contramodules, hence there is a natural morphism of graded k-vector spaces

$$\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M})/\mathfrak{m}\,\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M}) \longrightarrow \mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M}).$$

Now if $\mathfrak{L}=\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U}$ is a graded $\mathfrak{B}$-module freely generated by a free $\mathfrak{R}$-contramodule $\mathfrak{U}$, then $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M})\simeq\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{U},\mathfrak{M})$ and $\mathfrak{L}/\mathfrak{m}\mathfrak{L}\simeq\mathfrak{B}/\mathfrak{m}\mathfrak{B}\otimes_k\mathfrak{U}/\mathfrak{m}\mathfrak{U}$, hence $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M})\simeq\mathrm{Hom}_k(\mathfrak{U}/\mathfrak{m}\mathfrak{U},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M})$, and the desired isomorphism follows from Lemma 1.3.6 and the related result of Section 1.5.

Similarly, if $\mathfrak{M}=\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{V})$ is a graded $\mathfrak{B}$-module cofreely cogenerated by a free $\mathfrak{R}$-contramodule $\mathfrak{V}$, then $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{M})\simeq\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{L},\mathfrak{V})$ and $\mathfrak{M}/\mathfrak{m}\mathfrak{M}\simeq\mathrm{Hom}_k(\mathfrak{B}/\mathfrak{m}\mathfrak{B},\ \mathfrak{V}/\mathfrak{m}\mathfrak{V})$, hence $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M})\simeq\mathrm{Hom}_k(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{V}/\mathfrak{m}\mathfrak{V})$ and the desired isomorphism follows.

Parts (a) and (b) are proven. The proof of the second assertion of part (c), based on the natural isomorphism $\mathfrak{N}\otimes_{\mathfrak{B}}(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U})\simeq\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{U}$, is similar; and the first assertion is easy, since the functor of reduction modulo $\mathfrak{m}$ preserves tensor products and cokernels in the category of $\mathfrak{R}$-contramodules. ☐

Lemma 2.1.2. *An $\mathfrak{R}$-free graded module $\mathfrak{M}$ over an $\mathfrak{R}$-free graded algebra $\mathfrak{B}$ is projective (resp., injective) if and only if the graded module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the graded k-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ is projective (resp., injective).*

Proof. The "only if" part follows from the above description of the projective and injective $\mathfrak{R}$-free graded $\mathfrak{B}$-modules. Conversely, assume that $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is a projective $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$-module. Then it is the image of a homogeneous idempotent endomorphism of a $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$-module freely generated by some graded k-vector space U. The latter can be presented as $\mathfrak{U}/\mathfrak{m}\mathfrak{U}$, where $\mathfrak{U}$ is some free graded $\mathfrak{R}$-contramodule. Set $\mathfrak{F}=\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U}$. According to Lemma 2.1.1(a), the graded k-algebra $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{F}/\mathfrak{m}\mathfrak{F},\ \mathfrak{F}/\mathfrak{m}\mathfrak{F})$ can be obtained by reducing modulo $\mathfrak{m}$ the $\mathfrak{R}$-free graded algebra $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{F},\mathfrak{F})$.

Using Lemma 1.6.1, we we can lift our idempotent endomorphism of $\mathfrak{F}/\mathfrak{m}\mathfrak{F}$ to a homogeneous idempotent endomorphism of $\mathfrak{F}$. Let $\mathfrak{P}$ be the image of the latter endomorphism; it is a projective $\mathfrak{R}$-free graded $\mathfrak{B}$-module. The graded k-vector space $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M})$ can also be obtained by reducing the free graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{M})$; in particular, our isomorphism $\mathfrak{P}/\mathfrak{m}\mathfrak{P}\simeq\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ can be lifted to a morphism of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules $\mathfrak{P}\longrightarrow\mathfrak{M}$. The latter, being an isomorphism modulo $\mathfrak{m}$, is consequently itself an isomorphism. The case of injective graded modules is similar. ☐

Corollary 2.1.3. *The homological dimension of the exact category of $\mathfrak{R}$-free graded left $\mathfrak{B}$-modules does not exceed that of the abelian category of graded left $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$-modules.*

Proof. This follows immediately from the observation that the exact category of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules has enough projective or injective objects together with Lemma 2.1.2. ☐

2.2. **Absolute derived category of $\mathfrak{R}$-free CDG-modules.** Let $\mathfrak{U}$ and $\mathfrak{V}$ be graded $\mathfrak{R}$-contramodules endowed with homogeneous $\mathfrak{R}$-contramodule endomorphisms (differentials) $d_{\mathfrak{U}}\colon \mathfrak{U} \longrightarrow \mathfrak{U}$ and $d_{\mathfrak{V}}\colon \mathfrak{V} \longrightarrow \mathfrak{V}$ of degree 1. Then the graded $\mathfrak{R}$-contramodules $\mathfrak{U} \otimes^{\mathfrak{R}} \mathfrak{V}$ and $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{U}, \mathfrak{V})$ are endowed with their differentials d defined by the conventional rules $d(u \otimes v) = d_{\mathfrak{U}}(u) \otimes v + (-1)^{|u|} u \otimes d_{\mathfrak{V}}(v)$ and $d(f)(u) = d_{\mathfrak{V}}(f(u)) - (-1)^{|f|} f(d_{\mathfrak{U}}(u))$.

An *odd derivation* d (of degree 1) of an $\mathfrak{R}$-free graded algebra $\mathfrak{B}$ is a differential ($\mathfrak{R}$-contramodule endomorphism) such that the multiplication map $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{B} \longrightarrow \mathfrak{B}$ forms a commutative diagram with the differential d on $\mathfrak{B}$ and the induced differential on $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{B}$. An *odd derivation* $d_{\mathfrak{M}}$ of an $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{M}$ *compatible with* the derivation d on $\mathfrak{B}$ is a differential ($\mathfrak{R}$-contramodule endomorphism of degree 1) such that the action map $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$ forms a commutative diagram with $d_{\mathfrak{M}}$ and the differential on $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{M}$ induced by d and $d_{\mathfrak{M}}$. Odd derivations of $\mathfrak{R}$-free graded right $\mathfrak{B}$-modules are defined similarly. Alternatively, one requires the action map $\mathfrak{M} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{M})$ to commute with the differentials.

An $\mathfrak{R}$*-free CDG-algebra* is, by the definition, a CDG-algebra object in the tensor category of free $\mathfrak{R}$-contramodules; in other words, it is an $\mathfrak{R}$-free graded algebra endowed with an odd derivation $d\colon \mathfrak{B} \longrightarrow \mathfrak{B}$ of degree 1 and a curvature element $h \in \mathfrak{B}^2$ satisfying the conventional equations $d^2(b) = [h, b]$ for all $b \in \mathfrak{B}$ and $d(h) = 0$. Morphisms $\mathfrak{B} \longrightarrow \mathfrak{A}$ of $\mathfrak{R}$-free CDG-algebras are defined as pairs (f, a), with $f\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ being a morphism of $\mathfrak{R}$-free graded algebras and $a \in \mathfrak{A}^1$, satisfying the conventional equations (see [29, Section 3.1] or [31, Section 1.1]).

An $\mathfrak{R}$*-free left CDG-module* $\mathfrak{M}$ over $\mathfrak{B}$ is, by the definition, an $\mathfrak{R}$-free graded left $\mathfrak{B}$-module endowed with an odd derivation $d_{\mathfrak{M}}\colon \mathfrak{M} \longrightarrow \mathfrak{M}$ of degree 1 compatible with the derivation d on $\mathfrak{B}$ and satisfying the equation $d_{\mathfrak{M}}^2(x) = hx$ for all $x \in \mathfrak{M}$. The definition of an $\mathfrak{R}$*-free right CDG-module* $\mathfrak{N}$ over $\mathfrak{B}$ is similar; the only nonobvious difference is that the equation for the square of the differential has the form $d_{\mathfrak{N}}^2(y) = -yh$ for all $y \in \mathfrak{N}$.

$\mathfrak{R}$-free left (resp., right) CDG-modules over $\mathfrak{B}$ naturally form a DG-category, which we will denote by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ (resp., $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B}$). In fact, these DG-categories are enriched over the tensor category of (complexes of) $\mathfrak{R}$-contramodules, so the complexes of morphisms in $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B}$ are the underlying complexes of abelian groups for naturally defined complexes of $\mathfrak{R}$-contramodules.

The underlying graded $\mathfrak{R}$-contramodules of these complexes were defined in Section 2.1. Given two left CDG-modules $\mathfrak{L}$ and $\mathfrak{M} \in \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$, the differential in the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathfrak{M})$ is defined by the conventional formula $d(f)(x) = d_{\mathfrak{M}}(f(x)) - (-1)^{|f|} f(d_{\mathfrak{L}}(x))$; one easily checks that $d^2(f) = 0$. For any two right CDG-modules $\mathfrak{K}$ and $\mathfrak{N} \in \mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B}$, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{B}^{\mathrm{op}}}(\mathfrak{K}, \mathfrak{N})$ is defined in the similar way.

Passing to the zero cohomology of the complexes of morphisms, we construct the homotopy categories of $\mathfrak{R}$-free CDG-modules $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})$. Even though these are also naturally enriched over $\mathfrak{R}$-contramodules, we will mostly consider them as conventional categories with abelian groups of morphisms. Since the DG-categories $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B}$ have shifts, twists, and infinite direct sums and products, their homotopy categories are triangulated categories with infinite direct sums and products.

Remark 2.2.1. Our reason for hesitating to work with triangulated categories enriched over $\mathfrak{R}$-contramodules *as* our main setting is that we would like to be able to take the Verdier quotient categories freely and without always worrying about the existence of adjoint functors to the localization functor. The problem is that the Verdier localization acts on morphisms as a kind of inductive limit, and inductive limits of $\mathfrak{R}$-contramodules are *not* well-behaved; in particular, they do not commute with the forgetful functors to abelian groups. When the localization functor *does* have an adjoint, though, the inductive limits involved in the quotient category construction are *stabilizing* inductive limits, which have all the good properties, of course.

So we generally consider triangulated categories with abelian groups of morphisms; alternatively, our triangulated categories can be viewed as being $\mathfrak{R}$-*linear*, i. e., having (abstract, nontopological) $\mathfrak{R}$-modules of morphisms. The triangulated categories that we are most interested in will be eventually presented as appropriate resolution subcategories of the homotopy categories and consequently endowed with DG-enhancements; these will be naturally enriched over (complexes of) $\mathfrak{R}$-contramodules, and in fact, even *free* $\mathfrak{R}$-contramodules. In our notation, these will be spoken of as the "derived functors Ext" defined on our triangulated categories and taking values in the homotopy category of free $\mathfrak{R}$-contramodules.

The tensor product $\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M}$ of a right $\mathfrak{R}$-free CDG-module $\mathfrak{N}$ and a left $\mathfrak{R}$-free CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ is a complex of $\mathfrak{R}$-contramodules obtained by endowing the graded $\mathfrak{R}$-contramodule $\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M}$ constructed in Section 2.1 with the conventional differential $d(y\otimes x)=d_{\mathfrak{N}}(y)\otimes x+(-1)^{|y|}y\otimes d_{\mathfrak{M}}(x)$. The tensor product of $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ is a triangulated functor of two arguments

$$\otimes_{\mathfrak{B}}\colon H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})\times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}),$$

where $H^0(\mathfrak{R}\text{–}\mathsf{contra})$ denotes, by an abuse of notation, the homotopy category of complexes of $\mathfrak{R}$-contramodules.

An $\mathfrak{R}$-free left CDG-module over $\mathfrak{B}$ is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of absolutely acyclic $\mathfrak{R}$-free CDG-modules is called the *absolute derived category* of $\mathfrak{R}$-free left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The absolute derived category of $\mathfrak{R}$-free right CDG-modules over $\mathfrak{B}$, denoted by $\mathsf{D}^{\mathsf{abs}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})$, is defined similarly (see [29, Sections 1.2 and 3.3] or [30, Section 3.2]).

An $\mathfrak{R}$-free left CDG-module over $\mathfrak{B}$ is said to be *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ and closed under infinite products. The quotient cateogry of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-free CDG-modules is called the *contraderived* *category* of $\mathfrak{R}$-free left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The (similarly defined) contraderived category of $\mathfrak{R}$-free right CDG-modules over $\mathfrak{B}$ is denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})$.

An $\mathfrak{R}$-free left CDG-module over $\mathfrak{B}$ is said to be *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-free CDG-modules is called the *coderived category* of $\mathfrak{R}$-free left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The coderived category of $\mathfrak{R}$-free right CDG-modules over $\mathfrak{B}$ is denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})$.

Denote by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}} \subset \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ whose underlying $\mathfrak{R}$-free graded $\mathfrak{B}$-modules are projective. Similarly, let $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}} \subset \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ be the full DG-subcategory formed by all the $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ whose underlying $\mathfrak{R}$-free graded $\mathfrak{B}$-modules are injective. The corresponding homotopy categories are denoted by $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$, respectively.

Lemma 2.2.2. (a) *Let $\mathfrak{P}$ be a CDG-module from $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$. Then $\mathfrak{P}$ is contractible (i. e., represents a zero object in $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$) if and only if the CDG-module $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over the CDG-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ over k is contractible.*

(b) *Let $\mathfrak{J}$ be a CDG-module from $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$. Then $\mathfrak{J}$ is contractible if and only if the CDG-module $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ over the CDG-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ is contractible.*

Proof. According to Lemma 2.1.1(a-b), we have $\mathfrak{R}$-free DG-algebras $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{P})$ and $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{J},\mathfrak{J})$ whose reductions modulo $\mathfrak{m}$ are naturally isomorphic to the DG-algebras $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\ \mathfrak{P}/\mathfrak{m}\mathfrak{P})$ and $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{J}/\mathfrak{m}\mathfrak{J},\ \mathfrak{J}/\mathfrak{m}\mathfrak{J})$. If, e. g., the CDG-module $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ is contractible, then the DG-algebra $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\ \mathfrak{P}/\mathfrak{m}\mathfrak{P})$ over k is acyclic, and it follows by virtue of Lemma 1.3.3 that the $\mathfrak{R}$-free DG-algebra $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{P})$ is contractible as a complex of $\mathfrak{R}$-contramodules. Hence, in particular, the complex $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{P})$ is acyclic and $\mathfrak{P}$ is a contractible CDG-module over $\mathfrak{B}$. $\square$

Theorem 2.2.3. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Then*

(a) *for any CDG-module $\mathfrak{P} \in H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and any contraacyclic $\mathfrak{R}$-free left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{M})$ is contractible;*

(b) *for any coacyclic $\mathfrak{R}$-free left CDG-module $\mathfrak{L}$ over $\mathfrak{B}$ and any CDG-module $\mathfrak{J} \in H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathfrak{J})$ is contractible.*

Proof. By the first assertions of Lemma 2.1.1(a-b), the complexes $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P},\mathfrak{N})$ and $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{K},\mathfrak{J})$ are complexes of free $\mathfrak{R}$-contramodules for any $\mathfrak{R}$-free CDG-module $\mathfrak{N}$ or $\mathfrak{K}$ over $\mathfrak{B}$. To any short exact sequence of $\mathfrak{R}$-free CDG-modules $\mathfrak{N}$ or $\mathfrak{K}$, short exact sequences of such complexes of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{B}}$ correspond; and the totalization of a short exact sequence of complexes of free $\mathfrak{R}$-contramodules is a contractible complex of $\mathfrak{R}$-contramodules.

Finally, to infinite products of the CDG-modules $\mathfrak{N}$ or infinite direct sums of the CDG-modules $\mathfrak{K}$, our functors $\mathrm{Hom}_{\mathfrak{B}}$ assign infinite products of complexes of $\mathfrak{R}$-contramodules, and to cones of morphisms of the CDG-modules $\mathfrak{N}$ or $\mathfrak{K}$ they assign the (co)cones of morphisms of complexes of $\mathfrak{R}$-contramodules. The class of contractible complexes of $\mathfrak{R}$-contramodules is closed under both operations. (Cf. [29, Theorem 3.5 and Remark 3.5], and Theorem 4.2.5 below.) ☐

Theorem 2.2.4. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Assume that the exact category of $\mathfrak{R}$-free graded left $\mathfrak{B}$-modules has finite homological dimension. Then the compositions of natural functors $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ are equivalences of triangulated categories.*

Proof. The proof is completely analogous to that in the case of a CDG-algebra over a field or a CDG-ring [29, Theorem 3.6], and is a particular case of the general scheme of [29, Remark 3.6]. The only aspect of these arguments that needs some additional comments here is the constructions of the $\mathfrak{R}$-free CDG-module $G^+(\mathfrak{L})$ over $\mathfrak{B}$ freely generated by an $\mathfrak{R}$-free graded $\mathfrak{B}$-module $\mathfrak{L}$ and the $\mathfrak{R}$-free CDG-module $G^-(\mathfrak{L})$ cofreely cogenerated by $\mathfrak{L}$.

The free graded $\mathfrak{R}$-contramodule $G^+(\mathfrak{L})$ is defined by the rule $G^+(\mathfrak{L})^i = \mathfrak{L}^i \oplus \mathfrak{L}^{i-1}$, the elements of $G^+(\mathfrak{L})^i$ are denoted formally by $x + dy$, where $x \in \mathfrak{L}^i$ and $y \in \mathfrak{L}^{i-1}$, and the differential and the left action of $\mathfrak{B}$ in $G^+(\mathfrak{L})$ are defined by the formulas from [29, proof of Theorem 3.6] (expressing the CDG-module axioms). Similarly, one sets $G^-(\mathfrak{L})^i = \mathfrak{L}^{i+1} \oplus \mathfrak{L}^i$ as a graded $\mathfrak{R}$-contramodule, and defines the differential and the action of $\mathfrak{B}$ in $G^-(\mathfrak{L})$ by the formulas from [29]. ☐

Corollary 2.2.5. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Assume that the exact category of $\mathfrak{R}$-free graded left $\mathfrak{B}$-modules has finite homological dimension. Then an $\mathfrak{R}$-free left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ is absolutely acyclic if and only if the CDG-module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the CDG-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ over the field k is absolutely acyclic.*

Proof. The "only if" part is clear, because the functor of reduction modulo $\mathfrak{m}$ preserves exact triples of $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$. To prove "if", notice that by Theorems 2.2.3–2.2.4, $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ with projective underlying graded $\mathfrak{B}$-modules and absolutely acyclic $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$ form a semiorthogonal decomposition of the homotopy category of $\mathfrak{R}$-free CDG-modules $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. So it suffices to show that for any CDG-module $\mathfrak{P} \in H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ the complex $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{P}, \mathfrak{M})$ is acyclic (as a complex of abelian groups).

By Lemma 2.1.1(a), this complex is the underlying complex of abelian groups to a complex of free $\mathfrak{R}$-contramodules, and reducing this complex of $\mathfrak{R}$-contramodules modulo $\mathfrak{m}$ one obtains the complex of k-vector spaces $\mathrm{Hom}_{\mathfrak{B}/\mathfrak{m}\mathfrak{B}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\, \mathfrak{M}/\mathfrak{m}\mathfrak{M})$. Since the latter complex is acyclic, the desired assertion follows from Lemma 1.3.3. Alternatively, one can use Theorem 2.2.4 together with Lemma 2.2.2. ☐

The following theorem is to be compared with Theorem 3.2.2 below.

Theorem 2.2.6. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Then the composition of natural functors* $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ *is an equivalence of triangulated categories whenever the graded k-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ satisfies the condition* ($**$) *from* [29, *Section* 3.8]. *The composition of natural functors* $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{inj}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ *is an equivalence of triangulated categories whenever the graded k-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ satisfies the condition* ($*$) *from* [29, *Section* 3.7].

Proof. The semiorthogonality being known from Theorem 2.2.3, it remains to show the existence of resolutions. These are provided by the constructions of [29, Sections 3.7–3.8] applied to $\mathfrak{R}$-free CDG-modules over $\mathfrak{B}$. The injectivity/projectivity properties of the $\mathfrak{R}$-free graded $\mathfrak{B}$-modules so obtained follow from Lemma 2.1.2. $\square$

Clearly, the assertion of Corollary 2.2.5 holds with the absolute acyclicity property replaced with the contra- or coacyclicity whenever the corresponding assertion of Theorem 2.2.6 holds.

Let $f = (f, a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ be a morphism of $\mathfrak{R}$-free CDG-algebras. Then any $\mathfrak{R}$-free graded module over $\mathfrak{A}$ can be endowed with a graded $\mathfrak{B}$-module structure via f, and any homogeneous morphism (of any degree) between graded $\mathfrak{A}$-modules can be also considered as a homogeneous morphism (of the same degree) between graded $\mathfrak{B}$-modules. With any $\mathfrak{R}$-free left CDG-module $(\mathfrak{M}, d_{\mathfrak{M}})$ over $\mathfrak{A}$ one can associate an $\mathfrak{R}$-free left CDG-module $(\mathfrak{M}, d'_{\mathfrak{M}})$ over $\mathfrak{B}$ with the modified differential $d'_{\mathfrak{M}}(x) = d_{\mathfrak{M}}(x) + ax$. Similarly, for any $\mathfrak{R}$-free right CDG-module $(\mathfrak{N}, d_{\mathfrak{N}})$ over $\mathfrak{A}$ the formula $d'_{\mathfrak{N}}(y) = d_{\mathfrak{N}}(y) - (-1)^{|y|}ya$ defines a modified differential on $\mathfrak{N}$ making $(\mathfrak{N}, d'_{\mathfrak{N}})$ an $\mathfrak{R}$-free right CDG-module over $\mathfrak{B}$. We have constructed the DG-functors of restriction of scalars $R_f\colon \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A} \longrightarrow \mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{B}$; passing to the homotopy categories, we obtain the triangulated functors

$$R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$$

and $H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A}) \longrightarrow H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{B})$.

2.3. **Semiderived category of $\mathfrak{R}$-free wcDG-modules.** A *weakly curved differential graded algebra*, or *wcDG-algebra* over $\mathfrak{R}$ is, by the definition, an $\mathfrak{R}$-free CDG-algebra $(\mathfrak{A}, d, h)$ with the curvature element h belonging to $\mathfrak{m}\mathfrak{A}^2$. A *morphism of wcDG-algebras* over $\mathfrak{R}$ (*wcDG-morphism*) $\mathfrak{B} \longrightarrow \mathfrak{A}$ is a morphism (f, a) between $\mathfrak{B}$ and $\mathfrak{A}$ considered as $\mathfrak{R}$-free CDG-algebras such that the change-of-connection element a belongs to $\mathfrak{m}\mathfrak{A}^1$. So the reduction modulo $\mathfrak{m}$ of a wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$ is a DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ over the field k, and the reduction of a wcDG-algebra morphism is a conventional DG-algebra morphism.

CDG-modules over a wcDG-algebra will be referred to as *wcDG-modules*. All the above definitions, constructions, and notation related to $\mathfrak{R}$-free CDG-modules will be applied to the particular case of $\mathfrak{R}$-free wcDG-modules as well.

An $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over a wcDG-algebra $\mathfrak{A}$ is said to be *semiacyclic* if the complex of k-vector spaces $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ (which is in fact a DG-module over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$) is acyclic. In particular, it follows from Lemma 1.3.3 that when

$\mathfrak{A}$ itself is an $\mathfrak{R}$-free DG-algebra (i. e., the curvature element h of $\mathfrak{A}$ vanishes), an $\mathfrak{R}$-free CDG-module $\mathfrak{M}$ over $\mathfrak{A}$ is semiacyclic if and only if $\mathfrak{M}$ is contractible as a complex of free $\mathfrak{R}$-contramodules.

Clearly, the property of semiacyclicity of an $\mathfrak{R}$-free wcDG-module over $\mathfrak{A}$ is preserved by shifts, cones and homotopy equivalences (i. e., isomorphisms in $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$). The class of semiacyclic $\mathfrak{R}$-free wcDG-modules is also closed with respect to infinite direct sums and products. The quotient category of the homotopy category $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of semiacyclic $\mathfrak{R}$-free wcDG-modules is called the *semiderived category* of $\mathfrak{R}$-free left wcDG-modules over $\mathfrak{A}$ and denoted by $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The semiderived category of $\mathfrak{R}$-free right wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A})$ is defined similarly.

Every absolutely acyclic $\mathfrak{R}$-free wcDG-module, and in fact every contraacyclic and every coacyclic $\mathfrak{R}$-free wcDG-module is semiacyclic. Indeed, the functor of reduction modulo $\mathfrak{m}$ preserves the absolute acyclicity, contra- and coacyclicity properties, and any contraacyclic or coacyclic DG-module over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is acyclic.

Notice that any acyclic complex of free $\mathfrak{R}$-contramodules is contractible when (the category of contramodules over) $\mathfrak{R}$ has finite homological dimension. In this case, we call semiacyclic $\mathfrak{R}$-free wcDG-modules over $\mathfrak{A}$ over simply *acyclic*, and refer to the semiderived category of $\mathfrak{R}$-free wcDG-modules as their *derived category*.

An $\mathfrak{R}$-free wcDG-module $\mathfrak{P}$ over $\mathfrak{A}$ is called *homotopy projective* if the complex $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P},\mathfrak{M})$ is acyclic for any semiacyclic $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$. Similarly, an $\mathfrak{R}$-free wcDG-module $\mathfrak{J}$ over $\mathfrak{A}$ is called *homotopy injective* if the complex $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{M},\mathfrak{J})$ is acyclic for any semiacyclic $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$. The full triangulated subcategories in $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ formed by the homotopy projective and homotopy injective $\mathfrak{R}$-free wcDG-modules are denoted by $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{proj}}$ and $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{inj}}$, respectively. (Notice the difference with the notation for graded projective/injective CDG-modules introduced in Section 2.2.)

Lemma 2.3.1. (a) *A wcDG-module $\mathfrak{P}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ is homotopy projective if and only if the DG-module $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ over the field k is homotopy projective.*

(b) *A wcDG-module $\mathfrak{J}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ is homotopy injective if and only if the DG-module $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is homotopy injective.*

Proof. We will prove the "if" assertions here. Then it will follow from the proof of Theorem 2.3.2 below that any homotopy projective/injective $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ is homotopy equivalent to a graded projective/injective $\mathfrak{R}$-free wcDG-module with a homotopy projective/injective reduction modulo $\mathfrak{m}$; hence the reduction $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ of the wcDG-module $\mathfrak{M}$, being homotopy equivalent to a homotopy projective/injective DG-module, is itself homotopy projective/injective.

Indeed, let $\mathfrak{N}$ be a semiacyclic wcDG-module over $\mathfrak{A}$. By Lemma 2.1.1(a-b), the complexes $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P},\mathfrak{N})$ and $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{N},\mathfrak{J})$ are the underlying complexes of abelian groups to complexes of free $\mathfrak{R}$-contramodules whose reductions modulo $\mathfrak{m}$ are the complexes of k-vector spaces $\mathrm{Hom}_{\mathfrak{A}/\mathfrak{m}\mathfrak{A}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\mathfrak{N}/\mathfrak{m}\mathfrak{N})$ and $\mathrm{Hom}_{\mathfrak{A}/\mathfrak{m}\mathfrak{A}}(\mathfrak{N}/\mathfrak{m}\mathfrak{N},\mathfrak{J}/\mathfrak{m}\mathfrak{J})$.

Since the latter two complexes are acyclic by assumption, so are the former two complexes (see Lemma 1.3.3). $\square$

Introduce the notation $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})_{\mathsf{proj}}$ for the intersection $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{proj}} \cap H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$, i. e., the homotopy category of homotopy projective $\mathfrak{R}$-free left CDG-modules over $\mathfrak{A}$ whose underlying graded $\mathfrak{A}$-modules are also projective. Similarly, let $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})_{\mathsf{inj}} = H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{inj}} \cap H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$ denote the homotopy category of homotopy injective $\mathfrak{R}$-free left CDG-modules over $\mathfrak{A}$ whose underlying graded $\mathfrak{A}$-modules are injective.

Theorem 2.3.2. (a) *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the compositions of functors*
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{proj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$$
and
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})_{\mathsf{proj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$$
are equivalences of triangulated categories.

(b) *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the compositions of functors*
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})_{\mathsf{inj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$$
and
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})_{\mathsf{inj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}).$$
are equivalences of triangulated categories.

Proof. It suffices to construct for any $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$ two closed morphisms of $\mathfrak{R}$-free wcDG-modules $\mathfrak{P} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{J}$ with semiacyclic cones such that the wcDG-module $\mathfrak{P}$ is homotopy projective and the wcDG-module $\mathfrak{J}$ is homotopy injective. In fact, we will have $\mathfrak{P} \in \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ with $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ homotopy projective and $\mathfrak{J} \in \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ with $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ homotopy injective over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. Then we will use the "if" assertions of Lemma 2.3.1, and have proven the "only if" assertions.

To obtain the wcDG-modules $\mathfrak{P}$ and $\mathfrak{J}$, we will use a curved version of the bar- and cobar-resolutions. Consider the bigraded $\mathfrak{B}$-module whose component of the internal grading $n \geqslant 1$ is the graded $\mathfrak{B}$-module $\mathfrak{B}^{\otimes^{\mathfrak{R}} n} \otimes^{\mathfrak{R}} \mathfrak{M}$ with the action of $\mathfrak{B}$ given by the rule $b(b_1 \otimes \cdots \otimes b_n \otimes x) = (-1)^{|b|(n-1)} bb_1 \otimes b_2 \otimes \cdots \otimes b_n \otimes x$ for b, $b_s \in \mathfrak{B}$ and $x \in \mathfrak{M}$. Define the differentials ∂, d, and δ on this $\mathfrak{B}$-module by the rules
$$\begin{aligned}\partial(b_1 \otimes \cdots \otimes b_n \otimes x) &= b_1b_2 \otimes b_3 \otimes \cdots b_n \otimes x - b_1 \otimes b_2b_3 \otimes b_4 \otimes \cdots \otimes b_n \otimes x \\ &\quad + \cdots + (-1)^n b_1 \otimes \cdots \otimes b_{n-2} \otimes b_{n-1}b_n \otimes x + (-1)^{n+1} b_1 \otimes \cdots \otimes b_{n-1} \otimes b_n x,\end{aligned}$$
$$\begin{aligned}(-1)^{n-1} d(b_1 \otimes \cdots \otimes b_n \otimes x) &= d(b_1) \otimes b_2 \otimes \cdots \otimes b_n \otimes x \\ &\quad + (-1)^{|b_1|} b_1 \otimes d(b_2) \otimes \cdots \otimes b_n \otimes x + \cdots + (-1)^{|b_1|+\cdots+|b_n|} b_1 \otimes \cdots b_n \otimes d(x),\end{aligned}$$
and
$$\begin{aligned}\delta(b_1 \otimes \cdots \otimes b_n \otimes x) &= b_1 \otimes h \otimes b_2 \otimes \cdots \otimes b_n \otimes x - \cdots \\ &\quad + (-1)^n b_1 \otimes \cdots \otimes b_{n-1} \otimes h \otimes b_n \otimes x + (-1)^{n+1} b_1 \otimes \cdots \otimes b_n \otimes h \otimes x,\end{aligned}$$

and endow it with the total differential $\partial+d+\delta$. Totalizing our bigraded $\mathfrak{B}$-module by taking infinite direct sums (in the category of $\mathfrak{R}$-contramodules) along the diagonals where the difference $i = |b_1|+\dots+|b_n|+|x|-n+1$ is constant, we obtain the desired wcDG-module $\mathfrak{P}$ over $\mathfrak{B}$. The closed morphism of wcDG-modules $\mathfrak{P}\longrightarrow\mathfrak{M}$ is defined by the rules $b\otimes x\longmapsto bx$ and $b_1\otimes\dots\otimes b_n\otimes x\longmapsto 0$ for $n\geqslant 2$.

Similarly, consider the bigraded $\mathfrak{B}$-module whose component of the internal graded $n\geqslant 1$ is the graded $\mathfrak{B}$-module $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B}^{\otimes^{\mathfrak{R}} n},\,\mathfrak{M})$ with the action of $\mathfrak{B}$ given by the rule $(bf)(b_1,\dots,b_n)=(-1)^{|b|(|f|+|b_1+\dots+|b_n|+n-1)}f(b_1,\dots,b_nb)$, where $f\colon\mathfrak{B}^{\otimes^{\mathfrak{R}} n}\longrightarrow\mathfrak{M}$ is a homogeneous graded $\mathfrak{R}$-contramodule morphism of degree $|f|$, while b, $b_s\in\mathfrak{B}$ and $f(b_1,\dots,b_n)\in\mathfrak{M}$. Define the differentials ∂, d, and δ on this $\mathfrak{B}$-module by the rules

$$(\partial f)(b_1,\dots,b_n)=(-1)^{|f||b_1|}b_1f(b_2,\dots,b_n)-f(b_1b_2,b_3,\dots,b_n)\\ +f(b_1,b_2b_3,b_4,\dots,b_n)-\dots+(-1)^{n-1}f(b_1,\dots,b_{n-2},b_{n-1}b_n),$$

$$(-1)^{n-1}(df)(b_1,\dots,b_n)=d(f(b_1,\dots,b_n))-(-1)^{|f|}f(db_1,b_2,\dots,b_n)\\ -(-1)^{|f|+|b_1|}f(b_1,db_2,\dots,b_n)-\dots-(-1)^{|f|+|b_1|+\dots+|b_{n-1}|}f(b_1,\dots,b_{n-1},db_n),$$

and

$$(\delta f)(b_1,\dots,b_n)=-f(h,b_1,\dots,b_n)\\ +f(b_1,h,b_2,\dots,b_n)-\dots+(-1)^nf(b_1,\dots,b_{n-1},h,b_n)$$

and endow it with the total differential $\partial+d+\delta$. Totalizing our bigraded $\mathfrak{B}$-module by taking infinite products along the diagonals where the sum $i=|f|+n-1$ is constant, we obtain the desired wcDG-module $\mathfrak{J}$ over $\mathfrak{B}$. The closed morphism of wcDG-modules $\mathfrak{M}\longrightarrow\mathfrak{J}$ is defined by the rule $x\longmapsto f_x$, $f_x(b)=(-1)^{|x||b|}bx$ and $f_x(b_1,\cdots,b_n)=0$ for $n\geqslant 2$, where $x\in\mathfrak{M}$.

The reductions $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ and $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ of $\mathfrak{P}$ and $\mathfrak{J}$ modulo $\mathfrak{m}$ are the conventional bar- and cobar-resolutions of the DG-module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ over the field k. Hence DG-module $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ is homotopy projective, the DG-module $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ is homotopy injective, and the cones of the morphisms $\mathfrak{P}/\mathfrak{m}\mathfrak{P}\longrightarrow\mathfrak{M}/\mathfrak{m}\mathfrak{M}\longrightarrow\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ are acyclic (see [29, proofs of Theorems 1.4–1.5]). □

Notice that it follows from the above arguments that for any homotopy projective $\mathfrak{R}$-free wcDG-module $\mathfrak{P}$ and semiacyclic $\mathfrak{R}$-free wcDG-module $\mathfrak{N}$ over $\mathfrak{A}$ the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P},\mathfrak{N})$ is contractible. Similarly, for any semiacyclic $\mathfrak{R}$-free wcDG-module $\mathfrak{N}$ and homotopy injective $\mathfrak{R}$-free wcDG-module $\mathfrak{J}$ over $\mathfrak{A}$ the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{N},\mathfrak{J})$ is contractible (cf. Theorem 4.3.1 below).

Theorem 2.3.3. *Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$. Assume that the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is cofibrant (in the standard model structure on the category of DG-algebras over k). Then an $\mathfrak{R}$-free wcDG-module over $\mathfrak{A}$ is semiacyclic if and only if it is absolutely acyclic. So the semiderived category of $\mathfrak{R}$-free wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ over $\mathfrak{A}$ coincides with their absolute derived category $\mathsf{D}^{\mathsf{abs}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$.*

Proof. Clearly, the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ has finite homological dimension, so Corollary 2.2.5 is applicable in view of Corollary 2.1.3. By [29, Theorem 9.4], a DG-module over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is acyclic if and only if it is absolutely acyclic, so the assertion follows. $\square$

Lemma 2.3.4. *Let $\mathfrak{P}$ be a homotopy projective $\mathfrak{R}$-free left wcDG-module over a wcDG-algebra $\mathfrak{A}$, and let $\mathfrak{N}$ be a semiacyclic $\mathfrak{R}$-free right wcDG-module over $\mathfrak{A}$. Then the tensor product $\mathfrak{N}\otimes_{\mathfrak{A}}\mathfrak{P}$ is a contractible complex of $\mathfrak{R}$-contramodules.*

Proof. It follows from Theorem 2.3.2(a) that any homotopy projective $\mathfrak{R}$-free wcDG-module $\mathfrak{P}$ over $\mathfrak{A}$ is homotopy equivalent to a wcDG-module with a projective underlying $\mathfrak{R}$-free graded $\mathfrak{B}$-module; so we can assume that $\mathfrak{P}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}}$. Then, by Lemma 2.1.1(c), the tensor product $\mathfrak{N}\otimes_{\mathfrak{A}}\mathfrak{P}$ is a complex of free $\mathfrak{R}$-contramodules whose reduction modulo $\mathfrak{m}$ is isomorphic to the complex of k-vector spaces $(\mathfrak{N}/\mathfrak{m}\mathfrak{N})\otimes_{\mathfrak{A}/\mathfrak{m}\mathfrak{A}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P})$. The latter complex is acyclic, since the DG-module $\mathfrak{N}/\mathfrak{m}\mathfrak{N}$ is acyclic and the DG-module $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ is homotopy flat over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ by Lemma 2.3.1(a) and the results of [29, Section 1.6]. It remains to apply Lemma 1.3.3. $\square$

The left derived functor of tensor product of $\mathfrak{R}$-free wcDG-modules

$$\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}),$$

where $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ denotes, by an abuse of notation, the homotopy category of complexes of free $\mathfrak{R}$-contramodules, is constructed by restricting the functor $\otimes_{\mathfrak{A}}$ to either of the full subcategories $H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}})_{\mathsf{proj}}$ or $H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\subset H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$. Here $H^0(\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}$ denotes the homotopy category of homotopy projective $\mathfrak{R}$-free right wcDG-modules over $\mathfrak{A}$ with projective underlying graded $\mathfrak{A}$-modules. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 2.1.1(c) and factorizes through the Cartesian product of semiderived categories $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ by Theorem 2.3.2(a) and Lemma 2.3.4.

Similarly, the right derived functor of homomorphisms of $\mathfrak{R}$-free wcDG-modules

$$\operatorname{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}}\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\operatorname{Hom}_{\mathfrak{A}}$ to either of the full subcategories $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}})^{\mathsf{op}}_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ or $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{inj}})_{\mathsf{inj}}\subset H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$.

Let $(f,a)\colon\mathfrak{B}\longrightarrow\mathfrak{A}$ be a morphism of wcDG-algebras over $\mathfrak{R}$. Then we have the induced functor $R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$. Notice that, by the definition of a wcDG-morphism, this functor agrees with the functor $R_{f/\mathfrak{m}f}\colon H^0(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod})\longrightarrow H^0(\mathfrak{B}/\mathfrak{m}\mathfrak{B}\text{–}\mathsf{mod})$ induced on the homotopy categories of DG-modules over DG-algebras over k by the DG-algebra morphism $f/\mathfrak{m}f\colon\mathfrak{B}/\mathfrak{m}\mathfrak{B}\longrightarrow\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. Hence the functor R_f preserves semiaciclicity of wcDG-modules and therefore induces a triangulated functor

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow\mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$$

on the semiderived categories.

The triangulated functor $\mathbb{I}R_f$ has adjoints on both sides. The DG-functor $E_f\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ is defined on the level of graded modules by the rule $\mathfrak{N} \longmapsto \mathfrak{A}\otimes_{\mathfrak{B}}\mathfrak{N}$; the differential on $\mathfrak{A}\otimes_{\mathfrak{B}}\mathfrak{N}$ induced by the differentials on $\mathfrak{A}$ and $\mathfrak{N}$ is modified to obtain the differential on $E_f(\mathfrak{N})$ using the change-of-connection element a. Restricting the induced triangulated functor $E_f\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ to $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})_{\mathsf{proj}}$, composing it with the localization functor $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, and taking into account Theorem 2.3.2(a), we construct the left derived functor

$$\mathbb{L}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}),$$

which is left adjoint to the functor $\mathbb{I}R_f$.

Similarly, the DG-functor $E^f\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ is defined on the level of graded modules by the rule $\mathfrak{N} \longmapsto \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{A},\mathfrak{N})$; the change-of-connection element a is used to modify the differential on $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{A},\mathfrak{N})$ induced by the differentials on $\mathfrak{A}$ and $\mathfrak{N}$. Restricting the induced triangulated functor E^f to $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})_{\mathsf{inj}}$, composing it with the localization functor to $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and using Theorem 2.3.2(b), we obtain the right derived functor

$$\mathbb{R}E^f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}),$$

which is right adjoint to the functor $\mathbb{I}R_f$.

Theorem 2.3.5. *The functor* $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *is an equivalence of triangulated categories whenever the DG-algebra morphism* $f/\mathfrak{m}f\colon \mathfrak{B}/\mathfrak{m}\mathfrak{B} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ *is a quasi-isomorphism.*

Proof. It suffices to check that the adjunction morphisms for the functors $\mathbb{L}E_f$, $\mathbb{I}R_f$, $\mathbb{R}E^f$ are isomorphisms. However, a morphism in a semiderived category of $\mathfrak{R}$-free wcDG-modules is an isomorphism whenever it becomes an isomorphism in the derived category of DG-modules after the reduction modulo $\mathfrak{m}$. Our three functors make commutative diagrams with the similar functors $\mathbb{L}E_{f/\mathfrak{m}f}$, $\mathbb{I}R_{f/\mathfrak{m}f}$, $\mathbb{R}E^{f/\mathfrak{m}f}$ for DG-modules over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ and $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ and the functors of reduction modulo $\mathfrak{m}$, and the adjunctions agree. Finally, it remains to use the DG-algebra version of the desired assertion, which is well-known [29, Theorem 1.7]. $\square$

Remark 2.3.6. Unlike in the situation of DG-algebras over a field or DG-rings [29, Theorem 1.7], the converse assertion to Theorem 2.3.5 is *not* true for wcDG-algebras over a pro-Artinian topological local ring $\mathfrak{R}$. Moreover, there exist wcDG-algebras $\mathfrak{A}$ with the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ having a nonzero cohomology k-algebra, while the semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ vanishes. A counterexample, essentially from [22], will be discussed below in Example 5.3.5.

2.4. **$\mathfrak{R}$-cofree graded modules.** A *graded* $\mathfrak{R}$*-comodule* is a family of $\mathfrak{R}$-comodules indexed by elements of the grading group Γ. A *cofree graded* $\mathfrak{R}$*-comodule* is a graded $\mathfrak{R}$-comodule whose grading components are cofree $\mathfrak{R}$-comodules.

The operations of contratensor product $\odot_{\mathfrak{R}}$, contrahomomorphisms $\mathrm{Ctrhom}_{\mathfrak{R}}$, and comodule homomorphisms $\mathrm{Hom}_{\mathfrak{R}}$ are extended to graded $\mathfrak{R}$-contramodules and graded $\mathfrak{R}$-comodules in the following way. The components of $\mathfrak{P}\odot_{\mathfrak{R}}\mathcal{M}$ are the direct sums of the contratensor products of the appropriate components of $\mathfrak{P}$ and $\mathcal{M}$, i. e., $(\mathfrak{P}\odot_{\mathfrak{R}}\mathcal{M})^n=\bigoplus_{i+j=n}\mathfrak{P}^i\odot_{\mathfrak{R}}\mathcal{M}^j$. The components of $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M})$ are the comodule direct products of the comodules Ctrhom between the appropriate components of $\mathfrak{P}$ and $\mathcal{M}$, that it $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M})^n=\prod_{j-i=n}\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P}^i,\mathcal{M}^j)$. The components of $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{M})$ are the direct products of the contramodules of homomorphisms between the appropriate components of $\mathcal{L}$ and $\mathcal{M}$, i. e., $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{M})^n$ $=\prod_{j-i=n}\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{M})$.

Similarly, the components of the graded version of the cotensor product $\mathcal{N}\,\square_{\mathfrak{R}}\,\mathcal{M}$ are the direct sums of the cotensor products of the appropriate components of $\mathfrak{N}$ and $\mathfrak{M}$, while the components of the graded $\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M},\mathfrak{P})$ are the direct products of the contramodules $\mathrm{Cohom}_{\mathfrak{R}}$ between the appropriate components of $\mathcal{M}$ and $\mathfrak{P}$. Notice that all of our operations produced free (graded) $\mathfrak{R}$-contramodules or cofree (graded) $\mathfrak{R}$-comodules when receiving free (graded) $\mathfrak{R}$-contramodules and cofree (graded) $\mathfrak{R}$-comodules as their inputs (see Section 1.4–1.7).

Recall that the category of (cofree) $\mathfrak{R}$-comodules is a module category with respect to the operation $\odot_{\mathfrak{R}}$ over the tensor category of (free) $\mathfrak{R}$-contramodules. The same applies to the categories of graded comodules and contramodules.

An $\mathfrak{R}$-*cofree graded left module* $\mathcal{M}$ over an $\mathfrak{R}$-free graded algebra $\mathfrak{B}$ is, by the definition, a graded left module over $\mathfrak{B}$ in the module category of cofree $\mathfrak{R}$-contramodules, i. e., a cofree graded $\mathfrak{R}$-comodule endowed with an (associative and unital) homogeneous $\mathfrak{B}$-action map $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathcal{M}$. $\mathfrak{R}$-*cofree graded right modules* $\mathcal{N}$ over $\mathfrak{B}$ are defined in the similar way. Alternatively, one can define $\mathfrak{B}$-module structures on cofree graded $\mathfrak{R}$-comodules in terms of the action maps $\mathcal{M}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M})$, and similarly for $\mathcal{N}$.

The category of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules is enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules, so the abelian groups of morphisms between two $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules $\mathcal{L}$ and $\mathcal{M}$ is the underlying abelian group of the degree-zero component of a certain naturally defined (not necessarily free) graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{B}}(\mathcal{L},\mathcal{M})$. This $\mathfrak{R}$-contramodule is constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{M})\rightrightarrows\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{L},\ \mathcal{M})$ induced by the actions of $\mathfrak{B}$ in $\mathcal{L}$ and $\mathcal{M}$. For the sign rule, see [29, Section 1.1].

The *tensor product* $\mathfrak{N}\otimes_{\mathfrak{B}}\mathcal{M}$ of an $\mathfrak{R}$-free graded right $\mathfrak{B}$-module $\mathfrak{N}$ and an $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-module $\mathcal{M}$ is a (not necessarily cofree) graded $\mathfrak{R}$-comodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{M}\rightrightarrows\mathfrak{N}\odot_{\mathfrak{R}}\mathcal{M}$ induced by the actions of $\mathfrak{B}$ in $\mathfrak{N}$ and $\mathcal{M}$. The graded $\mathfrak{R}$-comodule $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathcal{M})$ from an $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{L}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-module $\mathcal{M}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\mathcal{M})\rightrightarrows\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{L},\ \mathcal{M})\simeq\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M}))$ induced by the $\mathfrak{B}$-action morphisms $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{L}\longrightarrow\mathfrak{L}$ and $\mathcal{M}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M}))$.

The additive category of $\mathfrak{R}$-cofree graded (left or right) $\mathfrak{B}$-modules has a natural exact category structure: a short sequence of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules is said to be exact if it is (split) exact as a short sequence of cofree graded $\mathfrak{R}$-comodules. The exact category of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules admits infinite direct sums and products, which are preserved by the forgetful functors to the category of cofree graded $\mathfrak{R}$-comodules and preserve exact sequences.

If a morphism of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules has an $\mathfrak{R}$-cofree kernel (resp., cokernel) in the category of graded $\mathfrak{R}$-comodules, then this kernel (resp., cokernel) is naturally endowed with an $\mathfrak{R}$-cofree graded $\mathfrak{B}$-module structure, making it the kernel (resp., cokernel) of the same morphism in the additive category of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules.

There are enough projective objects in the exact category of $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-modules; these are the direct summands of the graded $\mathfrak{B}$-modules $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{U}$ freely generated by cofree graded $\mathfrak{R}$-comodules $\mathcal{U}$. Similarly, there are enough injectives in this exact category, and these are the direct summands of the graded $\mathfrak{B}$-modules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{V})$ cofreely cogenerated by cofree graded $\mathfrak{R}$-comodules $\mathcal{V}$.

Proposition 2.4.1. *The exact categories of $\mathfrak{R}$-free graded left $\mathfrak{B}$-modules and of $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-modules are naturally equivalent. The equivalence is provided by the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ of comodule-contramodule correspondence over $\mathfrak{R}$, defined in Section* 1.5*, which transform free graded $\mathfrak{R}$-contramodules with a $\mathfrak{B}$-module structure into cofree graded $\mathfrak{R}$-comodules with a $\mathfrak{B}$-module structure and back.*

Proof. The assertion follows from the fact that the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ define mutually inverse equivalences between the module categories $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}$ and $\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}$ of free $\mathfrak{R}$-contramodules and cofree $\mathfrak{R}$-comodules over the tensor category $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}$. In fact, there is a natural isomorphism $\Phi_{\mathfrak{R}}(\mathfrak{P}\otimes^{\mathfrak{R}}\mathfrak{Q})\simeq\mathfrak{P}\odot_{\mathfrak{R}}\Phi_{\mathfrak{R}}(\mathfrak{Q})$ for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$. The functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ also transform the functor $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},-)$ from a fixed free $\mathfrak{R}$-contramodule $\mathfrak{P}$ to varying free $\mathfrak{R}$-contramodules into the functor $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},-)$ from the same $\mathfrak{R}$-contramodule $\mathfrak{P}$ to varying cofree $\mathfrak{R}$-comodules. In fact, there is a natural isomorphism $\Psi_{\mathfrak{R}}(\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{M}))\simeq\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\Psi_{\mathfrak{R}}(\mathcal{M}))$ for any $\mathfrak{R}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule $\mathcal{M}$ (see Sections 1.5–1.6). ☐

The equivalence $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ between the categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules is also an equivalence of categories enriched over graded $\mathfrak{R}$-contramodules. Besides, for any $\mathfrak{R}$-free graded right $\mathfrak{B}$-module $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N}\otimes_{\mathfrak{B}}\Phi_{\mathfrak{R}}(\mathfrak{M})\simeq\Phi_{\mathfrak{R}}(\mathfrak{N}\otimes_{\mathfrak{B}}\mathfrak{M})$; and for any $\mathfrak{R}$-free graded left $\mathfrak{B}$-module $\mathfrak{L}$ and any $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-module $\mathcal{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\Psi_{\mathfrak{R}}(\mathcal{M}))\simeq\Psi_{\mathfrak{R}}(\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathcal{M}))$.

Furthermore, the equivalence $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ transforms graded $\mathfrak{B}$-modules $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U}$ freely generated by free graded $\mathfrak{R}$-contramodules $\mathfrak{U}$ into graded $\mathfrak{B}$-modules $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{U}$ freely generated by cofree graded $\mathfrak{R}$-comodules $\mathcal{U}$, with $\mathcal{U}=\Phi_{\mathfrak{R}}(\mathfrak{U})$, and graded $\mathfrak{B}$-modules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{V})$ cofreely cogenerated by free graded $\mathfrak{R}$-contramodules

$\mathfrak{V}$ into graded $\mathfrak{B}$-modules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathcal{V})$ cofreely cogenerated by cofree graded $\mathfrak{R}$-comodules $\mathcal{V}$, with $\Psi_{\mathfrak{R}}(\mathcal{V}) = \mathfrak{V}$.

Finally, the equivalence $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ between the exact categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules transforms the reduction functor $\mathfrak{P} \longmapsto \mathfrak{P}/\mathfrak{m}\mathfrak{P}$ into the reduction functor $\mathcal{M} \longmapsto {}_{\mathfrak{m}}\mathcal{M}$ (both taking values in the abelian category of graded $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$-modules). This follows from the results of Section 1.5.

All the results of Section 2.1 have their analogues for $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules, which can be, at one's choice, either proven directly in the similar way to the proofs in 2.1, or deduced from the results in 2.1 using Proposition 2.4.1.

2.5. **Absolute derived category of $\mathfrak{R}$-cofree CDG-modules.** Let $\mathcal{U}$ and $\mathcal{V}$ be graded $\mathfrak{R}$-comodules endowed with homogeneous $\mathfrak{R}$-comodule endomorphisms (differentials) $d_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathcal{U}$ and $d_{\mathcal{V}}\colon cV \longrightarrow \mathcal{V}$ of degree 1. Then the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{U}, \mathcal{V})$ is endowed with the differential d defined by the conventional rule $d(f)(u) = d_{\mathcal{V}}(f(u)) - (-1)^{|f|} f(d_{\mathcal{U}}(u))$.

Let $\mathfrak{W}$ be a graded $\mathfrak{R}$-contramodule endowed with a differential $d_{\mathfrak{W}}$. Then the graded $\mathfrak{R}$-comodules $\mathfrak{W} \odot_{\mathfrak{R}} \mathcal{U}$ and $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{W}, \mathcal{V})$ are endowed with their differentials d defined by the conventional rules $d(w \odot u) = d_{\mathfrak{W}}(w) \odot u + (-1)^{|w|} w \odot d_{\mathcal{U}}(u)$ and $d(f)(w) = d_{\mathcal{V}}(f(w)) - (-1)^{|f|} f(d_{\mathfrak{W}}(w))$.

Let $\mathfrak{B}$ be an $\mathfrak{R}$-free graded algebra endowed with an odd derivation d of degree 1. An *odd derivation* $d_{\mathcal{M}}$ of an $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-module $\mathcal{M}$ *compatible with* the derivation d on $\mathfrak{B}$ is a differential ($\mathfrak{R}$-comodule endomorphism of degree 1) such that the action map $\mathfrak{B} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathcal{M}$ forms a commutative diagram with $d_{\mathcal{M}}$ and the differential on $\mathfrak{B} \odot_{\mathfrak{R}} \mathcal{M}$ induced by d and $d_{\mathcal{M}}$. Odd derivations of $\mathfrak{R}$-cofree graded right $\mathfrak{B}$-modules are defined similarly. Alternatively, one requires the action map $\mathcal{M} \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathcal{M})$ to commute with the differentials; the adjunction of $\odot_{\mathfrak{R}}$ and $\mathrm{Ctrhom}_{\mathfrak{R}}$ transforms one equation into the other.

An *$\mathfrak{R}$-cofree left CDG-module* $\mathcal{M}$ over an $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$ is, by the definition, an $\mathfrak{R}$-cofree graded left $\mathfrak{B}$-module endowed with an odd derivation $d_{\mathcal{M}}\colon \mathcal{M} \longrightarrow \mathcal{M}$ of degree 1 compatible with the derivation d on $\mathfrak{B}$ and satisfying the equation $d_{\mathcal{M}}^2(x) = hx$ for all $x \in \mathcal{M}$. *$\mathfrak{R}$-cofree right CDG-modules* $\mathcal{N}$ over $\mathfrak{B}$ are defined similarly, except for the sign. Here the element $h \in \mathfrak{B}^2$ can be viewed as a homogeneous morphism $\mathfrak{R} \longrightarrow \mathfrak{B}$ of degree 2; the notation $x \longmapsto hx$ is interpreted as the composition $\mathcal{M} \simeq \mathfrak{R} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathfrak{B} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathcal{M}$ defining a homogeneous endomorphism of degree 2 of the graded $\mathfrak{R}$-comodule $\mathcal{M}$.

$\mathfrak{R}$-cofree left (resp., right) CDG-modules over $\mathfrak{B}$ form a DG-category, which we will denote by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ (resp., $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B}$). In fact, these DG-categories are enriched over the tensor category of (complexes of) $\mathfrak{R}$-contramodules, so the complexes of morphisms in them are the underlying complexes of abelian groups for naturally defined complexes of $\mathfrak{R}$-contramodules, which we will denote by $\mathrm{Hom}_{\mathfrak{B}}(\mathcal{L}, \mathcal{M})$. The underlying graded $\mathfrak{R}$-contramodules of these complexes were defined in Section 2.4, and the differentials in them are defined in the conventional way.

Passing to the zero cohomology of the complexes of morphisms, we construct the homotopy categories of $\mathfrak{R}$-cofree CDG-modules $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and

$H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B})$. These we will mostly view as conventional categories with abelian groups of morphisms (see Remark 2.2.1). Since the DG-categories $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B}$ have shifts, twists, and infinite direct sums and products, their homotopy categories are triangulated categories with infinite direct sums and products.

The tensor product $\mathfrak{N}\otimes_{\mathfrak{B}}\mathcal{M}$ of an $\mathfrak{R}$-free right CDG-module $\mathfrak{N}$ and an $\mathfrak{R}$-cofree left CDG-module $\mathcal{M}$ over $\mathfrak{B}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\mathfrak{N}\otimes_{\mathfrak{B}}\mathcal{M}$ constructed in Section 2.4 with the conventional tensor product differential. The tensor product of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$ is a triangulated functor of two arguments

$$\otimes_{\mathfrak{B}}\colon H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{B})\times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}),$$

where $H^0(\mathfrak{R}\text{–}\mathsf{comod})$ denotes, by an abuse of notation, the homotopy category of complexes of $\mathfrak{R}$-comodules. The complex $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathcal{M})$ from an $\mathfrak{R}$-free left CDG-module $\mathfrak{L}$ to an $\mathfrak{R}$-cofree left CDG-module $\mathcal{M}$ over $\mathfrak{B}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathcal{M})$ constructed in Section 2.4 with the conventional Hom differential. The Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$ is a triangulated functor of two arguments

$$\operatorname{Hom}_{\mathfrak{B}}\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

An $\mathfrak{R}$-cofree left CDG-module over $\mathfrak{B}$ is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ by the thick subcategory of absolutely acyclic $\mathfrak{R}$-cofree CDG-modules is called the *absolute derived category* of $\mathfrak{R}$-cofree left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The absolute derived category of $\mathfrak{R}$-cofree right CDG-modules over $\mathfrak{B}$, denoted by $\mathsf{D}^{\mathsf{abs}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B})$, is defined similarly.

An $\mathfrak{R}$-cofree left CDG-module over $\mathfrak{B}$ is said to be *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$ and closed under infinite products. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-cofree CDG-modules is called the *contraderived category* of $\mathfrak{R}$-cofree left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The (similarly defined) contraderived category of $\mathfrak{R}$-cofree right CDG-modules over $\mathfrak{B}$ is denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B})$.

An $\mathfrak{R}$-cofree left CDG-module over $\mathfrak{B}$ is said to be *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-cofree CDG-modules is called the *coderived category* of $\mathfrak{R}$-cofree left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The coderived category of $\mathfrak{R}$-cofree right CDG-modules over $\mathfrak{B}$ is denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B})$.

As above, we denote by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}} \subset \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-cofree CDG-modules whose underlying $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules are projective. Similarly, $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}} \subset \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ is the full DG-subcategory formed by all the $\mathfrak{R}$-cofree CDG-modules whose underlying $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules are injective. The corresponding homotopy categories are denoted by $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})$ and $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})$.

The functors $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ from Section 2.4 define an equivalence of DG-categories $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \simeq \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ (and similarly for right CDG-modules). Being also an equivalence of exact categories, this correspondence identifies absolutely acyclic (resp., contraacyclic, coacyclic) $\mathfrak{R}$-free CDG-modules with absolutely acyclic (resp., contraacyclic, coacyclic) $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$. So an equivalence of the absolute derived categories $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$, an equivalence of the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$, and an equivalence of the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ are induced. The above equivalence of DG-categories also identifies $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ with $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}$ and $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ with $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}$.

For any $\mathfrak{R}$-free right CDG-module $\mathfrak{N}$ and $\mathfrak{R}$-free left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ the isomorphism $\mathfrak{N} \otimes_{\mathfrak{B}} \Phi_{\mathfrak{R}}(\mathfrak{M}) \simeq \Phi_{\mathfrak{R}}(\mathfrak{N} \otimes_{\mathfrak{B}} \mathfrak{M})$ from Section 2.4 is an isomorphism of complexes of $\mathfrak{R}$-comodules. Similarly, for any $\mathfrak{R}$-free left CDG-module $\mathfrak{L}$ and $\mathfrak{R}$-cofree left CDG-module $\mathcal{M}$ over $\mathfrak{B}$ the isomorphism $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L}, \Psi_{\mathfrak{R}}(\mathcal{M})) \simeq \Psi_{\mathfrak{R}}(\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathcal{M}))$ is an isomorphism of complexes of $\mathfrak{R}$-contramodules. All the results of Section 2.2 have their analogues for $\mathfrak{R}$-cofree CDG-modules over $\mathfrak{B}$.

Let $f = (f, a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ be a morphism of $\mathfrak{R}$-free CDG-algebras. Then with any $\mathfrak{R}$-cofree left CDG-module $(\mathcal{M}, d_{\mathcal{M}})$ over $\mathfrak{A}$ one can associate an $\mathfrak{R}$-cofree left CDG-module $(\mathcal{M}, d'_{\mathcal{M}})$ over $\mathfrak{B}$ with the graded $\mathfrak{B}$-module structure on $\mathcal{M}$ defined via f and the modified differential $d'_{\mathcal{M}}$ constructed in terms of a. A similar procedure applies to right CDG-modules. So we obtain the DG-functors of restriction of scalars $R_f\colon \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{A} \longrightarrow \mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B}$; passing to the homotopy categories, we have the triangulated functors

$$R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$$

and $H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{A}) \longrightarrow H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{B})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ identify these functors with the functors R_f for $\mathfrak{R}$-free CDG-modules constructed in Section 2.2.

2.6. **Semiderived category of $\mathfrak{R}$-cofree wcDG-modules.** As above, we refer to $\mathfrak{R}$-cofree CDG-modules over a wcDG-algebra $\mathfrak{A}$ as $\mathfrak{R}$-*cofree wcDG-modules*. An $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ over $\mathfrak{A}$ is said to be *semiacyclic* if the complex of vector spaces ${}_{\mathfrak{m}}\mathcal{M}$ (which is in fact a DG-module over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$) is acyclic. In particular, it follows from Lemma 1.4.3 that when $\mathfrak{A}$ is an $\mathfrak{R}$-free DG-algebra (i. e., $h = 0$), an $\mathfrak{R}$-cofree CDG-module over $\mathfrak{A}$ is semiacyclic if and only if it is is contractible as a complex of cofree $\mathfrak{R}$-comodules.

Clearly, the property of semiacyclicity of an $\mathfrak{R}$-cofree wcDG-module over $\mathfrak{A}$ is preserved by shifts, cones, homotopy equivalences, infinite direct sums, and infinite products. The quotient category of the homotopy category $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$ by the thick subcategory of semiacyclic $\mathfrak{R}$-cofree wcDG-modules is called the *semiderived category* of $\mathfrak{R}$-cofree left wcDG-modules over $\mathfrak{A}$ and denoted by $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$. The semiderived category of $\mathfrak{R}$-cofree right wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–cof}}\text{–}\mathfrak{A})$ is defined similarly.

Notice that any acyclic complex of cofree $\mathfrak{R}$-comodules is contractible when (the category of comodules over) $\mathfrak{R}$ has finite homological dimension. In this case, we call semiacyclic $\mathfrak{R}$-cofree wcDG-modules over $\mathfrak{A}$ simply *acyclic*, and refer to the semiderived category of $\mathfrak{R}$-cofree wcDG-modules as their *derived category*.

An $\mathfrak{R}$-cofree wcDG-module $\mathcal{P}$ over $\mathfrak{A}$ is called *homotopy projective* if the complex $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{P},\mathcal{M})$ is acyclic for any semiacyclic $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ over $\mathfrak{A}$. Similarly, an $\mathfrak{R}$-cofree wcDG-module $\mathcal{J}$ over $\mathfrak{A}$ is called *homotopy injective* if the complex $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{M},\mathcal{J})$ is acyclic for any semiacyclic $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$. The full triangulated subcategories in $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$ formed by the homotopy projective and homotopy injective $\mathfrak{R}$-cofree wcDG-modules are denoted by $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})_{\mathsf{proj}}$ and $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})_{\mathsf{inj}}$, respectively. The intersections $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})_{\mathsf{proj}}\cap H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}})$ and $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})_{\mathsf{inj}}\cap H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}})$ are denoted by $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}})_{\mathsf{proj}}$ and $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}})_{\mathsf{inj}}$.

The equivalence of homotopy categories $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\simeq H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$ identifies semiacyclic $\mathfrak{R}$-free wcDG-modules with semiacyclic $\mathfrak{R}$-cofree wcDG-modules, and therefore induces an equivalence of the semiderived categories $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\simeq\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$. It also follows that the equivalence of homotopy categories identifies homotopy projective (resp., injective) $\mathfrak{R}$-free wcDG-modules with homotopy projective (resp., injective) $\mathfrak{R}$-cofree wcDG-modules.

All the results of Section 2.3 have their analogues for $\mathfrak{R}$-cofree wcDG-modules, which can be, at one's choice, either proven directly in the similar way to the proofs in 2.3, or deduced from the results in 2.3 using the above observations about the functors $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$. In particular, the following assertions hold.

Lemma 2.6.1. (a) *A wcDG-module $\mathcal{P}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}}$ is homotopy projective if and only if the DG-module ${}_{\mathfrak{m}}\mathcal{P}$ over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ over the field k is homotopy projective.*

(b) *A wcDG-module $\mathcal{J}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}}$ is homotopy injective if and only if the DG-module ${}_{\mathfrak{m}}\mathcal{J}$ over the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is homotopy injective.* ☐

Theorem 2.6.2. (a) *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the compositions of functors*

$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})_{\mathsf{proj}}\longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})\longrightarrow\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$$

and

$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}})_{\mathsf{proj}}\longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})\longrightarrow\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}})$$

are equivalences of triangulated categories.

(b) *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the compositions of functors*
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})_{\mathsf{inj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$$
and
$$H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})_{\mathsf{inj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}).$$
are equivalences of triangulated categories. □

Lemma 2.6.3. (a) *Let $\mathcal{P}$ be a homotopy projective $\mathfrak{R}$-cofree left wcDG-module over a wcDG-algebra $\mathfrak{A}$, and let $\mathfrak{N}$ be a semiacyclic $\mathfrak{R}$-free right wcDG-module over $\mathfrak{A}$. Then the tensor product $\mathfrak{N}\otimes_{\mathfrak{A}}\mathcal{P}$ is a contractible complex of $\mathfrak{R}$-comodules.*

(b) *Let $\mathfrak{Q}$ be a homotopy projective $\mathfrak{R}$-free right wcDG-module and $\mathcal{M}$ be a semiacyclic $\mathfrak{R}$-cofree left wcDG-module over a wcDG-algebra $\mathfrak{A}$. Then the tensor product $\mathfrak{Q}\otimes_{\mathfrak{A}}\mathcal{M}$ is a contractible complex of $\mathfrak{R}$-comodules.*

(c) *Let $\mathfrak{P}$ be a homotopy projective $\mathfrak{R}$-free left wcDG-module over a wcDG-algebra $\mathfrak{A}$, and let $\mathcal{M}$ be a semiacyclic $\mathfrak{R}$-cofree left wcDG-module over $\mathfrak{A}$. Then the complex of $\mathfrak{R}$-comodules $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P},\mathcal{M})$ is contractible.*

(d) *Let $\mathcal{J}$ be a homotopy injective $\mathfrak{R}$-cofree left wcDG-module and $\mathfrak{L}$ be a semiacyclic $\mathfrak{R}$-free left wcDG-module over a wcDG-algebra $\mathfrak{A}$. Then the complex of $\mathfrak{R}$-comodules $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{L},\mathcal{J})$ is contractible.* □

The left derived functor of tensor product of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree wcDG-modules
$$\mathrm{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}),$$
where $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ denotes, by an abuse of notation, the homotopy category of complexes of cofree $\mathfrak{R}$-comodules, is constructed by restricting the functor $\otimes_{\mathfrak{A}}$ to either of the full subcategories $H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})_{\mathsf{proj}}$ or $H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \subset H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The equivalences of categories $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ transform this functor into the derived functor $\mathrm{Tor}^{\mathfrak{A}}$ of tensor product of $\mathfrak{R}$-free wcDG-modules over $\mathfrak{A}$ defined in Section 2.3.

Similarly, the right derived functor of Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree wcDG-modules
$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$
is constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{A}}$ to either of the full subcategories $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})^{\mathsf{op}}_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ or $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})_{\mathsf{inj}} \subset H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The right derived functor of homomorphisms of $\mathfrak{R}$-cofree wcDG-modules
$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}}\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$
is constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{A}}$ to either of the full subcategories $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})^{\mathsf{op}}_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ or $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})_{\mathsf{inj}} \subset H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The equivalences of categories $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ transform these two functors into each other and the derived functor $\mathrm{Ext}_{\mathfrak{A}}$ of homomorphisms of $\mathfrak{R}$-free wcDG-modules defined in Section 2.3.

Let $(f,a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ be a morphism of wcDG-algebras over $\mathfrak{R}$. Then the functor $R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ induces a triangulated functor

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}).$$

The functor $\mathbb{I}R_f$ has adjoints on both sides. The DG-functor $E_f\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}$ is defined on the level of graded modules by the rule $\mathfrak{N} \longmapsto \mathfrak{A}\otimes_{\mathfrak{B}}\mathfrak{N}$; the differential on $\mathfrak{A}\otimes_{\mathfrak{B}}\mathfrak{N}$ induced by the differentials on $\mathfrak{A}$ and $\mathfrak{N}$ is modified to obtain the differential on $E_f(\mathfrak{N})$ using the element a. Restricting the induced functor between the homotopy categories to $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})_{\mathsf{proj}}$, we construct the left derived functor

$$\mathbb{L}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}),$$

which is left adjoint to the functor $\mathbb{I}R_f$.

Similarly, the DG-functor $E^f\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}$ is defined on the level of graded modules by the rule $\mathfrak{N}\longmapsto \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{A},\mathfrak{N})$; the element a is used to modify the differential on $\mathrm{Hom}_{\mathfrak{B}}(\mathfrak{A},\mathfrak{N})$ induced by the differentials on $\mathfrak{A}$ and $\mathfrak{N}$. Restricting the induced functor between the homotopy categories to $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})_{\mathsf{inj}}$, we obtain the right derived functor

$$\mathbb{R}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}),$$

which is right adjoint to the functor $\mathbb{I}R_f$.

The equivalences of semiderived categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform the $\mathfrak{R}$-cofree wcDG-module restriction- and extension-of-scalars functors $\mathbb{L}E_f$, $\mathbb{I}R_f$, $\mathbb{R}E^f$ defined above into the $\mathfrak{R}$-free wcDG-module restriction- and extension-of-scalars functors $\mathbb{L}E_f$, $\mathbb{I}R_f$, $\mathbb{R}E^f$ defined in Section 2.3.

Theorem 2.6.4. *The functor* $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *is an equivalence of triangulated categories whenever the DG-algebra morphism* $f/\mathfrak{m}f\colon \mathfrak{B}/\mathfrak{m}\mathfrak{B}\longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ *is a quasi-isomorphism.* ☐

Just as in the $\mathfrak{R}$-free wcDG-module situation of Theorem 2.3.5, the converse statement to Theorem 2.6.4 is not true.

3. $\mathfrak{R}$-Free and $\mathfrak{R}$-Cofree CDG-Contramodules and CDG-Comodules

3.1. **$\mathfrak{R}$-free graded coalgebras and contra/comodules.** An *$\mathfrak{R}$-free graded coalgebra* $\mathfrak{C}$ is, by the definition, a graded coalgebra object in the tensor category of free $\mathfrak{R}$-contramodules. In other words, it is a free graded $\mathfrak{R}$-contramodule endowed with a (coassociative, noncocommutative) homogeneous comultiplication map $\mathfrak{C}\longrightarrow \mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{C}$ and a homogeneous counit map $\mathfrak{C}\longrightarrow\mathfrak{R}$ satisfying the conventional axioms.

An *$\mathfrak{R}$-free graded left comodule* $\mathfrak{M}$ over $\mathfrak{C}$ is a graded left comodule over $\mathfrak{C}$ in the tensor category of free $\mathfrak{R}$-contramodules, i. e., a free graded $\mathfrak{R}$-contramodule endowed with a (coassociative and counital) homogeneous $\mathfrak{C}$-coaction map $\mathfrak{M}\longrightarrow\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$. *$\mathfrak{R}$-free graded right comodules* $\mathfrak{N}$ over $\mathfrak{C}$ are defined in the similar way.

An $\mathfrak{R}$-*free graded left contramodule* $\mathfrak{P}$ over $\mathfrak{C}$ is a free graded $\mathfrak{R}$-contramodule endowed with a $\mathfrak{C}$-*contraaction* map $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$, which must be a morphism of graded $\mathfrak{R}$-contramodules satisfying the conventional contraassociativity and counit axioms. This can be rephrased by saying that an $\mathfrak{R}$-free $\mathfrak{C}$-contramodule is an object of the opposite category to the category of graded $\mathfrak{C}$-comodules in the module category $(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})^{\mathsf{op}}$ over the tensor category $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}$ [28, Section 0.2.4].

Specifically, the contraassociativity axiom asserts that the two compositions $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{C},\ \mathfrak{P}) \rightrightarrows \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ of the maps induced by the contraaction and comultiplication maps with the contraaction map must be equal to each other. The counit axiom claims that the composition $\mathfrak{P} \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$ of the map induced by the counit map with the contraaction map must be equal to the identity endomorphism of $\mathfrak{P}$. For the details, including the sign rule, see [29, Section 2.2].

In fact, the categories of $\mathfrak{R}$-free graded $\mathfrak{C}$-comodules and $\mathfrak{C}$-contramodules are enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules. So the abelian group of morphisms between two $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules $\mathfrak{L}$ nad $\mathfrak{M}$ is the underlying abelian group of the degree-zero component of the (not necessarily free) graded $\mathfrak{R}$-contramodule $\operatorname{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{M})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{L},\mathfrak{M}) \rightrightarrows \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{L},\ \mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M})$ induced by the coactions of $\mathfrak{C}$ in $\mathfrak{L}$ and $\mathfrak{M}$. For the sign rules, which are different for the left and the right comodules, see [29, Section 2.1].

Similarly, the abelian group of morphisms between two $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ is the underlying abelian group of the zero-degree component of the (not necessarily free) graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q}) \rightrightarrows \operatorname{Hom}^{\mathfrak{R}}(\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\mathfrak{Q})$ induced by the contraactions of $\mathfrak{C}$ in $\mathfrak{P}$ and $\mathfrak{Q}$.

The *contratensor product* $\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P}$ of an $\mathfrak{R}$-free right $\mathfrak{C}$-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-free left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is the (not necessarily free) graded $\mathfrak{R}$-contramodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\otimes^{\mathfrak{R}}\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \rightrightarrows \mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{P}$ defined in terms of the $\mathfrak{C}$-coaction in $\mathfrak{N}$, the $\mathfrak{C}$-contraaction in $\mathfrak{P}$, and the evaluation map $\mathfrak{C}\otimes^{\mathfrak{R}}\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$. For further details, see [28, Section 0.2.6]; for the sign rule, see [29, Section 2.2].

The *cotensor product* $\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathfrak{M}$ of an $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ is a (not necessarily free) graded $\mathfrak{R}$-contramodule constructed as the kernel of the pair of morphisms $\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{M} \rightrightarrows \mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$. The graded $\mathfrak{R}$-contramodule of *cohomomorphisms* $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ to an $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is constructed as the cokernel of the pair of morphisms $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M},\ \mathfrak{P}) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{M},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})) \rightrightarrows \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{M},\mathfrak{P})$ induced by the $\mathfrak{C}$-coaction in $\mathfrak{M}$ and the $\mathfrak{C}$-contraaction in $\mathfrak{P}$.

For any free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ and any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$, the free graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N},\mathfrak{U})$ has a natural left $\mathfrak{C}$-contramodule structure provided by the map $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N},\mathfrak{U})) \simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C},\ \mathfrak{U}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N},\mathfrak{U})$ induced by the coaction map $\mathfrak{N} \longrightarrow \mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C}$. For any $\mathfrak{R}$-free right

$\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-free left $\mathfrak{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P},\,\mathfrak{U})\simeq\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{P},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{N},\mathfrak{U}))$.

The additive categories of $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodules and $\mathfrak{C}$-comodules have natural exact category structures: a short sequence of $\mathfrak{R}$-free graded $\mathfrak{C}$-contra/comodules is said to be exact if it is (split) exact as a short sequence of free graded $\mathfrak{R}$-contramodules. The exact category of $\mathfrak{R}$-free graded $\mathfrak{C}$-comodules admits infinite direct sums, while the exact category of $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodules admits infinite products. Both of these are preserved by the forgetful functors to the category of free graded $\mathfrak{R}$-contramodules and preserve exact sequences.

If a morphism of $\mathfrak{R}$-free graded $\mathfrak{C}$-contra/comodules has an $\mathfrak{R}$-free kernel (resp., cokernel) in the category of graded $\mathfrak{R}$-contramodules, then this kernel (resp., cokernel) is naturally endowed with an $\mathfrak{R}$-free graded $\mathfrak{C}$-contra/comodule structure, making it the kernel (resp., cokernel) of the same morphism in the additive category of $\mathfrak{R}$-free graded $\mathfrak{C}$-contra/comodules.

There are enough projective objects in the exact category of $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodules; these are the direct summands of the graded $\mathfrak{C}$-contramodules $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ freely generated by free graded $\mathfrak{R}$-contramodules $\mathfrak{U}$. Similarly, there are enough injective objects in the exact category of $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules; these are the direct summands of the graded $\mathfrak{C}$-comodules $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V}$ cofreely cogenerated by free graded $\mathfrak{R}$-contramodules $\mathfrak{V}$.

For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{C}}(\mathfrak{L},\,\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V})\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{L},\mathfrak{V})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{Q}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{C}}(\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U}),\mathfrak{Q})\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{U},\mathfrak{Q})$ [28, Sections 1.1.1–2 and 3.1.1–2].

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\odot_{\mathfrak{C}}\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})\simeq\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{U}$. The proof following [28, Section 5.1.1] requires considering non-$\mathfrak{R}$-free $\mathfrak{C}$-contramodules (see Section 4.4); alternatively, a direct argument can be made based on the idea of the proof of [28, Lemma 1.1.2].

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\,\square_{\mathfrak{C}}\,(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V})\simeq\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{V}$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V},\,\mathfrak{P})\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{V},\mathfrak{P})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{M},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U}))\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{M},\mathfrak{U})$ [28, Sections 1.2.1 and 3.2.1].

Lemma 3.1.1. (a) *Let $\mathfrak{F}$ be a projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule and $\mathfrak{Q}$ be an arbitrary $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule. Then the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{F},\mathfrak{Q})$ is free, and the functor of reduction modulo $\mathfrak{m}$ induces an isomorphism of graded k-vector spaces*

$$\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{F},\mathfrak{Q})/\mathfrak{m}\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{F},\mathfrak{Q})\simeq\operatorname{Hom}^{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{F}/\mathfrak{m}\mathfrak{F},\,\mathfrak{Q}/\mathfrak{m}\mathfrak{Q}).$$

(b) *Let $\mathfrak{L}$ be an arbitrary $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule and $\mathfrak{J}$ be an injective $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule. Then the graded $\mathfrak{R}$-contramodule* $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{J})$ *is free, and the functor of reduction modulo* $\mathfrak{m}$ *induces an isomorphism of graded k-vector spaces*

$$\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{J})/\mathfrak{m}\,\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{J}) \simeq \mathrm{Hom}_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{J}/\mathfrak{m}\mathfrak{J}).$$

(c) *For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, there is a natural isomorphism of graded k-vector spaces*

$$(\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P})/\mathfrak{m}(\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P}) \simeq (\mathfrak{N}/\mathfrak{m}\mathfrak{N})\odot_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P}).$$

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{F}$, the graded $\mathfrak{R}$-contramodule $\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{F}$ is free.

(d) *Let $\mathfrak{N}$ be an arbitrary $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule and $\mathfrak{J}$ be an injective $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule. Then the graded $\mathfrak{R}$-contramodule $\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathfrak{J}$ is free, and there is a natural isomorphism of graded k-vector spaces*

$$(\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathfrak{J})/\mathfrak{m}(\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathfrak{J}) \simeq (\mathfrak{N}/\mathfrak{m}\mathfrak{N})\,\square_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}\,(\mathfrak{J}/\mathfrak{m}\mathfrak{J}).$$

(e) *For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, there is a natural isomorphism of graded k-vector spaces*

$$\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})/\mathfrak{m}\,\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P}) \simeq \mathrm{Cohom}_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{M}/\mathfrak{m}\mathfrak{M},\ \mathfrak{P}/\mathfrak{m}\mathfrak{P}).$$

For any injective $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{J}$ and any $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, the graded $\mathfrak{R}$-contramodule $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{J},\mathfrak{P})$ *is free. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and any projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{F}$, the graded $\mathfrak{R}$-contramodule* $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{F})$ *is free.*

Proof. We will spell out the proofs of parts (b-d); part (a) is similar to (b), and (e) is similar to (c) (see also the proof of Lemma 2.1.1). Clearly, $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is a graded coalgebra over k, and the reduction modulo $\mathfrak{m}$ takes $\mathfrak{R}$-free $\mathfrak{C}$-comodules to comodules over $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ and $\mathfrak{R}$-free $\mathfrak{C}$-contramodules to contramodules over $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$, since the functor of reduction modulo $\mathfrak{m}$ preserves the tensor products of $\mathfrak{R}$-contramodules and the internal Hom from a free $\mathfrak{R}$-contramodule.

For any $\mathfrak{R}$-free left $\mathfrak{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, the reduction functor induces a natural morphism of graded k-vector spaces

$$\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})/\mathfrak{m}\,\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{P}/\mathfrak{m}\mathfrak{P},\ \mathfrak{Q}/\mathfrak{m}\mathfrak{Q}).$$

Now if $\mathfrak{F}=\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ is a graded $\mathfrak{C}$-contramodule freely generated by a free $\mathfrak{R}$-contramodule $\mathfrak{U}$, then $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{F},\mathfrak{Q})\simeq\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{U},\mathfrak{Q})$ and $\mathfrak{F}/\mathfrak{m}\mathfrak{F}\simeq\mathrm{Hom}_k(\mathfrak{C}/\mathfrak{m}\mathfrak{C},\ \mathfrak{U}/\mathfrak{m}\mathfrak{U})$, hence $\mathrm{Hom}^{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{F}/\mathfrak{m}\mathfrak{F},\mathfrak{Q}/\mathfrak{m}\mathfrak{Q})\simeq\mathrm{Hom}_k(\mathfrak{U}/\mathfrak{m}\mathfrak{U},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M})$ and the desired isomorphism (b) follows.

Analogously, for any $\mathfrak{R}$-free left $\mathfrak{C}$-comodules $\mathfrak{L}$ and $\mathfrak{M}$, the functor of reduction modulo $\mathfrak{m}$ induces a natural morphism of graded k-vector spaces

$$\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{M})/\mathfrak{m}\,\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{M}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}}(\mathfrak{L}/\mathfrak{m}\mathfrak{L},\ \mathfrak{M}/\mathfrak{m}\mathfrak{M}),$$

and the rest of the argument is as above.

To prove the first assertion of part (c), it suffices to notice that the functor of reduction modulo $\mathfrak{m}$ preserves the internal Hom from a free $\mathfrak{R}$-contramodule, the tensor products, and the cokernels in the category of $\mathfrak{R}$-contramodules. The second assertion is similar to the above.

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$, there is a natural morphism of k-vector spaces

$$(\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{M})/\mathfrak{m}(\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{M}) \longrightarrow (\mathfrak{N}/\mathfrak{m}\mathfrak{N}) \mathbin{\square}_{\mathfrak{C}/\mathfrak{m}\mathfrak{C}} (\mathfrak{M}/\mathfrak{m}\mathfrak{M})$$

induced by the composition $\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{M} \longrightarrow \mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{M} \simeq \mathfrak{N}/\mathfrak{m}\mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{M}/\mathfrak{m}\mathfrak{M}$. If $\mathfrak{J} = \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{V}$ is a graded $\mathfrak{C}$-comodule cofreely cogenerated by a free $\mathfrak{R}$-contramodule $\mathfrak{V}$, then $\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{J} \simeq \mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{V}$ and $\mathfrak{V}/\mathfrak{m}\mathfrak{V} \simeq \mathfrak{C}/\mathfrak{m}\mathfrak{C} \otimes_k \mathfrak{V}/\mathfrak{m}\mathfrak{V}$, hence $(\mathfrak{N}/\mathfrak{m}\mathfrak{N}) \mathbin{\square}_{\mathfrak{C}} (\mathfrak{J}/\mathfrak{m}\mathfrak{J}) \simeq (\mathfrak{N}/\mathfrak{m}\mathfrak{N}) \otimes_k (\mathfrak{V}/\mathfrak{m}\mathfrak{V})$ and the isomorphism (d) follows. $\square$

Lemma 3.1.2. (a) *An $\mathfrak{R}$-free graded comodule $\mathfrak{M}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is injective if and only if the graded comodule $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the graded k-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is injective.*

(b) *An $\mathfrak{R}$-free graded contramodule $\mathfrak{P}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is projective if and only if the graded contramodule $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over the graded k-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is projective.*

Proof. Similar to the proof of Lemma 2.1.2. $\square$

Recall that the homological dimensions of the abelian categories of graded left comodules, graded right comodules, and graded contramodules over a graded coalgebra C over a field k are equal to each other [29, Section 4.5]. This number is called the *homological dimension* of a graded coalgebra C.

Corollary 3.1.3. *The homological dimension of the exact categories of $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules, $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodules, and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodules does not exceed that of the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over k.* $\square$

Given a projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, the $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\Phi_{\mathfrak{C}}(\mathfrak{P})$ is defined as $\Phi_{\mathfrak{C}}(\mathfrak{P}) = \mathfrak{C} \odot_{\mathfrak{C}} \mathfrak{P}$. Here the contratensor product is $\mathfrak{R}$-free by Lemma 3.1.1(c), and is endowed with a left $\mathfrak{C}$-comodule structure as an $\mathfrak{R}$-free cokernel of a morphism of $\mathfrak{R}$-free left $\mathfrak{C}$-comodules. Given an injective $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$, the $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\Psi_{\mathfrak{C}}(\mathfrak{M})$ is defined as $\Psi_{\mathfrak{C}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}, \mathfrak{M})$. Here the Hom of $\mathfrak{C}$-comodules is $\mathfrak{R}$-free by Lemma 3.1.1(b), and is endowed with a left $\mathfrak{C}$-contramodule structure as an $\mathfrak{R}$-free kernel of a morphism of $\mathfrak{R}$-free left $\mathfrak{C}$-contramodules.

Proposition 3.1.4. *The functors $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ are mutually inverse equivalences between the additive categories of projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodules and injective $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules.*

Proof. One has $\Phi_{\mathfrak{C}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U})) = \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{U}$ and $\Psi_{\mathfrak{C}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{V}) = \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{V})$. $\square$

3.2. **Contra/coderived category of $\mathfrak{R}$-free CDG-contra/comodules.** Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. A homogeneous $\mathfrak{R}$-contramodule endomorphism (differential) $d\colon \mathfrak{C} \longrightarrow \mathfrak{C}$ (of degree 1) is said to be an *odd coderivation* of $\mathfrak{C}$ if it forms a commutative diagram with the comultiplication map $\mathfrak{C} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C}$ and the induced differential on $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C}$.

An *odd coderivation* $d_{\mathfrak{M}}$ of an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ *compatible with* the coderivation d on $\mathfrak{C}$ is a differential ($\mathfrak{R}$-contramodule endomorphism of degree 1) such that the coaction map $\mathfrak{M} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ forms a commutative diagram with $d_{\mathfrak{M}}$ and the differential on $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ induced by d and $d_{\mathfrak{M}}$. Odd coderivations of $\mathfrak{R}$-free graded right $\mathfrak{B}$-modules are defined similarly.

An *odd contraderivation* $d_{\mathfrak{P}}$ of an $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ *compatible with* the coderivation d on $\mathfrak{C}$ is a differential ($\mathfrak{R}$-contramodule endomorphism of degree 1) such that the contraaction map $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ forms a commutative diagram with $d_{\mathfrak{P}}$ and the differential on $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ induced by d and $d_{\mathfrak{P}}$.

Given an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$, the free graded $\mathfrak{R}$-contramodule $\mathfrak{C}^* = \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{R})$ is endowed with an $\mathfrak{R}$-free graded algebra structure provided by the multiplication map $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{R}) \otimes^{\mathfrak{R}} \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{R}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C}, \mathfrak{R}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{R})$ and the unit map $\mathfrak{R} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{R})$ induced by the comultiplication and counit morphisms of $\mathfrak{C}$.

An $\mathfrak{R}$-free graded left comodule $\mathfrak{M}$ over $\mathfrak{C}$ is endowed with a graded left $\mathfrak{C}^*$-module structure provided by the action map $\mathfrak{C}^* \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{C}^* \otimes^{\mathfrak{R}} \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$. $\mathfrak{R}$-free graded right comodules over $\mathfrak{C}$ are endowed with graded right $\mathfrak{C}^*$-module structures in the similar way. An $\mathfrak{R}$-free graded left contramodule $\mathfrak{P}$ over $\mathfrak{C}$ is endowed with a graded left $\mathfrak{C}^*$-module structure provided by the action map $\mathfrak{C}^* \otimes^{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. For the left-right and sign rules, see [29, Section 4.1]. Following [29], will denote the multiplication in $\mathfrak{C}^*$ and its action in comodules and contramodules over $\mathfrak{C}$ by the $*$ symbol.

An *$\mathfrak{R}$-free CDG-coalgebra* $\mathfrak{C}$ is an $\mathfrak{R}$-free graded coalgebra endowed with an odd coderivation d of degree 1 and a homogeneous $\mathfrak{R}$-contramodule morphism $h\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ of degree 2 (that is, h vanishes on all the components of $\mathfrak{C}$ except $\mathfrak{C}^{-2}$) satisfying the equations $d^2(c) = h * c - c * h$ and $h(d(c)) = 0$ for all $c \in \mathfrak{C}$. Morphisms $\mathfrak{C} \longrightarrow \mathfrak{D}$ of $\mathfrak{R}$-free CDG-coalgebras are defined as pairs (f, a), with $f\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ being a morphism of $\mathfrak{R}$-free graded coalgebras and $a\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ a homogeneous $\mathfrak{R}$-contramodule morphism of degree 1, satisfying the conventional equations [29, Section 4.1].

An *$\mathfrak{R}$-free left CDG-comodule* $\mathfrak{M}$ over $\mathfrak{C}$ is an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule endowed with an odd coderivation $d_{\mathfrak{M}}\colon \mathfrak{M} \longrightarrow \mathfrak{M}$ of degree 1 compatible with the coderivation d on $\mathfrak{C}$ and satisfying the equation $d_{\mathfrak{M}}^2(x) = h * x$ for all $x \in \mathfrak{M}$. The definition of an *$\mathfrak{R}$-free right CDG-comodule* $\mathfrak{N}$ over $\mathfrak{C}$ is similar; the only difference is that the equation for the square of the differential has the form $d_{\mathfrak{N}}^2(y) = -y * h$ for all $y \in \mathfrak{N}$. An *$\mathfrak{R}$-free left CDG-contramodule* $\mathfrak{P}$ over $\mathfrak{C}$ is an $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule endowed with an odd contraderivation $d_{\mathfrak{P}}\colon \mathfrak{P} \longrightarrow \mathfrak{P}$ of degree 1 compatible with the coderivation d on $\mathfrak{C}$ and satisfying the equation $d_{\mathfrak{P}}^2(p) = h * p$ for all $p \in \mathfrak{P}$.

An $\mathfrak{R}$-*free DG-coalgebra* $\mathfrak{C}$ is an $\mathfrak{R}$-free CDG-coalgebra with $h = 0$. A morphism of $\mathfrak{R}$-free DG-coalgebras $\mathfrak{C} \longrightarrow \mathfrak{D}$ is a morphism $f = (f, a)$ between $\mathfrak{C}$ and $\mathfrak{D}$ considered as $\mathfrak{R}$-free CDG-coalgebras such that $a = 0$. A *DG-comodule* or *DG-contramodule* over an $\mathfrak{R}$-free DG-coalgebra $\mathfrak{C}$ is the same thing as a CDG-comodule or CDG-contramodule over $\mathfrak{C}$ considered as a CDG-coalgebra.

Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. $\mathfrak{R}$-free left (resp., right) CDG-comodules over $\mathfrak{C}$ naturally form a DG-category, which we will denote by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ (resp., $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}$). The similar DG-category of $\mathfrak{R}$-free left CDG-contramodules over $\mathfrak{C}$ will be denoted by $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$. In fact, these DG-categories are enriched over the tensor category of (complexes of) $\mathfrak{R}$-contramodules, so the complexes of morphisms in $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$, $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}$, and $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$ are the underlying complexes of abelian groups for naturally defined complexes of $\mathfrak{R}$-contramodules.

The underlying graded $\mathfrak{R}$-contramodules of these complexes were defined in Section 3.1. Given two left CDG-comodules $\mathfrak{L}$ and $\mathfrak{M} \in \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$, the differential in the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L}, \mathfrak{M})$ is defined by the conventional formula $d(f)(x) = d_{\mathfrak{M}}(f(x)) - (-1)^{|f|} f(d_{\mathfrak{L}}(x))$; one easily checks that $d^2(f) = 0$. The complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}^{\mathrm{op}}}(\mathfrak{K}, \mathfrak{N})$ for any two right CDG-comodules $\mathfrak{K}$ and $\mathfrak{N} \in \mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}$ and the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P}, \mathfrak{Q})$ for any two left CDG-contramodules $\mathfrak{P}$ and $\mathfrak{Q} \in \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$ are defined in the similar way.

Passing to the zero cohomology of the complexes of morphisms, we construct the homotopy categories of $\mathfrak{R}$-free CDG-comodules and CDG-contramodules $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$, and $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$. Even though these are also naturally enriched over $\mathfrak{R}$-contramodules, we will mostly consider them as conventional categories with abelian groups of morphisms (see Remark 2.2.1). Since the DG-categories $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$, $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}$, and $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$ have shifts and twists, their homotopy categories are triangulated. The DG- and homotopy categories of $\mathfrak{R}$-free CDG-comodules also have infinite direct sums, while the DG- and homotopy categories of $\mathfrak{R}$-free CDG-contramodules have infinite products.

The *contratensor product* $\mathfrak{N} \odot_{\mathfrak{C}} \mathfrak{P}$ of an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-contramodules obtained by endowing the graded $\mathfrak{R}$-contramodule $\mathfrak{N} \odot_{\mathfrak{C}} \mathfrak{P}$ constructed in Section 3.1 with the conventional tensor product differential $d(y \odot p) = d_{\mathfrak{N}}(y) \odot p + (-1)^{|y|} y \odot d_{\mathfrak{P}}(p)$. The contratensor product of $\mathfrak{R}$-free right CDG-comodules and $\mathfrak{R}$-free left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\odot_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

The *cotensor product* $\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{M}$ of an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{M}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-contramodules obtained by endowing the graded $\mathfrak{R}$-contramodule $\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathfrak{M}$ with the conventional tensor product differential. The cotensor product of $\mathfrak{R}$-free CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\square_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

The complex $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$ from an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{M}$ to an $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-contramodules obtained by endowing the graded $\mathfrak{R}$-contramodule $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$ constructed in Section 3.1 with the conventional Hom differential. The Cohom between $\mathfrak{R}$-free left CDG-comodules and CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

An $\mathfrak{R}$-free left CDG-comodule over $\mathfrak{C}$ is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-comodules over $\mathfrak{C}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-free CDG-comodules is called the *coderived category* of $\mathfrak{R}$-free left CDG-comodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The coderived category of $\mathfrak{R}$-free right CDG-comodules over $\mathfrak{C}$, denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$, is defined similarly (see [29, Sections 1.2 and 4.2] or [28, Sections 0.2.2 and 2.1]).

An $\mathfrak{R}$-free left CDG-contramodule over $\mathfrak{C}$ is called *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-contramodules over $\mathfrak{C}$ and closed under infinite products. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-free CDG-contramodules is called the *contraderived category* of $\mathfrak{R}$-free left CDG-contramodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ (see [29, Sections 1.2 and 4.2] or [28, Sections 0.2.5 and 4.1]).

Denote by $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}\subset\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-free CDG-contramodules over $\mathfrak{C}$ whose underlying $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodules are projective. Similarly, denote by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}\subset\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-free CDG-comodules over $\mathfrak{C}$ whose underlying $\mathfrak{R}$-free graded $\mathfrak{C}$-comodules are injective. The corresponding homotopy categories are denoted by $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$, respectively.

Lemma 3.2.1. (a) *Let $\mathfrak{P}$ be a CDG-contramodule from $\mathfrak{C}\text{–}\mathsf{contra}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$. Then $\mathfrak{P}$ is contractible (i. e., represents a zero object in $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$) if and only if the CDG-contramodule $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over k is contractible.*

(b) *Let $\mathfrak{M}$ be a CDG-comodule from $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$. Then $\mathfrak{M}$ is contractible if and only if the CDG-comodule $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is contractible.*

Proof. See the proof of Lemma 2.2.2. ☐

Theorem 3.2.2. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *for any CDG-contramodule $\mathfrak{P}\in H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and any contraacyclic $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$, the complex of $\mathfrak{R}$-contramodules* $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$ *is contractible;*

(b) *for any coacyclic $\mathfrak{R}$-free left CDG-comodule $\mathfrak{L}$ over $\mathfrak{C}$ and any CDG-comodule $\mathfrak{M}\in H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$, the complex of $\mathfrak{R}$-contramodules* $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{M})$ *is contractible;*

(c) *the composition of natural functors $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})\longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})\longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ is an equivalence of triangulated categories;*

(d) *the composition of natural functors* $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *is an equivalence of triangulated categories.*

Proof. Parts (a-b) follow from the first assertions of Lemma 3.1.1(a-b) together with the observations that the total complex of a short exact sequence of complexes of free $\mathfrak{R}$-contramodules is contractible and the infinite products of complexes of $\mathfrak{R}$-contramodules preserve contractibility (cf. Theorem 4.5.2 below).

Parts (c-d) can be proven in the way similar to the proof of [29, Theorem 4.4(c-d)], or alternatively deduced from [29, Remark 3.7]. The key observations are that the class of projective $\mathfrak{R}$-free $\mathfrak{C}$-contramodules is closed under infinite products, and the class of injective $\mathfrak{R}$-free $\mathfrak{C}$-comodules is closed under infinite direct sums. □

Corollary 3.2.3. *For any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, an $\mathfrak{R}$-free CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$ is contraacyclic if and only if the CDG-contramodule $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ over the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over the field k is contraacyclic. An $\mathfrak{R}$-free CDG-comodule $\mathfrak{M}$ over $\mathfrak{C}$ is coacyclic if and only if the CDG-comodule $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over k is coacyclic.*

Proof. See the proof of Corollary 2.2.5. □

Given a CDG-contramodule $\mathfrak{P} \in \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$, the graded left $\mathfrak{C}$-comodule $\Phi_{\mathfrak{C}}(\mathfrak{P}) = \mathfrak{C} \odot_{\mathfrak{C}} \mathfrak{P}$ is endowed with a CDG-comodule structure with the conventional tensor product differential. Conversely, given a CDG-comodule $\mathfrak{M} \in \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$, the graded left $\mathfrak{C}$-contramodule $\Psi_{\mathfrak{C}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}, \mathfrak{M})$ is endowed with a CDG-contramodule structure with the conventional Hom differential. One easily checks that $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ are mutually inverse equivalences between the DG-categories $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ and $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ (see Proposition 3.1.4).

Corollary 3.2.4. *The derived functors* $\mathbb{L}\Phi_{\mathfrak{C}} \colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathbb{R}\Psi_{\mathfrak{C}} \colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *defined by identifying* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *with* $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ *and* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *with* $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$ *are mutually inverse equivalences between the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. □

An $\mathfrak{R}$-free CDG-contra/comodule over $\mathfrak{C}$ is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category of $\mathfrak{R}$-free CDG-contra/comodules over $\mathfrak{C}$ containing the totalizations of short exact sequences of $\mathfrak{R}$-free CDG-contra/comodules. The quotient category of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$, or $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ by the thick subcategory of absolutely acyclic CDG-contra/comodules is called the *absolute derived category* of (left or right) CDG-contra/comodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, $\mathsf{D}^{\mathsf{abs}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$, or $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$, respectively.

The following result is to be compared with Theorem 2.2.4.

Theorem 3.2.5. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then whenever the exact category of $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodules has finite homological dimension, the* *classes of contraacyclic and absolutely acyclic $\mathfrak{R}$-free left CDG-contramodules over $\mathfrak{C}$ coincide, so* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) = \mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$. *Whenever the exact category of $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules has finite homological dimension, the classes of coacyclic and absolutely acyclic $\mathfrak{R}$-free left CDG-comodules over $\mathfrak{C}$ coincide, so* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) = \mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$.

Proof. See [29, Theorem 4.5 or Remark 3.6]. A more general (but also much more difficult) argument can be found in [31, Theorem 1.6]. $\square$

The right derived functor of homomorphisms of $\mathfrak{R}$-free CDG-contramodules

$$\operatorname{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\operatorname{Hom}^{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 3.1.1(a) and factorizes through the Cartesian product of contraderived categories by Theorem 3.2.2(a) and (c).

The right derived functor of homomorphisms of $\mathfrak{R}$-free CDG-comodules

$$\operatorname{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\operatorname{Hom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 3.1.1(b) and factorizes through the Cartesian product of coderived categories by Theorem 3.2.2(b) and (d).

The left derived functor of contratensor product of $\mathfrak{R}$-free CDG-comodules and CDG-contramodules

$$\operatorname{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\odot_{\mathfrak{C}}$ to the full subcategory $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 3.1.1(c) and factorizes through the coderived category in the first argument because the contratensor product with a projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule preserves short exact sequences and infinite direct sums of $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodules.

The right derived functor of cotensor product of $\mathfrak{R}$-free CDG-comodules

$$\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\square_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})$ or $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. Here $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}\text{–}\mathfrak{C}$ denotes the DG-category of $\mathfrak{R}$-free right CDG-comodules over $\mathfrak{C}$ with injective underlying $\mathfrak{R}$-free graded $\mathfrak{C}$-comodules, and $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}\text{–}\mathfrak{C})$ is the corresponding homotopy category. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 3.1.1(d) and factorizes through the Cartesian product of coderived categories because the cotensor product with an injective $\mathfrak{R}$-free graded $\mathfrak{C}$-comodule preserves short exact sequences and infinite direct sums of $\mathfrak{R}$-free graded $\mathfrak{C}$-comodules.

The left derived functor of cohomomorphisms of $\mathfrak{R}$-free CDG-comodules and CDG-contramodules

$$\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\mathrm{Cohom}_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ or $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$. The restriction takes values in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ by Lemma 3.1.1(e) and factorizes through the Cartesian product of the coderived and contraderived categories for the reasons similar to the ones explained above.

Proposition 3.2.6. (a) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *from Corollary* 3.2.4 *transforms the left derived functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *into the right derived functors* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *and* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

(b) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *transforms the right derived functor* $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *into the left derived functor* $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

Proof. For any projective $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{P}$ and any $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$, there is a natural isomorphism of complexes of free $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\Phi_{\mathfrak{C}}(\mathfrak{P}),\mathfrak{Q}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$. For any $\mathfrak{R}$-free left CDG-comodule $\mathfrak{L}$ and any injective $\mathfrak{R}$-free left CDG-comodule $\mathfrak{M}$ over $\mathfrak{C}$, there is a natural isomorphism of complexes of free $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{L},\Psi_{\mathfrak{C}}(\mathfrak{M})) \simeq \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathfrak{M})$. For any $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and any projective $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$, there is a natural isomorphism of complexes of free $\mathfrak{R}$-contramodules $\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P} \simeq \mathfrak{N}\,\square_{\mathfrak{C}}\,\Phi_{\mathfrak{C}}(\mathfrak{P})$. (Cf. [28, Section 5.6].) ☐

Let $f=(f,a)\colon \mathfrak{C}\longrightarrow\mathfrak{D}$ be a morphism of $\mathfrak{R}$-free CDG-coalgebras. Then any $\mathfrak{R}$-free graded contramodule (resp., comodule) over $\mathfrak{C}$ can be endowed with a graded $\mathfrak{D}$-contramodule (resp., $\mathfrak{D}$-comodule) structure via f, and any homogeneous morphism (of any degree) between graded $\mathfrak{C}$-contramodules (resp., $\mathfrak{C}$-comodules) can be also considered as a homogeneous morphism (of the same degree) between graded $\mathfrak{D}$-contramodules (resp., $\mathfrak{D}$-comodules).

With any $\mathfrak{R}$-free left CDG-contramodule $(\mathfrak{P},d_{\mathfrak{P}})$ over $\mathfrak{C}$ one can associate an $\mathfrak{R}$-free left CDG-contramodule $(\mathfrak{P},d'_{\mathfrak{P}})$ over $\mathfrak{D}$ with the modified differential $d'_{\mathfrak{P}}(p)=d_{\mathfrak{P}}(p)+a{*}p$. Similarly, fo any $\mathfrak{R}$-free left CDG-comodule $(\mathfrak{M},d_{\mathfrak{M}})$ over $\mathfrak{C}$ the formula $d'_{\mathfrak{M}}(x)=d_M(x)+a*x$ defines a modified differential on $\mathfrak{M}$ making $(\mathfrak{M},d'_{\mathfrak{M}})$ an $\mathfrak{R}$-free left CDG-comodule over $\mathfrak{D}$. Finally, with any $\mathfrak{R}$-free right CDG-comodule $(\mathfrak{N},d_{\mathfrak{N}})$ over $\mathfrak{C}$ one associate an $\mathfrak{R}$-free right CDG-comodule $(\mathfrak{N},d'_{\mathfrak{N}})$ with the modified differential $d'_{\mathfrak{N}}(y)=d_N(y)-(-1)^{|y|}y*a$.

We have constructed the DG-functors of contrarestriction of scalars $R^f\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}$, and corestriction of scalars $R_f\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$

and $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C} \longrightarrow \mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{D}$. Passing to the homotopy categories, we obtain the triangulated functors $R^f\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $R_f\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. Since the contra/corestriction of scalars clearly preserves contra/coacyclicity, we have the induced functors on the contra/coderived categories

$$\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$$

and

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}).$$

The triangulated functor $\mathbb{I}R^f$ has a left adjoint. The DG-functor $E^f\colon \mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ is defined on the level of graded contramodules by the rule $\mathfrak{Q} \longmapsto \mathrm{Cohom}_{\mathfrak{D}}(\mathfrak{C},\mathfrak{Q})$ (see Lemma 3.1.1(e)); the differential on $\mathrm{Cohom}_{\mathfrak{D}}(\mathfrak{C},\mathfrak{Q})$ induced by the differentials on $\mathfrak{C}$ and $\mathfrak{Q}$ is modified to obtain the differential on $E^f(\mathfrak{Q})$ using the change-of-connection linear function a. Passing to the homotopy categories and taking into account Theorem 3.2.2(c), we obtain the left derived functor of contraextension of scalars

$$\mathbb{L}E^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}),$$

which is left adjoint to the functor $\mathbb{I}E^f$.

Similarly, the triangulated functor $\mathbb{I}R_f$ has a right adjoint. The DG-functor $E_f\colon \mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ is defined on the level of graded comodules by the rule $\mathfrak{N} \longmapsto \mathfrak{C}\,\square_{\mathfrak{D}}\,\mathfrak{N}$ (see Lemma 3.1.1(d)); the change-of-connection linear function a is used to modify the differential on $\mathfrak{C}\,\square_{\mathfrak{D}}\,\mathfrak{N}$ induced by the differentials on $\mathfrak{C}$ and $\mathfrak{N}$. Passing to the homotopy categories and taking into account Theorem 3.2.2(d), we obtain the right derived functor of coextension of scalars

$$\mathbb{R}E_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}).$$

Proposition 3.2.7. *The equivalences of triangulated categories $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ and $\mathbb{L}\Phi_{\mathfrak{D}} = \mathbb{R}\Psi_{\mathfrak{D}}^{-1}$ from Corollary 3.2.4 transform the left derived functor $\mathbb{L}E^f$ into the right derived functor $\mathbb{R}E_f$ and back.*

Proof. See [29, Section 5.4] or [28, Section 7.1.4 and Corollary 8.3.4]. ☐

Theorem 3.2.8. (a) *Let $f = (f,a)\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ be a morphism of $\mathfrak{R}$-free CDG-coalgebras. Then the functor $\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ is an equivalence of triangulated categories if and only if the functor $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ is such an equivalence.*

(b) *A morphism $f = (f,a)$ of $\mathfrak{R}$-free CDG-coalgebras has the above two equivalent properties* (a) *provided that the morphism $f/\mathfrak{m}f = (f/\mathfrak{m}f,\, a/\mathfrak{m}a)\colon \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{D}/\mathfrak{m}\mathfrak{D}$ of CDG-coalgebras over k has the similar properties.*

Proof. To prove part (a), it suffices to notice that, according to Proposition 3.2.7, the two triangulated functors in question are the adjoints on different sides to the same functor $\mathbb{L}E^f = \mathbb{R}E_f$. To prove part (b), consider the adjunction morphisms for the functors $\mathbb{L}E^f$ and $\mathbb{I}R^f$, or alternatively, for the functors $\mathbb{I}R_f$ and $\mathbb{R}E_f$. The functors of reduction modulo $\mathfrak{m}$ transform these into the adjunction morphisms for the similar functors related to the morphism of CDG-algebras $f/\mathfrak{m}f$. By Corollary 3.2.3, if the latter are isomorphisms, then so are the former. $\square$

3.3. **$\mathfrak{R}$-cofree graded contra/comodules.** Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. An $\mathfrak{R}$-*cofree graded left comodule* $\mathcal{M}$ over $\mathfrak{C}$ is a graded left $\mathfrak{C}$-comodule in the module category of cofree $\mathfrak{R}$-comodules over the tensor category of free $\mathfrak{R}$-contramodules. In other words, it is a cofree graded $\mathfrak{R}$-comodule endowed with a (coassociative and counital) homogeneous $\mathfrak{C}$-coaction map $\mathcal{M} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$. $\mathfrak{R}$-*free graded right comodules* $\mathcal{N}$ over $\mathfrak{C}$ are defined in the similar way.

An $\mathfrak{R}$-*cofree graded left contramodule* $\mathcal{P}$ over is a cofree graded $\mathfrak{R}$-comodule endowed with a $\mathfrak{C}$-contraaction map $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}) \longrightarrow \mathcal{P}$, which must be a morphism of graded $\mathfrak{R}$-comodules satisfying the conventional contraassociativity and counit axioms. This can be rephrased by saying that an $\mathfrak{R}$-cofree $\mathfrak{C}$-contramodule is an object of the opposite category to the category of graded $\mathfrak{C}$-comodules in the module category $(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})^{\mathsf{op}}$ over the tensor category $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}$ (see Section 1.6 and [28, Section 0.2.4 or 3.1.1]).

Specifically, the contraassociativity axiom asserts that the two compositions $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P})) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C},\ \mathcal{P}) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}) \longrightarrow \mathcal{P}$ of the maps induced by the contraaction and comultiplication maps with the contraaction map must be equal to each other. The counit axiom claims that the composition $\mathcal{P} \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}) \longrightarrow \mathcal{P}$ of the map induced by the counit map with the contraaction map must be equal to the identity endomorphism of $\mathcal{P}$. For the sign rule, see [29, Section 2.2].

The categories of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodules and $\mathfrak{C}$-contramodules are enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules. So the abelian group of morphisms between two $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodules $\mathcal{L}$ and $\mathcal{M}$ is the underlying abelian group of the degree-zero component of the (not necessarily free) graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L}, \mathcal{M})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L}, \mathcal{M}) \rightrightarrows \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M})$ induced by the coactions of $\mathfrak{C}$ in $\mathcal{L}$ and $\mathcal{M}$. For the sign rules, which are different for the left and the right comodules, see [29, Section 2.1]. Similarly, the abelian group of morphisms between two $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodules $\mathcal{P}$ and $\mathcal{Q}$ is the underlying abelian group of the zero-degree component of the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{C}}(\mathcal{P}, \mathcal{Q})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{P}, \mathcal{Q}) \rightrightarrows \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}), \mathcal{Q})$ induced by the contraactions of $\mathfrak{C}$ in $\mathcal{P}$ and $\mathcal{Q}$.

The graded $\mathfrak{R}$-comodule $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L}, \mathcal{M})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L}, \mathcal{M}) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M})$, one of which is induced by the coaction morphism $\mathcal{M} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$, while the other is constructed in terms of the coaction morphism $\mathfrak{L} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{L}$ in the following way. A given element of $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L}, \mathcal{M})$ is composed with the adjunction morphism $\mathcal{M} \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M})$, resulting in an element of $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L}, \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M})) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M})$. The latter element is composed with the morphism of $\mathfrak{C}$-coaction in $\mathfrak{L}$. Similarly, the graded $\mathfrak{R}$-comodule $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathcal{Q})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{Q}) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\mathcal{Q}))$, one of which is induced by the contraaction morphism $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$, while the other is constructed in terms of the contraaction morphism $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{Q}) \longrightarrow \mathcal{Q}$ in the following way. A given element of $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{Q})$ is composed with the evaluation morphism $\mathfrak{C}\otimes^{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$, resulting in an element of $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\ \mathcal{Q}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\ \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{Q}))$. The latter element is composed with the morphism of $\mathfrak{C}$-contraaction in $\mathcal{Q}$.

The *contratensor product* $\mathfrak{N}\odot_{\mathfrak{C}}\mathcal{P}$ of an $\mathfrak{R}$-free right $\mathfrak{C}$-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-cofree left $\mathfrak{C}$-contramodule $\mathcal{P}$ is the (not necessarily cofree) graded $\mathfrak{R}$-comodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathfrak{N}\odot_{\mathfrak{R}}\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})) \rightrightarrows \mathfrak{N}\odot_{\mathfrak{R}}\mathcal{P}$ defined in terms of the $\mathfrak{C}$-coaction in $\mathfrak{N}$, the $\mathfrak{C}$-contraaction in $\mathfrak{P}$, and the evaluation map $\mathfrak{C}\odot_{\mathfrak{R}}\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}) \longrightarrow \mathcal{P}$. The latter can be obtained from the natural isomorphism $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}),\ \mathcal{P}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}),\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}))$ (see Section 1.6). For further details, see [28, Section 0.2.6]; for the sign rule, see [29, Section 2.2].

The *cotensor product* $\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathcal{M}$ of an $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ is a graded $\mathfrak{R}$-comodule constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathfrak{N}\odot_{\mathfrak{R}}\mathcal{M} \rightrightarrows \mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$. The graded $\mathfrak{R}$-comodule of *cohomomorphisms* $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathcal{P})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ is constructed as the cokernel of the pair of morphisms $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M},\ \mathcal{P}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{M},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{C}}(\mathfrak{M},\mathcal{P})$ induced by the $\mathfrak{C}$-coaction in $\mathfrak{M}$ and the $\mathfrak{C}$-contraaction in $\mathcal{P}$. The graded $\mathfrak{R}$-contramodule of *cohomomorphisms* $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$ from an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ is constructed as the cokernel of the pair of morphisms $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M},\ \mathcal{P}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{M},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathcal{M},\mathcal{P})) \longrightarrow \mathrm{Hom}_{\mathfrak{R}}(\mathcal{M},\mathcal{P})$ induced by the $\mathfrak{C}$-coaction in $\mathcal{M}$ and the $\mathfrak{C}$-contraaction in $\mathcal{P}$.

For any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$ and any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$, the cofree graded $\mathfrak{R}$-comodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N},\mathcal{U})$ has a natural left $\mathfrak{C}$-contramodule structure provided by the map $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N},\mathcal{U})) \simeq \mathrm{Ctrhom}_{\mathfrak{C}}(\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C},\ \mathcal{U}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{C}}(\mathfrak{N},\mathcal{U})$ induced by the $\mathfrak{C}$-coaction in $\mathfrak{N}$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P},\ \mathcal{U}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N},\mathcal{U}))$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{N}\odot_{\mathfrak{C}}\mathcal{P},\ \mathcal{U}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathcal{P},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N},\mathcal{U}))$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathcal{M},\ \mathcal{U}) \simeq \mathrm{Cohom}^{\mathfrak{C}}(\mathcal{M},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{N},\mathcal{U}))$.

The additive categories of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contramodules and $\mathfrak{C}$-comodules have natural exact category structures: a short sequence of $\mathfrak{R}$-cofree graded

$\mathfrak{C}$-contra/comodules is said to be exact is it is (split) exact as a short sequence of cofree graded $\mathfrak{R}$-comodules. The exact category of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodules admits infinite direct sums, while the exact category of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contramodules admits infinite products. Both operations are preserved by the forgetful functors to the category of cofree graded $\mathfrak{R}$-comodules and preserve exact sequences.

If a morphism of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contra/comodules has an $\mathfrak{R}$-cofree kernel (resp., cokernel) in the category of graded $\mathfrak{R}$-comodules, then this kernel (resp., cokernel) is naturally endowed with an $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contra/comodule structure, making it the kernel (resp., cokernel) of the same morphism in the additive category of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contra/comodules.

There are enough projective objects in the exact category of $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodules; these are the direct summands of the graded $\mathfrak{C}$-contramodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U})$ freely generated by cofree graded $\mathfrak{R}$-comodules $\mathcal{U}$. Similarly, there are enough injective objects in the exact category of $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodules; these are the direct summands of the graded $\mathfrak{C}$-comodules $\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}$ cofreely cogenerated by cofree graded $\mathfrak{R}$-comodules $\mathcal{V}$.

For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{L}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{V}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L}, \mathcal{V})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{V}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L}, \mathcal{V})$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U}), \mathcal{Q}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{U}, \mathcal{Q})$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Hom}^{\mathfrak{C}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U}), \mathcal{Q})) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{U}, \mathcal{Q})$ [28, Lemma 1.1.2].

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N} \odot_{\mathfrak{C}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U}) \simeq \mathfrak{N} \odot_{\mathfrak{R}} \mathcal{U}$ [28, Section 5.1.1]. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N} \square_{\mathfrak{C}} (\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathfrak{N} \odot_{\mathfrak{C}} \mathcal{V}$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ and any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{V}, \mathcal{P}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{V}, \mathcal{P})$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V},\ \mathcal{P}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{V}, \mathcal{P})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M}, \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U})) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{M}, \mathcal{U})$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M}, \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U})) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{M}, \mathcal{U})$ [28, Sections 1.2.1 and 3.2.1].

All the results of Section 3.1 have their analogues for $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contramodules and $\mathfrak{C}$-comodules, which can be, at one's choice, either proven directly in the similar way to the proofs in 3.1, or deduced from the results in 3.1 using Proposition 3.3.2 below. In particular, we have the following constructions.

Given a projective $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$, the $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\Phi_{\mathfrak{C}}(\mathcal{P})$ is defined as $\Phi_{\mathfrak{C}}(\mathcal{P}) = \mathfrak{C} \odot_{\mathfrak{C}} \mathcal{P}$. Here the contratensor product is endowed with a left $\mathfrak{C}$-comodule structure as an $\mathfrak{R}$-cofree cokernel of a morphism of $\mathfrak{R}$-cofree $\mathfrak{C}$-comodules. Given an injective $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$, the $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\Psi_{\mathfrak{C}}(\mathcal{M})$ is defined as $\Psi_{\mathfrak{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{C}, \mathcal{M})$. Here the Hom of $\mathfrak{C}$-comodules is endowed with a left $\mathfrak{C}$-contramodule structure as an $\mathfrak{R}$-cofree kernel of a morphism of $\mathfrak{R}$-cofree left $\mathfrak{C}$-contramodules.

Proposition 3.3.1. *The functors $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ are mutually inverse equivalences between the additive categories of projective $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodules and injective $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodules.*

Proof. One has $\Phi_{\mathfrak{C}}(\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U})) \simeq \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{U}$ and $\Phi_{\mathfrak{C}}(\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{V})$. $\square$

Proposition 3.3.2. (a) *The exact categories of $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodules and of $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodules are naturally equivalent. The equivalence is provided by the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ of comodule-contramodule correspondence over $\mathfrak{R}$, defined in Section* 1.5, *which transfrom free graded $\mathfrak{R}$-contramodules with a $\mathfrak{C}$-contramodule structure into cofree graded $\mathfrak{R}$-comodules with a $\mathfrak{C}$-contramodule structure and back.*

(b) *The exact categories of $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodules and of $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodules are naturally equivalent. The equivalence is provided by the functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$, which transform free graded $\mathfrak{R}$-contramodules with a $\mathfrak{C}$-comodule structure into cofree graded $\mathfrak{R}$-comodules with a $\mathfrak{C}$-comodule structure and back.*

Proof. See the proof of Proposition 2.4.1. $\square$

The equivalence $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ between the categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contra/comodules is also an equivalence of categories enriched over graded $\mathfrak{R}$-contramodules. Besides, for any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ and $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L}, \Psi_{\mathfrak{R}}(\mathcal{M})) \simeq \Psi_{\mathfrak{R}}(\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L}, \mathcal{M}))$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P}, \Psi_{\mathfrak{R}}(\mathcal{Q})) \simeq \Phi_{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P}, \mathcal{Q}))$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N} \odot_{\mathfrak{C}} \Phi_{\mathfrak{R}}(\mathfrak{P}) \simeq \Phi_{\mathfrak{R}}(\mathfrak{N} \odot_{\mathfrak{C}} \mathfrak{P})$.

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \Psi_{\mathfrak{R}}(\mathcal{M}) \simeq \Psi_{\mathfrak{R}}(\mathfrak{N} \mathbin{\square}_{\mathfrak{C}} \mathcal{M})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\Phi_{\mathfrak{R}}(\mathfrak{M}), \Phi_{\mathfrak{R}}(\mathfrak{P})) \simeq \mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M}, \mathfrak{P})$ and a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M}, \Phi_{\mathfrak{R}}(\mathfrak{P})) \simeq \Phi_{\mathfrak{R}}(\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M}, \mathfrak{P}))$.

Furthermore, the equivalences $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform graded $\mathfrak{C}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U})$ freely generated by free graded $\mathfrak{R}$-contramodules $\mathfrak{U}$ into graded $\mathfrak{C}$-contramodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{U})$ freely generated by cofree graded $\mathfrak{R}$-comodules $\mathcal{U}$, with $\Psi_{\mathfrak{R}}(\mathcal{U}) = \mathfrak{U}$, and graded $\mathfrak{C}$-comodules $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{V}$ cofreely cogenerated by free graded $\mathfrak{R}$-contamodules $\mathfrak{V}$ into graded $\mathfrak{C}$-comodules $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$ cofreely cogenerated by cofree graded $\mathfrak{R}$-comodules $\mathcal{V}$, with $\mathcal{V}=\Phi_{\mathfrak{R}}(\mathfrak{V})$.

In particular, the equivalences of additive categories $\Phi_{\mathfrak{C}}=\Psi_{\mathfrak{C}}^{-1}$ from Propositions 3.1.4 and 3.3.1 form a commutative diagram with (the appropriate restriction of) the equivalence of exact categories $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ from Proposition 3.3.2.

Finally, the equivalences $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ between the exact categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contra/comodules trasforms the reduction functors $\mathfrak{P}\longmapsto\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ and $\mathfrak{M}\longmapsto\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ into the reduction functors $\mathcal{P}\longmapsto{}_{\mathfrak{m}}\mathcal{P}$ and $\mathcal{M}\longmapsto{}_{\mathfrak{m}}\mathcal{M}$, respectively, the functors taking values in the abelian category of $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$-contra/comodules. This follows from the results of Sections 1.5–1.6.

3.4. **Contra/coderived category of $\mathfrak{R}$-cofree CDG-contra/comodules.** Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. Suppose that it is endowed with an odd coderivation d of degree 1. An *odd coderivation* $d_{\mathcal{M}}$ of an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{M}$ *compatible with* the coderivation d on $\mathfrak{C}$ is a differential ($\mathfrak{R}$-comodule endomorphism of degree 1) such that the coaction map $\mathcal{M}\longrightarrow\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ form a commutative diagram with $d_{\mathcal{M}}$ and the differential on $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ induced by d and $d_{\mathcal{M}}$ (see Section 2.5). Odd coderivations of $\mathfrak{R}$-cofree graded right $\mathfrak{C}$-comodules are defined similarly.

An *odd contraderivation* $d_{\mathcal{P}}$ of an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ *compatible with* the coderivation d on $\mathfrak{C}$ is a differential ($\mathfrak{R}$-comodule endomorphism of degree 1) such that the contraaction map $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})\longrightarrow\mathcal{P}$ forms a commutative diagram with $d_{\mathcal{P}}$ and the differential on $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$ induced by d and $d_{\mathcal{P}}$.

An $\mathfrak{R}$-cofree graded comodule $\mathcal{M}$ over $\mathfrak{C}$ is endowed with the structure of a graded left module over the $\mathfrak{R}$-free graded algebra $\mathfrak{C}^*$ (see Section 3.2) provided by the action map $\mathfrak{C}^*\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathfrak{C}^*\odot_{\mathfrak{R}}(\mathfrak{C}^*\odot_{\mathfrak{R}}\mathcal{M})\simeq(\mathfrak{C}^*\otimes^{\mathfrak{R}}\mathfrak{C})\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathcal{M}$. $\mathfrak{R}$-cofree graded right comodules over $\mathfrak{C}$ are endowed with graded right $\mathfrak{C}^*$-module structures in the similar way. An $\mathfrak{R}$-cofree graded left contramodule $\mathcal{P}$ over $\mathfrak{C}$ is endowed with a graded left $\mathfrak{C}^*$-module structure provided by the action map $\mathfrak{C}^*\odot_{\mathfrak{R}}\mathcal{P}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})\longrightarrow\mathcal{P}$. For the sign rules, see [29, Section 4.1].

An $\mathfrak{R}$-*cofree left CDG-comodule* $\mathcal{M}$ over $\mathfrak{C}$ is an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule endowed with an odd coderivation $\partial_{\mathcal{M}}\colon\mathcal{M}\longrightarrow\mathcal{M}$ of degree 1 compatible with the coderivation d on $\mathfrak{C}$ and satisfying the equation $d_{\mathcal{M}}^2(x)=h*x$ for all $x\in\mathcal{M}$. The definition of an $\mathfrak{R}$-*cofree right CDG-comodule* $\mathcal{N}$ over $\mathfrak{C}$ is similar; the only difference is that the equation for the square of the differential has the form $d_{\mathcal{N}}^2(y)=-y*h$ for all $y\in\mathcal{N}$. An $\mathfrak{R}$-*cofree left CDG-contramodule* $\mathcal{P}$ over $\mathfrak{C}$ is an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule endowed with an odd contraderivation $d_{\mathcal{P}}\colon\mathcal{P}\longrightarrow\mathcal{P}$ of degree 1 compatible with the coderivation d on $\mathfrak{C}$ and satisfying the equation $d_{\mathcal{P}}^2(p)=h*p$ for all $p\in\mathcal{P}$. Here the element $h\in\mathfrak{C}^{*2}$ is viewed as a homogeneous morphism $\mathfrak{R}\longrightarrow\mathfrak{C}^*$ of degree 2 and the notation $x\longmapsto h*x$ is interpreted as the composition $\mathcal{M}\simeq\mathfrak{R}\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathfrak{C}^*\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathcal{M}$ defining a homogeneous endomorphism of degree 2 of the graded $\mathfrak{R}$-comodule $\mathcal{M}$; and similarly for $\mathcal{N}$ and $\mathcal{P}$.

$\mathfrak{R}$-cofree left (resp., right) CDG-comodules over $\mathfrak{C}$ naturally form a DG-category, which we will denote by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ (resp., $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{C}$). The similar DG-category of $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ will be denoted by $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$.

In fact, these DG-categories are enriched over the tensor category of (complexes of) $\mathfrak{R}$-contramodules, so the complexes of morphisms in $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$, $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{C}$, and $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$ are the underlying complexes of abelian groups for naturally defined complexes of $\mathfrak{R}$-contramodules. The underlying graded $\mathfrak{R}$-contramodules of these complexes were defined in Section 3.3, and the differentials in them are defined in the conventional way.

Passing to the zero cohomology of the complexes of morphisms, we construct the homotopy categories of $\mathfrak{R}$-cofree CDG-comodules and CDG-contramodules $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$, $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{C})$, and $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$. These we will mostly view as conventional categories with abelian groups of morphisms (see Remark 2.2.1). Since the DG-categories $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$, $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{C}$, and $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$ have shifts and twists, their homotopy categories are triangulated. The DG- and homotopy categories of $\mathfrak{R}$-cofree CDG-comodules also have infinite direct sums, while the DG- and homotopy categories of $\mathfrak{R}$-cofree CDG-contramodules have infinite products.

The complex $\operatorname{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathcal{M})$ from an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{L}$ to an $\mathfrak{R}$-cofree left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\operatorname{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathcal{M})$ constructed in Section 3.3 with the conventional Hom differential. The Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\operatorname{Hom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

Similarly, the complex $\operatorname{Hom}_{\mathfrak{C}}(\mathfrak{P},\mathcal{Q})$ from an $\mathfrak{R}$-free left CDG-contramodule $\mathfrak{P}$ to an $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{Q}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\operatorname{Hom}^{\mathfrak{C}}(\mathfrak{L},\mathcal{M})$ with the conventional Hom differential. The Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\operatorname{Hom}^{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The *contratensor product* $\mathfrak{N}\odot_{\mathfrak{C}}\mathcal{P}$ of an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{P}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $N\odot_{\mathfrak{C}}\mathcal{P}$ with the conventional tensor product differential. The contratensor product of $\mathfrak{R}$-free right CDG-comodules and $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\odot_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The cotensor product $\mathfrak{N}\mathbin{\square}_{\mathfrak{C}}\mathcal{M}$ of an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-cofree left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\mathfrak{N}\mathbin{\square}_{\mathfrak{C}}\mathcal{M}$ defined in Section 3.3 with the conventional tensor product differential. The cotensor product of $\mathfrak{R}$-free right CDG-comodules and $\mathfrak{R}$-cofree left CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\square_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The complex $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathcal{P})$ from an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{M}$ to an $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{P}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-comodules obtained by endowing the graded $\mathfrak{R}$-comodule $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathcal{P})$ with the conventional Hom differential. The Cohom from $\mathfrak{R}$-free CDG-comodules to $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The complex $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$ from an $\mathfrak{R}$-cofree left CDG-comodule $\mathcal{M}$ to an $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{P}$ over $\mathfrak{C}$ is a complex of $\mathfrak{R}$-contramodules obtained by endowing the graded $\mathfrak{R}$-contramodule $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$ with the conventional Hom differential. The Cohom from $\mathfrak{R}$-cofree CDG-comodules to $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

An $\mathfrak{R}$-cofree left CDG-comodule over $\mathfrak{C}$ is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-cofree CDG-comodules over $\mathfrak{C}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-cofree CDG-comodules is called the *coderived category* of $\mathfrak{R}$-cofree left CDG-comodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The coderived category of $\mathfrak{R}$-cofree right CDG-comodules over $\mathfrak{C}$, denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}\text{–}\mathfrak{C})$, is defined similarly.

An $\mathfrak{R}$-cofree left CDG-contramodule over $\mathfrak{C}$ is called *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ and closed under infinite products. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-cofree CDG-contramodules is called the *contraderived category* of $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$.

Denote by $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}\subset\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ whose underlying $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contramodules are projective. Similarly, denote by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}\subset\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ the full DG-subcategory formed by all the $\mathfrak{R}$-cofree CDG-comodules whose underlying $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodules are injective. The corresponding homotopy categories are denoted by $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})$ and $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})$.

The functors $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ from Section 3.3 define equivalences of DG-categories $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\simeq\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ (and similarly for right CDG-comodules), and $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}\simeq\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$. Being also equivalences of exact categories, these correspondences idenfity coacyclic $\mathfrak{R}$-free CDG-comodules with coacyclic $\mathfrak{R}$-cofree CDG-comodules and contraacyclic $\mathfrak{R}$-free CDG-contramodules with contraacyclic $\mathfrak{R}$-cofree CDG-contramodules. So an equivalence of the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})\simeq\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and an equivalence of the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})\simeq\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ are induced. The above equivalences of DG-categories also identify $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$ with $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}$ and $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$ with $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}$. The comparison isomorphisms of graded $\mathfrak{R}$-comodules and $\mathfrak{R}$-contramodules from Section 3.3 describing the compatibilities of the functors

$\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ with the operations $\operatorname{Hom}_{\mathfrak{C}}$, $\operatorname{Hom}^{\mathfrak{C}}$, $\odot_{\mathfrak{C}}$, $\square_{\mathfrak{C}}$, $\operatorname{Cohom}_{\mathfrak{C}}$ on $\mathfrak{R}$-(co)free graded $\mathfrak{C}$-comodules and $\mathfrak{C}$-contramodules all become isomorphisms of complexes of $\mathfrak{R}$-comodules and $\mathfrak{R}$-contramodules when applied to $\mathfrak{R}$-(co)free CDG-comodules and CDG-contramodules over $\mathfrak{C}$.

All the results of Section 3.2 have their analogues for $\mathfrak{R}$-cofree CDG-comodules and CDG-contramodules, which can be, at one's choice, either proven directly in the similar way to the proofs in 3.2, or deduced from the results in 3.2 using the above observations about the functors $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$. In particular, we have the following results.

Theorem 3.4.1. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *for any CDG-contramodule $\mathcal{P} \in H^0(\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and any contraacyclic $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{Q}$ over $\mathfrak{C}$, the complex of $\mathfrak{R}$-contramodules $\operatorname{Hom}^{\mathfrak{C}}(\mathcal{P},\mathcal{Q})$ is contractible;*

(b) *for any coacyclic $\mathfrak{R}$-cofree left CDG-comodule $\mathcal{L}$ over $\mathfrak{C}$ and any CDG-comodule $\mathcal{M} \in H^0(\mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}})$, the complex of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{C}}(\mathcal{L},\mathcal{M})$ is contractible;*

(c) *the composition of natural functors $H^0(\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ is an equivalence of triangulated categories;*

(d) *the composition of natural functors $H^0(\mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ is an equivalence of triangulated categories.* ☐

Given a CDG-contramodule $\mathcal{P} \in \mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$, the graded left $\mathfrak{C}$-comodule $\Phi_{\mathfrak{C}}(\mathcal{P}) = \mathfrak{C} \odot_{\mathfrak{C}} \mathcal{P}$ is endowed with a CDG-comodule structure with the conventional tensor product differential. Conversely, given a CDG-comodule $\mathcal{M} \in \mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$, the graded left $\mathfrak{C}$-contramodule $\Psi_{\mathfrak{C}}(\mathcal{M}) = \operatorname{Hom}_{\mathfrak{C}}(\mathfrak{C},\mathcal{M})$ is endowed with a CDG-contramodule structure with the conventional Hom differential. One easily checks that $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ are mutually inverse equivalences between the DG-categories $\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$ and $\mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$ (see Proposition 3.3.1).

Corollary 3.4.2. *The derived functors $\mathbb{L}\Phi_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and $\mathbb{R}\Psi_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ defined by identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ with $H^0(\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ with $H^0(\mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}})$ are mutually inverse equivalences between the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$.* ☐

The equivalences of DG-categories $\Phi_{\mathfrak{C}} = \Psi_{\mathfrak{C}}^{-1}$ form a commutative diagram with the (appropriate restrictions of) the equivalences of DG-categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$, hence the derived functors $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ commute with the triangulated functors induced by $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ (cf. the construction of functors $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ in Section 4.5 below).

The right derived functor of homomorphisms of $\mathfrak{R}$-cofree CDG-contramodules

$$\operatorname{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\operatorname{Hom}^{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$. Similarly, the right derived functor of Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-contramodules

$$\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{comod}^{\mathsf{cofr}})$$

is constructed by restricting the functor $\mathrm{Hom}^{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{fr}}_{\mathsf{proj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform these two functors into each other and the derived functor $\mathrm{Ext}^{\mathfrak{C}}$ of homomorphisms of $\mathfrak{R}$-free CDG-contramodules defined in Section 3.2.

Analogously, the right derived functor of homomorphisms of $\mathfrak{R}$-cofree CDG-comodules

$$\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}_{\mathsf{inj}}) \subset H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}})$. Similarly, the right derived functor of Hom between $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-comodules

$$\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{comod}^{\mathsf{cofr}})$$

is constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}_{\mathsf{inj}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform these two functors into each other and the derived functor $\mathrm{Ext}_{\mathfrak{C}}$ of homomorphisms of $\mathfrak{R}$-free CDG-comodules defined in Section 3.2.

The left derived functor of contratensor product of $\mathfrak{R}$-free CDG-comodules and $\mathfrak{R}$-cofree CDG-contramodules

$$\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}\text{--}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{comod}^{\mathsf{cofr}})$$

is constructed by restricting the functor $\odot_{\mathfrak{C}}$ to the full subcategory $H^0(\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}\mathfrak{C}) \times H^0(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}}_{\mathsf{proj}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform this functor into the derived functor $\mathrm{Ctrtor}^{\mathfrak{C}}$ of contratensor product of $\mathfrak{R}$-free CDG-comodules and CDG-contramodules constructed in Section 3.2.

The right derived functor of cotensor product of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree CDG-comodules

$$\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}\text{--}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{comod}^{\mathsf{cofr}})$$

is constructed by restricting the functor $\square_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}\text{--}\mathfrak{C}) \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}_{\mathsf{inj}})$ or $H^0(\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}_{\mathsf{inj}}\text{--}\mathfrak{C}) \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}\text{--}\mathfrak{C}) \times H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{cof}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform this functor into the derived functor $\mathrm{Cotor}^{\mathfrak{C}}$ of cotensor product of $\mathfrak{R}$-free CDG-comodules constructed in Section 3.2.

The left derived functor of cohomomorphisms from $\mathfrak{R}$-free CDG-comodules to $\mathfrak{R}$-cofree CDG-contramodules

$$\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{--}\mathsf{comod}^{\mathsf{cofr}})$$

is constructed by restricting the functor $\mathrm{Cohom}_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}}_{\mathsf{inj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}})$ or $H^0(\mathfrak{C}\text{--}\mathsf{comod}^{\mathfrak{R}\text{--}\mathsf{fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{--}\mathsf{contra}^{\mathfrak{R}\text{--}\mathsf{cof}}_{\mathsf{proj}}) \subset$

$H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})$. Similarly, the left derived functor of cohomomorphisms of $\mathfrak{R}$-cofree CDG-comodules and CDG-contramodules

$$\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

is constructed by restricting the functor $\mathrm{Cohom}_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})$ or $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}})$ $\subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform these two functors into each other and the derived functor $\mathrm{Coext}_{\mathfrak{C}}$ of homomorphisms of $\mathfrak{R}$-free CDG-comodules and CDG-contramodules constructed in Section 3.2.

Proposition 3.4.3. (a) *The equivalences of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *from Corollaries* 3.2.4 *and* 3.4.2 *transform the left derived functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *into the right derived functors* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *and* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$.

(b) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *transforms the left derived functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *into the right derived functors* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *and* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

(c) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *transforms the right derived functor* $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *into the left derived functor* $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. □

Let $f = (f,a)\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ be a morphism of $\mathfrak{R}$-free CDG-coalgebras. Then with any $\mathfrak{R}$-cofree left CDG-contramodule $(\mathfrak{P}, d_{\mathfrak{P}})$ over $\mathfrak{C}$ one can associate an $\mathfrak{R}$-cofree left CDG-contramodule $(\mathfrak{P}, d'_{\mathfrak{P}})$ over $\mathfrak{D}$ with the graded $\mathfrak{C}$-contramodule structure on $\mathfrak{P}$ defined via f and the modified differential $d'_{\mathfrak{P}}$ constructed in terms of a. Similar procedures apply to left and right CDG-comodules.

So we obtain the DG-functors of contrarestriction of scalars $R^f\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}} \longrightarrow \mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}$, and corestriction of scalars $R_f\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}} \longrightarrow \mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}$ and $\mathsf{comod}^{\mathfrak{R}\text{–cof}}\text{–}\mathfrak{C} \longrightarrow \mathsf{comod}^{\mathfrak{R}\text{–cof}}\text{–}\mathfrak{D}$. Passing to the homotopy categories, we have the triangulated fuctors $R^f\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})$ and $R_f\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})$. The equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_R^{-1}$ identify these functors with the functors R^f and R_f for $\mathfrak{R}$-free CDG-contramodules and CDG-comodules constructed in Section 3.2.

Since the contra/corestriction of scalars clearly preserves contra/coacyclicity, we have the induced functors on the contra/coderived categories

$$\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}}),$$

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}).$$

The triangulated functor $\mathbb{I}R^f$ has a left adjoint. The DG-functor $E^f\colon \mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}$ is defined on the level of graded contramodules by the rule $\mathfrak{Q} \longmapsto \operatorname{Cohom}_{\mathfrak{D}}(\mathfrak{C},\mathfrak{Q})$; the differential on $\operatorname{Cohom}_{\mathfrak{D}}(\mathfrak{C},\mathfrak{Q})$ induced by the differentials on $\mathfrak{C}$ and $\mathfrak{Q}$ is modified to obtain the differential on $E^f(\mathfrak{Q})$ using the linear function a. Passing to the homotopy categories and taking into account Theorem 3.4.1(c), we obtain the left derived functor

$$\mathbb{L}E^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}),$$

which is left adjoint to the functor $\mathbb{I}R^f$.

Similarly, the triangulated functor $\mathbb{I}R^f$ has a right adjoint. The DG-functor $E_f\colon \mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ is defined on the level of graded comodules by the rule $\mathfrak{N} \longmapsto \mathfrak{C}\,\square_{\mathfrak{D}}\,\mathfrak{N}$; the linear function a is used to modify the differential on $\mathfrak{C}\,\square_{\mathfrak{D}}\,\mathfrak{N}$ induced by the differentials on $\mathfrak{C}$ and $\mathfrak{N}$. Passing to the homotopy categories and taking into account Theorem 3.4.1(d), we obtain the right derived functor

$$\mathbb{R}E_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}).$$

Proposition 3.4.4. *The equivalences of triangulated categories $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ and $\mathbb{L}\Phi_{\mathfrak{D}} = \mathbb{R}\Psi_{\mathfrak{D}}^{-1}$ from Corollary 3.4.2 transform the left derived functor $\mathbb{L}E^f$ into the right derived functor $\mathbb{R}E_f$ and back.* $\square$

The equivalences of contra/coderived categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform the $\mathfrak{R}$-cofree CDG-contra/comodule contra/corestriction- and contra/coextension-of-scalars functors $\mathbb{L}E^f$, $\mathbb{I}R^f$, $\mathbb{I}R_f$, $\mathbb{R}E_f$ defined above into the $\mathfrak{R}$-free CDG-contra/comodule contra/corestriction- and contra/coextension-of-scalars functors $\mathbb{L}E^f$, $\mathbb{I}R^f$, $\mathbb{I}R_f$, $\mathbb{R}E_f$ defined in Section 3.2.

4. Non-$\mathfrak{R}$-Free and Non-$\mathfrak{R}$-Cofree wcDG-Modules, CDG-Contramodules, and CDG-Comodules

4.1. **Non-$\mathfrak{R}$-free and non-$\mathfrak{R}$-cofree graded modules.** Let $\mathfrak{B}$ be an $\mathfrak{R}$-free graded algebra. An *$\mathfrak{R}$-contramodule graded left module* $\mathfrak{M}$ over $\mathfrak{B}$ is, by the definition, a graded left $\mathfrak{B}$-module in the module category of $\mathfrak{R}$-contramodules over the tensor category of free $\mathfrak{R}$-contramodules. In other words, it is a graded $\mathfrak{R}$-contramodule endowed with an (associative and unital) homogeneous $\mathfrak{B}$-action map $\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{M} \longrightarrow \mathfrak{M}$. *$\mathfrak{R}$-contramodule graded right modules* $\mathfrak{N}$ over $\mathfrak{B}$ are defined in the similar way.

Alternatively, one can define $\mathfrak{B}$-module structures on graded $\mathfrak{R}$-contramodules in terms of the action maps $\mathfrak{M} \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{M})$, and similarly for $\mathfrak{N}$. The latter point of view may be preferable in that the functor $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},-)$ is exact, while the functor $\mathfrak{B}\otimes^{\mathfrak{R}}-$ is only right exact (see Remark 1.9.3).

In fact, the category of $\mathfrak{R}$-contramodule graded (left or right) $\mathfrak{B}$-modules is enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules, so the abelian group of morphisms between two $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $\mathfrak{L}$ and $\mathfrak{M}$ is the underlying abelian group of the degree-zero component of the graded $\mathfrak{R}$-contramodule

$\mathrm{Hom}_{\mathfrak B}(\mathfrak L,\mathfrak M)$ constructed as the kernel of the pair of morphisms of graded $\mathfrak R$-contramodules $\mathrm{Hom}^{\mathfrak R}(\mathfrak L,\mathfrak M)\rightrightarrows\mathrm{Hom}^{\mathfrak R}(\mathfrak B\otimes^{\mathfrak R}\mathfrak L,\,\mathfrak M)\simeq\mathrm{Hom}^{\mathfrak R}(\mathfrak L,\mathrm{Hom}^{\mathfrak R}(\mathfrak B,\mathfrak M))$ induced by the actions of $\mathfrak B$ in $\mathfrak L$ and $\mathfrak M$. The *tensor product* $\mathfrak N\otimes_{\mathfrak B}\mathfrak M$ of an $\mathfrak R$-contramodule graded right $\mathfrak B$-module $\mathfrak N$ and an $\mathfrak R$-contramodule graded left $\mathfrak B$-module $\mathfrak M$ is a graded $\mathfrak R$-contramodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak R$-contramodules $\mathfrak N\otimes^{\mathfrak R}\mathfrak B\otimes^{\mathfrak R}\mathfrak M\rightrightarrows\mathfrak N\otimes^{\mathfrak R}\mathfrak M$.

The category of $\mathfrak R$-contramodule graded $\mathfrak B$-modules is abelian with infinite direct sums and products; and the forgetful functor from it to the category of graded $\mathfrak R$-contramodules is exact and preserves both infinite direct sums and products. There are enough projective objects in the abelian category of $\mathfrak R$-contramodule graded $\mathfrak B$-modules; these are the same as the projective objects in the exact subcategory of $\mathfrak R$-free graded $\mathfrak B$-modules. The latter exact subcategory in the abelian category of $\mathfrak R$-contramodule graded $\mathfrak B$-modules is closed under extensions and infinite directs sums and products. Infinite products of $\mathfrak R$-contramodule graded $\mathfrak B$-modules are exact functors (because infinite products of $\mathfrak R$-contramodules are); infinite direct sums are not, in general (because infinite direct sums of $\mathfrak R$-contramodules are not).

For any graded $\mathfrak R$-contramodule $\mathfrak U$ and any $\mathfrak R$-contramodule graded left $\mathfrak B$-module $\mathfrak M$, there is a natural isomorphism of graded $\mathfrak R$-contramodules $\mathrm{Hom}_{\mathfrak B}(\mathfrak B\otimes^{\mathfrak R}\mathfrak U,\,\mathfrak M)\simeq\mathrm{Hom}^{\mathfrak R}(\mathfrak U,\mathfrak M)$ [28, Lemma 1.1.2]. For any graded $\mathfrak R$-contramodule $\mathfrak V$, the $\mathfrak R$-contramodule graded left $\mathfrak B$-module $\mathrm{Hom}^{\mathfrak R}(\mathfrak B,\mathfrak V)$ has a similar property [28, Section 3.1.1]. Given an $\mathfrak R$-contramodule graded left $\mathfrak B$-module $\mathfrak N$, an epimorphism onto in from a projective $\mathfrak R$-free graded $\mathfrak B$-module can be obtained as the composition $\mathfrak B\otimes^{\mathfrak R}\mathfrak U\longrightarrow\mathfrak B\otimes^{\mathfrak R}\mathfrak N\longrightarrow\mathfrak N$, where $\mathfrak U$ is a free graded $\mathfrak R$-contramodule and $\mathfrak U\longrightarrow\mathfrak N$ is an epimorphism of graded $\mathfrak R$-contramodules. For any $\mathfrak R$-contramodule graded right $\mathfrak B$-module $\mathfrak N$ and any graded $\mathfrak R$-contramodule $\mathfrak U$, there is a natural isomorphism of $\mathfrak R$-contramodules $\mathfrak N\otimes_{\mathfrak B}(\mathfrak B\otimes^{\mathfrak R}\mathfrak U)\simeq\mathfrak N\otimes^{\mathfrak R}\mathfrak U$ [28, Lemma 1.2.1].

An $\mathfrak R$*-comodule graded left module* $\mathcal M$ over $\mathfrak B$ is, by the definition, a graded $\mathfrak B$-module in the module category of $\mathfrak R$-comodules over the tensor category of free $\mathfrak R$-contramodules. In other words, it is a graded $\mathfrak R$-comodule endowed with an (associative and unital) homogeneous $\mathfrak B$-action map $\mathfrak B\odot_{\mathfrak R}\mathcal M\longrightarrow\mathcal M$. $\mathfrak R$*-comodule graded right modules* $\mathcal N$ over $\mathfrak B$ are defined in the similar way.

Alternatively, one can define $\mathfrak B$-module structures on graded $\mathfrak R$-comodules in terms of the action maps $\mathcal M\longrightarrow\mathrm{Ctrhom}_{\mathfrak R}(\mathfrak B,\mathcal M)$, and similarly for $\mathcal N$. The former point of view may be preferable in that the functor $\mathfrak B\odot_{\mathfrak R}-$ is exact, while the functor $\mathrm{Ctrhom}_{\mathfrak R}(\mathfrak B,-)$ is only left exact (see Remark 1.9.3).

In fact, the category of $\mathfrak R$-comodule graded $\mathfrak B$-modules is enriched over the tensor category of (graded) $\mathfrak R$-contramodules, so the abelian group of morphisms between two $\mathfrak R$-comodule graded $\mathfrak B$-modules $\mathcal L$ and $\mathcal M$ is the underlying abelian group of the degree-zero component of the graded $\mathfrak R$-contramodule $\mathrm{Hom}_{\mathfrak B}(\mathcal L,\mathcal M)$ constructed as the kernel of the pair of morphisms of graded $\mathfrak R$-contramodules $\mathrm{Hom}_{\mathfrak R}(\mathcal L,\mathcal M)\rightrightarrows\mathrm{Hom}_{\mathfrak R}(\mathfrak B\odot_{\mathfrak R}\mathcal L,\,\mathfrak M)\simeq\mathrm{Hom}_{\mathfrak R}(\mathcal L,\mathrm{Ctrhom}_{\mathfrak R}(\mathfrak B,\mathfrak M))$. The *tensor product* $\mathfrak N\otimes_{\mathfrak B}\mathcal M$ of an $\mathfrak R$-contramodule graded right $\mathfrak B$-module $\mathfrak N$ and an $\mathfrak R$-comodule graded left $\mathfrak B$-module $\mathcal M$ is a graded $\mathfrak R$-comodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{M} \rightrightarrows \mathfrak{N}\odot_{\mathfrak{R}}\mathcal{M}$. The graded $\mathfrak{R}$-comodule $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\mathcal{M})$ from an $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{L}$ to an $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\mathcal{M}) \rightrightarrows \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{L},\ \mathcal{M}) \simeq \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M}))$.

The contratensor product $\mathcal{K}\odot_{\mathfrak{R}}\mathfrak{Q}$ of an $\mathfrak{R}$-comodule $\mathcal{K}$ and an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is set to be equal to the contratensor product $\mathfrak{Q}\odot_{\mathfrak{R}}\mathcal{K}$ as defined in Section 1.5. This operation is extended to graded $\mathfrak{R}$-comodules and $\mathfrak{R}$-contramodules by taking infinite direct sums of $\mathfrak{R}$-comodules along the diagonals of the bigrading (as usually).

The *tensor product* $\mathcal{N}\otimes_{\mathfrak{B}}\mathcal{M}$ of an $\mathfrak{R}$-comodule graded right $\mathfrak{B}$-module $\mathcal{N}$ and and $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$ is a graded $\mathfrak{R}$-comodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $(\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{B})\square\mathcal{M} \simeq \mathcal{N}\square_{\mathfrak{R}}(\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{M}) \rightrightarrows \mathcal{N}\square_{\mathfrak{R}}\mathcal{M}$ (see Lemma 1.7.1). The graded $\mathfrak{R}$-contramodule $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L},\mathfrak{M})$ from an $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{L}$ to an $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{M}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{L},\mathfrak{M}) \rightrightarrows \operatorname{Cohom}_{\mathfrak{R}}(\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{L},\ \mathfrak{M}) \simeq \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{L},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{M}))$.

The category of $\mathfrak{R}$-comodule graded $\mathfrak{B}$-modules is abelian with infinite direct sums and products; and the forgetful functor from it to the category of graded $\mathfrak{R}$-comodules is exact and preserves both infinite direct sums and products. There are enough injective objects in the abelian category of $\mathfrak{R}$-comodule graded $\mathfrak{B}$-modules; these are the same as the injective objects in the exact subcategory of $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules. The latter exact subcategory in the abelian category of $\mathfrak{R}$-comodule graded $\mathfrak{B}$-modules is closed under extensions and infinite direct sums and products. Filtered inductive limits of $\mathfrak{R}$-comodule graded $\mathfrak{B}$-modules are exact functors (because filtered inductive limits of $\mathfrak{R}$-comodules are); infinite products are not, in general (because infinite products of $\mathfrak{R}$-comodules are not).

For any graded $\mathfrak{R}$-comodule $\mathcal{V}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{L}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{V})) \simeq \operatorname{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{V})$. Given an $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$, a monomorphism from it into an injective $\mathfrak{R}$-cofree graded $\mathfrak{B}$-module can be obtained as the composition $\mathcal{M}\longrightarrow\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M})\longrightarrow\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{V})$, where $\mathcal{V}$ is a cofree graded $\mathfrak{R}$-comodule and $\mathcal{M}\longrightarrow\mathcal{V}$ is a monomorphism of graded $\mathfrak{R}$-comodules.

For any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ and $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $(\mathfrak{U}\otimes^{\mathfrak{R}}\mathfrak{B})\otimes_{\mathfrak{B}}\mathcal{M}\simeq\mathfrak{U}\odot_{\mathfrak{R}}\mathcal{M}$. For any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ and $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{U},\ \mathcal{M}) \simeq \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{U},\mathcal{M})$. Similarly, for any $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{L}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{V}))\simeq\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\mathcal{V})$.

For any graded $\mathfrak{R}$-comodule $\mathcal{U}$, the $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{U}$ has similar properties. In particular, for any graded $\mathfrak{R}$-comodule $\mathcal{U}$ and $\mathfrak{R}$-comodule graded right $\mathfrak{B}$-module $\mathcal{N}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathcal{N}\otimes_{\mathfrak{B}}(\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{U})\simeq\mathcal{N}\square_{\mathfrak{R}}\mathcal{U}$. For any graded $\mathfrak{R}$-comodule $\mathcal{U}$ and $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{M}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules

$\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{B} \odot_{\mathfrak{R}} \mathcal{U},\, \mathfrak{M}) \simeq \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{U}, \mathfrak{M})$. For any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{L}$ and graded $\mathfrak{R}$-contramodule $\mathfrak{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L}, \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{V})) \simeq \operatorname{Cohom}_{\mathfrak{R}}(\mathcal{L}, \mathfrak{V})$.

For any $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{M}$, the graded $\mathfrak{R}$-comodule $\Phi_{\mathfrak{R}}(\mathfrak{M}) = \mathcal{C}(\mathfrak{R}) \odot_{\mathfrak{R}} \mathfrak{M}$ has a natural $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module structure. The similar construction applies to right $\mathfrak{B}$-modules. For any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$, the graded $\mathfrak{R}$-contramodule $\Psi_{\mathfrak{R}}(\mathcal{M}) = \operatorname{Hom}_{\mathfrak{R}}(\mathcal{C}(\mathfrak{R}), \mathcal{M})$ has a natural $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module structure (see the proof of Proposition 2.4.1).

The functors $\Phi_{\mathfrak{R}}$ and $\Psi_{\mathfrak{R}}$ between the abelian categories of $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-modules are adjoint to each other. Their restrictions to the exact subcategories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded $\mathfrak{B}$-modules provide the equivalence $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ between these exact categories defined in Section 2.4.

For any $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{L}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$ there are natural isomorphisms of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \Psi_{\mathfrak{R}}(\mathcal{M})) \simeq \Psi_{\mathfrak{R}}(\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathcal{M})) \simeq \operatorname{Hom}_{\mathfrak{B}}(\Phi_{\mathfrak{R}}(\mathfrak{L}), \mathcal{M}))$. For any $\mathfrak{R}$-contramodule graded right $\mathfrak{B}$-module $\mathfrak{N}$ and any $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module $\mathfrak{M}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N} \otimes_{\mathfrak{B}} \Phi_{\mathfrak{R}}(\mathfrak{M}) \simeq \Phi_{\mathfrak{R}}(\mathfrak{N} \otimes_{\mathfrak{B}} \mathfrak{M})$.

For any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-modules $\mathcal{L}$ and $\mathcal{M}$, there is a natural morphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L}, \Psi_{\mathfrak{R}}(\mathcal{M})) \longrightarrow \operatorname{Hom}_{\mathfrak{B}}(\mathcal{L}, \mathcal{M})$, which is an isomorphism whenever one of the graded $\mathfrak{B}$-modules $\mathcal{L}$ and $\mathcal{M}$ is $\mathfrak{R}$-cofree. For any $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-modules $\mathfrak{L}$ and $\mathfrak{M}$, there is a natural morphism of graded $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\Phi_{\mathfrak{R}}(\mathfrak{L}), \mathfrak{M}) \longrightarrow \operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathfrak{M})$, which is an isomorphism whenever one of the graded $\mathfrak{B}$-modules $\mathfrak{L}$ and $\mathfrak{M}$ is $\mathfrak{R}$-cofree. For any $\mathfrak{R}$-contramodule graded right $\mathfrak{B}$-module $\mathfrak{N}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module $\mathcal{M}$, there is a natural morphism of graded $\mathfrak{R}$-comodules $\mathfrak{N} \otimes_{\mathfrak{B}} \mathcal{M} \longrightarrow \Phi_{\mathfrak{R}}(\mathfrak{N}) \otimes_{\mathfrak{B}} \mathcal{M}$, which is an isomorphism whenever either the graded $\mathfrak{B}$-module $\mathfrak{N}$ is $\mathfrak{R}$-free, or the graded $\mathfrak{B}$-module $\mathcal{M}$ is $\mathfrak{R}$-cofree.

4.2. **Contra/coderived category of CDG-modules.** The definitions of

- odd derivations of $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule $\mathfrak{B}$-modules compatible with a given odd derivation of an $\mathfrak{R}$-free graded algebra $\mathfrak{B}$,
- $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule left and right CDG-modules over an $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$,
- the complexes of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathfrak{M})$ for given $\mathfrak{R}$-contramodule left CDG-modules $\mathfrak{L}$ and $\mathfrak{M}$ over $\mathfrak{B}$,
- the complexes of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L}, \mathcal{M})$ for given $\mathfrak{R}$-comodule left CDG-modules $\mathcal{L}$ and $\mathcal{M}$ over $\mathfrak{B}$,
- the complex of $\mathfrak{R}$-comodules $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathcal{M})$ for a given $\mathfrak{R}$-contramodule left CDG-module $\mathfrak{L}$ and $\mathfrak{R}$-comodule left CDG-module $\mathcal{M}$ over $\mathfrak{B}$,
- the complex of $\mathfrak{R}$-contramodules $\operatorname{Hom}_{\mathfrak{B}}(\mathcal{L}, \mathfrak{M})$ for a given $\mathfrak{R}$-comodule left CDG-module $\mathcal{L}$ and $\mathfrak{R}$-contramodule left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$,
- the complex of $\mathfrak{R}$-contramodules $\mathfrak{N} \otimes_{\mathfrak{B}} \mathfrak{M}$ for a given $\mathfrak{R}$-contramodule right CDG-module $\mathfrak{N}$ and $\mathfrak{R}$-contramodule left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$,

- the complex of $\mathfrak{R}$-comodules $\mathfrak{N} \otimes_{\mathfrak{B}} \mathcal{M}$ for a given $\mathfrak{R}$-contramodule right CDG-module $\mathfrak{N}$ and $\mathfrak{R}$-comodule left CDG-module $\mathcal{M}$ over $\mathfrak{B}$,
- the complex of $\mathfrak{R}$-comodules $\mathcal{N} \otimes_{\mathfrak{B}} \mathcal{M}$ for a given $\mathfrak{R}$-comodule right CDG-module $\mathcal{N}$ and $\mathfrak{R}$-comodule left CDG-module $\mathcal{M}$ over $\mathfrak{B}$,
- the $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule CDG-modules over $\mathfrak{B}$ obtained by restriction of scalars via a morphism of $\mathfrak{R}$-free CDG-algebras $\mathfrak{B} \longrightarrow \mathfrak{A}$ from $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule CDG-modules over $\mathfrak{A}$

repeat the similar definitions for $\mathfrak{R}$-free and $\mathfrak{R}$-cofree $\mathfrak{B}$-modules given in Sections 2.2 and 2.5 *verbatim* (with the definitions and constructions of Section 4.1 being used in place of those from Sections 2.1 and 2.4 as applicable), so there is no need to spell them out here again. We restrict ourselves to introducing the new notation for our new and more general classes of objects.

The DG-categories of $\mathfrak{R}$-contramodule left and right CDG-modules over $\mathfrak{B}$ are denoted by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–ctr}}\text{–}\mathfrak{B}$, and their homotopy categories are $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ and $H^0(\mathsf{mod}^{\mathfrak{R}\text{–ctr}}\text{–}\mathfrak{B})$. Similarly, the DG-categories of $\mathfrak{R}$-comodule left and right CDG-modules over $\mathfrak{B}$ are denoted by $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}$ and $\mathsf{mod}^{\mathfrak{R}\text{–co}}\text{–}\mathfrak{B}$, and their homotopy categories are $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$ and $H^0(\mathsf{mod}^{\mathfrak{R}\text{–co}}\text{–}\mathfrak{B})$. The tensor products of CDG-modules over $\mathfrak{B}$ are triangulated functors of two arguments

$$\otimes_{\mathfrak{B}}\colon H^0(\mathsf{mod}^{\mathfrak{R}\text{–ctr}}\text{–}\mathfrak{B}) \times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}),$$
$$\otimes_{\mathfrak{B}}\colon H^0(\mathsf{mod}^{\mathfrak{R}\text{–ctr}}\text{–}\mathfrak{B}) \times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}),$$
$$\otimes_{\mathfrak{B}}\colon H^0(\mathsf{mod}^{\mathfrak{R}\text{–co}}\text{–}\mathfrak{B}) \times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The Hom from $\mathfrak{R}$-contramodule to $\mathfrak{R}$-comodule CDG-modules over $\mathfrak{B}$ is a triangulated functor

$$\mathrm{Hom}_{\mathfrak{B}}\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})^{\mathsf{op}} \times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}),$$

and the Hom from $\mathfrak{R}$-comodule to $\mathfrak{R}$-contramodule CDG-modules over $\mathfrak{B}$ is a triangulated functor

$$\mathrm{Hom}_{\mathfrak{B}}\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})^{\mathsf{op}} \times H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

Given a morphism of $\mathfrak{R}$-free CDG-algebras $f = (f, a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$, the functors of restriction of scalars are denoted by

$$R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}),$$
$$R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}),$$

and similarly for right CDG-modules.

An $\mathfrak{R}$-contramodule left CDG-module over $\mathfrak{B}$ is said to be *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-contramodule CDG-modules over $\mathfrak{B}$ and closed under infinite products. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-contramodule CDG-modules is called the *contraderived category* of $\mathfrak{R}$-contramodule left CDG-modules over

$\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. The contraderived category of $\mathfrak{R}$-contramodule right CDG-modules over $\mathfrak{B}$, denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}\text{–}\mathfrak{B})$, is defined similarly.

An $\mathfrak{R}$-comodule left CDG-module over $\mathfrak{B}$ is said to be *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-comodule CDG-modules over $\mathfrak{B}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-comodule CDG-modules is called the *coderived category* of $\mathfrak{R}$-comodule left CDG-modules over $\mathfrak{B}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$. The coderived category of $\mathfrak{R}$-comodule right CDG-modules over $\mathfrak{B}$, denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{B})$, is defined similarly.

It follows from the next theorem, among other things, that our terminology is not ambigous: an $\mathfrak{R}$-free CDG-module over $\mathfrak{B}$ is contraacyclic in the sense of Section 2.2 if and only if it is contraacyclic as an $\mathfrak{R}$-contramodule CDG-module, in the sense of the above definition. Similarly, an $\mathfrak{R}$-cofree CDG-module over $\mathfrak{B}$ is coacyclic in the sense of Section 2.5 if and only if it is coacyclic as an $\mathfrak{R}$-comodule CDG-module, in the sense of the above definition.

Theorem 4.2.1. *For any $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$, the functors $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ induced by the natural embeddings of DG-categories $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ and $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}$ are equivalences of triangulated categories.*

Proof. We follow the idea of [31, proof of Theorem 1.5 and Remark 1.5]. Let $\mathfrak{M}$ be an $\mathfrak{R}$-contramodule left CDG-module over $\mathfrak{B}$. As explained in Section 4.1, there exists a surjective morphism onto the $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-module $\mathfrak{M}$ from an $\mathfrak{R}$-free (and even projective) $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-module $\mathfrak{P}_0$. Let $G^+(\mathfrak{P}_0)$ be the $\mathfrak{R}$-contramodule CDG-module over $\mathfrak{B}$ freely generated by $\mathfrak{P}_0$ (see the proof of Theorem 2.2.4); then there is a surjective closed morphism of CDG-modules $G^+(\mathfrak{P}_0) \longrightarrow \mathfrak{M}$. Applying the same procedure to the kernel of the latter morphism, etc., we obtain a left resolution of $\mathfrak{M}$ by $\mathfrak{R}$-free CDG-modules and closed morphisms $\cdots \longrightarrow G^+(\mathfrak{P}_1) \longrightarrow G^+(\mathfrak{P}_0) \longrightarrow \mathfrak{M} \longrightarrow 0$.

Totalizing this complex of $\mathfrak{R}$-free CDG-modules by taking infinite products along the diagonals, we get a closed morphism of $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{P} \longrightarrow \mathfrak{M}$ with an $\mathfrak{R}$-free CDG-module $\mathfrak{P}$. It follows from the next lemma that the cone of this morphism is a contraacyclic $\mathfrak{R}$-contramodule CDG-module.

Lemma 4.2.2. *For any bounded above exact sequence of $\mathfrak{R}$-contramodule CDG-modules over $\mathfrak{B}$ and closed morphisms between them $\cdots \longrightarrow \mathfrak{K}_2 \longrightarrow \mathfrak{K}_1 \longrightarrow \mathfrak{K}_0 \longrightarrow 0$, the total CDG-module $\mathfrak{L}$ of the complex of CDG-modules $\mathfrak{K}_\bullet$, constructed by taking infinite products along the diagonals, is a contraacyclic $\mathfrak{R}$-contramodule CDG-module.*

Proof. Let $\mathfrak{L}_n$ denote the totalizations of the finite quotient complexes of canonical filtration of the exact complex $\mathfrak{K}_\bullet$. Clearly, the $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{L}_n$ are contraacyclic. Consider the "telescope" short sequence of $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{L} \longrightarrow \prod_n \mathfrak{L}_n \longrightarrow \prod_n \mathfrak{L}_n$, the second morphism being constructed in terms of the identity morphisms $\mathfrak{L}_n \longrightarrow \mathfrak{L}_n$ and the natural closed surjections

$\mathfrak{L}_{n+1} \longrightarrow \mathfrak{L}_n$. Forgetting the differentials (that is considering our short sequence as a sequence of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules), we discover that this short sequence is a product of telescope sequences related to stabilizing projective systems of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules. The latter being always split exact, our short sequence of $\mathfrak{R}$-contramodule CDG-modules is also exact; and $\mathfrak{L}_n$ being contraacyclic, it follows that $\mathfrak{L}$ is contraacyclic, too. $\square$

By [29, Lemma 1.6], it follows that the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ is equivalent to the quotient category of the homotopy category of $\mathfrak{R}$-free left CDG-modules over $\mathfrak{B}$ by the thick subcategory of $\mathfrak{R}$-free CDG-modules *contraacyclic as $\mathfrak{R}$-contramodule CDG-modules.* It remains to show that any $\mathfrak{R}$-free CDG-module contraacyclic as an $\mathfrak{R}$-contramodule CDG-module is also contraacyclic as an $\mathfrak{R}$-free CDG-module.

In fact, we will prove that any closed morphism from an $\mathfrak{R}$-free CDG-module $\mathfrak{P}$ to a contraacyclic $\mathfrak{R}$-contramodule CDG-module $\mathfrak{L}$ factorizes through a contraacyclic $\mathfrak{R}$-free CDG-module $\mathfrak{F}$ as a morphism in the homotopy category $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. Indeed, consider the class of all $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{L}$ satisfying the above condition with respect to all $\mathfrak{R}$-free CDG-modules $\mathfrak{P}$. We will check that this class of CDG-modules is closed under cones and infinite products, and contains the total CDG-modules of short exact sequences of $\mathfrak{R}$-contramodule CDG-modules.

Let $\mathfrak{L}_\alpha$ be a family of $\mathfrak{R}$-contramodule CDG-modules, $\mathfrak{P}$ be an $\mathfrak{R}$-free CDG-module, and $\mathfrak{P} \longrightarrow \prod_\alpha \mathfrak{L}_\alpha$ be a morphism in $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. Assuming that each component $\mathfrak{P} \longrightarrow \mathfrak{L}_\alpha$ of our morphism factorizes through a contraacyclic $\mathfrak{R}$-free CDG-module $\mathfrak{F}_\alpha$, the morphism $\mathfrak{P} \longrightarrow \prod_\alpha \mathfrak{L}_\alpha$ factorizes though the CDG-module $\prod_\alpha \mathfrak{F}_\alpha$, which is also a contraacyclic $\mathfrak{R}$-free CDG-module. This proves the closedness under infinite products.

Now let us reformulate the property of CDG-modules that we are interested in as follows: an $\mathfrak{R}$-contramodule CDG-module $\mathfrak{L}$ belongs to our class of CDG-modules if and only if for any closed morphism $\mathfrak{P} \longrightarrow \mathfrak{L}$ into $\mathfrak{L}$ from an $\mathfrak{R}$-free CDG-module $\mathfrak{P}$ there exists a closed morphism of $\mathfrak{R}$-free CDG-modules $\mathfrak{Q} \longrightarrow \mathfrak{P}$ whose cone is a contraacyclic $\mathfrak{R}$-free CDG-module and whose composition with the morphism $\mathfrak{P} \longrightarrow \mathfrak{L}$ is homotopic to zero. Assume that $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{K}$ and $\mathfrak{M}$ have this property, and let $\mathfrak{K} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{M}$ be a distinguished triangle in $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$.

Given a closed morphism $\mathfrak{P} \longrightarrow \mathfrak{L}$ with an $\mathfrak{R}$-free CDG-module $\mathfrak{P}$, consider the composition $\mathfrak{P} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{M}$ and find a closed morphism of $\mathfrak{R}$-free CDG-modules $\mathfrak{T} \longrightarrow \mathfrak{P}$ whose cone is a contraacyclic $\mathfrak{R}$-free CDG-module and such that the composition $\mathfrak{T} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{L} \longrightarrow \mathfrak{M}$ is homotopic to zero. Then the composition $\mathfrak{T} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{L}$ factorizes through $\mathfrak{K}$ in the homotopy category. Find a closed morphism of $\mathfrak{R}$-free CDG-modules $\mathfrak{Q} \longrightarrow \mathfrak{T}$ whose cone is a contraacyclic $\mathfrak{R}$-free CDG-module and such that the composition $\mathfrak{Q} \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{K}$ is homotopic to zero. Then the composition $\mathfrak{Q} \longrightarrow \mathfrak{T} \longrightarrow \mathfrak{P}$ provides the desired closed morphism of $\mathfrak{R}$-free CDG-modules with a contraacyclic $\mathfrak{R}$-free cone annihilating the morphism $\mathfrak{P} \longrightarrow \mathfrak{L}$. This proves the closedness with respect to cones.

Finally, let $\mathfrak{M}$ be the total CDG-module of a short exact sequence of $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{U} \longrightarrow \mathfrak{V} \longrightarrow \mathfrak{W}$. Let $\mathfrak{P} \longrightarrow \mathfrak{M}$ be a closed morphism into $\mathfrak{M}$ from an $\mathfrak{R}$-free CDG-module $\mathfrak{P}$. It remains to construct a closed morphism $\mathfrak{Q} \longrightarrow \mathfrak{P}$ of $\mathfrak{R}$-free CDG-modules whose cone is a contraacyclic $\mathfrak{R}$-free CDG-module and whose composition with the morphism $\mathfrak{P} \longrightarrow \mathfrak{M}$ is homotopic to zero.

As a graded $\mathfrak{R}$-contramodule $\mathfrak{B}$-module, $\mathfrak{M}$ is the direct sum of three modules $\mathfrak{U}[1]$, $\mathfrak{V}$, and $\mathfrak{W}[-1]$; so any graded $\mathfrak{B}$-module morphism $\mathfrak{N} \longrightarrow \mathfrak{M}$ into $\mathfrak{M}$ from an $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-module $\mathfrak{N}$ can be viewed as a triple of graded $\mathfrak{B}$-module morphisms $f\colon \mathfrak{N} \longrightarrow \mathfrak{U}[1]$, $g\colon \mathfrak{N} \longrightarrow \mathfrak{V}$, and $h\colon \mathfrak{N} \longrightarrow \mathfrak{W}[-1]$.

Lemma 4.2.3. *Let $\mathfrak{N} \longrightarrow \mathfrak{M}$ be a closed morphism of $\mathfrak{R}$-contramodule CDG-modules represented by a triple (f, g, h) as above. Then whenever the morphism $h\colon \mathfrak{N} \longrightarrow \mathfrak{W}[-1]$ can be lifted to a morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $t\colon \mathfrak{N} \longrightarrow \mathfrak{V}[-1]$, the morphism $\mathfrak{N} \longrightarrow \mathfrak{M}$ is homotopic to zero.*

Proof. See [31, Lemma 1.5.E(b)]. □

Lemma 4.2.4. *Let $\mathfrak{N} \longrightarrow \mathfrak{M}$ be a morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules with the components (f, g, h). Let $G^+(\mathfrak{N}) \longrightarrow \mathfrak{M}$ be the induced closed morphism of $\mathfrak{R}$-contramodule CDG-modules over $\mathfrak{B}$; denote its components by $(\tilde f, \tilde g, \tilde h)$. Then the morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $\tilde h\colon G^+(\mathfrak{N}) \longrightarrow W[-1]$ can be lifted to a morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $G^+(\mathfrak{N}) \longrightarrow \mathfrak{V}[-1]$ whenever the morphism $h\colon \mathfrak{N} \longrightarrow \mathfrak{W}[-1]$ can be lifted to a morphism $\mathfrak{N} \longrightarrow \mathfrak{V}[-1]$.*

Proof. See [31, Lemma 1.5.F]. □

Let $\mathfrak{N}$ be an $\mathfrak{R}$-free graded $\mathfrak{B}$-module mapping surjectively onto the fibered product of the morphisms of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $\mathfrak{P} \longrightarrow \mathfrak{W}[-1]$ and $\mathfrak{V}[-1] \longrightarrow \mathfrak{W}[-1]$. Then $\mathfrak{N} \longrightarrow \mathfrak{P}$ is a surjective morphism of $\mathfrak{R}$-free graded $\mathfrak{B}$-modules; consider the induced surjective closed morphism of $\mathfrak{R}$-free CDG-modules $G^+(\mathfrak{N}) \longrightarrow \mathfrak{P}$ over $\mathfrak{B}$. Let $\mathfrak{T}$ be the kernel of the latter morphism and $\mathfrak{Q}$ be the cone of the closed embedding $\mathfrak{T} \longrightarrow G^+(\mathfrak{N})$. Then there is a natural closed morphism of $\mathfrak{R}$-free CDG-modules $\mathfrak{Q} \longrightarrow \mathfrak{P}$. Its cone, being the total CDG-module of a short exact sequence of $\mathfrak{R}$-free CDG-modules $\mathfrak{T} \longrightarrow \mathfrak{Q} \longrightarrow G^+(\mathfrak{N})$ over $\mathfrak{B}$, is a contraacyclic $\mathfrak{R}$-free CDG-module.

Consider the composition $\mathfrak{Q} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{W}[-1]$; it is a morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $G^+(\mathfrak{N}) \oplus \mathfrak{T}[1] \longrightarrow \mathfrak{W}[-1]$ vanishing on $\mathfrak{T}[1]$. The morphism $G^+(\mathfrak{N}) \longrightarrow \mathfrak{W}[-1]$ is the component $\tilde h$ of the closed morphism $G^+(\mathfrak{N}) \longrightarrow \mathfrak{M}$ induced by the morphism of $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-modules $\mathfrak{N} \longrightarrow \mathfrak{M}$ equal to the composition $\mathfrak{N} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{M}$. By the construction, the component $h\colon \mathfrak{N} \longrightarrow \mathfrak{W}[-1]$ of the latter morphism lifts to an $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-module morphism $\mathfrak{N} \longrightarrow \mathfrak{V}[-1]$. By Lemma 4.2.4, the $\mathfrak{R}$-contramodule graded $\mathfrak{B}$-module morphism $\tilde h\colon G^+(\mathfrak{N}) \longrightarrow \mathfrak{W}[-1]$ can be also lifted into $\mathfrak{V}[-1]$. Hence the same applies to the composition $\mathfrak{Q} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{W}[-1]$; and by Lemma 4.2.3 it follows that the composition $\mathfrak{Q} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{M}$ is homotopic to zero.

We have proven the first assertion of Theorem; the proof of the second one is analogous up to duality. □

Notice that when the pro-Artinian topological local ring $\mathfrak{R}$ has finite homological dimension (see Section 1.9), the above argument also proves that the embeddings of DG-categories $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ and $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}$ induce also equivalences of the absolute derived categories $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$. Here the absolute derived categories $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ are defined as the quotient categories of the homotopy categories $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ by the thick subcategories of *absolutely acyclic* $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule CDG-modules, i. e., the minimal thick subcategories containing the totalizations of short exact sequences of $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule CDG-modules, respectively.

As a particular case of the above definitions in the case $\mathfrak{B} = \mathfrak{R}$, we have the classes of *contraacyclic* and *absolutely acyclic* complexes of $\mathfrak{R}$-contramodules, and similarly, *coacyclic* and *absolutely acyclic* complexes of $\mathfrak{R}$-comodules. The corresponding quotient categories of the homotopy categories $H^0(\mathfrak{R}\text{–}\mathsf{contra})$ and $H^0(\mathfrak{R}\text{–}\mathsf{comod})$ are the *contraderived category* of complexes of $\mathfrak{R}$-contramodules $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, the *absolute derived category* of complexes of $\mathfrak{R}$-contramodules $\mathsf{D}^{\mathsf{abs}}(\mathfrak{R}\text{–}\mathsf{contra})$, the *coderived category* of complexes of $\mathfrak{R}$-comodules $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and the *absolute derived category* of complexes of $\mathfrak{R}$-comodules $\mathsf{D}^{\mathsf{abs}}(\mathfrak{R}\text{–}\mathsf{comod})$.

When $\mathfrak{R}$ has finite homological dimension, the quotient categories $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ and $\mathsf{D}^{\mathsf{abs}}(\mathfrak{R}\text{–}\mathsf{contra})$ coincide with each other and with the conventional derived category of (complexes of) $\mathfrak{R}$-contramodules $\mathsf{D}(\mathfrak{R}\text{–}\mathsf{contra})$, and similarly, the quotient categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{abs}}(\mathfrak{R}\text{–}\mathsf{comod})$ coincide with each other and with the conventional derived category of (complexes of) $\mathfrak{R}$-comodules $\mathsf{D}(\mathfrak{R}\text{–}\mathsf{comod})$ [28, Remark 2.1].

Theorem 4.2.5. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Then* (a) *for any CDG-module $\mathfrak{P} \in H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and any contraacyclic $\mathfrak{R}$-contramodule left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$, the complex of $\mathfrak{R}$-contramodules* $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{P}, \mathfrak{M})$ *is contraacyclic;*

(b) *for any coacyclic $\mathfrak{R}$-comodule left CDG-module $\mathfrak{L}$ over $\mathfrak{B}$ and any CDG-module $\mathfrak{J} \in H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})$, the complex of $\mathfrak{R}$-contramodules* $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{L}, \mathfrak{J})$ *is contraacyclic.*

Proof. To prove part (a), notice that the functor $\operatorname{Hom}_{\mathfrak{B}}(\mathfrak{P}, -)$ takes short exact sequences and infinite products of $\mathfrak{R}$-contramodule CDG-modules to short exact sequences and infinite products of complexes of $\mathfrak{R}$-contramodules. It also takes cones of closed morphisms of $\mathfrak{R}$-contramodule CDG-modules to cones of closed morphisms of complexes of $\mathfrak{R}$-contramodules. The proof of part (b) is similar up to duality (cf. [29, Theorem 3.5 and Remark 3.5]; see also Theorem 2.2.3). □

Corollary 4.2.6. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra such that the exact category of $\mathfrak{R}$-(co)free graded left $\mathfrak{B}$-modules has finite homological dimension (cf. Proposition* 2.4.1*) and Corollary* 2.1.3*). Then the functors* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *and* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *induced by the natural* *embeddings of DG-categories* $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ *and* $\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}$ *are equivalences of triangulated categories.*

Proof. It follows from Theorems 2.2.3(a) and 2.2.4 that the exotic derived categories $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ coincide. Similarly one can show that the derived categories of the second kind $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ coincide (see Section 2.5). So it remains to apply Theorem 4.2.1. □

Corollary 4.2.7. *Let* $\mathfrak{B}$ *be an* $\mathfrak{R}$*-free CDG-algebra such that the exact category of* $\mathfrak{R}$*-(co)free graded left* $\mathfrak{B}$*-modules has finite homological dimension. Then the derived functors* $\mathbb{L}\Phi_{\mathfrak{R}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathbb{R}\Psi_{\mathfrak{R}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *defined by identifying* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *with* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *with* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *(see Corollary* 4.2.6*) are mutually inverse equivalences of triangulated categories.*

Proof. Follows from the results of Section 2.5. □

4.3. **Semiderived category of wcDG-modules.** Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$. An $\mathfrak{R}$*-contramodule* (left or right) *wcDG-module* over $\mathfrak{A}$ is, by the definition, an $\mathfrak{R}$-contramodule CDG-module over $\mathfrak{A}$ considered as an $\mathfrak{R}$-free CDG-algebra. $\mathfrak{R}$*-comodule wcDG-modules* over $\mathfrak{A}$ are defined similarly. All the definitions, constructions, and notation of Section 4.2 related to CDG-modules will be applied to wcDG-modules as a particular case.

An $\mathfrak{R}$-contramodule left wcDG-module over $\mathfrak{A}$ is called *semiacyclic* if it belongs to the minimal thick subcategory of $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ containing both the contraacyclic $\mathfrak{R}$-contramodule wcDG-modules and the semiacyclic $\mathfrak{R}$-free wcDG-modules. Equivalently, an $\mathfrak{R}$-contramodule wcDG-module is semiacyclic if the equivalence of contraderived categories from Theorem 4.2.1 assigns a semiacyclic $\mathfrak{R}$-free wcDG-module to it. It is clear from the latter definition that the class of semiacyclic $\mathfrak{R}$-contramodule CDG-modules is closed under infinite products. It is important for these arguments that the class of semiacyclic $\mathfrak{R}$-free wcDG-modules contains the class of contraacyclic $\mathfrak{R}$-free wcDG-modules.

The *semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ of $\mathfrak{R}$-contramodule left wcDG-modules over $\mathfrak{A}$ is defined as the quotient category of the homotopy category $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ by the thick subcategory of semiacyclic $\mathfrak{R}$-contramodule wcDG-modules. It is obvious from the definition that the embedding of DG-categories $\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ induces an equivalence of semiderived categories $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. The semiderived category of $\mathfrak{R}$-contramodule right wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}\text{–}\mathfrak{A})$ over $\mathfrak{A}$ is defined similarly.

Analogously, an $\mathfrak{R}$-comodule left wcDG-module over $\mathfrak{A}$ is called *semiacyclic* if it belongs to the minimal thick subcategory of $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ containing both the coacyclic $\mathfrak{R}$-comodule wcDG-modules and the semiacyclic $\mathfrak{R}$-cofree wcDG-modules. Equivalently, an $\mathfrak{R}$-comodule wcDG-module is semiacyclic if the equivalence of coderived categories from Theorem 4.2.1 assigns a semiacyclic $\mathfrak{R}$-cofree wcDG-module to it. It is clear from the latter definition that the class of semiacyclic $\mathfrak{R}$-comodule wcDG-modules is closed under infinite direct sums. It is important for these arguments that the class of semiacyclic $\mathfrak{R}$-cofree wcDG-modules contains the class of coacyclic $\mathfrak{R}$-cofree wcDG-modules.

The *semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$ of $\mathfrak{R}$-comodule left wcDG-modules over $\mathfrak{A}$ is defined as the quotient category of the homotopy category $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$ by the thick subcategory of semiacyclic $\mathfrak{R}$-comodule wcDG-modules. It is obvious from the definition that the embedding of DG-categories $\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}} \longrightarrow \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}$ induces an equivalence of semiderived categories $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$. The semiderived category of $\mathfrak{R}$-comodule right wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–co}}\text{–}\mathfrak{A})$ is defined similarly.

When $\mathfrak{A}$ is actually a DG-algebra (i. e., $h = 0$), an $\mathfrak{R}$-contramodule wcDG-module over $\mathfrak{A}$ is semiacyclic if and only if its underlying complex of $\mathfrak{R}$-contramodules is contraacyclic. This follows from the discussion in Section 2.3 together with the facts that the forgetful functor takes contraacyclic $\mathfrak{R}$-contramodule wcDG-modules to contraacyclic complexes a complex of free $\mathfrak{R}$-contramodules, and a complex of free $\mathfrak{R}$-contramodules is contraacyclic (with respect to the class of complexes of arbitrary $\mathfrak{R}$-contramodules) if and only if it is contractible.

Similarly, an $\mathfrak{R}$-comodule wcDG-module over $\mathfrak{A}$ is semiacyclic if and only if its underlying complex of $\mathfrak{R}$-comodules is coacyclic (see Section 2.6). These assertions explain the "semiderived category" terminology (cf. [28]).

When the topological local ring $\mathfrak{R}$ has finite homological dimension, the semiacyclic $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule wcDG-modules over $\mathfrak{A}$ can be simply called *acyclic*, and the semiderived categories of $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule wcDG-modules over $\mathfrak{A}$ can be simply called their *derived categories*.

Theorem 4.3.1. *Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$. Then*

(a) *for any wcDG-module $\mathfrak{P} \in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}})_{\mathsf{proj}}$ and any semiacyclic $\mathfrak{R}$-contramodule left wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, the complex of $\mathfrak{R}$-contramodules* $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P}, \mathfrak{M})$ *is contraacyclic;*

(b) *for any semiacyclic $\mathfrak{R}$-comodule left wcDG-module $\mathfrak{L}$ over $\mathfrak{A}$ and any wcDG-module $\mathfrak{J} \in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}})_{\mathsf{inj}}$, the complex of $\mathfrak{R}$-contramodules* $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{L}, \mathfrak{J})$ *is contraacyclic;*

(c) *the composition of natural functors $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}})_{\mathsf{proj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ is an equivalence of triangulated categories;*

(d) *the composition of natural functors $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}})_{\mathsf{inj}} \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$ is an equivalence of triangulated categories.*

Proof. Part (a): it suffices to consider the cases when $\mathfrak{M}$ is a contraacyclic $\mathfrak{R}$-contramodule wcDG-module or $\mathfrak{M}$ is a semiacyclic $\mathfrak{R}$-free wcDG-module. The former case holds for any $\mathfrak{P} \in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}})$ by Theorem 4.2.5(a). In the latter case, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P}, \mathfrak{M})$ is even contractible; see the remarks after the proof of Theorem 2.3.2.

The proof of part (b) is similar up to duality. Parts (c) and (d) follow from Theorems 2.3.2(a) and 2.6.2(b), respectively. $\square$

Proposition 4.3.2. *The derived functors* $\mathbb{L}\Phi_{\mathfrak{R}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathbb{R}\Psi_{\mathfrak{R}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *defined by identifying* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *with* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *with* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *are mutually inverse equivalences between the semiderived categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *of* $\mathfrak{R}$*-contramodule and* $\mathfrak{R}$*-comodule wcDG-modules.*

Proof. Follows from the results of Section 2.6. ☐

One uses the equivalences of categories $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ in order to extend the constructions and results of Sections 2.3 and 2.6 related to $\mathfrak{R}$-free and $\mathfrak{R}$-cofree wcDG-modules over $\mathfrak{A}$ to arbitrary $\mathfrak{R}$-contramodule and $\mathfrak{R}$-comodule wcDG-modules. Let us state some of the assertions which one can obtain in this way.

Corollary 4.3.3. *Assume that the DG-algebra* $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ *is cofibrant (in the standard model structure on the category of DG-algebras over* k*). Then an* $\mathfrak{R}$*-contramodule wcDG-module over* $\mathfrak{A}$ *is semiacyclic if and only if it is contraacyclic, that is* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$*. Similarly, an* $\mathfrak{R}$*-comodule wcDG-module over* $\mathfrak{A}$ *is semiacyclic if and only if it is coacyclic, that is* $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$*. Assuming additionally that* $\mathfrak{R}$ *has finite homological dimension, an* $\mathfrak{R}$*-contramodule or* $\mathfrak{R}$*-comodule wcDG-module over* $\mathfrak{A}$ *is semiacyclic if and only if it is absolutely acyclic.*

Proof. Follows from Theorem 2.3.3. ☐

In line with our usual notation, let $H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}$ denote the homotopy category of homotopy projective $\mathfrak{R}$-cofree right wcDG-modules over $\mathfrak{A}$ with projective underlying $\mathfrak{R}$-cofree graded $\mathfrak{A}$-modules.

Lemma 4.3.4. *Let* $\mathfrak{A}$ *be a wcDG-algebra over* $\mathfrak{R}$*. Then*

(a) *for any wcDG-module* $\mathfrak{Q} \in H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}$ *and any semiacyclic* $\mathfrak{R}$*-comodule left wcDG-module* $\mathcal{M}$ *over* $\mathfrak{A}$*, the complex of* $\mathfrak{R}$*-comodules* $\mathfrak{Q} \otimes_{\mathfrak{A}} \mathcal{M}$ *is coacyclic;*

(b) *for any wcDG-module* $\mathcal{Q} \in H^0(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}$ *and any semiacyclic* $\mathfrak{R}$*-comodule left wcDG-module* $\mathcal{M}$ *over* $\mathfrak{A}$*, the complex of* $\mathfrak{R}$*-comodules* $\mathcal{Q} \otimes_{\mathfrak{A}} \mathcal{M}$ *is coacyclic;*

(c) *for any wcDG-module* $\mathcal{P} \in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{proj}})_{\mathsf{proj}}$ *and any semiacyclic* $\mathfrak{R}$*-contramodule left wcDG-module* $\mathfrak{M}$ *over* $\mathfrak{A}$*, the complex of* $\mathfrak{R}$*-contramodules* $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{P}, \mathfrak{M})$ *is contraacyclic;*

(d) *for any wcDG-module* $\mathfrak{J} \in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}})_{\mathsf{inj}}$ *and any semiacyclic* $\mathfrak{R}$*-comodule left wcDG-module* $\mathcal{L}$ *over* $\mathfrak{A}$*, the complex of* $\mathfrak{R}$*-contramodules* $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{L}, \mathfrak{J})$ *is contraacyclic.*

Proof. The proof is similar to that of Theorem 4.3.1(a-b). The cases of a coacyclic $\mathfrak{R}$-comodule wcDG-module $\mathcal{M}$ or $\mathcal{L}$, or a contraacyclic $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$, are straightforward in view of the results of Section 4.1. The cases of a semiacyclic $\mathfrak{R}$-cofree wCDG-module $\mathcal{M}$ or $\mathcal{L}$, or a semiacyclic $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$, can be dealt with using the techniques of Sections 2.3 and 2.6. Alternatively, the isomorphisms $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{P}, \mathfrak{M}) \simeq \mathrm{Hom}_{\mathfrak{A}}(\Psi_{\mathfrak{R}}(\mathcal{P}), \mathfrak{M})$ and $\mathrm{Hom}_{\mathfrak{A}}(\mathcal{L}, \mathfrak{J}) \simeq \mathrm{Hom}_{\mathfrak{A}}(\mathcal{L}, \Phi_{\mathfrak{R}}(\mathfrak{J}))$

from Section 4.1 reduce parts (c) and (d) to Theorem 4.3.1(a-b). The isomorphism $\mathcal{Q}\otimes_{\mathfrak{A}}\mathcal{M}\simeq\Phi_{\mathfrak{R}}(\mathcal{Q})\otimes_{\mathfrak{A}}\mathcal{M}$ reduces part (b) to part (a); and part (a) in the case of a semiacyclic $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ is Lemma 2.6.3(b). $\square$

Remark 4.3.5. The analogues of the assertions of Lemma do *not* hold for the other tensor product and Hom operations on wcDG-modules over $\mathfrak{A}$. In particular, the tensor product of a projective $\mathfrak{R}$-free wcDG-module $\mathfrak{Q}$ and a semiacyclic $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$ is *not* in general a contraacyclic complex of $\mathfrak{R}$-contramodules (because this is not true for contraacyclic $\mathfrak{R}$-contramodule wcDG-modules $\mathfrak{M}$). The tensor product of a projective $\mathfrak{R}$-cofree wcDG-module $\mathcal{Q}$ and a semiacyclic $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$ is *not* in general a coacyclic complex of $\mathfrak{R}$-comodules. The Hom from a projective $\mathfrak{R}$-free wcDG-module $\mathfrak{P}$ to a semiacyclic $\mathfrak{R}$-comodule wcDG-module $\mathcal{M}$ is *not* in general a coacyclic complex of $\mathfrak{R}$-comodules (because this is not true for coacyclic $\mathfrak{R}$-comodule wcDG-modules $\mathcal{M}$). The Hom from a semiacyclic $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{L}$ to an injective $\mathfrak{R}$-cofree wcDG-module $\mathcal{J}$ is *not* in general a coacyclic complex of $\mathfrak{R}$-comodules (because this is not true for contraacyclic $\mathfrak{R}$-contramodule wcDG-modules $\mathfrak{L}$).

Restricting the functor $\otimes_{\mathfrak{A}}$ from Section 4.2 to either of the full subcategories $H^0(\mathsf{mod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-cof}}_{\mathsf{proj}})_{\mathsf{proj}}$ or $H^0(\mathsf{mod}^{\mathfrak{R}\text{-cof}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})\subset H^0(\mathsf{mod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{A})\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{comod})\longrightarrow\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and identifying the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ with the homotopy category $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ (see, e. g., Theorem 4.5.2(d) below), we construct the double-sided derived functor

$$\operatorname{Tor}^{\mathfrak{A}}\colon\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}).$$

Using the above identifications of the semiderived categories and the constructions of the derived functors $\operatorname{Tor}^{\mathfrak{A}}$ from Sections 2.3 and 2.6, we obtain the left derived functors

$$\operatorname{Tor}^{\mathfrak{A}}\colon\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{-ctr}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}),$$
$$\operatorname{Tor}^{\mathfrak{A}}\colon\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{-ctr}}\text{–}\mathfrak{A})\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}).$$

The latter functor can be also constructed by restricting the functor $\otimes_{\mathfrak{A}}$ from Section 4.2 to the full subcategory $H^0(\mathsf{mod}^{\mathfrak{R}\text{-fr}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{comod})\longrightarrow\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and identifying $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ with $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. The three functors $\operatorname{Tor}^{\mathfrak{A}}$ are transformed into each other by the equivalences of categories $\mathbb{L}\Phi_{\mathfrak{R}}=\mathbb{R}\Psi_{\mathfrak{R}}^{-1}$ from Proposition 4.3.2.

Similarly, restricting the functor $\operatorname{Hom}_{\mathfrak{A}}$ from Section 4.2 to either of the full subcategories $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-cof}}_{\mathsf{proj}})^{\mathsf{op}}_{\mathsf{proj}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-ctr}})$ or $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-fr}}_{\mathsf{inj}})_{\mathsf{inj}}\subset H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra})\longrightarrow\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with the homotopy category $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ (see, e. g., Theorem 4.5.2(c)), we construct the double-sided derived functor

$$\operatorname{Ext}_{\mathfrak{A}}\colon\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{-ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}).$$

From the constructions of the derived functors $\mathrm{Ext}^{\mathfrak{A}}$ in Sections 2.3 and 2.6 we obtain the right derived functors

$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}),$$
$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}),$$
$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}),$$

The second of the four can be also constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{A}}$ from Section 4.2 to the full subcategory $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})^{\mathsf{op}}_{\mathsf{proj}} \times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. Similarly, the third functor can be constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{A}}$ from Section 4.2 to the full subcategory $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})_{\mathsf{inj}}$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The four functors $\mathrm{Ext}_{\mathfrak{A}}$ are transformed into each other by the equivalences of categories $\mathbb{L}\Phi_{\mathfrak{R}} = \mathbb{R}\Psi_{\mathfrak{R}}^{-1}$.

Let $(f,a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ be a morphism of wcDG-algebras over $\mathfrak{R}$. Then we have the induced functor $R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. This functors obviously takes contraacyclic $\mathfrak{R}$-contramodule wcDG-modules over $\mathfrak{A}$ to contraacyclic $\mathfrak{R}$-contramodule wcDG-modules over $\mathfrak{B}$; it was explained in Section 2.3 that it takes semiacyclic $\mathfrak{R}$-free wcDG-modules over $\mathfrak{A}$ to semiacyclic $\mathfrak{R}$-free wcDG-modules over $\mathfrak{B}$. Hence it takes semiacyclic $\mathfrak{R}$-contramodule wcDG-modules to semiacyclic $\mathfrak{R}$-contramodule wcDG-modules, and therefore induces a triangulated functor

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

The constructions of Section 2.3 provide, via the identification of the semiderived categories of $\mathfrak{R}$-free and arbitrary $\mathfrak{R}$-contramodule CDG-modules, the left and right adjoint functors to $\mathbb{I}R_f$

$$\mathbb{L}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}),$$
$$\mathbb{R}E^f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, the functor $R_f\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ takes coacyclic $\mathfrak{R}$-comodule wcDG-modules over $\mathfrak{A}$ to coacyclic $\mathfrak{R}$-comodule wcDG-modules over $\mathfrak{B}$, and semiacyclic $\mathfrak{R}$-cofree wcDG-modules over $\mathfrak{A}$ to semiacyclic $\mathfrak{R}$-cofree wcDG-modules over $\mathfrak{B}$. Hence it also takes semiacyclic $\mathfrak{R}$-comodule wcDG-modules to semiacyclic $\mathfrak{R}$-comodule wcDG-modules, and therefore induces a triangulated functor

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The constructions of Section 2.6 provide the left and right adjoint functors

$$\mathbb{L}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}),$$
$$\mathbb{R}E^f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The equivalences of semiderived categories $\mathbb{L}\Phi_{\mathfrak{R}} = \mathbb{R}\Psi_{\mathfrak{R}}^{-1}$ transform the $\mathfrak{R}$-comodule wcDG-module restriction- and extension-of-scalars functors into the similar $\mathfrak{R}$-contramodule wcDG-module functors.

Corollary 4.3.6. *The functors* $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ *and* $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$ *are equivalences of triangulated categories whenever the DG-algebra morphism* $f/\mathfrak{m}f\colon \mathfrak{B}/\mathfrak{m}\mathfrak{B} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ *is a quasi-isomorphism.*

Proof. Follows from Theorems 2.3.5 and/or 2.6.4. ☐

4.4. **Non-$\mathfrak{R}$-free graded contramodules and non-$\mathfrak{R}$-cofree graded comodules.** Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. An $\mathfrak{R}$-*comodule graded left* $\mathfrak{C}$-*comodule* $\mathcal{M}$ is, by the definition, a graded left $\mathfrak{C}$-comodule in the module category of $\mathfrak{R}$-comodules over the tensor category of free $\mathfrak{R}$-contramodules. In other words, it is a graded $\mathfrak{R}$-comodule endowed with a (coassociative and counital) homogeneous $\mathfrak{C}$-coaction map $\mathcal{M} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$. $\mathfrak{R}$-*comodule graded right* $\mathfrak{C}$-*comodules* $\mathcal{N}$ are defined in the similar way.

An $\mathfrak{R}$-*contaramodule graded left* $\mathfrak{C}$-*contramodule* $\mathfrak{P}$ is, by the definition, a graded left $\mathfrak{C}$-contramodule in the "internal Hom category" of $\mathfrak{R}$-contramodules over the tensor category of free $\mathfrak{R}$-contramodules. In other words, it is a graded $\mathfrak{R}$-contramodule endowed with a (contraassociative and counital) homogeneous $\mathfrak{C}$-contraaction map $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow \mathfrak{P}$. For the details, see Section 3.1 and the references therein.

The categories of $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodules and $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules are enriched over the tensor category of (graded) $\mathfrak{R}$-contramodules. So the abelian group of morphisms between two $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ is the underlying abelian group of the degree-zero component of the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q}) \rightrightarrows \mathrm{Hom}^{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\mathfrak{Q})$. Similarly, the abelian group of morphisms between two $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodules $\mathcal{L}$ and $\mathcal{M}$ is the underlying abelian group of the degree-zero component of the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\mathcal{M})$ constructed as the kernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\mathcal{M}) \rightrightarrows \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L},\ \mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M})$.

The graded $\mathfrak{R}$-comodule $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathcal{M})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ to an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\mathcal{M}) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L},\ \mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M})$ constructed in the way explained in Section 3.3. The graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathcal{Q})$ from an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ is defined as the kernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{P},\mathcal{Q}) \rightrightarrows \mathrm{Ctrhom}_{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}),\mathcal{Q})$ constructed in the way explained in Section 3.3.

The *contratensor product* $\mathcal{N}\odot_{\mathfrak{C}}\mathfrak{P}$ of an $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$ and an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is the graded $\mathfrak{R}$-comodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-comodules $\mathcal{N}\odot_{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \rightrightarrows \mathcal{N}\odot_{\mathfrak{R}}\mathfrak{P}$ defined in terms of the $\mathfrak{C}$-coaction in $\mathcal{N}$, the $\mathfrak{C}$-contraaction in $\mathfrak{P}$, and the evaluation map $\mathfrak{C}\otimes^{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})\longrightarrow\mathfrak{P}$ (see Section 4.1 for the definition of $\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{P}$). The *contratensor product* $\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P}$ of an $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and and $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is the graded $\mathfrak{R}$-contramodule constructed as the cokernel of the pair of morphisms of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\otimes^{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})\rightrightarrows\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{P}$ defined in terms of the $\mathfrak{C}$-coaction in $\mathfrak{N}$, the $\mathfrak{C}$-contraaction in $\mathfrak{P}$, and the evaluation map $\mathfrak{C}\otimes^{\mathfrak{R}}\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})\longrightarrow\mathfrak{P}$.

The *cotensor product* $\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathcal{M}$ of an $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ is a graded $\mathfrak{R}$-comodule constructed as the kernel of the pair of morphisms $\mathfrak{N}\odot_{\mathfrak{R}}\mathcal{M}\rightrightarrows\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$. Similarly one defines the cotensor product $\mathcal{N}\,\square_{\mathfrak{C}}\,\mathfrak{M}$ of an $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$ and an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$. The graded $\mathfrak{R}$-contramodule of *cohomomorphisms* $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$ from an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ to an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is constructed as the cokernel of the pair of morphisms $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M},\ \mathfrak{P})\simeq\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{M},\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}))\rightrightarrows\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{M},\mathfrak{P})$ induced by the $\mathfrak{C}$-coaction in $\mathfrak{M}$ and the $\mathfrak{C}$-contraaction in $\mathfrak{P}$. The graded $\mathfrak{R}$-contramodule of *cohomomorphisms* $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$ from an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ to an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ is constructed as the cokernel of the pair of morphisms $\mathrm{Hom}_{\mathfrak{R}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M},\ \mathcal{P})\simeq\mathrm{Hom}_{\mathfrak{R}}(\mathcal{M},\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}))\rightrightarrows\mathrm{Hom}_{\mathfrak{R}}(\mathcal{M},\mathcal{P})$.

The *cotensor product* $\mathcal{N}\,\square_{\mathfrak{C}}\,\mathcal{M}$ of an $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$ and an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ is a graded $\mathfrak{R}$-comodule constructed as the kernel of the pair of morphisms $\mathcal{N}\square_{\mathfrak{R}}\mathcal{M}\rightrightarrows(\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{C})\square_{\mathfrak{R}}\mathcal{M}\simeq\mathcal{N}\square_{\mathfrak{R}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M})$ (see Lemma 1.7.1). The graded $\mathfrak{R}$-contramodule of *cohomomorphisms* $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathfrak{P})$ from an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ to an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is constructed as the cokernel of the pair of morphisms $\mathrm{Cohom}_{\mathfrak{R}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M},\mathfrak{P})\simeq\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M},\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}))\rightrightarrows\mathrm{Cohom}_{\mathfrak{R}}(\mathcal{M},\mathfrak{P})$ induced by the $\mathfrak{C}$-coaction in $\mathcal{M}$ and the $\mathfrak{C}$-contraaction in $\mathfrak{P}$.

The category of $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodules is abelian and admits infinite products; the forgetful functor from it to the category of graded $\mathfrak{R}$-contramodules is exact and preserves infinite products. There are enough projective objects in the abelian category of $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodules; these are the same as the projective objects in the exact subcategory of $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodules. The latter exact subcategory in the abelian category of $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodules is closed under extensions and infinite products.

The category of $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules is abelian and admits infinite directs sums; the forgetful functor from it to the category of $\mathfrak{R}$-comodules is exact and preserves infinite direct sums. There are enough injective objects in the abelian category of $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules; these are the same as the injective objects in the exact subcategory of $\mathfrak{R}$-cofree graded $\mathfrak{C}$-contramodules. The latter exact subcategory in the abelian category of $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules is closed under extensions and infinite direct sums.

Remark 4.4.1. $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules and $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules can be defined in what is our usual way in this paper. The reason we do not consider these is because the kernels of morphisms of $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules and the cokernels of morphisms of $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules are not well-behaved, the functors of contramodule tensor product with a free $\mathfrak{R}$-contramodule and Ctrhom from a free $\mathfrak{R}$-contramodule being not exact (see Remark 1.9.3). So we do not know of any reason for the categories of $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules and $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules to be even abelian. $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules are reasonably behaved, though, when the pro-Artinian topological ring $\mathfrak{R}$ has homological dimension 1 (see Remark 1.2.1 and Section 1.9).

For any graded $\mathfrak{R}$-contramodule $\mathfrak{V}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$, the contratensor product $\mathfrak{V} \odot_{\mathfrak{R}} \mathcal{M}$ has a natural left $\mathfrak{C}$-comodule structure. For any graded $\mathfrak{R}$-comodule $\mathcal{V}$ and any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$, the contratensor product $\mathfrak{M} \odot_{\mathfrak{R}} \mathcal{V}$ has a natural left $\mathfrak{C}$-comodule structure. The same applies to right $\mathfrak{C}$-comodules. For any graded $\mathfrak{R}$-comodule $\mathcal{U}$ and any $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$, the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{U})$ has a natural left $\mathfrak{C}$-contramodule structure provided by the map $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{U})) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C},\ \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{U})$. For any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ and any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$, the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{N}, \mathfrak{U})$ has a natural left $\mathfrak{C}$-contramodule structure.

The left $\mathfrak{C}$-contramodule $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U})$, where $\mathfrak{U}$ is a graded $\mathfrak{R}$-contramodule, is called the left $\mathfrak{C}$-contramodule *induced* from the graded $\mathfrak{R}$-contramodule $\mathfrak{U}$. The left $\mathfrak{C}$-comodule $\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}$, where $\mathcal{V}$ is a graded $\mathfrak{R}$-comodule, is called the left $\mathfrak{C}$-comodule *coinduced* from the graded $\mathfrak{R}$-comodule $\mathcal{V}$.

For any $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$, $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, and graded $\mathfrak{R}$-comodule $\mathcal{U}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{N} \odot_{\mathfrak{C}} \mathfrak{P},\ \mathcal{U}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{U}))$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$, $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, and graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{N} \odot_{\mathfrak{C}} \mathfrak{P},\ \mathfrak{U}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P}, \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{N}, \mathfrak{U}))$. For any $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$, $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$, and graded $\mathfrak{R}$-comodule $\mathcal{U}$ such that the cotensor product $\mathcal{N} \,\square_{\mathfrak{C}}\, \mathfrak{M}$ is a cofree graded $\mathfrak{R}$-comodule there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{N} \,\square_{\mathfrak{C}}\, \mathfrak{M},\ \mathcal{U}) \simeq \mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M}, \mathrm{Hom}_{\mathfrak{R}}(\mathcal{N}, \mathcal{U}))$.

For any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{L}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathrm{Hom}_{\mathfrak{R}}(\mathcal{L}, \mathcal{V})$. For any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{Q}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U}), \mathfrak{Q}) \simeq \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{U}, \mathfrak{Q})$ [28, Sections 1.1.1–2 and 3.1.1–2]. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{L}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\ \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{V}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{L}, \mathcal{V})$. For any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{Q}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathrm{Hom}^{\mathfrak{C}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U}), \mathcal{Q}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{U}, \mathcal{Q})$. For any $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathcal{N} \odot_{\mathfrak{C}} \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{U}) \simeq \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{U}$. For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathfrak{N}\odot_{\mathfrak{C}}\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})\simeq\mathfrak{N}\otimes^{\mathfrak{R}}\mathfrak{U}$ [28, Section 5.1.1].

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathfrak{N}\,\square_{\mathfrak{C}}\,(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V})\simeq\mathfrak{N}\odot_{\mathfrak{R}}\mathcal{V}$. For any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $(\mathfrak{V}\otimes^{\mathfrak{R}}\mathfrak{C})\,\square_{\mathfrak{C}}\,\mathcal{M}\simeq\mathfrak{V}\odot_{\mathfrak{R}}\mathcal{M}$. For any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$ and any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V},\ \mathfrak{P})\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{V},\mathfrak{P})$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{M},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U}))\simeq\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{M},\mathfrak{U})$. For any graded $\mathfrak{R}$-comodule $\mathcal{V}$ and any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V},\ \mathcal{P})\simeq\operatorname{Hom}_{\mathfrak{R}}(\mathcal{V},\mathcal{P})$. For any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ and any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathcal{M},\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{U}))\simeq\operatorname{Hom}_{\mathfrak{R}}(\mathcal{M},\mathcal{U})$ [28, Sections 1.2.1 and 3.2.1].

For any $\mathfrak{R}$-comodule graded right $\mathfrak{C}$-comodule $\mathcal{N}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\mathcal{N}\,\square_{\mathfrak{C}}\,(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V})\simeq\mathcal{N}\,\square_{\mathfrak{R}}\,\mathcal{V}$. For any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ and any graded $\mathfrak{R}$-comodule $\mathcal{V}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V},\ \mathfrak{P})\simeq\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{V},\mathfrak{P})$. For any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ and any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\operatorname{Cohom}_{\mathfrak{C}}(\mathcal{M},\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U}))\simeq\operatorname{Cohom}_{\mathfrak{R}}(\mathcal{M},\mathfrak{U})$.

Given an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, an epimorphism onto it from a projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodule can be obtained as the composition $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})\longrightarrow\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})\longrightarrow\mathfrak{P}$, where $\mathfrak{U}$ is a free graded $\mathfrak{R}$-contramodule and $\mathfrak{U}\longrightarrow\mathfrak{P}$ is an epimorphism of graded $\mathfrak{R}$-contramodules. Given an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$, a monomorphism from it into an injective $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodule can be obtained as the composition $\mathcal{M}\longrightarrow\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$, where $\mathcal{V}$ is a cofree graded $\mathfrak{R}$-comodule and $\mathcal{M}\longrightarrow\mathcal{V}$ is a monomorphism of graded $\mathfrak{R}$-comodules.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be $\mathfrak{R}$-free graded coalgebras. An $\mathfrak{R}$-*comodule graded* $\mathfrak{D}$-$\mathfrak{C}$-*bicomodule* $\mathcal{K}$ is a graded $\mathfrak{R}$-comodule endowed with commuting graded left $\mathfrak{D}$-comodule and graded right $\mathfrak{R}$-comodule structures. Equivalently, $\mathcal{K}$ should be endowed with a coassociative and counital homogeneous $\mathfrak{D}$-$\mathfrak{C}$-*bicoaction* map $\mathcal{K}\longrightarrow\mathfrak{D}\odot_{\mathfrak{R}}\mathcal{K}\odot_{\mathfrak{R}}\mathfrak{C}$.

Given an $\mathfrak{R}$-comodule graded $\mathfrak{D}$-$\mathfrak{C}$-bicomodule $\mathcal{K}$ and an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, the contratensor product $\mathcal{K}\odot_{\mathfrak{C}}\mathfrak{P}$ is endowed with a graded left $\mathfrak{D}$-comodule structure as the cokernel of a pair of morphisms of $\mathfrak{R}$-comodule graded left $\mathfrak{D}$-comodules. Given an $\mathfrak{R}$-comodule graded $\mathfrak{D}$-$\mathfrak{C}$-bicomodule $\mathcal{K}$ and an $\mathfrak{R}$-comodule graded left $\mathfrak{D}$-comodule $\mathcal{M}$, the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}_{\mathfrak{D}}(\mathcal{K},\mathcal{M})$ is endowed with a graded left $\mathfrak{C}$-contramodule structure as the kernel of a pair of morphisms of $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodules.

Lemma 4.4.2. *Let $\mathcal{K}$ be an $\mathfrak{R}$-comodule graded $\mathfrak{D}$-$\mathfrak{C}$-bicomodule. Then the functor taking an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ to the $\mathfrak{R}$-comodule graded left $\mathfrak{D}$-comodule $\mathcal{K}\odot_{\mathfrak{C}}\mathfrak{P}$ is naturally left adjoint to the functor taking an $\mathfrak{R}$-comodule graded left $\mathfrak{D}$-comodule $\mathcal{M}$ to the $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule* $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{K},\mathcal{M})$. *Moreover, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules* $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{K}\odot_{\mathfrak{C}}\mathfrak{P},\ \mathcal{M})\simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathrm{Hom}_{\mathfrak{D}}(\mathcal{K},\mathcal{M}))$.

Proof. According to Section 1.5, there is a natural isomorphism of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\mathcal{K}\odot_{\mathfrak{R}}\mathfrak{P},\ \mathcal{M})\simeq \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathrm{Hom}_{\mathfrak{R}}(\mathcal{K},\mathcal{M}))$. It is straightforward to check that a graded $\mathfrak{R}$-comodule map $\mathcal{K}\odot_{\mathfrak{R}}\mathfrak{P}\longrightarrow\mathcal{M}$ factorizes through $\mathcal{K}\odot_{\mathfrak{C}}\mathfrak{P}$ if and only if the corresponding graded $\mathfrak{R}$-contramodule map $\mathfrak{P}\longrightarrow\mathrm{Hom}_{\mathfrak{R}}(\mathcal{K},\mathcal{M})$ is a $\mathfrak{C}$-contramodule morphism, and a graded $\mathfrak{R}$-contramodule map $\mathfrak{P}\longrightarrow\mathrm{Hom}_{\mathfrak{R}}(\mathcal{K},\mathcal{M})$ factorizes through $\mathrm{Hom}_{\mathfrak{D}}(\mathcal{K},\mathcal{M})$ if and only if the corresponding graded $\mathfrak{R}$-comodule map $\mathcal{K}\odot_{\mathfrak{R}}\mathfrak{P}\longrightarrow\mathcal{M}$ is a $\mathfrak{D}$-comodule morphism. (Cf. [28, Section 5.1.2].) $\square$

Set $\mathcal{C}(\mathfrak{R},\mathfrak{C})=\mathcal{C}(\mathfrak{R})\odot_{\mathfrak{R}}\mathfrak{C}$. Obviously, the graded $\mathfrak{R}$-comodule $\mathcal{C}(\mathfrak{R},\mathfrak{C})$ has a natural $\mathfrak{R}$-comodule graded $\mathfrak{C}$-bicomodule structure. For any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$, set $\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{P})=\mathcal{C}(\mathfrak{R},\mathfrak{C})\odot_{\mathfrak{C}}\mathfrak{P}$ and $\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{M})=\mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C}),\ \mathcal{M})$. According to Lemma 4.4.2, $\Phi_{\mathfrak{R},\mathfrak{C}}$ and $\Psi_{\mathfrak{R},\mathfrak{C}}$ are adjoint functors between the abelian categories of $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodules and $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodules.

One easily checks that for any projective $\mathfrak{R}$-free graded left $\mathfrak{C}$-contramodule $\mathfrak{F}$ there is a natural isomorphism of $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules $\Phi_{\mathfrak{R}}\Phi_{\mathfrak{C}}(\mathfrak{F})\simeq\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{F})$. Similarly, for any injective $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-comodule $\mathcal{J}$ there is a natural isomorphism of $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodules $\Psi_{\mathfrak{R}}\Psi_{\mathfrak{C}}(\mathcal{J})\simeq\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{J})$ (see Sections 3.1 and 3.3 for the definitions of $\Phi_{\mathfrak{R}}=\Psi_{\mathfrak{R}}^{-1}$ and $\Phi_{\mathfrak{C}}=\Psi_{\mathfrak{C}}^{-1}$). So the functors $\Phi_{\mathfrak{R},\mathfrak{C}}$ and $\Psi_{\mathfrak{R},\mathfrak{C}}$ restrict to mutually inverse equivalences between the additive categories of projective $\mathfrak{R}$-free left $\mathfrak{C}$-contramodules and injective $\mathfrak{R}$-cofree left $\mathfrak{C}$-comodules.

The equivalence $\Phi_{\mathfrak{R}}=\Psi_{R}^{-1}$ between the exact categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree graded right $\mathfrak{C}$-comodules is constructed in the way similar to the case of left $\mathfrak{C}$-comodules (see Proposition 3.3.2(b)). For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\Phi_{\mathfrak{R}}(\mathfrak{N})\odot_{\mathfrak{C}}\mathfrak{P}\simeq\Phi_{\mathfrak{R}}(\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P})$.

For any $\mathfrak{R}$-free graded right $\mathfrak{C}$-comodule $\mathfrak{N}$ and any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$, there is a natural isomorphism of graded $\mathfrak{R}$-comodules $\Phi_{\mathfrak{R}}(\mathfrak{N})\square_{\mathfrak{C}}\mathcal{M}\simeq\mathfrak{N}\square_{\mathfrak{C}}\mathcal{M}$. For any $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{M}$ and any $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$, there is a natural isomorphism of graded $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\Phi_{\mathfrak{R}}(\mathfrak{M}),\mathfrak{P})\simeq\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$. For any $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathcal{M}$ and any $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathcal{P}$, there is a natural isomorphsm of graded $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\Psi_{\mathfrak{R}}(\mathcal{P}))\simeq\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$.

4.5. **Contra/coderived category of CDG-contra/comodules.** The definitions of

- odd contra- and coderivations of $\mathfrak{R}$-contramodule $\mathfrak{C}$-contramodules and $\mathfrak{R}$-comodule $\mathfrak{C}$-comodules compatible with a given odd derivation of an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$,
- $\mathfrak{R}$-contramodule CDG-contramodules and $\mathfrak{R}$-comodule CDG-comodules over an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$,
- the complexes of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$ for given $\mathfrak{R}$-contramodule left CDG-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ over $\mathfrak{C}$,
- the complexes of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\mathcal{M})$ for given $\mathfrak{R}$-comodule left CDG-comodules $\mathcal{L}$ and $\mathcal{M}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-comodules $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathcal{Q})$ for an $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ and $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{Q}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-comodules $\mathrm{Hom}_{\mathfrak{C}}(\mathfrak{L},\mathcal{M})$ for an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{L}$ and $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-comodules $\mathcal{N}\odot_{\mathfrak{C}}\mathfrak{P}$ for an $\mathfrak{R}$-comodule right CDG-comodule $\mathcal{N}$ and $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-contramodules $\mathfrak{N}\odot_{\mathfrak{C}}\mathfrak{P}$ for an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-comodules $\mathfrak{N}\,\square_{\mathfrak{C}}\,\mathcal{M}$ for an $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ and $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-comodules $\mathcal{N}\square_{\mathfrak{C}}\mathcal{M}$ for an $\mathfrak{R}$-comodule right CDG-comodule $\mathcal{N}$ and $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathfrak{M},\mathfrak{P})$ for an $\mathfrak{R}$-free left CDG-comodule $\mathfrak{M}$ and $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathcal{P})$ for an $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ and $\mathfrak{R}$-cofree left CDG-contramodule $\mathcal{P}$ over $\mathfrak{C}$,
- the complex of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{M},\mathfrak{P})$ for an $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ and $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$,
- the $\mathfrak{R}$-contramodule CDG-contramodules or $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{D}$ obtained by contra- or corestriction of scalars via a morphism of $\mathfrak{R}$-free CDG-coalgebras $\mathfrak{C}\longrightarrow\mathfrak{D}$ from $\mathfrak{R}$-contramodule CDG-contramodules or $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{C}$

repeat the similar definition for $\mathfrak{R}$-free CDG-contramodules and $\mathfrak{R}$-cofree CDG-comodules given in Sections 3.2 and 3.4 *verbatim* (with the definitions and constructions of Section 4.4 being used in place of those from Sections 3.1 and 3.3 as applicable), so there is no need to spell them out here again. We restrict ourselves to introducing the new notation for our new classes of objects.

The DG-categories of $\mathfrak{R}$-comodule left and right CDG-comodules over $\mathfrak{C}$ are denoted by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}$ and $\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C}$, and their homotopy categories are $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ and $H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$. Similarly, the DG-category of $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is denoted by $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}$, and its homotopy category is $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. The Hom from $\mathfrak{R}$-contramodule to $\mathfrak{R}$-cofree CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Hom}^{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The Hom from $\mathfrak{R}$-free to $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Hom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The contratensor product of $\mathfrak{R}$-comodule right CDG-comodules and $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\odot_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The contratensor product of $\mathfrak{R}$-free right CDG-comodules and $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\odot_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

The cotensor product of $\mathfrak{R}$-free right CDG-comodules and $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\square_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The cotensor product of $\mathfrak{R}$-comodule left and right CDG-comodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\square_{\mathfrak{C}}\colon H^0(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}).$$

The Cohom from $\mathfrak{R}$-free left CDG-comodules to $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

The Cohom from $\mathfrak{R}$-comodule left CDG-comodules to $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

The Cohom from $\mathfrak{R}$-comodule left CDG-comodules to $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is a triangulated functor of two arguments

$$\mathrm{Cohom}_{\mathfrak{C}}\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})\longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}).$$

Given a morphism of $\mathfrak{R}$-free CDG-coalgebras $f=(f,a)\colon\mathfrak{C}\longrightarrow\mathfrak{D}$, the functors of contra- and corestriction of scalars are denoted by

$$R^f\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})\longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$$

and

$$R_f\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})\longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

An $\mathfrak{R}$-contramodule left CDG-contramodule over $\mathfrak{C}$ is called *contraacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-contramodule CDG-contramodules over $\mathfrak{C}$ and closed under infinite products. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ by the thick subcategory of contraacyclic $\mathfrak{R}$-contramodule CDG-contramodules is called the *contraderived category* of $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$.

An $\mathfrak{R}$-comodule left CDG-comodule over $\mathfrak{C}$ is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ containing the totalizations of short exact sequences of $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{C}$ and closed under infinite direct sums. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ by the thick subcategory of coacyclic $\mathfrak{R}$-comodule CDG-comodules is called the *coderived category* of $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$. The coderived category of $\mathfrak{R}$-comodule right CDG-comodules over $\mathfrak{C}$, denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$, is defined similarly.

It follows from the next theorem, among other things, that our terminology is not ambigous: an $\mathfrak{R}$-free CDG-contramodule over $\mathfrak{C}$ is contraacyclic in the sense of Section 3.2 if and only if it is contraacyclic as an $\mathfrak{R}$-contramodule CDG-contramodule, in the sense of the above definition. Similarly, an $\mathfrak{R}$-cofree CDG-comodule over $\mathfrak{C}$ is coacyclic in the sense of Section 3.4 if and only if it is coacyclic as an $\mathfrak{R}$-comodule CDG-comodules, in the sense of the above definition.

Theorem 4.5.1. *For any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, the functors $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ induced by the natural embeddings of DG-categories $\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ and $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}$ are equivalences of triangulated categories.*

Proof. The argument from the proof of Theorem 4.2.1 is applicable, but due to the fact that there are always enough projective resolutions for the contraderived category of CDG-contramodules (and injective resolutions for the coderived category of CDG-comodules), there is a much simpler proof. The desired assertions follow immediately from the next Theorem 4.5.2 together with Theorems 3.2.2(c) and 3.4.1(d). ☐

Theorem 4.5.2. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *for any CDG-contramodule $\mathfrak{P} \in H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ and any contraacyclic $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$ is contraacyclic;*

(b) *for any coacyclic $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{L}$ over $\mathfrak{C}$ and any CDG-comodule $\mathcal{M} \in H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})$, the complex of $\mathfrak{R}$-contramodules $\mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\mathcal{M})$ is contraacyclic;*

(c) *the composition of natural functors $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ is an equivalence of triangulated categories;*

(d) *the composition of natural functors $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ is an equivalence of triangulated categories.*

Proof. Part (a) holds, since the functor $\mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},-)$ takes short exact sequences and infinite products of $\mathfrak{R}$-contramodule CDG-contramodules to short exact sequences and infinite products of complexes of $\mathfrak{R}$-contramodules. The proof of part (b) is similar up to duality (cf. [29, Theorem 4.4, Theorem 3.5 and Remark 3.5]).

To prove part (c), we will need to use the construction of the CDG-contramodule $G^+(\mathfrak{F})$ over $\mathfrak{C}$ freely generated by a given $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule

$\mathfrak{F}$ (cf. the proof of Theorem 2.2.4). The graded $\mathfrak{R}$-contramodule $G^+(\mathfrak{F})$ is defined by the rule $G^+(\mathfrak{F})^i = \mathfrak{F}^i \oplus \mathfrak{F}^{i-1}$; the elements of $G^+(\mathfrak{F})^i$ are denoted formally by $x + d_G y$, where $x \in \mathfrak{F}^i$ and $y \in \mathfrak{F}^{i-1}$.

To define the left contraaction of $\mathfrak{C}$ in $G^+(\mathfrak{F})$, present an arbitrary element of degree i component $\mathrm{Hom}^{\mathfrak{R},i}(\mathfrak{C}, G^+(\mathfrak{F}))$ of the graded $\mathfrak{R}$-contramodule $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, G^+(\mathfrak{F}))$ in the form $c \longmapsto f(c) + d_G(g(c))$, where $f \in \mathrm{Hom}^{\mathfrak{R},i}(\mathfrak{C},\mathfrak{F})$ and $g \in \mathrm{Hom}^{\mathfrak{R},\,i-1}(\mathfrak{C},\mathfrak{F})$. Set $\pi_{G^+(\mathfrak{F})}(f) = \pi_{\mathfrak{F}}(f)$ and $\pi_{G^+(\mathfrak{F})}(d_G \circ g) = d(\pi_{\mathfrak{F}}(g)) + (-1)^{|g|}\pi_{\mathfrak{F}}(g \circ d_{\mathfrak{C}})$, where $\pi_{\mathfrak{F}}\colon \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{F}) \longrightarrow \mathfrak{F}$ and $\pi_{G^+(\mathfrak{F})}\colon \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\, G^+(\mathfrak{F})) \longrightarrow G^+(\mathfrak{F})$ denote the $\mathfrak{C}$-contraaction map and $d_{\mathfrak{C}}\colon \mathfrak{C} \longrightarrow \mathfrak{C}$ is the differential in $\mathfrak{C}$. Finally, define the differential in $G^+(\mathfrak{F})$ by the rule $d(x + d_G(y)) = h * y + d_G(x)$, where $h\colon \mathfrak{C}^{-2} \longrightarrow \mathfrak{R}$ is the curvature linear function of $\mathfrak{C}$.

There is a natural morphism of graded $\mathfrak{C}$-contramodules $\mathfrak{F} \longrightarrow G^+(\mathfrak{F})$ defined by the rule $x \longmapsto x = x + d_G(0)$. The cokernel of this morphism is isomorphic to $\mathfrak{F}[-1]$ as an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule. Any morphism of graded $\mathfrak{C}$-contramodules $p\colon \mathfrak{F} \longrightarrow \mathfrak{P}$ from $\mathfrak{F}$ to an $\mathfrak{R}$-contramodule CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$ factorizes uniquely as the composition of the natural embedding $\mathfrak{F} \longrightarrow G^+(\mathfrak{F})$ and a closed morphism of CDG-contramodules $G^+(\mathfrak{F}) \longrightarrow \mathfrak{P}$; the latter is given by the rule $x + d_G y \longmapsto p(x) + d_{\mathfrak{P}}(p(y))$, where $d_{\mathfrak{P}}$ is the differential in $\mathfrak{P}$.

Now let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule left CDG-contramodule over $\mathfrak{C}$. As explained in Section 4.4, there exists a surjective morphism onto the $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodule $\mathfrak{P}$ from a projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodule $\mathfrak{F}_0$. The above construction then provides a surjective closed morphism $G^+(\mathfrak{F}) \longrightarrow \mathfrak{P}$ onto $\mathfrak{P}$ from a CDG-contramodule $G^+(\mathfrak{F}) \in H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$. Applying the same procedure to the kernel of the latter morphism, etc., we obtain a left resolution of $\mathfrak{P}$ by $\mathfrak{R}$-free CDG-contramodules with projective underlying graded $\mathfrak{C}$-contramodules $\dotsb \longrightarrow G^+(\mathfrak{F}_1) \longrightarrow G^+(\mathfrak{F}_0) \longrightarrow \mathfrak{P} \longrightarrow 0$.

Totalizing this complex of CDG-contramodules from $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ by taking infinite products along the diagonals, we get a closed morphism of $\mathfrak{R}$-contramodule CDG-contramodules $\mathfrak{F} \longrightarrow \mathfrak{P}$ with $\mathfrak{F} \in H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ (cf. the proof of Theorem 3.2.2). It remains to use the $\mathfrak{C}$-contramodule analogue of Lemma 4.2.2.

We have proven part (c). The proof of part (d) is analogous up to the duality, so we restrict ourselves to writing down the construction of the $\mathfrak{R}$-comodule left CDG-comodule $G^-(\mathfrak{L})$ cofreely cogenerated by an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathfrak{L}$. The graded $\mathfrak{R}$-comodule $G^-(\mathfrak{L})$ is defined by the rule $G^-(\mathfrak{L})^i = \mathfrak{L}^{i+1} \oplus \mathfrak{L}^i$; the elements of $G^+(\mathfrak{L})^i$ are denoted formally by $d_G^{-1}x + py$, where $x \in \mathfrak{L}^{i+1}$ and $y \in \mathfrak{L}^i$.

We will need to use Sweedler's notation for the coaction maps: given an $\mathfrak{R}$-comodule left $\mathfrak{C}$-comodule $\mathcal{M}$, the left $\mathfrak{C}$-coaction map $\mathcal{M} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$ is denoted by by $z \longmapsto z_{(-1)} \odot z_{(0)}$, where $z,\, z_{(0)} \in \mathcal{M}$, $z_{(-1)} \in \mathfrak{C}$, and $c \odot z$ is a symbolic notation for an element of $\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$. Then the $\mathfrak{C}$-coaction in $G^+(\mathfrak{L})$ is expressed in terms of the $\mathfrak{C}$-coaction in $\mathfrak{L}$ and the differential $d_{\mathfrak{C}}$ in $\mathfrak{C}$ by the rules $d_G^{-1}(x) \longmapsto (-1)^{|x_{(-1)}|} x_{(-1)} \odot d_G^{-1}(x_{(0)})$ and $py \longmapsto (-1)^{|y_{(-1)}|} d_{\mathfrak{C}}(y_{(-1)}) \odot d_G^{-1}(y_{(0)}) + y_{(-1)} \odot p(y_{(0)})$. The differential in $G^-(\mathfrak{L})$ is $d(d_G^{-1}x + py) = d_G^{-1}(h * y) + px$.

There is a natural morphism of graded $\mathfrak{C}$-comodules $G^-(\mathcal{L}) \longrightarrow \mathcal{L}$ defined by the rule $d_G^{-1}x + py \longmapsto y$. The cokernel of this morphism is isomorphic to $\mathcal{L}[1]$ as an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule. Any morphism of graded $\mathfrak{C}$-comodules $f\colon \mathcal{M} \longrightarrow \mathcal{L}$ from an $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$ to $\mathcal{L}$ factorizes uniquely as the composition of a closed morphism of CDG-comodules $\mathcal{M} \longrightarrow G^-(\mathcal{L})$ and the natural surjection $G^-(\mathcal{L}) \longrightarrow \mathcal{L}$; the former is given by the formula $z \longmapsto d_G^{-1}(f(d_{\mathcal{M}}(z))) + p(f(z))$ (cf. [29, proof of Theorem 3.6]). $\square$

Given an $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$, the $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{P}) = \mathcal{C}(\mathfrak{R},\mathfrak{C}) \odot_{\mathfrak{C}} \mathfrak{P}$ is endowed with a CDG-comodule structure with the conventional tensor product differential (where the differential on $\mathcal{C}(\mathfrak{R},\mathfrak{C}) = \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{C}(\mathfrak{R})$ is induced by the differential on $\mathfrak{C}$). Similarly, given an $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$, the $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C}),\, \mathcal{M})$ is endowed with a CDG-contramodule structure with the conventional Hom differential.

For any CDG-contramodule $\mathfrak{F} \in \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}}$, the natural isomorphism $\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{F}) \simeq \Phi_{\mathfrak{R}}\Phi_{\mathfrak{C}}(\mathfrak{F})$ from Section 4.4 agrees with the CDG-comodule structures on both sides. Similarly, for any CDG-comodule $\mathcal{J} \in \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}$, the natural isomorphism $\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{J}) \simeq \Psi_{\mathfrak{R}}\Psi_{\mathfrak{C}}(\mathcal{J})$ agrees with the CDG-contramodule structures on both sides (see Sections 3.2 and 3.4 for the definitions).

Corollary 4.5.3. *The derived functors* $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *defined by identifying* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *with* $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})$ *and* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *with* $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}})$ *are mutually inverse equivalences between the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *and the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$. $\square$

Using the equivalence of triangulated categories from Theorem 4.5.1 and the construction of the derived functor $\mathrm{Ext}^{\mathfrak{C}}$ from Section 3.2, we obtain the right derived functor

$$\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}).$$

The same functor can be constructed by restricting the functor $\mathrm{Hom}^{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{proj}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with the homotopy category $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

Similarly, from Theorem 4.5.1 and the construction of the derived functor $\mathrm{Ext}_{\mathfrak{C}}$ in Section 3.4 we obtain the right derived functor

$$\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}).$$

The same functor can be constructed by restricting the functor $\mathrm{Hom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}_{\mathsf{inj}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

Restricting the functor $\odot_{\mathfrak{C}}$ to the full subcategory $H^0(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-fr}}_{\mathsf{proj}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and identifying the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ with the homotopy category $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, we obtain the left derived functor

$$\operatorname{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}).$$

The functor obtained by restriction and composition factorizes through the Cartesian product of coderived and contraderived categories by Theorem 4.5.2(c) and because the contratensor product with a projective $\mathfrak{R}$-free CDG-contramodule preserves exact triples and infinite direct sums of $\mathfrak{R}$-comodule right CDG-comodules. The equivalences of categories from Theorem 4.5.1 together with the equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform the derived functor $\operatorname{Ctrtor}^{\mathfrak{C}}$ that we have constructed into the functors $\operatorname{Ctrtor}^{\mathfrak{C}}$ from Sections 3.2 and 3.4.

Restricting the functor $\square_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}})$ or $H^0(\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and identifying $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ with $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, we obtain the right derived functor

$$\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{-co}}\text{–}\mathfrak{C})\times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}).$$

Here $\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}}\text{–}\mathfrak{C}$ denotes the DG-category of $\mathfrak{R}$-cofree right CDG-comodules over $\mathfrak{C}$ with injective underlying $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodules, and $H^0(\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}}\text{–}\mathfrak{C})$ is the corresponding homotopy category. From Theorem 4.5.1 and the construction of the derived functor $\operatorname{Cotor}^{\mathfrak{C}}$ in Section 3.4 we obtain a triangulated functor

$$\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{-fr}}\text{–}\mathfrak{C})\times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}).$$

The latter functor can be also obtained by restricting the functor $\square_{\mathfrak{C}}$ to the full subcategory $H^0(\mathsf{comod}^{\mathfrak{R}\text{-fr}}_{\mathsf{inj}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}}) \subset H^0(\mathsf{comod}^{\mathfrak{R}\text{-fr}}\text{–}\mathfrak{C})\times H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$, and identifying $\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{comod})$ with $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. The equivalences of categories from Theorem 4.5.1 together with the equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform the above two derived functors $\operatorname{Ctrtor}^{\mathfrak{C}}$ into each other and the functor $\operatorname{Cotor}^{\mathfrak{C}}$ from Sections 3.2.

Restricting the functor $\operatorname{Cohom}_{\mathfrak{C}}$ to either of the full subcategories $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-cof}}_{\mathsf{inj}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-ctr}})$ or $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-fr}}_{\mathsf{proj}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, we obtain the left derived functor

$$\operatorname{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{-co}})^{\mathsf{op}}\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{-ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}).$$

From Theorem 4.5.1 and the constructions of the derived functors $\mathrm{Coext}_{\mathfrak{C}}$ in Sections 3.2 and 3.4 we obtain triangulated functors

$$\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}),$$
$$\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}).$$

The former of these two functors can be also obtained by restricting the functor $\mathrm{Cohom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, composing it with the localization functor $H^0(\mathfrak{R}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$, and identifying $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ with $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The latter derived functor can be similarly obtained by restricting the functor $\mathrm{Cohom}_{\mathfrak{C}}$ to the full subcategory $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}) \subset H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$. The equivalences of categories from Theorem 4.5.1 together with the equivalences of categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ transform these three derived functors $\mathrm{Coext}_{\mathfrak{C}}$ into each other and the derived functor $\mathrm{Coext}_{\mathfrak{C}}$ of cohomomorphisms from $\mathfrak{R}$-free CDG-comodules to $\mathfrak{R}$-cofree CDG-comodules from Section 3.4 (taking values in $H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$).

Proposition 4.5.4. (a) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ *from Corollary* 4.5.3 *transforms the left derived functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *into the right derived functors* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}}\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *and* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}\times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$.

(b) *The equivalence of triangulated categories* $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ *transforms the right derived functor* $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})\times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *into the left derived functor* $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})\times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$.

Proof. For any $\mathfrak{R}$-contramodule left CDG-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ over $\mathfrak{C}$, there is a natural morphism of complexes of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{P}),\mathfrak{Q}) = \mathrm{Cohom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C})\odot_{\mathfrak{C}}\mathfrak{P},\mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C}),\mathfrak{Q}) \simeq \mathrm{Hom}^{\mathfrak{C}}(\mathfrak{P},\mathfrak{Q})$, which is an isomorphism whenever either of the graded $\mathfrak{C}$-contramodules $\mathfrak{P}$ or $\mathfrak{Q}$ is a projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodule. For any $\mathfrak{R}$-comodule left CDG-comodules $\mathcal{L}$ and $\mathcal{M}$ over $\mathfrak{C}$, there is a natural morphism of complexes of $\mathfrak{R}$-contramodules $\mathrm{Cohom}_{\mathfrak{C}}(\mathcal{L},\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{M})) = \mathrm{Cohom}_{\mathfrak{C}}(\mathcal{L},\mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C}),\mathcal{M}) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(\mathcal{C}(\mathfrak{R},\mathfrak{C})\,\square_{\mathfrak{C}}\,\mathcal{L},\,\mathcal{M}) \simeq \mathrm{Hom}_{\mathfrak{C}}(\mathcal{L},\mathcal{M})$, which is an isomorphism whenever either of the graded $\mathfrak{C}$-comodules $\mathcal{L}$ or $\mathcal{M}$ is an injective $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodule.

For any $\mathfrak{R}$-comodule right CDG-comodule $\mathcal{N}$ and $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$, there is a natural morphism of complexes of $\mathfrak{R}$-comodules $\mathcal{N}\odot_{\mathfrak{C}}\mathfrak{P} \simeq (\mathcal{N}\,\square_{\mathfrak{C}}\,\mathcal{C}(\mathfrak{R},\mathfrak{C}))\odot_{\mathfrak{C}}\mathfrak{P} \longrightarrow \mathcal{N}\,\square_{\mathfrak{C}}\,(\mathcal{C}(\mathfrak{R},\mathfrak{C})\odot_{\mathfrak{C}}\mathfrak{P}) = \mathcal{N}\,\square_{\mathfrak{C}}\,\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{P})$, which is an isomorphism whenever either the graded right $\mathfrak{C}$-comodule $\mathcal{N}$ is an injective $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodule, or the graded left $\mathfrak{C}$-contramodule $\mathfrak{P}$ is a projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodule. (Cf. [29, Section 5.3] and Lemma 1.7.1; see [28, proof of Corollary 5.6] for further details.) ∎

Let $(f,a)\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ be a morphism of $\mathfrak{R}$-free CDG-algebras. Then the functor $R^f\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ obviously takes contraacyclic $\mathfrak{R}$-contramodule CDG-contramodules over $\mathfrak{C}$ to contraacyclic $\mathfrak{R}$-contramodule CDG-modules over $\mathfrak{D}$, and hence induces a triangulated functor

$$\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, the functor $R_f\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ takes coacyclic $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{C}$ to coacyclic $\mathfrak{R}$-comodule CDG-comodules over $\mathfrak{D}$, and hence induces a triangulated functor

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The constructions of Sections 3.2 and 3.4 together with Theorem 4.5.1 provide the left adjoint functor to $\mathbb{I}R^f$

$$\mathbb{L}E^f\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$$

and the right adjoint functor to $\mathbb{I}R_f$

$$\mathbb{R}E_f\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

Proposition 4.5.5. *The equivalences of triangulated categories $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ and $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{D}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{D}}^{-1}$ from Corollary 4.5.3 transform the left derived functor $\mathbb{L}E^f$ into the right derived functor $\mathbb{R}E_f$ and back.*

Proof. Follows from Proposition 3.2.7 or 3.4.4. ☐

5. Change of Coefficients and Compact Generation

5.1. **Change of coefficients for wcDG-modules.** Let $\eta\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ be a profinite morphism (see Section 1.8) of pro-Artinian topological local rings with the maximal ideals $\mathfrak{m}_{\mathfrak{R}}$ and $\mathfrak{m}_{\mathfrak{S}}$ and the residue fields $k_{\mathfrak{R}}$ and $k_{\mathfrak{S}}$. Notice that such a morphism is always local, i. e., $\eta(\mathfrak{m}_{\mathfrak{R}}) \subset \mathfrak{m}_{\mathfrak{S}}$; so η induces a finite field extension $\eta/\mathfrak{m}\colon k_{\mathfrak{R}} \longrightarrow k_{\mathfrak{S}}$.

We will apply the functors R^η, E^η, R_η, E_η to graded contramodules and comodules over $\mathfrak{R}$ and $\mathfrak{S}$ termwise. The natural (iso)morphisms from Section 1.8 hold for graded contramodules and comodules in the same form as in the ungraded case.

Let $\mathfrak{B}$ be an $\mathfrak{R}$-free graded algebra. Then the free graded $\mathfrak{S}$-contramodule $E^\eta(\mathfrak{B})$ has a natural graded algebra structure provided by the multiplication map $E^\eta(\mathfrak{B})\otimes^{\mathfrak{S}} E^\eta(\mathfrak{B}) \simeq E^\eta(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{B}) \longrightarrow E^\eta(\mathfrak{B})$ induced by the multiplication in $\mathfrak{B}$ and the unit map $\mathfrak{S} \longrightarrow E^\eta(\mathfrak{B})$ induced by the unit map $\mathfrak{R} \longrightarrow \mathfrak{B}$.

Let $\mathfrak{M}$ be an $\mathfrak{R}$-contramodule graded left $\mathfrak{B}$-module. Then the graded $\mathfrak{S}$-contramodule $E^\eta(\mathfrak{M})$ has a natural graded left $E^\eta(\mathfrak{B})$-module structure provided by the action map $E^\eta(\mathfrak{B}) \otimes^{\mathfrak{S}} E^\eta(\mathfrak{M}) \simeq E^\eta(\mathfrak{B}\otimes^{\mathfrak{R}}\mathfrak{M}) \longrightarrow E^\eta(\mathfrak{M})$. The same module structure can be defined in terms of the action map $E^\eta(\mathfrak{M}) \longrightarrow E^\eta \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},\mathfrak{M}) \simeq \operatorname{Hom}^{\mathfrak{S}}(E^\eta(\mathfrak{B}), E^\eta(\mathfrak{M}))$. The similar construction applies to right $\mathfrak{B}$-modules.

Let $\mathcal{M}$ be an $\mathfrak{R}$-comodule graded left $\mathfrak{B}$-module. Then the graded $\mathfrak{S}$-comodule $E_\eta(\mathcal{M})$ has a natural graded left $E^\eta(\mathfrak{B})$-module structure provided by the action map $E^\eta(\mathfrak{B})\odot_{\mathfrak{S}} E_\eta(\mathcal{M})\simeq E_\eta(\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{M})\longrightarrow E_\eta(\mathcal{M})$. The same module structure can be defined in terms of the action map $E_\eta(\mathcal{M})\longrightarrow E_\eta\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{B},\mathcal{M})\simeq \operatorname{Ctrhom}_{\mathfrak{S}}(E^\eta(\mathfrak{B}),E^\eta(\mathcal{M}))$. The similar construction applies to right $\mathfrak{B}$-modules.

Let $\mathfrak{N}$ be an $\mathfrak{S}$-contramodule graded left $E^\eta(\mathfrak{B})$-module. Then the graded $\mathfrak{R}$-contramodule $R^\eta(\mathfrak{N})$ has a natural graded left $\mathfrak{B}$-module structure provided by the action map $\mathfrak{B}\otimes^{\mathfrak{R}} R^\eta(\mathfrak{N})\longrightarrow R^\eta(E^\eta(\mathfrak{B})\otimes^{\mathfrak{S}}\mathfrak{N})\longrightarrow R^\eta(\mathfrak{N})$. The same module structure can be defined in terms of the action map $R^\eta(\mathfrak{N})\longrightarrow R^\eta\operatorname{Hom}^{\mathfrak{S}}(E^\eta(\mathfrak{B}),\mathfrak{N})\simeq \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B},R^\eta(\mathfrak{N}))$. The similar construction applies to right $E^\eta(\mathfrak{B})$-modules.

Let $\mathcal{N}$ be an $\mathfrak{S}$-comodule graded left $E^\eta(\mathfrak{B})$-module. Then the graded $\mathfrak{R}$-comodule $R_\eta(\mathcal{N})$ has a natural graded left $\mathfrak{B}$-module structure provided by the action map $\mathfrak{B}\odot_{\mathfrak{R}} R^\eta(\mathcal{N})\simeq R_\eta(E^\eta(\mathfrak{B})\odot_{\mathfrak{S}}\mathcal{N})\longrightarrow R^\eta(\mathcal{N})$. The same module structure can be defined in terms of the action map $R_\eta(\mathcal{N})\longrightarrow R_\eta\operatorname{Ctrhom}_{\mathfrak{S}}(E^\eta(\mathfrak{B}),\mathcal{N})\longrightarrow \operatorname{Ctrhom}_{\mathfrak{S}}(\mathfrak{B},R_\eta(\mathcal{N}))$. The similar construction applies to right $E^\eta(\mathfrak{B})$-modules.

Now let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Then the $\mathfrak{S}$-free graded algebra $E^\eta(\mathfrak{B})$ has a natural CDG-algebra structure with the differential and the curvature element induced by the differential and the curvature element of $\mathfrak{B}$. The above constructions E^η, E_η, R^η, R_η for graded $\mathfrak{B}$- and $E^\eta(\mathfrak{B})$-modules assign CDG-modules to CDG-modules, defining DG-functors

$$E^\eta\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}\longrightarrow E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}},$$
$$E_\eta\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\longrightarrow E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}}$$

and

$$R^\eta\colon E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}\longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}},$$
$$R_\eta\colon E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}}\longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}.$$

The DG-functor E^η is left adjoint to the DG-functor R^η, and the DG-functor E_η is right adjoint to the DG-functor R_η. Passing to the homotopy categories, we obtain the induced triangulated functors.

Clearly, the functor R^η takes short exact sequences and infinite products of $\mathfrak{S}$-contramodule CDG-modules to short exact sequences and infinite products of $\mathfrak{R}$-contramodule CDG-modules, hence it takes contraacyclic $\mathfrak{S}$-contramodule CDG-modules to contraacyclic $\mathfrak{R}$-contramodule CDG-modules and therefore induces a triangulated functor

$$\mathbb{I}R^\eta\colon \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}})\longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, the functor R_η takes short exact sequences and infinite direct sums of $\mathfrak{S}$-comodule CDG-modules to short exact sequences and infinite direct sums of $\mathfrak{R}$-comodule CDG-modules, hence it takes coacyclic $\mathfrak{S}$-comodule CDG-modules to coacyclic $\mathfrak{R}$-comodule CDG-modules and induces a triangulated functor

$$\mathbb{I}R^\eta\colon \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}})\longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The functor $E^\eta\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})\longrightarrow H^0(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ takes short exact sequences and infinite products of $\mathfrak{R}$-free CDG-modules to short exact sequences and infinite products of $\mathfrak{S}$-free CDG-modules; hence it takes contraacyclic $\mathfrak{R}$-free

CDG-modules to contraacyclic $\mathfrak{S}$-free CDG-modules and induces a triangulated functor $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. Using Theorem 4.2.1, we obtain the left derived functor

$$\mathbb{L}E^\eta\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}),$$

which is left adjoint to the functor $\mathbb{I}R^\eta$. Similarly, the functor $E_\eta\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ takes coacyclic $\mathfrak{R}$-cofree CDG-modules to coacyclic $\mathfrak{S}$-cofree CDG-modules and induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. Using Theorem 4.2.1, we obtain the right derived functor

$$\mathbb{R}E_\eta\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{B})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}}),$$

which is right adjoint to the functor $\mathbb{I}R_\eta$.

Finally, let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$; then $E^\eta(\mathfrak{A})$ is a wcDG-algebra over $\mathfrak{S}$. Clearly, for any $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, one has $E^\eta(\mathfrak{M})/\mathfrak{m}_\mathfrak{S}E^\eta(\mathfrak{M}) \simeq E^{\eta/\mathfrak{m}}(\mathfrak{M}/\mathfrak{m}_\mathfrak{R}\mathfrak{M})$, and the DG-modules $\mathfrak{M}/\mathfrak{m}_\mathfrak{R}\mathfrak{M}$ over $\mathfrak{A}/\mathfrak{m}_\mathfrak{R}\mathfrak{A}$ and $E^{\eta/\mathfrak{m}}(\mathfrak{M}/\mathfrak{m}_\mathfrak{R}\mathfrak{M})$ over $E^{\eta/\mathfrak{m}}(\mathfrak{A}/\mathfrak{m}\mathfrak{A}) \simeq E^\eta(\mathfrak{A})/\mathfrak{m}_\mathfrak{S}E^\eta(\mathfrak{A})$ are acyclic simultaneously. Hence an $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$ is semiacyclic if and only if the $\mathfrak{S}$-free wcDG-module $E^\eta(\mathfrak{M})$ over $E^\eta(\mathfrak{A})$ is semiacyclic, and the functor E^η induces a triangulated functor $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{fr}})$. Identifying the semiderived categories of $\mathfrak{R}$- or $\mathfrak{S}$-free wcDG-modules with the semiderived categories of $\mathfrak{R}$- or $\mathfrak{S}$-contramodule wcDG-modules, we obtain the left derived functor

$$\mathbb{L}E^\eta\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, for any $\mathfrak{R}$-comodule wcDG-module $\mathcal{M}$ over $\mathfrak{A}$, one has ${}_{\mathfrak{m}_\mathfrak{S}}E_\eta(\mathcal{M}) \simeq E_{\eta/\mathfrak{m}}({}_{\mathfrak{m}_\mathfrak{R}}\mathcal{M})$, hence an $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ over $\mathfrak{A}$ is semiacyclic if and only if the $\mathfrak{S}$-cofree wcDG-module $E^\eta(\mathcal{M})$ over $\mathfrak{A}$ is semiacyclic. The functor E_η induces a triangulated functor $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. Identifying the semiderived categories of $\mathfrak{R}$- or $\mathfrak{S}$-cofree wcDG-modules with the semiderived categories of $\mathfrak{R}$- or $\mathfrak{S}$-comodule wcDG-modules, we obtain the right derived functor

$$\mathbb{R}E_\eta\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

Proposition 5.1.1. *The equivalences of triangulated categories $\mathbb{L}\Phi_\mathfrak{R} = \mathbb{R}\Psi_\mathfrak{R}^{-1}$ and $\mathbb{L}\Phi_\mathfrak{S} = \mathbb{R}\Psi_\mathfrak{S}^{-1}$ from Proposition 4.3.2 transform the left derived functor $\mathbb{L}E^\eta$ into the right derived functor $\mathbb{R}E_\eta$ and back.*

Proof. For any $\mathfrak{R}$-contramodule left wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, the natural morphism of graded $\mathfrak{S}$-comodules $\Phi_\mathfrak{S}E^\eta(\mathfrak{M}) \longrightarrow E_\eta\Phi_\mathfrak{R}(\mathfrak{M})$ is a closed morphism of $\mathfrak{S}$-comodule left wcDG-modules over $E^\eta(\mathfrak{A})$. Similarly, for any $\mathfrak{R}$-comodule left wcDG-module $\mathcal{M}$ over $\mathfrak{A}$, the natural morphism of graded $\mathfrak{S}$-contramodules $E^\eta\Psi_\mathfrak{R}(\mathcal{M}) \longrightarrow \Psi_\mathfrak{S}E_\eta(\mathcal{M})$ is a closed morphism of $\mathfrak{S}$-contramodule left wcDG-modules over $E^\eta(\mathfrak{A})$ (cf. Proposition 1.8.1). ☐

In order to define the functors induced by R^η and R_η on the semiderived categories of wcDG-modules, we will need to prove the following lemma first.

Lemma 5.1.2. (a) *The triangulated functor of contrarestriction of coefficients* $R^\eta\colon H^0(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *takes semiacyclic $\mathfrak{S}$-contramodule wcDG-modules to semiacyclic $\mathfrak{R}$-contramodule wcDG-modules.*

(b) *The triangulated functor corestriction of coefficients* $R_\eta\colon H^0(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *takes semiacyclic $\mathfrak{S}$-comodule wcDG-modules to semiacyclic $\mathfrak{R}$-comodule wcDG-modules.*

Proof. Part (a): in view of the semiorthogonal decomposition of Theorem 4.3.1(a, c) and the adjunction of R^η and E^η, the desired assertion is equivalent to the functor $E^\eta\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}})$ taking $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}_{\mathsf{proj}})_{\mathsf{proj}}$ to $H^0(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}_{\mathsf{proj}})_{\mathsf{proj}}$. It is clear that the functor E^η takes $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}_{\mathsf{proj}})$ to $H^0(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}_{\mathsf{proj}})$, and it remains to use Lemma 2.3.1(a) in order to check the preservation of homotopy projectivity (which is obviously preserved by the functor $E^{\eta/\mathfrak{m}}$). The proof of part (b) is analogous up to duality. ☐

According to Lemma 5.1.2, the functor R^η induces a triangulated functor of contrarestriction of coefficients

$$\mathbb{I}R^\eta\colon \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, the functor R_η induces a triangulated functor of corestriction of coefficients

$$\mathbb{I}R_\eta\colon \mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The functor $\mathbb{I}R^\eta$ is right adjoint to the functor $\mathbb{L}E^\eta$, and the functor $\mathbb{I}R_\eta$ is left adjoint to the functor $\mathbb{R}E_\eta$. In view of Proposition 5.1.1, identifying $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ with $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ and $\mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{ctr}})$ with $\mathsf{D}^{\mathsf{si}}(E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{S}\text{–}\mathsf{co}})$ allows one to view the functors $\mathbb{I}R^\eta$ and $\mathbb{I}R_\eta$ as the adjoints on two sides to the same triangulated functor $\mathbb{L}E^\eta = \mathbb{R}E_\eta$.

5.2. **Change of coefficients for CDG-contra/comodules.** Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. Then the free graded $\mathfrak{S}$-contramodule $E^\eta(\mathfrak{C})$ has a natural graded coalgebra structure provided by the comultiplication map $E^\eta(\mathfrak{C}) \longrightarrow E^\eta(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{C}) \simeq E^\eta(\mathfrak{C})\otimes^{\mathfrak{S}}E^\eta(\mathfrak{C})$ induced by the comultiplication in $\mathfrak{C}$ and the counit map $E^\eta(\mathfrak{C}) \longrightarrow \mathfrak{S}$ induced by the counit map $\mathfrak{C} \longrightarrow \mathfrak{R}$.

Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule. Then the graded $\mathfrak{S}$-contramodule $E^\eta(\mathfrak{P})$ has a natural left $E^\eta(\mathfrak{C})$-contramodule structure provided by the contraaction map $\operatorname{Hom}^{\mathfrak{S}}(E^\eta(\mathfrak{C}), E^\eta(\mathfrak{P})) \simeq E^\eta \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P}) \longrightarrow E^\eta(\mathfrak{P})$.

Let $\mathcal{P}$ be an $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule. Then the cofree graded $\mathfrak{S}$-comodule $E_\eta(\mathcal{P})$ has a natural left $E^\eta(\mathfrak{C})$-contramodule structure provided by the contraaction map $\operatorname{Ctrhom}^{\mathfrak{S}}(E^\eta(\mathfrak{C}), E_\eta(\mathcal{P})) \simeq E_\eta \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P}) \longrightarrow E_\eta(\mathcal{P})$.

Let $\mathcal{M}$ be an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule. Then the graded $\mathfrak{S}$-comodule $E_\eta(\mathcal{M})$ has a natural left $E^\eta(\mathfrak{C})$-comodule structure provided by the coaction map $E_\eta(\mathcal{M}) \longrightarrow E_\eta(\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}) \simeq E^\eta(\mathfrak{C})\odot_{\mathfrak{S}} E_\eta(\mathcal{M})$. The similar construction applies to right $\mathfrak{C}$-comodules.

Let $\mathfrak{M}$ be an $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule. Then the free graded $\mathfrak{S}$-contramodule $E^\eta(\mathfrak{M})$ has a natural left $E^\eta(\mathfrak{C})$-comodule structure provided by the coaction map

$E^\eta(\mathfrak{M}) \longrightarrow E^\eta(\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}) \simeq E^\eta(\mathfrak{C})\otimes^{\mathfrak{S}} E^\eta(\mathfrak{M})$. The similar construction applies to right $\mathfrak{C}$-comodules.

Let $\mathfrak{Q}$ be an $\mathfrak{S}$-contramodule graded left $E^\eta(\mathfrak{C})$-contramodule. Then the graded $\mathfrak{R}$-contramodule $R^\eta(\mathfrak{Q})$ has a natural graded left $\mathfrak{C}$-contramodule structure provided by the contraaction map $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, R^\eta(\mathfrak{Q})) \simeq R^\eta \operatorname{Hom}^{\mathfrak{S}}(E^\eta(\mathfrak{C}),\mathfrak{Q}) \longrightarrow R^\eta(\mathfrak{Q})$.

Let $\mathcal{N}$ be an $\mathfrak{S}$-comodule graded left $E^\eta(\mathfrak{C})$-comodule. Then the graded $\mathfrak{R}$-comodule $R_\eta(\mathcal{N})$ has a natural graded left $\mathfrak{C}$-comodule structure provided by the coaction map $R_\eta(\mathcal{N}) \longrightarrow R_\eta(E^\eta(\mathfrak{C})\odot_{\mathfrak{S}}\mathcal{N}) \simeq \mathfrak{C}\odot_{\mathfrak{R}} R_\eta(\mathcal{N})$. The similar construction applies to right $E^\eta(\mathfrak{C})$-comodules.

Now let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then the $\mathfrak{S}$-free graded coalgebra $E^\eta(\mathfrak{C})$ has a natural CDG-coalgebra structure with the differential and the curvature linear function induced by the differential and the curvature linear function of $\mathfrak{C}$. The above constructions E^η, E_η, R^η, R_η assign CDG-contra/comodules to CDG-contra/comodules, defining DG-functors

$$\begin{aligned} E^\eta\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}} &\longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{ctr}},\\ E_\eta\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}} &\longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{cof}},\\ E_\eta\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}} &\longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{co}},\\ E^\eta\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}} &\longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{fr}} \end{aligned}$$

and

$$\begin{aligned} R^\eta\colon E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{ctr}} &\longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}},\\ R_\eta\colon E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{co}} &\longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}. \end{aligned}$$

The DG-functor $E^\eta\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}} \longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{ctr}}$ is left adjoint to the DG-functor R^η, and the DG-functor $E_\eta\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}} \longrightarrow E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{co}}$ is right adjoint to the DG-functor R_η. Passing to the homotopy categories, we obtain the induced triangulated functors.

Clearly, the functor R^η takes short exact sequences and infinite products of $\mathfrak{S}$-contramodule CDG-contramodules to short exact sequences and infinite products of $\mathfrak{R}$-contramodule CDG-contramodules, hence it takes contraacyclic $\mathfrak{S}$-contramodule CDG-contramodules to contraacyclic $\mathfrak{R}$-contramodule CDG-contramodules and induces a triangulated functor

$$\mathbb{I}R^\eta\colon \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

Similarly, the functor R_η takes coacyclic $\mathfrak{S}$-comodule CDG-comodules to coacyclic $\mathfrak{R}$-comodule CDG-comodules and therefore induces a triangulated functor

$$\mathbb{I}R_\eta\colon \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}).$$

The functor $E^\eta\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{fr}})$ takes short exact sequences and infinite products of $\mathfrak{R}$-free CDG-contramodules to short exact sequences and infinite products of $\mathfrak{S}$-free CDG-contramodules; hence it takes contraacyclic $\mathfrak{R}$-free CDG-contramodules to contraacyclic $\mathfrak{S}$-free CDG-contramodules and induces a triangulated functor $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{fr}})$. Using Theorem 4.5.1, we obtain the left derived functor

$$\mathbb{L}E^\eta\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{ctr}}).$$

Similarly, the functor $E_\eta\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ takes coacyclic $\mathfrak{R}$-cofree CDG-comodules to coacyclic $\mathfrak{S}$-cofree CDG-comodules and induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{cof}})$. Using Theorem 4.5.1, we obtain the right derived functor

$$\mathbb{R}E_\eta\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{co}}).$$

The functor $E_\eta\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{cof}})$ takes short exact sequences and infinite products of $\mathfrak{R}$-cofree CDG-contramodules to short exact sequences and infinite products of $\mathfrak{S}$-cofree CDG-contramodules. Hence it takes contraacyclic $\mathfrak{R}$-cofree CDG-contramodules to contraacyclic $\mathfrak{S}$-cofree CDG-contramodules and induces a triangulated functor

$$\mathbb{I}E_\eta\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(E^\eta(\mathfrak{C})\text{–}\mathsf{contra}^{\mathfrak{S}\text{–}\mathsf{cof}}).$$

Similarly, the functor $E^\eta\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ takes coacyclic $\mathfrak{R}$-free CDG-modules to coacyclic $\mathfrak{S}$-free CDG-modules and induces a triangulated functor

$$\mathbb{I}E^\eta\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(E^\eta(\mathfrak{C})\text{–}\mathsf{comod}^{\mathfrak{S}\text{–}\mathsf{fr}}).$$

Proposition 5.2.1. (a) *The equivalences of triangulated categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ and $\Phi_{\mathfrak{S}} = \Psi_{\mathfrak{S}}^{-1}$ from Section* 3.4 *together with the equivalences of categories from Theorem* 4.5.1 *transform the left derived functor $\mathbb{L}E^\eta$ into the induced functor $\mathbb{I}E_\eta$ and the induced functor $\mathbb{I}E_\eta$ into the right derived functor $\mathbb{R}E_\eta$.*

(b) *The equivalences of triangulated categories $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ and $\mathbb{L}\Phi_{E^\eta(\mathfrak{C})} = \mathbb{R}\Psi_{E^\eta(\mathfrak{C})}^{-1}$ from Corollaries* 3.2.4 *and* 3.4.2 *together with the equivalence of categories from Theorem* 4.5.1 *transform the left derived functor $\mathbb{L}E^\eta$ into the induced functor $\mathbb{I}E^\eta$ and the right derived functor $\mathbb{R}E_\eta$ into the induced functor $\mathbb{I}E_\eta$.*

(c) *The equivalences of triangulated categories $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ and $\mathbb{L}\Phi_{\mathfrak{S},E^\eta(\mathfrak{C})} = \mathbb{R}\Psi_{\mathfrak{S},E^\eta(\mathfrak{C})}^{-1}$ from Corollary* 4.5.3 *transform the left derived functor $\mathbb{L}E^\eta$ into the right derived functor $\mathbb{R}E_\eta$.*

Proof. Part (c): Notice that the functor E_η takes the $\mathfrak{R}$-cofree graded $\mathfrak{C}$-bicomodule $\mathcal{C}(\mathfrak{R},\mathfrak{C})$ to the $\mathfrak{S}$-cofree graded $E^\eta(\mathfrak{C})$-bicomodule $\mathcal{C}(\mathfrak{S},E^\eta(\mathfrak{C}))$. Furthermore, for any $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{P}$ over $\mathfrak{C}$ there is a natural closed morphism of $\mathfrak{S}$-comodule left CDG-comodules $\Phi_{\mathfrak{S},E^\eta(\mathfrak{C})}E^\eta(\mathfrak{P}) \longrightarrow E_\eta\Phi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{P})$ over $E^\eta(\mathfrak{C})$, which is an isomorphism whenever $\mathfrak{P}$ is a projective $\mathfrak{R}$-free graded $\mathfrak{C}$-contramodule. Similarly, for any $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{M}$ over $\mathfrak{C}$ there is a natural closed morphism of $\mathfrak{S}$-contramodule left CDG-contramodules $E^\eta\Psi_{\mathfrak{R},\mathfrak{C}}(\mathcal{M}) \longrightarrow \Psi_{\mathfrak{S},E^\eta(\mathfrak{C})}E_\eta(\mathcal{M})$ over $E^\eta(\mathfrak{C})$, which is an isomorphism whenever $\mathcal{M}$ is an injective $\mathfrak{R}$-cofree graded $\mathfrak{C}$-comodule. $\square$

5.3. **Compact generator for wcDG-modules.** Denote by $\kappa\colon \mathfrak{R} \longrightarrow k$ the natural surjection from a pro-Artinian topological local ring $\mathfrak{R}$ to its residue field k. Given a CDG-algebra B over k, we denote by $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$, respectively, the contraderived and coderived category of left CDG-modules over B.

Theorem 5.3.1. *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra. Then*

(a) *the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ is generated, as a triangulated category with infinite products, by the image of the triangulated functor $\mathbb{I}R^{\kappa}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}/\mathfrak{m}\mathfrak{B}\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$;*

(b) *the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ is generated, as a triangulated category with infinite direct sums, by the image of the triangulated functor $\mathbb{I}R_{\kappa}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{B}/\mathfrak{m}\mathfrak{B}\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$.*

Proof. Part (a): by Theorem 4.2.1, any object of $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ can be represented by an $\mathfrak{R}$-free left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$. For any $n \geqslant 0$, denote by $\overline{\mathfrak{m}^n} \subset \mathfrak{R}$ the topological closure of the n-th power of the ideal $\mathfrak{m} \subset \mathfrak{R}$; so we have $\mathfrak{R} = \overline{\mathfrak{m}^0} \supset \mathfrak{m} = \overline{\mathfrak{m}^1} \supset \overline{\mathfrak{m}^2} \supset \overline{\mathfrak{m}^3} \supset \cdots$ Applying to $\mathfrak{M}$ the contraextension and subsequently the contrarestriction of scalars for the morphism of pro-Artinian local rings $\mathfrak{R} \longrightarrow \mathfrak{R}/\overline{\mathfrak{m}^n}$, we obtain a sequence of closed morphisms of $\mathfrak{R}$-contramodule CDG-modules $\mathfrak{M}/\mathfrak{m}\mathfrak{M} \longleftarrow \mathfrak{M}/\overline{\mathfrak{m}^2}\mathfrak{M} \longleftarrow \mathfrak{M}/\overline{\mathfrak{m}^3}\mathfrak{M} \longleftarrow \cdots$ over $\mathfrak{B}$. Since $\mathfrak{M}$ is $\mathfrak{R}$-free and $\mathfrak{m}$ is topologically nilpotent, the projective limit of this sequence coincides with $\mathfrak{M}$. Since the contraaction morphism $\mathfrak{m}[[\overline{\mathfrak{m}^n}\mathfrak{M}]] \longrightarrow \mathfrak{M}$ lands in $\overline{\mathfrak{m}^{n+1}}\mathfrak{M}$, all the kernels of closed morphisms of CDG-modules in our sequence have their graded $\mathfrak{R}$-contramodule structures obtained by the contrarestriction of scalars from k-vector space structures, i. e., they belong to the image of R^{κ}. Hence the CDG-modules $\mathfrak{M}/\overline{\mathfrak{m}^n}\mathfrak{M}$ belong to the triangulated subcategory in $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ generated by the image of R^{κ}. It remains to notice that the telescope short sequence $0 \longrightarrow \mathfrak{M} \longrightarrow \prod_n \mathfrak{M}/\overline{\mathfrak{m}^n}\mathfrak{M} \longrightarrow \prod_n \mathfrak{M}/\overline{\mathfrak{m}^n}\mathfrak{M} \longrightarrow 0$ is exact, since it is exact as a sequence of abelian groups (the morphisms $\mathfrak{M}/\overline{\mathfrak{m}^{n+1}}\mathfrak{M} \longrightarrow \mathfrak{M}/\overline{\mathfrak{m}^n}\mathfrak{M}$ being surjective and the forgetful functor from $\mathfrak{R}\text{–}\mathsf{contra}$ to abelian groups commuting with the infinite products).

The proof of part (b) is analogous up to duality (and even somewhat simpler). ☐

Given a DG-algebra A over the field k, we denote by $\mathsf{D}(A\text{–}\mathsf{mod})$ the (conventional) derived category of left DG-modules over A.

Corollary 5.3.2. *Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$. Then*

(a) *the semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ is generated, as a triangulated category with infinite products, by the image of the triangulated functor $\mathbb{I}R^{\kappa}\colon \mathsf{D}(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$;*

(b) *the semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ is generated, as a triangulated category with infinite direct sums, by the image of the triangulated functor $\mathbb{I}R_{\kappa}\colon \mathsf{D}(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$.*

Proof. Follows from Theorem 5.3.1. ☐

Any DG-algebra A over a field k can be also considered, at one's choice, as a left or a right DG-module over itself. We will denote this DG-module simply by A.

Theorem 5.3.3. *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the $\mathfrak{R}$-comodule left wcDG-module $\mathbb{I}R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A})$ over $\mathfrak{A}$ is a compact generator of the semiderived category of $\mathfrak{R}$-comodule left wcDG-modules $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$.*

Proof. Let us show that $\mathbb{I}R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A}) \in \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ is a compact object. Since the DG-module $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is a compact object of $\mathsf{D}(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod})$ and the functor $\mathbb{I}R_\kappa$ is left adjoint to the functor $\mathbb{R}E_\kappa$, it suffices to check that the functor $\mathbb{R}E_\kappa$ preserves infinite direct sums. The latter assertion is true for any profinite morphism $\eta\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ in place of κ. It suffices to identify $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ with $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and notice that the DG-functor $E_\eta\colon \mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow E^\eta(\mathfrak{A})\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ preserves infinite direct sums, as does the localization functor $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. (Similarly, the functor $\mathbb{L}E^\eta$ preserves infinite products.)

More explicitly, the object $\mathbb{I}R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A}) \in \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ represents the functor assigning to an $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ over $\mathfrak{A}$ the degree-zero cohomology group of the DG-module ${}_{\mathfrak{m}}\mathcal{M}$ over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. By the definition of the semiderived category of $\mathfrak{R}$-cofree wcDG-modules, an object $\mathcal{M}$ annihilated, together with all of its shift, by this cohomological functor, vanishes in $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. Alternatively, one can use Theorem 5.3.1(b) together with the fact that the DG-module $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ generates $\mathsf{D}(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod})$ in order to show that $\mathbb{I}R_\eta(\mathfrak{A}/\mathfrak{m}\mathfrak{A})$ generates $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$. ☐

Corollary 5.3.4. *For any wcDG-algebra $\mathfrak{A}$ over $\mathfrak{R}$, the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ of $\mathfrak{R}$-contramodule left wcDG-modules over $\mathfrak{A}$ has a single compact generator. So do the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ of $\mathfrak{R}$-free left wcDG-modules over $\mathfrak{A}$ and the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ of $\mathfrak{R}$-cofree left wcDG-modules over $\mathfrak{A}$.*

Proof. All the three categories are naturally equivalent to the one whose compact generator was constructed in Theorem 5.3.3. Use the construction of Theorem 4.2.1 to obtain the object of $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ corresponding to our $\mathfrak{R}$-comodule wcDG-module $\mathbb{I}R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A})$, and the construction of the functor $\Psi_{\mathfrak{R}}$ from Sections 2.4–2.6 and Proposition 4.3.2 to obtain the corresponding object of $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. ☐

Example 5.3.5. Let $\mathfrak{R}$ be a pro-Artinian topological local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. Let ϵ be an element of $\mathfrak{m} \setminus \overline{\mathfrak{m}^2} \subset \mathfrak{R}$. Then there exists an open ideal $\overline{\mathfrak{m}^2} \subset \mathfrak{J} \subset \mathfrak{R}$ such that the quotient ring $S = \mathfrak{R}/\mathfrak{J}$ is a module of length 2 over itself and has its maximal ideal m generated by the class $0 \neq \bar\epsilon \in S$ of the element ϵ. Denote the natural surjections between our local rings by $\eta\colon \mathfrak{R} \longrightarrow S$, $\sigma\colon S \longrightarrow k$, and $\kappa\colon \mathfrak{R} \longrightarrow k$.

Consider the $\mathfrak{R}$-free graded algebra $\mathfrak{A}$ with the components $\mathfrak{A}^i = \mathfrak{R}$ for all i divisible by 2 and $\mathfrak{A}^i = 0$ otherwise, the multiplication maps in $\mathfrak{A}$ being the identity maps $\mathfrak{R} \otimes^{\mathfrak{R}} \mathfrak{R} = \mathfrak{R} \longrightarrow \mathfrak{R}$. Let $x \in \mathfrak{A}^2$ denote the element corresponding to $1 \in \mathfrak{R}$ (so $\mathfrak{A} = \mathfrak{R}[x, x^{-1}]$ if 2 has an infinite order in the grading group Γ and $\mathfrak{A} = \mathfrak{R}[x]/(x^n - 1)$ if $2 \in \Gamma$ is an element of order n; in particular, $\mathfrak{A} = \mathfrak{R}$ and $x = 1$ if $2 = 0$ in Γ). Define the wcDG-algebra structure on $\mathfrak{A}$ with $d = 0$ and $h = \epsilon x$. Set $B = E^\eta(\mathfrak{A})$.

It was noticed in [22, proof of Proposition 3.7] that the S-contramodule wcDG-module $R^\sigma(B/mB)$ (or, which is essentially the same, the S-comodule wcDG-module

$R_\sigma(B/mB))$ over B is absolutely acyclic. Indeed, consider the algebra B as an S-contra/comodule graded module over itself and apply the construction G^+ of the freely generated wcDG-module (see the proof of Theorem 2.2.4) to it.

Taking the tensor product of the exact triple of S-modules $k \longrightarrow S \longrightarrow k$ with the algebra B over S, we obtain an exact triple of S-contra/comodule graded B-modules $B/mB \longrightarrow B \longrightarrow B/mB$; applying the construction G^+, we get an exact triple of S-contra/comodule CDG-modules and closed morphisms $G^+(B/mB) \longrightarrow G^+(B) \longrightarrow G^+(B/mB)$. Since the quotient module B/mB has a natural structure of S-contra/comodule CDG-module over B, there is a natural closed morphism of CDG-modules $G^+(B/mB) \longrightarrow B/mB$; hence the induced exact triple of CDG-modules and closed morphisms $B/mB \longrightarrow M \longrightarrow G^+(B/mB)$.

The closed morphism $B/mB \longrightarrow M$ is homotopic to zero, the contracting homotopy being given by the rule $b \longmapsto x^{-1}d_G(b)$. The CDG-modules $G^+(L)$ being always contractible (see [31, proof of Theorem 1.4]), it follows that both the S-contra/comodule CDG-modules B/mB and M are absolutely acyclic. Consequently, the $\mathfrak{R}$-contramodule wcDG-module $R^\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A}) = R^\eta R^\sigma(B/mB)$ and the $\mathfrak{R}$-comodule wcDG-module $R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A}) = R_\eta R_\sigma(B/mB)$ are absolutely acyclic, too.

Notice that the graded algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is a "graded field", i. e., every graded module over it is free. Since the differential on $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is zero, it follows easily (cf. [22, proof of Proposition 5.10]) that every acyclic DG-module over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is contractible. Consequently, the DG-module $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ generates the homotopy category $H^0(\mathfrak{A}/\mathfrak{m}\mathfrak{A}\text{–}\mathsf{mod})$ considered either as a triangulated category with infinite direct sums, or as a triangulated category with infinite products. By Theorem 5.3.1, it follows that both the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ vanish. Thus $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) = 0 = \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ (cf. Example 6.6.1 below).

Furthermore, every short exact sequence of $\mathfrak{R}$-free graded modules over $\mathfrak{A}$ splits, as does every short exact sequence of $\mathfrak{R}$-cofree graded modules; so $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) = H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) = H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$. Using Theorem 4.2.1, we conclude that $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) = 0 = H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$.

In particular, the wcDG-algebra morphisms like $\mathfrak{A} \longrightarrow 0$ or $\mathfrak{A} \longrightarrow \mathfrak{A} \oplus \mathfrak{A}$, etc., induce equivalences of the semiderived categories of wcDG-modules, while not being quasi-isomorphisms modulo $\mathfrak{m}$ at all (cf. Remark 2.3.6).

Remark 5.3.6. The notion of compactness in application to the triangulated categories we are dealing with in this paper in inherently ambigous, because these categories (and their DG-enhancements) can be naturally viewed as being enriched over $\mathfrak{R}$-contramodules. The problem is that the definition of compactness involves considering infinite direct sums of the groups/modules of morphisms in the category, and the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ does not preserve infinite direct sums.

In the above discussion, as indeed everywhere in this paper, we presume the conventional notion of compactness of triangulated categories with abelian groups of morphisms (so the contramodule enrichment is ignored). The following example illustrates the difference.

Let $\mathfrak{A}$ be an $\mathfrak{R}$-free DG-algebra (i. e., a wcDG-algebra with $h = 0$); consider $\mathfrak{A}$ as a left (wc)DG-module over itself. Then the $\mathfrak{R}$-cofree wcDG-module $\mathbb{I}R_\kappa(\mathfrak{A}/\mathfrak{m}\mathfrak{A})$ over $\mathfrak{A}$ has a right resolution by direct sums of copies of $\Phi_{\mathfrak{R}}(\mathfrak{A})$, so $\mathfrak{A}$ generates $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ as a triangulated category with infinite direct sums. Besides, for any $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, the complex $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A},\mathfrak{M})$ computes Hom in the semiderived category of wcDG-modules (see Lemma 2.3.1 and Theorem 4.3.1); this complex also coincides with the complex of $\mathfrak{R}$-contramodules underlying the DG-module $\mathfrak{M}$. Suppose $\mathfrak{R}$ has finite homological dimension; then a wcDG-module $\mathfrak{M}$ is (semi)acyclic whenever the the complex $\mathfrak{M} = \mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A},\mathfrak{M})$ is acyclic.

Furthermore, the functor $\mathrm{Hom}_{\mathfrak{A}}(\mathfrak{A},-)$ transforms infinite direct sums in $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ (represented by infinite direct sums in $H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}_{\mathsf{proj}})$; cf. the definition of functor $\mathrm{Ext}_{\mathfrak{A}}$ in Section 4.3) into infinite direct sums in the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ (represented by infinite direct sums in $H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{proj}})$). When $\mathfrak{R}$ has homological dimension 1, infinite direct sums in $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–}\mathsf{contra})$ even commute with the passage to the $\mathfrak{R}$-contramodules of cohomology (see Remark 1.2.1 and Section 1.9). Still, these do not commute with the forgetful functor to $\mathfrak{R}\text{–}\mathsf{mod}$, and so the wcDG-module $\mathfrak{A}$ over $\mathfrak{A}$ is *not* compact in our sense (as one can see already in the simplest case $\mathfrak{A} = \mathfrak{R} = k[[\epsilon]]$).

5.4. **Compact generators for CDG-co/contramodules.** Given a CDG-coalgebra C over the field k, we denote by $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ and $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, respectively, the contraderived category of left CDG-contramodules and the coderived category of left CDG-comodules over C.

Theorem 5.4.1. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ is generated, as a triangulated category with infinite products, by the image of the triangulated functor $\mathbb{I}R^\kappa\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}/\mathfrak{m}\mathfrak{C}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$;*

(b) *the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ is generated, as a triangulated category with infinite direct sums, by the image of the triangulated functor $\mathbb{I}R_\kappa\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}/\mathfrak{m}\mathfrak{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$.*

Proof. Similar to the proof of Theorem 5.3.1. ☐

Lemma 5.4.2. (a) *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra. Then any $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodule is the union of its $\mathfrak{R}$-comodule graded $\mathfrak{C}$-subcomodules that have finite length as graded $\mathfrak{R}$-comodules (in particular, these have a finite number of nonzero grading components only).*

(b) *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then any $\mathfrak{R}$-comodule CDG-comodule over $\mathfrak{C}$ is the union of its $\mathfrak{R}$-comodule CDG-subcomodules whose underlying graded $\mathfrak{R}$-comodules have finite length.*

Proof. Part (a): the key observation is that the functor of contratensor product of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules $\odot_{\mathfrak{R}}$ commutes with the inductive limits in the comodule argument. In addition, filtered inductive limits are exact in $\mathfrak{R}\text{–}\mathsf{comod}$, as is the functor $\mathfrak{C}\odot_{\mathfrak{R}}-$. Let $\mathcal{M}$ be an $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule; pick an $\mathfrak{R}$-subcomodule of finite length $\mathcal{V}\subset\mathcal{M}$ and consider the full preimage $\mathcal{L}\subset\mathcal{M}$ of $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}\subset\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ under the $\mathfrak{C}$-coaction map $\mathcal{M}\longrightarrow\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$. It follows from the counit axiom for $\mathcal{M}$ that $\mathcal{L}$ is contained in $\mathcal{V}$, hence $\mathcal{L}$ is an $\mathfrak{R}$-comodule of finite length. Furthermore, $\mathcal{L}$ is a $\mathfrak{C}$-subcomodule in $\mathcal{M}$, because the $\mathfrak{C}$-coaction map $\mathcal{M}\longrightarrow\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ is a $\mathfrak{C}$-comodule morphism (the coassociativity axiom for the coaction) and $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$ is a $\mathfrak{C}$-subcomodule in $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$. Finally, $\mathcal{M}$ is the filtered inductive limit of its $\mathfrak{C}$-subcomodules $\mathcal{L}$ indexed by all the $\mathfrak{R}$-subcomodules $\mathcal{V}\subset\mathcal{M}$ of finite length, since $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ is the inductive limit of $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$ and inductive limits commute with fibered products in $\mathfrak{R}\text{–}\mathsf{comod}$.

Part (b): Let $\mathcal{M}$ be a left CDG-comodule over $\mathfrak{C}$ and $\mathcal{L}\subset\mathcal{M}$ be a graded $\mathfrak{C}$-subcomodule having finite length as a graded $\mathfrak{R}$-comodule. Then $\mathcal{L}+d_{\mathcal{M}}(\mathcal{L})\subset\mathcal{M}$ is a CDG-subcomodule of $\mathcal{M}$ with the same property. $\square$

It follows from Lemma 5.4.2 that having finite length as a graded $\mathfrak{R}$-comodule or as a graded $\mathfrak{C}$-comodule (or even as a CDG-comodule over $\mathfrak{C}$) are equivalent properties for an $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodule (or an $\mathfrak{R}$-comodule CDG-comodule over $\mathfrak{C}$). Therefore, we will simply call the graded comodules (resp., CDG-comodules) with this property the $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodules (resp., CDG-comodules over $\mathfrak{C}$) *of finite length*.

The DG-subcategory of $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}$ formed by the CDG-comodules of finite length will be denoted by $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}}$; the corresponding homotopy category is $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}})$. The quotient category of $H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}})$ by its minimal thick subcategory containing the total CDG-comodules of short exact sequences of CDG-comodules of finite length and closed morphisms between them is called the *absolute derived category* of $\mathfrak{R}$-comodule left CDG-comodules of finite length over $\mathfrak{C}$ and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}})$.

Theorem 5.4.3. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *the triangulated functor $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}})\longrightarrow\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}})$ induced by the embedding of DG-categories $\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}_{\mathsf{fin}}\longrightarrow\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}}$ is fully faithful; and*

(b) *the image of this functor (or, more precisely, a set of representatives of the isomorphisms classes in the image) is a set of compact generators of the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}})$.*

Proof. The proof of part (a) is similar to that of [29, Theorem 3.11.1]. There are two ways to prove part (b): either one can use the general argument from [29, proof of Theorem 3.11.2] (due to D. Arinkin), or alternatively the assertion can be deduced, using Theorem 5.4.1(b), from the similar result for CDG-comodules over CDG-coalgebras over fields [29, Section 5.5]. $\square$

Corollary 5.4.4. *For any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})$ of $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ is compactly generated. So are the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–fr}})$ and $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–cof}})$ and the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})$.*

Proof. All the five mentioned categories are naturally equivalent to the one whose compact generators were constructed in Theorem 5.4.3. See Theorem 4.5.1 and Corollaries 3.2.4, 3.4.2, and 4.5.3. $\square$

6. Bar and Cobar Duality

6.1. **Bar- and cobar-constructions.** The bar-construction for nonaugmented $\mathfrak{R}$-free CDG-algebras and the cobar-construction for noncoaugmented $\mathfrak{R}$-free CDG-coalgebras are based on the following lemma.

Lemma 6.1.1. (a) *If $\mathfrak{B}$ is a nonzero $\mathfrak{R}$-free graded algebra, then the unit map $\mathfrak{R} \longrightarrow \mathfrak{B}$ is the embedding of a direct summand in the category of free graded $\mathfrak{R}$-contramodules.*

(b) *If $\mathfrak{C}$ is a nonzero $\mathfrak{R}$-free graded coalgebra, then the counit map $\mathfrak{C} \longrightarrow \mathfrak{R}$ is the projection onto a direct summand in the category of free graded $\mathfrak{R}$-contramodules.*

Proof. Part (a): reducing the unit map $\mathfrak{R} \longrightarrow \mathfrak{B}$ modulo $\mathfrak{m}$, we obtain the unit map $k \longrightarrow \mathfrak{B}/\mathfrak{m}\mathfrak{B}$ of the graded k-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$. If the latter map is zero, it follows that $\mathfrak{B}/\mathfrak{m}\mathfrak{B} = 0$ and $\mathfrak{B} = 0$. Otherwise, pick a homogeneous k-linear map $\bar v\colon \mathfrak{B}/\mathfrak{m}\mathfrak{B} \longrightarrow k$ such that the composition $k \longrightarrow \mathfrak{B}/\mathfrak{m}\mathfrak{B} \longrightarrow k$ is the identity map, and lift $\bar v$ to a homogeneous morphism of graded $\mathfrak{R}$-contramodules $v\colon \mathfrak{B} \longrightarrow \mathfrak{R}$. Then the composition $\mathfrak{R} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{R}$ is invertible (see the proof of Lemma 1.3.3).

Part (b): reducing the counit map $\mathfrak{C} \longrightarrow \mathfrak{R}$ modulo $\mathfrak{m}$, we obtain the counit map $\mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow k$ of the graded coalgebra $\mathfrak{C}$ over k. If the latter map is zero, it follows that $\mathfrak{C}/\mathfrak{m}\mathfrak{C} = 0$ and $\mathfrak{C} = 0$. Otherwise, pick a homogeneous k-linear map $\bar w\colon k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C}$ such that the composition $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow k$ is the identity map, and continue to argue as above. $\square$

Let $\mathfrak{U}$ be a free graded $\mathfrak{R}$-contramodule. Then the infinite direct sum of tensor powers $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ in the category of free graded $\mathfrak{R}$-contramodules has a natural structure of $\mathfrak{R}$-free graded algebra with the multiplication given by the conventional rule $(u_1 \otimes \dots \otimes u_j)(u_{j+1} \otimes \dots \otimes u_n) = u_1 \otimes \dots \otimes u_j \otimes u_{j+1} \otimes \dots \otimes u_n$ and the unit element provided by the embedding of the component $\mathfrak{R} = \mathfrak{U}^{\otimes 0} \longrightarrow \bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$. The same infinite direct sum of tensor powers also has a natural $\mathfrak{R}$-free graded coalgebra structure with the comultiplication $u_1 \otimes \dots \otimes u_n \longmapsto \sum_{j=0}^n (u_1 \otimes \dots \otimes u_j) \otimes (u_{j+1} \otimes \dots \otimes u_n)$ and the counit map being the projection onto the component $\mathfrak{U}^{\otimes 0} = \mathfrak{R}$.

Lemma 6.1.2. (a) *Odd derivations of degree 1 on the $\mathfrak{R}$-free graded algebra $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ are determined by their restrictions to the component $\mathfrak{U}^{\otimes 1} \simeq \mathfrak{U}$. Conversely, any homogeneous $\mathfrak{R}$-contramodule morphism $\mathfrak{U} \longrightarrow \bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ of degree 1 gives rise to an odd derivation of degree 1 on $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$.*

(b) *Odd coderivations of degree 1 on the $\mathfrak{R}$-free graded coalgebra $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ are determined by their projections to the component $\mathfrak{U}^{\otimes 1} \simeq \mathfrak{U}$. Conversely, any homogeneous $\mathfrak{R}$-contramodule morphism $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \mathfrak{U}$ of degree 1 gives rise to an odd coderivation of degree 1 on $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$.*

Proof. Straightforward and similar to the graded k-(co)algebra case. In the case (b), it is essential that the natural map $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \prod_{n=0}^\infty \mathfrak{U}^{\otimes n}$ is injective (since $\mathfrak{U}$ is a *free* graded $\mathfrak{R}$-contramodule) and no coderivation of $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ can raise the tensor degree by more than 1 (since no map $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \mathfrak{U}$ does). $\square$

A graded coalgebra D without counit over a field k is called *conilpotent* if it is the union $D = \bigcup_n \ker(D \to D^{\otimes n+1})$ of the kernels of the iterated comultiplication maps. A graded coalgebra C over k endowed with a coaugmentation (morphism of coalgebras) $\bar{w}\colon k \longrightarrow C$ is called *conilpotent* if the graded coalgebra without counit $D/\bar{w}(k)$ is conilpotent. One can easily see that a conilpotent graded coalgebra has a unique coaugmentation.

The graded tensor coalgebra $\bigoplus_{n=0}^\infty U^{\otimes n}$ over k is conilpotent for any graded vector space U. For the reasons that are clear from the following lemma, this coalgebra can be called the *conilpotent graded coalgebra cofreely cogenerated by* the graded vector space U. More generally, the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n \mathfrak{U}^{\otimes n}$ is the $\mathfrak{R}$-free graded coalgebra with conilpotent reduction modulo $\mathfrak{m}$ cofreely cogenerated by the free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$.

On the other hand, the $\mathfrak{R}$-free graded algebra $\bigoplus_n \mathfrak{U}^{\otimes n}$ is freely generated, just as an $\mathfrak{R}$-free graded algebra, by the free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$.

Lemma 6.1.3. (a) *Let $\mathfrak{B}$ be an $\mathfrak{R}$-free graded algebra and $\mathfrak{U}$ be a free graded $\mathfrak{R}$-contramodule. Then morphisms of $\mathfrak{R}$-free graded algebras $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \mathfrak{B}$ are determined by their restrictions to the component $\mathfrak{U}^{\otimes 1} \simeq \mathfrak{U}$. Conversely, any homogeneous $\mathfrak{R}$-contramodule morphism $\mathfrak{U} \longrightarrow \mathfrak{B}$ of degree* 0 *gives rise to a morphism of $\mathfrak{R}$-free graded algebras $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \mathfrak{B}$.*

(b) *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra and $\mathfrak{U}$ be a free graded $\mathfrak{R}$-contramodule. Then morphisms of $\mathfrak{R}$-free graded coalgebras $\mathfrak{C} \longrightarrow \bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ are determined by their projections to the component $\mathfrak{U}^{\otimes 1} \simeq \mathfrak{U}$. Conversely, if the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent with the coaugmentation $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C}$, then a homogeneous $\mathfrak{R}$-contramodule morphism $\mathfrak{C} \longrightarrow \mathfrak{U}$ of degree* 0 *gives rise to a morphism of $\mathfrak{R}$-free graded coalgebras $\mathfrak{C} \longrightarrow \bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$ if and only if the composition $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{U}/\mathfrak{m}\mathfrak{U}$ vanishes.*

Proof. We will only prove part (b), as the proof of part (a) is similar but much simpler. Since the map $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n} \longrightarrow \prod_{n=0}^\infty \mathfrak{U}^{\otimes n}$ is injective, a morphism $\mathfrak{C} \longrightarrow \bigoplus_{n=2}^\infty \mathfrak{U}^{\otimes n}$ is determined by its projections to $\mathfrak{U}^{\otimes n}$. For an $\mathfrak{R}$-free graded coalgebra morphism, the component $\mathfrak{C} \longrightarrow \mathfrak{U}^{\otimes n}$ is equal to the composition of the iterated comultiplication map $\mathfrak{C} \longrightarrow \mathfrak{C}^{\otimes n}$ and the n-th tensor power $\mathfrak{C}^{\otimes n} \longrightarrow \mathfrak{U}^{\otimes n}$ of the component $\mathfrak{C} \longrightarrow \mathfrak{U}$. This proves the first assertion; to prove the "only if" part of the second one, it suffices to notice that it holds for $\mathfrak{R} = k$ and apply the reduction modulo $\mathfrak{m}$.

Assuming that $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent, let us show that an $\mathfrak{R}$-contramodule morphism $\mathfrak{C} \longrightarrow \mathfrak{U}$ of degree 0 for which the composition $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{U}/\mathfrak{m}\mathfrak{U}$ is zero gives rise to an $\mathfrak{R}$-free graded coalgebra morphism into $\bigoplus_{n=0}^\infty \mathfrak{U}^{\otimes n}$. For any family of free $\mathfrak{R}$-contramodules $\mathfrak{V}_\alpha$, one has $\bigoplus_\alpha \mathfrak{V}_\alpha = \varprojlim_{\mathfrak{J}} \bigoplus_\alpha \mathfrak{V}_\alpha/\mathfrak{J}\mathfrak{V}_\alpha$, where the projective limit is taken over all open ideals $\mathfrak{I} \subset \mathfrak{R}$. Hence it suffices to consider the case of a discrete Artinian local ring R in place of $\mathfrak{R}$.

Let m be the maximal ideal of R; let C be an R-free graded coalgebra such that C/mC is conilpotent; let U be a free graded R-module endowed with a graded R-module morphism $C \longrightarrow U$ such that the composition $k \longrightarrow C/mC \longrightarrow U/mU$ vanishes. Given an element $c \in C$, we have to show that the composition $C \longrightarrow C^{\otimes n+1} \longrightarrow U^{\otimes n+1}$ annihilates c for n large enough.

Let $N \subset C$ be a free R-submodule with one generator such that the quotient module C/N is free and the image of the map $N/mN \longrightarrow C/mC$ coincides with the image of $k \longrightarrow C/mC$ (see the proof of Lemma 6.1.1(b)). Then the image of the composition $N \longrightarrow C \longrightarrow U$ is contained in mU. Pick an integer $i \geqslant 1$ such that c is annihilated by the composition $C \longrightarrow C/mC \longrightarrow ((C/mC)/k)^{\otimes i+1}$ of the natural surjection with the iterated comultiplication map. Then the image of c in $C^{\otimes i+1}$ belongs to the sum of $mC^{\otimes i+1}$ and $\sum_{j=0}^{i} C^{\otimes j} \otimes_R N \otimes_R C^{\otimes j-i}$.

Repeat this procedure by applying the high enough iterated comultiplication maps simultaneously to, e. g., the first and the last tensor factors in $C^{\otimes i+1}$, etc. Proceeding in this way, we can find an integer $l \geqslant 1$ such that the image of c in $C^{\otimes l+1}$ belongs to the sum of $m^r C^{\otimes l+1}$ and all the tensor products of $l+1$ factors, r of which are the submodules N and the remaining ones are the whole of C, for any given $r \geqslant 1$. If $m^r = 0$, it follows that the image of c in $U^{\otimes l+1}$ vanishes. $\square$

The following constructions repeat those of [29, Section 6.1]; the only difference is that k-vector spaces are replaced with free $\mathfrak{R}$-contramodules.

Let $\mathfrak{B} = (\mathfrak{B}, d, h)$ be an $\mathfrak{R}$-free CDG-algebra; we assume that $\mathfrak{B} \neq 0$. Let $v\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ be a homogeneous retraction onto the image of the unit map $\mathfrak{R} \longrightarrow \mathfrak{B}$, i. e., a morphism of graded $\mathfrak{R}$-contramodules such that the composition $\mathfrak{R} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{R}$ is the identity map. Set $\mathfrak{V} = \ker v \subset \mathfrak{B}$, so $\mathfrak{B} = \mathfrak{R} \oplus \mathfrak{V}$ as a graded $\mathfrak{R}$-contramodule. Using this direct sum decomposition, we can split the multiplication map $m\colon \mathfrak{V} \otimes^{\mathfrak{R}} \mathfrak{V} \longrightarrow \mathfrak{B}$, the differential $d\colon \mathfrak{V} \longrightarrow \mathfrak{B}$, and the curvature element $h \in \mathfrak{B}$ into the components $m = (m_{\mathfrak{V}}, m_{\mathfrak{R}})$, $d = (d_{\mathfrak{V}}, d_{\mathfrak{R}})$, and $h = (h_{\mathfrak{V}}, h_{\mathfrak{R}})$, where $m_{\mathfrak{V}}\colon \mathfrak{V} \otimes^{\mathfrak{R}} \mathfrak{V} \longrightarrow \mathfrak{V}$, $m_{\mathfrak{R}}\colon \mathfrak{V} \otimes^{\mathfrak{R}} \mathfrak{V} \longrightarrow \mathfrak{R}$, $d_{\mathfrak{V}}\colon \mathfrak{V} \longrightarrow \mathfrak{V}$, $d_{\mathfrak{R}}\colon \mathfrak{V} \longrightarrow \mathfrak{R}$, $h_{\mathfrak{V}} \in \mathfrak{V}$, and $h_{\mathfrak{R}} \in \mathfrak{R}$. Notice that the restrictions of m and d to the direct summands $\mathfrak{R} \otimes^{\mathfrak{R}} \mathfrak{V}$, $\mathfrak{V} \otimes^{\mathfrak{R}} \mathfrak{R}$, $\mathfrak{R} \otimes^{\mathfrak{R}} \mathfrak{R}$, and $\mathfrak{R}$ are uniquely determined by the axioms of a graded algebra and its derivation. One has $h_{\mathfrak{R}} = 0$ for the dimension reasons unless $2 = 0$ in Γ.

Set $\mathfrak{B}_+ = \mathfrak{B}/\mathfrak{R}$. Let $\operatorname{Bar}(\mathfrak{B}) = \bigoplus_{n=0}^{\infty} \mathfrak{B}_+^{\otimes n}$ be the tensor coalgebra of the free graded $\mathfrak{R}$-contramodule $\mathfrak{B}_+$. The $\mathfrak{R}$-free coalgebra $\operatorname{Bar}(\mathfrak{B})$ is endowed with the induced grading $|b_1 \otimes \cdots \otimes b_n| = |b_1| + \cdots + |b_n|$, the tensor grading n, and the total grading $|b_1| + \cdots + |b_n| - n$. All the gradings here are understood as direct sum decompositions in $\mathfrak{R}\text{--}\mathsf{contra}^{\mathsf{free}}$. In the sequel, the total grading on $\operatorname{Bar}(\mathfrak{B})$ will be generally presumed. Equivalently, one can define the $\mathfrak{R}$-free (totally) graded coalgebra $\operatorname{Bar}(\mathfrak{B})$ as the tensor coalgebra of the free graded $\mathfrak{R}$-contramodule $\mathfrak{B}_+[1]$.

Let d_{Bar} be the odd coderivation of degree 1 on $\operatorname{Bar}(\mathfrak{B})$ whose compositions with the projection $\operatorname{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{B}_+$ are given by the rules $b_1 \otimes \cdots \otimes b_n \longmapsto 0$ for $n \geqslant 3$, $b_1 \otimes b_2 \longmapsto (-1)^{|b_1|+1} m_{\mathfrak{V}}(b_1 \otimes b_2)$, $b \longmapsto -d_{\mathfrak{V}}(b)$, and $1 \longmapsto h_{\mathfrak{V}}$, where $\mathfrak{B}_+$ is identified with $\mathfrak{V}$ and $1 \in \mathfrak{B}_+^{\otimes 0}$. Let $h_{\mathrm{Bar}}\colon \mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{R}$ be the linear function given by the formulas $h_{\mathrm{Bar}}(b_1\otimes\cdots b_n) = 0$ for $n \geqslant 3$, $h_{\mathrm{Bar}}(b_1\otimes b_2) = (-1)^{|b_1|+1}h_{\mathfrak{R}}(b_1\otimes b_2)$, $h_{\mathrm{Bar}}(b) = -d_{\mathfrak{R}}(b)$, and $h_{\mathrm{Bar}}(1) = h_{\mathfrak{R}}$. Then the $\mathfrak{R}$-free graded coalgebra $\mathrm{Bar}(\mathfrak{B})$ endowed with the coderivation d_{Bar} and the curvature linear function h_{Bar} is a CDG-coalgebra. We will denote it $\mathrm{Bar}_v(\mathfrak{B})$ and call the *bar-construction* of an $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$ endowed with a homogeneous retraction $v\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ of the unit map.

Given an $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$, changing a retraction $v\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ to another one $v'\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ given by the formula $v'(b) = v(b) + \alpha(b)$ leads to an isomorphism of $\mathfrak{R}$-free CDG-coalgebras $(\mathrm{id}, a)\colon \mathrm{Bar}_v(\mathfrak{B}) \longrightarrow \mathrm{Bar}_{v'}(\mathfrak{B})$, where the linear function $\alpha\colon \mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{R}$ of degree 1 is obtained as the composition of the natural projection $\mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{B}_+^{\otimes 1} \simeq \mathfrak{B}_+$ and the linear function $\alpha\colon \mathfrak{B}_+ \longrightarrow \mathfrak{R}$ of degree 0.

A morphism of $\mathfrak{R}$-free CDG-algebras $(f, a)\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is said to be *weakly strict* if the element $a \in \mathfrak{A}^1$ belongs to $\mathfrak{m}\mathfrak{A}_1$. In particular, if $\mathfrak{A}$ and $\mathfrak{B}$ are wcDG-algebras, then, by the definition, (f, a) is a wcDG-algebra morphism if and only if it is a weakly strict CDG-algebra morphism.

To a weakly strict isomorphism of $\mathfrak{R}$-free CDG-algebras $(\mathrm{id}, a)\colon (\mathfrak{B}, d', h') \longrightarrow (\mathfrak{B}, d, h)$ one can assign an isomorphism of the corresponding bar-constructions of the form $(f_{\mathrm{Bar}}, a_{\mathrm{Bar}})\colon \mathrm{Bar}_v(\mathfrak{B}, d', h') \longrightarrow \mathrm{Bar}_v(\mathfrak{B}, d, h)$ constructed as follows. Let $a_{\mathfrak{V}} \in \mathfrak{V}$ and $a_{\mathfrak{R}} \in \mathfrak{R}$ be the components of the element $a \in \mathfrak{B}$ with respect to the direct sum decomposition $\mathfrak{B} = \mathfrak{R} \oplus \mathfrak{V}$.

Then the automorphism f_{Bar} of the $\mathfrak{R}$-free graded coalgebra $\mathrm{Bar}(\mathfrak{B})$ is defined by the rule that the composition of f_{Bar} with the projection $\mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{B}_+^{\otimes 1} \simeq \mathfrak{B}_+$ is equal to the sum of the same projection and minus the composition of the projection $\mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{B}_+^{\otimes 0} \simeq \mathfrak{R}$ with the map $a_{\mathfrak{V}}\colon \mathfrak{R} \longrightarrow \mathfrak{B}_+$. A unique such automorphism f_a exists by Lemma 6.1.3(b). The linear function $a_{\mathrm{Bar}}\colon \mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{R}$ is equal to the composition of the projection $\mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{R}$ with the map $a_{\mathfrak{R}}\colon \mathfrak{R} \longrightarrow \mathfrak{R}$. Notice that $a_{\mathfrak{R}}$ and a_{Bar} can be only nonzero when $1 = 0$ in Γ, which can only happen when $2 = 0$ in $\mathfrak{R}$ (see [30, Section 1.1]).

Consequently, there is a functor from the category of $\mathfrak{R}$-free CDG-algebras and weakly strict morphisms between them to the category of $\mathfrak{R}$-free CDG-coalgebras assigning to a CDG-algebra $\mathfrak{B}$ its bar-construction $\mathrm{Bar}_v(\mathfrak{B})$. This functor takes wcDG-algebras $\mathfrak{A}$ over $\mathfrak{R}$ to $\mathfrak{R}$-free CDG-coalgebras $\mathfrak{C} = \mathrm{Bar}(\mathfrak{A})$ whose reductions $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ are *conilpotent CDG-coalgebras* in the sense of [29, Sections 6.1 and 6.4].

Recall the definition of the latter notion: a CDG-coalgebra C over a field k is called *conilpotent* if it is conilpotent as graded coalgebra and the homogeneous coaugmentation morphism $\bar w\colon k \longrightarrow C$ satisfies the equations $d \circ \bar w = 0 = h \circ \bar w$ of compatibility with the CDG-coalgebra structure. The definition of the category of conilpotent CDG-coalgebras requires a little care when the field k has characteristic 2: a morphism of conilpotent CDG-coalgebras $(f, a)\colon C \longrightarrow D$ is a morphism of CDG-coalgebras such that $a \circ \bar w = 0$.

Let $\mathfrak{C} = (\mathfrak{C}, d, h)$ be an $\mathfrak{R}$-free CDG-coalgebra; we assume that $\mathfrak{C} \neq 0$. Let $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be a homogeneous section of the counit map $\mathfrak{C} \longrightarrow \mathfrak{R}$, i. e., a morphism of graded $\mathfrak{R}$-contramodules such that the composition $\mathfrak{R} \longrightarrow \mathfrak{C} \longrightarrow \mathfrak{R}$ is the identity map. Set $\mathfrak{W} = \operatorname{coker} w$, so $\mathfrak{C} = \mathfrak{R} \oplus \mathfrak{W}$ as a graded $\mathfrak{R}$-contramodule. Using this direct sum decomposition, we can split the comultiplication map $\mu\colon \mathfrak{C} \longrightarrow \mathfrak{W} \otimes^{\mathfrak{R}} \mathfrak{W}$, the differential $d\colon \mathfrak{C} \longrightarrow \mathfrak{W}$, and the curvature linear function $h\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ into the components $\mu = (\mu_{\mathfrak{W}}, \mu_{\mathfrak{R}})$, $d = (d_{\mathfrak{W}}, d_{\mathfrak{R}})$, and $h = (h_{\mathfrak{W}}, h_{\mathfrak{R}})$, where $\mu_{\mathfrak{W}}\colon \mathfrak{W} \longrightarrow \mathfrak{W} \otimes^{\mathfrak{R}} \mathfrak{W}$, $\mu_{\mathfrak{R}} \in \mathfrak{W} \otimes^{\mathfrak{R}} \mathfrak{W}$, $d_{\mathfrak{W}}\colon \mathfrak{W} \longrightarrow \mathfrak{W}$, $d_{\mathfrak{R}} \in \mathfrak{W}$, $h_{\mathfrak{W}}\colon \mathfrak{W} \longrightarrow \mathfrak{R}$, and $h_{\mathfrak{R}} \in \mathfrak{R}$. Notice that compositions of μ and d with the projections of $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C}$ onto $\mathfrak{R} \otimes^{\mathfrak{R}} \mathfrak{W}$, $\mathfrak{W} \otimes^{\mathfrak{R}} \mathfrak{R}$, $\mathfrak{R} \otimes^{\mathfrak{R}} \mathfrak{R}$ and $\mathfrak{C}$ onto $\mathfrak{R}$ are uniquely determined by the axioms of a graded coalgebra and a coderivation. One has $h_{\mathfrak{R}} = 0$ for the dimension reasons if $2 \neq 0$ in Γ.

Set $\mathfrak{C}_+ = \ker(\mathfrak{C} \to \mathfrak{R})$ to be the kernel of the counit map. Let $\operatorname{Cob}(\mathfrak{C}) = \bigoplus_{n=0}^{\infty} \mathfrak{C}_+^{\otimes n}$ be the tensor algebra of the free graded $\mathfrak{R}$-contramodule $\mathfrak{C}_+$. The $\mathfrak{R}$-free algebra $\operatorname{Cob}(\mathfrak{C})$ is endowed with the induced grading $|c_1 \otimes \cdots \otimes c_n| = |c_1| + \cdots + |c_n|$, the tensor grading n, and the total grading $|c_1| + \cdots + |c_n| + n$. All the gradings here are understood as direct sum decompositions in $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}}$. In the sequel, the total grading on $\operatorname{Cob}(\mathfrak{C})$ will be generally presumed. Equivalently, one can define the $\mathfrak{R}$-free (totally) graded coalgebra $\operatorname{Cob}(\mathfrak{C})$ as the tensor coalgebra of the free graded $\mathfrak{R}$-contramodule $\mathfrak{C}_+[-1]$.

Let d_{Cob} be the odd derivation of degree 1 on $\operatorname{Cob}(\mathfrak{B})$ whose restriction to $\mathfrak{C}_+ \subset \operatorname{Cob}(\mathfrak{C})$ is given by the formula $d(c) = (-1)^{|c_{(1,\mathfrak{W})}|+1} c_{(1,\mathfrak{W})} \otimes c_{(2,\mathfrak{W})} - d_{\mathfrak{W}}(c) + h_{\mathfrak{W}}(c)$, where $\mathfrak{C}_+$ is identified with $\mathfrak{W}$ and $\mu_{\mathfrak{W}}(c) = c_{(1,\mathfrak{W})} \otimes c_{(2,\mathfrak{W})}$. Let $h_{\mathrm{Cob}} \in \operatorname{Cob}(\mathfrak{C})$ be the element given by the formula $h_{\mathrm{Cob}} = (-1)^{|\mu_{(1,\mathfrak{R})}|+1} \mu_{(1,\mathfrak{R})} \otimes \mu_{(2,\mathfrak{R})} - d_{\mathfrak{R}} + h_{\mathfrak{R}}$, where $\mu_{\mathfrak{R}} = \mu_{(1,\mathfrak{R})} \otimes \mu_{(2,\mathfrak{R})}$. Then the $\mathfrak{R}$-free graded algebra $\operatorname{Cob}(\mathfrak{C})$ endowed with the derivation d_{Cob} and the curvature element h_{Cob} is a CDG-algebra. We will denote it $\operatorname{Cob}_w(\mathfrak{C})$ and call the *cobar-construction* of an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ endowed with a homogeneous section $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ of the counit map.

Given an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, changing a section $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ to another one $w'\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ given by the rule $w'(1) = w(1) + \alpha$ leads to an isomorphism of $\mathfrak{R}$-free CDG-algebras $(\mathrm{id}, a)\colon \operatorname{Cob}_{w'}(\mathfrak{C}) \longrightarrow \operatorname{Cob}_w(\mathfrak{C})$, where $a \in \operatorname{Cob}(\mathfrak{C})$ is the element of degree 1 corresponding to $\alpha \in \mathfrak{C}_+ \subset \operatorname{Cob}(\mathfrak{C})$.

To an isomorphism of $\mathfrak{R}$-free CDG-coalgebras $(\mathrm{id}, a)\colon (\mathfrak{C}, d, h) \longrightarrow (\mathfrak{C}, d', h')$ one can assign an isomorphism of the corresponding cobar-constructions of the form $(f_{\mathrm{Cob}}, a_{\mathrm{Cob}})\colon \operatorname{Cob}_w(\mathfrak{C}, d, h) \longrightarrow \operatorname{Cob}_w(\mathfrak{C}, d', h')$ constructed as follows. Let $a_{\mathfrak{W}}\colon \mathfrak{W} \longrightarrow \mathfrak{R}$ and $a_{\mathfrak{R}} \in \mathfrak{R}$ be the components of the linear function $a\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ with respect to the direct sum decomposition $\mathfrak{C} = \mathfrak{R} \oplus \mathfrak{W}$. Then the automorphism f_{Cob} of the $\mathfrak{R}$-free graded algebra $\operatorname{Cob}(\mathfrak{C})$ is given by the rule $c \longmapsto c - a_{\mathfrak{W}}(c)$, where $c \in \mathfrak{C}_+^{\otimes 1} \simeq \mathfrak{C}_+$. The element $a_{\mathrm{Cob}} \in \operatorname{Cob}(\mathfrak{C})$ is equal to $a_{\mathfrak{R}} \in \mathfrak{R} \simeq \mathfrak{C}_+^{\otimes 0}$. Notice that $a_{\mathfrak{R}}$ and a_{Cob} are always zero unless $1 = 0$ in Γ, which implies $2 = 0$ in $\mathfrak{R}$.

Consequently, there is a functor from the category of $\mathfrak{R}$-free CDG-coalgebras to the category of $\mathfrak{R}$-free CDG-algebras assigning to a CDG-coalgebra $\mathfrak{C}$ its cobar-construction $\operatorname{Cob}_w(\mathfrak{C})$. When the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is *coaugmented* [29, Section 6.1], one can pick a section $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ such that its reduction $\bar{w}\colon k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is the coaugmentation. This makes $\mathfrak{C} \longmapsto \operatorname{Cob}_w(\mathfrak{C})$ a functor from the category of $\mathfrak{R}$-free CDG-coalgebras $(\mathfrak{C}, d, h)$ with coaugmented (and, in particular, conilpotent) reductions $(\mathfrak{C}/\mathfrak{m}\mathfrak{C},\ d/\mathfrak{m}d,\ h/\mathfrak{m}h)$ to the category of wcDG-algebras over $\mathfrak{R}$.

Here we recall that a CDG-coalgebra C over k is said to be *coaugmented* if it is endowed with a morphism of CDG-coalgebras $(\bar{w}, 0)\colon (k, 0, 0) \longrightarrow (C, d, h)$, or equivalently, a morphism of graded coalgebras $\bar{w}\colon k \longrightarrow C$ such that $d\circ\bar{w} = 0 = h\circ\bar{w}$. A morphism of coaugmented CDG-coalgebras $(f, a)\colon C \longrightarrow D$ is a morphism of CDG-coalgebras such that $a \circ \bar{w} = 0$ (a condition nontrivial in characteristic 2 only).

6.2. **Twisting cochains.** Let $\mathfrak{C} = (\mathfrak{C}, d_{\mathfrak{C}}, h_{\mathfrak{C}})$ be an $\mathfrak{R}$-free CDG-coalgebra and $\mathfrak{B} = (\mathfrak{B}, d_{\mathfrak{B}}, h_{\mathfrak{B}})$ be an $\mathfrak{R}$-free CDG-algebra. Introduce a CDG-algebra structure on the graded $\mathfrak{R}$-contramodule of homogeneous $\mathfrak{R}$-contramodule homomorphisms $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ in the following way. The multiplication in $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ is defined as the composition $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}) \otimes^{\mathfrak{R}} \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C},\ \mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{B}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$, the second map being induced by the comultiplication in $\mathfrak{C}$ and the multiplication in $\mathfrak{B}$. The left-right and sign rule is $(fg)(c) = (-1)^{|g||c_{(1)}|} f(c_{(1)})g(c_{(2)})$. The differential is given by the conventional rule $d(f)(c) = d_{\mathfrak{B}}(f(c)) - (1)^{|f|} f(d_{\mathfrak{C}}(c))$. The curvature element in $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ is defined by the formula $h(c) = \varepsilon(c)h_{\mathfrak{B}} - h_{\mathfrak{C}}(c)e$, where $\varepsilon\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ is the counit map and $e \in \mathfrak{B}$ is the unit element.

A homogeneous $\mathfrak{R}$-contramodule map $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$ is called a *twisting cochain* if it satisfies the equation $\tau^2 + d\tau + h = 0$ with respect to the above-defined CDG-algebra structure on $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ (see [29, Section 6.2] and the references therein). Given a morphism of $\mathfrak{R}$-free CDG-algebras $(f, a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ and a twisting cochain $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$, one constructs the twisting cochain $(f, a) \circ \tau\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ by the rule $(f, a) \circ \tau = f \circ \tau + a \circ \varepsilon_{\mathfrak{C}}$. Given a morphism of $\mathfrak{R}$-free CDG-coalgebras $(g, a)\colon \mathfrak{D} \longrightarrow \mathfrak{C}$ and a twisting cochain $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$, one constructs the twisting cochain $\tau \circ (g, a)\colon \mathfrak{D} \longrightarrow \mathfrak{B}$ by the rule $\tau \circ (g, a) = \tau \circ g - e_{\mathfrak{B}} \circ a$.

Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra and $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be a homogeneous section of the counit map. Then the composition $\tau = \tau_{\mathfrak{C},w}\colon \mathfrak{C} \longrightarrow \mathrm{Cob}(\mathfrak{C})$ of the maps $\mathfrak{C} \longrightarrow \mathfrak{W} \simeq \mathfrak{C}_+ \simeq \mathfrak{C}_+^{\otimes 1} \longrightarrow \mathrm{Cob}(\mathfrak{C})$ is a twisting cochain for $\mathfrak{C}$ and $\mathrm{Cob}_w(\mathfrak{C})$. Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra and $v\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ be a homogeneous retraction of the unit map. Then minus the composition $\mathrm{Bar}(\mathfrak{B}) \longrightarrow \mathfrak{B}_+^{\otimes 1} \simeq \mathfrak{B}_+ \simeq \mathfrak{V} \longrightarrow \mathfrak{B}$ is a twisting cochain $\tau = \tau_{\mathfrak{B},v}\colon \mathrm{Bar}_v(\mathfrak{B}) \longrightarrow \mathfrak{B}$ for $\mathrm{Bar}_v(\mathfrak{B})$ and $\mathfrak{B}$.

Let $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$ be a twisting cochain for an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ and an $\mathfrak{R}$-free CDG-algebra $\mathfrak{B}$. Then for any $\mathfrak{R}$-free left CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ there is a natural structure of $\mathfrak{R}$-free left CDG-comodule over $\mathfrak{C}$ on the tensor product $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$. Namely, the coaction of $\mathfrak{C}$ in $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ is induced by the left coaction of $\mathfrak{C}$ in itself, while the differential on $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ is given by the formula $d(c \otimes x) = d(c) \otimes x + (-1)^{|c|} c \otimes d(x) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)})x$, where $c \longmapsto c_{(1)} \otimes c_{(2)}$ denotes the comultiplication in $\mathfrak{C}$, while $b \otimes x \longmapsto bx$ is the left action of $\mathfrak{B}$ in $\mathfrak{M}$. We will denote the free graded $\mathfrak{R}$-contramodule $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ with this CDG-comodule structure by $\mathfrak{C} \otimes^{\tau} \mathfrak{M}$.

Furthermore, for any $\mathfrak{R}$-free left CDG-comodule $\mathfrak{N}$ over $\mathfrak{C}$ there is a natural structure of $\mathfrak{R}$-free left CDG-module over $\mathfrak{B}$ on the tensor product $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{N}$. Namely, the action of $\mathfrak{B}$ in $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{N}$ is induced by the left action of $\mathfrak{B}$ in itself, while the differential on $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{N}$ is given by the formula $d(b \otimes y) = d(b) \otimes y + (-1)^{|b|} b \otimes d(y) - (-1)^{|b|} b\tau(y_{(-1)}) \otimes y_{(0)}$, where $y \longmapsto y_{(-1)} \otimes y_{(0)}$ denotes the left coaction of $\mathfrak{C}$

in $\mathfrak{N}$, while $b \otimes b' \longmapsto bb'$ is the multiplication in $\mathfrak{B}$. We will denote the free graded $\mathfrak{R}$-contramodule $\mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{N}$ with this CDG-module structure by $\mathfrak{B} \otimes^{\tau} \mathfrak{N}$.

The correspondences assigning to an $\mathfrak{R}$-free CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ the $\mathfrak{R}$-free CDG-comodule $\mathfrak{C} \otimes^{\tau} \mathfrak{M}$ over $\mathfrak{C}$ and to an $\mathfrak{R}$-free CDG-comodule $\mathfrak{N}$ over $\mathfrak{C}$ the $\mathfrak{R}$-free CDG-module $\mathfrak{B} \otimes^{\tau} \mathfrak{N}$ over $\mathfrak{B}$ can be extended to DG-functors whose action on morphisms is given by the standard formulas $f_*(c \otimes x) = (-1)^{|f||c|} c \otimes f_*(x)$ and $g_*(b \otimes y) = (-1)^{|g||b|} b \otimes g_*(y)$. The DG-functor $\mathfrak{C} \otimes^{\tau} -\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ is right adjoint to the DG-functor $\mathfrak{B} \otimes^{\tau} -\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$.

Similarly, for any $\mathfrak{R}$-free right CDG-module $\mathfrak{M}$ over $\mathfrak{B}$ there is a natural structure of right CDG-comodule over $\mathfrak{C}$ on the tensor product $\mathfrak{M} \otimes^{\mathfrak{R}} \mathfrak{C}$. The coaction of $\mathfrak{C}$ in $\mathfrak{M} \otimes^{\mathfrak{R}} \mathfrak{C}$ is induced by the right coaction of $\mathfrak{C}$ in itself, and the differential on $\mathfrak{M} \otimes^{\mathfrak{R}} \mathfrak{C}$ is given by the formula $d(x \otimes c) = d(x) \otimes c + (-1)^{|x|} x \otimes d(c) - (-1)^{|x|} x\tau(c_{(1)}) \otimes c_{(2)}$. We will denote the free graded $\mathfrak{R}$-contramodule $\mathfrak{M} \otimes^{\mathfrak{R}} \mathfrak{C}$ with this CDG-comodule structure by $\mathfrak{M} \otimes^{\tau} \mathfrak{C}$. For any $\mathfrak{R}$-free right CDG-comodule $\mathfrak{N}$ over $\mathfrak{C}$ there is a natural structure of right CDG-module over $\mathfrak{B}$ on the tensor product $\mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{B}$. Namely, the action of $\mathfrak{B}$ in $\mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{B}$ is induced by the right action of $\mathfrak{B}$ in itself, and the differential on $\mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{B}$ is given by the formula $d(y \otimes b) = d(y) \otimes b + (-1)^{|y|} y \otimes d(b) + (-1)^{|y_{(0)}|} y_{(0)} \otimes \tau(y_{(1)}b$, where $y \longmapsto y_{(0)} \otimes y_{(1)}$ denotes the right coaction of $\mathfrak{C}$ in $\mathfrak{N}$. We will denote the free graded $\mathfrak{R}$-contramodule $\mathfrak{N} \otimes^{\mathfrak{R}} \mathfrak{B}$ with this CDG-module structure by $\mathfrak{N} \otimes^{\tau} \mathfrak{B}$.

For any $\mathfrak{R}$-contramodule left CDG-module $\mathfrak{P}$ over $\mathfrak{B}$ there is a natural structure of left CDG-contramodule over $\mathfrak{C}$ on the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$. The contraaction of $\mathfrak{C}$ in $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ is induced by the right coaction of $\mathfrak{C}$ in itself as explained in Sections 3.1 and 4.4 (for the sign rule, see [29, Section 2.2]). The differential on $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ is given by the formula $d(f)(c) = d(f(c)) - (-1)^{|f|} f(d(c)) + (-1)^{|f||c_{(1)}|} \tau(c_{(1)}) f(c_{(2)})$ for $f \in \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$. Here the third summand is interpreted as the $\mathfrak{R}$-contamodule morphism $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ corresponding to the $\mathfrak{R}$-contramodule morphism $\mathfrak{C} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ defined as the composition $\mathfrak{C} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{C} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathfrak{P}$. We will denote the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ with this CDG-contramodule structure by $\operatorname{Hom}^{\tau}(\mathfrak{C}, \mathfrak{P})$.

For any $\mathfrak{R}$-contramodule left CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$ there is a natural structure of left CDG-module over $\mathfrak{B}$ on the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$. The action of $\mathfrak{B}$ in $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ is induced by the right action of $\mathfrak{B}$ in itself (see Section 2.1). The differential on $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ is given by the formula $d(f)(b) = d(f(b)) - (-1)^{|f|} f(d(b)) - \pi(c \mapsto (-1)^{|f|+|c||b|} f(\tau(c)b)$, where π denotes the contraaction map $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$. Here the third summand is interpreted as the $\mathfrak{R}$-contramodule morphism $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ corresponding to the $\mathfrak{R}$-contramodule morphism $\mathfrak{B} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$ defined as the composition $\mathfrak{B} \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}) \otimes^{\mathfrak{R}} \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$, where the morphism $\mathfrak{B} \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ corresponds to the composition $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{B} \longrightarrow \mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{B} \longrightarrow \mathfrak{B}$. We denote the graded $\mathfrak{R}$-contramodule $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ with this CDG-module structure by $\operatorname{Hom}^{\tau}(\mathfrak{B}, \mathfrak{Q})$.

The correspondences assigning to an $\mathfrak{R}$-contramodule CDG-module $\mathfrak{P}$ over $\mathfrak{B}$ the $\mathfrak{R}$-contramodule CDG-contramodule $\mathrm{Hom}^\tau(\mathfrak{C},\mathfrak{P})$ over $\mathfrak{C}$ and to an $\mathfrak{R}$-contramodule CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$ the $\mathfrak{R}$-contramodule CDG-module $\mathrm{Hom}^\tau(\mathfrak{B},\mathfrak{Q})$ over $\mathfrak{B}$ can be extended to DG-functors whose action on morphisms is given by the standard formula $g_*(f) = g\circ f$ for $f\colon \mathfrak{C}\longrightarrow\mathfrak{P}$ or $f\colon\mathfrak{B}\longrightarrow\mathfrak{Q}$. The DG-functor $\mathrm{Hom}^\tau(\mathfrak{C},-)\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}\longrightarrow\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ is left adjoint to the DG-functor $\mathrm{Hom}^\tau(\mathfrak{B},-)\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}\longrightarrow\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$.

Analogously, for any $\mathfrak{R}$-comodule left CDG-module $\mathcal{M}$ over $\mathfrak{B}$ there is a natural structore of $\mathfrak{R}$-comodule left CDG-comodule over $\mathfrak{B}$ on the contratensor product $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$. Namely, the coaction of $\mathfrak{C}$ in $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ is induced by the left coaction of $\mathfrak{C}$ in itself, and the differential on $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ is defined in terms of the differentials on $\mathfrak{C}$ and $\mathcal{M}$, the twisting cochain τ, the comultiplication in $\mathfrak{C}$, and the action of $\mathfrak{B}$ in $\mathcal{M}$ by the formula above. We will denote the graded $\mathfrak{R}$-comodule $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ with this CDG-comodule structure by $\mathfrak{C}\odot^\tau\mathcal{M}$.

For any $\mathfrak{R}$-comodule left CDG-comodule $\mathcal{N}$ over $\mathfrak{C}$ there is a natural structure of $\mathfrak{R}$-comodule left CDG-module over $\mathfrak{B}$ on the contratensor product $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{N}$. The action of $\mathfrak{B}$ in $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{N}$ is induced by the left action of $\mathfrak{B}$ in itself, and the differential on $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{N}$ is defined in terms of the differentials on $\mathfrak{B}$ and $\mathcal{N}$, the twisting cochain τ, the multiplication in $\mathfrak{B}$, and the coaction of $\mathfrak{C}$ in $\mathcal{N}$ by the formula above. We will denote the graded $\mathfrak{R}$-comodule $\mathfrak{B}\odot_{\mathfrak{R}}\mathcal{N}$ with this CDG-module structure by $\mathfrak{B}\odot^\tau\mathcal{N}$.

The correspondences assigning to an $\mathfrak{R}$-comodule CDG-module $\mathcal{M}$ over $\mathfrak{B}$ the $\mathfrak{R}$-comodule CDG-comodule $\mathfrak{C}\odot^\tau\mathcal{M}$ over $\mathfrak{C}$ and to an $\mathfrak{R}$-comodule CDG-comodule $\mathcal{N}$ over $\mathfrak{C}$ the $\mathfrak{R}$-comodule CDG-module $\mathfrak{B}\odot^\tau\mathcal{N}$ over $\mathfrak{B}$ can be extended to DG-functors whose action on morphisms is given by the standard formulas above. The DG-functor $\mathfrak{C}\odot^\tau -\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\longrightarrow\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}$ is right adjoint to the DG-functor $\mathfrak{B}\odot^\tau\colon \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\longrightarrow\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}$.

Similarly, for any $\mathfrak{R}$-comodule right CDG-module $\mathcal{M}$ over $\mathfrak{B}$ there is a natural structure of right CDG-comodule over $\mathfrak{C}$ on the contratensor product $\mathcal{M}\odot_{\mathfrak{R}}\mathfrak{C}$ (see Section 4.1 for the definition of such contratensor product). The coaction of $\mathfrak{C}$ in $\mathcal{M}\odot_{\mathfrak{R}}\mathfrak{C}$ is induced by the right coaction of $\mathfrak{C}$ in itself, and the differential is given by the formula above. We will denote the graded $\mathfrak{R}$-comodule $\mathcal{M}\odot_{\mathfrak{R}}\mathfrak{C}$ with this CDG-comodule structure by $\mathcal{M}\odot^\tau\mathfrak{C}$. For any $\mathfrak{R}$-comodule right CDG-comodule $\mathcal{N}$ over $\mathfrak{C}$ there is a natural structure of right CDG-module over $\mathfrak{B}$ on the contratensor product $\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{B}$. The action of $\mathfrak{B}$ in $\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{B}$ is induced by the right action of $\mathfrak{B}$ in itself, and the differential is given by the formula above. We will denote the graded $\mathfrak{R}$-comodule $\mathcal{N}\odot_{\mathfrak{C}}\mathfrak{B}$ with this CDG-module structure by $\mathcal{N}\odot^\tau\mathfrak{B}$.

For any $\mathfrak{R}$-cofree left CDG-module $\mathcal{P}$ over $\mathfrak{B}$ there is a natural structure of left CDG-contramodule over $\mathfrak{C}$ on the cofree graded $\mathfrak{R}$-comodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$. The contraaction of $\mathfrak{C}$ in $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$ is induced by the right coaction of $\mathfrak{C}$ in itself as explained in Section 3.3. The differential on $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$ is defined in terms of the differentials on $\mathfrak{C}$ and $\mathcal{P}$, the twisting cochain τ, the comultiplication in $\mathfrak{C}$, and the action of $\mathfrak{B}$ in $\mathcal{P}$ by the formula above. The twisting term is constructed as the $\mathfrak{R}$-comodule morphism $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$ corresponding to the $\mathfrak{R}$-comodule morphism $\mathfrak{C} \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ defined as the composition $\mathfrak{C} \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{C} \odot_{\mathfrak{R}} \mathrm{Ctrhom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{B} \otimes^{\mathfrak{R}} \mathfrak{C} \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \mathfrak{B} \odot_{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathfrak{P}$. We will denote the cofree graded $\mathfrak{R}$-comodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ with this CDG-contramodule structure by $\mathrm{Ctrhom}^{\tau}(\mathfrak{C}, \mathfrak{P})$.

For any $\mathfrak{R}$-cofree left CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$ there is a natural structure of left CDG-module over $\mathfrak{B}$ on the cofree graded $\mathfrak{R}$-comodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$. The action of $\mathfrak{B}$ in $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ is induced by the right action of $\mathfrak{B}$ in itself (see Section 2.4). The differential on $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ is defined in terms of the differentials on $\mathfrak{B}$ and $\mathfrak{Q}$, the twisting cochain τ, the multiplication in $\mathfrak{B}$, and the contraaction of $\mathfrak{C}$ in $\mathfrak{Q}$ by the formula above. The twisting term is constructed as the $\mathfrak{R}$-comodule morphism $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ corresponding to the $\mathfrak{R}$-comodule morphism $\mathfrak{B} \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$ defined as the composition $\mathfrak{B} \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}) \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q}) \longrightarrow \mathfrak{Q}$, where the morphism $\mathfrak{B} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B})$ was defined above. The natural morphism $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}) \odot_{\mathfrak{R}} \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q})$ corresponds to the composition $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C} \otimes^{\mathfrak{R}} \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}),\ \mathfrak{Q}) \simeq \mathrm{Ctrhom}_{\mathfrak{R}}(\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{B}), \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q}))$. We will denote the cofree graded $\mathfrak{R}$-comodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{B}, \mathfrak{Q})$ with this CDG-module structure by $\mathrm{Ctrhom}^{\tau}(\mathfrak{B}, \mathfrak{Q})$.

The correspondences assigning to an $\mathfrak{R}$-cofree CDG-module $\mathfrak{P}$ over $\mathfrak{B}$ the $\mathfrak{R}$-cofree CDG-contramodule $\mathrm{Ctrhom}^{\tau}(\mathfrak{C}, \mathfrak{P})$ over $\mathfrak{C}$ and to an $\mathfrak{R}$-cofree CDG-contramodule $\mathfrak{Q}$ over $\mathfrak{C}$ the $\mathfrak{R}$-cofree CDG-module $\mathrm{Ctrhom}^{\tau}(\mathfrak{B}, \mathfrak{Q})$ over $\mathfrak{B}$ can be extended to DG-functors whose action on morphisms is given by the standard formula above. The DG-functor $\mathrm{Ctrhom}^{\tau}(\mathfrak{C}, -)\colon \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}$ is left adjoint the DG-functor $\mathrm{Ctrhom}^{\tau}(\mathfrak{B}, -)\colon \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$.

6.3. **Conilpotent duality.** Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$ and $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra such that the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over k is conilpotent with the coaugmentation map $\bar{w}\colon k \longrightarrow \mathfrak{C}$. Let $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ be a twisting cochain such that the composition $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ vanishes.

Theorem 6.3.1. *Assume that the twisting cochain $\bar{\tau} = \tau/\mathfrak{m}\tau\colon \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is* acyclic *in the sense of* [29, *Section* 6.5]. *Then*

(a) *the functors $\mathfrak{C} \otimes^{\tau} -\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathfrak{A} \otimes^{\tau} -\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ induce functors $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, which are mutually inverse equivalences of triangulated categories;*

(b) *the functors $\mathrm{Hom}^{\tau}(\mathfrak{C}, -)\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathrm{Hom}^{\tau}(\mathfrak{A}, -)\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ induce functors $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, which are mutually inverse equivalences of triangulated categories;*

(c) *the functors $\mathfrak{C} \odot^{\tau} -\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ and $\mathfrak{A} \odot^{\tau} -\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ induce functors $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ and $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$, which are mutually inverse equivalences of triangulated categories;*

(d) *the functors* $\mathrm{Ctrhom}^\tau(\mathfrak{C},-)\colon H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathrm{Ctrhom}^\tau(\mathfrak{A},-)\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *induce functors* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$*, which are mutually inverse equivalences of triangulated categories;*

(e) *the above equivalences of triangulated categories form a commutative diagram with the equivalences of categories* $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ *of Sections* 2.6 *and* 3.4*,* $\mathbb{L}\Phi_{\mathfrak{R}} = \mathbb{R}\Psi_{\mathfrak{R}}^{-1}$ *of Proposition* 4.3.2*,* $\mathbb{L}\Phi_{\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{C}}^{-1}$ *of Corollaries* 3.2.4 *and* 3.4.2*,* $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ *of Corollary* 4.5.3*, the equivalences of semiderived categories from Section* 4.3*, and the equivalences of contra/coderived categories from Theorem* 4.5.1*.*

Proof. The assertions about the existence of induced functors in parts (a-d) do not depend on the acyclicity assumption on $\bar\tau$; the assertions about the induced functors being equivalences of categories do. Part (a): the functor $\mathfrak{A}\otimes^\tau -$ takes coacyclic $\mathfrak{R}$-free CDG-comodules over $\mathfrak{C}$ to contractible wcDG-modules over $\mathfrak{A}$. Indeed, whenever $\mathfrak{N}$ is the total CDG-comodule of an short exact sequence of $\mathfrak{R}$-free CDG-comodules, $\mathfrak{A}\otimes^\tau\mathfrak{N}$ is the total wcDG-module of a short exact sequence of wcDG-modules that is split as a short exact sequence of graded $\mathfrak{A}$-modules. Furthermore, the functor $\mathfrak{C}\otimes^\tau -$ takes semiacyclic $\mathfrak{R}$-free wcDG-modules $\mathfrak{M}$ to contractible CDG-comodules $\mathfrak{N} = \mathfrak{C}\otimes^\tau\mathfrak{M}$. Indeed, the $\mathfrak{R}$-free graded $\mathfrak{C}$-comodule $\mathfrak{N}$ is injective, and the CDG-comodule $\mathfrak{N}/\mathfrak{m}\mathfrak{N}$ over $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is contractible by [29, proofs of Theorems 6.3 and 6.5], so it remains to apply Lemma 3.2.1. The cone of the adjunction map $\mathfrak{A}\otimes^\tau\mathfrak{C}\otimes^\tau\mathfrak{M} \longrightarrow \mathfrak{M}$ is semiacyclic for any $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$, because the reduction modulo $\mathfrak{m}$ of this cone is acyclic by [29, Theorem 6.5(a)]. The cone of the adjunction map $\mathfrak{N} \longrightarrow \mathfrak{C}\otimes^\tau\mathfrak{A}\otimes^\tau\mathfrak{N}$ is coacyclic for any $\mathfrak{R}$-free CDG-comodule $\mathfrak{N}$ by the same result from [29] and according to Corollary 3.2.3.

Part (b): the functor $\mathrm{Hom}^\tau(\mathfrak{A},-)$ takes contraacyclic $\mathfrak{R}$-contramodule CDG-contramodules to contraacyclic $\mathfrak{R}$-contramodule wcDG-modules, since it preserves short exact sequences and infinite products of $\mathfrak{R}$-contramodule CDG-contramodules. The functor $\mathrm{Hom}^\tau(\mathfrak{C},-)$ takes contraacyclic $\mathfrak{R}$-contramodule wcDG-modules to contraacyclic $\mathfrak{R}$-cotramodule CDG-contramodules for the same reason. It also takes semiacyclic $\mathfrak{R}$-free wcDG-modules to contractible CDG-contramodules, for the reasons explained in the proof of part (a). Hence the induced adjoint functors $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ and $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ exist. To check that they are mutually inverse equivalences, it suffices to consider them as functors $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, restricting both constructions to $\mathfrak{R}$-free wcDG-modules and $\mathfrak{R}$-free CDG-contramodules. Then one can argue as in the proof of part (a) above.

The proof of parts (c) and (d) are similar to the proofs of parts (b) and (a), respectively. To prove part (e), notice that for any $\mathfrak{R}$-free wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, there is a natural isomorphism of $\mathfrak{R}$-cofree CDG-comodules $\Phi_{\mathfrak{R}}(\mathfrak{C}\otimes^\tau\mathfrak{M}) \simeq \mathfrak{C}\odot^\tau\Phi_{\mathfrak{R}}(\mathfrak{M})$ over $\mathfrak{C}$. For any $\mathfrak{R}$-cofree wcDG-module $\mathcal{P}$ over $\mathfrak{A}$, there is a natural isomorphism of $\mathfrak{R}$-free CDG-contramodules $\Psi_{\mathfrak{R}}(\mathrm{Cohom}^\tau(\mathfrak{C},\mathcal{P})) \simeq \mathrm{Hom}^\tau(\mathfrak{C},\Psi_{\mathfrak{R}}(\mathcal{P}))$ over $\mathfrak{C}$. For any $\mathfrak{R}$-free wcDG-module $\mathfrak{P}$ over $\mathfrak{A}$, the functors $\Phi_{\mathfrak{C}} = \Psi_{\mathfrak{C}}^{-1}$ transform the CDG-contramodule $\mathrm{Hom}^\tau(\mathfrak{C},\mathfrak{P}) \in \mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{fr}}$ into the CDG-comodule $\mathfrak{C}\otimes^\tau\mathfrak{P} \in \mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{fr}}$ and back. For any $\mathfrak{R}$-cofree wcDG-module $\mathcal{M}$ over $\mathfrak{A}$, the functors $\Phi_{\mathfrak{C}} = \Psi_{\mathfrak{C}}^{-1}$ transform the CDG-contramodule $\mathrm{Ctrhom}^\tau(\mathfrak{C}, \mathcal{M}) \in \mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$ into the CDG-comodule $\mathfrak{C}\odot^\tau \mathcal{M} \in \mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$ and back. For any $\mathfrak{R}$-contramodule wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$, there is a natural isomorphism of $\mathfrak{R}$-comodule CDG-comodules $\Phi_{\mathfrak{R},\mathfrak{C}}(\mathrm{Hom}^\tau(\mathfrak{C}, \mathfrak{M})) \simeq \mathfrak{C}\odot^\tau \Phi_{\mathfrak{R}}(\mathfrak{M})$. For any $\mathfrak{R}$-comodule wcDG-module $\mathcal{P}$ over $\mathfrak{A}$, there is a natural isomorphism of $\mathfrak{R}$-contramodule CDG-contramodules $\Psi_{\mathfrak{R},\mathfrak{C}}(\mathfrak{C}\odot^\tau \mathcal{P}) \simeq \mathrm{Hom}^\tau(\mathfrak{C}, \Psi_{\mathfrak{R}}(\mathcal{P}))$. ☐

Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$, $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ be a homogeneous retraction onto the unit map, $\mathfrak{C} = \mathrm{Bar}_v(\mathfrak{A})$ be the corresponding $\mathfrak{R}$-free CDG-coalgebra, and $\tau_{\mathfrak{A},v}\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ be the natural twisting cochain.

Corollary 6.3.2. *All the assertions of Theorem* 6.3.1 *hold for the twisting cochain* $\tau = \tau_{\mathfrak{A},v}$.

Proof. The twisting cochain $\bar\tau_{\mathfrak{A},v} = \tau_{\mathfrak{A},v}/\mathfrak{m}\tau_{\mathfrak{A},v}$ is acyclic, as one can see by comparing [29, Theorems 6.3 and 6.5], or more directly from [29, Theorem 6.10(a)]. ☐

Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra; suppose the reduction $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is a coaugmented CDG-coalgebra over k with the coaugmentation $\bar w\colon k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C}$. Let $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be a homogeneous section of the counit map lifting the coaugmentation $\bar w$. Consider the corresponding wcDG-algebra $\mathfrak{A} = \mathrm{Cob}_w(\mathfrak{C})$ over $\mathfrak{R}$, and let $\tau_{\mathfrak{C},w}\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ be the natural twisting cochain.

Corollary 6.3.3. *All the assertions of Theorem* 6.3.1 *hold for the twisting cochain* $\tau = \tau_{\mathfrak{C},w}$, *provided that the CDG-coalgebra* $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ *is conilpotent.*

Proof. The twisting cochain $\bar\tau_{\mathfrak{C},w} = \tau_{\mathfrak{C},w}/\mathfrak{m}\tau_{\mathfrak{C},w}$ is acyclic by the definition (see also [29, Theorem 6.4]). ☐

6.4. **Nonconilpotent duality.** Let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra such that the graded k-algebra $\mathfrak{B}/\mathfrak{m}\mathfrak{B}$ has finite left homological dimension. Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra, $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$ be a twisting cochain, and $\bar\tau\colon \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{B}/\mathfrak{m}\mathfrak{B}$ be its reduction modulo $\mathfrak{m}$.

Theorem 6.4.1. *Assume one (or, equivalently, all) of the functors* $\mathfrak{C}/\mathfrak{m}\mathfrak{C} \otimes^{\bar\tau} -$, $\mathfrak{B}/\mathfrak{m}\mathfrak{B} \otimes^{\bar\tau} -$, $\mathrm{Hom}^{\bar\tau}(\mathfrak{C}/m\mathfrak{C}, -)$, *and/or* $\mathrm{Hom}^{\bar\tau}(\mathfrak{B}/\mathfrak{m}\mathfrak{B}, -)$ *induce equivalences between the triangulated categories* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}/\mathfrak{m}\mathfrak{B}\text{–}\mathsf{mod})$, $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}/\mathfrak{m}\mathfrak{C}\text{–}\mathsf{comod})$, *and/or* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}/\mathfrak{m}\mathfrak{C}\text{–}\mathsf{contra})$ *(see* [29, *Theorems* 6.7 *and* 6.8]*). Then*

(a) *the functors* $\mathfrak{C} \otimes^\tau -\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathfrak{B} \otimes^\tau -\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *induce functors* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, *which are mutually inverse equivalences of triangulated categories;*

(b) *the functors* $\mathrm{Hom}^\tau(\mathfrak{C}, -)\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *and* $\mathrm{Hom}^\tau(\mathfrak{B}, -)\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *induce functors* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *and* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, *which are mutually inverse equivalences of triangulated categories;*

(c) *the functors* $\mathfrak{C} \odot^\tau -\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathfrak{B} \odot^\tau -\colon H^0(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *induce functors* $\mathsf{D}^{\mathsf{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow$

$\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D^{co}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$, *which are mutually inverse equivalences of triangulated categories;*

(d) *the functors* $\mathrm{Ctrhom}^\tau(\mathfrak{C},-)\colon H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathrm{Ctrhom}^\tau(\mathfrak{B},-)\colon H^0(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *induce functors* $\mathsf{D^{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D^{abs}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$, *which are mutually inverse equivalences of triangulated categories;*

(e) *the above equivalences of triangulated categories form a commutative diagram with the equivalences of categories* $\Phi_\mathfrak{R} = \Psi_\mathfrak{R}^{-1}$ *of Sections* 2.5 *and* 3.4, $\mathbb{L}\Phi_\mathfrak{R} = \mathbb{R}\Psi_\mathfrak{R}^{-1}$ *of Corollary* 4.2.7, $\mathbb{L}\Phi_\mathfrak{C} = \mathbb{R}\Psi_\mathfrak{C}^{-1}$ *of Corollaries* 3.2.4 *and* 3.4.2, $\mathbb{L}\Phi_{\mathfrak{R},\mathfrak{C}} = \mathbb{R}\Psi_{\mathfrak{R},\mathfrak{C}}^{-1}$ *of Corollary* 4.5.3, *the equivalences of triangulated categories from Corollary* 4.2.6, *and the equivalences of contra/coderived categories from Theorem* 4.5.1.

Proof. The assertions about the existence of induced functors in parts (a-d) do not depend on the assumption on $\bar\tau$; the assertions about them being equivalences of categories do. Both functors in part (a) take coacyclic objects to contractible ones, since they transform infinite direct sums into infinite direct sums and short exact sequences into short exact sequences whose underlying sequences of graded objects are split exact. Both functors in part (c) take coacyclic objects to coacyclic objects, for the similar reasons. Analogously, both functors in part (d) take contraacyclic objects to contractible ones, and the functors in part (b) preserve contraacyclicity. It follows that the induced adjoint functors exist in all cases.

To prove that the adjunction morphisms in (a) are equivalences, one reduces them modulo $\mathfrak{m}$ and uses the assumption of Theorem together with Corollaries 2.2.5 and 3.2.3. To demonstrate the same assertion in the case (b), it suffices to restrict both functors to $\mathfrak{R}$-free CDG-modules and CDG-contramodules (see Theorems 4.2.1 and 4.5.1) and apply the same argument as in part (a).

Parts (c) and (d) are similar; and the proof of part (e) is identical to that of Theorem 6.3.1(e). ☐

Let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra and $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be a homogeneous section of the counit map. Consider the corresponding $\mathfrak{R}$-free CDG-algebra $\mathfrak{B} = \mathrm{Cob}_w(\mathfrak{C})$, and let $\tau_{\mathfrak{C},w}\colon \mathfrak{C} \longrightarrow \mathfrak{B}$ be the natural twisting cochain.

Corollary 6.4.2. *All the assertions of Theorem* 6.4.1 *are true for the twisting cochain* $\tau = \tau_{\mathfrak{C},w}$.

Proof. The assumption of Theorem 6.4.1 holds in this case by [29, Theorem 6.7]. ☐

Notice that by comparing Corollaries 6.3.3 and 6.4.2 one can conclude that $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) = \mathsf{D^{abs}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ and $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) = \mathsf{D^{abs}}(\mathfrak{C}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ when $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent, $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ reduces to the coaugmentation, and $\mathfrak{A} = \mathrm{Cob}_w(\mathfrak{C})$. In fact, this is a particular case of Theorem 2.3.3.

6.5. **Transformation of functors under Koszul duality.** Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$ and $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra with a conilpotent reduction $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$. Let $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ be a twisting cochain such that the twisting cochain $\bar\tau = \tau/\mathfrak{m}\tau$ is acyclic. Notice that by the right version of Theorem 6.3.1 the functors $\mathfrak{M} \longmapsto \mathfrak{M}\otimes^\tau\mathfrak{C}$

and $\mathfrak{N} \longmapsto \mathfrak{N} \otimes^\tau \mathfrak{A}$ induce an equivalence of triangulated categories $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$, while the functors $\mathcal{M} \longmapsto \mathcal{M} \odot^\tau \mathfrak{C}$ and $\mathcal{N} \longmapsto \mathcal{N} \odot^\tau \mathfrak{A}$ induce an equivalence of triangulated categories $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$.

The left derived functor

$$\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$

is obtained by switching the left and right sides in the construction of the derived functor $\operatorname{Tor}^{\mathfrak{A}}$ from Section 4.3.

The following theorem shows how Koszul duality transforms the functor $\operatorname{Tor}^{\mathfrak{A}}$ into the functor $\operatorname{Ctrtor}^{\mathfrak{C}}$.

Theorem 6.5.1. (a) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.3 *into the functor* $\operatorname{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.2.

(b) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 2.6 *into the functor* $\operatorname{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 3.4.

(c) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *into the functor* $\operatorname{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C}) \times \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 4.5.

The following theorem shows how Koszul duality transforms the functor $\operatorname{Tor}^{\mathfrak{A}}$ into the functor $\operatorname{Cotor}^{\mathfrak{C}}$.

Theorem 6.5.2. (a) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.3 *into the functor* $\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.2.

(b) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 2.6 *into the functor* $\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 3.4.

(c) *The equivalences of categories* $\mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \simeq \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$ *and* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *transform the functor* $\operatorname{Tor}^{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 4.3 *into the functor* $\operatorname{Cotor}^{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C}) \times \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 4.5.

The following theorem shows how Koszul duality transforms the functor $\operatorname{Ext}_{\mathfrak{A}}$ into the functor $\operatorname{Ext}^{\mathfrak{C}}$.

Theorem 6.5.3. (a) *The equivalence of categories* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *transforms the functor* $\operatorname{Ext}_{\mathfrak{A}}\colon \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow$

$H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.3 *into the functor* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.5.

(b) *The equivalence of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transforms the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.6 *into the functor* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.4.

(c) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 2.6 *into the functor* $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 3.4.

The following theorem shows how Koszul duality transforms the functor $\mathrm{Ext}_{\mathfrak{A}}$ into the functor $\mathrm{Ext}_{\mathfrak{C}}$.

Theorem 6.5.4. (a) *The equivalence of categories* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *transforms the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.3 *into the functor* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.2.

(b) *The equivalence of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *transforms the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.3 *into the functor* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.5.

(c) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 2.6 *into the functor* $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 3.4.

The following theorem shows how Koszul duality transforms the functor $\mathrm{Ext}_{\mathfrak{A}}$ into the functor $\mathrm{Coext}_{\mathfrak{C}}$.

Theorem 6.5.5. (a) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.3 *into the functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.2.

(b) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 2.6 *into the functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 3.4.

(c) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 2.6 *into the functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ *from Section* 3.4.

(d) *The equivalences of categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *transform the functor* $\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}}$

$\times\, \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.3 *into the functor* $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}})^{\mathsf{op}} \times \mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$ *from Section* 4.5.

Proof of Theorems 6.5.1–6.5.5. The proofs of all the sixteen assertions are very similar to each other, so will only discuss two randomly chosen ones, namely, Theorem 6.5.2(c) and Theorem 6.5.3(a).

Let $\mathcal{M}$ be an $\mathfrak{R}$-comodule left wcDG-module over $\mathfrak{A}$ and $\mathcal{N}$ be an $\mathfrak{R}$-comodule right CDG-comodule over $\mathfrak{C}$. Then there is a natural isomorphism of complexes of $\mathfrak{R}$-comodules $\mathcal{N}\,\square_{\mathfrak{C}}\,(\mathfrak{C}\odot^{\tau}\mathcal{M}) \simeq (\mathcal{N}\odot^{\tau}\mathfrak{A})\otimes_{\mathfrak{A}}\mathcal{M}$. Indeed, both complexes are isomorphic to the complex $\mathcal{N}\,\square_{\mathfrak{R}}\,\mathcal{M}$ with the differential induced by the differentials on $\mathcal{N}$ and $\mathcal{M}$ being twisted using the twisting cochain τ. Notice that $\mathfrak{C}\odot^{\tau}\mathcal{M} \in \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–cof}}_{\mathsf{inj}}$ whenever $\mathcal{M}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–cof}}$ and $\mathcal{N}\odot^{\tau}\mathfrak{A}\in H^0(\mathsf{mod}^{\mathfrak{R}\text{–cof}}_{\mathsf{proj}}\text{–}\mathfrak{A})_{\mathsf{proj}}$ whenever $\mathcal{N}\in\mathsf{comod}^{\mathfrak{R}\text{–cof}}\text{–}\mathfrak{C}$. The latter assertion follows from the adjunction of the functors $-\odot^{\tau}\mathfrak{A}$ and $-\odot^{\tau}\mathfrak{C}$ together with the fact that the latter functor takes semiacyclic $\mathfrak{R}$-cofree wcDG-modules to contractible $\mathfrak{R}$-cofree CDG-comodules (see the proof of Theorem 6.3.1(a)) and the semiorthogonal decomposition of Theorem 2.6.2(a).

Let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule left wcDG-module over $\mathfrak{A}$ and $\mathfrak{Q}$ be an $\mathfrak{R}$-contramodule left CDG-contramodule over $\mathfrak{C}$. Then there is a natural isomorphism of complexes of $\mathfrak{R}$-contramodules $\mathrm{Hom}^{\mathfrak{C}}(\mathrm{Hom}^{\tau}(\mathfrak{C},\mathfrak{P}),\mathfrak{Q}) \simeq \mathrm{Hom}_{\mathfrak{A}}(\mathfrak{P},\mathrm{Hom}^{\tau}(\mathfrak{A},\mathfrak{Q}))$. Indeed, both complexes are isomorphic to the complex $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q})$ with the differential induced by the differentials on $\mathfrak{P}$ and $\mathfrak{Q}$ being twisted by τ. Notice that $\mathrm{Hom}^{\tau}(\mathfrak{C},\mathfrak{P}) \in \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–fr}}_{\mathsf{proj}}$ whenever $\mathfrak{P}\in\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}$ and $\mathrm{Hom}^{\tau}(\mathfrak{A},\mathfrak{Q})\in H^0(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}}_{\mathsf{inj}})_{\mathsf{inj}}$ whenever $\mathfrak{Q}\in\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–fr}}$. The proof of the latter assertion is similar to the above. (See [29, proof of Theorem 6.9.1] for further details.) ☐

Let $\mathfrak{A}$ and $\mathfrak{B}$ be wcDG-algebras over $\mathfrak{R}$, let $\mathfrak{C}$ and $\mathfrak{D}$ be $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ and $\mathfrak{D}/\mathfrak{m}\mathfrak{D}$, and let $\tau\colon\mathfrak{C}\longrightarrow\mathfrak{A}$ and $\sigma\colon\mathfrak{D}\longrightarrow\mathfrak{B}$ be twisting cochains with acyclic reductions $\tau/\mathfrak{m}\tau$ and $\sigma/\mathfrak{m}\sigma$. Let $f=(f,\xi)\colon\mathfrak{A}\longrightarrow\mathfrak{B}$ be a morphism of wcDG-algebras and $g=(g,\eta)\colon\mathfrak{C}\longrightarrow\mathfrak{D}$ be a morphism of CDG-coalgebras. Assume that (f,ξ) and (g,η) make a commutative diagram with τ and σ, i. e., $\sigma\circ(g,\eta)=(f,\xi)\circ\tau$, or explicitly, $\sigma\circ g-f\circ\tau=e_{\mathfrak{B}}\circ\eta+\xi\circ\varepsilon_{\mathfrak{C}}$.

Proposition 6.5.6. (a) *The equivalences of triangulated categories* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\simeq\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})$ *and* $\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\simeq\mathsf{D^{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})$ *transform the functor* $\mathbb{I}R_f\colon\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ *into the functor* $\mathbb{R}E_g\colon\mathsf{D^{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})\longrightarrow\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})$ *and the functor* $\mathbb{L}E_f\colon\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})\longrightarrow\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–fr}})$ *into the functor* $\mathbb{I}R_g\colon\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})\longrightarrow\mathsf{D^{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–fr}})$.

(b) *The equivalences of triangulated categories* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})\simeq\mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})$ *and* $\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})\simeq\mathsf{D^{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})$ *transform the functor* $\mathbb{I}R_f\colon\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})\longrightarrow\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ *into the functor* $\mathbb{L}E^g\colon\mathsf{D^{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})\longrightarrow\mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})$ *and the functor* $\mathbb{R}E^f\colon\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})\longrightarrow\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–ctr}})$ *into the functor* $\mathbb{I}R^g\colon\mathsf{D^{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})\longrightarrow\mathsf{D^{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–ctr}})$.

(c) *The equivalences of triangulated categories* $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})\simeq\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}})$ *and* $\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})\simeq\mathsf{D^{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–co}})$ *transform the functor* $\mathbb{I}R_f\colon\mathsf{D^{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–co}})$

$\longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *into the functor* $\mathbb{R}E_g\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *and the functor* $\mathbb{L}E_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *into the functor* $\mathbb{I}R_g\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$.

(d) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and* $\mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *transform the functor* $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *into the functor* $\mathbb{L}E^g\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *and the functor* $\mathbb{R}E^f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *into the functor* $\mathbb{I}R^g\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$.

Proof. See [29, proof of Proposition 6.9]. ☐

Remark 6.5.7. The analogues of the results of this section also hold for the nonconilpotent Koszul duality theory of Section 6.4. In order to formulate them, though, one needs to define the Tor, Ext, and derived (co)extension-of-scalars functors acting in the co/contra/absolute derived categories of CDG-modules. This can be done, at least, for a CDG-algebra $\mathfrak{B}$ such that the categories of $\mathfrak{R}$-(co)free graded left/right $\mathfrak{B}$-modules have finite homological dimension, using the results of Theorem 2.2.4 and Corollary 4.2.6. (Cf. [29, Section 3.12 and Theorem 6.9.2].)

6.6. **Examples.** Here we use the results of Sections 6.3 and 6.5 in order to compute the remaining two of the three examples mentioned in the Introduction (for the first one, see Example 5.3.5). We will only discuss the $\mathfrak{R}$-contramodules of morphisms and the complexes $\mathrm{Ext}_{\mathfrak{A}}$ of free $\mathfrak{R}$-contramodules in the exotic derived categories of wcDG-modules over certain wcDG-algebras $\mathfrak{A}$.

According to the results of Section 7.3 below, the semiderived categories of strictly unital wc A_∞-modules over the wcDG-algebra $\mathfrak{A}$ viewed as a strictly unital wc A_∞-algebra are equivalent to the semiderived categories of wcDG-modules, and the complexes of free $\mathfrak{R}$-contramodules Ext in them are isomorphic in the homotopy category. Moreover, in both examples the wcDG-algebra $\mathfrak{A}$ is actually augmented, so one can interpret the same $\mathfrak{R}$-contramodules of morphisms and complexes of free $\mathfrak{R}$-contramodules $\mathrm{Ext}_{\mathfrak{A}}$ as being related to the semiderived categories of nonunital wc A_∞-modules over the augmentation ideal of $\mathfrak{A}$ viewed as a nonunital wc A_∞-algebra (see Sections 7.1–7.2).

Example 6.6.1. Let $\mathfrak{R}$ be a pro-Artinian topological local ring with the maximal ideal $\mathfrak{m}$, and let $\epsilon \in \mathfrak{m}$ be an element. Consider the $\mathfrak{R}$-free graded algebra $\mathfrak{A} = \mathfrak{R}[x] = \bigoplus_{n=0}^{\infty} \mathfrak{R}x^n$, where the direct sum is taken in the category of free graded $\mathfrak{R}$-contramodules and $\deg x = 2$. Define the wcDG-algebra structure on $\mathfrak{A}$ with $d = 0$ and $h = \epsilon x$ (cf. Example 5.3.5).

Let $\mathfrak{C}$ be the $\mathfrak{R}$-free DG-coalgebra such that the $\mathfrak{R}$-free DG-algebra $\mathfrak{C}^*$ is isomorphic to the graded algebra $\mathfrak{R}[y]/(y^2)$ (i. e., the direct sum of $\mathfrak{R}$ and $\mathfrak{R}y$) with $\deg y = -1$ and $d(y) = \epsilon$. Then the wcDG-algebra $\mathfrak{A}$ is isomorphic to $\mathrm{Cob}_w(\mathfrak{C})$, where $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ is defined by the rule that $w^*\colon \mathfrak{C}^* \longrightarrow \mathfrak{R}$ takes y to 0.

Let N be any ($\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule) DG-contramodule or DG-comodule over $\mathfrak{C}$; equivalently, it can be viewed as a DG-module over $\mathfrak{C}^*$. Then the action of the element $y \in \mathfrak{C}^*$ provides a contracting homotopy for the endomorphism of multiplication with ϵ on N. By Corollary 6.3.3, it follows that all the $\mathfrak{R}$-modules of morphisms in the $\mathfrak{R}$-linear triangulated category $\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ are annihilated by the multiplication with ϵ.

By Theorem 2.3.3 and Corollaries 4.2.6–4.2.7, the same applies to the $\mathfrak{R}$-linear triangulated categories $\mathsf{D^{abs}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D^{abs}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ and $\mathsf{D^{ctr}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D^{co}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$.

Furthermore, by Theorem 6.5.3 or 6.5.4, it follows that the endomorphisms of multiplication with ϵ are homotopic to zero on the complexes of free $\mathfrak{R}$-contramodules $\mathrm{Ext}_{\mathfrak{A}}$ between $\mathfrak{R}$-comodule or $\mathfrak{R}$-contramodule wcDG-modules over $\mathfrak{A}$.

Example 6.6.2. Let $\mathfrak{R}$ be a pro-Artinian topological local ring and $\mathfrak{C}$ be an ungraded $\mathfrak{R}$-free coalgebra considered as an $\mathfrak{R}$-free CDG-coalgebra concentrated entirely in grading zero with the trivial CDG-coalgebra structure $d = 0 = h$. Assume that the reduced coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent and pick a section $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ of the counit map such that $w/\mathfrak{m}w = \bar{w}$ is the coaugmentation of $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$.

Let $\mathfrak{N}$ be an $\mathfrak{R}$-free $\mathfrak{C}$-comodule viewed as a CDG-comodule concentrated in degree zero and with zero differential. By [28, Remark 4.1], one has $\mathrm{Hom}_{\mathsf{D^{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})}(\mathfrak{N},\mathfrak{N}) \simeq \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{N},\mathfrak{N})$. The same applies to $\mathfrak{R}$-contramodule $\mathfrak{C}$-contramodules, $\mathfrak{R}$-comodule $\mathfrak{C}$-comodules, etc. In particular, for $\mathfrak{N} = \mathfrak{C}$ one has $\mathrm{Ext}_{\mathfrak{C}}(\mathfrak{N},\mathfrak{N}) \simeq \mathfrak{C}^*$ (by the definition and since $\mathfrak{N} \in \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}_{\mathsf{inj}}$).

Set $\mathfrak{A} = \mathrm{Cob}_w(\mathfrak{C})$ and $\mathfrak{M} = \mathfrak{A} \otimes^{\tau_{\mathfrak{C},w}} \mathfrak{N}$. By Corollary 6.3.3, it follows that $\mathrm{Hom}_{\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})}(\mathfrak{M},\mathfrak{M}) \simeq \mathrm{Hom}_{\mathfrak{C}}(\mathfrak{N},\mathfrak{N})$. In particular, for $\mathfrak{N} = \mathfrak{C}$ we have $\mathrm{Ext}_{\mathfrak{A}}(\mathfrak{M},\mathfrak{M}) \simeq \mathfrak{C}^*$ by Theorem 6.5.4(a).

To point out a specific example, let $\mathfrak{C}^*$ be the Clifford algebra with one generator $\mathfrak{R}[y]/(y^2 - \epsilon)$, where $\epsilon \in \mathfrak{R}$. Being by the definition a free $\mathfrak{R}$-module with two generators endowed with an $\mathfrak{R}$-algebra structure, it is clearly an algebra in the tensor category of (finitely generated) free $\mathfrak{R}$-contramodules (since the tensor categories of finitely generated free $\mathfrak{R}$-modules and finitely generated free $\mathfrak{R}$-contramodules are equivalent). One easily recovers from it the dual $\mathfrak{R}$-free coalgebra $\mathfrak{C}$. Let $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be the section of the counit map given by the rule $w^*(y) = 0$.

Then $\mathfrak{A} = \mathrm{Cob}_w(\mathfrak{C})$ is the $\mathfrak{R}$-free graded algebra $\mathfrak{R}[x]$ with $\deg x = 1$ endowed with the wcDG-algebra structure with $d = 0$ and $h = \epsilon x^2$. Thus there is wcDG-module structure on the free graded $\mathfrak{A}$-module $\mathfrak{M}$ with two generators in degree 0 such that the $\mathfrak{R}$-module $\mathrm{Hom}_{\mathsf{D^{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})}(\mathfrak{M},\mathfrak{M}[i])$ is isomorphic to the two-dimensional $\mathfrak{R}$-free algebra $\mathfrak{C}^*$ when $i = 0$ and vanishes otherwise. More precisely, the complex of free $\mathfrak{R}$-contramodules $\mathrm{Ext}_{\mathfrak{A}}(\mathfrak{M},\mathfrak{M})$ is homotopy equivalent to $\mathfrak{C}^*$.

6.7. **Duality between algebras and coalgebras.** Let $\mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$ denote the category of $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions modulo $\mathfrak{m}$. The objects of this category are $\mathfrak{R}$-free CDG-coalgebras $\mathfrak{C}$ such that the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ over k is conilpotent. Morphisms $\mathfrak{C} \longrightarrow \mathfrak{D}$ in $\mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$ are $\mathfrak{R}$-free CDG-coalgebra morphisms $(f,a)\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ whose reductions $(f/\mathfrak{m}f,\ a/\mathfrak{m}a)\colon \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{D}/\mathfrak{m}\mathfrak{D}$ are conilpotent CDG-coalgebra morphisms; in other words, the composition

$k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow k$ of the coaugmentation and the reduced change-of-connection maps is requred to be zero (a condition only nontrivial when $1 = 0$ in Γ and $2 = 0$ in $\mathfrak{R}$). The category of nonzero wcDG-algebras over $\mathfrak{R}$ will be denoted by $\mathfrak{R}\text{–}\mathsf{alg}_{\mathsf{wcdg}}^+$.

It follows from Lemma 6.1.3 that the functor $\mathfrak{A} \longmapsto \mathrm{Bar}_v(\mathfrak{A})$ acting from the category $\mathfrak{R}\text{–}\mathsf{alg}_{\mathsf{wcdg}}^+$ to the category $\mathfrak{R}\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{k\text{–}\mathsf{cnlp}}$ is right adjoint to the functor $\mathfrak{C} \longmapsto \mathrm{Cob}_w(\mathfrak{C})$ acting in the opposite direction.

Indeed, let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$ and $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra with conilpotent reduction. Let $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ be a homogeneous retraction onto the unit map, and let $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ be a homogeneous section of the counit map such that $\bar w = w/\mathfrak{m}w\colon k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is the coaugmentation. Then both wcDG-algebra morphisms $\mathrm{Cob}_w(\mathfrak{C}) \longrightarrow \mathfrak{A}$ and CDG-coalgebra morphisms $\mathfrak{C} \longrightarrow \mathrm{Bar}_v(\mathfrak{A})$ correspond bijectively to twisting cochains $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ for which the composition $k \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ vanishes. The former bijection assigns to a morphism $(f,a)\colon \mathrm{Cob}_w(\mathfrak{C}) \longrightarrow \mathfrak{A}$ the twisting cochain $(f,a)\circ\tau_{\mathfrak{C},w}$, and the latter one assigns to a morphism $(g,a)\colon \mathfrak{C} \longrightarrow \mathrm{Bar}_v(\mathfrak{A})$ the twisting cochain $\tau_{\mathfrak{A},v}\circ(g,a)$.

More generally, let $\mathfrak{B}$ be an $\mathfrak{R}$-free CDG-algebra and $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-coalgebra. Let $v\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ be a homogeneous retraction and $w\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ be a homogeneous section. Then CDG-coalgebra morphisms $\mathrm{Cob}_w(\mathfrak{C}) \longrightarrow \mathfrak{B}$ correspond bijectively to twisting cochains $\tau\colon \mathfrak{C} \longrightarrow \mathfrak{B}$.

Let us call a wcDG-algebra morphism $(f,a)\colon \mathfrak{B} \longrightarrow \mathfrak{A}$ a *semi-isomorphism* if the DG-algebra morphism $f/\mathfrak{m}f\colon \mathfrak{B}/\mathfrak{m}\mathfrak{B} \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is a quasi-isomorphism. The definition of the equivalence relation on CDG-coalgebras with conilpotent reductions requires a little more effort.

Let C be a coaugmented CDG-coalgebra over k with the coaugmentation $\bar w\colon k \longrightarrow C$. An *admissible filtration* F on C [17] (see also [29, Section 6.10]) is an increasing filtration $F_0C \subset F_1C \subset F_2C \subset \cdots$ that is preserved by the differential, compatible with the comultiplication, and such that $F_0C = \bar w(k)$ and $C = \bigcup_n F_nC$. The compatibility with the comultiplication means, as usually, that $\mu(F_nC) \subset \sum_{p+q=n} F_pC \otimes_k F_qC$, where $\mu\colon C \longrightarrow C\otimes_k C$ is the comultiplication map.

A coaugmented CDG-coalgebra C over k admits an admissible filtration if and only if it is conilpotent. A conilpotent CDG-coalgebra C has a canonical (maximal) admissible filtration F given by the rule $F_nC = \ker(C \to (C/\bar w(k))^{\otimes n+1})$; for any other admissible filtration G on C, one has $G_nC \subset F_nC$.

Let F be an admissible filtration on a conilpotent CDG-coalgebra C. Then the associated quotient coalgebra $\mathrm{gr}_F C = \bigoplus_n F_nC/F_{n-1}C$ with the induced differential is a DG-coalgebra over k (since it is assumed that $h\circ\bar w = 0$). A morphism of conilpotent CDG-coalgebras $(g,a)\colon C \longrightarrow D$ is said to be a *filtered quasi-isomorphism* if there exist admissible filtrations F on C and D such that $g(F_nC) \subset F_nD$ and the induced morphisms of complexes $F_nC/F_{n-1}C \longrightarrow F_nD/F_{n-1}D$ are quasi-isomorphisms.

A morphism of $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions $(f,a)\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is called a *filtered semi-isomorphism* if its reduction $(f/\mathfrak{m}f,\ a/\mathfrak{m}a)\colon \mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{D}/\mathfrak{m}\mathfrak{D}$ is a filtered quasi-isomorphism. In other words, there should exist admissible filtrations F on the CDG-coalgebras $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ and $\mathfrak{D}/\mathfrak{m}\mathfrak{D}$ over k such that the above conditions are satisfied. The classes of semi-isomorphisms of wcDG-algebras over $\mathfrak{R}$ and filtered semi-isomorphisms of $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions will be denoted by $\mathsf{Seis} \subset \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}$ and $\mathsf{FSeis} \subset \mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$.

Let us make a warning that the class of filtered semi-isomorphisms FSeis is *not* closed under compositions, fractions, or retractions of morphisms in $\mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$ (while the class of semi-isomorphisms $\mathsf{Seis} \subset \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}$ is, of course, closed under these operations).

Theorem 6.7.1. *The functors* $\mathrm{Bar}_v\colon \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}} \longrightarrow \mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$ *and* $\mathrm{Cob}_w\colon \mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}} \longrightarrow \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}$ *induce functors between the localized categories*

$$\mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}[\mathsf{Seis}^{-1}] \longrightarrow \mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}[\mathsf{FSeis}^{-1}]$$

and

$$\mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}[\mathsf{FSeis}^{-1}] \longrightarrow \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}[\mathsf{Seis}^{-1}],$$

which are mutually inverse equivalences of categories.

Proof. It follows from the arguments in the proof of [29, Theorem 6.10] that the functor Bar_v takes Seis to FSeis and the functor Cob_w takes FSeis to Seis. Furthermore, the adjunction morphisms $\mathrm{Cob}_v(\mathrm{Bar}_w(\mathfrak{A})) \longrightarrow \mathfrak{A}$ belong to Seis and the adjunction morphisms $\mathfrak{C} \longrightarrow \mathrm{Bar}_w(\mathrm{Cob}_v(\mathfrak{C}))$ belong to FSeis for all wcDG-algebras $\mathfrak{A} \in \mathfrak{R}\text{–}\mathsf{alg}^+_{\mathsf{wcdg}}$ and all $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions $\mathfrak{C} \in \mathfrak{R}\text{–}\mathsf{coalg}^{k\text{–}\mathsf{cnlp}}_{\mathsf{cdg}}$. ☐

7. Strictly Unital Weakly Curved A_∞-Algebras

7.1. **Nonunital wc A_∞-algebras.** Let $\mathfrak{A}$ be a free graded $\mathfrak{R}$-contramodule. Consider the tensor coalgebra $\mathfrak{C} = \bigoplus_{n=0}^\infty \mathfrak{A}[1]^{\otimes n}$; it is an $\mathfrak{R}$-free graded coalgebra. Its reduction $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is the tensor coalgebra $\bigoplus_{n=0}^\infty \mathfrak{A}/\mathfrak{m}\mathfrak{A}[1]^{\otimes n}$. Being a conilpotent graded k-coalgebra, $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ has a unique coaugmentation $k \simeq \mathfrak{A}/\mathfrak{m}\mathfrak{A}[1]^{\otimes 0} \longrightarrow \bigoplus_n \mathfrak{A}/\mathfrak{m}\mathfrak{A}[1]^{\otimes n}$, which we will denote by $\bar w$.

A *nonunital wc* A_∞*-algebra* structure on $\mathfrak{A}$ is, by the definition, a structure of $\mathfrak{R}$-free DG-coalgebra with a conilpotent reduction on the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$, i. e., an odd coderivation $d\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n}$ of degree 1 such that $d^2 = 0$ and $d/\mathfrak{m}d \circ \bar w = 0$. A nonunital wc A_∞-algebra $\mathfrak{A}$ over $\mathfrak{R}$ is a free graded $\mathfrak{R}$-contramodule endowed with the mentioned structure.

Since a coderivation of $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$ is uniquely determined by its composition with the projection $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{A}[1]^{\otimes 1} \simeq \mathfrak{A}[1]$, a nonunital wc A_∞-algebra structure on $\mathfrak{A}$ can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $m_n\colon \mathfrak{A}^{\otimes n} \longrightarrow \mathfrak{A}$, $n = 0, 1, 2, \ldots$ of degree $2 - n$ (see Lemma 6.1.2(b)). For the sign rule (here and below), see [29, Section 7.1]. The sequence of maps m_n must satisfy the *weak curvature* condition $m_0/\mathfrak{m}m_0 = 0$ (i. e., $m_0 \in \mathfrak{m}\mathfrak{A}^2$) and a sequence of quadratic equations corresponding to the equation $d^2 = 0$ on the coderivation d.

A morphism of nonunital wc A_∞-algebras $f\colon \mathfrak{A}\longrightarrow\mathfrak{B}$ over $\mathfrak{R}$ is, by the definition, a morphism of $\mathfrak{R}$-free DG-coalgebras $\bigoplus_n \mathfrak{A}[1]^{\otimes n}\longrightarrow\bigoplus_n \mathfrak{B}[1]^{\otimes n}$ (such a morphism is always compatible with the coaugmentations modulo $\mathfrak{m}$).

Since an $\mathfrak{R}$-free graded coalgebra morphism into $\bigoplus_n \mathfrak{B}[1]^{\otimes n}$ is uniquely determined by its composition with the projection $\bigoplus_n \mathfrak{B}[1]^{\otimes n}\longrightarrow\mathfrak{B}[1]$, a morphism of nonunital wc A_∞-algebras $f\colon\mathfrak{A}\longrightarrow\mathfrak{B}$ can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $f_n\colon \mathfrak{A}^{\otimes n}\longrightarrow\mathfrak{B}$, $n=0, 1, 2, \dots$ of degree $1-n$ (see Lemma 6.1.3(b)). The sequence of maps f_n must satisfy the *weak change of connection* condition $f_0/\mathfrak{m}f_0=0$ (i. e., $f_0\in\mathfrak{m}\mathfrak{B}^1$) and a sequence of polynomial equations corresponding to the equation $d\circ f=f\circ d$ on the morphism f. (With respect to f_0, these are even formal power series rather than polynomials.)

Lemma 7.1.1. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra endowed with an odd coderivation d of degree 1. Then*

(a) *for any free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$, odd coderivations of degree 1 on the $\mathfrak{R}$-free graded left $\mathfrak{C}$-comodule $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V}$ compatible with the coderivation d on $\mathfrak{C}$ are determined by their compositions with the map $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V}\longrightarrow\mathfrak{V}$ induced by the counit of $\mathfrak{C}$. Conversely, any homogeneous $\mathfrak{R}$-contramodule morphism $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V}\longrightarrow\mathfrak{V}$ of degree 1 gives rise to an odd coderivation of degree 1 on $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{V}$ compatible with the coderivation d on $\mathfrak{C}$;*

(b) *for any graded $\mathfrak{R}$-contramodule $\mathfrak{U}$, odd contraderivations of degree 1 on the $\mathfrak{R}$-contramodule graded left $\mathfrak{C}$-contramodule $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ compatible with the coderivation d on $\mathfrak{C}$ are determined by their compositions with the map $\mathfrak{U}\longrightarrow\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ induced by the counit of $\mathfrak{C}$. Conversely, any homogeneous $\mathfrak{R}$-contramodule morphism $\mathfrak{U}\longrightarrow\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ of degree 1 gives rise to an odd contraderivation of degree 1 on $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$ compatible with the coderivation d on $\mathfrak{C}$;*

(c) *for any graded $\mathfrak{R}$-comodule $\mathcal{V}$, odd coderivations of degree 1 on the $\mathfrak{R}$-comodule graded left $\mathfrak{C}$-comodule $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$ compatible with the coderivation d on $\mathfrak{C}$ are determined by their compositions with the map $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}\longrightarrow\mathcal{V}$ induced by the counit of $\mathfrak{C}$. Conversely, any homogeneous $\mathfrak{R}$-comodule morphism $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}\longrightarrow\mathcal{V}$ of degree 1 gives rise to an odd coderivation of degree 1 on $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$ compatible with the coderivation d on $\mathfrak{C}$;*

(d) *for any cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$, odd contraderivations of degree 1 on the $\mathfrak{R}$-cofree graded left $\mathfrak{C}$-contramodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{V})$ compatible with the coderivation d on $\mathfrak{C}$ are determined by their compositions with the map $\mathcal{V}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{V})$ induced by the counit of $\mathfrak{C}$. Conversely, any homogeneous $\mathfrak{R}$-comodule morphism $\mathcal{V}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{V})$ of degree 1 gives rise to an odd contraderivation of degree 1 on $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{V})$ compatible with the coderivation d on $\mathfrak{C}$.*

Proof. Straightforward from the definitions. □

Let $\mathfrak{A}$ be a nonunital wc A_∞-algebra over $\mathfrak{R}$ and $\mathfrak{M}$ be a free graded $\mathfrak{R}$-contramodule. A structure of *nonunital $\mathfrak{R}$-free left wc A_∞-module* over $\mathfrak{A}$ on $\mathfrak{M}$ is, by the definition, a structure of DG-comodule over the DG-coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$ on the injective $\mathfrak{R}$-free graded left comodule $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$ over the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$. By Lemma 7.1.1(a), a nonunital left wc A_∞-module structure on $\mathfrak{M}$ can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $l_n\colon \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$, $n = 0, 1, \dots$ of degree $1 - n$. The sequence of maps l_n must satisfy a system of nonhomogeneous quadratic equations corresponding to the equation $d^2 = 0$ on the coderivation d on $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$. *Nonunital $\mathfrak{R}$-free right wc* A_∞*-modules* over $\mathfrak{A}$ are defined similarly (see [29, Section 7.1]).

The complex of morphisms between nonunital $\mathfrak{R}$-free left wc A_∞-modules $\mathfrak{L}$ and $\mathfrak{M}$ over $\mathfrak{A}$ is, by the definition, the complex of morphisms between the $\mathfrak{R}$-free left DG-comodules $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{L}$ and $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$ over the $\mathfrak{R}$-free DG-coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$. Since an $\mathfrak{R}$-free graded $\mathfrak{C}$-comodule morphism into $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ is determined by its composition with the map $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$ induced by the counit of $\mathfrak{C}$, a morphism of nonunital $\mathfrak{R}$-free left wc A_∞-modules $f\colon \mathfrak{L} \longrightarrow \mathfrak{M}$ of degree i can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $f_n\colon \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{L} \longrightarrow \mathfrak{M}$, $n = 0, 1, \dots$ of degree $i - n$ (see Section 3.1). Any sequence of homogeneous $\mathfrak{R}$-contramodule maps f_n of the required degrees corresponds to a (not necessarily closed) morphism of nonunital $\mathfrak{R}$-free wc A_∞-modules f.

Let $\mathfrak{P}$ be a graded $\mathfrak{R}$-contramodule. A structure of *nonunital $\mathfrak{R}$-contramodule left wc* A_∞*-module* over $\mathfrak{A}$ on $\mathfrak{P}$ is, by the definition, a structure of DG-contramodule over the DG-coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$ on the induced $\mathfrak{R}$-contramodule graded left contramodule $\mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{A}[1]^{\otimes n}, \mathfrak{P})$ over the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$. By Lemma 7.1.1(b), a nonunital left wc A_∞-module structure on $\mathfrak{P}$ can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $p_n\colon \mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{P})$, $n = 0, 1, \dots$ of degree $1 - n$. The sequence of maps p_n must satisfy a sequence of nonhomogeneous quadratic equations corresponding to the equation $d^2 = 0$ on the contraderivation d on $\mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{A}[1]^{\otimes n}, \mathfrak{P})$.

The complex of morphisms between nonunital $\mathfrak{R}$-contramodule left wc A_∞-modules $\mathfrak{P}$ and $\mathfrak{Q}$ over $\mathfrak{A}$ is, by the definition, the complex of morphisms between the $\mathfrak{R}$-free left DG-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{A}[1]^{\otimes n}, \mathfrak{P})$ and $\mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{A}[1]^{\otimes n}, \mathfrak{Q})$ over the $\mathfrak{R}$-free DG-coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$. Since an $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-comodule morphism from $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ is determined by the its composition with the map $\mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ induced by the counit of $\mathfrak{C}$, a morphism of nonunital $\mathfrak{R}$-contramodule left wc A_∞-modules $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ of degree i can be viewed as a sequence of $\mathfrak{R}$-contramodule morphisms $f^n\colon \mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{Q})$ of degree $i - n$ (see Section 4.4). Any sequence of homogeneous $\mathfrak{R}$-contramodule maps f^n of the required degrees corresponds to a (not necessarily closed) morphism of nonunital $\mathfrak{R}$-contramodule wc A_∞-modules f.

For any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, the functors $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ of Section 3.2 provide an equivalence between the DG-category of $\mathfrak{R}$-free left CDG-comodules over $\mathfrak{C}$ which, considered as graded $\mathfrak{C}$-comodules, are cofreely cogenerated by free graded $\mathfrak{R}$-contramodules, and the DG-category of $\mathfrak{R}$-free left CDG-contramodules over $\mathfrak{C}$ which, considered as graded $\mathfrak{C}$-contramodules, are freely generated by free graded $\mathfrak{R}$-contramodules. So our terminology is consistent: a nonunital $\mathfrak{R}$-free left wc A_∞-module $\mathfrak{M}$ over $\mathfrak{A}$ with the structure maps $l_n\colon \mathfrak{A}^{\otimes n}\otimes^{\mathfrak{R}}\mathfrak{M}\longrightarrow\mathfrak{M}$ can be equivalently viewed as a nonunital $\mathfrak{R}$-contramodule wc A_∞-module with a free underlying graded $\mathfrak{R}$-contramodule and the structure maps $p_n\colon\mathfrak{M}\longrightarrow\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n},\mathfrak{M})$ given by the rule $p_n(x)(a_1\otimes\dots\otimes a_n)=(-1)^{|x|(|a_1|+\dots+|a_n|)}l_n(a_1\otimes\dots\otimes a_n\otimes x)$. Similarly, a morphism of nonunital $\mathfrak{R}$-free left wc A_∞-modules $\mathfrak{L}\longrightarrow\mathfrak{M}$ over $\mathfrak{A}$ with the components $f_n\colon\mathfrak{A}^{\otimes n}\otimes^{\mathfrak{R}}\mathfrak{L}\longrightarrow\mathfrak{M}$ can be equivalently viewed as a morphism of nonunital $\mathfrak{R}$-contramodule wc A_∞-modules $\mathfrak{L}\longrightarrow\mathfrak{M}$ with the components $f^n\colon\mathfrak{L}\longrightarrow\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n},\mathfrak{M})$ given by the rule $f^n(x)(a_1\otimes\dots\otimes a_n)=(-1)^{|x|(|a_1|+\dots+|a_n|)}f_n(a_1\otimes\dots\otimes a_n\otimes x)$.

Let $\mathcal{M}$ be a graded $\mathfrak{R}$-comodule. A structure of *nonunital $\mathfrak{R}$-comodule left wc A_∞-module* over $\mathfrak{A}$ on $\mathcal{M}$ is, by the definition, a structure of DG-comodule over the DG-coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$ on the coinduced $\mathfrak{R}$-comodule graded left comodule $\bigoplus_n\mathfrak{A}[1]^{\otimes n}\odot_{\mathfrak{R}}\mathcal{M}$ over the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$. By Lemma 7.1.1(c), a nonunital left wc A_∞-module structure on $\mathcal{M}$ can be viewed as a sequence of $\mathfrak{R}$-comodule morphisms $l_n\colon\mathfrak{A}^{\otimes n}\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathcal{M}$, $n=0$, 1, … of degree $1-n$. The sequence of maps l_n must satisfy a sequence of nonhomogeneous quadratic equations corresponding to the equation $d^2=0$ on the coderivation d on $\bigoplus_n\mathfrak{A}[1]^{\otimes n}\odot_{\mathfrak{R}}\mathcal{M}$. *Nonunital $\mathfrak{R}$-comodule right wc A_∞-modules* over $\mathfrak{A}$ are defined similarly.

The complex of morphisms between nonunital $\mathfrak{R}$-comodule left wc A_∞-modules $\mathcal{L}$ and $\mathcal{M}$ over $\mathfrak{A}$ is, by the definition, the complex of morphisms between the $\mathfrak{R}$-comodule left DG-comodules $\bigoplus_n\mathfrak{A}[1]^{\otimes n}\odot_{\mathfrak{R}}\mathcal{L}$ and $\bigoplus_n\mathfrak{A}[1]^{\otimes n}\odot_{\mathfrak{R}}\mathcal{M}$ over the $\mathfrak{R}$-free DG-coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$. Since an $\mathfrak{R}$-comodule graded $\mathfrak{C}$-comodule morphism into $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ is determined by its composition with the map $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}\longrightarrow\mathcal{M}$ induced by the counit of $\mathfrak{C}$, a morphism of nonunital $\mathfrak{R}$-comodule left wc A_∞-modules $f\colon\mathcal{L}\longrightarrow\mathcal{M}$ of degree i can be viewed as a sequence of $\mathfrak{R}$-comodule morphisms $f_n\colon\mathfrak{A}^{\otimes n}\odot_{\mathfrak{R}}\mathcal{L}\longrightarrow\mathcal{M}$, $n=0$, 1, … of degree $i-n$ (see Section 4.4). Any sequence of homogeneous $\mathfrak{R}$-comodule maps f_n of the required degrees corresponds to a (not necessarily closed) morphism of nonunital $\mathfrak{R}$-comodule wc A_∞-modules f.

For any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$, the functors $\Phi_{\mathfrak{C}}$ and $\Psi_{\mathfrak{C}}$ of Section 3.4 provide an equivalence between the DG-category of $\mathfrak{R}$-cofree left CDG-comodules over $\mathfrak{C}$ which, considered as graded $\mathfrak{C}$-comodules, are cofreely cogenerated by cofree graded $\mathfrak{R}$-comodules, and the DG-category of $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ which, considered as graded $\mathfrak{C}$-contramodules, are freely generated by cofree graded $\mathfrak{R}$-comodules. So one can alternatively define a *nonunital $\mathfrak{R}$-cofree left wc A_∞-module* $\mathcal{P}$ over $\mathfrak{A}$ as a cofree graded $\mathfrak{R}$-comodule for which a structure of DG-contramodule over the DG-coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$ is given on the projective $\mathfrak{R}$-cofree graded contramodule $\mathrm{Ctrhom}_{\mathfrak{R}}(\bigoplus_n\mathfrak{A}[1]^{\otimes n},\mathcal{P})$. By Lemma 7.1.1(d), such a structure can be viewed as a sequence of $\mathfrak{R}$-comodule maps $p_n\colon\mathcal{P}\longrightarrow\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{A}^{\otimes n},\mathcal{P})$, $n=0$, 1, … of degree $1-n$. The maps p_n are related to the above maps l_n by the above rule.

Furthermore, one can alternatively define the complex of morphisms between nonunital $\mathfrak{R}$-cofree left wc A_∞-modules $\mathcal{P}$ and $\mathcal{Q}$ over $\mathfrak{A}$ as the complex of morphisms between left DG-contramodules $\mathrm{Hom}_{\mathfrak{R}}(\bigoplus_n\mathfrak{A}[1]^{\otimes n},\mathcal{P})$ and $\mathrm{Hom}_{\mathfrak{R}}(\bigoplus_n\mathfrak{A}[1]^{\otimes n},\mathcal{Q})$ over the $\mathfrak{R}$-free DG-coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$. Thus a (not necessarily closed) morphism of nonunital $\mathfrak{R}$-cofree left wc A_∞-modules $f\colon \mathcal{P} \longrightarrow \mathcal{Q}$ is the same thing as a sequence of $\mathfrak{R}$-comodule maps $f^n\colon \mathcal{P} \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{A}^{\otimes n},\, \mathcal{Q})$ of degree $i-n$ (see Section 3.3). The maps f^n are related to the above maps f_n by the above rule.

Proposition 7.1.2. *The equvalences of DG-categories $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ from Section* 3.4 *and the DG-functors $\Phi_{\mathfrak{R},\mathfrak{C}}$, $\Psi_{\mathfrak{R},\mathfrak{C}}$ from Section* 4.5 *restrict to an equivalence $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ between the DG-categories of nonunital $\mathfrak{R}$-free left wc A_∞-modules and nonunital $\mathfrak{R}$-cofree left wc A_∞-modules over $\mathfrak{A}$ making a commutative diagram with the forgetful functors and the equivalence $\Phi_{\mathfrak{R}} = \Psi_{\mathfrak{R}}^{-1}$ between the categories of free graded $\mathfrak{R}$-contramodules and cofree graded $\mathfrak{R}$-comodules from Proposition* 1.5. ☐

By the definition, the category of nonunital wc A_∞-algebras over $\mathfrak{R}$ is isomorphic to the full subcategory of the category of $\mathfrak{R}$-free DG-coalgebras consisting of those DG-coalgebras $\mathfrak{C}$ whose underlying graded coalgebras are the tensor coalgebras $\bigoplus_n \mathfrak{U}^{\otimes n}$ and whose reductions $\mathfrak{U}/\mathfrak{m}\mathfrak{U}$ are conilpotent DG-coalgebras. Analogously, the DG-category of nonunital $\mathfrak{R}$-free wc A_∞-modules over $\mathfrak{A}$ is isomorphic to the full subcategory in the category of $\mathfrak{R}$-free DG-comodules $\mathfrak{N}$ over $\mathfrak{C}$ consisting of those DG-comodules whose underlying graded comodules are the graded comodules $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$ (and similarly for nonunital $\mathfrak{R}$-cofree wc A_∞-modules). The following lemmas provide a characterization formulated entirely in terms of the reductions modulo $\mathfrak{m}$.

Lemma 7.1.3. (a) *An $\mathfrak{R}$-free graded comodule $\mathfrak{N}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is cofreely cogenerated by a free graded $\mathfrak{R}$-contramodule $\mathfrak{V}$ (i. e., is isomorphic to $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$) if and only if the graded comodule $\mathfrak{N}/\mathfrak{m}\mathfrak{N}$ over the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is isomorphic to $\mathfrak{C}/\mathfrak{m}\mathfrak{C}\otimes_k \mathfrak{V}/\mathfrak{m}\mathfrak{V}$.*

(b) *An $\mathfrak{R}$-free graded contramodule $\mathfrak{Q}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is freely generated by a free graded $\mathfrak{R}$-contramodule $\mathfrak{U}$ (i. e., is isomorphic to* $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{U})$*) if and only if the graded contramodule $\mathfrak{Q}/\mathfrak{m}\mathfrak{Q}$ over the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is isomorphic to* $\mathrm{Hom}_k(\mathfrak{C}/\mathfrak{m}\mathfrak{C},\, \mathfrak{U}/\mathfrak{m}\mathfrak{U})$.

(c) *An $\mathfrak{R}$-cofree graded comodule $\mathcal{N}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is cofreely cogenerated by a cofree graded $\mathfrak{R}$-comodule $\mathcal{V}$ (i. e., is isomorphic to $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{V}$) if and only if the graded comodule ${}_{\mathfrak{m}}\mathcal{N}$ over the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is isomorphic to $\mathfrak{C}/\mathfrak{m}\mathfrak{C}\otimes_k {}_{\mathfrak{m}}\mathcal{V}$.*

(d) *An $\mathfrak{R}$-cofree graded contramodule $\mathcal{Q}$ over an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ is freely generated by a cofree graded $\mathfrak{R}$-comodule $\mathcal{U}$ (i. e., is isomorphic to* $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{U})$*) if and only if the graded contramodule ${}_{\mathfrak{m}}\mathcal{Q}$ over the graded coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is isomorphic to* $\mathrm{Hom}_k(\mathfrak{C}/\mathfrak{m}\mathfrak{C},\, {}_{\mathfrak{m}}\mathcal{U})$.

Proof. One lifts an isomorphism of the reductions to a morphism of $\mathfrak{C}$-contra/comodules using the projectivity/injectivity properties of the (co)freely (co)generated contra/comodules in the exact categories of $\mathfrak{R}$-(co)free graded $\mathfrak{C}$-contra/comodules. Then it remains to use Lemma 1.3.1 or 1.4.1. See the proofs of Lemmas 3.1.2, 2.1.2, and Lemma 7.1.4 below for further details. ☐

Lemma 7.1.4. *Let $\mathfrak{C}$ be an $\mathfrak{R}$-free graded coalgebra and $\mathfrak{U}$ be a free graded $\mathfrak{R}$-contramodule. Then $\mathfrak{C}$ is isomorphic to the $\mathfrak{R}$-free graded coalgebra $\bigoplus_n \mathfrak{U}^{\otimes n}$ if and only if the graded k-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is isomorphic to $\bigoplus_n (\mathfrak{U}/\mathfrak{m}\mathfrak{U})^{\otimes n}$.*

Proof. The "only if" part is obvious. To prove the "if", consider the composition $\mathfrak{C} \longrightarrow \mathfrak{C}/\mathfrak{m}\mathfrak{C} \simeq \bigoplus_n (\mathfrak{U}/\mathfrak{m}\mathfrak{U})^{\otimes n} \longrightarrow (\mathfrak{U}/\mathfrak{m}\mathfrak{U})^{\otimes 1} \simeq \mathfrak{U}/\mathfrak{m}\mathfrak{U}$ and lift it to a graded $\mathfrak{R}$-contramodule morphism $\mathfrak{C} \longrightarrow \mathfrak{U}$. Since the isomorphism of graded k-coalgebras $\mathfrak{C}/\mathfrak{m}\mathfrak{C} \simeq \bigoplus_n (\mathfrak{U}/\mathfrak{m}\mathfrak{U})^{\otimes n}$, as any morphism of conilpotent graded coalgebras over k, commutes with the coaugmentations, it follows from Lemma 6.1.3(b) that the map $\mathfrak{C} \longrightarrow \mathfrak{U}$ gives rise to a morphism of $\mathfrak{R}$-free graded coalgebras $\mathfrak{C} \longrightarrow \bigoplus_n \mathfrak{U}^{\otimes n}$. By the uniqueness assertion of the same Lemma applied to the case of graded coalgebras over k, the morphism $\mathfrak{C} \longrightarrow \bigoplus_n \mathfrak{U}^{\otimes n}$ reduces to the isomorphism $\mathfrak{C}/\mathfrak{m}\mathfrak{C} \simeq \bigoplus_n (\mathfrak{U}/\mathfrak{m}\mathfrak{U})^{\otimes n}$ that we started from. It remains to apply Lemma 1.3.1 in order to conclude that the morphism $\mathfrak{C} \longrightarrow \bigoplus_n \mathfrak{U}^{\otimes n}$ is an isomorphism. ☐

Let $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of wc A_∞-algebras over $\mathfrak{R}$ corresponding to a morphism of $\mathfrak{R}$-free DG-coalgebras $\mathfrak{C} = \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{D} = \bigoplus_n \mathfrak{B}[1]^{\otimes n}$. Let $\mathfrak{M}$ be an $\mathfrak{R}$-free wc A_∞-module over $\mathfrak{B}$ corresponding to an $\mathfrak{R}$-free DG-comodule $\mathfrak{N} = \bigoplus_n \mathfrak{B}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$ over $\mathfrak{D}$. Then the structure of wc A_∞-module over $\mathfrak{A}$ on $\mathfrak{M}$ obtained by the restriction of scalars with respect to the morphism f from the given structure of wc A_∞-module over $\mathfrak{B}$ corresponds, by the definition, to the $\mathfrak{R}$-free DG-comodule structure $E_f(\mathfrak{N}) = \mathfrak{C} \mathbin{\square}_{\mathfrak{D}} \mathfrak{N}$ on the $\mathfrak{R}$-free graded $\mathfrak{C}$-comodule $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$ (see Section 3.2). The structure maps l'_n of the new structure of wc A_∞-module over $\mathfrak{A}$ on $\mathfrak{M}$ are expressed in terms of the structure maps l_n of the original structure of wc A_∞-module over $\mathfrak{B}$ and the structure maps f_n of the wc A_∞-morphism $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ as certain sums of compositions (with signs) depending linearly on l_n and polynomially on f_n (and even as formal power series on f_0). Clearly, the restriction of scalars with respect to a morphism of wc A_∞-algebras is naturally a DG-functor (since the coextension of scalars E_f with respect to a morphism of $\mathfrak{R}$-free DG-coalgebras is).

Analogously, let $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule wc A_∞-module over $\mathfrak{B}$ corresponding to an $\mathfrak{R}$-contramodule DG-contramodule $\mathfrak{Q} = \mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{B}[1]^{\otimes n}, \mathfrak{P})$ over $\mathfrak{D}$. Then the structure of wc A_∞-module over $\mathfrak{A}$ on $\mathfrak{P}$ obtained by the restriction of scalars with respect to the morphism $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ from the given structure of wc A_∞-module over $\mathfrak{B}$ on $\mathfrak{P}$ corresponds, by the definition, to the $\mathfrak{R}$-contramodule DG-contramodule structure $E^f(\mathfrak{Q}) = \mathrm{Cohom}_{\mathfrak{D}}(\mathfrak{C}, \mathfrak{Q})$ on the $\mathfrak{R}$-contramodule graded $\mathfrak{C}$-contramodule $\mathrm{Hom}^{\mathfrak{R}}(\bigoplus_n \mathfrak{A}[1]^{\otimes n}, \mathfrak{P})$. One can easily see that this construction agrees with the previous one in the case of $\mathfrak{R}$-free wc A_∞-modules (cf. Proposition 3.2.7). The restriction-of-scalars functors for $\mathfrak{R}$-comodule and $\mathfrak{R}$-cofree wc A_∞-modules are constructed in the similar way (see Section 3.4).

Remark 7.1.5. The assignment of the underlying graded $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule to an $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule wc A_∞-module over a wc A_∞-algebra $\mathfrak{A}$ over $\mathfrak{R}$ does *not* have good functorial properties.

From the point of view of graded contramodules or comodules over the tensor coalgebra $\mathfrak{C} = \bigoplus_n \mathfrak{A}[1]^{\otimes n}$, to recover, say, a free graded $\mathfrak{R}$-contramodule $\mathfrak{M}$ from the $\mathfrak{R}$-free graded $\mathfrak{C}$-comodule $\mathfrak{N} = \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ one has to use the construction of cotensor product with $\mathfrak{R}$ over $\mathfrak{C}$, that is $\mathfrak{M} = \mathfrak{R} \mathbin{\square}_{\mathfrak{C}} \mathfrak{N}$. This depends on the structure of right $\mathfrak{C}$-comodule on $\mathfrak{R}$, which is defined in terms of the coaugmentation map $\mathfrak{R} =$

$\mathfrak{A}[1]^{\otimes 0} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n} = \mathfrak{C}$. The problem is, such coaugmentations over $\mathfrak{R}$ are *not* preserved by $\mathfrak{R}$-free DG-coalgebra morphisms (or even isomorphisms) $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$, because of the presence of the change-of-connection component $f_0 \in \mathfrak{m}\mathfrak{B}^1$.

Accordingly, the functor of forgetting the wc A_∞-module structure is well-defined for wc A_∞-modules over a *fixed* wc A_∞-algebra $\mathfrak{A}$ (as, e. g., in the setting of Proposition 7.1.2), but such functors do *not* form commutative diagrams with the restriction-of-scalars functors for (iso)morphisms of wc A_∞-algebras. This is a phenomenon specific to the wc A_∞ situation; it does not occur for wcDG-modules over $\mathfrak{R}$-free wcDG-algebras or A_∞-modules over A_∞-algebras over fields. The described problem also does not manifest itself for the forgetful functors acting from the DG-categories of wc A_∞-modules and *strict* morphisms between them (see Section 7.3).

The situtation is even worse for the assignment of the underlying free graded $\mathfrak{R}$-module to a wc A_∞-algebra over $\mathfrak{R}$, which is *not* a functor on the category of wc A_∞-algebras at all. It is a functor, however, on the category of wc A_∞-algebras and morphisms f between them for which $f_0 = 0$, and it is also a functor on the category of wc A_∞-algebra morphisms f such that $f_i = 0$ for $i \geqslant 2$.

In general, it is only the reduction modulo $\mathfrak{m}$ of the underlying free graded $\mathfrak{R}$-contramodule of a wc A_∞-algebra that is a well-behaved operation (which even produces a complex of vector spaces). Similarly, the functors of reduction modulo $\mathfrak{m}$ of the underlying free graded $\mathfrak{R}$-contramodules or cofree graded $\mathfrak{R}$-modules of $\mathfrak{R}$-(co)free wc A_∞-modules are well-behaved (and produce complexes).

7.2. **Strictly unital wc A_∞-algebras.** Let $\mathfrak{A}$ be a wc A_∞-algebra over $\mathfrak{R}$. An element $1 \in \mathfrak{A}^0$ (or equivalently, the corresponding morphism of graded $\mathfrak{R}$-contramodules $\mathfrak{R} \longrightarrow \mathfrak{A}$) is called a *strict unit* if the compositions of the two maps $\mathfrak{A} \longrightarrow \mathfrak{A} \otimes^{\mathfrak{R}} \mathfrak{A}$ induced by the unit map with the map $m_2\colon \mathfrak{A} \otimes^{\mathfrak{R}} \mathfrak{A} \longrightarrow \mathfrak{A}$ are equal to the identity endomorphism of $\mathfrak{A}$, while the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \longrightarrow \mathfrak{A}^{\otimes n}$ induced by the unit map with the map $m_n\colon \mathfrak{A}^{\otimes n} \longrightarrow \mathfrak{A}$ all vanish for $n \geqslant 1$, $n \neq 2$. It follows from the condition on m_2 that a strict unit is unique if it exists. (See [23, Section 2.3.2] or [29, Section 7.2].)

A *strictly unital wc A_∞-algebra* over $\mathfrak{R}$ is a nonunital wc A_∞-algebra that has a strict unit. A morphism of strictly unital wc A_∞-algebras $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is a morphism of nonunital wc A_∞-algebras for which $f_1(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$ (i. e., the unit maps $\mathfrak{R} \longrightarrow \mathfrak{A}$ and $\mathfrak{R} \longrightarrow \mathfrak{B}$ form a commutative diagram with f_1) and the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \longrightarrow \mathfrak{A}^{\otimes n}$ induced by $1_{\mathfrak{A}}$ with the map $f_n\colon \mathfrak{A}^{\otimes n} \longrightarrow \mathfrak{B}$ vanish for all $n \geqslant 2$.

Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra over $\mathfrak{R}$. A *strictly unital $\mathfrak{R}$-contramodule left wc A_∞-module* $\mathfrak{P}$ over $\mathfrak{A}$ is a nonunital left wc A_∞-module such that the composition of the structure map p_1 with the map $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ induced by the unit map is the identity endomorphism of $\mathfrak{P}$, while the compositions of the map $p_n\colon \mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{P})$ with the n maps $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{P}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n-1}, \mathfrak{P})$ induced by the counit map all vanish for $n \geqslant 2$. In the case of an $\mathfrak{R}$-free left wc A_∞+module $\mathfrak{M}$, one can rewrite these equations equivalently in terms of the maps l_n, requiring that the composition of the map $\mathfrak{M} \longrightarrow \mathfrak{A} \otimes^{\mathfrak{R}} \mathfrak{M}$ induced by the unit map with the map $l_1\colon \mathfrak{A} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$ be the identity endomorphism of $\mathfrak{M}$, while the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M}$ induced by the unit map with the map $l_n\colon \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$ vanish for all $n \geqslant 2$. *Strictly unital $\mathfrak{R}$-free right wc* A_∞*-modules* over $\mathfrak{A}$ are defined similarly.

The complex of morphisms between strictly unital $\mathfrak{R}$-contramodule left wc A_∞-modules $\mathfrak{P}$ and $\mathfrak{Q}$ over $\mathfrak{A}$ is a complex of $\mathfrak{R}$-contramodules defined as the subcomplex of the complex of morphisms between $\mathfrak{P}$ and $\mathfrak{Q}$ as nonunital wc A_∞-modules consisting of all morphisms $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ such that the compositions of the map $f^n\colon \mathfrak{P} \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{Q})$ with the n maps $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n}, \mathfrak{Q}) \longrightarrow \mathrm{Hom}^{\mathfrak{R}}(\mathfrak{A}^{\otimes n-1}, \mathfrak{Q})$ induced by the counit map all vanish for $n \geqslant 1$. For $\mathfrak{R}$-free left wc A_∞-modules $\mathfrak{L}$ and $\mathfrak{M}$, one can rewrite these equations equivalently in terms of the maps f_n, requiring that the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \otimes^{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{P}$ induced by the unit map with the map $f_n\colon \mathfrak{A}^{\otimes n} \otimes^{\mathfrak{R}} \mathfrak{P} \longrightarrow \mathfrak{Q}$ vanish for all $n \geqslant 1$.

A *strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-module* $\mathcal{M}$ over $\mathfrak{A}$ is a nonunital left wc A_∞-module such that the composition of the map $\mathcal{M} \longrightarrow \mathfrak{A} \odot_{\mathfrak{R}} \mathcal{M}$ induced by the unit map with the structure map l_1 is the identity endomorphism of $\mathcal{M}$, while the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathfrak{A}^{\otimes n} \odot_{\mathfrak{R}} \mathcal{M}$ induced by the unit map with the map $l_n\colon \mathfrak{A}^{\otimes n} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathcal{M}$ all vanish for $n \geqslant 2$. In the case of an $\mathfrak{R}$-cofree left wc A_∞-module $\mathcal{P}$, one can rewrite these equations equivalently in terms of the maps p_n, just as it was done above. *Strictly unital $\mathfrak{R}$-comodule right wc* A_∞*-modules* over $\mathfrak{A}$ are defined similarly.

The complex of morphisms between strictly unital $\mathfrak{R}$-comodule left wc A_∞-modules $\mathcal{L}$ and $\mathcal{M}$ over $\mathfrak{A}$ is a complex of $\mathfrak{R}$-contramodules defined as the subcomplex of the complex of morphisms between $\mathcal{L}$ and $\mathcal{M}$ as nonunital wc A_∞-modules consisting of all morphisms $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ such that the compositions of the n maps $\mathfrak{A}^{\otimes n-1} \odot_{\mathfrak{R}} \mathcal{L} \longrightarrow \mathfrak{A}^{\otimes n} \odot_{\mathfrak{R}} \mathcal{L}$ induced by the unit map with the map $f_n\colon \mathfrak{A}^{\otimes n} \odot_{\mathfrak{R}} \mathcal{L} \longrightarrow \mathcal{M}$ vanish for all $n \geqslant 1$. For $\mathfrak{R}$-cofree left wc A_∞-modules $\mathcal{P}$ and $\mathcal{Q}$, one can rewrite these equations equivalently in terms of the maps f^n, as it was done above.

The following theorems show that strictly unital wc A_∞-algebras and wc A_∞-modules are related to CDG-coalgebras and CDG-contra/comodules in the same way as nonunital wc A_∞-algebras and wc A_∞-modules are, by the definition, related to DG-coalgebras and DG-contra/comodules. The basic idea goes back to the paper [27]; our exposition here follows that in [29, Section 7.2], where similar results were obtained for A_∞-algebras and A_∞-modules.

Theorem 7.2.1. *The category of nonzero strictly unital wc* A_∞*-algebras over $\mathfrak{R}$ is naturally equivalent to the subcategory of the category of $\mathfrak{R}$-free CDG-coalgebras formed by the CDG-algebras $\mathfrak{C}$ whose reductions $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ are coaugmented CDG-coalgebras with the underlying graded coalgebra isomorphic to $\bigoplus_n U^{\otimes n}$ for some graded k-vector space U, and $\mathfrak{R}$-free CDG-coalgebra morphisms $\mathfrak{C} \longrightarrow \mathfrak{D}$ whose reductions $\mathfrak{C}/\mathfrak{m}\mathfrak{C} \longrightarrow \mathfrak{D}/\mathfrak{m}\mathfrak{D}$ are morphisms of coaugmented CDG-coalgebras over k.*

In other words, the category of strictly unital wc A_∞-algebras over $\mathfrak{R}$ is equivalent to the full subcategory in the category of $\mathfrak{R}$-free CDG-coalgebras with conilpotent reductions $\mathfrak{R}\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{k\text{–}\mathsf{cnlp}}$ consisting of the CDG-coalgebras $\mathfrak{C}$ whose reductions $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$

have their underlying conilpotent graded k-coalgebras cofreely cogenerated by graded k-vector spaces.

Proof of Theorem 7.2.1. The construction is based on the following A_∞ generalization of Lemma 6.1.1(a).

Lemma 7.2.2. *If $\mathfrak{A}$ is a nonzero strictly unital wc A_∞-algebra over $\mathfrak{R}$, then the unit map $\mathfrak{R} \longrightarrow \mathfrak{A}$ is the embedding of a direct summand in the category of free graded $\mathfrak{R}$-contramodules.*

Proof. Reducing modulo $\mathfrak{m}$, we obtain a strictly unital A_∞-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ over the field k. In particular, the reduction $k \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ of the unit map $\mathfrak{R} \longrightarrow \mathfrak{A}$ is a unit element of the (nonassociative) algebra structure on $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ given by the operation m_2. Hence if the map $k \longrightarrow \mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is zero, then $\mathfrak{A}/\mathfrak{m}\mathfrak{A} = 0$ and $\mathfrak{A} = 0$. The rest of the argument is the same as in the proof of Lemma 6.1.1(a). ☐

Let $\mathfrak{A}$ be a free graded $\mathfrak{R}$-contramodule and $e_\mathfrak{A}\colon \mathfrak{R} \longrightarrow \mathfrak{A}$ be a graded $\mathfrak{R}$-contramodule morphism with an $\mathfrak{R}$-free cokernel (i. e., an embedding of a direct summand in the category of free graded $\mathfrak{R}$-contramodules); denote the cokernel by $\mathfrak{A}_+ = \mathfrak{A}/\mathfrak{R}$. Then there is a surjective morphism of $\mathfrak{R}$-free graded coalgebras $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{A}_+[1]^{\otimes n}$. Denote by $\mathfrak{K}_\mathfrak{A}$ the kernel of this morphism; then $\mathfrak{K}_\mathfrak{A}$ is the direct sum of the kernel of the morphisms $\mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{A}_+[1]^{\otimes n}$ taken in the category of free graded $\mathfrak{R}$-contramodules. Denote by $\varkappa_\mathfrak{A}\colon \mathfrak{K}_\mathfrak{A} \longrightarrow \mathfrak{R}$ the $\mathfrak{R}$-contramodule morphism of the total degree 1 vanishing on all the components of tensor degree $n \neq 1$ and defined on the component of tensor degree 1 by the condition that the composition $\mathfrak{R} \longrightarrow \mathfrak{K}_\mathfrak{A} \longrightarrow \mathfrak{R}$ of the emdedding $\mathfrak{R} \longrightarrow \mathfrak{K}_\mathfrak{A}$ induced by the map $\mathfrak{R} \longrightarrow \mathfrak{A}$ with the map $\varkappa_\mathfrak{A}$ must be the identity map. Let $\theta_\mathfrak{A}\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{R}$ be any $\mathfrak{R}$-contramodule morphism of the (total) degree 1 extending the map $\varkappa_\mathfrak{A}$.

Lemma 7.2.3. (a) *Let $\mathfrak{A}$ be a wc A_∞-algebra with the wc A_∞structure $d\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n}$. Then an embedding of a direct summand in the category of graded $\mathfrak{R}$-contramodules $e_\mathfrak{A}\colon \mathfrak{R} \longrightarrow \mathfrak{A}$ is a strict unit of the A_∞-structure on $\mathfrak{A}$ if and only if the following two conditions hold:*

- (i) *the odd coderivation of degree 1 on the tensor coalgebra $\bigoplus_n \mathfrak{A}[1]^{\otimes n}$ given by the formula $d'(c) = d(c) + \theta_\mathfrak{A} * c - (-1)^{|c|} c * \theta_\mathfrak{A}$ (the notation of Section 3.2) preserves the $\mathfrak{R}$-subcontramodule $\mathfrak{K}_\mathfrak{A} \subset \bigoplus_n \mathfrak{A}[1]^{\otimes n}$, and*
- (ii) *the $\mathfrak{R}$-contramodule map $h'\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{R}$ of degree 2 given by the formula $h'(c) = \theta_\mathfrak{A}(d(c)) + \theta_\mathfrak{A}^2(c)$ vanishes in the restriction to $\mathfrak{K}_\mathfrak{A}$.*

(b) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be strictly unital wc A_∞algebras over $\mathfrak{R}$ with the units $e_\mathfrak{A}\colon \mathfrak{R} \longrightarrow \mathfrak{A}$ and $e_\mathfrak{B}\colon \mathfrak{R} \longrightarrow \mathfrak{B}$, and let $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of nonunital wc A_∞-algebras. Then f is a morphism of strictly unital wc A_∞-algebras if and only if*

- (i) *the morphism of $\mathfrak{R}$-free DG-coalgebras $f\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{B}[1]^{\otimes n}$ takes $\mathfrak{K}_\mathfrak{A}$ into $\mathfrak{K}_\mathfrak{B}$, and*
- (ii) *the composition of the map $f|_{\mathfrak{K}_\mathfrak{A}}\colon \mathfrak{K}_\mathfrak{A} \longrightarrow \mathfrak{K}_\mathfrak{B}$ with the linear function $\varkappa_\mathfrak{B}$ on $\mathfrak{K}_\mathfrak{B}$ is equal to the linear function $\varkappa_\mathfrak{A}$ on $\mathfrak{K}_\mathfrak{A}$.*

Proof. Part (a): Clearly, the condition (i) does not depend on the choice of a linear function $\theta_{\mathfrak{A}}$ (for a given $e_{\mathfrak{A}}$); the condition (ii) does not depend on the choice of $\theta_{\mathfrak{A}}$ assuming that the condition (i) is fulfilled. The condition (i) is equivalent to the vanishing of the composition of the differential d' with the composition of the projections $\bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{A}[1]^{\otimes 1} \simeq \mathfrak{A}[1] \longrightarrow \mathfrak{A}_+[1]$.

The equations of the strict unit prescribe the values of certain maps landing in $\mathfrak{A}$. The above vanishing condition is equivalent to these equations being true modulo $e_{\mathfrak{A}}(\mathfrak{R})$. Furthermore, assume that the linear function $\theta_{\mathfrak{A}}$ is chosed in such a way that it vanishes on the components of the tensor degree different from 1. Then the choice of $\theta_{\mathfrak{A}}$ is equivalent to the choice of a direct sum decomposition $\mathfrak{A} \simeq \mathfrak{A}_+ \oplus \mathfrak{R}$. The condition (ii) is equivalent to the equations on the strict unit being true in the projection to the component $\mathfrak{R}$.

Part (b): The condition (i) is equivalent to the vanishing of the composition $\mathfrak{K}_{\mathfrak{A}} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{B}[1]^{\otimes n} \longrightarrow \mathfrak{B}[1]^{\otimes 1} \simeq \mathfrak{B}[1]$. The equations of a morphism of strictly unital wc A_∞-algebras prescribe the values of certain maps landing in $\mathfrak{B}$. The above vanishing condition is equivalent to these equations being true modulo $e_{\mathfrak{B}}(\mathfrak{R})$. Assuming that this condition holds, the maps we are interested in actually land in $e_{\mathfrak{B}}(\mathfrak{R}) \subset \mathfrak{B}$. The condition (ii) means that they take the prescribed values there. $\square$

Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra. Pick a homogeneous retraction $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ of the unit map (i. e., a graded $\mathfrak{R}$-contramodule morphism left inverse to $e_{\mathfrak{A}}$). Define the $\mathfrak{R}$-contramodule morphism $\theta_A\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{R}$ of degree 1 by the rule that $\theta_{\mathfrak{A}} = v$ on the components of the tensor degree 1 and 0 on the components of all other tensor degrees. Then the $\mathfrak{R}$-contramodule morphism $\theta_{\mathfrak{A}}$ extends the $\mathfrak{R}$-contramodule morphism $\varkappa_{\mathfrak{A}}\colon \mathfrak{K}_{\mathfrak{A}} \longrightarrow \mathfrak{R}$. The odd coderivation $d'\colon \bigoplus_n \mathfrak{A}_[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n}$ corresponding to $\theta_{\mathfrak{A}}$ takes $\mathfrak{K}_{\mathfrak{A}}$ to $\mathfrak{K}_{\mathfrak{B}}$ and therefore induced an odd coderivation $d\colon \bigoplus_n \mathfrak{A}_+[1]^{\otimes n} \longrightarrow \mathfrak{A}_+[1]^{\otimes n}$ on the tensor coalgebra $\bigoplus_n \mathfrak{A}_+[1]^{\otimes n}$. Similarly, the morphism $h'\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{R}$ corresponding to $\theta_{\mathfrak{A}}$ annihilates $\mathfrak{K}_{\mathfrak{A}}$ and therefore induces a morphism $h\colon \bigoplus_n \mathfrak{A}_+[1] \longrightarrow \mathfrak{R}$.

The *bar-construction* $\mathrm{Bar}_v(\mathfrak{A}) = (\bigoplus_n \mathfrak{A}_+[1]^{\otimes n},\, d,\, h)$ of a strictly unital wc A_∞-algebra $\mathfrak{A}$ is the $\mathfrak{R}$-free CDG-coalgebra corresponding to $\mathfrak{A}$ under the desired equivalences of categories. The CDG-coalgebra $\mathrm{Bar}_v(\mathfrak{A})/\mathfrak{m}\,\mathrm{Bar}_v(\mathfrak{A})$ over k is coaugmented (and therefore, conilpotent) because $m_0 \in \mathfrak{m}\mathfrak{A}$.

Let $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of strictly unital wc A_∞-algebras. Let $v_{\mathfrak{A}}\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ and $v_{\mathfrak{B}}\colon \mathfrak{B} \longrightarrow \mathfrak{R}$ be homogeneous retractions of the unit maps, and let $\theta_{\mathfrak{A}}$ and $\theta_{\mathfrak{B}}$ be the corresponding linear functions of degree 1 on $\mathfrak{R}$-free graded tensor coalgebras. The morphism of tensor coalgebras $f\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \bigoplus_n \mathfrak{B}[1]^{\otimes n}$ takes $\mathfrak{K}_{\mathfrak{A}}$ into $\mathfrak{K}_{\mathfrak{B}}$, so it induces a morphism of $\mathfrak{R}$-free graded coalgebras $f\colon \bigoplus_n \mathfrak{A}_+[1] \longrightarrow \bigoplus_n \mathfrak{B}_+[1]$. The $\mathfrak{R}$-contramodule morphism $\theta_{\mathfrak{B}} \circ f - \theta_{\mathfrak{A}}\colon \bigoplus_n \mathfrak{A}[1]^{\otimes n} \longrightarrow \mathfrak{R}$ annihilates $\mathfrak{K}_{\mathfrak{A}}$, so it induces an $\mathfrak{R}$-contramodule morphism $a_f\colon \bigoplus_n \mathfrak{A}_+[1] \longrightarrow \mathfrak{R}$.

The pair (f, a_f) is the morphism of $\mathfrak{R}$-free CDG-coalgebras $\mathrm{Bar}_v(\mathfrak{A}) \longrightarrow \mathrm{Bar}_v(\mathfrak{B})$ corresponding to the morphism of strictly unital wc A_∞-algebras $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$. The morphism $(f/\mathfrak{m}f, a_f/\mathfrak{m}a_f)$ is a morphism of coaugmented/conilpotent CDG-coalgebras (i. e., $a_f/\mathfrak{m}a_f \circ \bar{w} = 0$) because $f_0 \in \mathfrak{m}\mathfrak{B}$ and $\theta_{\mathfrak{A}}/\mathfrak{m}\theta_{\mathfrak{A}} \circ \bar{w} = 0 = \theta_{\mathfrak{B}}/\mathfrak{m}\theta_{\mathfrak{B}} \circ \bar{w}$.

Alternatively, one can use any linear functions $\theta_{\mathfrak{A}}$ of degree 1 extending the linear functions $\kappa_{\mathfrak{A}}$ and satisfying the condition $\theta_{\mathfrak{A}}/\mathfrak{m}\theta_{\mathfrak{A}}\circ\bar w=0$ in this construction of the functor Bar_v.

To obtain the inverse functor, assign to an $\mathfrak{R}$-free CDG-coalgebra $(\mathfrak{D},d_{\mathfrak{D}},h_{\mathfrak{D}})$ the $\mathfrak{R}$-free CDG-algebra $(\mathfrak{C},d_{\mathfrak{C}})$ constructed as follows. Adjoin to $\mathfrak{D}$ a single "cofree in the conilpotent sense" cogenerator of degree -1, obtaining an $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ endowed with an $\mathfrak{R}$-free graded coalgebra morphism $\mathfrak{C}\longrightarrow\mathfrak{D}$ and an $\mathfrak{R}$-contramodule morphism $\theta\colon\mathfrak{C}\longrightarrow\mathfrak{R}$ of degree 1 characterized by the following property. The map $\mathfrak{C}\longrightarrow\prod_{n=1}^\infty\mathfrak{D}^{\otimes n}[n-1]$ whose components are obtained by composing the iterated comultiplication maps $\mathfrak{C}\longrightarrow\mathfrak{C}^{\otimes 2n-1}$ with the tensor products of the maps $\mathfrak{C}\longrightarrow\mathfrak{D}$ at the odd positions and the maps θ at the even ones should be equal to the composition of an isomorphism $\mathfrak{C}\simeq\bigoplus_{n=1}^\infty\mathfrak{D}^{\otimes n}[n-1]$ and the natural embedding of the infinite direct sum of free graded $\mathfrak{R}$-contramodules into their infinite product.

Define the odd coderivation $d'_{\mathfrak{C}}$ on the $\mathfrak{R}$-free graded coalgebra $\mathfrak{C}$ by the conditions that $d'_{\mathfrak{C}}$ must preserve the kernel of the morphism of $\mathfrak{R}$-free graded coalgebras $\mathfrak{C}\longrightarrow\mathfrak{D}$ and induce the coderivation $d_{\mathfrak{D}}$ on $\mathfrak{D}$, and that the equation $\theta\circ d'_{\mathfrak{C}}=\theta^2+h_{\mathfrak{D}}$ must hold (where $h_{\mathfrak{D}}$ is considered as a morphism $\mathfrak{C}\longrightarrow\mathfrak{R}$). Finally, set $d_{\mathfrak{C}}(c)=d'_{\mathfrak{C}}(c)-\theta*c+(-1)^{|c|}c*\theta$ for all $c\in\mathfrak{C}$.

Restricting this procedure to $\mathfrak{R}$-free CDG-algebras with conilpotent reductions whose underlying graded coalgebras are cotensor coalgebras of free graded $\mathfrak{R}$-contramodules, we obtain a functor recovering the $\mathfrak{R}$-free DG-coalgebra $\bigoplus_n\mathfrak{A}[1]^{\otimes n}$ from the $\mathfrak{R}$-free CDG-coalgebra $\mathrm{Bar}_v(\mathfrak{A})=\bigoplus_n\mathfrak{A}_+[1]^{\otimes n}$. ☐

Theorem 7.2.4. *Let $\mathfrak{A}$ be a nonzero strictly unital wc A_∞-algebra over $\mathfrak{R}$ and $\mathfrak{D}$ be the corresponding $\mathfrak{R}$-free CDG-coalgebra. Then*

(a) *the DG-category of $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ is naturally equivalent to the DG-category of $\mathfrak{R}$-free left CDG-comodules over $\mathfrak{D}$;*

(b) *the DG-category of $\mathfrak{R}$-contramodule left wc A_∞-modules over $\mathfrak{A}$ is naturally equivalent to the DG-category of $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{D}$;*

(c) *the DG-category of $\mathfrak{R}$-comodule left wc A_∞-modules over $\mathfrak{A}$ is naturally equivalent to the DG-category of $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{D}$;*

(d) *the DG-category of $\mathfrak{R}$-cofree left wc A_∞-modules over $\mathfrak{A}$ is naturally equivalent to the DG-category of $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{D}$;*

(e) *the above equivalences of DG-categories form a commutative diagram with the natural embeddings of the DG-categories of strictly unital wc A_∞-modules into those of nonunital wc A_∞-modules over $\mathfrak{A}$, the equivalences of DG-categories from Section* 7.1 *and Proposition* 7.1.2*, and the similar equivalences between DG-categories of CDG-comodules and CDG-contramodules over $\mathfrak{D}$.*

Proof. The proof is based on the following two lemmas.

Lemma 7.2.5. *Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra over $\mathfrak{R}$; set $\mathfrak{C}=\bigoplus_n\mathfrak{A}[1]^{\otimes n}$ and $\mathfrak{D}=\bigoplus_n\mathfrak{A}_+[1]^{\otimes n}$. Let $\mathfrak{K}_{\mathfrak{A}}\subset\mathfrak{C}$ be the kernel of the natural morphism of $\mathfrak{R}$-free* *graded coalgebras* $\mathfrak{C} \longrightarrow \mathfrak{D}$*, let* $\theta_{\mathfrak{A}}\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ *be an* $\mathfrak{R}$*-contramodule morphism constructed in the proof of Theorem 7.2.1, and let* $d'\colon \mathfrak{C} \longrightarrow \mathfrak{C}$ *be the corresponding odd coderivation from Lemma 7.2.3. Then*

(a) *a nonunital* $\mathfrak{R}$*-free left wc* A_∞*-module* $\mathfrak{M}$ *over* $\mathfrak{A}$ *with the wc* A_∞*-module structure* $d\colon \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ *is a strictly unital* $\mathfrak{R}$*-free wc* A_∞*-module over* $\mathfrak{A}$ *if and only if the odd coderivation* $d'(z) = d(z) + \theta_{\mathfrak{A}} * z$ *of degree* 1 *on* $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ *compatible with the odd coderivation* d' *on* $\mathfrak{C}$ *preserves the graded* $\mathfrak{R}$*-subcontramodule* $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{M} \subset \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$*;*

(b) *a nonunital* $\mathfrak{R}$*-contramodule left wc* A_∞*-module* $\mathfrak{P}$ *over* $\mathfrak{A}$ *with the wc* A_∞*-module structure* $d\colon \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ *is a strictly unital* $\mathfrak{R}$*-contramodule wc* A_∞*-module over* $\mathfrak{A}$ *if and only if the odd contraderivation* $d'(q) = d(q) + \theta_{\mathfrak{A}} * q$ *of degree* 1 *on* $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$ *compatible with the odd coderivation* d' *on* $\mathfrak{C}$ *preserves the graded* $\mathfrak{R}$*-subcontramodule* $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{D}, \mathfrak{P}) \subset \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P})$*;*

(c) *a nonunital* $\mathfrak{R}$*-comodule left wc* A_∞*-module* $\mathcal{M}$ *over* $\mathfrak{A}$ *with the wc* A_∞*-module structure* $d\colon \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$ *is a strictly unital* $\mathfrak{R}$*-comodule wc* A_∞*-module over* $\mathfrak{A}$ *if and only if the odd coderivation* $d'(z) = d(z) + \theta_{\mathfrak{A}} * z$ *of degree* 1 *on* $\mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$ *compatible with the odd coderivation* d' *on* $\mathfrak{C}$ *preserves the graded* $\mathfrak{R}$*-subcomodule* $\mathfrak{K}_{\mathfrak{A}} \odot_{\mathfrak{R}} \mathcal{M} \subset \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$*;*

(d) *a nonunital* $\mathfrak{R}$*-cofree left wc* A_∞*-module* $\mathcal{P}$ *over* $\mathfrak{A}$ *with the wc* A_∞*-module structure* $d\colon \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}) \longrightarrow \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P})$ *is a strictly unital* $\mathfrak{R}$*-cofree wc* A_∞*-module over* $\mathfrak{A}$ *if and only if the odd contraderivation* $d'(q) = d(q) + \theta_{\mathfrak{A}} * q$ *of degree* 1 *on* $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P})$ *compatible with the odd coderivation* d' *on* $\mathfrak{C}$ *preserves the graded* $\mathfrak{R}$*-subcontramodule* $\operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{D}, \mathcal{P}) \subset \operatorname{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P})$.

Proof. Part (a): the coderivation d' takes $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{M}$ into $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{M}$ if and only if its composition $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{M}$ with the natural embedding and the morphism induced by the counit of $\mathfrak{C}$ vanishes. Part (b): the contraderivation d' takes $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{D}, \mathfrak{P})$ into $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{D}, \mathfrak{P})$ if and only if its composition $\mathfrak{P} \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{K}_{\mathfrak{A}}, \mathfrak{P})$ with the morphism induced by the counit of $\mathfrak{C}$ and the natural surjection vanishes. In both cases, it is straightforward to check that the condition is equivalent to the definition of a strictly unital wc A_∞-module. Parts (c) and (d) are similar. $\square$

Lemma 7.2.6. *Let* $\mathfrak{A}$ *be a strictly unital wc* A_∞*-algebra over* $\mathfrak{R}$*; set* $\mathfrak{C} = \bigoplus_n \mathfrak{A}[1]^{\otimes n}$ *and* $\mathfrak{D} = \bigoplus_n \mathfrak{A}_+[1]^{\otimes n}$*, and let* $\mathfrak{K}_{\mathfrak{A}} \subset \mathfrak{C}$ *be the kernel of the natural morphism* $\mathfrak{C} \longrightarrow \mathfrak{D}$*. Then*

(a) *for strictly unital* $\mathfrak{R}$*-free left wc* A_∞*-modules* $\mathfrak{L}$ *and* $\mathfrak{M}$ *over* $\mathfrak{A}$*, a (not necessarily closed) morphism of nonunital* $\mathfrak{R}$*-free wc* A_∞*-modules* $f\colon \mathfrak{L} \longrightarrow \mathfrak{M}$ *is a morphism of strictly unital* $\mathfrak{R}$*-free wc* A_∞*-modules if and only if the morphism of* $\mathfrak{R}$*-free CDG-comodules* $f\colon \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{L} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ *over* $\mathfrak{C}$ *takes* $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{L}$ *into* $\mathfrak{K}_{\mathfrak{A}} \otimes^{\mathfrak{R}} \mathfrak{M}$*;*

(b) *for strictly unital* $\mathfrak{R}$*-contramodule left wc* A_∞*-modules* $\mathfrak{P}$ *and* $\mathfrak{Q}$ *over* $\mathfrak{A}$*, a (not necessarily closed) morphism of nonunital* $\mathfrak{R}$*-contramodule wc* A_∞*-modules* $f\colon \mathfrak{L} \longrightarrow \mathfrak{M}$ *is a morphism of strictly unital* $\mathfrak{R}$*-contramodule wc* A_∞*-modules if and only if the morphism of* $\mathfrak{R}$*-contramodule CDG-contramodules* $f\colon \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{P}) \longrightarrow \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{Q})$ *over* $\mathfrak{C}$ *takes* $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{D}, \mathfrak{P})$ *into* $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{D}, \mathfrak{Q})$*;*

(c) *for strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-modules* $\mathcal{L}$ *and* $\mathcal{M}$ *over* $\mathfrak{A}$*, a (not necessarily closed) morphism of nonunital* $\mathfrak{R}$*-comodule wc* A_∞*-modules* $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ *is a morphism of strictly unital* $\mathfrak{R}$*-comodule wc* A_∞*-modules if and only if the morphism of* $\mathfrak{R}$*-comodule CDG-comodules* $f : \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{L} \longrightarrow \mathfrak{C} \odot_{\mathfrak{R}} \mathcal{M}$ *over* $\mathfrak{C}$ *takes* $\mathfrak{K}_{\mathfrak{A}} \odot_{\mathfrak{R}} \mathcal{L}$ *into* $\mathfrak{K}_{\mathfrak{A}} \odot_{\mathfrak{R}} \mathcal{M}$*;*

(d) *for strictly unital* $\mathfrak{R}$*-cofree left wc* A_∞*-modules* $\mathcal{P}$ *and* $\mathcal{Q}$ *over* $\mathfrak{A}$*, a (not necessarily closed) morphism of nonunital* $\mathfrak{R}$*-cofree wc* A_∞*-modules* $f\colon \mathcal{P} \longrightarrow \mathcal{Q}$ *is a morphism of strictly unital* $\mathfrak{R}$*-cofree wc* A_∞*-modules if and only if the morphism of* $\mathfrak{R}$*-cofree CDG-contramodules* $f\colon \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{P}) \longrightarrow \mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C}, \mathcal{Q})$ *over* $\mathfrak{C}$ *takes* $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{D}, \mathcal{P})$ *into* $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{D}, \mathcal{Q})$*.*

Proof. Similar to the proofs of the three previous lemmas. ☐

Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra, $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ be a homogeneous retraction of the unit map, and $\theta_{\mathfrak{A}}\colon \mathfrak{C} \longrightarrow \mathfrak{R}$ be the corresponding linear function of degree 1. Part (a): let $\mathfrak{M}$ be a strictly unital $\mathfrak{R}$-free left wc A_∞-module over $\mathfrak{A}$. Set $d\colon \mathfrak{D} \otimes^{\mathfrak{R}} \mathfrak{M} \longrightarrow \mathfrak{D} \otimes^{\mathfrak{R}} \mathfrak{M}$ to be the map induced by the coderivation d' on $\mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$. Then $(\mathfrak{D} \otimes^{\mathfrak{R}} \mathfrak{M},\ d)$ is the $\mathfrak{R}$-free left CDG-comodule over the $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{D} = \mathrm{Bar}_v(\mathfrak{A})$ corresponding to the strictly unital wc A_∞-module $\mathfrak{M}$.

Let $f\colon \mathfrak{L} \longrightarrow \mathfrak{M}$ be a (not necessarily closed) morphism of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$. Then the morphism of $\mathfrak{R}$-free DG-comodules $f\colon \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{L} \longrightarrow \mathfrak{C} \otimes^{\mathfrak{R}} \mathfrak{M}$ induces a morphism of $\mathfrak{R}$-free CDG-comodules $f\colon \mathfrak{D} \otimes^{\mathfrak{R}} \mathfrak{L} \longrightarrow \mathfrak{D} \otimes^{\mathfrak{R}} \mathfrak{M}$. This provides the desired DG-functor in one direction; the inverse DG-functor can be constructed as the coextension of scalars with respect to the natural morphism of $\mathfrak{R}$-free CDG-coalgebras $(p, \theta_{\mathfrak{A}})\colon (\mathfrak{C}, d) \longrightarrow (\mathfrak{D}, d, h)$, where $p\colon \mathfrak{C} \longrightarrow \mathfrak{D}$ is the natural surjection. The proofs of parts (b-d) are similar. Part (e) is a particular case of (the underived versions of) Propositions 3.2.7, 3.4.4, and 4.5.5. ☐

A morphism of strictly unital wc A_∞-algebras $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is called *strict* if $f_n = 0$ for all $n \neq 1$. An *augmented* strictly unital wc A_∞-algebra $\mathfrak{A}$ is a strictly unital wc A_∞-algebra endowed with a morphism of strictly unital wc A_∞-algebras $\mathfrak{A} \longrightarrow \mathfrak{R}$, where $\mathfrak{R}$ is endowed with its only strictly unital wc A_∞-algebra structure in which $1 \in \mathfrak{R}$ is the strict unit. An augmented strictly unital wc A_∞-algebra is *strictly augmented* if the augmentation morphism is strict. A morphism of (strictly) augmented strictly unital wc A_∞-algebras is a morphism of strictly unital wc A_∞-algebras forming a commutative diagram with the augmentation morphisms [29, Section 7.2].

The categories of augmented strictly unital wc A_∞-algebras, strictly augmented strictly unital wc A_∞-algebras, and nonunital wc A_∞-algebras are naturally equivalent. The equivalence of the latter two categories is provided by the functor of formal adjoining of a strict unit, and the equivalences of the former two ones can be obtained from the equivalence between the categories of $\mathfrak{R}$-free DG-coalgebras $\mathfrak{C}$ with coaugmented reductions and $\mathfrak{R}$-free CDG-coalgebras endowed with a CDG-coalgebra morphisms $\mathfrak{C} \longrightarrow \mathfrak{R}$ compatible with the coaugmentations of the reductions. The DG-category of ($\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule) strictly unital wc A_∞-modules over an augmented strictly unital wc A_∞-algebra $\mathfrak{A}$ is equivalent to the DG-category of nonunital wc A_∞-modules over the corresponding nonunital wc A_∞-algebra.

Now let $\mathfrak{A}$ be a nonzero wcDG-algebra over $\mathfrak{R}$ and $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ be a homogeneous retraction of the unit map. The construction of the strictly unital wc A_∞-algebra structure corresponding to the given wcDG-algebra structure on $\mathfrak{A}$ can be recovered from the bar-construction of Section 6.1. Explicitly, set $m_0 = h$, $m_1(a) = d(a)$, $m_2(a_1 \otimes a_2) = a_1a_2$, and $m_n = 0$ for $n \geqslant 3$; then the CDG-coalgebra structure $\mathrm{Bar}_v(\mathfrak{A})$ on the $\mathfrak{R}$-free graded tensor coalgebra $\bigoplus_n \mathfrak{A}_+[1]^{\otimes n}$ defined in 6.1 coincides with the CDG-coalgebra structure $\mathrm{Bar}_v(\mathfrak{A})$ constructed in Theorem 7.2.1. This defines a natural faithful functor from the category of wcDG-algebras to the category of wc A_∞-algebras over $\mathfrak{R}$.

Similarly, the construction of the natural strictly unital wc A_∞-module structure on any $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule wcDG-module over $\mathfrak{A}$ can be recovered from the construction of the twisting cochain $\tau = \tau_{\mathfrak{A},v}$ and the functors $\mathrm{Hom}^\tau(\mathrm{Bar}_v(\mathfrak{A}), -)$ and $\mathrm{Bar}_v(\mathfrak{A}) \odot^\tau -$ from Section 6.2. Explicitly, given an $\mathfrak{R}$-contramodule left wcDG-module $\mathfrak{P}$ over $\mathfrak{A}$, define the structure of $\mathfrak{R}$-contramodule left wc A_∞-module over $\mathfrak{A}$ on $\mathfrak{P}$ by the rules $p_0(q) = d(q)$, $p_1(q)(a) = (-1)^{|a||q|}aq$, and $p_n = 0$ for $n \geqslant 2$. Given an $\mathfrak{R}$-comodule left wcDG-module $\mathcal{M}$ over $\mathfrak{A}$, define the structure of $\mathfrak{R}$-comodule left wc A_∞-module over $\mathfrak{A}$ on $\mathcal{M}$ by the rules $l_0(x) = d(x)$, $l_1(a \otimes x) = ax$, and $l_n = 0$ for $n \geqslant 2$. Similar constructions apply to $\mathfrak{R}$-free and $\mathfrak{R}$-cofree left wcDG-modules, and to $\mathfrak{R}$-free and $\mathfrak{R}$-comodule right wcDG-modules. These constructions define natural faithful DG-functors from the DG-categories of wcDG-modules over $\mathfrak{A}$ to the categories of wc A_∞-modules over $\mathfrak{A}$.

7.3. **Semiderived category of wc A_∞-modules.** Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra over $\mathfrak{R}$. A (not necessarily closed) morphism of strictly unital $\mathfrak{R}$-contramodule left wc A_∞-modules $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ is said to be *strict* if one has $f^n = 0$ and $(df)^n = 0$ for all $n \geqslant 1$. A (not necessarily closed) morphism of strictly unital $\mathfrak{R}$-comodule left wc A_∞-modules $f\colon \mathcal{L} \longrightarrow \mathcal{M}$ is said to be *strict* if one has $f_n = 0$ and $(df)_n = 0$ for all $n \geqslant 1$ [29, Section 7.3].

The strictness property of morphisms of strictly unital $\mathfrak{R}$-free or $\mathfrak{R}$-cofree wc A_∞-modules can be equivalently defined by the similar conditions imposed on the components f_n and $(df)_n$, or f^n and $(df)^n$, respectively. The categories of strictly unital wc A_∞-modules and strict morphisms between them are DG-subcategories of the DG-categories of strictly unital wc A_∞-modules and their (wc A_∞) morphisms. Both DG-categories of strictly unital wc A_∞-modules (from any of the four classes) with wc A_∞-morphisms or strict morphisms between them admit shifts, twists, and infinite direct sums and products.

A closed strict morphism of strictly unital ($\mathfrak{R}$-contramodule, $\mathfrak{R}$-free, $\mathfrak{R}$-comodule, or $\mathfrak{R}$-cofree) wc A_∞-modules is called a *strict homotopy equivalence* if it is a homotopy equivalence in the DG-category of strictly unital wc A_∞-modules (of the respective class) and strict morphisms between them. A short sequence $K \longrightarrow L \longrightarrow M$ of strictly unital ($\mathfrak{R}$-contramodule, $\mathfrak{R}$-free, $\mathfrak{R}$-comodule, or $\mathfrak{R}$-cofree) wc A_∞-modules and strict morphisms between them is said to be *exact* if $K \longrightarrow L \longrightarrow M$ is a short exact sequence of graded $\mathfrak{R}$-contramodules or $\mathfrak{R}$-comodules. The total strictly unital wc A_∞-module of such an exact triple is constructed in the obvious way.

For any strictly unital wc A_∞-algebra $\mathfrak{A}$, the graded k-vector space $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ has a natural structure of strictly unital A_∞-algebra. For any $\mathfrak{R}$-contramodule strictly unital left wc A_∞-module $\mathfrak{P}$ over $\mathfrak{A}$, the graded k-vector space $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ has a natural structure of strictly unital left A_∞-module over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. For any $\mathfrak{R}$-comodule strictly unital left wc A_∞-module $\mathcal{M}$ over $\mathfrak{A}$, the graded k-vector space ${}_\mathfrak{m}\mathcal{M}$ has a natural structure of strictly unital left A_∞-module over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. In particular, there are natural differentials $d = m_0/\mathfrak{m}m_0$ on $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$, $d = p_0/\mathfrak{m}p_0$ on $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$, and $d = l_0/\mathfrak{m}l_0$ on ${}_\mathfrak{m}\mathcal{M}$, making $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$, $\mathfrak{P}/\mathfrak{m}\mathfrak{P}$ and ${}_\mathfrak{m}\mathcal{M}$ complexes of k-vector spaces.

A strictly unital $\mathfrak{R}$-free wc A_∞-module $\mathfrak{M}$ over $\mathfrak{A}$ is said to be *semi-acyclic* if the strictly unital A_∞-module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ (i. e., the complex of vector spaces $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ with the differential $d = l_0/\mathfrak{m}l_0$) is acyclic. A closed morphism of strictly unital $\mathfrak{R}$-free wc A_∞-modules $f\colon \mathfrak{L} \longrightarrow \mathfrak{M}$ over $\mathfrak{A}$ is called a *semi-isomorphism* if the morphism of strictly unital A_∞-modules $f/\mathfrak{m}f\colon \mathfrak{L}/\mathfrak{m}\mathfrak{L} \longrightarrow \mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is a quasi-isomorphism (i. e., the morphism of complexes of k-vector spaces $f_0/\mathfrak{m}f_0\colon \mathfrak{L}/\mathfrak{m}\mathfrak{L} \longrightarrow \mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is a quasi-isomorphism).

Similarly, a strictly unital $\mathfrak{R}$-cofree wc A_∞-module $\mathcal{P}$ over $\mathfrak{A}$ is said to be *semi-acyclic* if the strictly unital A_∞-module ${}_\mathfrak{m}\mathcal{P}$ over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ (i. e., the complex of vector spaces ${}_\mathfrak{m}\mathcal{P}$ with the differential $d = p_0/\mathfrak{m}p_0$) is acyclic. A closed morphism of strictly unital $\mathfrak{R}$-cofree wc A_∞-modules $f\colon \mathcal{P} \longrightarrow \mathcal{Q}$ over $\mathfrak{A}$ is called a *semi-isomorphism* if the morphism of strictly unital A_∞-modules ${}_\mathfrak{m}f\colon {}_\mathfrak{m}\mathcal{P} \longrightarrow {}_\mathfrak{m}\mathcal{Q}$ over $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is a quasi-isomorphism (i. e., the morphism of complexes of k-vector spaces ${}_\mathfrak{m}f^0\colon {}_\mathfrak{m}\mathcal{P} \longrightarrow {}_\mathfrak{m}\mathcal{Q}$ is a quasi-isomorphism).

The following theorems provide a wealth of equivalent definitions of the *semiderived category of strictly unital wc* A_∞*-modules* over a strictly unital wc A_∞-algebra $\mathfrak{A}$ over a pro-Artinian topological local ring $\mathfrak{R}$.

Theorem 7.3.1. (a) *The following six definitions of the* semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ of strictly unital $\mathfrak{R}$-free left wc A_∞-modules *over* $\mathfrak{A}$ *are equivalent, i. e., lead to naturally isomorphic (triangulated) categories:*

(I) *the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ and their (wc A_∞) morphisms;*

(II) *the localization of the category of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ and their closed morphisms by the class of semi-isomorphisms;*

(III) *the localization of the category of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ and their closed morphisms by the class of strict homotopy equivalences;*

(IV) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ and strict morphisms between them by the thick subcategory of semi-acyclic $\mathfrak{R}$-free wc A_∞-modules;*

(V) *the localization of the category of strictly unital $\mathfrak{R}$-free left wc A_∞-modules over $\mathfrak{A}$ and closed strict morphisms between them by the class of strict semi-isomorphisms;*

(VI) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-free left wc* A_∞*-modules over* $\mathfrak{A}$ *and strict morphisms between them by its minimal thick subcategory containing all the total strictly unital wc* A_∞*-modules of short exact sequences of strictly unital $\mathfrak{R}$-free left wc* A_∞*-modules over* $\mathfrak{A}$ *with closed strict morphisms between them.*

(b) *The similar six definitions of the* semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ of strictly unital $\mathfrak{R}$-cofree left wc A_∞-modules *over* $\mathfrak{A}$ *(obtained from the above six definitions by replacing $\mathfrak{R}$-free wc* A_∞*-modules with $\mathfrak{R}$-cofree ones) are equivalent.*

Theorem 7.3.2. (a) *The following two definitions of the* semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ of strictly unital $\mathfrak{R}$-contramodule left wc A_∞-modules *over* $\mathfrak{A}$ *are equivalent, i. e., lead to naturally isomorphic triangulated categories:*

(I) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-contramodule left wc* A_∞*-modules over* $\mathfrak{A}$ *and their (wc* A_∞*) morphisms by its minimal triangulated subcategory containing the total strictly unital wc* A_∞*-modules of short exact sequences of strictly unital $\mathfrak{R}$-contramodule left wc* A_∞*-modules over* $\mathfrak{A}$ *with closed strict morphisms between them, and closed with respect to infinite products;*

(II) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-contramodule left wc* A_∞*-modules over* $\mathfrak{A}$ *and strict morphisms between them by its minimal thick subcategory containing all the total strictly unital wc* A_∞*-modules of short exact sequences of strictly unital $\mathfrak{R}$-contramodule left wc* A_∞*-modules over* $\mathfrak{A}$ *with closed strict morphisms between them, and closed with respect to infinite products.*

(b) *The following two definitions of the* semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ of strictly unital $\mathfrak{R}$-comodule left wc A_∞-modules *over* $\mathfrak{A}$ *are equivalent, i. e., lead to naturally isomorphic triangulated categories:*

(I) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-modules over* $\mathfrak{A}$ *and their (wc* A_∞*) morphisms by its minimal triangulated subcategory containing the total strictly unital wc* A_∞*-modules of short exact sequences of strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-modules over* $\mathfrak{A}$ *with closed strict morphisms between them, and closed with respect to infinite direct sums;*

(II) *the quotient category of the homotopy category of the DG-category of strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-modules over* $\mathfrak{A}$ *and strict morphisms between them by its minimal thick subcategory containing all the total strictly unital wc* A_∞*-modules of short exact sequences of strictly unital $\mathfrak{R}$-comodule left wc* A_∞*-modules over* $\mathfrak{A}$ *with closed strict morphisms between them, and closed with respect to infinite direct sums.*

Let $\mathfrak{A}$ be a nonzero strictly unital wc A_∞-algebra over $\mathfrak{R}$, let $v\colon \mathfrak{A} \longrightarrow \mathfrak{R}$ be a homogeneous retraction of the unit map, and let $\mathfrak{C} = \mathrm{Bar}_v(\mathfrak{A})$ be the corresponding CDG-coalgebra structure on the $\mathfrak{R}$-free graded tensor coalgebra $\bigoplus_n \mathfrak{A}_+[1]^{\otimes n}$. Denote by $w\colon \mathfrak{R} \longrightarrow \mathfrak{C}$ the natural section $\mathfrak{R} \simeq \mathfrak{A}[1]^{\otimes 0} \longrightarrow \bigoplus_n \mathfrak{A}[1]^{\otimes n}$ of the counit map, and set $\mathfrak{U}=\mathrm{Cob}_w(\mathfrak{C})$. The wcDG-algebra $\mathfrak{U}$ is called the *enveloping wcDG-algebra* of a strictly unital wc A_∞-algebra $\mathfrak{A}$. The adjunction map $\mathfrak{C}\longrightarrow\mathrm{Bar}_v(\mathrm{Cob}_w(\mathfrak{C}))$ provides a natural morphism of strictly unital wc A_∞-algebras $\mathfrak{A}\longrightarrow\mathfrak{U}$.

Theorem 7.3.3. *The following triangulated categories are naturally equivalent:*

(a) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of strictly unital* $\mathfrak{R}$*-free left wc* A_∞*-modules over* $\mathfrak{A}$*;*

(b) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *of strictly unital* $\mathfrak{R}$*-contramodule left wc* A_∞*-modules over* $\mathfrak{A}$*;*

(c) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *of strictly unital* $\mathfrak{R}$*-comodule left wc* A_∞*-modules over* $\mathfrak{A}$*;*

(d) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of strictly unital* $\mathfrak{R}$*-cofree left wc* A_∞*-modules over* $\mathfrak{A}$*;*

(e) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of strictly unital* $\mathfrak{R}$*-free left wcDG-modules over* $\mathfrak{U}$*;*

(f) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *of strictly unital* $\mathfrak{R}$*-contramodule left wcDG-modules over* $\mathfrak{U}$*;*

(g) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *of strictly unital* $\mathfrak{R}$*-comodule left wcDG-modules over* $\mathfrak{U}$*;*

(h) *the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of strictly unital* $\mathfrak{R}$*-cofree left wcDG-modules over* $\mathfrak{U}$*;*

(i) *the contraderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *of strictly unital* $\mathfrak{R}$*-contramodule left wcDG-modules over* $\mathfrak{U}$*;*

(j) *the coderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *of strictly unital* $\mathfrak{R}$*-comodule left wcDG-modules over* $\mathfrak{U}$*;*

(k) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of strictly unital* $\mathfrak{R}$*-free left wcDG-modules over* $\mathfrak{U}$*;*

(l) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of strictly unital* $\mathfrak{R}$*-cofree left wcDG-modules over* $\mathfrak{U}$*;*

(m) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of* $\mathfrak{R}$*-free left CDG-contramodules over* $\mathfrak{C}$*;*

(n) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *of* $\mathfrak{R}$*-contramodule left CDG-contramodules over* $\mathfrak{C}$*;*

(o) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of* $\mathfrak{R}$*-cofree left CDG-contramodules over* $\mathfrak{C}$*;*

(p) *the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of* $\mathfrak{R}$*-free left CDG-comodules over* $\mathfrak{C}$*;*

(q) *the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ *of* $\mathfrak{R}$*-comodule left CDG-comodules over* $\mathfrak{C}$*;*

(r) *the coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of* $\mathfrak{R}$*-cofree left CDG-comodules over* $\mathfrak{C}$*;*

(s) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of* $\mathfrak{R}$*-free left CDG-contramodules over* $\mathfrak{C}$*;*

(t) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of* $\mathfrak{R}$*-cofree left CDG-contramodules over* $\mathfrak{C}$*;*

(u) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})$ *of* $\mathfrak{R}$*-free left CDG-comodules over* $\mathfrak{C}$*;*

(v) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *of* $\mathfrak{R}$*-cofree left CDG-comodules over* $\mathfrak{C}$*.*

Proof of Theorems 7.3.1–7.3.3. The equivalence of all the constructions in the three theorems with the exception of (s–v) of Theorem 7.3.3 holds in the much greater generality of CDG-comodules and CDG-contramodules over an arbitrary $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ with conilpotent reduction $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$.

Specifically, the DG-categories of strictly unital $\mathfrak{R}$-free, $\mathfrak{R}$-contramodule, $\mathfrak{R}$-comodule, and $\mathfrak{R}$-cofree left wc A_∞-modules over $\mathfrak{A}$ with their wc A_∞-morphisms are described as the DG-categories of left CDG-contra/comodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-contra/comodules, respectively, cofreely cogenerated by free graded $\mathfrak{R}$-contramodules, induced from graded $\mathfrak{R}$-contramodules, coinduced from graded $\mathfrak{R}$-comodules, and freely generated by cofree graded $\mathfrak{R}$-comodules.

Furthermore, let $\mathfrak{C}$ be an $\mathfrak{R}$-free CDG-algebra. A morphism of $\mathfrak{R}$-free left CDG-comodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-comodules cofreely cogenerated by free graded $\mathfrak{R}$-contramodules $f\colon \mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{L}\longrightarrow\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$ can be called *strict* if both f and df, viewed as morphisms of graded $\mathfrak{C}$-comodules, can be obtained from morphisms of free graded $\mathfrak{R}$-contramodules by applying the functor $\mathfrak{C}\otimes^{\mathfrak{R}}-$. Using Lemma 7.1.1(a), one can show that the DG-category of such $\mathfrak{R}$-free CDG-comodules $\mathfrak{C}\otimes^{\mathfrak{R}}\mathfrak{M}$ and strict morphisms between them is isomorphic to the DG-category of $\mathfrak{R}$-free left CDG-modules $\mathfrak{M}$ over the $\mathfrak{R}$-free CDG-algebra $\mathfrak{U}=\mathrm{Cob}_w(\mathfrak{C})$; the equivalence is provided by the DG-functor $\mathfrak{M}\longmapsto\mathfrak{C}\otimes^{\tau_{\mathfrak{C},w}}\mathfrak{M}$.

Similarly, the DG-categories of $\mathfrak{R}$-contramodule left CDG-contramodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-contramodules $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{P})$ induced from graded $\mathfrak{R}$-contramodules $\mathfrak{P}$, $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-comodules $\mathfrak{C}\odot_{\mathfrak{R}}\mathcal{M}$ coinduced from graded $\mathfrak{R}$-comodules $\mathcal{M}$, $\mathfrak{R}$-cofree left CDG-contramodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-contramodules $\mathrm{Ctrhom}_{\mathfrak{R}}(\mathfrak{C},\mathcal{P})$ freely generated by cofree graded $\mathfrak{R}$-comodules $\mathcal{P}$, and strict morphisms between them are described as the DG-categories of $\mathfrak{R}$-contramodule, $\mathfrak{R}$-comodule, and $\mathfrak{R}$-cofree left CDG-modules over $\mathfrak{U}$, respectively.

When the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is coaugmented (e. g., conilpotent), we pick a homogeneous section $w\colon\mathfrak{R}\longrightarrow\mathfrak{C}$ of the counit map $\mathfrak{C}\longrightarrow\mathfrak{R}$ so that $w/\mathfrak{m}w=\bar w$ be the coaugmentation map; then $\mathfrak{U}$ is a wcDG-algebra over $\mathfrak{R}$.

So the definition (IV) of Theorem 7.3.1(a) is that of the semiderived category $\mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$, and the definition (VI) is that of the absolute derived category $\mathsf{D}^{\mathsf{abs}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}})$. As mentioned above, the definition (I) is that of the homotopy category of CDG-comodules over $\mathfrak{C}$ with the underlying graded comodules cofreely cogenerated by free graded $\mathfrak{R}$-contramodules.

Now the equivalence of (I) and (IV) follows essentially from Corollary 6.3.3, and the equivalence of (I) and (VI) is Corollary 6.4.2 (for the latter argument we do not even need the CDG-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ to be conilpotent). Alternatively, the equivalence of

(IV) and (VI) is a particular case of Theorem 2.3.3 (cf. the remarks after the proof of Corollary 6.4.2).

In the rest of the proof of Theorem 7.3.1, we will need to use the following result ([15, Proposition III.4.2 and Lemma III.4.3]).

Lemma 7.3.4. *Let* DG *be a DG-category with shifts and cones. Then the category obtained from the category of closed morphisms* $Z^0(\mathsf{DG})$ *by inverting formally all morphisms that are homotopy equivalences in* DG *(i. e., represent isomorphisms in* $H^0(\mathsf{DG})$*) is naturally isomorphic to* $H^0(\mathsf{DG})$*. More precisely, it suffices to invert in* $Z^0(\mathsf{DG})$ *all the morphisms of the form* $s_X = (\mathrm{id}_X, 0)\colon X \oplus \mathrm{cone}(\mathrm{id}_X) \longrightarrow X$*, where* X *are objects of* DG*, in order to obtain the homotopy category* $H^0(\mathsf{DG})$*.*

Proof. Clearly, the morphisms s_X are isomorphisms in $H^0(\mathsf{DG})$, so one only has to check that inverting such morphisms leads to any two morphisms in $Z^0(\mathsf{DG})$ that are equal to each other in $H^0(\mathsf{DG})$ becoming equal in the localized category. A closed morphism $X \longrightarrow Y$ in DG is homotopic to zero if and only if it can be factorized as the composition of the canonical morphism $c_X\colon X \longrightarrow \mathrm{cone}(\mathrm{id}_X)$ and a closed morphism $\mathrm{cone}(\mathrm{id}_X) \longrightarrow Y$. Let i_X, $j_X\colon X \longrightarrow X \oplus \mathrm{cone}(\mathrm{id}_X)$ be the two closed morphisms defined by the rules $i_X = (\mathrm{id}_X, 0)$ and $j_X = (\mathrm{id}_X, c_X)$. Then two closed morphisms f, $g\colon X \longrightarrow Y$ are homotopic to each other if and only if there exists a closed morphism $q\colon X \oplus \mathrm{cone}(\mathrm{id}_X) \longrightarrow Y$ such that $f = q \circ i_X$ and $g = q \circ j_X$. Now both compositions $s_X \circ i_X$ and $s_X \circ j_X$ are equal to id_X, so inverting the morphism s_X makes i_X equal to j_X and f equal to g. □

The classes of semi-isomorphisms and homotopy equivalences in the DG-category of strictly unital $\mathfrak{R}$-(co)free wc A_∞-modules and wc A_∞-morphisms coincide. The appropriate generalization of this holds for any $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ with conilpotent reduction and follows from the similar assertion for conilpotent CDG-coalgebras over k [29, proof of Theorem 7.3.1] and Lemma 3.2.1.

So the equivalence of the definitions (I) and (II) in Theorem 7.3.1(a) follows from the first assertion of Lemma 7.3.4, and the equivalence of (I) and (III) is clear from the second one. The equivalence of (IV) and (V) also follows from the first assertion of the lemma, as a strict morphism is a strict semi-isomorphism if and only if its cone is semi-acyclic. The latter two deductions do not even require $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ to be conilpotent. The proof of Theorem 7.3.1(b) is similar.

The proof of Theorem 7.3.2 is based on the next lemma.

Lemma 7.3.5. *Let* $\mathfrak{C}$ *be an* $\mathfrak{R}$*-free CDG-algebra. Then*

(a) *the quotient category of the homotopy category of* $\mathfrak{R}$*-contramodule left CDG-contramodules over* $\mathfrak{C}$ *with the underlying graded* $\mathfrak{C}$*-contramodules induced from graded* $\mathfrak{R}$*-contramodules by its minimal triangulated subcategory which contains the total CDG-contramodules of the short exact sequences of CDG-contramodules over* $\mathfrak{C}$ *whose underlying short exact sequences of graded* $\mathfrak{C}$*-contramodules are induced from short exact sequences of graded* $\mathfrak{R}$*-contramodules, and is closed under infinite products, is equivalent to the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$ *of* $\mathfrak{R}$*-contramodule left CDG-contramodules over* $\mathfrak{C}$*;*

(b) *the quotient category of the homotopy category of $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{C}$ with the underlying graded $\mathfrak{C}$-comodules coinduced from graded $\mathfrak{R}$-comodules by its minimal triangulated subcategory which contains the total CDG-comodules of the short exact sequences of CDG-comodules over $\mathfrak{C}$ whose underlying short exact sequences of graded $\mathfrak{C}$-comodules are induced from short exact sequences of graded $\mathfrak{R}$-comodules, and is closed under infinite direct sums, is equivalent to the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})$ of $\mathfrak{R}$-comodule left CDG-comodules over $\mathfrak{C}$.*

Proof. This is a much simpler version of [28, Theorem 5.5]. Part (a): let us show that the natural functor from the homotopy category of CDG-contramodules over $\mathfrak{C}$ whose underlying graded $\mathfrak{C}$-contramodules are freely generated by free graded $\mathfrak{R}$-contramodules to the quotient category of the homotopy category of CDG-contramodules with induced underlying graded $\mathfrak{C}$-contramodules that we are interested in is an equivalence of categories. Then it will remain to take into account the version Theorem 4.5.2(c) with projective graded $\mathfrak{C}$-contramodules replaced by free ones. The semiorthogonality being provided by part (a) of the same Theorem, we only have to show that the natural functor from the homotopy category of CDG-contramodules with free underlying graded $\mathfrak{C}$-contramodules to our quotient category of the homotopy category of the homotopy category of CDG-contramodules with induced underlying graded $\mathfrak{C}$-contramodules is essentially surjective.

Applying the construction from the beginning of the proof of Theorem 4.2.1 to $\mathfrak{R}$-contramodule left CDG-modules over the $\mathfrak{R}$-free CDG-algebra $\mathfrak{U} = \mathrm{Cob}_w(\mathfrak{C})$, we conclude that any $\mathfrak{R}$-contramodule left CDG-contramodule over $\mathfrak{C}$ with an induced underlying graded $\mathfrak{C}$-contramodule admits a left resolution by $\mathfrak{R}$-free CDG-contramodules with free underlying graded $\mathfrak{C}$-contramodules and closed morphisms between these such that the underlying complex of graded $\mathfrak{C}$-contramodules is induced from a complex of graded $\mathfrak{R}$-contramodules. Now it remains to use (the appropriate version of) Lemma 4.2.2. The proof of part (b) is similar. ☐

The definition (II) of Theorem 7.3.2(a) is that of the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})$, while according to Lemma 7.3.5(a) the definition (I) is that of the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})$. The two categories are equivalent by Corollary 6.4.2. This argument does not even depend on the assumption about $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ being conilpotent. The proof of Theorem 7.3.2(b) is similar.

For the purposes of the proof of Theorem 7.3.3, we will use the definitions (I) in Theorems 7.3.1 and 7.3.2 as setting the meaning of items (a-d). Items (e) and (h) are the constructions (IV) of Theorem 7.3.1, items (k) and (l) are the constructions (VI) of the same Theorem, and items (i) and (j) of Theorem 7.3.3 are the constructions (II) of Theorem 7.3.2.

So the equivalence of (a), (e), and (k) is provided by Theorem 7.3.1(a), the equivalence of (d), (h), and (l) is provided by Theorem 7.3.1(b). The equivalence of (b) and (i) is Theorem 7.3.2(a), and the equivalence of (c) and (j) is Theorem 7.3.2(b).

The equivalence of (a) and (p) is essentially Theorem 3.2.2(d), and the equivalence of (d) and (o) is Theorem 3.4.1(c). The equivalence of (b) and (n) is Lemma 7.3.5(a), and the equivalence of (c) and (q) is Lemma 7.3.5(b).

The equivalence of (e–h) is the results of Section 2.6, Proposition 4.3.2, and the basic results of Section 4.3. The equivalence of (e) and (k) is Theorem 2.3.3, and the equivalence of (h) and (l) is similar. The equivalence of (f) and (i) and the equivalence of (g) and (j) are Corollary 4.3.3. Alternatively, the equivalence of (i) and (k) and the equivalence of (j) and (l) are Corollary 4.2.6. The equivalence of (i) and (j) is Corollary 4.2.7, and the equivalence of (k) and (l) was established in Section 2.5.

The equivalence of the respective items in (e), (f), (g), (h) and (p), (n), (q), (o) is Corollary 6.3.3. The equivalence of the respective items in (i), (j), (k), (l) and (n), (q), (p), (o) is Corollary 6.4.2.

The equivalence of (m–r) is Corollaries 3.2.4 and 3.4.2, the results of Sections 3.2 and 3.4, and Corollary 4.5.3. The equivalence of (m) and (s) is Theorem 3.2.5 together with Corollary 3.1.3, and so is the equivalence of (p) and (u). The equivalence of (o) and (t) and the equivalence of (r) and (v) are similar. ☐

Let $\mathfrak{A}$ be a wcDG-algebra over $\mathfrak{R}$; it can be considered also as a strictly unital wc A_∞-algebra, and wcDG-modules over it can be viewed also as strictly unital wc A_∞-modules (see Section 7.2). It follows from Corollary 6.3.2 that the semiderived category of ($\mathfrak{R}$-free, $\mathfrak{R}$-contramodule, $\mathfrak{R}$-comodule, or $\mathfrak{R}$-cofree) left wcDG-modules over $\mathfrak{A}$ is equivalent to the semiderived category of strictly unital left wc A_∞-modules over $\mathfrak{A}$ (from the same class), so our notation is consistent.

When the pro-Artinian topological local ring $\mathfrak{R}$ has finite homological dimension, the semiderived categories of strictly unital ($\mathfrak{R}$-free, $\mathfrak{R}$-contramodule, $\mathfrak{R}$-comodule, or $\mathfrak{R}$-cofree) wc A_∞-modules over $\mathfrak{A}$ can be called simply their *derived categories* (see Sections 2.3, 2.6, and 4.3 for the discussion).

Corollary 7.3.6. *For any strictly unital wc* A_∞*-algebra* $\mathfrak{A}$ *over a pro-Artinian topological local ring* $\mathfrak{R}$*, the semiderived category* $\mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \simeq \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}})$ *has a single compact generator.*

Proof. Follows from Theorem 5.3.3 applied to the enveloping wcDG-algebra $\mathfrak{U}$ (see also Theorem 5.4.3). ☐

Let $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a morphism of strictly unital wc A_∞-algebras over $\mathfrak{R}$. Then the constructions of the restriction of scalars from Section 7.1 for nonunital ($\mathfrak{R}$-free, $\mathfrak{R}$-contramodule, $\mathfrak{R}$-comodule, or $\mathfrak{R}$-cofree) wc A_∞-modules with respect to the morphism f take strictly unital wc A_∞-modules over $\mathfrak{B}$ to strictly unital wc A_∞-modules over $\mathfrak{A}$. The induced restriction-of-scalars functors on the DG-categories of strictly unital wc A_∞-modules can be also obtained as the contra/coextension-of-scalars functors related to the morphism of $\mathfrak{R}$-free CDG-coalgebras $g\colon \mathfrak{C} = \mathrm{Cob}_w(\mathfrak{A}) \longrightarrow \mathrm{Cob}_w(\mathfrak{B}) = \mathfrak{D}$ assigned to the morphism $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ by the construction of Theorem 7.2.1.

Furthermore, the restrictions of these DG-functors to the DG-categories of strictly unital wc A_∞-modules and strict morphisms between them can be identified with the restriction-of-scalars functors with respect to the morphism of the enveloping wcDG-algebras $F\colon \mathfrak{U} = \mathrm{Bar}_v(\mathrm{Cob}_w(\mathfrak{A})) \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(\mathfrak{B})) = \mathfrak{V}$ induced by g.

The functors $E_g\colon \mathfrak{D}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}_{\mathsf{inj}}^{\mathfrak{R}\text{–}\mathsf{fr}}$ and $R_F\colon \mathfrak{V}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}} \longrightarrow \mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}$ induce the same triangulated functor on the semiderived categories of strictly unital $\mathfrak{R}$-free left wc A_∞-modules, which we denote by

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}),$$

and similarly for strictly unital $\mathfrak{R}$-free right wc A_∞-modules. The functors $E^g\colon \mathfrak{D}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ and $R_F\colon \mathfrak{V}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}} \longrightarrow \mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}$ induce the same triangulated functor on the semiderived categories of strictly unital $\mathfrak{R}$-contramodule left wc A_∞-modules, which we denote by

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}).$$

The functors $E_g\colon \mathfrak{D}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}} \longrightarrow \mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}$ and $R_F\colon \mathfrak{V}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}} \longrightarrow \mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}$ induce the same triangulated functor on the semiderived categories of strictly unital $\mathfrak{R}$-comodule left wc A_∞-modules, which we denote by

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}),$$

and similarly for strictly unital $\mathfrak{R}$-comodule right wc A_∞-modules. The functors $E^g\colon \mathfrak{D}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{C}\text{–}\mathsf{contra}_{\mathsf{proj}}^{\mathfrak{R}\text{–}\mathsf{cof}}$ and $R_F\colon \mathfrak{V}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}} \longrightarrow \mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}$ induce the same triangulated functor on the semiderived categories of strictly unital $\mathfrak{R}$-cofree left wc A_∞-modules, which we denote by

$$\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{B}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{cof}}).$$

The functors $\mathbb{I}R_f$ are identified by the equivalences of categories from Theorem 7.3.3. When $\mathfrak{A}$ is a wcDG-algebra over $\mathfrak{R}$, the above functors $\mathbb{I}R_f$ are the same restriction-of-scalars functors that were constructed in Sections 2.3, 2.6, and 4.3.

The triangulated functors $\mathbb{I}R_f$ have the left and right adjoint functors $\mathbb{L}E_f$ and $\mathbb{R}E^f$ that can be constructed either as the functors of co- and contrarestriction of scalars $\mathbb{I}R_g$ and $\mathbb{I}R^g$ on the level of the co- and contraderived categories of CDG-comodules and CDG-contramodules over $\mathfrak{C}$ and $\mathfrak{D}$ (see Sections 3.2, 3.4, and 4.5), or as the functors of extension and coextension of scalars $\mathbb{L}E_F$ and $\mathbb{R}E^F$ on the level of the semiderived categories of wcDG-modules over $\mathfrak{U}$ and $\mathfrak{V}$ (cf. Proposition 6.5.6).

A morphism of (strictly unital) wc A_∞-algebras $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is called a *semi-isomorphism* if the morphism of (strictly unital) A_∞-algebras $f/\mathfrak{m}f\colon \mathfrak{A}/\mathfrak{m}\mathfrak{A} \longrightarrow \mathfrak{B}/\mathfrak{m}\mathfrak{B}$ over the field k is a quasi-isomorphism (i. e., the morphism of complexes of k-vector spaces $f_1/\mathfrak{m}f_1$ is a quasi-isomorphism).

Theorem 7.3.7. *The triangulated functors $\mathbb{I}R_f$ are equivalences of categories whenever a morphism of strictly unital wc A_∞-algebras $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is a semi-isomorphism.*

Proof. Can be deduced easily either from Theorem 3.2.8(b), or from Corollary 4.3.6 (see [29, proof of Theorem 7.3.2] for further details). ☐

Let $\mathfrak{A}$ be a strictly unital wc A_∞-algebra over $\mathfrak{R}$, let $\mathfrak{C} = \mathrm{Bar}_v(\mathfrak{A})$ be the corresponding $\mathfrak{R}$-free CDG-coalgebra, and let $\mathfrak{U} = \mathrm{Cob}_w(\mathfrak{C})$ be the enveloping wcDG-algebra.

All the above results about strictly unital left wc A_∞-modules over $\mathfrak{A}$ apply to strictly unital right wc A_∞-modules as well, e. g., because one can pass to the opposite $\mathfrak{R}$-free (C)DG-coalgebras and wcDG-algebras (see [29, Section 4.7] and [30, Section 1.2] for the sign rules) and such a passage is compatible with the functors Bar_v and Cob_w.

In particular, the semiderived categories of strictly unital $\mathfrak{R}$-free and $\mathfrak{R}$-comodule right wc A_∞-modules $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A})$ and $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A})$ over $\mathfrak{A}$ are defined and naturally equivalent to the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C})$ and $\mathsf{D}^{\mathsf{co}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C})$ and to the semiderived categories $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{U})$ and $\mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{U})$, respectively.

The functor

$$\mathrm{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

can be defined either as the functor $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or as the functor $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$; the two points of view are equivalent by Proposition 3.2.6(b) (see [29, Section 7.3] for further details). Alternatively, the functor $\mathrm{Tor}^{\mathfrak{A}}$ can be defined as the functor $\mathrm{Tor}^{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{U}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Tor}^{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.1(a) and/or 6.5.2(a).

The functor

$$\mathrm{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{A}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$

can be defined either as the functor $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, or as the functor $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$; the two points of view are equivalent by Proposition 3.4.3(c). Alternatively, the functor $\mathrm{Tor}^{\mathfrak{A}}$ can be defined as the functor $\mathrm{Tor}^{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}\text{–}\mathfrak{U}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Tor}^{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.1(b) and/or 6.5.2(b).

The functor

$$\mathrm{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$

can be defined as the functor $\mathrm{Ctrtor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, or alternatively as the functor $\mathrm{Tor}^{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{U}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Tor}^{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.1(c).

The functor

$$\mathrm{Tor}^{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{A}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$

can be defined either as the functor $\mathrm{Cotor}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{C}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, or alternatively as the functor $\mathrm{Tor}^{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}\text{–}\mathfrak{U}) \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$ The two definitions are equivalent and agree with the definition of the functor $\mathrm{Tor}^{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.2(c).

The above four functors $\mathrm{Tor}^{\mathfrak{A}}$ are transformed into each other by the equivalences of categories from Theorem 7.3.3(a-d) and the equivalence of categories $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}} \simeq \mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}$ from Proposition 1.5.1.

The functor

$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

can be defined either as the functor $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or as the functor $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or as the functor $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$; the three points of view are equivalent by Proposition 3.2.6(a) (see [29, Section 7.3] for further details). Alternatively, the functor $\mathrm{Ext}_{\mathfrak{A}}$ can be defined as the functor $\mathrm{Ext}_{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Ext}_{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.3(a), 6.5.4(a), and/or 6.5.5(a).

The functor

$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

can be defined either as the functor $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or as the functor $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or as the functor $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$; the three points of view are equivalent by Proposition 3.4.3(b). Alternatively, the functor $\mathrm{Ext}_{\mathfrak{A}}$ can be defined as the functor $\mathrm{Ext}_{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Ext}_{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.3(b), 6.5.4(b), and/or 6.5.5(b).

The functor

$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$$

can be defined either as the functor $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, or as the functor $\mathrm{Ext}^{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$, or as the functor $\mathrm{Ext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{fr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{cof}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$; the three points of view are equivalent by Proposition 3.4.3(a). Alternatively, the functor $\mathrm{Ext}_{\mathfrak{A}}$ can be defined as the functor $\mathrm{Ext}_{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Ext}_{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.3(c), 6.5.4(c), and/or 6.5.5(c).

The functor

$$\mathrm{Ext}_{\mathfrak{A}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{A}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$$

can be defined as the functor $\mathrm{Coext}_{\mathfrak{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{C}\text{–}\mathsf{comod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(\mathfrak{C}\text{–}\mathsf{contra}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$, or alternatively as the functor $\mathrm{Ext}_{\mathfrak{U}}\colon \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{co}})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{si}}(\mathfrak{U}\text{–}\mathsf{mod}^{\mathfrak{R}\text{–}\mathsf{ctr}}) \longrightarrow H^0(\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}})$. The two definitions are equivalent and agree with the definition of the functor $\mathrm{Ext}_{\mathfrak{A}}$ for a wcDG-algebra $\mathfrak{A}$ by Theorem 6.5.5(d).

The above four functors $\mathrm{Ext}_{\mathfrak{A}}$ are transformed into each other by the equivalences of categories from Theorem 7.3.3(a-d) and the equivalence of categories $\mathfrak{R}\text{–}\mathsf{contra}^{\mathsf{free}} \simeq \mathfrak{R}\text{–}\mathsf{comod}^{\mathsf{cofr}}$ from Proposition 1.5.1.

Appendix A. Projective Limits of Artinian Modules

The following results are well-known and easy to prove in the case of projective systems indexed by countable sets. They are certainly known to the specialists in the general case as well. We include their proofs in this appendix, because we have not succeeded in finding a suitable reference.

A.1. **Main lemma.** Here is our main technical assertion.

Lemma A.1.1. *Let (R_α) be a filtered projective system of (noncommutative) rings and surjective morphisms between them and (M_α) be a projective system of Artinian R_α-modules. Then the first derived functor of projective limit vanishes on (M_α), that is $\varprojlim^1_\alpha M_\alpha = 0$.*

Proof. First of all, it suffices to consider the case when the indices α form a filtered (directed) poset, rather than a filtered category [1, Proposition I.8.1.6]. Furthermore, the derived functors $\varprojlim^n_\alpha$ computed in the abelian category of projective systems of R_α-modules agree with the ones computed in the category of projective systems of abelian groups (so the assertion of Lemma is unambigous). Indeed, given an index α_0 and an R_{α_0}-module N_0, consider the projective system (N_α) with $N_\alpha = N_0$ for $\alpha \geqslant \alpha_0$ and $N_\alpha = 0$ otherwise. Then infinite products of projective systems of this form are adjusted both to the projective limits of R_α-modules and abelian groups.

In order to compute $\varprojlim^1_\alpha M_\alpha$, it suffices to embed (M_α) into an injective projective system of R_α-modules (L_α); denoting by (K_α) the cokernel of this embedding, we have $\varprojlim^1_\alpha M_\alpha = \operatorname{coker}(\varprojlim_\alpha L_\alpha \to \varprojlim_\alpha K_\alpha)$. Replacing one of the modules M_α with the image of the map $M_\beta \longrightarrow M_\alpha$ into it from some module M_β (and all the modules M_γ with $\gamma > \alpha$ with the related full preimage under the map $M_\gamma \to M_\alpha$) does not affect the projective limits we are interested in.

Iterating this procedure over a well-ordered set of steps and passing to the projective limits of projective systems as needed still does not change $\varprojlim_\alpha M_\alpha$ and $\varprojlim_\alpha K_\alpha$, because projective limits commute with projective limits. At every component with a given index α such a projective limit stabilizes due to the Artinian condition. When the possibilities to iterate the procedure nontrivially are exhausted, we will obtain a projective system M'_α of Artinian R_α-modules and surjective maps between them such that $\varprojlim_\alpha M'_\alpha = \varprojlim_\alpha M_\alpha$ and $\varprojlim^1_\alpha M'_\alpha = \varprojlim^1_\alpha M_\alpha$. So we can assume M_α to be a filtered projective system of Artinian modules and surjective maps between them.

Let (k_α) be an element of the projective limit $\varprojlim_\alpha K_\alpha$; we want to lift it to an element of $\varprojlim_\alpha L_\alpha$. For this purpose, we will apply Zorn's lemma to the following poset X. Its elements are subsets D in the set of all indices $\{\alpha\}$ endowed with chosen preimages $l_\alpha \in L_\alpha$ of the elements $k_\alpha \in K_\alpha$ for all $\alpha \in D$. The elements l_α must satisfy the following condition. For any finite subset $S \subset D$ there should exist an index β (not necessarily belonging to D) and a preimage $l'_\beta \in L_\beta$ of the element $k_\beta \in K_\beta$ such that $\beta > \alpha$ for all $\alpha \in S$ and the map $L_\beta \longrightarrow L_\alpha$ takes l'_β to l_α.

Clearly, X contains the unions of all its linearly ordered subsets. It remains to check that for any $(D; l_\alpha) \in X$ and $\gamma \notin D$ one can find a preimage $l_\gamma \in L_\gamma$ of the element k_γ so as to make $(D \cup \{\gamma\};\, l_\alpha, l_\gamma)$ a new element of X.

Notice that, given a finite set of indices S, preimages l_α of the elements k_α for all $\alpha \in S$, and an index $\beta > \alpha$ for all $\alpha \in S$, the possibility to choose an appropriate element l'_β as above does not depent on the choice of β. Indeed, suppose $\beta' > \beta > \alpha$ for all $\alpha \in S$. Then, having an appropriate element $l'_{\beta'} \in L_{\beta'}$, one can take its image under the map $L_{\beta'} \longrightarrow L_\beta$, obtaining an appropriate element in L_β. Conversely, given an appropriate element $l'_\beta \in L_\beta$, one can pick any preimage $l''_{\beta'}$ of $k_{\beta'}$ in $L_{\beta'}$, take its image l''_β under the map $L_{\beta'} \longrightarrow L_\beta$, consider the difference $l'_\beta - l''_\beta$ as an element of M_β, lift it to $M_{\beta'}$, and add the result to $l''_{\beta'}$, obtaining the desired element $l'_{\beta'}$.

Now let $S \subset D$ be a finite subset, let β be an index greater than all the elements of S, and $l'_\beta \in L_\beta$ be an appropriate preimage of the element k_β, as above. Consider the set $P_S \subset L_\gamma$ of all the preimages l_γ of the element k_γ for which the pair $(S \cup \{\gamma\}; l_\alpha, l_\gamma)$ belongs to X. First of all, let us show that the set P_S is nonempty. Indeed, let δ be any index greater than both β and γ. As we have seen, the element l'_β can be lifted to a preimage $l'_\delta \in L_\delta$ of the element k_δ that would be appropriate for $(S; l_\alpha)$. Set l_γ to be the image of l_δ under the map $L_\delta \longrightarrow L_\gamma$.

Furthermore, P_S is an affine R_γ-submodule (i. e., an additive coset by a conventional R_γ-submodule) of L_γ. Indeed, one can easily see that P_S is closed under linear those linear combinations with coefficients in R_γ in which the sum of all coefficients is equal to 1. Finally, if S', $S'' \subset D$ are two finite subsets and $S = S' \cup S''$, then the affine submodule $P_S \subset L_\gamma$ is contained in the intersection $P_{S'} \cap P_{S''}$. In addition, all the affine submodules P_S are contained in the coset of L_γ modulo M_γ corresponding to the element $k_\gamma \in K_\gamma$.

Since we assume the R_γ-module M_γ to be Artinian, it follows that the intersection of all P_S is nonempty.

Therefore, Zorn's lemma provides us with a system of preimages $l_\alpha \in L_\alpha$ of the elements k_α defined for all indices α and forming an element of the set X, i. e., satisfying the compatibility condition formulated above. Clearly, it follows that the map $L_\beta \to L_\alpha$ takes l_β to l_α for all $\beta > \alpha$; so Lemma is proven. □

A.2. **Pro-Artinian rings.** The following two corollaries of the main lemma demonstrate that a pro-Artinian topological ring is a well-behaved notion.

Corollary A.2.1. *Let (R_α) be a filtered projective system of (right) Artinian (noncommutative) rings and surjective morphisms between them, and let $\mathfrak{R}$ be its projective limit. Then the projection map $\mathfrak{R} \longrightarrow R_\alpha$ is surjective for each α.*

Proof. Consider the short exact sequence of projective systems $\ker(R_\beta \to R_\alpha) \longrightarrow R_\beta \longrightarrow R_\alpha$ of right R_β-modules, indexed by all $\beta > \alpha$ with α fixed, pass to the projective limits, and apply Lemma A.1.1. □

Corollary A.2.2. *Let $\mathfrak{R}$ be a complete separated (noncommutative) topological ring with a base of neighborhoods of zero formed by open ideals with right Artinian quotient rings. Let $\mathfrak{J} \subset \mathfrak{R}$ be a closed ideal. Then the quotient ring $\mathfrak{R}/\mathfrak{J}$ is complete in the quotient topology (and has all the other properties assumed above for $\mathfrak{R}$).*

Proof. Consider the short exact sequence of projective systems $(\mathfrak{J}+\mathfrak{I})/\mathfrak{I} \longrightarrow \mathfrak{R}/\mathfrak{I} \longrightarrow \mathfrak{R}/(\mathfrak{J}+\mathfrak{I})$ of right modules over the Artinian rings $\mathfrak{R}/\mathfrak{I}$, indexed by all the open ideals $\mathfrak{I} \subset \mathfrak{R}$. Passing to the projective limits and applying Lemma A.1.1, we obtain the desired isomorphism $\mathfrak{R}/\mathfrak{J} \simeq \varprojlim_{\mathfrak{I}} \mathfrak{R}/(\mathfrak{J}+\mathfrak{I})$. ☐

A.3. **Pro-Artinian modules.** The following result will be useful for us when dealing with comodules over pro-Artinian topological rings.

Corollary A.3.1. *Let $\mathfrak{R}$ be (noncommutative) topological ring. Then the functor of projective limit acting from the category of pro-objects in the abelian category of discrete $\mathfrak{R}$-modules of finite length (or discrete Artinian $\mathfrak{R}$-modules) to the abelian category of (nontopological) $\mathfrak{R}$-modules is exact and conservative.*

Proof. Given a morphism between the pro-objects M and N represented by filtered projective systems (M_α) and (N_β), one can consider the set of all triples (α, β, f), where $f\colon M_\alpha \longrightarrow N_\beta$ is a morphism of modules representing the morphism of pro-objects $M \longrightarrow N_\beta$. This is a filtered poset in the natural order. Replacing the index sets $\{\alpha\}$ and $\{\beta\}$ with the poset $\{(\alpha,\beta,f)\}$, one can assume the pro-objects M and N to be represented by projective systems (M_γ) and (N_γ) indexed by the same filtered poset $\{\gamma\}$ and the morphism $M \longrightarrow N$ to be represented by a morphism of projective systems $M_\gamma \longrightarrow N_\gamma$. The kernel and cokernel of the morphism of pro-objects $M \longrightarrow N$ are then represented by the (terms-wise) kernel and cokernel of this morphism of projective systems.

Now the assertion that the projective limit functor is exact follows straightforwardly from Lemma A.1.1 (it suffices to consider the constant projective system of rings $R_\gamma = R$). To prove conservativity, notice that any pro-object represented by a projective system (M_α) is also represented by its maximal projective subsystem $M'_\alpha \subset M_\alpha$ with surjective morphisms $M'_\beta \longrightarrow M'_\alpha$ (because M_α are Artinian, so M'_α is the image of the morphism into M_α from some M_δ). Now if $M'_\beta \neq 0$ for some β, then $\varprojlim_{\gamma>\beta} M'_\gamma \longrightarrow M'_\beta$ is a surjective morphism of R-modules by the same Lemma, hence $\varprojlim_\alpha M_\alpha \neq 0$ (cf. the proof of Corollary A.2.1). ☐

A.4. **Closed ideals.** The following result demonstrates that intersections of closed ideals in a pro-Artinian topological ring are well-behaved. It will be used in the proof of Lemma 1.6.1.

Corollary A.4.1. *Let $\mathfrak{R}$ be a (right) pro-Artinian topological ring and $\mathfrak{J}_\alpha$ be a filtered poset of its closed ideals ordered by inclusion. Then for any open ideal $\mathfrak{I} \subset \mathfrak{R}$ one has $\mathfrak{I} + \bigcap_\alpha \mathfrak{J}_\alpha = \bigcap_\alpha (\mathfrak{I} + \mathfrak{J}_\alpha)$. In particular, if the intersection of all $\mathfrak{J}_\alpha$ is contained in $\mathfrak{I}$, then there exists an index α such that $\mathfrak{J}_\alpha$ is contained in $\mathfrak{I}$.*

Proof. Since the family of ideals $\mathfrak{I}+\mathfrak{J}_\alpha$ is filtered and the ring $\mathfrak{R}/\mathfrak{I}$ is (right) Artinian, this family stabilizes, that is $\mathfrak{I}+\mathfrak{J}_\alpha = \mathfrak{I}'$ is the same open ideal in $\mathfrak{R}$ for all α large enough. Therefore, $\varprojlim_\alpha (\mathfrak{J}_\alpha + \mathfrak{I})/\mathfrak{I} = (\bigcap_\alpha (\mathfrak{J}_\alpha + \mathfrak{I}))/\mathfrak{I}$. Clearly, $\bigcap_\alpha \mathfrak{J}_\alpha = \varprojlim_\alpha \mathfrak{J}_\alpha$ and $\mathfrak{J}_\alpha = \varprojlim_{\mathfrak{K}} (\mathfrak{J}_\alpha + \mathfrak{K})/\mathfrak{K}$, where $\mathfrak{K}$ runs over all the open ideals in $\mathfrak{R}$; hence $\bigcap_\alpha \mathfrak{J}_\alpha = \varprojlim_{\alpha,\mathfrak{K}} (\mathfrak{J}_\alpha + \mathfrak{K})/\mathfrak{K}$. The map $(\mathfrak{J}_\alpha + \mathfrak{K})/\mathfrak{K} \longrightarrow (\mathfrak{J}_\alpha + \mathfrak{I})/\mathfrak{I}$ is well-defined and surjective for all $\mathfrak{K} \subset \mathfrak{I}$. Passing to the projective limit in $\mathfrak{K}$ and α, and using Lemma A.1.1, we conclude that the map $\bigcap_\alpha \mathfrak{J}_\alpha \longrightarrow (\bigcap_\alpha (\mathfrak{J}_\alpha + \mathfrak{I}))/\mathfrak{I}$ is surjective, as desired. □

Appendix B. Contramodules over a Complete Noetherian Ring

The aim of this appendix is to describe the abelian category of contramodules over a complete Noetherian local ring $\mathfrak{R}$ as a full subcategory of the category of $\mathfrak{R}$-modules formed by the $\mathfrak{R}$-modules with some special properties. In fact, we obtain a somewhat more general result applicable to any adic completion of a Noetherian ring.

This description of $\mathfrak{R}$-contramodules does not appear to be particularly useful for our purposes (and indeed, it is never used in the main body of the paper). However, it may present an independent interest, and may also help some readers to acquaint themselves with the notion of an $\mathfrak{R}$-contramodule.

B.1. **Formulation of theorem.** Recall that for any topological ring $\mathfrak{R}$ (in the sense of Section 1.1) there is a natural exact, conservative forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ from the abelian category of $\mathfrak{R}$-contramodules to the abelian category of modules over the ring $\mathfrak{R}$ viewed as an abstract (nontopological) ring.

The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ preserves infinite products, but may not preserve infinite direct sums. It is *not* in general a tensor functor with respect to the natural tensor category structures on $\mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{R}\text{–}\mathsf{mod}$, though it commutes with the internal Hom (see Section 1.2).

Let R be a Noetherian ring and $m \subset R$ be an ideal. Consider the m-adic completion $\mathfrak{R} = \varprojlim_n R/m^n$ of R and let $\mathfrak{m} = \varprojlim_n m/m^n \subset \mathfrak{R}$ be the extension of m in $\mathfrak{R}$. Endow the ring $\mathfrak{R}$ with the topology of the projective limit (i. e., the $\mathfrak{m}$-adic topology).

In particular, given a complete Noetherian local ring $\mathfrak{R}$ with the maximal ideal $\mathfrak{m}$, one can always take $R = \mathfrak{R}$ and $m = \mathfrak{m}$. We will be interested in the composition of the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ with the functor of restriction of scalars $\mathfrak{R}\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}$. By $\operatorname{Ext}^*_R(-,-)$ we denote the Ext functor computed in the abelian category of R-modules.

Theorem B.1.1. (1) *The forgetful functor* $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ *is fully faithful.*

(2) *The image of the functor* $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ *is the full abelian subcategory in* $R\text{–}\mathsf{mod}$ *consisting of all the* R*-modules* P *satisfying any of the following equivalent conditions:*

(a) *for any multiplicatively closed subset* $S \subset R$ *having a nonempty intersection with* m*, any* $R[S^{-1}]$*-module* L*, and any integer* $i \geqslant 0$ *one has* $\operatorname{Ext}^i_R(L,P) = 0$*;*

(b) *assuming m is a maximal ideal in R, for any multiplicatively closed subset $S \subset R$ and any integer $i \geqslant 0$ one has* $\operatorname{Ext}^i_R(R[S^{-1}], P) = 0$*, with the only exception of $S \cap m = \varnothing$ and $i = 0$;*

(c) *for any element $s \in m$ and any $i = 0$ or 1 one has* $\operatorname{Ext}^i_R(R[s^{-1}], P) = 0$.

The equivalent conditions (a-c) in part (2) are modelled after Jannsen's definition of *weakly l-complete abelian groups* in [19, Definition 4.6] or, which is the same, the definition of *Ext-p-complete abelian groups* in Bousfield–Kan [4, Sections VI.3–4]. (These are also closely related to Harrison's *co-torsion groups* [16].) The case of $\mathfrak{R} = k[[\epsilon]]$ was considered in [28, Remark A.1.1] and the comparison with Jannsen's definition in the l-adic case ($R = \mathbb{Z}$ and $\mathfrak{R} = \mathbb{Z}_l$) was mentioned in [28, Remark A.3].

The proof of the above theorem occupies Sections B.2–B.7.

B.2. **Hom and Ext into a contramodule.** Let us show that the underlying R-module of any $\mathfrak{R}$-contramodule $\mathfrak{P}$ satisfies the conditions (a-b).

Quite generally, let R be a ring, $\mathfrak{R}$ be a topological ring, and $R \longrightarrow \mathfrak{R}$ be a ring homomorphism. Let M be an R-module and $\mathfrak{P}$ be an $\mathfrak{R}$-contramodule. Then the set $\operatorname{Hom}_R(M, \mathfrak{P})$ of all R-module homomorphisms $M \longrightarrow \mathfrak{P}$ has a natural $\mathfrak{R}$-contramodule structure with the "pointwise" infinite summation operation $(\sum_\alpha r_\alpha f_\alpha)(x) = \sum_\alpha r_\alpha f_\alpha(x)$, where $f_\alpha \in \operatorname{Hom}_R(M, \mathfrak{P})$ are R-module homomorphisms and $r_\alpha \in \mathfrak{R}$ is any family of elements converging to zero (cf. Section 1.5). The R-module structure obtained by applying the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ to this $\mathfrak{R}$-contramodule structure coincides with the natural R-module structure on $\operatorname{Hom}_R(M, \mathfrak{P})$.

The $\mathfrak{R}$-contramodule structure on $\operatorname{Hom}_R(M, \mathfrak{P})$ is functorial with respect to R-module morphisms $M' \longrightarrow M''$ and $\mathfrak{R}$-contramodule morphisms $\mathfrak{P}' \longrightarrow \mathfrak{P}''$. It follows that for every $i \geqslant 0$ the functor $M \longmapsto \operatorname{Ext}^i_R(M, \mathfrak{P})$ factorizes through the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$, i. e., for any R-module M the R-module $\operatorname{Ext}^i_R(M, \mathfrak{P})$ has a natural $\mathfrak{R}$-contramodule structure. Indeed, the Ext module in question can be computed in terms of a left projective resolution of the R-module M, which makes it the cohomology (contra)module of a complex of $\mathfrak{R}$-contramodules.

On the other hand, if an element $s \in R$ acts invertibly in an R-module L, then its action in the R-module $\operatorname{Ext}^i_R(L, N)$ is also invertible for any R-module N and $i \geqslant 0$. Now if $\mathfrak{m}$ is a topologically nilpotent ideal in $\mathfrak{R}$ and the action of an element $s \in \mathfrak{m}$ in the $\mathfrak{R}$-contramodule $\mathfrak{Q} = \operatorname{Ext}^i_R(L, \mathfrak{P})$ is invertible, then $\mathfrak{m}\mathfrak{Q} = \mathfrak{Q}$ and $\mathfrak{Q} = 0$ by Lemma 1.3.1. This proves the property (a).

To check (b), it remains to consider the case when $S \cap m = \varnothing$. Then the elements of S become invertible in $\mathfrak{R}$, hence $\operatorname{Ext}^i_R(M, P) \simeq \operatorname{Ext}^i_{R[S^{-1}]}(M[S^{-1}], P)$ for any R-module M and $\mathfrak{R}$-module P. In particular, $\operatorname{Ext}^i_R(R[S^{-1}], P) \simeq \operatorname{Ext}^i_{R[S^{-1}]}(R[S^{-1}], P) \simeq P$ for $i = 0$ and 0 for $i > 0$.

B.3. **Ring of power series.** Let $x_j \in R$, $1 \leqslant j \leqslant n$, be a finite set of generators of the ideal $m \subset R$. Abusing notation, we will denote the images of x_j in $\mathfrak{R}$ also by x_j. Consider the ring $\mathfrak{T} = R[[t_j]]$ of formal power series in the variables t_j with coefficients in the ring R, and endow $\mathfrak{T}$ with the standard formal power series topology (i. e., the adic topology for the ideal generated by t_j).

There exists a unique continuous ring homomorphism $\tau\colon \mathfrak{T} \longrightarrow \mathfrak{R}$ equal to the natural map $R \longrightarrow \mathfrak{R}$ in the restriction to R and taking t_j to x_j. Since x_j generate $\mathfrak{m}$, the map τ is surjective and open, so the topology on $\mathfrak{R}$ is the quotient topology of the topology on $\mathfrak{T}$. In addition, the image under τ of the ideal $(t_j) \subset \mathfrak{T}$ generated by all the elements $t_j \in \mathfrak{T}$ is equal to $\mathfrak{m} \subset \mathfrak{R}$.

By [25, Theorem 8.12], the ideal $\mathfrak{J} = \ker\tau \subset \mathfrak{T} = R[[t_j]]$ is generated (as an abstract ideal in a nontopological ring) by the elements $x_j - t_j$. For completeness, let us give an independent proof of this claim. Clearly, $x_j - t_j \in \ker\tau$.

Lemma B.3.1. *For any Noetherian ring R and finite set of variables t_j, any ideal in the ring of formal power series $R[[t_j]]$ is closed in the (t_j)-adic topology.*

Proof. The assertion of Lemma can be equivalently rephrased by saying that an ideal in $R[[t_j]]$ is determined by its images in the quotient rings $R[[t_j]]/(t_j)^N$, where N runs over positive integers. In the case of a single variable t, it is clear from the standard proof of the Hilbert basis theorem for formal power series (see, e. g., [25, Theorem 3.3]) that an ideal in $R[[t]]$ is determined by its images in the rings $R[[t]]/(t^N)$. The general case is handled by induction: an ideal in $R[[t_1,\dots,t_n]]$ is determined by its images in $R[[t_2,\dots,t_n]][[t_1]]/(t_1^{N_1}) \simeq (R[t_1]/(t_1^{N_1}))[[t_2,\dots,t_n]]$, which in turn are determined by their images in $(R[[t_1,t_2]]/(t_1^{N_1},t_2^{N_2}))[[t_3,\dots,t_n]]$, etc. So finally an ideal in $R[[t_1,\dots,t_n]]$ is determined by its images in $R[[t_1,\dots,t_n]]/(t_j^{N_j})$, where N_1, …, N_n are positive integers. ☐

Hence it suffices to show that the images of the elements $x_j - t_j$ generate the image of $\mathfrak{J}$ in the quotient ring $\mathfrak{T}/(t_j)^N$ for each N. Since $\tau((t_j)^N) = \mathfrak{m}^N$, the ideal $\mathfrak{J} + (t_j)^N \subset \mathfrak{T}$ is the kernel of the ring homomorphism $\mathfrak{T} \longrightarrow \mathfrak{R}/\mathfrak{m}^N$ taking t_j to the images of x_j. As $\mathfrak{R}/\mathfrak{m}^N \simeq R/m^N$, it remains to show that the kernel of the homomorphism $R[[t_j]]/(t_j)^N \longrightarrow R/m^N$ is generated by $x_j - t_j$.

Since the monomial $t_1^{l_1}\cdots t_n^{l_n}$ is equal to $x_1^{l_1}\cdots x_n^{l_n}$ modulo the ideal generated by $x_j - t_j$, the ideal generated by $x_j - t_j$ in $R[[t_j]]/(t_j)^N$ contains m^N. We have reduced to the obvious assertion that the kernel of the map $(R/m^N)[t_j]/(t_j)^N \longrightarrow R/m^N$ is generated by $x_j - t_j$.

B.4. **Contramodules over a quotient ring.** The following lemma is an almost tautological restatement of the definitions.

Lemma B.4.1. (a) *Let $\mathfrak{U} \longrightarrow \mathfrak{V}$ be a continuous homomorphism of topological rings such that any family $X \longrightarrow \mathfrak{V}$ of elements of $\mathfrak{V}$ indexed by a set X and converging to* 0 *in the topology of $\mathfrak{V}$ can be lifted to a family $X \longrightarrow \mathfrak{U}$ converging to* 0 *in the topology of $\mathfrak{U}$. Then the functor of restriction of scalars $\mathfrak{V}$–$\mathsf{contra} \longrightarrow \mathfrak{U}$–$\mathsf{contra}$ is fully faithful.*

(b) *In the situation of part* (a)*, let $\mathfrak{J}$ be the kernel of the morphism $\mathfrak{U} \longrightarrow \mathfrak{V}$. Assume further that $f\colon Z \longrightarrow \mathfrak{J}$ is a family of elements converging to* 0 *in the topology of $\mathfrak{U}$ such that for any family of elements $g\colon X \longrightarrow \mathfrak{J}$ converging to* 0 *there exists* *a family of families of elements $h\colon Z\times X\longrightarrow\mathfrak{J}$ converging to 0 for every fixed $z\in Z$ such that $g(x)=\sum_{z\in Z}f(z)h(z,x)$ for any $x\in X$. Then the image of the functor of restriction of scalars $\mathfrak{V}$–$\mathsf{contra}\longrightarrow\mathfrak{U}$–$\mathsf{contra}$ consists precisely of those $\mathfrak{U}$-contramodules $\mathfrak{P}$ for which $\sum_{z\in Z}f(z)q(z)=0$ in $\mathfrak{P}$ for every family of elements $q\colon Z\longrightarrow\mathfrak{P}$.*

Proof. Given two $\mathfrak{V}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$, any morphism of $\mathfrak{U}$-contramodules $\mathfrak{P}\longrightarrow\mathfrak{Q}$ is also a morphism of $\mathfrak{V}$-contramodules provided that the map $\mathfrak{U}[[\mathfrak{P}]]\longrightarrow\mathfrak{V}[[\mathfrak{P}]]$ is surjective. The condition of part (a) means that the map $\mathfrak{U}[[X]]\longrightarrow\mathfrak{V}[[X]]$ is surjective for any set X, so the assertion of part (a) follows.

To prove part (b), notice that the sequence $0\longrightarrow\mathfrak{J}[[X]]\longrightarrow\mathfrak{U}[[X]]\longrightarrow\mathfrak{V}[[X]]\longrightarrow 0$ is exact for any set X in the assumptions of (a). One has to show that, for a $\mathfrak{U}$-contramodule $\mathfrak{P}$, the map $\mathfrak{J}[[\mathfrak{P}]]\longrightarrow\mathfrak{P}$ is zero provided that the family of elements $f(z)$ acts by zero in $\mathfrak{P}$. Let $g\colon\mathfrak{P}\longrightarrow\mathfrak{J}$ be an element of $\mathfrak{J}[[\mathfrak{P}]]$, i. e., a $\mathfrak{P}$-indexed family of elements of $\mathfrak{J}$ converging to 0. By the assumption of part (b), we have $g(p)=\sum_{z\in Z}f(z)h(z,p)$, hence $\sum_{p\in\mathfrak{P}}g(p)p=\sum_{z\in Z}f(z)\sum_{p\in\mathfrak{P}}h(z,p)p=0$. $\square$

Now we return to our ring homomorphism $\mathfrak{T}=R[[t_j]]\longrightarrow\mathfrak{R}$.

Corollary B.4.2. *The functor of restriction of scalars $\mathfrak{R}$–$\mathsf{contra}\longrightarrow\mathfrak{T}$–$\mathsf{contra}$ is fully faithful, and its image consists precisely of those $\mathfrak{T}$-contramodules in (the underlying $\mathfrak{T}$-module structure of) which the elements x_j-t_j act by zero.*

Proof. The topologies of $\mathfrak{T}$ and $\mathfrak{R}$ having countable bases of neighborhoods of zero, the condition (a) of Lemma B.4.1 clearly holds. It remains to check the condition (b) for the finite family of elements $f(j)=x_j-t_j$.

By the Artin–Rees lemma [25, Theorem 8.5] applied to the $\mathfrak{T}$-submodule $\mathfrak{J}\subset\mathfrak{T}$ and the ideal $(t_j)\subset\mathfrak{T}$, there exists an integer l such that $(t_j)^N\cap\mathfrak{J}\subset(t_j)^{N-l}\mathfrak{J}$ for all large enough N. It follows easily from this observation together with the fact that $\mathfrak{J}$ is the ideal generated by the elements x_j-t_j in $\mathfrak{T}$ that any family of elements of $\mathfrak{J}$ converging to 0 in the topology of $\mathfrak{T}$ can be presented as a linear combination of n families of elements converging to 0 in $\mathfrak{T}$ with the coefficients x_j-t_j. $\square$

B.5. **Contramodules over $R[[t]]$.** The results of this subsection and the next one do not depend on the Noetherianness assumption on a ring R. Let $R[[t]]$ be the ring of formal power series in one variable t with coefficients in R, endowed with the t-adic topology; and let $R[t]$ be the ring of polynomials.

Lemma B.5.1. *The forgetful functor $R[[t]]$–$\mathsf{contra}\longrightarrow R[t]$–$\mathsf{mod}$ identifies the category of $R[[t]]$-contramodules with the full subcategory of the category of $R[t]$-modules consisting of all the modules P with the following property. For any sequence of elements $p_i\in P$, $i\geqslant 0$, there exists a unique sequence of elements $q_i\in P$ such that $q_i=p_i+tq_{i+1}$ for all $i\geqslant 0$.*

Proof. First of all, the category of $R[[t]]$-contramodules is equivalent to the category of R-modules P endowed with the operation assigning to every sequence of elements

$p_i \in P$, $\ i \geqslant 0$, an element $\sum_{i=0}^\infty t^ip_i \in P$ and satisfying following equations of linearity, unitality, and associativity:

$$\textstyle\sum_{i=0}^\infty t^i(r'p'_i+r''p''_i) = r'\sum_{i=0}^\infty t^ip'_i + r''\sum_{i=0}^\infty t^ip''_i$$

for any p'_i, $p''_i \in P$ and r', $r'' \in R$;

$$\textstyle\sum_{i=0}^\infty t^ip_i = p_0$$

when $p_1=p_2=\dots=0$ in P, and

$$\textstyle\sum_{i=0}^\infty t^i\sum_{j=0}^\infty t^jp_{ij} = \sum_{n=0}^\infty t^n\sum_{i+j=n}p_{ij}$$

for any $p_{ij} \in P$, $\ i \geqslant 0$, $\ j \geqslant 0$. Morphisms of $R[[t]]$-contramodules correspond to R-module morphisms $f\colon P \longrightarrow Q$ such that $\sum_{i=0}^\infty t^if(p_i) = f(\sum_{i=0}^\infty t^ip_i)$ in Q for any sequence of elements $p_i \in P$.

Indeed, the $R[[t]]$-contramodule infinite summation operations restricted to the case of the sequence of coefficients t^i clearly have the above properties. Conversely, given an R-module P endowed with the operation of infinite summation with the coefficients t^i as above, one defines

$$\textstyle\sum_\alpha(\sum_{i=0}^\infty g_{\alpha,i}t^i)p_\alpha = \sum_{i=0}^\infty t^i\sum_\alpha g_{\alpha,i}p_\alpha$$

for any family of elements $g_\alpha = \sum_{i=0}^\infty g_{\alpha,i}t^i$, $\ g_{\alpha,i} \in R$, converging to 0 in $R[[t]]$ (so $g_{\alpha,i}=0$ for all but a finite number of indices α, for every fixed i), and any $p_\alpha \in P$.

Let us check that such infinite summations with the coefficients g_α are associative provided that the infinite summations with the coefficients t^i were associative and linear. We have $\sum_\alpha g_\alpha(t)\sum_\beta h_{\alpha\beta}(t)p_{\alpha\beta} = \sum_{i=0}^\infty t^i\sum_\alpha g_{\alpha,i}\sum_{j=0}^\infty t^j\sum_\beta h_{\alpha\beta,j}p_{\alpha\beta} = \sum_{i=0}^\infty t^i\sum_{j=0}^\infty t^j\sum_\alpha g_{\alpha,i}\sum_\beta h_{\alpha\beta,j}p_{\alpha\beta} = \sum_{n=0}^\infty t^n\sum_{i+j=n}\sum_{\alpha,\beta}g_{\alpha,i}h_{\alpha\beta,j}p_{\alpha\beta} = \sum_{n=0}^\infty t^n \sum_{\alpha,\beta}k_{\alpha\beta,n}p_{\alpha\beta}$, where $g_\alpha(t) = \sum_{i=0}^\infty g_{\alpha,i}t^i$, $\ h_{\alpha\beta}(t) = \sum_{j=0}^\infty h_{\alpha\beta,j}t^j$, and $k_{\alpha\beta}(t) = g_\alpha(t)h_{\alpha\beta}(t) = \sum_{n=0}^\infty k_{\alpha\beta,n}(t)t^n$. It is clear that any morphism preserving the operations of infinite summation with the coefficients t^i preserves also the operations of infinite summation with the coeffcients g_α.

Now assume that the infinite summation operations with the coefficients t^i are defined on an R-module P. The action of the operator t on P is then provided by the obvious rule $tp = \sum_{i=0}^\infty t^ip_i$, where $p_1=p$ and $p_i=0$ for $i\ne 1$.

Notice that any sequence of elements $q_i \in P$ such that $q_i = tq_{i+1}$ for $i \geqslant 0$ is zero. Indeed, one has $\sum_{i=0}^\infty t^iq_{i+n} = \sum_{i=0}^\infty t^itq_{i+n+1} = \sum_{i=0}^\infty t^{i+1}q_{i+n+1} = -q_n + \sum_{i=0}^\infty t^iq_{i+n}$, hence $q_n=0$ for all $n \geqslant 0$ (cf. the proof of Lemma 1.3.1).

It follows that for any given sequence $p_i \in P$, a sequence $q_i \in P$ such that $q_i = p_i + tq_{i+1}$ is unique if it exists. To prove the existence, set $q_n = \sum_{i=0}^\infty t^ip_{n+i}$.

Conversely, assuming the existence and uniqueness property for the sequence q_i such that $q_i = p_i + tq_{i+1}$, set $\sum_{i=0}^\infty t^ip_i = q_0$. Let us check the equations of linearity, unitality, and associativity.

Given two sequences p'_i and $p''_i \in P$, and two elements r', $r'' \in R$, let q'_i and $q''_i \in P$ be the sequences such that $q_i^{(s)} = p_i^{(s)} + tq_{i+1}^{(s)}$. Then the sequence $q_i = r'q'_i + r''q''_i$ satisfies the equation $q_i = (r'p'_i + r''p''_i) + tq_{i+1}$, hence $\sum_{i=0}^\infty t^i(r'p'_i + r''p''_i) = q_0 = r'q'_0 + r''q''_0 = r'\sum_{i=0}^\infty t^ip'_i + r''\sum_{i=0}^\infty t^ip''_i$.

Given a sequence $p_i \in P$ such that $p_i = 0$ for $i > 0$, set $q_0 = p_0$ and $q_i = 0$ for $i > 0$. Then the equation $q_i = p_i + tq_{i+1}$ holds for all i, so $\sum_{i=0}^\infty t^i p_i = q_0 = p_0$.

Finally, given a biindexed sequence $p_{ij} \in P$, find sequences $q_{ij} \in P$ such that $q_{ij} = p_{ij} + tq_{i,j+1}$ for i, $j \geqslant 0$. Let $u_n \in P$ be a sequence satisfying $u_n = q_{n,0} + tu_{n+1}$; set $v_n = q_{0,n} + q_{1,n-1} + \dots + q_{n-1,1} + u_n = q_{0,n} + q_{1,n-1} + \dots + q_{n-1,1} + q_{n,0} + tu_{n+1}$. Then we have $v_n - tv_{n+1} = (q_{0,n} - tq_{0,n+1}) + \dots + (q_{n,0} - tq_{n+1,0}) = p_{0,n} + p_{1,n-1} + \dots + p_{n-1,1} + p_{n,0}$. Hence $\sum_{i=0}^\infty t^i \sum_{j=0}^\infty t^j p_{ij} = \sum_{i=0}^\infty t^i q_{i,0} = u_{00} = v_{00} = \sum_{n=0}^\infty t^n \sum_{i+j=n} p_{ij}$.

We have described the image of the functor $R[[t]]\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ on the level of objects. It remains to show that it is surjective on morphisms. Indeed, let $\mathfrak{P}$ and $\mathfrak{Q}$ be $R[[t]]$-contramodules, and $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ be an $R[t]$-module morphism. Given a sequence $p_i \in P$, $i \geqslant 0$, find a sequence $u_i \in P$ such that $u_i = p_i + tu_{i+1}$. Then $f(u_i) = f(p_i) + tf(u_{i+1})$, hence $f(\sum_{i=0}^\infty t^i p_i) = f(u_0) = \sum_{i=0}^\infty t^i f(p_i)$. ☐

B.6. **Contramodules over** $R[[t_1, \dots, t_n]]$**.** Now we describe the category of contramodules over the ring of formal power series $R[[t_j]]$ in several variables t_j, where $j = 1, \dots, n$, with the (t_j)-adic topology on $R[[t_j]]$.

Lemma B.6.1. *The forgetful functor* $R[[t_j]]\text{–}\mathsf{contra} \longrightarrow R[t_j]\text{–}\mathsf{mod}$ *identifies the category of* $R[[t_j]]$*-contramodules with the full subcategory of the category of* $R[t_j]$*-modules with the following property. For any sequence of elements* $p_i \in P$, $i \geqslant 0$, *and any variable* t_j, $1 \leqslant j \leqslant n$, *there exists a unique sequence of elements* $q_i \in P$ *such that* $q_i = p_i + t_j q_{i+1}$ *for all* $i \geqslant 0$.

Proof. For a multiindex $a = (a_1, \dots, a_n)$, $a_i \geqslant 0$, denote by t^a the monomial $t_1^{a_1} \cdots t_n^{a_n}$. The category of $R[[t_j]]$-contramodules is equivalent to the category of R-modules P endowed with the operation assigning to every multiindexed sequence of elements $p_a \in P$ an element $\sum_a t^a p_a \in P$ satisfying the equations of linearity, unitality and associativity similar to those introduced in the proof of Lemma B.5.1. The proof is the same as in the case of one variable t.

Furthermore, the data of infinite summation operations with the coefficients t^a is equivalent to that of infinite summation operations with the coefficients t_j^i, for every fixed j, with the commutativity equation $\sum_{i'=0}^\infty t_{j'}^{i'} \sum_{i''=0}^\infty t_{j''}^{i''} p_{i'i''} = \sum_{i''=0}^\infty t_{j''}^{i''} \sum_{i'=0}^\infty t_{j'}^{i'} p_{i'i''}$ for every biindexed sequence $p_{i'i''} \in P$, i', $i'' \geqslant 0$, and every two variable numbers $1 \leqslant j', j'' \leqslant n$. Indeed, given the infinite summation operations with the coefficients t_j^i, for every fixed j, on an R-module P, one defines the infinite summation operations with the coefficients t^a on P by the rule $\sum_a t^a p_a = \sum_{a_1=0}^\infty t_1^{a_1} \sum_{a_2=0^\infty} t_2^{a_2} \cdots \sum_{a_n=0}^\infty t_n^{a_n} p_{a_1 \dots a_n}$. Clearly, any morphism preserving the infinite summation operations with the coefficients t_j^i preserves also the infinite summation operations with the coefficients t^a.

Finally, it remains to show that the infinite summation operations with the coefficients $t_{j'}^{i'}$ and $t_{j''}^{i''}$ recovered from an $R[t_j]$-module structure satisfying our condition commute with each other. For simplicity of notation, denote our variables $t_{j'}$ and $t_{j''}$ by s and t. Given a biindexed sequence $p_{ij} \in P$, i, $j \geqslant 0$, find a sequence q_{ij} such that $q_{ij} = p_{ij} + tq_{i,j+1}$ and a sequence u_{ij} such that $u_{ij} = q_{ij} + sq_{i+1,j}$. Similarly, find a sequence v_{ij} such that $v_{ij} = p_{ij} + sv_{i+1,j}$ and a sequence w_{ij} such that $w_{ij} = v_{ij} + tv_{i,j+1}$

for all $i,\, j \geqslant 0$. Set $w_{ij} - sw_{i+1,j} = z_{ij}$. Then we have $z_{ij} - tz_{i,j+1} = (w_{ij} - sw_{i+1,j}) - t(w_{i,j+1} - sw_{i+1,j+1}) = (w_{ij} - tw_{i,j+1}) - s(w_{i+1,j} - tw_{i+1,j+1}) = v_{ij} - sv_{i+1,j} = p_{ij}$. It follows that $z_{ij} = q_{ij}$ and therefore $w_{ij} = u_{ij}$ for all $i,\, j \geqslant 0$. Now $\sum_{i=0}^\infty s^i \sum_{j=0}^\infty t^j p_{ij} = \sum_{i=0}^\infty s^i q_{i,0} = u_{0,0} = w_{0,0} = \sum_{j=0}^\infty t^j v_{0,j} = \sum_{j=0}^\infty t^j \sum_{i=0}^\infty s^i p_{ij}$. ☐

B.7. **The one-variable Ext condition.** The following lemma compares the condition (c) of Theorem B.1.1 with the conditions of Lemmas B.5.1–B.6.1.

Lemma B.7.1. *Let R be a ring, $s \in R$ be an element, and P be an R-module. Then one has $\operatorname{Ext}_R^1(R[s^{-1}], P) = 0$ if and only if for any sequence of elements $p_i \in P$, $i \geqslant 0$, there exists a sequence of elements $q_i \in P$ such that $q_i = p_i + sq_{i+1}$ for all $i \geqslant 0$. One has $\operatorname{Hom}_R(R[s^{-1}], P) = 0$ if and only if the sequence of elements q_i as above is unique for every (or some particular) sequence of elements p_i.*

Proof. Consider the free resolution $0 \longrightarrow \bigoplus_{i=0}^\infty Rf_i \longrightarrow \bigoplus_{i=0}^\infty Re_i \longrightarrow R[s^{-1}] \longrightarrow 0$ of the R-module $R[s^{-1}]$ with the maps taking f_i to $e_i - se_{i+1}$ and e_i to s^{-i}.

Then the map $\operatorname{Hom}_R(\bigoplus_i Re_i,\ P) \longrightarrow \operatorname{Hom}_R(\bigoplus_i Rf_i,\ P)$ computing the desired modules Hom_R and Ext_R^1 is identified with the map taking a sequence $(q_i) \in P$ to the sequence $p_i = q_i - sq_{i+1}$. ☐

Now we are in the position to finish the proof of Theorem B.1.1. The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ factorizes into the composition of three functors $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{T}\text{–}\mathsf{contra} \longrightarrow R[t_j]\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}$.

Denote by $\mathfrak{T}\text{–}\mathsf{contra}_0$ and $R[t_j]\text{–}\mathsf{mod}_0$ the full subcategories of $\mathfrak{T}\text{–}\mathsf{contra}$ and $R[t_j]\text{–}\mathsf{mod}$, respectively, consisting of those (contra)modules where the elements $x_j - t_j$ act by zero. Then the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ can be also decomposed as $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{T}\text{–}\mathsf{contra}_0 \longrightarrow R[t_j]\text{–}\mathsf{mod}_0 \longrightarrow R\text{–}\mathsf{mod}$.

The functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{T}\text{–}\mathsf{contra}_0$ is an equivalence of categories by Corollary B.4.2 and the functor $R[t_j]\text{–}\mathsf{mod}_0 \longrightarrow R\text{–}\mathsf{mod}$ is obviously an equivalence of categories. The functor $\mathfrak{T}\text{–}\mathsf{contra} \longrightarrow R[t_j]\text{–}\mathsf{mod}$ is fully faithful by Lemma B.6.1.

The restriction of the latter functor to the category $\mathfrak{T}\text{–}\mathsf{contra}_0$ identifies it with the full subcategory of $R[t_j]\text{–}\mathsf{mod}_0$ consisting of those modules where the action of the elements t_j satisfies the condition of Lemma B.6.1. Finally, this action coincides with the action of the elements x_j, and the condition on this action is a particular case of the condition (c) by Lemma B.7.1. ☐

B.8. **The Ext comparison.** The following theorem complements the assertion of Theorem B.1.1(1). We keep the notation of Section B.1. Let $\operatorname{Ext}_{\mathfrak{R}}^*(-,-)$ denote the Ext functor computed in the abelian category of $\mathfrak{R}$-contramodules.

Theorem B.8.1. *The map $\operatorname{Ext}_{\mathfrak{R}}^*(\mathfrak{P},\mathfrak{Q}) \longrightarrow \operatorname{Ext}_R^*(\mathfrak{P},\mathfrak{Q})$ induced by the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is an isomorphism for any $\mathfrak{R}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$.*

Proof. The map $\operatorname{Hom}_{\mathfrak{R}}(\mathfrak{P},\mathfrak{Q}) \longrightarrow \operatorname{Hom}_R(\mathfrak{P},\mathfrak{Q})$ is an isomorphism by Theorem B.1.1(1). Clearly, it suffices to prove that $\operatorname{Ext}_R^i(\mathfrak{P},\mathfrak{Q}) = 0$ for any projective $\mathfrak{R}$-contramodule $\mathfrak{P}$, any $\mathfrak{R}$-contramodule $\mathfrak{Q}$, and all $i > 0$. Propositions B.9.1 and B.10.1 below imply as much. ☐

B.9. **Free contramodules are flat modules.** The following result goes back at least to [10, Lemma 1.4 and Theorem].

Proposition B.9.1. *The free $\mathfrak{R}$-contramodule $\mathfrak{F} = \mathfrak{R}[[X]]$ generated by a set X is a flat R-module, and the tensor product $R/m \otimes_R \mathfrak{R}[[X]]$ is naturally isomorphic to the free R/m-module $R/m[X]$.*

Proof. For any $\mathfrak{R}$-contramodule $\mathfrak{P}$, one has $R/m \otimes_R \mathfrak{P} \simeq \mathfrak{R}/\mathfrak{m} \otimes_{\mathfrak{R}} \mathfrak{P}$, since $\mathfrak{m}$ is the extension of m in $\mathfrak{R}$, and $\mathfrak{R}/\mathfrak{m} \otimes_{\mathfrak{R}} \mathfrak{P} \simeq \mathfrak{P}/\mathfrak{m}\mathfrak{P}$, since any family of elements of $\mathfrak{m}$ converging to 0 in the topology of $\mathfrak{R}$ can be represented as a finite linear combination of families of elements of $\mathfrak{R}$ converging to 0 in the topology of $\mathfrak{R}$ with any given set of generators of the ideal $\mathfrak{m}$ as the coefficients. Clearly, $\mathfrak{R}[[X]]/\mathfrak{m}\mathfrak{R}[[X]] \simeq \mathfrak{R}/\mathfrak{m}[X] \simeq R/m[X]$, so the the second assertion of Proposition follows.

To prove the first assertion, consider the functor $M \longmapsto \mathfrak{R}[[X]] \otimes_R M$ acting from the category of finitely generated R-modules M to, e. g., the category of abelian groups. For any finitely generated R-module M, denote by $M_m\widehat{\ }$ the m-adic completion of M; so $M_m\widehat{\ }$ is an $\mathfrak{R}$-module endowed with a natural complete and separated topology. Let $M_m\widehat{\ }[[X]]$ denote the abelian group of all formal linear combinations of elements of X with coefficients from $M_m\widehat{\ }$ forming a family converging to 0. There is an natural map $\mathfrak{R}[[X]] \otimes_R M \longrightarrow M_m\widehat{\ }[[X]]$, which is clearly an isomorphism for finitely generated free (or projective) R-modules M.

The functor on the left hand side is obviously right exact; and one can see from the Artin–Rees lemma that the functor on the right hand side is exact. It follows that the morphism $\mathfrak{R}[[X]] \otimes_R M \longrightarrow M_m\widehat{\ }[[X]]$ is an isomorphism for any finitely generated R-module M, and the functor $M \longmapsto \mathfrak{R}[[X]] \otimes_R M$ is exact.

Alternatively, it suffices to notice that $\mathfrak{R}[[X]] \simeq \mathrm{Hom}_R(C_{\mathfrak{R}}, \bigoplus_X C_{\mathfrak{R}})$ and both $C_{\mathfrak{R}}$ and $\bigoplus_X C_{\mathfrak{R}}$ are injective R-modules [10]. Both assertions of Proposition can be obtained from this isomorphism. ☐

B.10. **All contramodules are relatively cotorsion modules.** The following proposition demonstrates the connection between our $\mathfrak{R}$-contramodules and Harrison's *co-torsion abelian groups* [16] or Enochs' *cotorsion R-modules* [10] (see [32, Section 1.4] for further details).

Proposition B.10.1. *For any flat R-module F such that $R/m \otimes_R F$ is a projective R/m-module, and any $\mathfrak{R}$-contramodule $\mathfrak{Q}$, the vanishing $\mathrm{Ext}^i_R(F, \mathfrak{Q}) = 0$ holds for all $i > 0$.*

Proof. Present an $\mathfrak{R}$-contramodule $\mathfrak{Q}$ as the quotient contramodule of a free $\mathfrak{R}$-contramodule $\mathfrak{Q}'$ by its $\mathfrak{R}$-subcontramodule $\mathfrak{Q}''$. Then one has $\mathfrak{Q}' \simeq \varprojlim_n \mathfrak{Q}'/\mathfrak{m}^n\mathfrak{Q}'$ and $\mathfrak{Q}'' \simeq \varprojlim_n \mathfrak{Q}''/\mathfrak{m}^n\mathfrak{Q}''$ [28, Lemma A.2.3 and Remark A.3]. This allows one to assume that $\mathfrak{Q} \simeq \varprojlim_n \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q}$ in the assertion of Proposition.

Since F is a flat R-module, there is a natural isomorphism $\mathrm{Ext}^*_R(F, \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q}) \simeq \mathrm{Ext}^*_{R/m^n}(F/m^nF, \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q})$. The following lemma is standard.

Lemma B.10.2. *Let A be an associative ring, $J \subset A$ be an ideal, and $n \geqslant 1$ be an integer such that $J^n = 0$. Suppose P is a flat left A-module such that P/JP is a projective left A/J-module. Then the A-module P is projective.*

Proof. The proof is similar to that of Lemma 2.1.2. One picks a free A-module G such that P/JP is the image of an idempotent endomorphism e of the A/J-module G/JG; notices that the associative ring homomorphism $\operatorname{Hom}_A(G,G) \longrightarrow \operatorname{Hom}_{A/J}(G/JG, G/JG)$ is surjective with the kernel $I = \operatorname{Hom}_A(G, JG) \subset \operatorname{Hom}_A(G,G)$ for which one has $I^n = 0$; lifts the idempotent element $e \in \operatorname{Hom}_{A/J}(G/JG,\ G/JG)$ to an idempotent element $f \in \operatorname{Hom}_A(G,G)$; considers the projective A-module fG; and lifts its natural morphism onto $fG/J(fG) \simeq e(G/JG) \simeq P/JP$ to an A-module morphism $fG \longrightarrow P$. For the cokernel C of the latter morphism, one has $C/JC = 0$, hence $C = 0$. Finally, one considers the exact sequence of A-modules $0 \longrightarrow K \longrightarrow fG \longrightarrow P \longrightarrow 0$, which remains exact after tensoring with A/J, since the A-module P is flat; hence $K/JK = 0$ and $K = 0$, so $P \simeq fG$. ☐

We have proven that F/m^nF is a projective R/m^n-module. It follows that $\operatorname{Ext}^i_R(F,\ \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q}) = 0$ for all $i > 0$ and all n. Besides, the natural map

$$\operatorname{Hom}_R(F,\ \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q}) \simeq \operatorname{Hom}_R(F/m^nF,\ \mathfrak{Q}/\mathfrak{m}^n\mathfrak{Q}) \\ \longrightarrow \operatorname{Hom}_R(F/m^nF,\ \mathfrak{Q}/\mathfrak{m}^{n-1}\mathfrak{Q}) \simeq \operatorname{Hom}_R(F,\ \mathfrak{Q}/\mathfrak{m}^{n-1}\mathfrak{Q})$$

is surjective. Now it remains to apply the following general result. ☐

Lemma B.10.3. *Let A be an associative ring, L be a left A-module, and M_α be a projective system of left A-modules. Assume that $\varprojlim^i_\alpha M_\alpha = 0$ and $\operatorname{Ext}^i_A(L, M_\alpha) = 0$ for all $i > 0$ and all α. Then there is a natural isomorphism $\operatorname{Ext}^*_A(L,\ \varprojlim_\alpha M_\alpha) \simeq \varprojlim^*_\alpha \operatorname{Hom}_A(L, M_\alpha)$. In particular, if M_α form a countable filtered projective system and the natural maps $\operatorname{Hom}_A(L, M_\alpha) \longrightarrow \operatorname{Hom}_A(L, M_\beta)$ are surjective for all $\alpha > \beta$, then $\operatorname{Ext}^i_A(L,\ \varprojlim_\alpha M_\alpha) = 0$ for all $i > 0$.*

Proof. Let $P_\bullet \longrightarrow L$ be a left projective resolution of the A-module L. Consider the complex of projective systems $\operatorname{Hom}_A(P_\bullet, M_\alpha)$. The condition $\operatorname{Ext}^i_A(L, M_\alpha) = 0$ for all $i > 0$ and all α implies that this complex is a right resolution of the projective system $\operatorname{Hom}_A(L, M_\alpha)$. The condition $\varprojlim^i_\alpha M_\alpha = 0$ for all $i > 0$ implies $\varprojlim^i_\alpha \operatorname{Hom}_A(P_j, M_\alpha) = 0$ for all $i > 0$ and all j. Now the complex $\varprojlim_\alpha \operatorname{Hom}_A(P_\bullet, M_\alpha) \simeq \operatorname{Hom}_A(P_\bullet,\ \varprojlim_\alpha M_\alpha)$ computes both $\operatorname{Ext}^*_A(L,\ \varprojlim_\alpha M_\alpha)$ and $\varprojlim^*_\alpha \operatorname{Hom}_A(L, M_\alpha)$. ☐

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Department of Mathematics and Laboratory of Algebraic Geometry, National Research University "Higher School of Economics", Moscow 117312, and Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127994, Russia

E-mail address: posic@mccme.ru