DEDUALIZING COMPLEXES AND MGM DUALITY

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ABSTRACT. We show that various derived categories of torsion modules and contramodules over the adic completion of a commutative ring by a weakly proregular ideal are full subcategories of the related derived categories of modules. By the work of Dwyer–Greenlees and Porta–Shaul–Yekutieli, this implies an equivalence between the (bounded or unbounded) conventional derived categories of the abelian categories of torsion modules and contramodules. Over the adic completion of a commutative ring by an arbitrary finitely generated ideal, we obtain an equivalence between the derived categories of complexes of modules with torsion and contramodule cohomology modules. We also define the notion of a *dedualizing complex* over a pair of noncocommutative cocoherent coalgebras or the adic completion of a commutative ring by a weakly proregular ideal and use these to construct equivalences between the conventional as well as certain exotic derived categories of the abelian categories of comodules/torsion modules and contramodules. The philosophy of derived co-contra correspondence is discussed in the introduction.

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0. INTRODUCTION

0.0. In its simplest and purest form, the *comodule-contramodule correspondence* is a natural equivalence between the additive categories of injective left comodules and projective left contramodules over the same coassociative coalgebra $\mathcal{C}$ over a field k. The equivalence of additive categories induces an equivalence between the homotopy categories of complexes of injective comodules and projective contramodules. One would like to view this equivalence as an equivalence between the derived categories

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of left comodules and left contramodules over $\mathcal{C}$, using complexes of injective co-modules and projective contramodules as resolutions. However, the former are right resolutions and the latter are left ones, so one has to work with doubly unbounded complexes. The categories of unbounded complexes of injective or projective objects are well-known to differ from the unbounded derived categories as the latter are conventionally defined. The reason is that an unbounded complex of injectives or projectives may be acyclic, yet not contractible.

Hence the relevance of the concepts of the *derived categories of the second kind*, or the *coderived* and *contraderived* categories, in the comodule-contramodule correspondence constructions. In well-behaved situations, the coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{A})$ of an abelian or exact category A is equivalent to the homotopy category $\mathsf{Hot}(\mathsf{A}_{\mathsf{inj}})$ of complexes of injective objects in A, while the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B})$ of a category B is equivalent to the homotopy category $\mathsf{Hot}(\mathsf{B}_{\mathsf{proj}})$ of complexes of projective objects in B. Thus the *derived co-contra correspondence* over a coalgebra $\mathcal{C}$ takes the form of an equivalence between the coderived category of the abelian category of left $\mathcal{C}$-comodules $\mathcal{C}$–comod and the contraderived category of the abelian category of left $\mathcal{C}$-contramodules $\mathcal{C}$–contra,

$$\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–contra}).$$

Certain acyclic complexes of comodules and contramodules survive in the coderived and contraderived categories, and an acyclic complex is sometimes assigned to an irreducible object by the derived co-contra correspondence [21, Sections 0.2.2, 0.2.5, and 0.2.6–7] (see also [22, Sections 4.4 and 5.2]).

0.1.   In more complicated relative situations mixing algebra and coalgebra features, derived co-contra correspondence theories can be often developed using the following guiding principles [25]:

   (i) depending on whether one's abelian/exact category is a category of comodule-like or contramodule-like objects, one takes the coderived or the contraderived category along the coalgebra variables;
   (ii) one takes the conventional unbounded derived category along the ring or algebra variables;
   (iii) over a ring or coalgebra of finite homological dimension, there is no difference between the derived, coderived, and contraderived categories, so one does not have to pay attention to the distinction.

The rules (i-iii) are sufficient to build most of the derived the comodule-contra-module and semimodule-semicontramodule correspondence theorems of the book [21] (see [21, Sections 0.3.7, 5.4–5.5, and 6.3]). However, there are several simple and important situations which they do not cover.

0.2.   In the papers [10, 12], the unbounded homotopy categories of complexes of projective and injective modules over certain coherent or Noetherian rings were studied; and in [9], an equivalence between the homotopy categories of complexes of projective and injective modules over a *ring* (or two rings, in the noncommutative case) *with*

a dualizing complex was obtained. In the paper [17] and the dissertation [15], these results were extended to complexes of flat and injective quasi-coherent sheaves on a Noetherian scheme; and in the preprint [6, Section 2.5], to matrix factorizations. Finally, in [24, Section 5.7] this duality is formulated as a commutative diagram of equivalences between *four* exotic derived categories of quasi-coherent sheaves and contraherent cosheaves on a semi-separated Noetherian scheme.

The related piece of philosophy appears to look as follows. A coalgebra has been "dualized" already, being a dual thing to an algebra or a ring; so a coalgebra $\mathcal{C}$ is a *dualizing complex over itself.* The conventional derived category of modules over a ring is tautologically equivalent to itself, but constructing an equivalence between the coderived and contraderived categories of modules over a (say, commutative Noetherian) ring requires a dualizing complex. Having a dualizing complex over a ring makes it "more like a coalgebra". Therefore,

(iv) given a dualizing complex for a set of ring or algebra variables, one can use the coderived and the contraderived category along these variables on the two sides of the derived co-contra correspondence.

0.3.   The opposite situation, when one considers the conventional unbounded derived categories of comodules and contramodules, is not as well-studied. The relation between the conventional derived categories of DG-comodules and DG-contramodules over an arbitrary DG-coalgebra over a field is discussed in (particularly, the postpublication `arXiv` version of) [22, Section 5.5], but it does not have the familiar form of an equivalence of triangulated categories.

In [24, Section 4.6], we construct an equivalence between the conventional (bounded or unbounded; and also absolute, etc.) derived categories of the abelian category of quasi-coherent sheaves and the exact category of contraherent cosheaves on a quasi-compact semi-separated scheme X,

$$\mathsf{D}^{\star}(X\text{--qcoh}) \simeq \mathsf{D}^{\star}(X\text{--ctrh}).$$

An explanation is that, generally, a scheme "mixes the ring and coalgebra variables" in such a way that the ring(s) are responsible for the pieces being glued, while the coalgebra (or, rather, coring) governs the gluing procedure [11]. In a quasi-compact quasi-separated scheme, the gluing procedure has finite homological dimension (the quasi-coherent sheaf cohomology being a functor of finite homological dimension); so the rule (iii) "along the gluing variables" applies. Still, one additionally has to explain how to pass from the sheaves to the cosheaves and back. Over a scheme, one takes the contraherent $\mathfrak{Hom}_X$ from or the contratensor product $\odot_X$ with the structure sheaf $\mathcal{O}_X$ of the scheme X [24, Sections 2.5–6].

0.4.   On the other hand, there is a stream of literature discussing torsion, completion, and duality theories for modules or sheaves over formal schemes. From the point of view elaborated in the above discussion, there seem to be *two* such duality theories that need to be property distinguished from each other.

One of them is concerned with dualizing complexes on formal schemes [31, Section 5] and, being formulated in the generality of unbounded complexes of infinitely generated modules, leads to a covariant equivalence between derived categories of the second kind, extending the results of [9, 15, 6] and [24, Section 5.7] to the realm of formal schemes. For affine Noetherian formal schemes, this "covariant Serre–Grothendieck duality" is formulated in [24, Section C.1] (see [24, Section C.5] for a noncommutative version); and for ind-affine ind-Noetherian ind-schemes (the ind-spectra of pro-Noetherian topological rings), in [24, Section D.2].

To demonstrate a precise assertion here, given a dualizing complex $\mathcal{D}^\bullet$ for a projective system $R_0 \longleftarrow R_1 \longleftarrow R_2 \longleftarrow \cdots$ of Noetherian commutative rings and surjective morphisms between them with the projective limit $\mathfrak{R} = \varprojlim_n R_n$, [24, Theorem D.2.5] claims that there is a natural equivalence between the coderived category of discrete modules and the contraderived category of contramodules over the topological ring $\mathfrak{R}$,

$$\mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–discr}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathfrak{R}\text{–contra}).$$

The other duality theory over adically complete rings and formal schemes is known as the *Matlis–Greenlees–May duality* [14, 5, 18]. The aim of the present paper is to formulate it explicitly as an equivalence between the conventional derived categories of the abelian categories of torsion modules and contramodules over the adic completions of certain commutative rings. Thus the MGM duality is viewed as a species of the "naïve derived co-contra correspondence" of [24, Section 4.6].

0.5. Let us devote a few paragraphs to a more substantive discussion of the issues involved. Let X be a scheme and $Z \subset X$ be a closed subscheme. Denote by $U = X \setminus Z$ the open complement to Z in X. Then the formal completion of X along Z can be viewed heuristically as the complement to U in X,

$$X_Z^\wedge = X \setminus U.$$

In particular, the quasi-coherent torsion sheaves on $X_Z^\wedge$ are, almost by the definition, those quasi-coherent sheaves on X whose restrictions vanish on U.

Passing to the triangulated categories and denoting by $k \colon U \longrightarrow X$ the open embedding morphism, we notice that under weak assumptions on k the inverse image functor $k^* \colon \mathsf{D}(X\text{–qcoh}) \longrightarrow \mathsf{D}(U\text{–qcoh})$ between the derived categories of quasi-coherent sheaves on X and U has a right adjoint functor of derived direct image $\mathbb{R}k_* \colon \mathsf{D}(U\text{–qcoh}) \longrightarrow \mathsf{D}(X\text{–qcoh})$. The composition $k^* \circ \mathbb{R}k_*$ is isomorphic to the identity functor on $\mathsf{D}(U\text{–qcoh})$, so k^* is a Verdier quotient functor and the functor $\mathbb{R}k_*$ is fully faithful. To be more precise, one would probably want to have the scheme U quasi-compact and the scheme X semi-separated, or otherwise the scheme X locally Noetherian, for the functor $\mathbb{R}k_*$ to be well-behaved; and under slightly stronger assumptions one can prove existence of a Neeman extraordinary inverse image functor $k^! \colon \mathsf{D}(X\text{–qcoh}) \longrightarrow \mathsf{D}(U\text{–qcoh})$ right adjoint to $\mathbb{R}k_*$ [16].

It follows immediately that the kernels of the functors k^* and $k^!$ are two equivalent subcategories in $\mathsf{D}(X\text{–qcoh})$ (a purely algebraic description of this picture can

be found in [18, Section 8]). Furthermore, the functor of inverse image of contraherent cosheaves $k^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(U\text{–}\mathsf{ctrh})$ has a left adjoint functor of derived direct image $\mathbb{L}k_!\colon \mathsf{D}(U\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$. The equivalences of derived categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathsf{D}(U\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(U\text{–}\mathsf{ctrh})$ from Section 0.3 transform the functor $\mathbb{R}k_*$ into the functor $\mathbb{L}k_!$, identifying the two functors $k^!$ and providing a rather explicit construction of Neeman's inverse image [24, Section 4.8].

This is not yet a promised equivalence between the derived categories of a pair of abelian or exact categories, however. E. g., one still has to prove that the kernel of the functor k^* is equivalent to the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh}_{Z\text{-tors}})$ of quasi-coherent torsion sheaves on $\widehat{X_Z}$, or that the triangulated functor $\mathsf{D}(X\text{–}\mathsf{qcoh}_{Z\text{-tors}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ induced by the embedding of abelian categories $X\text{–}\mathsf{qcoh}_{Z\text{-tors}} \longrightarrow X\text{–}\mathsf{qcoh}$ is fully faithful. For bounded derived categories D^b or D^+ in place of D and a Noetherian scheme X this is not difficult, as one can use the Artin–Rees lemma to show that injectives in $X\text{–}\mathsf{qcoh}_{Z\text{-tors}}$ are also injective in $X\text{–}\mathsf{qcoh}$; for the unbounded derived categories, the question appears to be more involved. One also has to identify the kernel of the functor $k^!$ with the derived category of the abelian or exact category of contramodules or contraherent cosheaves of contramodules on $\widehat{X_Z}$.

0.6.   Let us now continue the discussion from a different angle. When the scheme X is quasi-compact and quasi-separated, and its open subscheme $U \subset X$ is quasi-compact, the above heuristics suggest that the formal scheme $\widehat{X_Z}$, being the result of "subtracting" U from X, has the homological dimension of its "coalgebra variables" bounded, approximately, by the sum of the cardinality parameters of the open coverings of X and U, and consequently finite. To use another visual metaphor, one can say that the formal completion of X along Z is obtained by cutting from X a small tubular neighborhood around Z. The observation is that the scissors used to perform the cut tend to have finite homological dimension.

Once again, the Artin–Rees lemma, claiming that the completion functor is exact, would seem to provide an even stronger assertion for finitely generated modules, but we are interested in infinitely generated ones (even over a Noetherian ring). The precise conditions needed to prove such a result are a more technical issue. In the case of an affine scheme $X = \operatorname{Spec} R$ our heuristics seem to suggest that the open subscheme $U = X \setminus Z$ should be quasi-compact, i. e., the ideal $I \subset R$ defining the closed subscheme $Z \subset X$ should be finitely generated. In assumptions, admittedly, stronger than this, still weaker than the Noetherianity of R, a pair of such homological finite-dimensionality assertions on the comodule and contramodule sides is indeed proven in [18, Corollaries 4.28 and 5.27].

0.7.   No homological dimension condition, though, can automatically solve the problem of constructing an equivalence between the derived categories of two different abelian categories, even if these are the categories of comodules and contramodules over the same coalgebra-like algebraic structure. The piece of data that is missing here is what we call a *dedualizing complex*.

The coderived category of left comodules and the contraderived category of left contramodules over a coalgebra over a field are always equivalent to each other; but constructing an equivalence between the conventional derived categories of comodules and contramodules over an (co-Noetherian or cocoherent) coalgebra requires a dedualizing complex. A ring is a dedualizing complex over itself; having a dedualizing complex for a coalgebra makes it more like a ring. In other words,

(v) given a dedualizing complex for a set of coalgebra variables, one can have a "naive" derived co-contra correspondence with the conventional derived categories of comodules and contramodules along these variables on the two sides of a triangulated equivalence.

The definition of a dedualizing complex $\mathcal{B}^\bullet$ is, approximately, dual to that of a dualizing one. It has to be a finite complex of comodules or bicomodules. There being, generally speaking, no projective objects in comodule categories, one cannot ask $\mathcal{B}^\bullet$ to be a finite complex *of projective objects*, but it has to have *finite projective dimension* as a complex in the bounded derived category. It has to satisfy a unitality condition imposed on the graded ring of its endomorphisms in the derived category. Finally, there should be a finiteness condition similar or dual to the coherence condition on the cohomology sheaves of a dualizing complex on a Noetherian scheme. The latter one seems to be the hardest to formulate, and the approaches may vary.

Let us empasize that the dualizing and the dedualizing complexes play rather different roles in the respective theories. The purpose of the dualizing complex is to mitigate the infinite homological dimension problem, bridging the gap between the coderived and the contraderived category. The dedualizing complex is there to bridge the gap between the abelian categories of comodules and contramodules. In both cases, the problem to be solved by the choice of a specific complex appears due to a mismatch between the kind of abelian categories being considered and the kind of derived category constructions being applied. Between the coderived category of comodules and the contraderived category of contramodules (say, over a coalgebra over a field), there is no gap to be bridged.

0.8. We have yet to explain how a dedualizing complex on a formal scheme is to be obtained. For dualizing complexes on algebraic varieties X over a field k, the classical prescription is to consider the structure morphism $p\colon X \longrightarrow \operatorname{Spec} k$ and set $\mathcal{D}_X^\bullet = p^+ \mathcal{O}_{\operatorname{Spec} k}$, where p^+ denotes the Deligne extraordinary inverse image functor (i. e., f^+ is equal to $f^!$ for proper morphisms and to f^* for open embeddings).

A similar rule works for formal completions of algebraic varieties: given a scheme X with a dualizing complex $\mathcal{D}_X^\bullet$ and a closed subscheme $Z \subset X$, a dualizing complex on the formal scheme $\widehat{X_Z}$ can be constructed as the derived subcomplex with set-theoretic supports $\mathcal{D}_{\widehat{X_Z}}^\bullet = \mathbb{R}i^! \mathcal{D}_X^\bullet$, where i denotes the embedding $\widehat{X_Z} \longrightarrow X$. As to the *dedualizing* complex on $\widehat{X_Z}$, it is produced by applying the functor $\mathbb{R}i^!$ to the structure sheaf of the scheme X,

$$\mathcal{B}_{\widehat{X_Z}}^\bullet = \mathbb{R}i^! \mathcal{O}_X, \qquad i\colon \widehat{X_Z} \longrightarrow X.$$

Note that the formal completion $X_{\widehat{Z}} \subset X$ of a closed subscheme Z in a scheme X can be considered both as a kind of "closed subscheme" or "open subscheme" in X, depending on a point of view. The above construction of the dedualizing complex can be explained by saying that one would "prefer" to use the $*$-restriction of quasi-coherent sheaves in order to obtain a dedualizing complex on $X_{\widehat{Z}}$ from a dedualizing complex on X; but given that a dedualizing complex on a formal scheme must be a complex of quasi-coherent torsion sheaves, one is "forced" to use the !-restriction, which is a "satisfactory substitution" inasmuch as the natural morphism $X_{\widehat{Z}} \longrightarrow X$ can be viewed as a species of open embedding.

The complex $\mathcal{B}^{\bullet}_{X_{\widehat{Z}}}$ for the formal completion of an affine scheme $X = \operatorname{Spec} R$ along its closed subscheme $Z \subset X$ defined by a finitely generated ideal $I \subset R$ plays a crucial role in the MGM duality papers [5, 18]. Traditionally, it is viewed as an object of the derived category of R-modules represented by an explicit Koszul/telescope complex of infinitely generated free R-modules $\operatorname{Tel}^{\bullet}(R, \mathbf{s})$ concentrated in the cohomological degrees from 0 to m, where $\mathbf{s} = \{s_1, \ldots, s_m\}$ is a chosen set of generators of the ideal I, and having I-torsion cohomology modules. With the perspective of extending the MGM duality to nonaffine schemes in mind, certainly it is preferable to work with torsion modules and contramodules over the I-adic completion of R only, avoiding the use of any other R-modules such as the infinitely generated free ones. Dependence on a fixed set of generators $\mathbf{s}$ of the ideal I, which may not exist globally along a closed subscheme Z in a nonaffine scheme X, is also undesirable.

0.9.   Let us now describe the contents of this paper and our main results in some detail. For a commutative ring R with a *weaky proregular* finitely generated ideal $I \subset R$, Porta, Shaul, and Yekutieli construct an equivalence between (what they call) the full subcategories of *cohomologically I-torsion complexes* $\mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-tors}} \subset \mathsf{D}(R\text{–}\mathsf{mod})$ and *cohomologically I-adically complete complexes* $\mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-com}} \subset \mathsf{D}(R\text{–}\mathsf{mod})$ in the unbounded derived category of R-modules $\mathsf{D}(R\text{–}\mathsf{mod})$,

$$(1) \qquad\qquad \mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-tors}} \simeq \mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-com}}.$$

[18, Theorems 1.1 and 7.11]. Moreover, they consider the full abelian subcategory of I-torsion R-modules $R\text{–}\mathsf{mod}_{I\text{-tors}} \subset R\text{–}\mathsf{mod}$ in the abelian category of R-modules, and show that the full subcategory of cohomologically I-torsion complexes coincides with the full subcategory of complexes with I-torsion cohomology modules in $\mathsf{D}(R\text{–}\mathsf{mod})$,

$$\mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-tors}} = \mathsf{D}_{I\text{-tors}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}(R\text{–}\mathsf{mod})$$

[18, Corollary 4.32]; a similar result was earlier obtained in [5, Proposition 6.12].

Our first result in Section 1 is that the derived category of the abelian category of I-torsion R-modules is a full subcategory in $\mathsf{D}(R\text{–}\mathsf{mod})$ coinciding with the full subcategory of complexes with I-torsion cohomology modules,

$$\mathsf{D}_{I\text{-tors}}(R\text{–}\mathsf{mod}) = \mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-tors}}) \subset \mathsf{D}(R\text{–}\mathsf{mod}).$$

Furthermore, in Section 2 we define the full abelian subcategory of *I-contramodules* (or *I-contramodule R-modules*) $R\text{–}\mathsf{mod}_{I\text{-ctra}} \subset R\text{–}\mathsf{mod}$ and show that the derived

category of I-contramodule R-modules is a full subcategory of the derived category of R-modules,

$$\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-ctra}}) \subset \mathsf{D}(R\text{–}\mathsf{mod}).$$

Moreover, the full subcategory of cohomologically I-adically complete complexes coincides with the full subcategory of complexes with I-contramodule cohomology modules in $\mathsf{D}(R\text{–}\mathsf{mod})$ and with (the image of) the derived category of the abelian category of I-contramodule R-modules,

$$\mathsf{D}(R\text{–}\mathsf{mod})_{I\text{-com}} = \mathsf{D}_{I\text{-ctra}}(R\text{–}\mathsf{mod}) = \mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-ctra}}).$$

(concerning the first equality, cf. [5, Proposition 6.15]).

Combining these results, we obtain a natural equivalence between the (bounded or unbounded) conventional derived categories of the abelian categories of I-torsion and I-contramodule R-modules,

$$(2) \qquad\qquad \mathsf{D}^{\star}(R\text{–}\mathsf{mod}_{I\text{-tors}}) \simeq \mathsf{D}^{\star}(R\text{–}\mathsf{mod}_{I\text{-ctra}}).$$

Let us point out that the equivalence of triangulated subcategories (1) in the derived category $\mathsf{D}(R\text{–}\mathsf{mod})$ can be actually established for any finitely generated ideal I in a commutative ring R, as it follows from the above discussion in Section 0.5. This was shown already by Dwyer and Greenlees in [5, Theorem 2.1, Section 4.1, and Section 6], and we explain this anew in our Section 3. In fact, we prove that for any finitely generated ideal $I \subset R$ the full subcategories of complexes with torsion and contramodule cohomology modules are naturally equivalent,

$$(3) \qquad\qquad \mathsf{D}^{\star}_{I\text{-tors}}(R\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\star}_{I\text{-ctra}}(R\text{–}\mathsf{mod}).$$

However, the weak proregularity condition appears to be essential for constructing an equivalence between the derived categories of the abelian categories $R\text{–}\mathsf{mod}_{I\text{-tors}}$ and $R\text{–}\mathsf{mod}_{I\text{-ctra}}$ in the formula (2).

0.10. Notice that the formal completions of algebraic varieties over a field k at their closed points defined over k are precisely the ind-spectra of finitely cogenerated conilpotent cocommutative coalgebras over k. We give the MGM duality for (not necessarily cocommutative, but coassociative) coalgebras a separate treatment in Section 4. An explicit definition of a dedualizing complex as a complex of bicomodules satisfying a list of conditions first appears in this section. Given a dedualizing complex $\mathcal{B}^{\bullet}$ for a pair of cocoherent coalgebras $\mathcal{C}$ and $\mathcal{D}$, an equivalence between bounded or unbounded, conventional or absolute derived categories of left comodules over $\mathcal{C}$ and left contramodules over $\mathcal{D}$ is obtained,

$$\mathsf{D}^{\star}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\star}(\mathcal{D}\text{–}\mathsf{contra}).$$

In the final Section 5, we apply the same approach to the adic completions of commutative rings by weakly proregular ideals. Even though our definition of a dedualizing complex in this setting is not as neat as in the case of two coalgebras, it allows to obtain a second proof of the equivalence of derived categories (2). Moreover, while the proof based on the arguments of Section 3 applies to the conventional

derived categories only, the approach in Section 5 produces an equivalence of the absolute derived categories as well.

A brief summary of the definitions of exotic derived categories used in the main body of the paper is included in Appendix A. A formalism of derived functors needed for the constructions of the last two sections is developed in Appendix B. Finiteness conditions on comodules and contramodules over coalgebras over fields relevant for the purposes of Section 4 are considered in Appendix C.

For a general discussion of the abelian categories of contramodules, we refer to the survey paper [26]. In the role of an introduction to the coderived, contraderived and absolute derived categories the reader can use [21, Sections 0.2 and 2.1] or, better yet, [22, Sections 3–5]; for more advanced results, one can look into [6, Sections 1.3–1.6] or [24, Appendix A]. An overview exposition on the derived comodule-contramodule correspondence phenomenon can be found in the presentation [25].

1. Derived Category of Torsion Modules

Let R be a commutative ring and $I \subset R$ be a finitely generated ideal. An R-module M is said to be *I-torsion* if for every element $x \in M$ there exists an integer $n \geqslant 1$ such that $I^n x = 0$ in M. The full subcategory of I-torsion submodules in R–mod is denoted by R–mod$_{I\text{-tors}}$. Clearly, the subcategory R–mod$_{I\text{-tors}}$ is closed under the operations of passage to submodules, quotient modules, extensions, and infinite direct sums in R–mod. Therefore, R–mod$_{I\text{-tors}}$ is an abelian category with exact functors of infinite direct sum and its embedding R–mod$_{I\text{-tors}} \longrightarrow R$–mod is an exact functor preserving the infinite direct sums.

Following [18], we denote by $\Gamma_I(M)$ the maximal I-torsion submodule in an arbitrary R-module M. The functor $\Gamma_I \colon R$–mod $\longrightarrow R$–mod$_{I\text{-tors}}$ is left exact; it is the right adjoint functor to the embedding functor R–mod$_{I\text{-tors}} \longrightarrow R$–mod.

Let $s_1, \ldots, s_m \in R$ be a finite sequence of elements in R generating the ideal I. For simplicity of notation, we will denote the sequence $s_1, \ldots, s_j$ by a single letter $\mathbf{s}$. For any R-module M, consider the following Čech complex $C_{\mathbf{s}}^\bullet(M)$

$$\bigoplus_{j=1}^m M[s_j^{-1}] \longrightarrow \bigoplus_{j'<j''} M[s_{j'}^{-1}, s_{j''}^{-1}] \longrightarrow \cdots \longrightarrow M[s_1^{-1}, \ldots, s_m^{-1}].$$

We place the first term $\bigoplus_j M[s_j^{-1}]$ of the complex $C_{\mathbf{s}}^\bullet(M)$ in the cohomological degree 0, so that there is a natural morphism of complexes (coaugmentation or unit) $k \colon M \longrightarrow C_{\mathbf{s}}^\bullet(M)$, and denote by $C_{\mathbf{s}}^\bullet(M)^\sim$ the cocone of this morphism,

$$M \longrightarrow \bigoplus_{j=1}^m M[s_j^{-1}] \longrightarrow \bigoplus_{j'<j''} M[s_{j'}^{-1}, s_{j''}^{-1}] \longrightarrow \cdots \longrightarrow M[s_1^{-1}, \ldots, s_m^{-1}].$$

So the complex $C_\mathbf{s}^\bullet(M)^\sim$ also has its first term M at the cohomological degree 0. Obviously, there are natural isomorpisms of complexes $C_\mathbf{s}^\bullet(M) \simeq C_\mathbf{s}^\bullet(R) \otimes_R M$ and $C_\mathbf{s}^\bullet(M)^\sim \simeq C_\mathbf{s}^\bullet(R)^\sim \otimes_R M$. Furthermore, the complex $C_\mathbf{s}^\bullet(R)^\sim$ is isomorphic to the tensor product of the similar complexes corresponding to one-element sequences $\{s_j\}$ over all the elements of the sequence $s_1, \ldots, s_m$

$$C_\mathbf{s}^\bullet(R)^\sim \simeq C_{\{s_1\}}^\bullet(R)^\sim \otimes_R \cdots \otimes_R C_{\{s_m\}}^\bullet(R)^\sim.$$

Lemma 1.1. (a) *For any R-module M, all the cohomology modules $H^* C_\mathbf{s}^\bullet(M)^\sim$ of the complex $C_\mathbf{s}^\bullet(M)^\sim$ are I-torsion R-modules.*

(b) *For any R-module M, the natural morphism of complexes $C_\mathbf{s}^\bullet(M)^\sim \longrightarrow M$ induces an isomorphism of R-modules*

$$H^0 C_\mathbf{s}^\bullet(M)^\sim \simeq \Gamma_I(M).$$

(c) *For any I-torsion R-module M, the morphism of complexes $C_\mathbf{s}^\bullet(M)^\sim \longrightarrow M$ is an isomorphism.*

Proof. To prove part (a), notice that the complex $C_\mathbf{s}^\bullet(M)^\sim[s_j^{-1}]$ is contractible for every j, because $C_{\{s_j\}}^\bullet(R)^\sim[s_j^{-1}]$ is a contractible two-term complex $(R[s_j^{-1}] \to R[s_j^{-1}])$. To obtain part (b), recall that for any given element $s \in R$ the kernel of the map $M \longrightarrow M[s^{-1}]$ consists precisely of all the elements $x \in M$ for which there exists an integer $n \geqslant 1$ such that $s^n x = 0$. Part (c) is obvious, as the complex $C_\mathbf{s}^\bullet(M)$ vanishes entirely for any I-torsion R-module M. $\qquad\square$

Denote by X the affine scheme $\operatorname{Spec} R$, by $Z = \operatorname{Spec} R/I \subset X$ the corresponding closed subscheme in X, by $U = X \setminus Z$ its open complement, and by U_j the principal affine open subschemes $\operatorname{Spec} R[s_j^{-1}] \subset \operatorname{Spec} R$. The open subschemes $U_j \subset X$ form an affine open covering of the open subscheme $U \subset X$. Let $\mathcal{M}$ denote the quasi-coherent sheaf on X corresponding to the R-module M; then the Čech complex $C_\mathbf{s}^\bullet(M)$ computes the sheaf cohomology $H^*(U, \mathcal{M}|_U)$.

In particular, when R is a Noetherian ring and J is an injective R-module, the related quasi-coherent sheaf $\mathcal{J}$ on X is flasque and $\mathcal{J}|_U$ is an injective quasi-coherent sheaf on U [8, Lemma II.7.16 and Theorem II.7.18]. Hence it follows that the complex $C_\mathbf{s}^\bullet(J)$ is quasi-isomorphic to its zero cohomology module $H^0 C_\mathbf{s}^\bullet(J) \simeq \mathcal{J}(U)$ and the complex $C_\mathbf{s}^\bullet(J)^\sim$ is quasi-isomorphic to the two-term complex $\mathcal{J}(X) \longrightarrow \mathcal{J}(U)$, which also has its only cohomology group in degree 0. We have shown that $H^i C_\mathbf{s}^\bullet(J)^\sim = 0$ for $n > 0$. A direct algebraic proof of the this assertion (based on the same Matlis' classification of injective modules over Noetherian rings [13] as Hartshorne's argument in [8]) can be found in [18, Theorem 4.34].

According to one of the equivalent definitions, a finite sequence of elements $\mathbf{s}$ in a commutative ring R is said to be *weakly proregular* if the cohomology groups of the complex $C_\mathbf{s}^\bullet(J)^\sim$ vanish in the positive cohomological degrees for any injective R-module J (see [29, Theorem 3.2(ii)] or [18, Theorem 4.24]). According to [18, Corollary 6.2] (see also [29, Corollary 3.3]), the weak proregularity property of a finite sequence $\mathbf{s}$ in a commutative ring R only depends on the ideal I that this sequence generates, and in fact, even only on the ideal $\sqrt{I} \subset R$. We have seen that

any ideal I in a Noetherian ring R is weakly proregular. Another equivalent definition of weak proregularity will appear in Section 2.

Lemma 1.2. (a) *For any weakly proregular finite sequence of elements* $\mathbf{s}$ *in a commutative ring* R, *the complex* $C_{\mathbf{s}}^{\bullet}(M)^{\sim}$ *assigned to an* R-*module* M *computes the right derived functor* $\mathbb{R}^{*}\Gamma_{I}(M)$ *of the left exact functor* Γ_{I}, *viewed as taking values in the category* R–mod.

(b) *For any weakly proregular finitely generated ideal* I *in a commutative ring* R, *(the right derived functor* $\mathbb{R}^{*}\Gamma_{I}$ *of) the left exact functor* $\Gamma_{I}\colon R$–$\mathsf{mod} \longrightarrow R$–$\mathsf{mod}_{I\text{-tors}}$ *has finite homological dimension not exceeding the minimal number of generators of the ideal* I.

Proof. This is [29, Theorem 3.2(iii)] or [18, Corollaries 4.26 and 4.28]. Part (a): given an R-module M, consider its right injective resolution $J^{\bullet}$. Then the total complex of the bicomplex $C_{\mathbf{s}}^{\bullet}(J^{\bullet})^{\sim}$ is quasi-isomorphic both to the complex $\Gamma_{I}(J^{\bullet})$, since the morphism $\Gamma_{I}(J^{n}) \longrightarrow C_{\mathbf{s}}^{\bullet}(J^{n})^{\sim}$ is a quasi-isomorphism for every n, and to the complex $C_{\mathbf{s}}^{\bullet}(M)^{\sim}$, because the functor $N \longmapsto C_{\mathbf{s}}^{\bullet}(N)^{\sim}$ takes exact sequences of R-modules N to exact sequences of complexes $C_{\mathbf{s}}^{\bullet}(-)^{\sim}$. As the embedding functor R–$\mathsf{mod}_{I\text{-tors}} \longrightarrow R$–$\mathsf{mod}$ is exact and takes nonzero objects to nonzero objects, part (b) immediately follows. $\qquad\square$

For the most part of Sections 1–2, we will consider simultaneously and almost on equal footing a number of conventional and exotic derived category constructions introduced in [21, 22], and [24, Appendix A]. The definitions of these derived categories are briefly recalled in Appendix A to this paper. A reader unfamiliar with derived categories of the second kind will not loose much restricting his attention to the conventional derived categories D^{b}, D^{+}, D^{-}, and D, and forgetting about the rest. In Section 3, we will only work with the four conventional derived categories; and then return to a list of symbols including both the conventional and the absolute derived categories in the final Sections 4–5 and Appendix B.

Theorem 1.3. *Let* R *be a commutative ring and* $I \subset R$ *be a weakly proregular finitely generated ideal. Then for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, *or* abs, *the triangulated functor* $\mathsf{D}^{\star}(R$–$\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^{\star}(R$–$\mathsf{mod})$ *induced by the embedding of abelian categories* R–$\mathsf{mod}_{I\text{-tors}} \longrightarrow R$–$\mathsf{mod}$ *is fully faithful.*

Proof. In the case of the derived categories D^{b} or D^{+} over a Noetherian ring R, it suffices to notice that, according to the Artin–Rees lemma, the functor R–$\mathsf{mod}_{I\text{-tors}} \longrightarrow R$–$\mathsf{mod}$ preserves injectivity of objects (cf. [26, Proposition 2.2.2(a)]). For the other derived categories in our list, Lemma 1.2(b) essentially says that the question "has finite homological dimension" and therefore "reduces to finite complexes". A formal proof in the general case proceeds as follows.

Denote by R–$\mathsf{mod}_{I\text{-tors-adj}}$ the full subcategory of objects adjusted to Γ_{I} in R–mod; in other words, the subcategory R–$\mathsf{mod}_{I\text{-tors-adj}} \subset R$–$\mathsf{mod}$ consists of all the R-modules M for which $\mathbb{R}^{n}\Gamma_{I}(M) = 0$ for $n > 0$, or equivalently, $H^{n}C_{\mathbf{s}}^{\bullet}(M)^{\sim} = 0$ for $n > 0$. The full subcategory R–$\mathsf{mod}_{I\text{-tors-adj}}$ is closed under extensions, cokernels of injective

morphisms, and infinite direct sums in R–mod. Hence the subcategory R–$\mathsf{mod}_{I\text{-tors-adj}}$ inherits an exact category structure of the abelian category R–mod.

Moreover, by Lemma 1.2(b) any R-module has a finite right resolution of uniformly bounded length by objects of the subcategory R–$\mathsf{mod}_{I\text{-tors-adj}}$. It follows that for any symbol $\star$ on our list the triangulated functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors-adj}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ induced by the exact embedding R–$\mathsf{mod}_{I\text{-tors-adj}} \longrightarrow R$–mod is an equivalence of triangulated categories [24, Proposition A.5.6],

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors-adj}}) \simeq \mathsf{D}^\star(R\text{–}\mathsf{mod}).$$

Obviously, the restriction of functor Γ_I to the full exact subcategory R–$\mathsf{mod}_{I\text{-tors-adj}}$ $\subset$ R–mod is an exact functor $\Gamma_I \colon R$–$\mathsf{mod}_{I\text{-tors-adj}} \longrightarrow R$–$\mathsf{mod}_{I\text{-tors}}$. Applying the functor Γ_I to complexes of modules from the category R–$\mathsf{mod}_{I\text{-tors-adj}}$ termwise, we obtain the right derived functor

$$\mathbb{R}\Gamma_I \colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}).$$

As the passage to derived functors in the sense of Deligne [4, n$^{\mathrm{os}}$ 1.2.1–2] generally preserves adjoint pairs of functors, the derived functor $\mathbb{R}\Gamma_I$ is right adjoint to the triangulated functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ induced by the embedding of abelian categories R–$\mathsf{mod}_{I\text{-tors}} \longrightarrow R$–mod (see also [21, Lemma 8.3]).

Furthermore, by Lemma 1.1(c) the abelian subcategory R–$\mathsf{mod}_{I\text{-tors}} \subset R$–mod is contained in the exact subcategory R–$\mathsf{mod}_{I\text{-tors-adj}}$,

$$R\text{–}\mathsf{mod}_{I\text{-tors}} \subset R\text{–}\mathsf{mod}_{I\text{-tors-adj}}.$$

Therefore, the composition of adjoint functors

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$$

is isomorphic to the identity functor on $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$. It follows immediately that the functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ is fully faithful, while the functor $\mathbb{R}\Gamma_I \colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$ is a Verdier quotient functor. $\qquad\square$

Corollary 1.4. *Let R be a commutative ring and $I \subset R$ be a weakly proregular finitely generated ideal. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, the triangulated functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^*(R\text{–}\mathsf{mod})$ identifies the derived category $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$ with the full subcategory $\mathsf{D}^\star_{I\text{-tors}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$ consisting of all the complexes with the cohomology modules belonging to R–$\mathsf{mod}_{I\text{-tors}}$.*

Proof. This is [18, Corollary 4.32]; see also [5, Proposition 6.12]. It is obvious that the image of the functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ is contained in $\mathsf{D}^\star_{I\text{-tors}}(R\text{–}\mathsf{mod})$. To prove the converse inclusion, one notices that, according to the proof of Theorem 1.3, the image of the fully faithful functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ consists precisely of all the complexes fixed by the composition of adjoint functors $\mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$. Since the functor $\mathbb{R}\Gamma_I$ has finite homological dimension, a complex viewed as an object of the conventional derived category is fixed by this composition whenever its cohomology modules are. $\qquad\square$

2. Derived Category of Contramodules

An abelian group P with an additive operator $s\colon P \longrightarrow P$ is said to be an *s-contramodule* if for any sequence of elements p_0, p_1, p_2, $\ldots \in P$ there is a unique sequence of elements q_0, q_1, q_2, $\ldots \in P$ satisfying the infinite system of nonhomogeneous linear equations

$$(4) \qquad\qquad q_n = sq_{n+1} + p_n \qquad \text{for all } n \geqslant 0.$$

The *infinite summation operation with s-power coefficients* in an s-contramodule P is defined by the rule

$$\sum\nolimits_{n=0}^{\infty} s^n p_n = q_0 \in P.$$

Conversely, given an additive, associative, and unital infinite summation operation

$$(p_n)_{n=0}^{\infty} \longmapsto \sum\nolimits_{n=0}^{\infty} s^n p_n$$

in an abelian group P one can uniquely solve the system of equations (4) by setting

$$q_n = \sum\nolimits_{i=0}^{\infty} s^i p_{n+i}.$$

(see [26, Section 1.6] and [23, Section B.5]; cf. [24, Section C.2]).

A module P over a ring R with a chosen element $s \in R$ is said to be an *s-contramodule* if it is a contramodule with respect to the operator of multiplication with s. When s is a central element in R, this condition is equivalent the Ext group vanishing $\operatorname{Ext}_R^i(R[s^{-1}], P) = 0$ for $i = 0$ and 1 (notice that the R-module $R[s^{-1}]$ has projective dimension at most 1, so such Ext groups with $i \geqslant 2$ always vanish) [23, Lemma B.7.1].

Let R be a commutative ring, I be an ideal in R, and $s_j \in R$ be a set of generators of the ideal I. Then the property of an R-module P to be a contramodule with respect to all the elements s_j depends only on the ideal $I \subset R$ and not on the choice of a particular set of its generators. Indeed, it suffices to check this assertion for rings finitely generated over the ring of integers $\mathbb{Z}$, and for Noetherian rings it follows from [23, Theorem B.1.1] (see also [26, Section 2.2]). Another proof of this assertion will be obtained as a byproduct of the arguments below in this section.

So an R-module P is called an *I-contramodule*, or an *I-contramodule R-module*, if it is a contramodule with respect to some set of generators of the ideal I. Making use of the definition of contramodules in terms of Ext vanishing, one easily deduces the assertion that the property of an R-module to be an I-contramodule only depends on the radical $\sqrt{I} \subset R$ of the ideal I.

The full subcategory of I-contramodule R-modules $R\text{–mod}_{I\text{-ctra}} \subset R\text{–mod}$ is closed under the kernels and cokernels of morphisms, extensions, and infinite products in $R\text{–mod}$. Hence $R\text{–mod}_{I\text{-ctra}}$ is an abelian category with exact functors of infinite product and its embedding $R\text{–mod}_{I\text{-ctra}} \longrightarrow R\text{–mod}$ into the category of R-modules is an exact functor preserving the infinite products.

Let M and P be two R-modules. Then whenever *either* an R-module M is I-torsion, *or* an R-module P is an I-contramodule, the R-module $\operatorname{Hom}_R(M, P)$ is

an I-contramodule. Indeed, for any element $s \in I$ the infinite summation operations with s-power coefficients in $\mathrm{Hom}_R(M, P)$ are provided by the formula

$$\left(\sum_{n=0}^{\infty} s^n f_n \right)(x) = \sum_{n=0}^{\infty} f_n(s^n x),$$

where the sum in the right-hand side only has a finite number of nonvanishing summands, in the former case, and

$$\left(\sum_{n=0}^{\infty} s^n f_n \right)(x) = \sum_{n=0}^{\infty} s^n f_n(x),$$

where the sum in the right-hand side is the s-power infinite summation operation in the s-contramodule P, in the latter one. Here f_n are arbitrary elements of the R-module $\mathrm{Hom}_R(M, P)$ and x is an arbitrary element of M.

Assume that the ideal $I \subset R$ is finitely generated. In any R-module that is separated and complete in the I-adic topology, i. e., an R-module P for which the natural map $P \longrightarrow \varprojlim_n P/I^n P$ is an isomorphism, the infinite summation operation with s-power coefficients can be defined for any $s \in I$ as the I-adic limit of finite partial sums. So all the I-adically separated and complete R-modules are I-contramodules. Conversely, all the I-contramodule R-modules are I-adically complete, but they do not have to be I-adically separated (see [18, Example 4.33] or [26, Section 1.5] and the references therein). Still, one has $P/IP \neq 0$ for any I-contramodule R-module $P \neq 0$ (see [20, Corollary 2.5], [23, Lemma 1.3.1], or [26, Lemma 2.1(b)]).

The theory of "cohomologically I-adically complete complexes of R-modules" is developed in [18] using the I-adic completion functor $\Lambda_I \colon M \longmapsto \varprojlim_n M/I^n M$, which is neither left, nor right exact (and generally seems to be only well-behaved for flat R-modules M). We prefer to construct and use the functor left adjoint to the embedding functor of the full subcategory of I-contramodule R-modules.

Proposition 2.1. *Assume that the ideal $I \subset R$ is finitely generated. Then the embedding functor R–$\mathsf{mod}_{I\text{-ctra}} \longrightarrow R$–$\mathsf{mod}$ has a left adjoint functor*

$$\Delta_I \colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{I\text{-ctra}}.$$

Proof. We will present an explicit construction of the functor Δ_I. Let us first consider the case of an ideal I generated by a single element $s \in R$. To any R-module M we assign the R-module morphism

$$\phi_M^s \colon \prod_{n=1}^{\infty} M \longrightarrow \prod_{n=0}^{\infty} M$$

taking every sequence of elements $r_1, r_2, \ldots \in M$ to the sequence of elements

$$m_0 = -sr_1, \quad m_1 = r_1 - sr_2, \quad m_2 = r_2 - sr_3, \ldots$$

Denote by $\Delta_s(M)$ the cokernel of the morphism ϕ_M^s.

Let us show, first of all, that the R-module $\Delta_s(M)$ is an s-contramodule. The assertion only depends on the $\mathbb{Z}[s]$-module structure on M, so one can assume that $R = \mathbb{Z}[s]$. For any abelian group A, consider the $\mathbb{Z}[s]$-module $A[s]$ of polynomials in s with the coefficients in A. It is straightforward to compute that $\Delta_s(A[s]) = A[[s]]$ is the $\mathbb{Z}[s]$-module of formal Taylor power series in s with the coefficients in A;

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the isomorphism is provided by the map $\prod_{n=0}^{\infty} A[s] \longrightarrow A[[s]]$ taking any sequence of polynomials $m_0,\ m_1,\ m_2\ \dots$ to the power series $\sum_{n=0}^{\infty} s^n m_n(s)$. Clearly, any $\mathbb{Z}[s]$-module is the cokernel of a morphism of modules of the form $A[s]$, the functor Δ_s preserves cokernels, and the $\mathbb{Z}[s]$-modules $A[[s]]$ are s-contramodules. Since the class of s-contramodules is closed under cokernels, the assertion is proven.

Now let us show that the group of R-module morphisms $\operatorname{Hom}_R(M, P)$ is naturally isomorphic to $\operatorname{Hom}_R(\Delta_s(M), P)$ whenever an R-module P is an s-contramodule. The embedding $M \longrightarrow \prod_{n=0}^{\infty} M$ taking every element $m \in M$ to the sequence $m_0 = m$, $m_1 = m_2 = \cdots = 0$, induces a natural morphism $M \longrightarrow \Delta_s(M)$ for any R-module M. For any s-contramodule P, any R-linear map $f \colon M \longrightarrow P$ is extended to an R-linear map $g \colon \Delta_s(M) \longrightarrow P$ by the rule

$$g(m_0, m_1, m_2, \dots) = \sum_{n=0}^{\infty} s^n f(m_n),$$

where the summation sign in the right-hand side stands for the infinite summation operation in an s-contramodule P.

It remains to show that there exists no other R-linear map $\Delta_s(M) \longrightarrow P$ whose composition with the natural map $M \longrightarrow \Delta_s(M)$ is equal to the given R-linear map $f \colon M \longrightarrow P$. Indeed, let $h \colon \Delta_s(M) \longrightarrow P$ be such a map; we will denote its composition with the natural surjection $\prod_{n=0}^{\infty} M \longrightarrow \Delta_s(M)$ also by h. Set

$$q_n = h(m_n, m_{n+1}, m_{n+2}, \dots) \in P \qquad \text{for every } n \geqslant 0.$$

The map ϕ_M^s takes the sequence of elements $(r_1, r_2, \dots) = (m_{n+1}, m_{n+2}, \dots)$ to the sequence

$$\phi(m_{n+1}, m_{n+2}, \dots) = (0, m_{n+1}, m_{n+2}, \dots) - s(m_{n+1}, m_{n+2}, \dots),$$

so the elements q_n and $f(m_n) \in P$ satisfy the system of linear equations

$$q_n - f(m_n) = s q_{n+1} \qquad \text{for all } n \geqslant 0.$$

By the definition of an s-contramodule, the sequence of elements q_n is uniquely determined by the sequence $f(m_n)$.

We have constructed the functor $\Delta_I = \Delta_s$ for an ideal $I = (s)$ generated by a single element $s \in R$. To produce the functor Δ_I for an ideal $I = (s_j)_{j=1}^m$ generated by a finite sequence of elements $s_j \in R$, notice that for any two elements s and $t \in R$ the functor $\Delta_s \colon R\text{–mod} \longrightarrow R\text{–mod}_{(s)\text{-ctra}} \subset R\text{–mod}$ takes the full subcategory $R\text{–mod}_{(t)\text{-ctra}} \subset R\text{–mod}$ into itself. Indeed, the full subcategory $R\text{–mod}_{(t)\text{-ctra}}$ is closed under the infinite products and cokernels in $R\text{–mod}$.

Hence it is clear that the restriction of the functor Δ_s to the full subcategory $R\text{–mod}_{(t)\text{-ctra}} \subset R\text{–mod}$ provides a functor $R\text{–mod}_{(t)\text{-ctra}} \longrightarrow R\text{–mod}_{(s,t)\text{-ctra}}$ left adjoint to the embedding of full subcategory $R\text{–mod}_{(s,t)\text{-ctra}} \longrightarrow R\text{–mod}_{(t)\text{-ctra}}$. Composing the functors Δ_{s_j} over all the elements in a chosen finite set of generators of the ideal $I \subset R$, we obtain the desired functor $\Delta_I = \Delta_{s_1} \cdots \Delta_{s_m}$. $\qquad \square$

In the paper [18, Definition 5.1], the notation $\mathrm{Tel}^\bullet(R, s)$ is used for the following two-term complex of infinitely generated free R-modules sitting in the degrees 0 and 1

$$\bigoplus_{n=0}^\infty R\delta_n \longrightarrow \bigoplus_{n=0}^\infty R\delta_n$$

with the differential $d(\delta_0) = \delta_0$, $d(\delta_n) = \delta_{n-1} - s\delta_n$ for $n \geqslant 1$. Essentially the same complex can be found in [5, equation (6.7)]. Passing to the quotient modules of both the terms of this complex by the submodules $R\delta_0$ and redenoting $\epsilon_n = \delta_{n+1}$ in the left-hand side, we obtain a homotopy equivalent complex $T^\bullet(R, s)$ of the form

$$\bigoplus_{n=0}^\infty R\epsilon_n \longrightarrow \bigoplus_{n=1}^\infty R\delta_n$$

with the differential $d(\epsilon_0) = -s\delta_1$, $d(\epsilon_n) = \delta_n - s\delta_{n+1}$ for $n \geqslant 1$.

The telescope complex $\mathrm{Tel}^\bullet(R, s)$ is interesting because it is quasi-isomorphic to the two-term Čech complex $C^\bullet_{\{s\}}(R)^\sim = (R \to R[s^{-1}])$ from Section 1; the quasi-isomorphism $\mathrm{Tel}^\bullet(R, s) \longrightarrow C^\bullet_{\{s\}}(R)^\sim$ is given by the rules $\delta_0 \longmapsto 1$, $\delta_n \longmapsto 0$ in degree 0 and $\delta_n \longmapsto s^{-n}$ in degree 1. Setting $\mathrm{Tel}^\bullet(R, \mathbf{s}) = \mathrm{Tel}^\bullet(R, s_1) \otimes_R \cdots \otimes_R \mathrm{Tel}^\bullet(R, s_m)$ and $T^\bullet(R, \mathbf{s}) = T^\bullet(R, s_1) \otimes_R \cdots \otimes_R T^\bullet(R, s_m)$ for a finite sequence $s_1, \ldots, s_m \in R$, we have a quasi-isomorphism of finite complexes of flat R-modules

$$\mathrm{Tel}^\bullet(R, \mathbf{s}) \longrightarrow C^\bullet_{\mathbf{s}}(R)^\sim$$

and a homotopy equivalence of finite complexes of free R-modules

$$\mathrm{Tel}^\bullet(R, \mathbf{s}) \longrightarrow T^\bullet(R, \mathbf{s}).$$

Lemma 2.2. (a) *For any R-module M, all the homology modules of the complex $\mathrm{Hom}_R(\mathrm{Tel}^\bullet(R, \mathbf{s}), M)$ are I-contramodule R-modules.*

(b) *For any R-module M, there is a natural isomorphism of R-modules*

$$H_0 \mathrm{Hom}_R(\mathrm{Tel}^\bullet(R, \mathbf{s}), M) \simeq \Delta_I(M).$$

(c) *For any I-contramodule R-module P, the morphism of complexes $\mathrm{Tel}^\bullet(R, \mathbf{s}) \longrightarrow R$ induces a quasi-isomorphism of complexes of R-modules*

$$P \longrightarrow \mathrm{Hom}_R(\mathrm{Tel}^\bullet(R, \mathbf{s}), P).$$

Proof. Part (a): for any element $s \in R$, denote by $T^\bullet(R, s)'$ the subcomplex of the two-term complex $\mathrm{Tel}^\bullet(R, s)$ spanned by all the generators δ_n with the exception of the generator δ_0 in degree 0. The complex $T^\bullet(R, s)'$ a free R-module resolution of the R-module $R[s^{-1}]$. Consequently, an R-module P is an s-contramodule if and only if the two-term complex $\mathrm{Hom}_R(T^\bullet(R, s)', P)$ is acyclic, that is the R-module morphism $\mathrm{Hom}_R(T^1(R, s)', P) \longrightarrow \mathrm{Hom}_R(T^0(R, s)', P)$ is an isomorphism. It follows that a complex of R-modules $P^\bullet$ has s-contramodule cohomology modules if and only if the morphism of complexes $\mathrm{Hom}_R(T^1(R, s)', P^\bullet) \longrightarrow \mathrm{Hom}_R(T^0(R, s)', P^\bullet)$ is a quasi-isomorphism, that is the complex $\mathrm{Hom}_R(T^\bullet(R, s)', P^\bullet)$ is acyclic. Now for any R-module M and any $1 \leqslant j \leqslant m$ one has $\mathrm{Hom}_R(T^\bullet(R, s_j)', \mathrm{Hom}_R(\mathrm{Tel}^\bullet(R, \mathbf{s}), M)) \simeq \mathrm{Hom}_R(\mathrm{Tel}^\bullet(R, \mathbf{s}) \otimes_R T^\bullet(R, s_j)', M)$, and the complex $\mathrm{Tel}^\bullet(R, \mathbf{s}) \otimes_R T^\bullet(R, s_j)'$, being a complex of free R-modules quasi-isomorphic to the contractible complex $C^\bullet_{\mathbf{s}}(R)^\sim[s_j^{-1}]$ (see the proof of Lemma 1.1(a)), is also contractible.

Part (b): the functor Δ_s from the proof of Proposition 2.1 is isomorphic to the functor $M \longmapsto H_0 \operatorname{Hom}_R(T^\bullet(R, s), M)$ by construction, as the map ϕ^s_M is precisely the differential in the two-term complex $\operatorname{Hom}_R(T^\bullet(R, s), M)$. Consequently, the functor Δ_I is isomorphic to the functor $M \longmapsto H_0 \operatorname{Hom}_R(T^\bullet(R, \mathbf{s}), M)$ when the ideal I is generated by a sequence of elements $s_1, \ldots, s_m \in R$. The latter functor is isomorphic to the functor $M \longmapsto H_0 \operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), M)$, because the complexes $\operatorname{Tel}^\bullet(R, \mathbf{s})$ and $T^\bullet(R, \mathbf{s})$ are homotopy equivalent.

Part (c): all the terms of the complexes on the both sides of the quasi-isomorphism $\operatorname{Tel}^\bullet(R, \mathbf{s}) \longrightarrow C^\bullet_{\mathbf{s}}(R)^\sim$ have the property that all the modules Ext^n_R from them into any I-contramodule R-module P in the positive degrees $n > 0$ vanish, so the complexes $\operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), P)$ and $\operatorname{Hom}_R(C^\bullet_{\mathbf{s}}(R)^\sim, P)$ are quasi-isomorphic. Furthermore, all the terms of the complex $C^\bullet_{\mathbf{s}}(R)$ in the kernel of the natural morphism of complexes $C^\bullet_{\mathbf{s}}(R)^\sim \longrightarrow R$ have the property that all the modules Ext^n_R from them into any I-contramodule R-module P in all the degrees $n \geqslant 0$ vanish, so the complex $\operatorname{Hom}_R(C^\bullet_{\mathbf{s}}(R)^\sim, P)$ is quasi-isomorphic to P. $\qquad\square$

In particular, we have obtained another proof of the fact that the property of an R-module to be a contramodule with respect to each element s_j in a finite sequence $s_1, \ldots, s_m \in R$ only depends on the radical $\sqrt{I} \subset R$ of the ideal I generated by the elements $s_1, \ldots, s_m$ in R. Indeed, according to Lemma 2.2(b) together with the proof of Proposition 2.1, an R-module P is a contramodule with respect to $s_1, \ldots, s_m$ if and only if it has the form $H_0 \operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), M)$ for a certain R-module M. Since the complexes $\operatorname{Tel}^\bullet(R, \mathbf{s})$ for various sequences $\mathbf{s}$ corresponding to the same ideal $\sqrt{I}$ are homotopy equivalent [18, Theorem 6.1], it follows that the contramodule property only depends on the ideal $\sqrt{I}$. (Alternatively, one can use Lemma 2.2(a) and (c).)

A projective system of R-modules $M_1 \longleftarrow M_2 \longleftarrow M_3 \longleftarrow \cdots$ indexed by the positive integers is said to be *pro-zero* if for every $l \geqslant 1$ there exists $n > l$ such that the composition of maps $M_l \longleftarrow \cdots \longleftarrow M_n$ vanishes. Both the countable projective limit functor and its (first and only) derived functor vanish on pro-zero projective systems, $\varprojlim_n M_n = 0 = \varprojlim^1_n M_n$.

The two-term complex $\operatorname{Tel}^\bullet(R, s)$ is the union of its subcomplexes of free R-modules $\operatorname{Tel}^\bullet_n(R, s)$ generated by the symbols $\delta_0, \ldots, \delta_n$ in the degrees 0 and 1. Accordingly, the complex $\operatorname{Tel}^\bullet(R, \mathbf{s})$ is the union of its subcomplexes of finitely generated free R-modules $\operatorname{Tel}^\bullet_n(R, s_1) \otimes_R \cdots \otimes_R \operatorname{Tel}^\bullet_n(R, s_m)$. The dual complexes of finitely generated free R-modules $\operatorname{Hom}_R(\operatorname{Tel}^\bullet_n(R, \mathbf{s}), R)$, concentrated in the homological degrees from 0 to m, form a projective system. A finite sequence of elements $\mathbf{s}$ in a commutative ring R is called *weakly proregular* if the projective system of homology modules $H_i \operatorname{Hom}_R(\operatorname{Tel}^\bullet_n(R, \mathbf{s}), R)$ is pro-zero for every $i > 0$ (see [29, Definition 2.3] or [18, Definition 4.21 and Lemma 5.7]). The following lemma shows that this definition is equivalent to the one that we used in Section 1.

Lemma 2.3. *Let $M^1_\bullet \longleftarrow M^2_\bullet \longleftarrow M^3_\bullet \longleftarrow \cdots$ be a projective system of complexes of R-modules. Then the projective system of homology modules $H_i(M^1_\bullet) \longleftarrow$*

$H_i(M_\bullet^2) \longleftarrow H_i(M_\bullet^3) \longleftarrow \cdots$ *is pro-zero if and only if for any injective R-module J one has* $\varinjlim_n H^i \operatorname{Hom}_R(M_\bullet^n, J) = 0$.

Proof. This is essentially [29, Lemma 2.4] or [18, Theorem 4.24]. One has $H^i \operatorname{Hom}_R(M_\bullet^n, J) \simeq \operatorname{Hom}_R(H_i(M_\bullet^n), J)$, so the "only if" assertion is obvious. To prove the "if", pick an injective R-module morphism from the R-module $H_i(M_\bullet^l)$ to some injective R-module J. This morphism represents an element of $H^i \operatorname{Hom}_R(M_\bullet^l, J)$, which has to die in the inductive limit; so there exists $n > k$ such that the composition $H_i(M_\bullet^n) \longrightarrow H_i(M_\bullet^l) \longrightarrow J$ is a zero morphism. It follows that the morphism $H_i(M_\bullet^n) \longrightarrow H_i(M_\bullet^l)$ is zero. $\square$

Example 2.4. It was already mentioned in Section 1 that by [18, Corollary 6.2] the weak proregularity is a property of the radical $\sqrt{I} \subset R$ of the ideal I that a finite sequence $s_1, \ldots, s_m \in R$ generates and not of the sequence itself. We have also seen that any ideal in a Noetherian commutative ring is weakly proregular. Here is a simple observation providing a class of examples of weakly proregular ideals in non-Noetherian rings (cf. [18, Example 4.35 and Theorem 6.5]).

Let $R \longrightarrow T$ be a morphism of commutative rings such that T is a flat module over R. Let $\mathbf{s}$ be a finite sequence of elements in the ring R, and let I be the ideal generated by $\mathbf{s}$ in R. Then the ideal TI in the ring T is weakly proregular whenever the ideal I in the ring R is. In other words, the image of a finite sequence $\mathbf{s}$ is weakly proregular in T if a sequence $\mathbf{s}$ weakly proregular in R. Indeed, one has $H_i \operatorname{Hom}_T(\operatorname{Tel}_n^\bullet(T, \mathbf{s}), T) \simeq T \otimes_R H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), R)$, so for any given i the projective system $H_i \operatorname{Hom}_T(\operatorname{Tel}_n^\bullet(T, \mathbf{s}), T)$ is pro-zero whenever the projective system $H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), R)$ is.

Lemma 2.5. *For any weakly proregular finitely generated ideal I in a commutative ring R, the restriction of the functor Δ_I to the full subcategory of flat R-modules F in R–*mod *is an exact functor isomorphic to the functor of I-adic completion $\Lambda_I \colon F \longmapsto \varprojlim_n F/I^n F$.*

Proof. In view of Lemma 2.2(b), this is [18, computations (2–3) in the proof of Theorem 5.21]. The complex $\operatorname{Tel}_n^\bullet(R, \mathbf{s})$ is homotopy equivalent to the tensor product over j of the two-term complexes of free R-modules $(R \to s_j^{-n} R)$. Therefore, one has $H_0 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), M) \simeq M/(\mathbf{s}^n)M$ for any R-module M, where $(\mathbf{s}^n)$ denotes the ideal generated by the sequence of elements $s_1^n, \ldots, s_m^n$ in R.

Clearly, $\varprojlim_n M/(\mathbf{s}^n M) \simeq \varprojlim_n M/I^n M$ for any R-module M; it remains to check the isomorphism

$$H_0 \varprojlim_n \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), F) \simeq \varprojlim_n H_0 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), F)$$

for a flat R-module F. Since the terms of the subcomplexes $\operatorname{Tel}_n^\bullet(R, \mathbf{s})$ are direct summands of the respective terms of the ambient complex $\operatorname{Tel}^\bullet(R, \mathbf{s})$, the transition maps in the projective system of complexes $\operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), M)$ are surjective for any R-module M. Finally, if the projective system $H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), R)$ is pro-zero for every $i > 0$, then the projective systems

$$H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), F) \simeq H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), R) \otimes_R F$$

are also pro-zero for any flat R-module F and every $i > 0$. Consequently, $\varprojlim_n^1 H_i \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), F) = 0$ for all $i \in \mathbb{Z}$. $\qquad\square$

Example 2.6. It is clear from the above computation that for any ideal I generated by a finite sequence of elements $\mathbf{s}$ in R and any R-module M there is a natural surjective morphism of R-modules $\Delta_I(M) \longrightarrow \Lambda_I(M)$ with the kernel isomorphic to $\varprojlim_n^1 H_1 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, \mathbf{s}), M)$. The following example shows that this kernel may be nontrivial even for the R-module $M = R$ when the weak proregularity is not assumed.

Let the sequence $\mathbf{s}$ consist of a single element $s \in R$. Then the homology group $H_1 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, s), M)$ is isomorphic to the submodule of elements annihilated by s^n in M; for any $n > k$, the map $H_1 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, s), M) \longrightarrow H_1 \operatorname{Hom}_R(\operatorname{Tel}_l^\bullet(R, s), M)$ is the multiplication with s^{n-l}. Now let R be the commutative algebra over a field k generated by an infinite sequence of elements x_1, x_2, x_3 ... and an additional generator s with the relations $x_i x_j = 0$ and $s^i x_i = 0$ for all i, $j \geqslant 1$. Then the submodule of elements annihilated by s^n in R is generated by the elements x_1, ..., x_n, $s x_{n+1}$, $s^2 x_{n+2}$, ..., and the map of multiplication with s^{n-l} never annihilates this submodule entirely for $l \geqslant 1$, so the projective system $H_1 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, s), R)$ is not pro-zero and the one-element sequence $s \in R$ is not weakly proregular.

Futhermore, one can compute the R-module $\varprojlim_n^1 H_1 \operatorname{Hom}_R(\operatorname{Tel}_n^\bullet(R, s), R)$ as being isomorphic to the module of formal power series $W[[s]]$ in the varible s with the coefficient space W equal to the derived projective limit $\prod_i k x_i / \bigoplus_i k x_i$ of the projective system of vector spaces $k x_n \oplus k x_{n+1} \oplus \cdots$ and their obvious embeddings. The generators $x_i \in R$ act by zero in the R-module $W[[s]]$, while the generator $s \in R$ acts in the power series in s in the standard way.

Lemma 2.7. (a) *For any weakly proregular finite sequence of elements $\mathbf{s}$ in a commutative ring R, the complex $\operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), M)$ assigned to an R-module M computes the left derived functor $\mathbb{L}_* \Delta_I(M)$ of the right exact functor Δ_I, viewed as taking values in the category R–mod.*

(b) *For any weakly proregular finitely generated ideal I in a commutative ring R, (the left derived functor $\mathbb{L}_* \Delta_I$ of) the right exact functor $\Delta_I \colon R\text{–mod} \longrightarrow R\text{–mod}_{I\text{-ctra}}$ has finite homological dimension not exceeding the minimal number of generators of the ideal I.*

Proof. Part (a): according to [18, Theorem 5.21], one has $H_i \operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), F) = 0$ for any flat R-module F and all $i > 0$ (we have essentially explained as much already in the proof of Lemma 2.5). Taking also into account Lemma 2.2(b), the argument proceeds from this point in the (standard) way dual to the proof of Lemma 1.2(a) above. As the embedding $R\text{–mod}_{I\text{-ctra}} \longrightarrow R\text{–mod}$ is an exact functor taking nonzero objects to nonzero objects, part (b) follows. (Cf. [18, Corollaries 5.25 and 5.27].) $\qquad\square$

Lemma 2.8. *For any weakly proregular finitely generated ideal I in a commutative ring R, the derived functor of the functor Δ_I takes I-contramodules to themselves, i. e., there is a natural isomorphism $\Delta_I(P) \simeq P$ and one has $\mathbb{L}_n \Delta_I(P) = 0$ for all $n > 0$ and any I-contramodule R-module P.*

Proof. This is Lemma 2.2(c) together with Lemma 2.7(a). Notice that the isomorphism $\Delta_I(P) \simeq P$ follows already from the fact that the functor Δ_I is left adjoint to the fully faithful embedding functor $R\text{–mod}_{I\text{-ctra}} \longrightarrow R\text{–mod}$. It is only the second assertion of the lemma that depends on the weak proregularity assumption. $\square$

Now we are in the position to prove the main result of this section.

Theorem 2.9. *Let R be a commutative ring and $I \subset R$ be a weakly proregular finitely generated ideal. Then for any symbol $\star = b, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{ctr}, \text{ or } \mathsf{abs}$, the triangulated functor $\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–mod})$ induced by the embedding of abelian categories $R\text{–mod}_{I\text{-ctra}} \longrightarrow R\text{–mod}$ is fully faithful.*

Proof. The argument is similar to the proof of Theorem 1.3. In the case of a Noetherian ring R and the derived categories D^b or D^-, the assertion follows from [23, Propositions B.9.1 and B.10.1] (see also [26, Proposition 2.2.2(b)]). A proof of the general case proceeds as follows.

Consider the full subcategory $R\text{–mod}_{I\text{-ctra-adj}} \subset R\text{–mod}$ consisting of all the R-modules M for which $\mathbb{L}_n\Delta_I(M) = 0$ for all $n > 0$. The full subcategory $R\text{–mod}_{I\text{-ctra-adj}}$ is closed under extensions, kernels of surjective morphisms, and infinite products in $R\text{–mod}$; and, according to Lemma 2.7(b), every R-module has a finite left resolution of uniformly bounded length by objects from $R\text{–mod}_{I\text{-ctra-adj}}$. Hence the subcategory $R\text{–mod}_{I\text{-ctra-adj}}$ inherits the exact category structure of the abelian category $R\text{–mod}$, and for any symbol $\star$ on our list the triangulated functor $\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra-adj}}) \longrightarrow \mathsf{D}^\star(R\text{–mod})$ is an equivalence of triangulated categories [24, Proposition A.5.6],

$$\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra-adj}}) \simeq \mathsf{D}^\star(R\text{–mod}).$$

Obviously, the restriction of the functor Δ_I to the full exact subcategory $R\text{–mod}_{I\text{-ctra-adj}} \subset R\text{–mod}$ is an exact functor $\Delta_I\colon R\text{–mod}_{I\text{-ctra-adj}} \longrightarrow R\text{–mod}_{I\text{-ctra}}$. Applying the functor Δ_I to complexes of modules from the category $R\text{–mod}_{I\text{-ctra-adj}}$ termwise, we obtain the left derived functor

$$\mathbb{L}\Delta_I\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}}).$$

The derived functor $\mathbb{L}\Delta_I$ is left adjoint to the triangulated functor $\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–mod})$ induced by the embedding of abelian categories $R\text{–mod}_{I\text{-ctra}} \longrightarrow R\text{–mod}$ [21, Lemma 8.3].

Furthermore, the abelian subcategory $R\text{–mod}_{I\text{-ctra}} \subset R\text{–mod}$ is contained in the exact subcategory $R\text{–mod}_{I\text{-ctra-adj}}$,

$$R\text{–mod}_{I\text{-ctra}} \subset R\text{–mod}_{I\text{-ctra-adj}}$$

by Lemma 2.8, and the functor Δ_I takes I-contramodule R-modules to themselves. Therefore, the composition of adjoint functors

$$\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}})$$

is isomorphic to the identity functor on $\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}})$. It follows immediately that the functor $\mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–mod})$ is fully faithful, while the functor $\mathbb{L}\Delta_I\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}_{I\text{-ctra}})$ is a Verdier quotient functor. $\square$

Corollary 2.10. *Let R be a commutative ring and $I \subset R$ be a weakly proregular finitely generated ideal. Then for any symbol $\star = \mathsf{b}, +, -, $ or $\varnothing$, the triangulated functor $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{--}\mathsf{mod})$ identifies the derived category $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}})$ with the full subcategory $\mathsf{D}^\star_{I\text{-ctra}}(R\text{--}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{--}\mathsf{mod})$ consisting of all the complexes with the cohomology modules belonging to $R\text{--}\mathsf{mod}_{I\text{-ctra}}$.*

Proof. The proof is similar to that of Corollary 1.4. It is obvious that the image of the functor $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{--}\mathsf{mod})$ is contained in $\mathsf{D}^\star_{I\text{-ctra}}(R\text{--}\mathsf{mod})$. To prove the converse inclusion, notice that, according to the proof of Theorem 2.9, the image of the fully faithful functor $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{--}\mathsf{mod})$ consists precisely of all the complexes fixed by the composition of adjoint functors $\mathsf{D}^\star(R\text{--}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{--}\mathsf{mod})$. Since the functor $\mathbb{L}\Delta_I$ has finite homological dimension, by the way-out functor argument of [8, Proposition I.7.1] a complex viewed as an object of the conventional derived category is fixed by this composition whenever its cohomology modules are. $\qquad\square$

3. MGM DUALITY THEOREM

Let R be a commutative ring and $\mathbf{s}$ be a finite sequence of its elements $s_1, \ldots, s_m \in R$. We recall the constructions of complexes $C^\bullet_{\mathbf{s}}(R)$ and $C^\bullet_{\mathbf{s}}(R)^\sim$ from Section 1.

Lemma 3.1. *The tensor product complex $C^\bullet_{\mathbf{s}}(R) \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim$ is a contractible complex of R-modules.*

Proof. It suffices to show that for every term $C^i_{\mathbf{s}}(R)$ of the complex $C^\bullet_{\mathbf{s}}(R)$ the tensor product $C^i_{\mathbf{s}}(R) \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim$ is a contractible complex, because the complex $C^\bullet_{\mathbf{s}}(R) \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim$ can be obtained from the complexes $C^i_{\mathbf{s}}(R) \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim$ by iterating the operations of shift and cone in the homotopy category of complexes of R-modules. Hence it is enough to check that the complex $C^\bullet_{\mathbf{s}}(R)^\sim[s_j^{-1}]$ is contractible for every j. This was already done in the proof of Lemma 1.1(a). $\qquad\square$

Following [18, Section 8], let us endow the complex $C^\bullet_{\mathbf{s}}(R)$ with the Čech/singular cochain multiplication, making it a (noncommutative) DG-ring. The natural morphism of complexes $k \colon R \longrightarrow C^\bullet_{\mathbf{s}}(R)$ is a morphism of DG-rings with the image lying in the center of $C^\bullet_{\mathbf{s}}(R)$, making $C^\bullet_{\mathbf{s}}(R)$ a DG-algebra over R. The following lemma is our version of [18, Lemmas 4.29 and 7.9].

Lemma 3.2. (a) *The two morphisms of complexes*

$$C^\bullet_{\mathbf{s}}(R)^\sim \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim \rightrightarrows C^\bullet_{\mathbf{s}}(R)^\sim$$

induced by the natural morphism $C^\bullet_{\mathbf{s}}(R)^\sim \longrightarrow R$ are homotopy equivalences of complexes of R-modules.
(b) *The three morphisms of complexes*

$$C^\bullet_{\mathbf{s}}(R) \rightrightarrows C^\bullet_{\mathbf{s}}(R) \otimes_R C^\bullet_{\mathbf{s}}(R) \longrightarrow C^\bullet_{\mathbf{s}}(R)$$

provided by the unit and multiplication in the DG-algebra $C_{\mathbf{s}}^\bullet(R)$ are homotopy equivalences of complexes of R-modules.

Proof. Follows immediately from Lemma 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Our next goal is to rewrite the geometric arguments of Section 0.5 in the algebraic language. A key ingredient is the derived category $\mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod})$ of left DG-modules over the DG-ring $C_{\mathbf{s}}^\bullet(R)$, which is used in lieu of the derived category $\mathsf{D}^\star(U\text{–qcoh}) \simeq \mathsf{D}^\star(U\text{–ctrh})$ of quasi-coherent sheaves/contraherent cosheaves on the open subscheme $U = X \setminus Z$, where $X = \operatorname{Spec} R$ and $Z = \operatorname{Spec} R/I$.

The DG-ring $C_{\mathbf{s}}^\bullet(R)$ being positively cohomologically graded, derived categories of the second kind of DG-modules over it may differ from the conventional derived categories even for bounded DG-modules (cf. [22, Section 3.4, last paragraph of Section 0.4, and Example 6.6]). So we restrict our exposition to the four conventional derived categories D^b, D^+, D^-, and D. Notice that some, though not all, of the related geometric results of [24, Sections 4.6 and 4.8] are applicable to the absolute derived categories $\mathsf{D}^{\mathsf{abs}+}$, $\mathsf{D}^{\mathsf{abs}-}$, and $\mathsf{D}^{\mathsf{abs}}$ as well.

So let $\star$ be one of the conventional derived category symbols b, $+$, $-$, or $\varnothing$. The derived categories $\mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod})$ are defined by inverting the classes of quasi-isomorphisms in the homotopy categories of (respectively bounded) left DG-modules over $C_{\mathbf{s}}^\bullet(R)$. Denote by $k_*\colon \mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod})$ the functor of restriction of scalars with respect to the DG-ring morphism $k\colon R \longrightarrow C_{\mathbf{s}}^\bullet(R)$.

Proposition 3.3. (a) *The triangulated functor k_* has a left adjoint functor $k^*\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod})$ and a right adjoint functor $\mathbb{R}k^!\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod})$.*

(b) *The compositions $k^* \circ k_*$ and $\mathbb{R}k^! \circ k_*$ are isomorphic to the identity functors on the category $\mathsf{D}^\star(C_{\mathbf{s}}^\bullet(R)\text{–mod})$, functor k_* is fully faithful, and the functors k^* and $\mathbb{R}k^!$ are Verdier quotient functors.*

Proof. Part (a): the functor k^* is easy to define, as $C_{\mathbf{s}}^\bullet(R)$ is a finite complex of flat R-modules, so setting $k^*(M^\bullet) = C_{\mathbf{s}}^\bullet(R) \otimes_R M^\bullet$ suffices for any symbol $\star$. In the case of $\star = \varnothing$, both the adjoint functors $\mathbb{L}f^*$ and $\mathbb{R}f^!$ exist for any morphism of DG-rings f [22, Section 1.7]. To construct the derived functor $\mathbb{R}k^!$ for bounded derived categories $\mathsf{D}^\star$, notice that $C_{\mathbf{s}}^\bullet(R)$ is a finite complex of R-modules of projective dimension at most 1, so embedding a complex of R-modules $M^\bullet$ into a complex of injective R-modules $J^\bullet$, replacing $M^\bullet$ with the cocone $N^\bullet$ of the morphism $J^\bullet \longrightarrow J^\bullet/M^\bullet$ and setting $\mathbb{R}k^!(M^\bullet) = \operatorname{Hom}_R(C_{\mathbf{s}}^\bullet(R), N^\bullet)$ does the job. These constructions of the functors k_*, k^*, and $\mathbb{R}k^!$ are actually applicable to the derived categories with exotic symbols $\star = \mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs just as well.

To prove part (b), notice that by Lemma 3.2(b) the adjunction morphisms

$$k_*k^* \rightrightarrows k_*k^*k_*k^* \longrightarrow k_*k^*$$

are isomorphisms of functors on the category $\mathsf{D}^\star(R\text{–mod})$, $\ k_*k^* \simeq k_*k^*k_*k^*$. It follows that for any objects $M^\bullet$ and $N^\bullet \in \mathsf{D}^\star(R\text{–mod})$ the adjunction morphisms

$k^*N^\bullet \longrightarrow k^*k_*k^*N^\bullet \longrightarrow k^*N^\bullet$ induce isomorphisms of Hom modules

$$\mathrm{Hom}_{\mathsf{D}}(k^*M^\bullet, k^*N^\bullet) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(k^*M^\bullet, k^*k_*k^*N^\bullet) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(k^*M^\bullet, k^*N^\bullet)$$

in the derived category $\mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod})$. In other words, the adjunction morphisms induce isomorphisms of the functors represented by the objects $k^*N^\bullet$ and $k^*k_*k^*N^\bullet$ on the essential image of the functor k^* (viewed as a full subcategory in $\mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod})$). As the objects $k^*N^\bullet$ and $k^*k_*k^*N^\bullet$ also belong to this essential image, it follows that the adjunction morphisms

$$k_*N^\bullet \longrightarrow k^*k_*k^*N^\bullet \longrightarrow k_*N^\bullet$$

are isomorphisms of functors, $k^* \simeq k^*k_*k^*$. Hence the adjunction morphisms

$$k^*k_* \longrightarrow k^*k_*k^*k_* \rightrightarrows k^*k_*$$

are also isomorphisms. Applying the same argument with the roles of the two categories $\mathsf{D}^\star(R\text{–}\mathsf{mod})$ and $\mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod})$ switched, one can see that the adjunction morphisms

$$k_* \longrightarrow k_*k^*k_* \longrightarrow k_*$$

are also isomorphisms of functors, $k_* \simeq k_*k^*k_*$.

Finally, we observe that the restrictions of scalars are conservative functors between conventional derived categories, i. e., a morphism of DG-modules over $C_{\mathsf{s}}^\bullet(R)$ is a quasi-isomorphism whenever it is a quasi-isomorphism of DG-modules over R (cf. [21, Remark 8.4.3 and Section 8.4.4] for a discussion of the conservativity problem for functors between derived categories of the second kind). The functor k_* being conservative and the natural morphism $k_*k^*k_*B^\bullet \longrightarrow k_*B^\bullet$ being an isomorphism in $\mathsf{D}^\star(R\text{–}\mathsf{mod})$ for any DG-module $B^\bullet \in \mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod})$, we can conclude that the morphism $k^*k_*B^\bullet \longrightarrow B^\bullet$ is an isomorphism in $\mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod})$.

Hence it follows that the functor k_* is fully faithful, the functors k^* and $\mathbb{R}k^!$ are Verdier quotient functors, and the composition $\mathbb{R}k^! \circ k_*$ is isomorphic to the identity functor. Alternatively, the latter assertion can be proven directly in the way similar to the above argument. □

Now we are ready to prove our version of the MGM duality theorem for commutative rings with finitely generated ideals. For any commutative ring R with an ideal I generated by a finite sequence of elements $s_1, \ldots, s_m \in R$, consider the full subcategory of complexes with I-torsion cohomology modules

$$\mathsf{D}_{I\text{-}\mathsf{tors}}^\star(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$$

and the full subcategory of complexes with R-contramodule cohomology modules

$$\mathsf{D}_{I\text{-}\mathsf{ctra}}^\star(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$$

in the derived category of R-modules. The essential image of the fully faithful functor k_* from Proposition 3.3

$$k_*\mathsf{D}^\star(C_{\mathsf{s}}^\bullet(R)\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$$

is a third triangulated subcategory in $\mathsf{D}^\star(R\text{–}\mathsf{mod})$ that is of interest to us.

Theorem 3.4. *The subcategory $k_* \mathsf{D}^\star(C_\mathbf{s}^\bullet(R)\text{–mod})$ is the right orthogonal complement to the subcategory $\mathsf{D}^\star_{I\text{-tors}}(R\text{–mod})$ and the left orthogonal complement to the subcategory $\mathsf{D}^\star_{I\text{-ctra}}(R\text{–mod})$ in $\mathsf{D}^\star(R\text{–mod})$. The passage to the Verdier quotient category by the triangulated subcategory $k_* \mathsf{D}^\star(C_\mathbf{s}^\bullet(R)\text{–mod}) \subset \mathsf{D}^\star(R\text{–mod})$ establishes an equivalence between the triangulated categories $\mathsf{D}^\star_{I\text{-tors}}(R\text{–mod})$ and $\mathsf{D}^\star_{I\text{-ctra}}(R\text{–mod})$,*

$$\mathsf{D}^\star_{I\text{-tors}}(R\text{–mod}) \simeq \mathsf{D}^\star(R\text{–mod})/k_* \mathsf{D}^\star(C_\mathbf{s}^\bullet(R)\text{–mod}) \simeq \mathsf{D}^\star_{I\text{-ctra}}(R\text{–mod}).$$

Proof. It is clear from Proposition 3.3 that the passage to the quotient category by the image of the functor k_* establishes an equivalence between the kernels of the functors k^* and $\mathbb{R}k^!$,

$$\ker(k^*) \simeq \mathsf{D}^\star(R\text{–mod})/\operatorname{im} k_* \simeq \ker(\mathbb{R}k^!).$$

Furthermore, as it always happens with semiorthogonal decompositions, the adjunction morphisms $\mathrm{id} \longrightarrow k_*k^*$ and $k_*\mathbb{R}k^! \longrightarrow \mathrm{id}$ have functorial cones. The subcategory $\ker(k^*) \subset \mathsf{D}^\star(R\text{–mod})$ coincides with the image of the functor

$$\operatorname{cocone}(\mathrm{id} \to k_*k^*)\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}),$$

while the subcategory $\ker(\mathbb{R}k^!) \subset \mathsf{D}^\star(R\text{–mod})$ coincides with the image of the functor

$$\operatorname{cone}(k_*\mathbb{R}k^! \to \mathrm{id})\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}).$$

Just as the functors k_*k^* and $k_*\mathbb{R}k^!$, both the functors $\operatorname{cocone}(\mathrm{id} \to k_*k^*)$ and $\operatorname{cone}(k_*\mathbb{R}k^! \to \mathrm{id})$ are projectors on their respective images: each of them is naturally isomorphic to its composition with itself. Moreover, there are natural transformations

$$\operatorname{cocone}(\mathrm{id} \to k_*k^*) \longrightarrow \mathrm{id} \quad \text{and} \quad \mathrm{id} \longrightarrow \operatorname{cone}(k_*\mathbb{R}k^! \to \mathrm{id}).$$

A complex $M^\bullet \in \mathsf{D}^\star(R\text{–mod})$ belongs to the subcategory $\ker(k^*)$ if and only if the morphism $\operatorname{cocone}(\mathrm{id} \to k_*k^*)(M^\bullet) \longrightarrow M^\bullet$ is a quasi-isomorphism, while a complex $P^\bullet \in \mathsf{D}^\star(R\text{–mod})$ belongs to the subcategory $\ker(\mathbb{R}k^!)$ if and only if the morphism $P^\bullet \longrightarrow \operatorname{cone}(k_*\mathbb{R}k^! \to \mathrm{id})(P^\bullet)$ is a quasi-isomorphism.

In view of the constructions of the functors k^* and $\mathbb{R}k^!$ in the proof of part (a) of the proposition, the functor $\operatorname{cocone}(\mathrm{id} \to k_*k^*)$ is isomorphic to the functor of tensor product with the complex $C_\mathbf{s}^\bullet(R)^\sim$,

$$C_\mathbf{s}^\bullet(R)^\sim \otimes_R -\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}),$$

while the functor $\operatorname{cone}(k_*\mathbb{R}k^! \to \mathrm{id})$ is the functor of right derived homomorphisms from $C_\mathbf{s}^\bullet(R)^\sim$. The latter can be easily computed as the homomorphisms from the complex of free R-modules $\operatorname{Tel}^\bullet(R, \mathbf{s})$,

$$\operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), -)\colon \mathsf{D}^\star(R\text{–mod}) \longrightarrow \mathsf{D}^\star(R\text{–mod}).$$

To sum up, a complex $M^\bullet \in \mathsf{D}^\star(R\text{–mod})$ belongs to the subcategory $\ker(k^*)$ if and only if the natural map $C_\mathbf{s}^\bullet(R)^\sim \otimes_R M^\bullet \longrightarrow M^\bullet$ is a quasi-isomorphism, while a complex $P^\bullet \in \mathsf{D}^\star(R\text{–mod})$ belongs to the subcategory $\ker(\mathbb{R}k^!)$ if and only if the natural map $P^\bullet \longrightarrow \operatorname{Hom}_R(\operatorname{Tel}^\bullet(R, \mathbf{s}), P^\bullet)$ is a quasi-isomorphism.

Finally, we recall that $C_{\mathbf{s}}^\bullet(R)^\sim$ and $\mathrm{Tel}^\bullet(R,\mathbf{s})$ are finite complexes of R-modules, so Hartshorne's way-out functor argument of of [8, Propositions I.7.1 and I.7.3] applies. Hence the cohomology modules of the complex $C_{\mathbf{s}}^\bullet(R)^\sim \otimes_R M^\bullet$ are I-torsion R-modules for any complex of R-modules $M^\bullet$ by Lemma 1.1(a). Similarly, the cohomology modules of the complex $\mathrm{Hom}_R(\mathrm{Tel}^\bullet(R,\mathbf{s}), M^\bullet)$ are I-contramodule R-modules for any complex of R-modules $M^\bullet$ by Lemma 2.2(a). We have shown that the cohomology modules of any complex $M^\bullet \in \ker(k^*)$ are I-torsion R-modules, and the cohomology modules over any complex $P^\bullet \in \ker(\mathbb{R}k^!)$ are I-contramodule R-modules.

Conversely, it follows from the above that a complex of R-modules belongs to the subcategory $\ker(k^*)$ or $\ker(\mathbb{R}k^!)$ whenever its cohomology modules, viewed as one-term complexes, do. When M is an I-torsion R-module, the map $C_{\mathbf{s}}^\bullet(R)^\sim \otimes_R M \longrightarrow M$ is a quasi-isomorphism by Lemma 1.1(c). When P is an I-contramodule, the map $P \longrightarrow \mathrm{Hom}_R(T^\bullet(R,\mathbf{s}), P)$ is a quasi-isomorphism by Lemma 2.2(c). Therefore, the full subcategory $\ker(k^*) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$ consists precisely of all the complexes with I-torsion cohomology modules, and the full subcategory $\ker(\mathbb{R}k^!) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$ consists precisely of all the complexes with I-contramodule cohomology modules.

Notice that *no* weak proregularity assumption has been used in this proof (cf. [5, Propositions 6.12 and 6.15]). $\qquad\square$

When the ideal $I \subset R$ is weakly proregular, the MGM duality theorem takes the form promised in the formula (2) in Section 0.9 of the introduction.

Corollary 3.5. *Let R be a commutative ring and $I \subset R$ be a weakly proregular finitely generated ideal. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, the (respectively bounded or unbounded) derived categories $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$ and $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-ctra}})$ are naturally equivalent,*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \simeq \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-ctra}}).$$

Proof. Compare the results of Corollaries 1.4 and 2.10, and Theorem 3.4. Another proof of this equivalence of derived categories (in somewhat greater generality) will be obtained in Theorem 5.10 below. $\qquad\square$

4. Dedualizing Complexes of Bicomodules

We refer to the book [30] and the overview paper [26] for the definitions of coassociative coalgebras over fields, comodules and contramodules over them, and the related basic concepts. A discussion of *cosemisimple* and *conilpotent* coalgebras can be found in [30, Sections 9.0–1] and (with a view towards contramodules and the terminology similar to the one in this paper) in [21, Appendix A].

The definitions of *Artinian*, *co-Noetherian* and *cocoherent* coalgebras and *finitely cogenerated* and *finitely copresented* comodules, as well as generally all the notation, terminology, and basic background facts related to coalgebras, comodules, and contramodules that are used in this section, are contained in Appendix C below. These

concepts and terminology being not widely known, the reader is advised to glance into Appendix C before reading this section.

We start with several constructions and lemmas related to complexes of comodules and contramodules. These are purported to clear way to our key definition of a *dedualizing complex* of bicomodules over a pair of cocoherent coalgebras.

Let $\mathcal{C}$ and $\mathcal{D}$ be two coassociative coalgebras (with counits) over the same field k. Given a derived category symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs+}, \mathsf{abs-}$, or abs, we denote by $\mathsf{D}^\star(\mathcal{C}\text{–comod})$ and $\mathsf{D}^\star(\mathcal{D}\text{–contra})$ the corresponding (conventional or absolute) derived categories of the abelian categories $\mathcal{C}\text{–comod}$ and $\mathcal{D}\text{–contra}$ of left $\mathcal{C}$-comodules and left $\mathcal{D}$-contramodules (see Appendix A for the definitions).

For any subcoalgebra $\mathcal{E}$ in a coalgebra $\mathcal{C}$, the maximal $\mathcal{E}$-subcomodule functor $\mathcal{M} \longmapsto {}_\mathcal{E}\mathcal{M}$ acting from the category of left $\mathcal{C}$-comodules to the category of left $\mathcal{E}$-comodules is left exact. The abelian category $\mathcal{C}\text{–comod}$ has enough injective objects, which are precisely the direct summands of cofree $\mathcal{C}$-comodules. So one can identify the bounded below derived category $\mathsf{D}^+(\mathcal{C}\text{–comod})$ with the homotopy category of injective $\mathcal{C}$-comodules $\mathsf{Hot}^+(\mathcal{C}\text{–comod}_{\mathsf{inj}})$ and, applying the functor $\mathcal{M} \longmapsto {}_\mathcal{E}\mathcal{M}$ to complexes of injective $\mathcal{C}$-comodules terwise, obtain the right derived functor

$$\mathcal{M}^\bullet \longmapsto {}_\mathcal{E}^\mathbb{R}\mathcal{M}^\bullet \colon\ \mathsf{D}^+(\mathcal{C}\text{–comod}) \longrightarrow \mathsf{D}^+(\mathcal{E}\text{–comod}).$$

Lemma 4.1. *Let $\mathcal{C}$ be a left co-Noetherian coalgebra and $\mathcal{E} \subset \mathcal{C}$ be a subcoalgebra such that the quotient coalgebra without counit $\mathcal{C}/\mathcal{E}$ is conilpotent. Then a complex $\mathcal{L}^\bullet \in \mathsf{D}^+(\mathcal{C}\text{–comod})$ has finitely cogenerated $\mathcal{C}$-comodules of cohomology if and only if the complex ${}_\mathcal{E}^\mathbb{R}\mathcal{L}^\bullet \in \mathsf{D}^+(\mathcal{E}\text{–comod})$ has finitely cogenerated $\mathcal{E}$-comodules of cohomology.*

Proof. Notice that the cohomology $\mathcal{E}$-comodules of the complex ${}_\mathcal{E}^\mathbb{R}\mathcal{L}$ are finitely cogenerated for any finitely cogenerated left $\mathcal{C}$-comodule $\mathcal{L}$ (viewed as a one-term complex of left $\mathcal{C}$-comodules). Indeed, one can compute the derived category object ${}_\mathcal{E}^\mathbb{R}\mathcal{L}$ using a right resolution of the $\mathcal{C}$-comodule $\mathcal{L}$ by finitely cogenerated cofree left $\mathcal{C}$-comodules (which exists since the class of finitely cogenerated left comodules over a left co-Noetherian coalgebra $\mathcal{C}$ is closed under the passages to the cokernels of morphisms) and apply Lemma C.2(a). Since the class of finitely cogenerated left $\mathcal{E}$-comodules is also closed under the kernels, cokernels, and extensions, the desired assertion now follows by induction in the cohomological degree from Lemma C.2(d). $\qquad\square$

We recall from Appendix C that finitely copresented left comodules over a left cocoherent coalgebra $\mathcal{C}$ form an abelian category. Notice that this abelian category has enough injective objects, which are precisely the direct summands of finitely cogenerated cofree $\mathcal{C}$-comodules.

Lemma 4.2. *Let $\mathcal{C}$ be a left cocoherent coalgebra, and let $\mathcal{L}^\bullet$ be a bounded below complex of left $\mathcal{C}$-comodules with finitely copresented cohomology modules. Then there exists a bounded below complex of finitely cogenerated cofree left $\mathcal{C}$-comodules $\mathcal{J}^\bullet$ together with a quasi-isomorphism of complexes of left $\mathcal{C}$-comodules $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$.*

Proof. This is a standard step-by-step construction (cf. [27, Lemma 1.2] and the proof of Lemma B.1(c) below). $\qquad\square$

A finite complex of left $\mathcal{C}$-comodules $\mathcal{L}^\bullet$ is said to have *projective dimension* $\leqslant d$ if one has $\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(\mathcal{C}-\mathsf{comod})}(\mathcal{L}^\bullet, \mathcal{M}[n]) = 0$ for all left $\mathcal{C}$-comodules $\mathcal{M}$ and all the integers $n > d$. Similarly, a finite complex of left $\mathcal{D}$-contramodules $\mathfrak{Q}^\bullet$ is said to have *injective dimension* $\leqslant d$ if one has $\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(\mathcal{D}-\mathsf{contra})}(\mathfrak{P}, \mathfrak{Q}^\bullet[n]) = 0$ for all left $\mathcal{D}$-contramodules $\mathfrak{P}$ and all $n > d$.

We refer to Appendix C and [26, Section 3.1] for the definition and discussion of the operation of *contratensor product* of right $\mathcal{D}$-comodules and left $\mathcal{D}$-contramodules. The bounded above derived category $\mathsf{D}^-(\mathcal{D}-\mathsf{contra})$ is equivalent to the homotopy category $\mathsf{Hot}^-(\mathcal{D}-\mathsf{contra}_{\mathsf{proj}})$ of bounded above complexes of projective left $\mathcal{D}$-contramodules. Given a complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$ and a bounded above complex of left $\mathcal{D}$-contramodules $\mathfrak{P}^\bullet$, we denote by $\mathrm{Ctrtor}_*^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{P}^\bullet)$ the homology vector spaces

$$\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{P}^\bullet) = H^{-n}(\mathcal{N}^\bullet \odot_{\mathcal{D}} \mathfrak{F}^\bullet)$$

of the contratensor product of the complex $\mathcal{N}^\bullet$ with a bounded above complex of projective left $\mathcal{D}$-contramodules $\mathfrak{F}^\bullet$ quasi-isomorphic to the complex $\mathfrak{P}^\bullet$.

Here the bicomplex $\mathcal{N}^\bullet \odot_{\mathcal{D}} \mathfrak{F}^\bullet$ is presumed to be totalized by taking infinite direct sums along the diagonals. For any complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$, any bounded above complex of left $\mathcal{D}$-contramodules $\mathfrak{P}^\bullet$, and any k-vector space V there are natural isomorphisms of k-vector spaces

$$\mathrm{Hom}_k(\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{P}^\bullet), V) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{D}-\mathsf{contra})}(\mathfrak{P}^\bullet, \mathrm{Hom}_k(\mathcal{N}^\bullet, k)[n]).$$

A finite complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$ is said to have *contraflat dimension* $\leqslant d$ if one has $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{P}) = 0$ for all left $\mathcal{D}$-contramodules $\mathfrak{P}$ and all the integers $n > d$. The contraflat dimension of a finite complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$ is equal to the injective dimension of the finite complex of left $\mathcal{D}$-contramodules $\mathfrak{Q}^\bullet = \mathrm{Hom}_k(\mathcal{N}^\bullet, V)$ for any k-vector space $V \neq 0$.

Lemma 4.3. *If the coalgebra $\mathcal{D}$ is right cocoherent and left co-Noetherian, then the contraflat dimension of any finite complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$ does not exceed its projective dimension.*

Proof. Let d be the projective dimension of the complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$. For any finitely copresented right $\mathcal{D}$-comodule $\mathcal{L}$ there are natural isomorphisms of complexes of vector spaces

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{N}^\bullet, \mathcal{L}) \simeq \mathrm{Hom}^{\mathcal{D}}(\mathcal{L}^*, \mathcal{N}^{\bullet *}) \simeq (\mathcal{N}^\bullet \odot_{\mathcal{D}} \mathcal{L}^*)^*$$

(see Proposition C.9), implying natural isomorphisms of cohomology spaces

$$\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(\mathcal{D}-\mathsf{comod})}(\mathcal{N}^\bullet, \mathcal{L}[n]) \simeq \mathrm{Hom}_{\mathsf{D}^\mathsf{b}(\mathcal{D}-\mathsf{contra})}(\mathcal{L}^*, \mathcal{N}^{\bullet *}[n]) \simeq \mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathcal{L}^*)^*.$$

Since any finitely presented left $\mathcal{D}$-contramodule $\mathfrak{P}$ has the form $\mathcal{L}^*$ for a certain finitely copresented right $\mathcal{D}$-contramodule $\mathcal{L}$, it follows that the supremum of all integers n for which there exists a finitely presented left $\mathcal{D}$-contramodule $\mathfrak{P}$ with $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{P}) \neq 0$ does not exceed d.

Furthermore, by Lemma C.13(b) the functor of contratensor product $\odot_{\mathcal{D}}$ is isomorphic to the tensor product functor $\otimes_{\mathcal{D}^*}$ over the algebra $\mathcal{D}^*$ on the whole

categories of arbitrary right $\mathcal{D}$-comodules and left $\mathcal{D}$-contramodules. Besides, the free $\mathcal{D}$-contramodules are the direct summands of infinite products of copies of the $\mathcal{D}$-contramodule $\mathcal{D}^*$. Since the coalgebra $\mathcal{D}$ is left (co-Noetherian and consequently) cocoherent, the algebra $\mathcal{D}^*$ is right cocoherent, so infinite products of flat left $\mathcal{D}^*$-modules are flat. In particular, projective left $\mathcal{D}$-contramodules are flat as left modules over $\mathcal{D}^*$. It follows that the functor $\mathrm{Ctrtor}^{\mathcal{D}}$ is isomorphic to the derived functor $\mathrm{Tor}^{\mathcal{D}^*}$ of tensor product of (complexes of) $\mathcal{D}^*$-modules on the whole domain of definition of the former derived functor.

Finally, since the coalgebra $\mathcal{D}$ is left cocoherent, the algebra $\mathcal{D}^*$ is right coherent and the abelian category of left $\mathcal{D}^*$-modules is isomorphic to the abelian category of left $\mathcal{D}$-contramodules. Any left $\mathcal{D}^*$-module is a filtered inductive limit of finitely presented ones, and the functor of tensor product over $\mathcal{D}^*$ preserves filtered inductive limits. The homological dimension of the functor $\mathrm{Tor}^{\mathcal{D}}_*(\mathcal{N}^\bullet, -)$ on the abelian category of finitely presented left $\mathcal{D}^*$-modules does not exceed d, hence the homological dimension of this derived functor on the abelian category of arbitrary left $\mathcal{D}^*$-modules does not exceed d, either. $\qquad\square$

Now we finally come to the main definition of this section. Assume that the coalgebra $\mathcal{C}$ is left cocoherent and the coalgebra $\mathcal{D}$ is right cocoherent. A finite complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathcal{B}^\bullet$ is called a *dedualizing complex* for the pair of coalgebras $\mathcal{C}$ and $\mathcal{D}$ if the following conditions hold:

(i) the complex $\mathcal{B}^\bullet$ has finite projective dimension as a complex of left $\mathcal{C}$-comodules and finite contraflat dimension a complex of right $\mathcal{D}$-comodules;

(ii) the homothety maps $\mathcal{C}^* \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{comod}-\mathcal{D})}(\mathcal{B}^\bullet, \mathcal{B}^\bullet[*])$ and $\mathcal{D}^{*\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathcal{C}-\mathsf{comod})}(\mathcal{B}^\bullet, \mathcal{B}^\bullet[*])$ are isomorphisms of graded rings; and

(iii) the bicomodules of cohomology of the complex $\mathcal{B}^\bullet$ are finitely copresented left $\mathcal{C}$-comodules and finitely copresented right $\mathcal{D}$-comodules.

Here the notation $\mathsf{comod}-\mathcal{D}$ stands for the abelian category of right $\mathcal{D}$-comodules, and $\mathsf{D}^{\mathsf{b}}(\mathsf{comod}-\mathcal{D})$ is its bounded derived category. The dedualizing complex $\mathcal{B}^\bullet$ itself is viewed as an object of the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathcal{C}-\mathsf{comod}-\mathcal{D})$ of the abelian category $\mathcal{C}-\mathsf{comod}-\mathcal{D}$ of $\mathcal{C}$-$\mathcal{D}$-bicomodules.

The homothety maps are induced by the left action of the algebra $\mathcal{C}^*$ by right $\mathcal{D}$-comodule endomorphisms of (every term of) the complex $\mathcal{B}^\bullet$ and the right action of the algebra $\mathcal{D}^*$ by left $\mathcal{C}$-comodule endomorphisms of $\mathcal{B}^\bullet$.

We refer to the paper [27] and the references therein for a discussion of the classical notion of a dualizing complex over a pair of noncommutative rings, after which the above definition is largely modelled. A discussion of bicomodules can be found in [26, Section 2.6] and the references therein.

Example 4.4. Let R be a finitely generated commutative algebra over a field k and $I \subset R$ be a maximal ideal. Then the quotient algebras R/I^n are finite-dimensional, so their dual vector spaces are cocommutative coalgebras over k, as is their inductive limit $\mathcal{C} = \varinjlim_n (R/I^n)^*$. According to Examples C.12 or Lemma C.10(b), this

coalgebra is Artinian, and consequently, by Lemma C.3(a), co-Noetherian and co-coherent. The category of $\mathcal{C}$-comodules is isomorphic to the category of I-torsion R-modules in the sense of Section 1, $\mathcal{C}\text{–comod} \simeq R\text{–mod}_{I\text{-tors}}$. Moreover, the category of $\mathcal{C}$-contramodules is isomorphic to the category of I-contramodule R-modules as defined in Section 2, $\mathcal{C}\text{–contra} \simeq R\text{–mod}_{I\text{-ctra}}$ (see [26, Sections 2.1–2.3]).

Consider the derived functor of maximal I-torsion submodule $\mathbb{R}\Gamma_I\colon \mathsf{D}^+(R\text{–mod}) \longrightarrow \mathsf{D}^+(R\text{–mod}_{I\text{-tors}})$. According to Lemma 1.2(b), this functor takes $\mathsf{D}^{\mathsf{b}}(R\text{–mod})$ into $\mathsf{D}^{\mathsf{b}}(R\text{–mod}_{I\text{-tors}})$. Set $\mathcal{B}^\bullet = \mathbb{R}\Gamma_I(R)$. We claim that the complex $\mathcal{B}^\bullet$ (viewed as a complex of $\mathcal{C}$-$\mathcal{C}$-bicomodules in which the left and right coactions of $\mathcal{C}$ coincide) is a dedualizing complex for the pair of coalgebras $(\mathcal{C}, \mathcal{C})$.

Indeed, according to Lemma 4.3 it suffices to check the projective dimension condition in (i). The latter follows from Theorem 1.3 in the (easy) case $\star = \mathsf{b}$, as one has $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–comod})}(\mathcal{B}^\bullet, \mathcal{M}[*]) = \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–mod})}(\mathcal{B}^\bullet, \mathcal{M}[*])$ for any $\mathcal{C}$-comodule $\mathcal{M}$, while the complex $\mathcal{B}^\bullet$ clearly has finite projective dimension as a complex of R-modules, being isomorphic to the telescope complex $\mathrm{Tel}^\bullet(R, \mathbf{s})$ in the derived category $\mathsf{D}^{\mathsf{b}}(R\text{–mod})$ for any finite generating sequence $\mathbf{s}$ of the ideal $I \subset R$. The algebra $\mathcal{C}^*$ is the I-adic completion of the algebra R, so in view of Lemma 2.5 the property (ii) is implicit in Theorem 3.4, again in the case $\star = \mathsf{b}$ (see Example 5.8 below for further details).

Let us now prove the condition (iii). Let $I^n \subset I' \subset I$ be some ideal in R with the radical $\sqrt{I'} = I$, and let $\mathcal{E} = (R/I')^* \subset \mathcal{C}$ be the corresponding finite-dimensional subcoalgebra in $\mathcal{C}$. According to Lemma 4.1, it suffices to show that the complex $_\mathcal{E}^\mathbb{R}\mathcal{B}^\bullet$ has finite-dimensional cohomology comodules (it would be sufficient to consider the case $I' = I$, but we prefer a discussion in the larger generality). The functor Γ_I takes injective objects in $R\text{–mod}$ to injective objects in $R\text{–mod}_{I\text{-tors}}$, so the composition of derived functors $_\mathcal{E}^\mathbb{R}\mathbb{R}\Gamma_I$ is the derived functor of the composition of left exact functors $M \longmapsto {}_\mathcal{E}\Gamma_I(M)$.

The latter functor assigns to an R-module M its maximal R/I'-submodule, i. e., the submodule of all the elements annihilated by I'. It remains to notice that the functor $\mathrm{Ext}^*_R(R/I', M)$ takes finitely generated R-modules to cohomologically graded R-modules with finitely generated components, because it can be computed in terms of a left resolution of the R-module R/I' by finitely generated projective R-modules. Furthermore, any finitely generated R/I'-module is finite-dimensional.

For any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$ and any left $\mathcal{C}$-comodule $\mathcal{M}$, the k-vector space $\mathrm{Hom}_\mathcal{C}(\mathcal{K}, \mathcal{M})$ is endowed with the left $\mathcal{D}$-contramodule structure of a subcontramodule of the $\mathcal{D}$-contramodule $\mathrm{Hom}_k(\mathcal{K}, \mathcal{M})$. Similarly, for any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$ and any left $\mathcal{D}$-contramodule $\mathfrak{P}$, the contratensor product $\mathcal{K} \odot_\mathcal{D} \mathfrak{P}$ is endowed with the left $\mathcal{C}$-comodule structure of a quotient comodule of the left $\mathcal{C}$-comodule $\mathcal{K} \otimes_k \mathfrak{P}$. For any $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, any left $\mathcal{C}$-comodule $\mathcal{M}$, and any left $\mathcal{D}$-contramodule $\mathfrak{P}$, there is a natural adjunction isomorphism of k-vector spaces [26, Section 3.1]

$$\mathrm{Hom}_\mathcal{C}(\mathcal{K} \odot_\mathcal{D} \mathfrak{P}, \mathcal{M}) \simeq \mathrm{Hom}^\mathcal{D}(\mathfrak{P}, \mathrm{Hom}_\mathcal{C}(\mathcal{K}, \mathcal{M})).$$

The following theorem is the main result of this section.

Theorem 4.5. *Given a dedualizing complex $\mathcal{B}^\bullet$ for a left cocoherent coalgebra $\mathcal{C}$ and a right cocoherent coalgebra $\mathcal{D}$ over a field k, for any symbol $\star = \mathsf{b}, +, -,$ $\varnothing, \mathsf{abs}+, \mathsf{abs}-,$ or abs there is an equivalence of derived categories $\mathsf{D}^\star(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^\star(\mathcal{D}\text{–}\mathsf{contra})$ provided by mutually inverse functors $\mathbb{R}\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, -)$ and $\mathcal{B}^\bullet \odot_\mathcal{D}^\mathbb{L} -$.*

Proof. Assume for simplicity of notation that the complex $\mathcal{B}^\bullet$ is concentrated in nonpositive cohomological degrees. Let d be an integer greater or equal to both the projective dimension of the complex $\mathcal{B}^\bullet$ viewed as a complex of left $\mathcal{C}$-comodules and the contraflat dimension of $\mathcal{B}^\bullet$ as a complex of right $\mathcal{D}$-comodules.

To construct the image of a complex of left $\mathcal{C}$-comodules $\mathcal{M}^\bullet$ under the functor $\mathbb{R}\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, -)$, one has to choose an exact sequence of complexes of left $\mathcal{C}$-comodules $0 \longrightarrow \mathcal{M}^\bullet \longrightarrow \mathcal{J}^{0,\bullet} \longrightarrow \mathcal{J}^{1,\bullet} \longrightarrow \cdots$ with injective left $\mathcal{C}$-comodules $\mathcal{J}^{j,i}$. Then one applies the functor $\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, -)$ to every complex $0 \longrightarrow \mathcal{J}^{0,i} \longrightarrow \mathcal{J}^{1,i} \longrightarrow \mathcal{J}^{2,i} \longrightarrow \cdots$, obtaining a nonnegatively graded complex of left $\mathcal{D}$-contramodules $0 \longrightarrow \mathfrak{P}^{0,i} \longrightarrow \mathfrak{P}^{1,i} \longrightarrow \mathfrak{P}^{2,i} \longrightarrow \cdots$. According to the projective dimension condition on the dedualizing complex $\mathcal{B}^\bullet$, the complex $\mathfrak{P}^{\bullet,i}$ has zero cohomology contramodules at the cohomological degrees above d; so it is quasi-isomorphic to its canonical truncation complex $\tau_{\leqslant d}\mathfrak{P}^{\bullet,i}$. By the definition, one sets the object $\mathbb{R}\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, \mathcal{M}^\bullet)$ in the derived category $\mathsf{D}^\star(\mathcal{D}\text{–}\mathsf{contra})$ to be represented by the total complex of the bicomplex $\tau_{\leqslant d}\mathfrak{P}^{\bullet,\bullet}$ concentrated in the cohomological degrees $0 \leqslant j \leqslant d$ and $i \in \mathbb{Z}$.

Similarly, to construct the image of a complex of left $\mathcal{D}$-contramodules $\mathfrak{P}^\bullet$ under the functor $\mathcal{B}^\bullet \odot_\mathcal{D}^\mathbb{L} -$, one has to choose an exact sequence of complexes of left $\mathcal{D}$-contramodules $\cdots \longrightarrow \mathfrak{F}^{-1,\bullet} \longrightarrow \mathfrak{F}^{0,\bullet} \longrightarrow \mathfrak{P}^\bullet \longrightarrow 0$ with projective left $\mathcal{C}$-contramodules $\mathfrak{F}^{j,i}$. Then one applies the functor $\mathcal{B}^\bullet \odot_\mathcal{D} -$ to every complex $\cdots \longrightarrow \mathfrak{F}^{-2,i} \longrightarrow \mathfrak{F}^{-1,i} \longrightarrow \mathfrak{F}^{0,i} \longrightarrow 0$, obtaining a nonpositively graded complex of left $\mathcal{C}$-comodules $\cdots \longrightarrow \mathcal{M}^{-2,i} \longrightarrow \mathcal{M}^{-1,i} \longrightarrow \mathcal{M}^{0,i} \longrightarrow 0$. According to the contraflat dimension condition on the complex $\mathcal{B}^\bullet$, the complex $\mathcal{M}^{\bullet,i}$ has zero cohomology comodules at the cohomological degrees below $-d$; so it is quasi-isomorphic to its canonical truncation complex $\tau_{\geqslant -d}\mathcal{M}^{\bullet,i}$. One sets the object $\mathcal{B}^\bullet \odot_\mathcal{D}^\mathbb{L} \mathfrak{P}^\bullet$ in the derived category $\mathsf{D}^\star(\mathcal{C}\text{–}\mathsf{comod})$ to be represented by the total complex of the bicomplex $\tau_{\geqslant -d}(\mathcal{M}^{\bullet,\bullet})$ concentrated in the cohomological degrees $-d \leqslant j \leqslant 0$ and $i \in \mathbb{Z}$.

These constructions of two derived functors are but particulars cases of the construction of a derived functor of finite homological dimension spelled out in Appendix B. According to the results of that appendix, the above constructions produce well-defined triangulated functors $\mathbb{R}\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, -)\colon \mathsf{D}^\star(\mathcal{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^\star(\mathcal{D}\text{–}\mathsf{contra})$ and $\mathcal{B}^\bullet \odot_\mathcal{D}^\mathbb{L} -\colon \mathsf{D}^\star(\mathcal{D}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^\star(\mathcal{C}\text{–}\mathsf{comod})$ for any derived category symbol $\star = \mathsf{b},$ $+, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-,$ or abs. Moreover, the former functor is right adjoint to the latter one. All these assertions only depend on the first condition (i) in the definition of a dedualizing complex.

It remains to prove that the adjunction morphisms are isomorphisms. Since the total complexes of finite acyclic complexes of complexes are absolutely acyclic, in order to check that the morphism $\mathfrak{P}^\bullet \longrightarrow \mathbb{R}\operatorname{Hom}_\mathcal{C}(\mathcal{B}^\bullet, \mathcal{B}^\bullet \odot_\mathcal{D}^\mathbb{L} \mathfrak{P}^\bullet)$ is an isomorphism in the derived category $\mathsf{D}^\star(\mathcal{D}\text{–}\mathsf{contra})$ for all the $\star$-bounded complexes of left $\mathcal{D}$-contramodules $\mathfrak{P}^\bullet$ it suffices to consider the case of a one-term complex $\mathfrak{P}^\bullet = \mathfrak{P}$

corresponding to a single $\mathcal{D}$-contramodule $\mathfrak{P}$. Furthermore, since a morphism in $\mathsf{D}^{\mathsf{b}}(\mathcal{D}\text{–}\mathsf{contra})$ is an isomorphism whenever it is an isomorphism in $\mathsf{D}^{-}(\mathcal{D}\text{–}\mathsf{contra})$, one can view the one-term complex $\mathfrak{P}$ as an object of the bounded above derived category $\mathsf{D}^{-}(\mathcal{D}\text{–}\mathsf{contra})$ and replace it with a free $\mathcal{D}$-contramodule resolution $\mathfrak{F}^{\bullet}$ of the contramodule $\mathfrak{P}$. Applying the same totalization argument to the complex $\mathfrak{F}^{\bullet}$, the question reduces to proving that the adjunction morphism $\mathfrak{F} \longrightarrow \mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{B}^{\bullet} \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F})$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathcal{D}\text{–}\mathsf{contra})$ for any free left $\mathcal{D}$-contramodule $\mathfrak{F}$.

So let V be a k-vector space and $\mathfrak{F} = \operatorname{Hom}_{k}(\mathcal{D}, V)$ be the free left $\mathcal{D}$-contramodule generated by V; then one has $\mathcal{B}^{\bullet} \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F} = \mathcal{B}^{\bullet} \odot_{\mathcal{D}} \mathfrak{F} = \mathcal{B}^{\bullet} \otimes_{k} V$. By the condition (iii) together with Lemma 4.2, there exists a bounded below complex of finitely cogenerated cofree left $\mathcal{C}$-comodules $\mathcal{J}^{\bullet}$ together with a quasi-isomorphism of complexes of left $\mathcal{C}$-comodules $\mathcal{B}^{\bullet} \longrightarrow \mathcal{J}^{\bullet}$. We have to check that the natural map

$$\operatorname{Hom}_{k}(\mathcal{D}, V) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{J}^{\bullet} \otimes_{k} V)$$

is a quasi-isomorphism of complexes of left $\mathcal{D}$-contramodules.

The left-hand side is the projective limit of the vector spaces $\operatorname{Hom}_{k}(\mathcal{E}, V)$ over all the finite-dimensional subcoalgebras $\mathcal{E} \subset \mathcal{D}$, while the right-hand side is the projective limit of the complexes of vector spaces $\operatorname{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathcal{B}^{\bullet}, {}_{\mathcal{E}}\mathcal{J}^{\bullet} \otimes_{k} V)$. In particular, the map $\mathcal{D}^{*} \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{J}^{\bullet})$ is a morphism of complexes of profinite-dimensional topological vector spaces. Being a quasi-isomorphism of complexes of discrete/nontopological vector spaces (with the topologies forgotten) by the condition (ii), it is consequently also a quasi-isomorphism of complexes in the abelian category of profinite-dimensional topological vector spaces.

For any profinite-dimensional topological k-vector space K and any discrete k-vector space V one denotes by $K \otimes^{\widehat{}} V$ the projective limit

$$K \otimes^{\widehat{}} V = \varprojlim_{U} K/U \otimes_{k} V$$

taken over all the open subspaces $U \subset K$ [26, Sections 2.3-4]. Equivalently, one can set $W^{*} \otimes^{\widehat{}} V = \operatorname{Hom}_{k}(W, V)$ for any discrete k-vector spaces W and V. Both the abelian categories of dicrete and profinite-dimensional vector spaces being semisimple, the additive functor $\otimes^{\widehat{}}$ is exact. Now one has

$$\varprojlim_{\mathcal{E}} \operatorname{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathcal{B}^{\bullet}, {}_{\mathcal{E}}\mathcal{J}^{\bullet} \otimes_{k} V) \simeq \varprojlim_{\mathcal{E}} \operatorname{Hom}_{\mathcal{E}}({}_{\mathcal{E}}\mathcal{B}^{\bullet}, {}_{\mathcal{E}}\mathcal{J}^{\bullet}) \otimes^{\widehat{}} V,$$

since the complex of $\mathcal{E}$-comodules ${}_{\mathcal{E}}\mathcal{B}^{\bullet}$ is finite, while the terms of the complex ${}_{\mathcal{E}}\mathcal{J}^{\bullet}$ are finite-dimensional cofree $\mathcal{E}$-comodules. Finally, the morphism of complexes in question is obtained by applying the exact functor $-\otimes^{\widehat{}} V$ to the quasi-isomorphism of complexes $\mathcal{D}^{*} \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{J}^{\bullet})$.

Similarly, in order to prove that the adjunction morphism $\mathcal{B}^{\bullet} \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{M}^{\bullet})$ $\longrightarrow \mathcal{M}^{\bullet}$ is an isomorphism in the derived category $\mathsf{D}^{\star}(\mathcal{C}\text{–}\mathsf{comod})$ for any $\star$-bounded complex of left $\mathcal{C}$-comodules $\mathcal{M}^{\bullet}$, it suffices to check that this morphism is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–}\mathsf{comod})$ for any cofree left $\mathcal{C}$-comodule $\mathcal{J}$ viewed as a one-term complex in $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–}\mathsf{comod})$. Let $\mathcal{J} = \mathcal{C} \otimes_{k} V$ be a cofree left $\mathcal{C}$-comodule generated by a k-vector space V; then one has $\mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{J}) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{B}^{\bullet}, \mathcal{J}) = \operatorname{Hom}_{k}(\mathcal{B}^{\bullet}, V)$. Let $\mathcal{J}^{\bullet}$ be a bounded below complex of finitely cogenerated cofree right $\mathcal{D}$-comodules

endowed with a quasi-isomorphism of complexes of right $\mathcal{D}$-comodules $\mathcal{B}^\bullet \longrightarrow \mathcal{J}^\bullet$. Then $\operatorname{Hom}_k(\mathcal{J}^\bullet, V)$ is a bounded above complex of free left $\mathcal{D}$-contramodules quasi-isomorphic to $\operatorname{Hom}_k(\mathcal{B}^\bullet, V)$. We have to show that the map

$$\mathcal{B}^\bullet \odot_\mathcal{D} \operatorname{Hom}_k(\mathcal{J}^\bullet, V) \longrightarrow \mathcal{C} \otimes_k V$$

induced by the left $\mathcal{C}$-coaction in $\mathcal{B}^\bullet$ is a quasi-isomorphism (of complexes of left $\mathcal{C}$-comodules).

The functors on both sides of our map preserve infinite direct sums and inductive limits in the argument V, so it suffices to consider the case $V = k$. Passing to the dual vector spaces, we have to check that the map $\mathcal{C}^* \longrightarrow \operatorname{Hom}^\mathcal{D}(\mathcal{J}^{\bullet*}, \mathcal{B}^{\bullet*})$ is a quasi-isomorphism. The latter map is the composition $\mathcal{C}^* \longrightarrow \operatorname{Hom}_{\mathcal{D}^{\mathrm{op}}}(\mathcal{B}^\bullet, \mathcal{J}^\bullet) \longrightarrow \operatorname{Hom}^\mathcal{D}(\mathcal{J}^{\bullet*}, \mathcal{B}^{\bullet*})$ of the homothety map of the condition (ii) and the map induced by the dualization functor $\mathcal{N} \longmapsto \mathcal{N}^*$. It remains to apply Proposition C.9(b). $\qquad\square$

5. Dedualizing Complexes of Torsion Modules

Let I be a finitely generated ideal in a commutative ring R, and let $s_1, \ldots, s_m$ be a sequence of its generators, denoted by $\mathbf{s}$ for brevity. For any element $s \in R$, the complex $T^\bullet(R, s)$ from Section 2 is the union of its subcomplexes of free R-modules $T_n^\bullet(R, s)$ generated by the symbols $\epsilon_0, \ldots, \epsilon_{n-1}$ in degree 0 and $\delta_1, \ldots, \delta_n$ in degree 1. Accordingly, the complex $T^\bullet(R, \mathbf{s})$ is the union of its subcomplexes of finitely generated free R-modules $T_n^\bullet(R, s_1) \otimes_R \otimes \cdots \otimes_R T_n^\bullet(R, s_m)$. The complex $T_n^\bullet(R, \mathbf{s})$ is homotopy equivalent to the complex $\operatorname{Tel}_n^\bullet(R, \mathbf{s})$.

Let us denote the dual complexes by $K_\bullet^n(R, \mathbf{s}) = \operatorname{Hom}_R(T_n^\bullet(R, \mathbf{s}), R)$. In particular, $K_\bullet^1(R, \mathbf{s})$ is the conventional Koszul complex associated with the sequence of elements $s_1, \ldots, s_m \in R$, while the complex $K_\bullet^n(R, \mathbf{s})$ is quasi-isomorphic to the Koszul complex $K_\bullet^1(R, \mathbf{s}^n)$ associated with the sequence of elements $s_1^n, \ldots, s_m^n \in R$.

The Thomason–Trobaugh–Neeman theory of compactly generated triangulated categories is fairly widely known by now (see, e. g., [16] or [28]). The following one is an old result.

Proposition 5.1. *For any given $n \geqslant 1$, the complex $K_\bullet^n(R, \mathbf{s})$ is a compact generator of the triangulated subcategory $\mathsf{D}_{I\text{-tors}}(R\text{–mod}) \subset \mathsf{D}(R\text{–mod})$ of complexes with I-torsion cohomology modules in the derived category of R-modules. A complex of R-modules with I-torsion cohomology modules is a compact object of $\mathsf{D}_{I\text{-tors}}(R\text{–mod})$ if and only if it is a compact object of $\mathsf{D}(R\text{–mod})$.*

Proof. The first assertion goes back to [2, Proposition 6.1]; see also [28, Proposition 6.6]. It suffices to consider the case $n = 1$. One can easily see that the complexes $K_\bullet^n(R, \mathbf{s})$, viewed as objects of the homotopy category $\mathsf{Hot}(R\text{–mod})$, can be obtained from the complex $K_\bullet^1(R, \mathbf{s})$ using the operations of shift and cone. Let $M^\bullet$ be a complex of R-modules with I-torsion cohomology modules. Then the inductive limit of the complexes $\operatorname{Hom}_R(K_\bullet^n(R, \mathbf{s}), M^\bullet) \simeq T_n^\bullet(R, \mathbf{s}) \otimes_R M^\bullet$ is quasi-isomorphic to the complex $M^\bullet$ by Lemma 1.1(c). Hence the complex $M^\bullet$ is acyclic whenever all the

complexes $\mathrm{Hom}_R(K^n_\bullet(R, \mathbf{s}), M^\bullet)$ are. The complexes $K^n_\bullet(R, \mathbf{s})$ are compact objects in $\mathsf{D}_{I\text{-tors}}(R\text{–mod})$, since they are compact in $\mathsf{D}(R\text{–mod})$. This proves the first assertion. It follows that the compact objects of $\mathsf{D}_{I\text{-tors}}(R\text{–mod})$ are precisely those objects that can be obtained from $K^1_\bullet(R, \mathbf{s})$ using the operations of shift, cone, and the passage to a direct summand, implying the second one (cf. [19, Lemma 5.3]). $\qquad\square$

Lemma 5.2. (a) *For any I-torsion R-module M, the natural morphism of complexes of R-modules*

$$T^\bullet(R, \mathbf{s}) \otimes_R \mathrm{Hom}_R(T^\bullet(R, \mathbf{s}), M) \longrightarrow M$$

is a quasi-isomorphism.

(b) *For any I-contramodule R-module P, the natural morphism of complexes of R-modules*

$$P \longrightarrow \mathrm{Hom}_R(T^\bullet(R, \mathbf{s}),\ T^\bullet(R, \mathbf{s}) \otimes_R P)$$

is a quasi-isomorphism.

Proof. These computations are implicit in the proof of Theorem 3.4. In view of Lemma 1.1(c), part (a) generalizes to the assertion that for any R-module M the morphism

$$T^\bullet(R, \mathbf{s}) \otimes_R M \longrightarrow T^\bullet(R, \mathbf{s}) \otimes_R \mathrm{Hom}_R(T^\bullet(R, \mathbf{s}), M)$$

induced by the natural morphism of complexes $T^\bullet(R, \mathbf{s}) \longrightarrow R$ is a quasi-isomorphism. This is essentially [18, Lemma 7.6]. One shows that the morphism of complexes

$$T^\bullet_n(R, \mathbf{s}) \otimes_R M \longrightarrow \mathrm{Hom}_R(T^\bullet(R, \mathbf{s}),\ T^\bullet_n(R, \mathbf{s}) \otimes_R M)$$

is a quasi-isomorphism for every $n \geqslant 1$. A cocone of the latter morphism is a complex computing the R-modules of morphisms $\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(R\text{–mod})}(C^\bullet_\mathbf{s}(R),\ T^\bullet_n(R, \mathbf{s}) \otimes_R M[*])$ in the derived category $\mathsf{D}^\mathsf{b}(R\text{–mod})$. So it suffices to check that the R-modules $\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(R\text{–mod})}(C^i_\mathbf{s}(R),\ T^\bullet_n(R, \mathbf{s}) \otimes_R M[*])$ vanish for every $i \geqslant 0$. Indeed, the R-module $C^i_\mathbf{s}(R)$ is a finite direct sum of R-modules in each of which one of the elements s_j acts invertibly, while in the complex $T^\bullet_n(R, \mathbf{s}) \otimes_R M$ all the elements s^n_j act contractibly.

Similarly, in view of Lemma 2.2(c), part (b) generalizes to the assertion that for any R-module M the morphism

$$\mathrm{Hom}_R(T^\bullet(R, \mathbf{s}),\ T^\bullet(R, \mathbf{s}) \otimes_R M) \longrightarrow T^\bullet(R, \mathbf{s}) \otimes_R M$$

induced by the morphism of complexes $T^\bullet(R, \mathbf{s}) \longrightarrow R$ is a quasi-isomorphism. This is essentially [18, Lemma 7.2]. A cocone of the morphism $T^\bullet(R, \mathbf{s}) \otimes_R M \longrightarrow M$ is quasi-isomorphic to the complex $C^\bullet_\mathbf{s}(M)$, so it suffices to check that the complex $\mathrm{Hom}_R(T^\bullet(R, \mathbf{s}), C^\bullet_\mathbf{s}(M))$ is acyclic. Indeed, let us show that the complex $\mathrm{Hom}_R(T^\bullet(R, \mathbf{s}), C^i_\mathbf{s}(M))$ is acyclic for every $i \geqslant 0$. By Lemma 2.2(a), the cohomology modules of this complex are finite direct sums of R-modules each of which is simultaneously an s_j-contramodule and an $R[s^{-1}_j]$-module for one of the elements s_j. All such modules vanish. $\qquad\square$

Let $\mathfrak{R} = \varprojlim_n R/I^n$ denote the I-adic completion of the ring R. The ring $\mathfrak{R}$ is complete and separated topological ring in its projective limit topology, which

coincides with its I-adic topology. An I-torsion R-module is the same thing as a discrete $\mathfrak{R}$-module. For any set X, we denote by $\mathfrak{R}[[X]] = \varprojlim_n R/I^n[X]$ the R-module of all maps of sets $f \colon X \longrightarrow \mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$ (i. e., for any open subset $U \subset \mathfrak{R}$ one has $f(x) \in U$ for all but a finite number of elements $x \in X$). As any R-module that is separated and complete in its I-adic topology, $\mathfrak{R}[[X]]$ is an I-contramodule (see Section 2).

For the rest of the section we assume that the ideal $I \subset R$ is weakly proregular. Then the R-module $\mathfrak{R}[[X]]$ is called the *free I-contramodule R-module generated by the set X* (cf. [26, Section 2.1]). Part (a) of the next lemma justifies the terminology.

Lemma 5.3. (a) *For any I-contramodule R-module P there is a bijective correspondence between the R-module morphisms $\mathfrak{R}[[X]] \longrightarrow P$ and the maps of sets $X \longrightarrow P$. Free I-contramodule R-modules are projective objects in the abelian category R–$\mathsf{mod}_{I\text{-}\mathsf{ctra}}$, there are enough of them, and any projective I-contramodule R-module is a direct summand of a free one.*

(b) *For any set X and any integer $n \geqslant 1$, the morphisms of complexes*

$$C_{\mathbf{s}}^{\bullet}(R[X])^{\sim} \longrightarrow C_{\mathbf{s}}^{\bullet}(\mathfrak{R}[[X]])^{\sim} \quad \text{and} \quad T_n^{\bullet}(R,\mathbf{s}) \otimes_R R[X] \longrightarrow T_n^{\bullet}(R,\mathbf{s}) \otimes_R \mathfrak{R}[[X]]$$

induced by the completion map $R[X] \longrightarrow \mathfrak{R}[[X]]$ are quasi-isomorphisms.

(c) *For any set X, the morphism of complexes*

$$\operatorname{Hom}_R(T^{\bullet}(R,\mathbf{s}), R[X]) \longrightarrow \operatorname{Hom}_R(T^{\bullet}(R,\mathbf{s}), \mathfrak{R}[[X]])$$

induced by the completion map $R[X] \longrightarrow \mathfrak{R}[[X]]$ is a quasi-isomorphism.

Proof. Part (a): by Lemma 2.5, one has $\Delta_I(R[X]) \simeq \mathfrak{R}[[X]]$, so $\operatorname{Hom}_R(R[X], P) \simeq \operatorname{Hom}_R(\mathfrak{R}[[X]], P)$ for any R-contramodule P. The remaining assertions immediately follow. Parts (b-c): by Lemmas 2.5 and 2.7(a), the complex $\operatorname{Hom}_R(T^{\bullet}(R,\mathbf{s}), R[X])$ is quasi-isomorphic to the R-module $\mathfrak{R}[[X]]$ and the morphism of complexes $R[X] \longrightarrow \operatorname{Hom}_R(T^{\bullet}(R,\mathbf{s}), R[X])$ computes the completion morphism $R[X] \longrightarrow \mathfrak{R}[[X]]$. Now part (b) is provided by the proof of Lemma 5.2(a) and part (c) follows from Lemma 3.2(a). $\qquad\square$

It was shown in [32, Theorem 3.4] and [23, Proposition B.9.1] (see also [20, Theorem 1.5(2)] and [24, Proposition C.5.4]) that the free I-contramodule R-modules $\mathfrak{R}[[X]]$ are flat R-modules when the ring R is Noetherian. Part (c) of the following lemma is a weak generalization of this result to the weakly proregular case.

Lemma 5.4. (a) *For any finite complex of finitely generated projective R-modules $K^{\bullet}$ with I-torsion cohomology modules, the complex $\operatorname{Hom}_R(K^{\bullet}, R)$ also has I-torsion cohomology modules. Moreover, for any complex of R-modules $M^{\bullet}$, the complex $\operatorname{Hom}_R(K^{\bullet}, M^{\bullet})$ has I-torsion cohomology modules.*

(b) *For any finite complex of finitely generated projective R-modules $K^{\bullet}$ with I-torsion cohomology modules and any set X, the morphism of complexes of R-modules*

$$K^{\bullet} \otimes_R R[X] \longrightarrow K^{\bullet} \otimes_R \mathfrak{R}[[X]]$$

induced by the completion map $R[X] \longrightarrow \mathfrak{R}[[X]]$ is a quasi-isomorphism.

(c) *Let $K^\bullet \longrightarrow M^\bullet$ be a quasi-isomorphism between a complex of I-torsion R-modules $M^\bullet$ and a finite complex of finitely generated projective R-modules $K^\bullet$ with I-torsion cohomology modules. Then for any set X the induced morphism*

$$K^\bullet \otimes_R \mathfrak{R}[[X]] \longrightarrow M^\bullet \otimes_R \mathfrak{R}[[X]]$$

is a quasi-isomorphism of complexes of R-modules.

Proof. Part (a) does not depend on the weak proregularity assumption yet. To prove it, choose a finite sequence of elements $\mathbf{s}$ generating the ideal $I \subset R$ and an integer $n \geqslant 1$. By Proposition 5.1, the complex $K^\bullet$, viewed as an object of the homotopy category $\mathsf{Hot}(R\text{–}\mathsf{mod})$, is a direct summand of a complex obtained from the complex $K_\bullet^n(R, \mathbf{s})$ using the operations of shift and cone. Since the complex $K_\bullet^n(R, \mathbf{s})$ is self-dual up to a shift, the first assertion follows. Moreover, all the cohomology modules of the complex $K_\bullet^n(R, \mathbf{s})$ are annihilated by a large enough power of the ideal I, and consequently so are all the cohomology modules of the complex $K^\bullet$ and every cohomology module of the complex $\mathrm{Hom}_R(K^\bullet, M^\bullet)$.

By virtue of the same argument based on Proposition 5.1, part (b) follows from Lemma 5.3(b). To prove part (c), notice that the morphism $K^\bullet \otimes_R R[X] \longrightarrow M^\bullet \otimes_R R[X]$ is a quasi-isomorphism because the R-module $R[X]$ is flat, the morphism $M^\bullet \otimes_R R[X] \longrightarrow M^\bullet \otimes_R \mathfrak{R}[[X]]$ is an isomorphism of complexes of R-modules since $M^\bullet$ is a complex of I-torsion R-modules, and the morphism $K^\bullet \otimes_R R[X] \longrightarrow K^\bullet \otimes_R \mathfrak{R}[[X]]$ is a quasi-isomorphism by part (b). $\qquad\square$

Clearly, the abelian category of I-torsion R-modules $R\text{–}\mathsf{mod}_{I\text{-tors}}$ has enough injective objects. Moreover, the maximal I-torsion submodule $\Gamma_I(J)$ of any injective R-module J is an injective object of $R\text{–}\mathsf{mod}_{I\text{-tors}}$. There are enough injective objects of this particular form in $R\text{–}\mathsf{mod}_{I\text{-tors}}$, so any injective object of $R\text{–}\mathsf{mod}_{I\text{-tors}}$ a direct summand of the R-module $\Gamma_I(J)$ for a certain injective R-module J.

For a Noetherian ring R, it follows from the Artin–Rees lemma that any injective object of the category $R\text{–}\mathsf{mod}_{I\text{-tors}}$ is simultaneously an injective object of the category $R\text{–}\mathsf{mod}$. Part (b) of the next lemma can be sometimes used in lieu of this assertion in the case of a weakly proregular ring R.

Lemma 5.5. (a) *Let J be an injective R-module and $H = \Gamma_I(J)$ be its maximal I-torsion submodule. Then for any bounded above complex of projective R-modules $L^\bullet$ with I-torsion cohomology modules the morphism of complexes of R-modules*

$$\mathrm{Hom}_R(L^\bullet, H) \longrightarrow \mathrm{Hom}_R(L^\bullet, J)$$

induced by the embedding map $H \longrightarrow J$ is a quasi-isomorphism.

(b) *Let $L^\bullet \longrightarrow M^\bullet$ be a quasi-isomorphism between a complex of I-torsion R-modules $M^\bullet$ and a bounded above complex of projective R-modules $L^\bullet$ with I-torsion cohomology modules. Then for any injective object H of the abelian category of I-torsion R-modules the induced morphism*

$$\mathrm{Hom}_R(M^\bullet, H) \longrightarrow \mathrm{Hom}_R(L^\bullet, H)$$

is a quasi-isomorphism of complexes of R-modules.

Proof. Part (a) is implicit in the adjunction of derived functors from the proof of Theorem 1.3 together with the result of Corollary 1.4. To give a direct proof, notice that by the weak proregularity assumption on the ideal I the complex $C_{\mathbf{s}}^\bullet(J)^\sim$ is quasi-isomorphic to the R-module H and the morphism of complexes $C_{\mathbf{s}}^\bullet(J)^\sim \longrightarrow J$ computes the embedding morphism $H \longrightarrow J$. The quasi-isomorphism of complexes of R-modules $H \longrightarrow C_{\mathbf{s}}^\bullet(J)^\sim$ induces a quasi-isomorphism $\mathrm{Hom}_R(L^\bullet, H) \longrightarrow \mathrm{Hom}_R(L^\bullet, C_{\mathbf{s}}^\bullet(J)^\sim)$, so it remains to show that the complex $\mathrm{Hom}_R(L^\bullet, C_{\mathbf{s}}^\bullet(J))$ is acyclic. Here one argues as in the proof of Lemma 5.2(b) together with the proof of Lemma 2.2(a). For every $i \geqslant 0$, the cohomology modules of the complex $\mathrm{Hom}_R(L^\bullet, C_{\mathbf{s}}^\bullet(J))$ are I-contramodule R-modules, since the complex $L^\bullet \otimes_R T^\bullet(R, s)'$ is contractible for any $s \in I$ as a bounded above complex of projective R-modules quasi-isomorphic to the acyclic complex $L^\bullet[s^{-1}]$. Being at the same time isomorphic to finite direct sums of R-modules each of which is an $R[s_j^{-1}]$-module for one of the elements s_j, these cohomology modules have to vanish.

Part (b) is deduced in the way similar to the proof of Lemma 5.4(c). One can assume that $H = \Gamma_I(J)$ for some injective R-module J, as there are enough injective I-torsion R-modules of this form. Then the morphism $\mathrm{Hom}_R(M^\bullet, J) \longrightarrow \mathrm{Hom}_R(L^\bullet, J)$ is a quasi-isomorphism because the R-module J is injective, the morphism $\mathrm{Hom}_R(M^\bullet, H) \longrightarrow \mathrm{Hom}_R(M^\bullet, J)$ is an isomorphism, and the morphism $\mathrm{Hom}_R(L^\bullet, H) \longrightarrow \mathrm{Hom}_R(L^\bullet, J)$ is a quasi-isomorphism by part (a). $\square$

Remark 5.6. In other words, Lemma 5.5(b) simply means that the Ext modules $\mathrm{Ext}_R^i(M, H)$ computed in the abelian category of R-modules between an I-torsion R-module M and an injective I-torsion R-module H vanish for all $i > 0$. This is but a particular case of Theorem 1.3. Similarly, it follows from Theorem 2.9 that $\mathrm{Ext}_R^i(\mathfrak{R}[[X]], P) = 0$ for any I-contramodule R-module P and all $i > 0$. Taking into account the natural isomorphism $\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, N), J) \simeq \mathrm{Ext}_R^i(N, \mathrm{Hom}_R(M, J))$ for an injective R-module J and any R-modules M, N, together with the fact that the R-module $\mathrm{Hom}_R(M, J)$ is an I-contramodule for any I-torsion R-module M and any R-module J (see Section 2), one concludes that $\mathrm{Tor}_i^R(M, \mathfrak{R}[[X]]) = 0$ for any I-torsion R-module M and any set X. This is a stronger version of Lemma 5.4(c).

As in Section 4, a finite complex of I-torsion R-modules $L^\bullet$ is said to have *projective dimension* $\leqslant d$ if one has $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R-\mathsf{mod}_{I\text{-tors}})}(L^\bullet, M) = 0$ for all I-torsion R-modules M and all the integers $n > d$.

By Lemma 5.3(a), the bounded above derived category $\mathsf{D}^-(R-\mathsf{mod}_{I\text{-ctra}})$ is equivalent to the homotopy category of bounded above complexes of projective I-contramodule R-modules. Given a complex of I-torsion R-modules $M^\bullet$ and a bounded above complex of I-contramodule R-modules $P^\bullet$, we denote by $\mathrm{Ctrtor}_*^{(R,I)}(M^\bullet, P^\bullet)$ the homology R-modules

$$\mathrm{Ctrtor}_n^{(R,I)}(M^\bullet, P^\bullet) = H^{-n}(M^\bullet \otimes_R F^\bullet)$$

of the tensor product of the complex of I-torsion R-modules $M^\bullet$ with a bounded above complex of projective I-contramodule R-modules $F^\bullet$ quasi-isomorphic to the complex $P^\bullet$.

A finite complex of I-torsion R-modules $L^\bullet$ is said to have *contraflat dimension* $\leqslant d$ if one has $\mathrm{Ctrtor}_n^{(R,I)}(L^\bullet, P) = 0$ for all I-contramodule R-modules P and all the integers $n > d$. The next one is the main definition of this section.

A finite complex of I-torsion R-modules $B^\bullet$ is called a *dedualizing complex* for the ideal $I \subset R$ if the following conditions hold:

(i) the complex $B^\bullet$ has finite projective and contraflat dimensions as a complex of I-torsion R-modules;

(ii) the homothety map $\mathfrak{R} \longrightarrow \mathrm{Hom}_{\mathsf{D}^\mathsf{b}(R-\mathsf{mod}_{I\text{-tors}})}(B^\bullet, B^\bullet[*])$ is an isomorphism of graded rings;

(iii) for any finite complex of finitely generated projective R-modules $K^\bullet$ with I-torsion cohomology modules, the complex $\mathrm{Hom}_R(K^\bullet, B^\bullet)$ is a compact object of the derived category $\mathsf{D}(R-\mathsf{mod}_{I\text{-tors}})$.

The dedualizing complex $B^\bullet$ is viewed as an object of the derived category $\mathsf{D}^\mathsf{b}(R-\mathsf{mod}_{I\text{-tors}})$.

Theorem 5.7. *Let I be a weakly proregular finitely generated ideal in a commutative ring R. Then for any finite complex of I-torsion R-modules $B^\bullet$ satisfying the conditions (ii-iii) there is an involutive anti-equivalence on the full subcategory of compact objects in $\mathsf{D}(R-\mathsf{mod}_{I\text{-tors}})$ provided by a derived functor $\mathbb{R}\,\mathrm{Hom}_R(-, B^\bullet)$.*

Proof. By Theorem 1.3 and Corollary 1.4, one has $\mathsf{D}(R-\mathsf{mod}_{I\text{-tors}}) = \mathsf{D}_{I\text{-tors}}(R-\mathsf{mod}) \subset \mathsf{D}(R-\mathsf{mod})$. According to Proposition 5.1, any compact object of $\mathsf{D}_{I\text{-tors}}(R-\mathsf{mod})$ is also compact in $\mathsf{D}(R-\mathsf{mod})$, so the full subcategory of compact objects in $\mathsf{D}_{I\text{-tors}}(R-\mathsf{mod})$ is equivalent to the homotopy category of finite complexes of finitely generated projective R-modules with I-torsion cohomology modules (see [1, Theorem 2.8]). To compute the image of a compact object $M^\bullet \in \mathsf{D}(R-\mathsf{mod}_{I\text{-tors}})$ under the functor $\mathbb{R}\,\mathrm{Hom}_R(-, B^\bullet)$, one chooses a finite complex of finitely generated projective R-modules $K^\bullet$ together with a quasi-isomorphism of complexes of R-modules $K^\bullet \longrightarrow M^\bullet$ and applies the functor $\mathrm{Hom}_R(-, B^\bullet)$ to the complex $K^\bullet$. The natural isomorphism of complexes

$$\mathrm{Hom}_R(L^\bullet, \mathrm{Hom}_R(K^\bullet, B^\bullet)) \simeq \mathrm{Hom}_R(K^\bullet, \mathrm{Hom}_R(L^\bullet, B^\bullet))$$

for any complexes of R-modules $K^\bullet$ and $L^\bullet$ makes the contravariant endofunctor $\mathbb{R}\,\mathrm{Hom}_R(-, B^\bullet)$ on the category of compact objects in $\mathsf{D}(R-\mathsf{mod}_{I\text{-tors}})$ right adjoint to itself. It remains to show that the adjunction morphism $K^\bullet \longrightarrow \mathbb{R}\,\mathrm{Hom}_R(\mathrm{Hom}_R(K^\bullet, B^\bullet), B^\bullet)$ is an isomorphism in $\mathsf{D}(R-\mathsf{mod}_{I\text{-tors}})$ for any finite complex of finitely generated projective R-modules $K^\bullet$ with I-torsion cohomology modules.

Let $'B^\bullet$ be a bounded below complex of injective R-modules endowed with a quasi-isomorphism $B^\bullet \longrightarrow {}'B^\bullet$. Then it suffices to check that the natural morphism $K^\bullet \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(K^\bullet, B^\bullet), {}'B^\bullet) \simeq K^\bullet \otimes_R \mathrm{Hom}_R(B^\bullet, {}'B^\bullet)$ is a quasi-isomorphism of complexes of R-modules. By the condition (ii) together with Theorem 1.3, the complex $\mathrm{Hom}_R(B^\bullet, {}'B^\bullet)$ is quasi-isomorphic to the ring $\mathfrak{R}$, so we have to show that the morphism $K^\bullet \longrightarrow \mathfrak{R} \otimes_R K^\bullet$ is a quasi-isomorphism. But this is the particular case of Lemma 5.4(b) for a one-element set X. $\qquad\square$

Example 5.8. Let I be a weakly proregular finitely generated ideal in a commutative ring R. As in Example 4.4, consider the derived functor of maximal I-torsion submodule $\mathbb{R}\Gamma_I \colon \mathsf{D}^\mathsf{b}(R\text{–mod}) \longrightarrow \mathsf{D}^\mathsf{b}(R\text{–mod}_{I\text{-tors}})$. It is claimed that the complex $B^\bullet = \mathbb{R}\Gamma_I(R)$ is a dedualizing complex for the ideal $I \subset R$.

Indeed, one has $\mathrm{Hom}_{\mathsf{D}^\mathsf{b}(R\text{–mod}_{I\text{-tors}})}(B^\bullet, M[*]) \simeq \mathrm{Hom}_{\mathsf{D}^\mathsf{b}(R\text{–mod})}(B^\bullet, M[*])$ for any I-torsion R-module M by (the case $\star = \mathsf{b}$ of) Theorem 1.3, so the complex $B^\bullet$, having finite projective dimension as a complex of R-modules for the reason of being isomorphic to the complex $T^\bullet(R, \mathbf{s})$ in $\mathsf{D}^\mathsf{b}(R\text{–mod})$, also has finite projective dimension as a complex of I-torsion R-modules.

Furthermore, the quasi-isomorphism of complexes of R-modules $T^\bullet(R, \mathbf{s}) \longrightarrow B^\bullet$ induces a quasi-isomorphism of tensor products $T^\bullet(R, \mathbf{s}) \otimes_R \mathfrak{R}[[X]] \longrightarrow B^\bullet \otimes_R \mathfrak{R}[[X]]$. Indeed, the complex $B^\bullet$ being a complex of I-torsion R-modules, the morphism $B^\bullet \otimes_R R[X] \longrightarrow B^\bullet \otimes_R \mathfrak{R}[[X]]$ induced by the completion morphism $R[X] \longrightarrow \mathfrak{R}[[X]]$ is an isomorphism of complexes, while the morphism $T^\bullet(R, \mathbf{s}) \otimes_R R[X] \longrightarrow B^\bullet \otimes_R R[X]$ is a quasi-isomorphism since the R-module $R[X]$ is flat, and the morphism $T^\bullet(R, \mathbf{s}) \otimes_R R[X] \longrightarrow T^\bullet(R, \mathbf{s}) \otimes_R \mathfrak{R}[[X]]$ is a quasi-isomorphism by Lemma 5.3(b). Since $T^\bullet(R, \mathbf{s})$ is a complex of flat R-modules, it follows that the homology R-modules of the complex $T^\bullet(R, \mathbf{s}) \otimes_R P$ are isomorphic to the R-modules $\mathrm{Ctrtor}_*^{(R,I)}(B^\bullet, P)$ for any I-contramodule R-module P, so the complex $B^\bullet$ has finite contraflat dimension, too (cf. Lemma 5.4(c) and Remark 5.6). This proves the condition (i).

The condition (ii) is provided by Lemma 5.2(b) applied to the I-contramodule R-module $\mathfrak{R}$ together with Lemma 5.3(b) applied to a one-element set X. Finally, by Proposition 5.1 it suffices to prove the condition (iii) for one of the complexes $K^\bullet = K^n_\bullet(R, \mathbf{s})$, for which it follows from Lemma 1.1(c), as the complex $\mathrm{Hom}_R(K^n_\bullet(R, \mathbf{s}), T^\bullet(R, \mathbf{s}))$ is quasi-isomorphic to $T^\bullet_n(R, \mathbf{s}) \otimes_R C^\bullet_{\mathbf{s}}(R)^\sim$ and to $T^\bullet_n(R, \mathbf{s})$.

According to Lemma 5.4(a), the involutive anti-equivalence $\mathrm{Hom}_R(-, R)$ of the category of finite complexes of finitely generated projective R-modules takes its full subcategory of complexes with I-torsion cohomology modules into itself. One can say that the choice of the dedualizing complex $B^\bullet = \mathbb{R}\Gamma_I(R)$ corresponds to the choice of the anti-equivalence $\mathbb{R}\mathrm{Hom}_R(-, B^\bullet)$ on the subcategory of compact objects in $\mathsf{D}(R\text{–mod}_{I\text{-tors}})$ obtained as the restriction of the anti-equivalence $\mathbb{R}\mathrm{Hom}_R(-, R)$ on the category of compact objects in $\mathsf{D}(R\text{–mod})$.

Remark 5.9. One might wish to replace the condition (iii) in the above definition of a dedualizing complex with a more conventional finiteness condition by requiring, in the spirit of the condition (iii) of Section 4, the cohomology R-modules of the complex $B^\bullet$ to have finitely generated submodules of elements annihilated by I^n for all (or some) $n \geqslant 1$. However, the class of all I-torsion R-modules with finitely generated submodules of elements annihilated by I^n does not seem to have good enough homological properties to make such a definition reasonable even for a Noetherian ring R. Indeed, the following counterexample demonstrates that this class of R-modules is *not* closed under the passages to quotient modules (though it is, of course, preserved by the passages to arbitrary submodules).

Let $R = k[x, s]$ be the ring of polynomials in two variables over a field k with the ideal $I = (s)$. Consider the I-torsion R-module $k[x, s, s^{-1}]/sk[x, s]$ and the R-submodule M in it with the k-basis consisting of all the vectors $x^j s^{-i}$ with $i \leqslant j$. Then for any $n \geqslant 1$ the submodule of elements annihilated by s^n in M is a finitely generated free $k[x]$-module with the generators 1, xs^{-1}, ..., $x^{n-1}s^{-n+1}$, but the quotient R-module M/sM of the R-module M is an infinite-dimensional k-vector space with the basis $x^i s^{-i}$, $i \geqslant 0$, where both x and s act by zero.

Hence even though the argument of Example 4.4 still shows that the derived functor of maximal submodule annihilated by I^n takes the complex $B^\bullet = \mathbb{R}\Gamma_I(R)$ to a complex with finitely generated cohomology R/I^n-modules when the ring R is Noetherian, no analogue of Lemmas 4.1 or 4.2 seems to be applicable in this case.

The next theorem is the main result of this section and this paper.

Theorem 5.10. *Let I be a weakly proregular finitely generated ideal in a commutative ring R. Then, given a dedualizing complex $B^\bullet$ for the ideal $I \subset R$, for any symbol $\star = $ b, $+$, $-$, $\varnothing$, abs+, abs$-$, or abs there is an equivalence of derived categories $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-tors}}) \simeq \mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}})$ provided by mutually inverse functors $\mathbb{R}\operatorname{Hom}_R(B^\bullet, -)$ and $B^\bullet \otimes_R^{\mathbb{L}} -$.*

Proof. As in the proof of Theorem 4.5, we assume for simplicity of notation that the complex $B^\bullet$ is concentrated in nonpositive cohomological degrees. Let d be an integer greater or equal to both the projective and the contraflat dimension of $B^\bullet$ as a complex of I-torsion R-modules.

To construct the image of a complex of I-torsion R-modules $M^\bullet$ under the functor $\mathbb{R}\operatorname{Hom}_R(B^\bullet, -)$, choose an exact sequence of complexes of I-torsion R-modules $0 \longrightarrow M^\bullet \longrightarrow J^{0,\bullet} \longrightarrow J^{1,\bullet} \longrightarrow \cdots$ with the injective I-torsion R-modules $J^{j,i}$ (i. e., the R-modules $J^{j,i}$ should be injective objects in the abelian category $R\text{--}\mathsf{mod}_{I\text{-tors}}$). Applying the functor $\operatorname{Hom}_R(B^\bullet, -)$ to every complex $0 \longrightarrow J^{0,i} \longrightarrow J^{1,i} \longrightarrow J^{2,i} \longrightarrow \cdots$ with $i \in \mathbb{Z}$, one obtains a nonnegatively graded complex of I-contramodule R-modules $0 \longrightarrow P^{0,i} \longrightarrow P^{1,i} \longrightarrow P^{2,i} \longrightarrow \cdots$. According to the projective dimension condition on the dedualizing complex $B^\bullet$, the complex $P^{\bullet,i}$ has zero cohomology modules at the cohomological degrees above d; so it quasi-isomorphic to its canonical truncation complex $\tau_{\leqslant d} P^{\bullet,i}$. By the definition, set the object $\mathbb{R}\operatorname{Hom}_R(B^\bullet, M^\bullet)$ in the derived category $\mathsf{D}^\star(R\text{--}\mathsf{mod}_{I\text{-ctra}})$ be represented by the total complex of the bicomplex $\tau_{\leqslant d} P^{\bullet,\bullet}$ concentrated in the cohomological degrees $0 \leqslant j \leqslant d$ and $i \in \mathbb{Z}$ (or, when the symbol $\star$ presumes bounded complexes, the bicomplex $P^{\bullet,\bullet}$ is bounded in the respective sense along the grading i).

Similarly, to construct the image of a complex of I-contramodule R-modules P under the functor $B^\bullet \otimes_R^{\mathbb{L}} -$, one has to choose an exact sequence of complexes of I-contramodule R-modules $\cdots \longrightarrow F^{-1,\bullet} \longrightarrow F^{0,\bullet} \longrightarrow P^\bullet \longrightarrow 0$ with projective I-contramodule R-modules $F^{j,i}$. Then one applies the functor $B^\bullet \otimes_R -$ to every complex $\cdots \longrightarrow F^{-2,i} \longrightarrow F^{-1,i} \longrightarrow F^{0,i} \longrightarrow 0$ with $i \in \mathbb{Z}$, obtaining a nonpositively graded complex of I-torsion R-modules $\cdots \longrightarrow M^{-2,i} \longrightarrow M^{-1,i} \longrightarrow M^{0,i} \longrightarrow 0$. According to the contraflat dimension condition on the complex $B^\bullet$, the complex

$M^{\bullet,i}$ has zero cohomology modules at the cohomological degrees below $-d$; so it is quasi-isomorphic to its canonical truncation complex $\tau_{\geqslant -d}M^{\bullet,i}$. One sets the object $B^\bullet \otimes_R^{\mathbb{L}} P^\bullet$ in the derived category $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$ to be represented by the total complex of the bicomplex $\tau_{\geqslant -d}(M^{\bullet,\bullet})$ concentraded in the cohomological degrees $-d \leqslant j \leqslant 0$ and $i \in \mathbb{Z}$.

As in the proof of Theorem 4.5, these constructions of two derived functors are but particular cases of the construction a derived functor of finite homological dimension elaborated in Appendix B. According to the results of Appendix B, the above constructions produce well-defined triangulated functors

$$\mathbb{R}\operatorname{Hom}_R(B^\bullet, -)\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-ctra}})$$

and

$$B^\bullet \otimes_R^{\mathbb{L}} -\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$$

for any derived category symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-$, or abs. Moreover, the former functor is right adjoint to the latter one. All these assertions only depend on the condition (i) in the definition of a dedualizing complex.

It remains to prove that the adjunction morphisms are isomorphisms. Since the total complexes of finite acyclic complexes of complexes are absolutely acyclic, in order to check that the morphism $B^\bullet \otimes_R^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_R(B^\bullet, M^\bullet) \longrightarrow M^\bullet$ is an isomorphism in the derived category $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-tors}})$ for all the $\star$-bounded complexes of I-torsion R-modules $M^\bullet$ it suffices to consider the case of a one-term complex $M^\bullet = M$ corresponding to a single I-torsion R-module M. Furthermore, since a morphism in $\mathsf{D}^\mathsf{b}(R\text{–}\mathsf{mod}_{I\text{-tors}})$ is an isomorphism whenever it is an isomorphism in $\mathsf{D}^+(R\text{–}\mathsf{mod}_{I\text{-tors}})$, one can view the one-term complex M is an object of the derived category $\mathsf{D}^+(R\text{–}\mathsf{mod}_{I\text{-tors}})$ and replace it with an injective resolution $J^\bullet$ of the R-module M in the abelian category $R\text{–}\mathsf{mod}_{I\text{-tors}}$. Applying the same totalization argument to the complex $J^\bullet$, the question reduces to proving that the adjunction morphism $B^\bullet \otimes_R^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_R(B^\bullet, J) \longrightarrow J$ is an isomorphism in $\mathsf{D}^\mathsf{b}(R\text{–}\mathsf{mod}_{I\text{-tors}})$ for any injective I-torsion R-module J.

One has $\mathbb{R}\operatorname{Hom}_R(B^\bullet, J) = \operatorname{Hom}_R(B^\bullet, J)$. Let $P^\bullet$ be a bounded above complex of projective I-contramodule R-modules endowed with a quasi-isomorphism of complexes of I-contramodule R-modules $P^\bullet \longrightarrow \operatorname{Hom}_R(B^\bullet, J)$, so $B^\bullet \otimes_R^{\mathbb{L}} \operatorname{Hom}_R(B^\bullet, J) = B^\bullet \otimes_R^{\mathbb{L}} P^\bullet$. We have to show that the natural morphism $B^\bullet \otimes_R P^\bullet \longrightarrow J$ is a quasi-isomorphism of complexes of I-torsion R-modules. Let $\mathbf{s}$ be a finite sequence of elements generating the ideal $I \subset R$. By Lemma 1.1(c), it suffices to check that the induced morphism of complexes

$$T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet \otimes_R P^\bullet \longrightarrow T_n^\bullet(R, \mathbf{s}) \otimes_R J$$

is a quasi-isomorphism for every $n \geqslant 1$ (as the complex of flat R-modules $C_\mathbf{s}^\bullet(R)^\sim$ is quasi-isomorphic to the complex of free R-modules $T^\bullet(R, \mathbf{s})$, which is the inductive limit of complexes of finitely generated free R-modules $T_n^\bullet(R, \mathbf{s})$).

By the condition (iii) together with Lemma 5.4(a) and Proposition 5.1, there exists a finite complex of finitely generated projective R-modules $L^\bullet$ together with a quasi-isomorphism of complexes of R-modules $L^\bullet \longrightarrow T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet$. By Lemma 5.4(c),

the induced morphism of complexes of R-modules

$$L^\bullet \otimes_R P^\bullet \longrightarrow T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet \otimes_R P^\bullet$$

is a quasi-isomorphism. So is the morphism of complexes of R-modules $L^\bullet \otimes_R P^\bullet \longrightarrow L^\bullet \otimes_R \mathrm{Hom}_R(B^\bullet, J)$. Hence it remains to check that the morphism

$$L^\bullet \otimes_R \mathrm{Hom}_R(B^\bullet, J) \longrightarrow T_n^\bullet(R, \mathbf{s}) \otimes_R J$$

induced by the morphism $L^\bullet \longrightarrow T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet$ is a quasi-isomorphism.

One has $L^\bullet \otimes_R \mathrm{Hom}_R(B^\bullet, J) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(L^\bullet, B^\bullet), J)$. By Theorem 5.7, the natural morphism of complexes of R-modules

$$K_\bullet^n(R, \mathbf{s}) = \mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}), R) \longrightarrow \mathrm{Hom}_R(L^\bullet, B^\bullet)$$

is a quasi-isomorphism. Applying Lemma 5.5(b), we conclude that the induced morphism of complexes

$$\mathrm{Hom}_R(\mathrm{Hom}_R(L^\bullet, B^\bullet), J) \longrightarrow \mathrm{Hom}_R(K_\bullet^n(R, \mathbf{s}), J) \simeq T_n^\bullet(R, \mathbf{s}) \otimes_R J$$

is a quasi-isomorphism, too.

Similarly, in order to prove that the adjunction morphism $P^\bullet \longrightarrow \mathbb{R}\,\mathrm{Hom}_R(B^\bullet, B^\bullet \otimes_R^{\mathbb{L}} P^\bullet)$ is an isomorphism in the derived category $\mathsf{D}^\star(R\text{–}\mathsf{mod}_{I\text{-ctra}})$ for any $\star$-bounded complex of I-contramodule R-modules $P^\bullet$, it suffices to check that it is an isomorphism in $\mathsf{D}^\mathsf{b}(R\text{–}\mathsf{mod}_{I\text{-ctra}})$ for any free I-contramodule R-module $F = \mathfrak{R}[[X]]$ viewed as a one-term complex in $\mathsf{D}^\mathsf{b}(R\text{–}\mathsf{mod}_{I\text{-ctra}})$. One has $B^\bullet \otimes_R^{\mathbb{L}} \mathfrak{R}[[X]] \simeq B^\bullet \otimes_R R[X] = B^\bullet[X]$. Let $B^\bullet \longrightarrow E^\bullet$ and $B^\bullet[X] \longrightarrow E_X^\bullet$ be quasi-isomorphisms from the complexes $B^\bullet$ and $B^\bullet[X]$ to bounded below complexes of injective objects in $R\text{–}\mathsf{mod}_{I\text{-tors}}$. We have to show that the natural morphism $\mathfrak{R}[[X]] \longrightarrow \mathrm{Hom}_R(B^\bullet, E_X^\bullet)$ is a quasi-isomorphism of complexes of R-modules.

By Lemma 1.1(c), the morphism of complexes of I-torsion R-modules $T^\bullet(R, \mathbf{s}) \otimes_R B^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism. Hence the induced morphism

(5) $$\mathrm{Hom}_R(B^\bullet, E_X^\bullet) \longrightarrow \mathrm{Hom}_R(T^\bullet(R, \mathbf{s}) \otimes_R B^\bullet, E_X^\bullet)$$

is a quasi-isomorphism, too. The complex in the target of this morphism is the projective limit of the projective system of complexes $\mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet, E_X^\bullet)$ and termwise surjective morphisms between them. By the condition (iii), every complex $T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet$ is a compact object of the derived category $\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-tors}})$. Hence the X-indexed family of morphisms of complexes $E^\bullet \longrightarrow E_X^\bullet$ corresponding to the embeddings $B^\bullet \longrightarrow B^\bullet[X]$ induces a quasi-isomorphism

(6) $$\mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet, E^\bullet)[X] \longrightarrow \mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet, E_X^\bullet).$$

Furthermore, one has

$$\mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}) \otimes_R B^\bullet, E^\bullet) \simeq \mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}), \mathrm{Hom}_R(B^\bullet, E^\bullet)).$$

By the condition (ii), the natural morphism $\mathfrak{R} \longrightarrow \mathrm{Hom}_R(B^\bullet, E^\bullet)$ is a quasi-isomorphism, hence so is the induced morphism

$$\mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}), \mathfrak{R}) \longrightarrow \mathrm{Hom}_R(T_n^\bullet(R, \mathbf{s}), \mathrm{Hom}_R(B^\bullet, E^\bullet)).$$

By Lemma 5.4(a-b), the morphism of complexes

$$\operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R) \longrightarrow \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),\mathfrak{R})$$

induced by the completion morphism $R \longrightarrow \mathfrak{R}$ is a quasi-isomorphism, too. Passing to the composition and taking the direct sum over X, we obtain a quasi-isomorphism of complexes of R-modules

(7) $$\operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X]) \longrightarrow \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}) \otimes_R B^\bullet, E^\bullet)[X]$$

for every $n \geqslant 1$.

Both sides of the quasi-isomorphisms (6) and (7) naturally form projective systems of complexes and termwise surjective morphisms between them, and the quasi-isomorphisms form commutative diagrams with the morphisms in the projective systems. Therefore, the induced morphisms between the projective limits are also quasi-isomorphisms. Furthermore, the complexes $\operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X])$ have I-torsion cohomology modules, so the natural morphism $R[X] \longrightarrow \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X])$ extends naturally to a cohomology morphism

$$\mathfrak{R}[[X]] \longrightarrow H^0 \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X]).$$

Due to the weak proregularity condition, after the passage to the projective limit this provides a cohomology morphism

$$\mathfrak{R}[[X]] \longrightarrow H^0 \varprojlim_n \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X])$$

forming a commutative diagram with the cohomology morphisms induced by the morphism of complexes $\mathfrak{R}[[X]] \longrightarrow \operatorname{Hom}_R(B^\bullet, E_X^\bullet)$, the morphism of complexes (5), and the projective limits of the morphisms of complexes (6–7).

Finally, by Lemma 5.3(c) the morphism of complexes

$$\varprojlim_n \operatorname{Hom}_R(T_n^\bullet(R,\mathbf{s}),R[X]) \simeq \operatorname{Hom}_R(T^\bullet(R,\mathbf{s}),R[X]) \longrightarrow \operatorname{Hom}_R(T^\bullet(R,\mathbf{s}),\mathfrak{R}[[X]])$$

induced by the completion morphism $R[X] \longrightarrow \mathfrak{R}[[X]]$ is a quasi-isomorphism, while by Lemma 2.2(c) the morphism

$$\mathfrak{R}[[X]] \longrightarrow \operatorname{Hom}_R(T^\bullet(R,\mathbf{s}),\mathfrak{R}[[X]]$$

is a quasi-isomorphism. Comparing these observations finishes the proof. $\qquad\square$

APPENDIX A. EXOTIC DERIVED CATEGORIES

This appendix, consisting almost entirely of the definitions, is included for the benefit of the reader who may have no other sources handy when looking into this paper. Detailed expositions can be found in [21, 22, 6]; and the most relevant source for our present purposes is [24, Appendix A].

Given an additive category A, we denote by $\mathsf{Hot}(\mathsf{A})$ the homotopy category of complexes over A, by $\mathsf{Hot}^+(\mathsf{A})$ its full subcategory of bounded below complexes, by $\mathsf{Hot}^-(\mathsf{A}) \subset \mathsf{Hot}(\mathsf{A})$ the full subcategory of bounded above complexes, and by $\mathsf{Hot}^b(\mathsf{A})$ the full subcategory of complexes bounded on both sides.

For a general discussion of exact categories, we refer to the overview paper [3]. A sequence of objects and morphisms $E \longrightarrow F \longrightarrow G$ in an exact category E is said to be *exact* (at the term F) if it is composed of an admissible epimorphism $E \longrightarrow S$, a short exact sequence $0 \longrightarrow S \longrightarrow F \longrightarrow T \longrightarrow 0$, and an admissible monomorphism $T \longrightarrow G$. A complex $E^\bullet$ in an exact category E is said to be *acyclic* if it is homotopy equivalent to an exact complex, or equivalently, if it is a direct summand of an exact complex. For any symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, the quotient category of the homotopy category $\mathsf{Hot}^\star(\mathsf{E})$ by its thick subcategory of acyclic complexes is called the (conventional) *derived category* of E and denoted by $\mathsf{D}^\star(\mathsf{E})$.

The exotic derived category symbols are $\star = \mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs. The related exotic derived category $\mathsf{D}^\star(\mathsf{E})$ is defined as a certain quotient category of the corresponding homotopy category, which is $\mathsf{Hot}^+(\mathsf{E})$ when $\star = \mathsf{abs}+$, or $\mathsf{Hot}^-(\mathsf{E})$ when $\star = \mathsf{abs}-$, or $\mathsf{Hot}(\mathsf{E})$ in the remaining cases $\star = \mathsf{co}$, ctr, or abs. It is sometimes convenient to denote simply by $\mathsf{Hot}^\star(\mathsf{E})$ the (respectively bounded or unbounded) homotopy category corresponding to an exotic symbol $\star$.

A short exact sequence of complexes in an exact category E can be viewed as a bicomplex with three rows. As such, it has a total complex. A complex in an exact category E is called *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $\mathsf{Hot}(\mathsf{E})$ containing the totalizations of all the short exact sequences of complexes in E. The quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by its thick subcategory of absolutely acyclic complexes is denoted by $\mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ and called the *absolute derived category* of an exact category E.

The definitions of the absolute derived categories $\mathsf{D}^{\mathsf{abs}+}(\mathsf{E})$ and $\mathsf{D}^{\mathsf{abs}-}(\mathsf{E})$ of bounded above or below complexes in an exact category E are similar. A $\star$-bounded complex in E is said to be absolutely acyclic with respect to the class of $\star$-bounded complexes if it belongs to the minimal thick subcategory of the homotopy category $\mathsf{Hot}^\star(\mathsf{E})$ containing the totalizations of all the short exact sequences of $\star$-bounded complexes in E. Given a symbol $\star = \mathsf{abs}+$ or $\mathsf{abs}-$, the quotient category of the homotopy category $\mathsf{Hot}^\star(\mathsf{E})$ by its thick subcategory of complexes absolutely acyclic with respect to the class of $\star$-bounded complexes is denoted by $\mathsf{D}^\star(\mathsf{E})$.

Any bounded acyclic complex is absolutely acyclic (with respect to the class of bounded complexes). Any bounded above complex that is absolutely acyclic is also absolutely acyclic with respect to the class of bounded above complexes; and any bounded below complex that is absolutely acyclic is also absolutely acyclic with respect to the class of bounded below complexes [24, Lemma A.1.1].

Assume that arbitrary infinite direct sums exist in the exact category E and the class of short exact sequences is closed under infinite direct sums. Then a complex in the category E is called *coacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the absolutely acyclic complexes and closed under infinite direct sums. The quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by its thick subcategory of coacyclic complexes is denoted by $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$ and called the *coderived category* of the exact category E.

Similarly, assume that arbitrary infinite products exist in the exact category E and the class of short exact sequences is closed under infinite products. Then a complex in the category E is called *contraacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the absolutely acyclic complexes and closed under infinite products. The quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by its thick subcategory of contraacyclic complexes is denoted by $\mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ and called the *contraderived category* of the exact category E.

Any coacyclic complex in an exact category with exact functors of infinite direct sum is acyclic, and any contraacyclic complex in an exact category with exact functors of infinite product is acyclic, but the converse is not generally true [22, Examples 3.3]. So the coderived and the contraderived categories are "larger" than the conventional unbounded derived category; and the absolute derived category is larger still.

Any bounded below acyclic complex in an exact category is coacyclic [21, Lemma 2.1]. Any bounded above acyclic complex in an exact category is contraacyclic [21, Lemma 4.1]. Any acyclic complex in an exact category of finite homological dimension is absolutely acyclic [21, Remark 2.1].

The conventional derived categories (with the symbols $+$, $-$, or $\varnothing$) are also known as *derived categories of the first kind*. The coderived, contraderived, and absolute derived categories (with the symbols $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs) are known as *derived categories of the second kind*. The bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$ of an exact category E can be thought of as belonging to both classes at the same time.

Appendix B. Derived Functors of Finite Homological Dimension

The constructions of derived functors of finite homological dimension between various derived categories $\mathsf{D}^{\star}$ in Sections 1–2 are based on the technique of [24, Section A.5], which is not sufficient for the purposes of Sections 4–5. The more sophisticated technique required there is developed in this appendix.

For any additive category E, we denote by $\mathsf{C}^{+}(\mathsf{E})$ the DG-category of bounded below complexes in E. The additive category of bounded below complexes in E and closed morphisms of degree zero between them will be denoted by the same symbol $\mathsf{C}^{+}(\mathsf{E})$; it will be clear from the context which category structure on $\mathsf{C}^{+}(\mathsf{E})$ is presumed. When E is an exact category, the full subcategory $\mathsf{C}^{\geqslant 0}(\mathsf{E}) \subset \mathsf{C}^{+}(\mathsf{E})$ of all the complexes $0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \cdots$ in E (and closed morphisms of degree zero between them) has a natural exact category structure where a short sequence of complexes is exact if and only if it is termwise exact in E.

Let A be an exact category and $\mathsf{J} \subset \mathsf{A}$ be a full subcategory, closed under extensions and the passages to the cokernels of admissible monomorphisms in A and such that any object of A is the source of an admissible monomorphism into an object of J. Being closed under extensions, the full subcategory J inherits the exact category structure of the ambient category A. Notice that a closed morphism in $\mathsf{C}^{+}(\mathsf{J})$ is a quasi-isomorphism of complexes in the exact category J if and only if it is a quasi-isomorphism of complexes in A.

Let $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J})$ denote the full subcategory in $\mathsf{C}^{\geqslant 0}(\mathsf{J})$ consisting of all the complexes $0 \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ in J for which there exists an object $A \in \mathsf{A}$ together with a morphism $A \longrightarrow J^0$ such that the sequence $0 \longrightarrow A \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ is exact in the category A. By the definition, one has $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) = \mathsf{C}^{\geqslant 0}(\mathsf{J}) \cap \mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{A}) \subset \mathsf{C}^{\geqslant 0}(\mathsf{A})$. The full subcategory $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J})$ is closed under extensions and the passages to the cokernels of admissible monomorphisms in $\mathsf{C}^{\geqslant 0}(\mathsf{J})$, so it inherits the exact category structure.

Let B be another exact category; suppose that it contains the images of idempotent endomorphisms of its objects. Let $d \geqslant 0$ be an integer. Denote by $\mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d} \subset \mathsf{C}^{\geqslant 0}(\mathsf{B})$ the full subcategory consisting of all the complexes $0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$ in B such that the sequence $B^d \longrightarrow B^{d+1} \longrightarrow B^{d+2} \longrightarrow \cdots$ is exact in B. In particular, one has $\mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant 0} = \mathsf{C}_\mathsf{B}^{\geqslant 0}(\mathsf{B})$. Being closed under extensions and the passages to the cokernels of admissible monomorphisms, the full subcategory $\mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d}$ inherits the exact category structure of the category $\mathsf{C}^{\geqslant 0}(\mathsf{B})$. Denote also by $\mathsf{C}^{[0,d]}(\mathsf{B})$ the exact category of finite complexes $B^0 \longrightarrow \cdots \longrightarrow B^d$ in B. Then there is an exact functor of canonical truncation $\tau_{\leqslant d} \colon \mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d} \longrightarrow \mathsf{C}^{[0,d]}(\mathsf{B})$.

Suppose that we are given a DG-functor $\Psi \colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$ taking quasi-isomorphisms of complexes belonging to $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J})$ to quasi-isomorphisms of complexes in the exact category B. Suppose further that the restriction of the functor Ψ to the subcategory $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) \subset \mathsf{C}^+(\mathsf{J})$ is an exact functor between exact categories $\Psi \colon \mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) \longrightarrow \mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d}$. Composing the restriction of the functor Ψ with the functor of canonical truncation, we obtain an exact functor

$$\tau_{\leqslant d}\Psi \colon \mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) \longrightarrow \mathsf{C}^{[0,d]}(\mathsf{B}).$$

Our aim is to construct the right derived functor

$$\mathbb{R}\Psi \colon \mathsf{D}^\star(\mathsf{A}) \longrightarrow \mathsf{D}^\star(\mathsf{B})$$

acting between any bounded or unbounded, conventional or absolute derived categories $\mathsf{D}^\star$ with the symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-,$ or abs (see Appendix A or [24, Section A.1] for the definitions and [22, Section 3] for a discussion).

When the exact categories A and B have exact functors of infinite direct sum, the full subcategory $\mathsf{J} \subset \mathsf{A}$ is closed under infinite direct sums, and the functor Ψ preserves infinite direct sums, there will be also the derived functor $\mathbb{R}\Psi$ acting between the coderived categories D^{co} of the exact categories A and B. Similarly, when the exact categories B and A have exact functors of infinite product, the full subcategory $\mathsf{J} \subset \mathsf{A}$ is closed under infinite products, and the functor Ψ preserves infinite products, we will also have the derived functor $\mathbb{R}\Psi$ acting between the contraderived categories $\mathsf{D}^{\mathsf{ctr}}$ of the exact categories A and B.

We start with a couple of somewhat tedious but rather straightforward lemmas.

Lemma B.1. (a) *For any object E in the category A there exists an exact sequence $0 \longrightarrow E \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots$ in the category A with the objects J^j beloging to the full subcategory $\mathsf{J} \subset \mathsf{A}$.*

(b) *For any two morphisms $E \longrightarrow G$ and $F \longrightarrow G$ and two exact sequences $0 \longrightarrow E \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots$ and $0 \longrightarrow F \longrightarrow L^0 \longrightarrow L^1 \longrightarrow \cdots$ in the category*

A *there exists an exact sequence* $0 \longrightarrow G \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ *in the category* A *together with two morphisms of complexes* $K^\bullet \longrightarrow J^\bullet$ *and* $L^\bullet \longrightarrow J^\bullet$ *forming a commutative diagram with the morphisms* $E \longrightarrow G$, $F \longrightarrow G$, $E \longrightarrow K^\bullet$, *and* $F \longrightarrow L^\bullet$ *and such that the objects* J^j *belong to the full subcategory* J $\subset$ A.

(c) *For any exact sequence* $0 \longrightarrow E \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots$ *and any complex* $0 \longrightarrow L^0 \longrightarrow L^1 \longrightarrow L^2 \longrightarrow \cdots$ *in the category* A*, and for any morphism of complexes* $K^\bullet \longrightarrow L^\bullet$ *with a vanishing composition* $E \longrightarrow K^\bullet \longrightarrow L^\bullet$*, there exists a complex* $0 \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots$ *in the category* J *together with a quasi-isomorphism* $L^\bullet \longrightarrow J^\bullet$ *of complexes in the category* A *such that the composition* $K^\bullet \longrightarrow L^\bullet \longrightarrow J^\bullet$ *is a contractible morphism of complexes.*

(d) *For any short exact sequence* $0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow E^3 \longrightarrow 0$ *in the category* A *there exists an exact sequence of short exact sequences* $0 \longrightarrow E^\bullet \longrightarrow J^{\bullet,0} \longrightarrow J^{\bullet,1} \longrightarrow J^{\bullet,2} \longrightarrow \cdots$ *in the category* A *with the objects* $J^{k,j}$ *belonging to the full subcategory* J $\subset$ A.

Proof. Part (a) follows immediately by induction from the condition that any object of A is the source of an admissible monomorphism into an object of J. To prove part (b), one also proceeds step by step, starting from choosing an admissible monomorphism from the fibered coproduct of the objects G and $K^0 \oplus L^0$ over $E \oplus F$ into an object $J^0 \in$ J, then applying the same construction to the cokernels of the morphisms $E \longrightarrow K^0$, $F \longrightarrow L^0$, and $G \longrightarrow J^0$ and the admissible monomorphisms from the former two to the objects K^1 and L^1, etc.

The proof of part (c) is also a step-by-step construction procedure. Let Z^j denote the images of the morphisms $K^{j-1} \longrightarrow K^j$ in the exact sequence $0 \longrightarrow E \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots$. Then the morphism $K^0 \longrightarrow L^0$ factorizes through the admissible epimorphism $K^0 \longrightarrow Z^1$, so there is a morphism $Z^1 \longrightarrow L^0$. Consider the fibered coproduct of the objects K^1 and L^0 over the object Z^1 in the category A and choose an admissible monomorphism from it into an object $J^0 \in$ J. Then there are a natural admissible monomorphism $L^0 \longrightarrow J^0$ and a natural morphism $K^1 \longrightarrow J^0$. Denote by M^1 the fibered coproduct of the objects J^0 and L^1 over the object L^0 in the category A. Then there are a natural admissible monomorphism $L^1 \longrightarrow M^1$ and natural morphisms $J^0 \longrightarrow M^1 \longrightarrow L^2$ with a vanishing composition in the category A. The difference of the compositions $K^1 \longrightarrow L^1 \longrightarrow M^1$ and $K^1 \longrightarrow J^0 \longrightarrow M^1$ is annihilated by the composition with the morphism $K^0 \longrightarrow K^1$, so it factorizes through the admissible epimorphism $K^1 \longrightarrow Z^2$, providing a morphism $Z^2 \longrightarrow M^1$. Consider the fibered coproduct of the objects K^2 and M^1 over the object Z^2 in the category A and choose an admissible monomorphism from it into an object $J^1 \in$ J. Then there are a natural admissible monomorphism $M^1 \longrightarrow J^1$ and a morphism $K^2 \longrightarrow J^1$ in the category A. The composition $J^0 \longrightarrow M^1 \longrightarrow J^1$ provides a morphism $J^0 \longrightarrow J^1$ in the category J. Denote by M^2 the fibered coproduct of the objects J^1 and L^2 over the object M^1 in the category A, etc.

To check part (d), one applies part (a) to the exact category of short exact sequences in A and its full subcategory of short exact sequences in J. It suffices to show that the short exact sequence $0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow E^3 \longrightarrow 0$ is the source of an admissible

monomorphism into a short exact sequence of objects from J. For this purpose, one picks admissible monomorphisms $E^2 \longrightarrow J^1$ and $E^3 \longrightarrow J^3$ into objects J^1, $J^3 \in \mathsf{J}$; then there is an admissible monomorphism from our short exact sequence to the split short exact sequence $0 \longrightarrow J^1 \longrightarrow J^1 \oplus J^3 \longrightarrow J^3 \longrightarrow 0$. $\qquad\square$

Lemma B.2. (a) *For any complex $E^\bullet$ in the category A there exists an exact sequence of complexes*

$$0 \longrightarrow E^\bullet \longrightarrow J^{0,\bullet} \longrightarrow J^{1,\bullet} \longrightarrow J^{2,\bullet} \longrightarrow \cdots$$

in the category A with the objects $J^{j,i}$ belonging to the full subcategory $\mathsf{J} \subset \mathsf{A}$.

(b) *For any two morphisms of complexes $E^\bullet \longrightarrow G^\bullet$ and $F^\bullet \longrightarrow G^\bullet$ and two exact sequences of complexes $0 \longrightarrow E^\bullet \longrightarrow K^{0,\bullet} \longrightarrow K^{1,\bullet} \longrightarrow \cdots$ and $0 \longrightarrow F^\bullet \longrightarrow L^{0,\bullet} \longrightarrow L^{1,\bullet} \longrightarrow \cdots$ in the category A there exists an exact sequence of complexes $0 \longrightarrow G^\bullet \longrightarrow J^{0,\bullet} \longrightarrow J^{1,\bullet} \longrightarrow \cdots$ in the category A together with two morphisms of bicomplexes $K^{\bullet,\bullet} \longrightarrow J^{\bullet,\bullet}$ and $L^{\bullet,\bullet} \longrightarrow J^{\bullet,\bullet}$ forming a commutative diagram with the morphisms $E^\bullet \longrightarrow G^\bullet$, $E^\bullet \longrightarrow G^\bullet$, $E^\bullet \longrightarrow K^{\bullet,\bullet}$, and $F^\bullet \longrightarrow L^{\bullet,\bullet}$ and such that the objects $J^{j,i}$ belong to the full subcategory $\mathsf{J} \subset \mathsf{A}$.*

(c) *For any exact sequence of complexes $0 \longrightarrow E^\bullet \longrightarrow K^{0,\bullet} \longrightarrow K^{1,\bullet} \longrightarrow \cdots$ and any complex of complexes $0 \longrightarrow L^{0,\bullet} \longrightarrow L^{1,\bullet} \longrightarrow L^{2,\bullet} \longrightarrow \cdots$ in the category A, and for any morphism of bicomplexes $K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$ with a vanishing composition $E^\bullet \longrightarrow K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$, there exists a complex of complexes $0 \longrightarrow J^{0,\bullet} \longrightarrow J^{1,\bullet} \longrightarrow J^{2,\bullet} \longrightarrow \cdots$ in the category J together with a quasi-isomorphism $L^{\bullet,\bullet} \longrightarrow J^{\bullet,\bullet}$ of complexes along the first grading j in the exact category of complexes along the second grading i in the exact category A such that the composition $K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet} \longrightarrow J^{\bullet,\bullet}$ is a contractible morphism of complexes of complexes with a contracting homotopy $h\colon K^{j,i} \longrightarrow J^{j-1,i}$ defined for all $j \geqslant 0$ and $i \in \mathbb{Z}$.*

(d) *For any short exact sequence of complexes $0 \longrightarrow E^{1,\bullet} \longrightarrow E^{2,\bullet} \longrightarrow E^{3,\bullet} \longrightarrow 0$ in the category A there exists an exact sequence of short exact sequences of complexes $0 \longrightarrow E^{\bullet,\bullet} \longrightarrow J^{\bullet,0,\bullet} \longrightarrow J^{\bullet,1,\bullet} \longrightarrow \cdots$ in the category A with the objects $J^{k,j,i}$ belonging to the full subcategory $\mathsf{J} \subset \mathsf{A}$.*

(e) *For any exact complex $E^\bullet$ in the category A there exists an exact sequence of exact complexes $0 \longrightarrow E^\bullet \longrightarrow J^{0,\bullet} \longrightarrow J^{1,\bullet} \longrightarrow \cdots$ in the category A with the objects $J^{j,i}$ belonging to the full subcategory $\mathsf{J} \subset \mathsf{A}$ and the complexes $J^{j,\bullet}$ exact in the exact category J.*

(f) *If in any of the parts (a-e) the original data of a complex, a pair of morphisms of complexes, a morphism of bicomplexes, or a short exact sequence of complexes consisted of (bi)complex(es) bounded above, below, or on both sides along the grading i, then the bicomplex or the short exact sequence of bicomplexes whose existence is asserted can be also chosen to be bounded on the respective side(s) along the grading i.*

Proof. Parts (a-d) are obtained from parts (a-d) of Lemma B.1 by substituting the exact category of complexes in A and its full subcategory of complexes in J in place of the exact category A and its full subcategory J. One only has to show that any complex $E^\bullet$ in A is the source of an admissible monomorphism into a complex $J^\bullet$ with the terms from J. Here it suffices to pick admissible monomorphisms $E^i \longrightarrow K^i$ into

some objects $K^i \in \mathsf{J}$ and set $J^\bullet$ to be the split exact complex with the terms $J^i = K^i \oplus K^{i+1}$. The same construction of an admissible monomorphism of a complex in A into an exact complex in J allows to prove part (e). Part (f) is clear from the above argument; one only has to modify the construction of the admissible monomorphism $E^\bullet \longrightarrow J^\bullet$ slightly in the case of complexes bounded below or from both sides in order to keep the complex $J^\bullet$ within the boundaries of the complex $E^\bullet$. $\qquad\square$

Now we can construct the derived functor $\mathbb{R}\Psi$. Let $\star$ be one of the symbols b, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs; and let $\mathsf{C}^\star(\mathsf{A})$ denote the category of (respectively bounded or unbounded) complexes (and closed morphisms of degree zero between them) in the category A. Given a complex $E^\bullet \in \mathsf{C}^\star(\mathsf{A})$, we resolve it by a bicomplex $J^{\bullet,\bullet}$ with the terms belonging to the subcategory $\mathsf{J} \subset \mathsf{A}$ as in Lemma B.2(a), apply the functor $\tau_{\leqslant d}\Psi$ to every complex $J^{j,\bullet}$ obtaining a bicomplex $\tau_{\leqslant d}\Psi(J^{\bullet,\bullet})$ concentrated in the segment of degrees $[0,d]$ along the grading j, and totalize this bicomplex in order to obtain the complex $\mathbb{R}\Psi(E^\bullet) \in \mathsf{D}^\star(\mathsf{B})$. When the symbol $\star$ presumes bounded complexes, we choose the bicomplex $J^{\bullet,\bullet}$ to be bounded on the respective side(s) along the grading i, as in Lemma B.2(f). This defines the action of the functor $\mathbb{R}\Psi$ on the objects $E^\bullet$ of the category of complexes $\mathsf{C}^\star(\mathsf{A})$.

To construct the image of a morphism $E^\bullet \longrightarrow F^\bullet$ in the category of complexes $\mathsf{C}^\star(\mathsf{A})$ under the functor $\mathbb{R}\Psi$, one applies Lemma B.2(b) to the pair of morphisms of complexes $E^\bullet \longrightarrow F^\bullet$ and $\mathrm{id}: F^\bullet \longrightarrow F^\bullet$ together with the chosen resolutions $E^\bullet \longrightarrow K^{\bullet,\bullet}$ and $F^\bullet \longrightarrow L^{\bullet,\bullet}$ of the complexes $E^\bullet$ and $F^\bullet$ by bicomplexes $K^{\bullet,\bullet}$ and $L^{\bullet,\bullet}$ in the exact category J. Then one uses the condition that the functor Ψ takes quasi-isomorphisms in $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J})$ to quasi-isomorphisms in $\mathsf{C}^{\geqslant 0}(\mathsf{B})$ in order to conclude that the induced morphism between the totalizations of the bicomplexes $\tau_{\leqslant d}\Psi(L^{\bullet,\bullet})$ and $\tau_{\leqslant d}\Psi(J^{\bullet,\bullet})$ has an absolutely acyclic cone.

Proving that the morphism $\mathbb{R}\Psi(E^\bullet) \longrightarrow \mathbb{R}\Psi(F^\bullet)$ in the derived category $\mathsf{D}^\star(\mathsf{B})$ obtained in this way depends only on the original morphism $E^\bullet \longrightarrow F^\bullet$ in the category $\mathsf{C}^\star(\mathsf{A})$ and not on any additional choices, and also that compositions of morphisms in the category $\mathsf{D}^\star(\mathsf{B})$ are assigned to compositions of morphisms in the category of complexes $\mathsf{C}^\star(\mathsf{A})$, reduces to checking that a morphism of resolving bicomplexes $K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$ in the category J forming a commutative diagram with a zero morphism of complexes $E^\bullet \longrightarrow F^\bullet$ in the category A induces a vanishing morphism between the totalizations of bicomplexes $\tau_{\leqslant d}\Psi(K^{\bullet,\bullet}) \longrightarrow \tau_{\leqslant d}\Psi(L^{\bullet,\bullet})$ in the derived category $\mathsf{D}^\star(\mathsf{B})$. Here one has to apply Lemma B.2(c) and recall that the functor $\Psi \colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$, having a DG-functor structure, transforms contracting homotopies to contracting homotopies and preserves the equations they may satisfy.

A morphism of complexes in the category A is homotopic to zero if and only if it factorizes through a cone of an identity morphism of complexes, i. e., through a complex of the form $\cdots \longrightarrow E^{i-1} \oplus E^i \longrightarrow E^i \oplus E^{i+1} \longrightarrow \cdots$. Complexes of this form are easily seen to be taken to zero by the functor $\mathbb{R}\Psi$, which is therefore a well-defined functor from the homotopy category $\mathsf{Hot}^\star(\mathsf{A})$ to the derived category $\mathsf{D}^\star(\mathsf{B})$.

Checking that the functor $\mathbb{R}\Psi \colon \mathsf{Hot}^\star(\mathsf{A}) \longrightarrow \mathsf{D}^\star(\mathsf{B})$ so obtained is triangulated reduces to another application of Lemma B.2(b), guaranteeing any closed morphism

of complexes $E^\bullet \longrightarrow F^\bullet$ in the category A can be included into a commutative diagram with resolution morphisms $E^\bullet \longrightarrow K^{\bullet,\bullet}$ and $F^\bullet \longrightarrow L^{\bullet,\bullet}$, and a morphism of resolving bicomplexes $K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$ in the category J. Since the cone of the closed morphism of bicomplexes $K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$ (taken along the grading i) is a resolution of the cone of the closed morphism $E^\bullet \longrightarrow F^\bullet$, and the functor Ψ, being a DG-functor, preserves the cones of closed morphisms, the functor $\mathbb{R}\Psi$ takes cones to cones.

It remains to show that the functor $\mathbb{R}\Psi$ factorizes through the derived category $\mathsf{D}^\star(\mathsf{A})$. Here one has to consider derived categories of the first and of the second kind separately. One uses Lemma B.2(e) in the former case and Lemma B.2(d) in the latter one, together with the assumption that $\Psi \colon \mathsf{C}^{\geqslant 0}_\mathsf{A}(\mathsf{J}) \longrightarrow \mathsf{C}^{\geqslant 0}(\mathsf{B})$ is an exact functor between exact categories.

Notice that the derived functor $\mathbb{R}\Psi \colon \mathsf{D}^\star(\mathsf{A}) \longrightarrow \mathsf{D}^\star(\mathsf{B})$ that we have constructed does not depend on the choice of an integer d satisfying the condition that $\Psi(\mathsf{C}^{\geqslant 0}_\mathsf{A}(\mathsf{J}))$ is contained in $\mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d}$. Indeed, for any bicomplex $J^{\bullet,\bullet}$ with $J^{\bullet,i} \in \mathsf{C}^{\geqslant 0}_\mathsf{A}(\mathsf{J})$ for all $i \in \mathbb{Z}$ and any integer $d' \geqslant d$ a cone of the natural morphism between the totalizations of bicomplexes $\tau_{\leqslant d}\Psi(J^{\bullet,\bullet}) \longrightarrow \tau_{\leqslant d'}\Psi(J^{\bullet,\bullet})$ is absolutely acyclic.

In the rest of this appendix, our aim is to establish an adjunction property of the left and right derived functors provided by the above construction. For any additive category E, we denote by $\mathsf{C}^-(\mathsf{E})$ the DG-category of bounded above complexes in E. When E is an exact category, the subcategory $\mathsf{C}^{\leqslant 0}(\mathsf{E}) \subset \mathsf{C}^-(\mathsf{E})$ of all the complexes $\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow 0$ and closed morphisms of degree zero between them is endowed with a natural exact category structure where a short sequence of complexes is exact if and only if it is termwise exact in E.

Let A and B be two exact categories containing the images of idempotent endomorphisms of their objects. Let $\mathsf{J} \subset \mathsf{A}$ be a full subcategory, closed under extensions and the passages to the cokernels of admissible monomorphisms in A and such that any object of A is the source of an admissible monomorphism into an object of J. Similarly, let $\mathsf{P} \subset \mathsf{B}$ be a full subcategory, closed under extensions and the passages to the kernels of admissible epimorphisms and such that any object of B is the target of an admissible epimorphism from an object of P. The full subcategories J and P inherit the exact category structures of the ambient categories A and B.

Denote by $\mathsf{C}^{\leqslant 0}_\mathsf{P}(\mathsf{B})$ the full subcategory in $\mathsf{C}^{\leqslant 0}(\mathsf{P})$ consisting of all the complexes $\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0$ in P for which there exits an object $B \in \mathsf{B}$ together with a morphism $B^0 \longrightarrow P$ such that the sequence $\cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow B \longrightarrow 0$ is exact in B. The full subcategory $\mathsf{C}^{\leqslant 0}_\mathsf{B}(\mathsf{P})$ is closed under extensions and the passages to the kernels of admissible epimorphisms in the exact category $\mathsf{C}^{\leqslant 0}(\mathsf{P})$, so it inherits the exact category structure.

Let $\mathsf{C}^{\leqslant 0}(\mathsf{A})^{\geqslant -d} \subset \mathsf{C}^{\leqslant 0}(\mathsf{A})$ denote the full subcategory consisting of all the complexes $\cdots \longrightarrow A^{-1} \longrightarrow A^0 \longrightarrow 0$ in A such that the sequence $\cdots \longrightarrow A^{-d-2} \longrightarrow A^{-d-1} \longrightarrow A^{-d}$ is exact in A. Being closed under extensions and the passages to the kernels of admissible epimorphisms, the full subcategory $\mathsf{C}^{\leqslant 0}(\mathsf{A})^{\geqslant -d}$ inherits the exact category structure of the category $\mathsf{C}^{\leqslant 0}(\mathsf{A})$. Denote by $\mathsf{C}^{[-d,0]}(\mathsf{A})$ the exact category of finite

complexes $A^{-d} \longrightarrow \cdots \longrightarrow A^0$ in A. Then there is an exact functor of canonical truncation $\tau_{\geqslant -d}\colon \mathsf{C}^{\leqslant 0}(\mathsf{A})^{\geqslant -d} \longrightarrow \mathsf{C}^{[-d,0]}(\mathsf{A})$.

As above, let $\Psi\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$ be a DG-functor taking quasi-isomorphisms of complexes belonging to $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J})$ to quasi-isomorphisms of complexes in the exact category B. Assume that the restriction of the functor Ψ to the subcategory $\mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) \subset \mathsf{C}^+(\mathsf{J})$ is an exact functor between exact categories $\Psi\colon \mathsf{C}_\mathsf{A}^{\geqslant 0}(\mathsf{J}) \longrightarrow \mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant d}$. Then the above construction provides the right derived functors $\mathbb{R}\Psi\colon \mathsf{D}^\star(\mathsf{A}) \longrightarrow \mathsf{D}^\star(\mathsf{B})$.

Similarly, let $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$ be a DG-functor taking quasi-isomorphisms of complexes belonging to $\mathsf{C}_\mathsf{B}^{\leqslant 0}(\mathsf{P})$ to quasi-isomorphisms of complexes in the exact category A. Assume that the restriction of the functor Φ to the subcategory $\mathsf{C}_\mathsf{B}^{\leqslant 0}(\mathsf{P}) \subset \mathsf{C}^-(\mathsf{P})$ is an exact functor between exact categories $\Phi\colon \mathsf{C}_\mathsf{B}^{\leqslant 0}(\mathsf{P}) \longrightarrow \mathsf{C}^{\leqslant 0}(\mathsf{A})^{\geqslant -d}$. Composing the restriction of the functor Φ with the functor of canonical truncation, we obtain an exact functor

$$\tau_{\geqslant -d}\Phi\colon \mathsf{C}_\mathsf{B}^{\leqslant 0}(\mathsf{P}) \longrightarrow \mathsf{C}^{[-d,0]}(\mathsf{A}).$$

The construction dual to the above provides a left derived functor

$$\mathbb{L}\Phi\colon \mathsf{D}^\star(\mathsf{B}) \longrightarrow \mathsf{D}^\star(\mathsf{A})$$

for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs. When the exact categories A and B have exact functors of infinite direct sum (resp., product), the full subcategory $\mathsf{P} \subset \mathsf{B}$ is closed under infinite direct sums (resp., products), and the functor Φ preserves infinite direct sums (resp., products), there is also the derived functor $\mathbb{L}\Phi$ acting between the coderived categories D^{co} (resp., the contraderived categories $\mathsf{D}^{\mathsf{ctr}}$) of the exact categories B and A.

Denote by $\mathsf{C}(\mathsf{A})$ and $\mathsf{C}(\mathsf{B})$ the DG-categories of unbounded complexes in the additive categories A and B. Suppose that the DG-functors $\Psi\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$ and $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$, viewed as partially defined DG-functors between the DG-categories $\mathsf{C}(\mathsf{A})$ and $\mathsf{C}(\mathsf{B})$, are partially adjoint to each other. In other words, this means that for any complexes $J^\bullet \in \mathsf{C}^+(\mathsf{J})$ and $P^\bullet \in \mathsf{C}^-(\mathsf{P})$ there is an isomorphism

$$\mathrm{Hom}^\bullet_{\mathsf{C}(\mathsf{A})}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}^\bullet_{\mathsf{C}(\mathsf{B})}(P^\bullet, \Psi(J^\bullet))$$

of complexes of morphisms in the DG-categories $\mathsf{C}(\mathsf{A})$ and $\mathsf{C}(\mathsf{B})$, functorial in the complexes $J^\bullet$ and $P^\bullet$. Let us show that in this case the derived functor $\mathbb{L}\Phi$ is left adjoint to the derived functor $\mathbb{R}\Psi$.

Our approach consists in constructing the adjunction morphisms

$$\mathbb{L}\Phi \circ \mathbb{R}\Psi \longrightarrow \mathrm{id} \quad \text{and} \quad \mathrm{id} \longrightarrow \mathbb{R}\Psi \circ \mathbb{L}\Phi.$$

Since the functors Φ and Ψ, being DG-functors, commute with totalizations of finite complexes of complexes in $\mathsf{C}^-(\mathsf{P})$ and $\mathsf{C}^+(\mathsf{J})$, the question essentially reduces to the case of single objects of the categories A and B viewed as one-term complexes along the grading i. So in the reasoning below we ignore the (generally speaking, doubly unbounded) grading i altogether and concentrate on the (bounded on one side or finite) gradings created by the resolution procedures.

Let $0 \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ be a right resolution of an object $A \in \mathsf{A}$ by objects $J^j \in \mathsf{J}$. Denote by $B^1 \longrightarrow \cdots \longrightarrow B^d$ the finite complex $\tau_{\leqslant d}\Psi(J^\bullet)$ in the category B. Let $\cdots \longrightarrow P^{-1,\bullet} \longrightarrow P^{0,\bullet} \longrightarrow B^\bullet \longrightarrow 0$ be an exact sequences of complexes in the category B with the terms $P^{k,j} \in \mathsf{P}$ concentrated in the interval of gradings $0 \leqslant j \leqslant d$. Then the total complex of the bicomplex $\tau_{\geqslant -d}\Phi(P^{\bullet,\bullet})$, where both the functor Φ and the canonical truncation are applied to complexes along the grading k, represents the object $\mathbb{L}\Phi(\mathbb{R}\Psi(A))$ in the derived category $\mathsf{D}^\star(\mathsf{A})$.

The composition $P^{\bullet,\bullet} \longrightarrow B^\bullet \longrightarrow \Psi(J^\bullet)$ is a closed morphism from the total complex of the bicomplex $P^{\bullet,\bullet}$ to the complex $\Psi(J^\bullet)$ in the category B. Consider the corresponding closed morphism of complexes $\Phi(P^{\bullet,\bullet}) \longrightarrow J^\bullet$ in the category A. Since the bicomplex $P^{k,j}$ is concentrated in the cohomological gradings $0 \leqslant j \leqslant d$ and $k \leqslant 0$, while the complex $J^\bullet$ is concentrated in the cohomological gradings $j \geqslant 0$, the latter morphism factorizes through the admissible epimorphism between the totalizations of bicomplexes $\tau_{\geqslant -d}\Phi(P^{\bullet,\bullet}) \longrightarrow \Phi(P^{\bullet,\bullet})$ and the admissible monomorphism of complexes $\tau_{\leqslant d}J^\bullet \longrightarrow J^\bullet$. We have obtained a closed morphism $\tau_{\geqslant -d}\Phi(P^{\bullet,\bullet}) \longrightarrow \tau_{\leqslant d}J^\bullet$ of complexes in the category A representing the desired adjunction morphism $\mathbb{L}\Phi(\mathbb{R}\Psi(A)) \longrightarrow A$.

The adjunction morphism $\mathrm{id} \longrightarrow \mathbb{R}\Psi \circ \mathbb{L}\Phi$ of functors on the category $\mathsf{D}^\star(\mathsf{B})$ is constructed similarly. Checking that the compositions $\mathbb{R}\Psi \longrightarrow \mathbb{R}\Psi \circ \mathbb{L}\Phi \circ \mathbb{R}\Psi$ and $\mathbb{L}\Phi \longrightarrow \mathbb{L}\Phi \circ \mathbb{R}\Psi \circ \mathbb{L}\Phi$ are the identity morphisms is straightforward.

Appendix C. Coalgebras with Finiteness Conditions

This appendix contains a discussion of Artinian, co-Noetherian, and cocoherent coalgebras. Most of the results below are certainly not new; we present them here for the sake of completeness of the exposition.

Let $\mathcal{C}$ be a coassociative coalgebra (with counit) over a field k. For any k-vector space V the left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ is called the *cofree left $\mathcal{C}$-comodule cogenerated by V*. For any left $\mathcal{C}$-comodule $\mathcal{L}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{L},\, \mathcal{C} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{L}, V),$$

where for any two left $\mathcal{C}$-comodules $\mathcal{L}$ and $\mathcal{M}$ we denote by $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}, \mathcal{M})$ the k-vector space of all morphisms $\mathcal{L} \longrightarrow \mathcal{M}$ in the abelian category of left $\mathcal{C}$-comodules $\mathcal{C}$–comod. Hence cofree $\mathcal{C}$-comodules are injective objects in $\mathcal{C}$–comod. Cofree $\mathcal{C}$-comodules are sufficiently many, so any injective $\mathcal{C}$-comodule is a direct summand of a cofree one. In particular, the left $\mathcal{C}$-comodule $\mathcal{C}$ is called the *cofree $\mathcal{C}$-comodule with one cogenerator*, and finite direct sums of copies of $\mathcal{C}$ are the *finitely cogenerated cofree $\mathcal{C}$-comodules*.

A coassociative coalgebra is called *cosimple* if it has no nonzero proper subcoalgebras. The cosimple k-coalgebras are precisely the dual coalgebras to simple finite-dimensional k-algebras. A coassociative coalgebra $\mathcal{E}$ is called *cosemisimple* if it is a direct sum of cosimple coalgebras, or equivalently, if the category of left $\mathcal{E}$-comodules is semisimple, or if the category of right $\mathcal{E}$-comodules is semisimple.

A coassociative coalgebra without counit $\mathcal{C}'$ is called *conilpotent* if for any element $c' \in \mathcal{C}'$ there exists an integer $n \geqslant 1$ such that c' is annihilated by the iterated coaction map $\mathcal{C}' \longrightarrow \mathcal{C}'^{\otimes n+1}$. Any coassociative coalgebra $\mathcal{C}$ has a unique maximal cosemisimple subcoalgebra $\mathcal{C}^{ss} \subset \mathcal{C}$, which can be also defined as the (direct) sum of all cosimple subcoalgebras in $\mathcal{C}$, or as the minimal subcoalgebra $\mathcal{E} \subset \mathcal{C}$ for which the quotient coalgebra without counit $\mathcal{C}/\mathcal{E}$ is conilpotent [30, Sections 9.0–1].

For any subcoalgebra $\mathcal{E} \subset \mathcal{C}$ and any left $\mathcal{C}$-comodule $\mathcal{M}$, we denote by $_{\mathcal{E}}\mathcal{M}$ the maximal $\mathcal{C}$-subcomodule in $\mathcal{M}$ whose $\mathcal{C}$-comodule structure comes from an $\mathcal{E}$-comodule structure. In other words, $_{\mathcal{E}}\mathcal{M} \subset \mathcal{M}$ is the full preimage of the subspace $\mathcal{E} \otimes_k \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$ under the left coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$. The following assertion is a dual version of Nakayama's lemma for comodules.

Lemma C.1. *Let $\mathcal{E} \subset \mathcal{C}$ be a subcoalgebra such that the quotient coalgebra without counit $\mathcal{C}/\mathcal{E}$ is conilpotent (i. e., $\mathcal{E}$ contains the subcoalgebra $\mathcal{C}^{ss} \subset \mathcal{C}$). Then the subcomodule $_{\mathcal{E}}\mathcal{M}$ is nonzero for any nonzero left $\mathcal{C}$-comodule $\mathcal{M}$.*

Proof. It follows from the conilpotency condition that for every element $x \in \mathcal{M}$ there exists an integer $n \geqslant 1$ such that x is annihilated by the iterated coaction map $\mathcal{M} \longrightarrow (\mathcal{C}/\mathcal{E})^{\otimes n} \otimes_k \mathcal{M}$. Hence the coaction map $\mathcal{M} \longrightarrow \mathcal{C}/\mathcal{E} \otimes_k \mathcal{M}$ cannot be injective for a nonzero left $\mathcal{C}$-comodule $\mathcal{M}$. $\qquad\square$

A left $\mathcal{C}$-comodule is said to be *finitely cogenerated* if it can be embedded as a subcomodule into a finitely cogenerated cofree left $\mathcal{C}$-comodule. Obviously, any subcomodule of a finitely cogenerated cofree $\mathcal{C}$-comodule is finitely cogenerated. One easily checks that the class of finitely cogenerated left $\mathcal{C}$-comodules is closed under extensions in $\mathcal{C}$–comod.

Lemma C.2. (a) *For any finitely cogenerated left $\mathcal{C}$-comodule $\mathcal{L}$ and any subcoalgebra $\mathcal{E} \subset \mathcal{C}$, the left $\mathcal{E}$-comodule $_{\mathcal{E}}\mathcal{L}$ is finitely cogenerated.*

(b) *The cofree left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ with an infinite-dimensional vector space of cogenerators V over a nonzero coalgebra $\mathcal{C}$ is not finitely cogenerated.*

(c) *For any subcoalgebra $\mathcal{E} \subset \mathcal{C}$, a left $\mathcal{E}$-comodule $\mathcal{L}$ is finitely cogenerated if and only if it is finitely cogenerated as a left $\mathcal{C}$-comodule.*

(d) *Let $\mathcal{E} \subset \mathcal{C}$ be a subcoalgebra such that the quotient coalgebra without counit $\mathcal{C}/\mathcal{E}$ is conilpotent. Then a left $\mathcal{C}$-comodule $\mathcal{L}$ is finitely cogenerated if and only if the left $\mathcal{E}$-comodule $_{\mathcal{E}}\mathcal{L}$ is finitely cogenerated.*

(e) *A left $\mathcal{C}$-comodule $\mathcal{L}$ is finitely cogenerated if and only if the left $\mathcal{E}$-comodule $_{\mathcal{E}}\mathcal{L}$ for every cosimple subcoalgebra $\mathcal{E} \subset \mathcal{C}$ is a finite direct sum of copies of the irreducible left $\mathcal{E}$-comodule with the multiplicity of the irreducible left $\mathcal{E}$-comodule in $_{\mathcal{E}}\mathcal{L}$ divided by its multiplicity in the left $\mathcal{E}$-comodule $\mathcal{E}$ bounded by a single constant uniformly over all the cosimple subcoalgebras $\mathcal{E} \subset \mathcal{C}$.*

Proof. Part (a): obviously, for any injective morphism of left $\mathcal{C}$-comodules $\mathcal{L} \longrightarrow \mathcal{M}$, the induced morphism $_{\mathcal{E}}\mathcal{L} \longrightarrow {}_{\mathcal{E}}\mathcal{M}$ is also injective, so it remains to notice the natural isomorphism of $\mathcal{E}$-comodules $_{\mathcal{E}}(\mathcal{C} \otimes_k V) \simeq \mathcal{E} \otimes_k V$ for any k-vector space V. Now it suffices to pick any nonzero finite-dimensional subcoalgebra $\mathcal{E} \subset \mathcal{C}$ in order to deduce

part (b) from the latter isomorphism and part (a). Part (c) follows from the same isomorphism.

Part (d): a morphism of $\mathcal{C}$-comodules $\mathcal{L} \longrightarrow \mathcal{C} \otimes_k V$ is uniquely determined by its composition with the map $\mathcal{C} \otimes_k V \longrightarrow V$ induced by the counit map $\mathcal{C} \longrightarrow k$ of the coalgebra $\mathcal{C}$; and this composition can be an arbitrary k-linear map $\mathcal{L} \longrightarrow V$. Suppose that we are given an injective morphism of $\mathcal{E}$-comodules $_\varepsilon\mathcal{L} \longrightarrow \mathcal{E} \otimes_k V$, where V is a finite-dimensional vector space. Consider the composition $_\varepsilon\mathcal{L} \longrightarrow \mathcal{E} \otimes_k V \longrightarrow V$ and extend it arbitrarily to a k-linear map $\mathcal{L} \longrightarrow V$. The corresponding $\mathcal{C}$-comodule morphism $\mathcal{L} \longrightarrow \mathcal{C} \otimes_k V$ forms a commutative diagram with the injective morphism $_\varepsilon\mathcal{L} \longrightarrow \mathcal{E} \otimes_k V$ and the embeddings $_\varepsilon\mathcal{L} \longrightarrow \mathcal{L}$ and $\mathcal{E} \otimes_k V \longrightarrow \mathcal{C} \otimes_k V$. Denote by $\mathcal{K}$ the kernel of the morphism $\mathcal{L} \longrightarrow \mathcal{C} \otimes_k V$; then the submodule $\mathcal{K} \subset \mathcal{L}$ does not intersect the submodule $_\varepsilon\mathcal{L} \subset \mathcal{L}$, so one has $_\varepsilon\mathcal{K} = 0$. By Lemma C.1, it follows that $\mathcal{K} = 0$. To prove part (e), one applies part (d) to the subcoalgebra $\mathcal{C}^{ss} \subset \mathcal{C}$ and then decomposes $\mathcal{C}^{ss}$ into a direct sum of its cosimple subcoalgebras $\mathcal{E}$. $\qquad\square$

A $\mathcal{C}$-comodule is called *co-Noetherian* if all its quotient $\mathcal{C}$-comodules are finitely cogenerated. The class of co-Noetherian left $\mathcal{C}$-comodules is closed under subobjects, quotient objects, and extensions in the abelian category $\mathcal{C}$–comod, so co-Noetherian left $\mathcal{C}$-comodules form an abelian category. Given a subcoalgebra $\mathcal{E} \subset \mathcal{C}$, an $\mathcal{E}$-comodule is co-Noetherian if and only if it is co-Noetherian as a $\mathcal{C}$-comodule.

A $\mathcal{C}$-comodule is called *Artinian* if every descending chain of its subcomodules terminates. As the class of Artinian objects in any abelian category, the class of Artinian left $\mathcal{C}$-comodules is closed under subobjects, quotient objects, and extensions in the abelian category $\mathcal{C}$–comod, so Artinian left $\mathcal{C}$-comodules form an abelian category. Given a subcoalgebra $\mathcal{E} \subset \mathcal{C}$, an $\mathcal{E}$-comodule is Artinian if and only if it is Artinian as a $\mathcal{C}$-comodule.

Lemma C.3. (a) *Any Artinian $\mathcal{C}$-comodule is co-Noetherian.*

(b) *If the subcoalgebra $\mathcal{C}^{ss} \subset \mathcal{C}$ is finite-dimensional, then any co-Noetherian $\mathcal{C}$-comodule is Artinian.*

Proof. This is a subset of results of [7, Proposition 2.5]. Part (a): it suffices to show that any Artinian left $\mathcal{C}$-comodule $\mathcal{L}$ is finitely cogenerated. Pick a nonzero linear function $\phi_1 \colon \mathcal{L} \longrightarrow k$ and consider the related morphism of left $\mathcal{C}$-comodules $f_1 \colon \mathcal{L} \longrightarrow \mathcal{C}$. Let $\mathcal{L}_1 \subset \mathcal{L}$ denote the kernel of the morphism f_1. Pick a nonzero linear function $\mathcal{L}_1 \longrightarrow k$ and extend it to a linear function $\phi_2 \colon \mathcal{L} \longrightarrow k$. Consider the related morphism of left $\mathcal{C}$-comodules $f_2 \colon \mathcal{L} \longrightarrow \mathcal{C}$; let $\mathcal{L}_2 \subset \mathcal{L}_1$ denote the intersection of the kernels of the morphisms f_1 and f_2, etc. According to the descending chain condition, this process must terminate, which can only happen if the intersection of the kernels of the morphisms $f_1, \ldots, f_n$ is zero for some integer n. We have constructed an injective morphism of left $\mathcal{C}$-comodules $\mathcal{L} \longrightarrow \mathcal{C}^{\oplus n}$.

Part (b): it suffices to show that any descending chain of subcomodules $\mathcal{L} \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots$ with zero intersection $\bigcap_n \mathcal{L}_n = 0$ terminates in a finitely cogenerated left $\mathcal{C}$-comodule $\mathcal{L}$. Indeed, by Lemma C.2(a) together with the assumption of part (b) the subcomodule $_{\varrho^{ss}}\mathcal{L} \subset \mathcal{L}$ is finite-dimensional. Hence the chain of intersections

$_{\mathrm{\scriptstyle ess}}\mathcal{L} \cap \mathcal{L}_i$ stabilizes, and consequently, eventually vanishes, i. e., there exists n for which $_{\mathrm{\scriptstyle ess}}\mathcal{L} \cap \mathcal{L}_n = 0$. Then it follows from Lemma C.1 that $\mathcal{L}_n = 0$. $\qquad\square$

A *left contramodule* $\mathfrak{P}$ over a coassociative coalgebra $\mathcal{D}$ over a field k is a k-vector space endowed with a *left $\mathcal{D}$-contraaction* map $\mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ satisfying the appropriate *contraassociativity* and *contraunitality* equations. Specifically, the two maps $\mathrm{Hom}_k(\mathcal{D}, \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})) \simeq \mathrm{Hom}_k(\mathcal{D} \otimes_k \mathcal{D}, \, \mathfrak{P}) \rightrightarrows \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})$ induced by the comultiplication map $\mathcal{D} \longrightarrow \mathcal{D} \otimes_k \mathcal{D}$ and the contraaction map should have equal compositions with the contraaction map $\mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P}$,

$$\mathrm{Hom}_k(\mathcal{D}, \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})) \simeq \mathrm{Hom}_k(\mathcal{D} \otimes_k \mathcal{D}, \mathfrak{P}) \rightrightarrows \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P},$$

while the composition of the map $\mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})$ induced by the counit map $\mathcal{D} \longrightarrow k$ with the contraaction map should be equal to the identity map on the contramodule $\mathfrak{P}$,

$$\mathfrak{P} \longrightarrow \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P}.$$

The natural isomorphism $\mathrm{Hom}_k(U, \mathrm{Hom}_k(V, W)) \simeq \mathrm{Hom}_k(V \otimes_k U, \, W)$ connecting the tensor product and Hom functors on the category of k-vector spaces is presumed in the first equation.

Left $\mathcal{D}$-contramodules form an abelian category $\mathcal{D}$–contra with an exact forgetful functor to the category of k-vector spaces $\mathcal{D}$–contra $\longrightarrow$ k–vect, preserving infinite products but not infinite direct sums (see [26, Sections 1.1–1.2] and the references therein). For any right $\mathcal{D}$-comodule $\mathcal{N}$ and k-vector space V, the vector space $\mathrm{Hom}_k(\mathcal{N}, V)$ has a natural left $\mathcal{D}$-contramodule structure. In particular, the left $\mathcal{D}$-contramodule $\mathrm{Hom}_k(\mathcal{D}, V)$ is called the *free left $\mathcal{D}$-contramodule generated by V*. For any left $\mathcal{D}$-contramodule $\mathfrak{Q}$, there is a natural isomorphism

$$\mathrm{Hom}^{\mathcal{D}}(\mathrm{Hom}_k(\mathcal{D}, V), \mathfrak{Q}) \simeq \mathrm{Hom}_k(V, \mathfrak{Q}),$$

where for any two left $\mathcal{D}$-contramodules $\mathfrak{P}$ and $\mathfrak{Q}$ we denote by $\mathrm{Hom}^{\mathcal{D}}(\mathfrak{P}, \mathfrak{Q})$ the k-vector space of all morphisms $\mathfrak{P} \longrightarrow \mathfrak{Q}$ in the abelian category $\mathcal{D}$–contra. Hence free $\mathcal{D}$-contramodules are projective objects in $\mathcal{D}$–contra. There are enough of them, so any projective left $\mathcal{D}$-contramodule is a direct summand of a free one. The left $\mathcal{D}$-contramodule $\mathrm{Hom}_k(\mathcal{D}, k)$ is called the *free $\mathcal{D}$-contramodule with one generator*, and the $\mathcal{D}$-contramodules $\mathrm{Hom}_k(\mathcal{D}, k)$ with finite-dimensional k-vector spaces V are the *finitely generated free $\mathcal{D}$-contramodules*.

For any subcoalgebra $\mathcal{E} \subset \mathcal{D}$ and any left $\mathcal{D}$-contramodule $\mathfrak{P}$, we denote by $^{\mathcal{E}}\mathfrak{P}$ the maximal quotient $\mathcal{D}$-contramodule of $\mathfrak{P}$ whose $\mathcal{D}$-contramodule structure comes from an $\mathcal{E}$-contramodule structure. In other words, $^{\mathcal{E}}\mathfrak{P}$ is the cokernel of the composition $\mathrm{Hom}_k(\mathcal{D}/\mathcal{E}, \mathfrak{P}) \longrightarrow \mathfrak{P}$ of the embedding $\mathrm{Hom}_k(\mathcal{D}/\mathcal{E}, \mathfrak{P}) \longrightarrow \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})$ with the contraaction map $\mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. The following assertion is called the *Nakayama lemma for contramodules* over coalgebras over fields.

Lemma C.4. *Let $\mathcal{E} \subset \mathcal{D}$ be a subcoalgebra such that the quotient coalgebra without counit $\mathcal{D}/\mathcal{E}$ is conilpotent. Then the quotient contramodule $^{\mathcal{E}}\mathfrak{P}$ is nonzero for any nonzero left $\mathcal{D}$-contramodule $\mathfrak{P}$.*

Proof. This is [21, Lemma A.2.1]; see also [23, Lemma 1.3.1] and [26, Lemma 2.1]. $\square$

A left $\mathcal{D}$-contramodule is said to be *finitely generated* if it is a quotient contramodule of a finitely generated free left $\mathcal{D}$-contramodule. The class of finitely generated left $\mathcal{D}$-contramodules is closed under extensions and the passages to quotient objects.

Lemma C.5. (a) *For any finitely generated left $\mathcal{D}$-contramodule $\mathfrak{Q}$ and any subcoalgebra $\mathcal{E} \subset \mathcal{D}$, the left $\mathcal{E}$-contramodule ${}^{\mathcal{E}}\mathfrak{Q}$ is finitely generated.*

(b) *The free $\mathcal{D}$-contramodule $\mathrm{Hom}_k(\mathcal{D}, V)$ with an infinite-dimensional vector space of generators V over a nonzero coalgebra $\mathcal{D}$ is not finitely generated.*

(c) *For any subcoalgebra $\mathcal{E} \subset \mathcal{D}$, a left $\mathcal{E}$-contramodule is finitely generated if and only if it is finitely generated as a left $\mathcal{D}$-contramodule.*

(d) *Let $\mathcal{E} \subset \mathcal{D}$ be a subcoalgebra such that the quotient coalgebra without counit $\mathcal{D}/\mathcal{E}$ is conilpotent. Then a left $\mathcal{D}$-contramodule $\mathfrak{Q}$ is finitely generated if and only if the left $\mathcal{D}$-contramodule ${}^{\mathcal{D}}\mathfrak{Q}$ is finitely generated.*

(e) *A left $\mathcal{D}$-contramodule $\mathfrak{Q}$ is finitely generated if and only if the left $\mathcal{E}$-contramodule ${}^{\mathcal{E}}\mathfrak{Q}$ for every simple subcoalgebra $\mathcal{E} \subset \mathcal{D}$ is a finite direct sum of copies of the irreductive left $\mathcal{E}$-contramodule with the multiplicity of the irreductible left $\mathcal{E}$-contramodule in ${}^{\mathcal{E}}\mathfrak{Q}$ divided by its multiplicity in the left $\mathcal{E}$-contramodule $\mathcal{E}^* = \mathrm{Hom}_k(\mathcal{E}, k)$ bounded by a single constant uniformly over all the simple subcoalgebras $\mathcal{E} \subset \mathcal{D}$.*

Proof. The proof is dual-analogous to that of Lemma C.2. To prove parts (a-c), one notices the natural isomorphism ${}^{\mathcal{E}}\mathrm{Hom}_k(\mathcal{D}, V) \simeq \mathrm{Hom}_k(\mathcal{E}, V)$ for any subcoalgebra $\mathcal{E} \subset \mathcal{D}$ and k-vector space V. Part (d): given a surjective morphism of $\mathcal{E}$-contramodules $\mathrm{Hom}_k(\mathcal{E}, V) \longrightarrow {}^{\mathcal{E}}\mathfrak{Q}$ with a finite-dimensional vector space V, one considers the composition $V \longrightarrow \mathrm{Hom}_k(\mathcal{E}, V) \longrightarrow {}^{\mathcal{E}}\mathfrak{Q}$ and lifts it to a k-linear map $V \longrightarrow \mathfrak{Q}$. The corresponding morphism of $\mathcal{D}$-contramodules $\mathrm{Hom}_k(\mathcal{D}, V) \longrightarrow \mathfrak{Q}$ is surjective by, since one has ${}^{\mathcal{E}}\mathfrak{K} = 0$ for its cokernel $\mathfrak{K}$. To prove part (e), one applies part (d) to the subcoalgebra $\mathcal{D}^{\mathrm{ss}} \subset \mathcal{D}$ and applies [21, Lemma A.2.2] in order to decompose the $\mathcal{D}^{\mathrm{ss}}$-contramodule ${}^{\mathcal{D}^{\mathrm{ss}}}\mathfrak{Q}$ into a product of contramodules over the simple subcoalgebras $\mathcal{E} \subset \mathcal{D}^{\mathrm{ss}}$. $\square$

A left $\mathcal{D}$-contramodule is called *Noetherian* if all its subcontramodules are finitely generated. The class of Noetherian left $\mathcal{D}$-contramodules is closed under subobjects, quotient objects, and extensions in the abelian category $\mathcal{D}$–contra, so Noetherian left $\mathcal{D}$-contramodules form an abelian category. Given a subcoalgebra $\mathcal{E} \subset \mathcal{D}$, an $\mathcal{E}$-contramodule is Noetherian if and only if it is Noetherian as a $\mathcal{D}$-contramodule.

A $\mathcal{D}$-contramodule is called *co-Artinian* if every ascending chain of its subcomodules terminates. As the similar class of objects in any abelian category, the class of co-Artinian left $\mathcal{D}$-contramodules is closed under subobjects, quotient objects, and extensions in the abelian category $\mathcal{D}$–contra, so co-Artinian left $\mathcal{D}$-contramodules form an abelian category. Given a subcoalgebra $\mathcal{E} \subset \mathcal{D}$, an $\mathcal{E}$-contramodule is co-Artinian if and only if it is co-Artinian as a $\mathcal{D}$-contramodule.

Lemma C.6. (a) *Any co-Artinian $\mathcal{D}$-contramodule is Noetherian.*

(b) *If the subcoalgebra $\mathcal{D}^{\mathrm{ss}} \subset \mathcal{D}$ is finite-dimensional, then any Noetherian $\mathcal{D}$-contramodule is co-Artinian.*

Proof. Part (a): it suffices to show that any co-Artinian left $\mathcal{D}$-contramodule $\mathfrak{Q}$ is finitely generated. Pick an element $q_1 \in \mathfrak{Q}$ and consider the related morphism of left $\mathcal{D}$-contramodules $f_1 \colon \mathcal{D}^* = \operatorname{Hom}_k(\mathcal{D}, k) \longrightarrow \mathfrak{Q}$. Pick an element $q_2 \in \mathfrak{Q}$ outside of the image of f_1, consider the related morphism $f_2 \colon \mathcal{D}^* \longrightarrow \mathfrak{Q}$, pick an element $q_3 \in \mathfrak{Q}$ outside of the sum of the images of f_1 and f_2, etc. According to the ascending chain condition, this process must terminate, which means that the sum of the images of the morphisms $f_1, \ldots, f_n$ is the whole of $\mathfrak{Q}$ for some integer n. We have constructed as surjective morphism of left $\mathcal{D}$-contramodules $\mathcal{D}^{*\oplus n} \longrightarrow \mathfrak{Q}$.

Part (b): it suffices to show that an ascending chain of subcontramodules $\mathfrak{Q}_1 \subset \mathfrak{Q}_2 \subset \cdots \subset \mathfrak{Q}$ terminates in a finitely generated left $\mathcal{D}$-contramodule $\mathfrak{Q}$ provided that there is no proper subcontramodule in $\mathfrak{Q}$ containing all the subcontramodules $\mathfrak{Q}_n$. Indeed, by Lemma C.5(a) together with the assumption of part (b) the maximal quotient $\mathcal{D}^{\mathrm{ss}}$-contramodule ${}^{\mathcal{D}^{\mathrm{ss}}}\mathfrak{Q}$ of the $\mathcal{D}$-contramodule $\mathfrak{Q}$ is finite-dimensional. Hence the chain of the images of the subcontramodules $\mathfrak{Q}_n \subset \mathfrak{Q}$ in ${}^{\mathcal{D}^{\mathrm{ss}}}\mathfrak{Q}$ stabilizes, and consequently, eventually reaches the whole of ${}^{\mathcal{D}^{\mathrm{ss}}}\mathfrak{Q}$, i. e., there exists n for which the composition $\mathfrak{Q}_n \longrightarrow \mathfrak{Q} \longrightarrow {}^{\mathcal{D}^{\mathrm{ss}}}\mathfrak{Q}$ is surjective. Then one has ${}^{\mathcal{D}^{\mathrm{ss}}}(\mathfrak{Q}/\mathfrak{Q}_n) = 0$, and it follows from Lemma C.4 that $\mathfrak{Q}_n = \mathfrak{Q}$. $\qquad\square$

Example C.7. Let $\mathcal{C}$ be an infinite-dimensional cosemisimple coalgebra. Then the left $\mathcal{C}$-comodule $\mathcal{C}$ is co-Noetherian, but not Artinian. Similarly, the left $\mathcal{C}$-contramodule $\mathcal{C}^*$ is Noetherian, but not co-Artinian. It follows that the classes of Artinian and co-Noetherian left comodules over a coalgebra $\mathcal{D}$ coincide if and only if the classes of co-Artinian and Noetherian left contramodules over $\mathcal{D}$ coincide and if and only if the subcoalgebra $\mathcal{D}^{\mathrm{ss}} \subset \mathcal{D}$ is finite-dimensional.

A left $\mathcal{D}$-contramodule is said to be *finitely presented* if it is the cokernel of a morphism of finitely generated free left $\mathcal{D}$-contramodules. Clearly, the cokernel of a morphism from a finitely generated left $\mathcal{D}$-contramodule to a finitely presented one is finitely presented. It is easy to check that an extension of finitely presented left $\mathcal{D}$-contramodules is finitely presented.

A left $\mathcal{C}$-comodule is said to be *finitely copresented* if it is the kernel of a morphism of finitely cogenerated cofree $\mathcal{C}$-comodules. Clearly, the kernel of a morphism from a finitely copresented left $\mathcal{C}$-comodule to a finitely cogenerated one is finitely copresented; an extension of finitely copresented left $\mathcal{C}$-comodules is finitely copresented.

Lemma C.8. (a) *The cokernel of an injective morphism from a finitely copresented $\mathcal{C}$-comodule to a finitely cogenerated one is finitely cogenerated.*

(b) *The kernel of a surjective morphism from a finitely generated $\mathcal{D}$-contramodule to a finitely presented one is finitely generated.*

Proof. Part (a): let $\mathcal{L}$ be the kernel of a morphism of finitely cogenerated cofree $\mathcal{C}$-comodules $\mathcal{I} \longrightarrow \mathcal{J}$, let $\mathcal{M}$ be a finitely cogenerated $\mathcal{C}$-comodule, and let $\mathcal{L} \longrightarrow \mathcal{M}$ be an injective morphism with the cokernel $\mathcal{K}$. Denote by $\mathcal{N}$ the fibered coproduct of $\mathcal{C}$-comodules $\mathcal{I}$ and $\mathcal{M}$ over the $\mathcal{C}$-comodule $\mathcal{L}$; then there are exact sequences of $\mathcal{C}$-comodules $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{J}$ and $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{N} \longrightarrow \mathcal{K} \longrightarrow 0$.

Now the $\mathcal{C}$-comodule $\mathcal{N}$ is finitely cogenerated as an extension of finitely cogenerated $\mathcal{C}$-comodules; and the $\mathcal{C}$-comodule $\mathcal{K}$ is a direct summand of $\mathcal{N}$, because the $\mathcal{C}$-comodule $\mathcal{J}$ is injective. The proof of part (b) is analogous. $\qquad\square$

The dual vector space $\mathcal{D}^*$ to a coassociative coalgebra $\mathcal{D}$ has a natural structure of topological associative algebra. There is but a slight ambiguity in its definition in that one has to make a decision about the order of the factors in the multiplication operation, i. e., which one of the two opposite algebras is to be denoted by $\mathcal{D}^*$ and which one by $\mathcal{D}^{*\mathrm{op}}$. We prefer the convention according to which right $\mathcal{D}$-comodules $\mathcal{N}$ become discrete right $\mathcal{D}^*$-modules; then the dual vector space $\mathcal{N}^*$ is a left $\mathcal{D}^*$-module (see [26, Sections 1.3–4] for a further discussion). Any left $\mathcal{D}$-contramodule has an underlying structure of left $\mathcal{D}^*$-module (see [26, Section 2.3] and [21, Section A.1.2]).

One observes that a left $\mathcal{D}$-contramodule is finitely generated if and only if its underlying left $\mathcal{D}^*$-module is finitely generated. It follows that a left $\mathcal{D}$-contramodule is finitely presented if and only if its underlying left $\mathcal{D}^*$-module is.

Proposition C.9. (a) *The restrictions of the functor $\mathcal{L} \longmapsto \mathcal{L}^* = \mathrm{Hom}_k(\mathcal{L}, k)$ and the forgetful functor $\mathcal{D}$–contra $\longrightarrow \mathcal{D}^*$–mod provide an anti-equivalence between the additive category of finitely copresented right $\mathcal{D}$-comodules and the additive category of finitely presented left $\mathcal{D}$-contramodules, and an isomorphism between the latter category and the additive category of finitely presented left $\mathcal{D}^*$-modules.*

(b) *For any right $\mathcal{D}$-comodule $\mathcal{N}$ and any finitely copresented right $\mathcal{D}$-comodule $\mathcal{L}$, the functor $\mathcal{N} \longmapsto \mathcal{N}^* = \mathrm{Hom}_k(\mathcal{N}, k)$ and the forgetful functor $\mathcal{D}$–contra $\longrightarrow \mathcal{D}^*$–mod induce isomorphisms of the Hom spaces*

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(\mathcal{N}, \mathcal{L}) \simeq \mathrm{Hom}^{\mathcal{D}}(\mathcal{L}^*, \mathcal{N}^*) \simeq \mathrm{Hom}_{\mathcal{D}^*}(\mathcal{L}^*, \mathcal{N}^*)$$

in the categories of right $\mathcal{D}$-comodules, left $\mathcal{D}$-contramodules, and left $\mathcal{D}^*$-modules.

Proof. Since the functor Hom preserves kernels in its second argument and transforms cokernels in its first argument into kernels, it suffices to prove part (b) for finitely generated cofree right $\mathcal{D}$-comodules $\mathcal{L} = V \otimes_k \mathcal{D}$, where V is a finite-dimensional k-vector space. Then $\mathcal{L}^* \simeq \mathrm{Hom}_k(\mathcal{D}, V^*) \simeq \mathcal{D}^* \otimes_k V^*$ is a finitely generated free left $\mathcal{D}$-contramodule and a finitely generated free left $\mathcal{D}^*$-module. One easily computes $\mathrm{Hom}_{\mathcal{D}^{\mathrm{op}}}(\mathcal{N}, \mathcal{D} \otimes_k V) \simeq \mathrm{Hom}_k(\mathcal{N}, V)$, $\mathrm{Hom}^{\mathcal{D}}(\mathrm{Hom}_k(\mathcal{D}, V^*), \mathcal{N}^*) \simeq \mathrm{Hom}_k(V^*, \mathcal{N}^*)$, and $\mathrm{Hom}_{\mathcal{D}^*}(\mathcal{D}^* \otimes_k V^*, \mathcal{N}^*) \simeq \mathrm{Hom}_k(V^*, \mathcal{N}^*)$, implying part (b). Part (a) immediately follows from the same computation of Hom spaces. $\qquad\square$

Lemma C.10. (a) *A left $\mathcal{D}$-contramodule is co-Artinian if and only if it is a Noetherian left $\mathcal{D}^*$-module.*

(b) *A right $\mathcal{D}$-comodule $\mathcal{L}$ is Artinian if and only if dual vector space $\mathcal{L}^*$ is a Noetherian left $\mathcal{D}^*$-module.*

(c) *A right $\mathcal{D}$-comodule $\mathcal{L}$ is co-Noetherian provided that its dual vector space $\mathcal{L}^*$ is a Noetherian left $\mathcal{D}$-contramodule.*

Proof. Part (a): one notices that a $\mathcal{D}^*$-module is Noetherian if and only if any ascending chain of its finitely generated submodules terminates. Similarly, a $\mathcal{D}$-contramodule is co-Artinian if and only if any ascending chain of its finitely generated subcontramodules terminates. Finally, the classes of finitely generated $\mathcal{D}^*$-submodules and finitely generated $\mathcal{D}$-subcontramodules in any given $\mathcal{D}$-contramodule coincide.

Part (b) is again a subset of [7, Proposition 2.5]. To any descending chain of $\mathcal{D}$-subcomodules in $\mathcal{L}$ one can assign the ascending chain of their orthogonal complements, which are $\mathcal{D}^*$-submodules in $\mathcal{L}^*$. Conversely, in view of Proposition C.9(b), any finitely generated $\mathcal{D}^*$-submodule in $\mathcal{L}^*$ is the orthogonal complement to a certain $\mathcal{D}^*$-subcomodule in $\mathcal{L}$. Part (c): for any quotient comodule of $\mathcal{L}$, there is its dual subcontramodule in $\mathcal{L}^*$. It remains to notice that a right $\mathcal{D}$-comodule $\mathcal{N}$ is finitely cogenerated if and only if its dual left $\mathcal{D}$-contramodule $\mathcal{N}^*$ is finitely generated. $\square$

A finitely cogenerated left $\mathcal{C}$-comodule is called *cocoherent* if every its finitely cogenerated quotient comodule is finitely copresented. Using Lemma C.8(a), one can show that the class of cocoherent left $\mathcal{C}$-comodules is closed under the operations of the passage to the kernels, cokernels, and extensions in the abelian category $\mathcal{C}$–comod; so cocoherent left $\mathcal{C}$-comodules form an abelian category.

Analogously, a finitely presented left $\mathcal{D}$-contramodule is called *coherent* if every its finitely generated subcontramodule is finitely presented. Using Lemma C.8(b), one shows that the class of coherent left $\mathcal{D}$-contramodules is closed under the passages to the kernels, cokernels, and extensions in the abelian category $\mathcal{D}$–comod; so coherent left $\mathcal{D}$-contramodules form an abelian category.

Lemma C.11. (a) *A left $\mathcal{D}$-contramodule is coherent if and only if its underlying left $\mathcal{D}^*$-module is coherent. The abelian categories of coherent left $\mathcal{D}$-contramodules and coherent left $\mathcal{D}^*$-modules are isomorphic.*

(b) *A right $\mathcal{D}$-comodule $\mathcal{L}$ is cocoherent if and only if its dual left $\mathcal{D}^*$-module $\mathcal{L}^*$ is coherent. The abelian categories of cocoherent right $\mathcal{D}^*$-comodules and coherent left $\mathcal{D}^*$-modules are anti-equivalent.*

Proof. In view of Proposition C.9(a), it suffices to check the first assertion in each of the parts (a) and (b). In part (a), one uses the bijection between finitely generated $\mathcal{D}$-subcontramodules and finitely generated $\mathcal{D}^*$-submodules of a given $\mathcal{D}$-contramodule, together with the fact that a $\mathcal{D}$-contramodule is finitely presented if and only if it is finitely presented as a $\mathcal{D}^*$-module. In part (b), one uses the bijection between finitely cogenerated quotient $\mathcal{D}$-modules of $\mathcal{L}$ and finitely generated $\mathcal{D}^*$-submodules of $\mathcal{L}^*$, together with the fact that a $\mathcal{D}$-comodule is finitely copresented if and only if its dual $\mathcal{D}^*$-module is finitely presented. $\square$

A coalgebra $\mathcal{C}$ is called *left co-Noetherian* if any quotient comodule of a finitely cogenerated left $\mathcal{C}$-comodule is finitely cogenerated, or equivalently, if the left $\mathcal{C}$-comodule $\mathcal{C}$ is co-Noetherian. Over a left co-Noetherian coalgebra $\mathcal{C}$, finitely cogenerated left comodules form an abelian category. By Lemma C.2(c), any subcoalgebra of a left co-Noetherian coalgebra is left co-Noetherian.

A coalgebra $\mathcal{D}$ is called *right Artinian* if any finitely cogenerated right $\mathcal{D}$-comodule is Artinian, or equivalently, if the right $\mathcal{D}$-comodule $\mathcal{D}$ is Artinian, or if any finitely generated left $\mathcal{D}$-contramodule is co-Artinian, or if the left $\mathcal{D}$-contramodule $\mathcal{D}^* = \mathrm{Hom}_k(\mathcal{D}, k)$ is co-Artinian (see Lemma C.10(a-b) for a proof of the equivalence between the second and the fourth of these conditions). A coalgebra $\mathcal{D}$ is right Artinian if and only if its dual algebra $\mathcal{D}^*$ is left Noetherian. Any subcoalgebra of a right Artinian coalgebra is right Artinian.

According to Lemma C.3 and Example C.7, any left Artinian coalgebra $\mathcal{C}$ is left co-Noetherian, but the converse is not generally true. More precisely, a coalgebra $\mathcal{C}$ is left Artinian if and only if it is left co-Noetherian and its maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}} \subset \mathcal{C}$ is finite-dimensional.

Examples C.12. The functor $\mathcal{C} \longmapsto \mathcal{C}^*$ is an anti-equivalence between the category of coassociative coalgebras and the category of pro-finite-dimensional topological associative algebras, so one can describe coalgebras in terms of their dual topological algebras. In particular, the topological algebra of formal Taylor power series in commuting variables $k[[z_1, \ldots, z_m]]$ corresponds to a certain cocommutative coalgebra $\mathcal{C}$. The algebra $k[[z_1, \ldots, z_m]]$ is Noetherian, so the coalgebra $\mathcal{C}$ is Artinian. Hence all the subcoalgebras of $\mathcal{C}$ are Artinian (and consequently, co-Noetherian), too. These are precisely the coalgebras dual to the topological algebras of functions on the formal completions of algebraic varieties over k at their closed points defined over k. Given a field extension $k \subset \ell$, a coalgebra $\mathcal{C}$ over the field k is Artinian or co-Noetherian whenever the coalgebra $\ell \otimes_k \mathcal{C}$ over the field ℓ is. Hence it follows that all the coalgebras dual to the topological algebras of functions on the formal completions of varieties over k at their closed points are Artinian.

Moreover, there are many noncocommutative Artinian coalgebras, like, e. g., the coalgebra dual to the algebra of quantum formal power series $k\{\{z_1, \cdots, z_m\}\}$ with the relations $z_i z_j = q_{i,j} z_j z_i$ for all $i < j$, with any constants $q_{i,j} \in k^*$.

A coalgebra $\mathcal{D}$ is called *right cocoherent* if any finitely cogenerated quotient comodule of a finitely copresented right $\mathcal{D}$-comodule is finitely copresented, or equivalently, if the right $\mathcal{D}$-comodule $\mathcal{D}$ is cocoherent. Equivalently, a coalgebra $\mathcal{D}$ is right cocoherent if any finitely generated subcontramodule of a finitely presented left $\mathcal{D}$-contramodule is finitely presented, or if the right $\mathcal{D}$-contramodule $\mathcal{D}^*$ is finitely presented. Over a right coherent coalgebra $\mathcal{D}$, both the finitely copresented right $\mathcal{D}$-comodules and the finitely presented left $\mathcal{D}$-contramodules form abelian categories. A coalgebra $\mathcal{D}$ is right cocoherent if and only if its dual algebra $\mathcal{D}^*$ is left coherent (see Lemma C.11). Any left co-Noetherian coalgebra $\mathcal{C}$ is left cocoherent, and any finitely cogenerated left $\mathcal{C}$-comodule is finitely copresented.

The *contratensor product* $\mathcal{N} \odot_{\mathcal{D}} \mathfrak{P}$ of a right $\mathcal{D}$-comodule $\mathcal{N}$ and a left $\mathcal{D}$-contramodule $\mathfrak{P}$ [26, Section 3.1] is a k-vector space constructed as the cokernel of (the difference of) the pair of maps

$$\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \rightrightarrows \mathcal{N} \otimes_k \mathfrak{P},$$

one which is induced by the $\mathcal{D}$-contraaction in $\mathfrak{P}$, while the other one is the composition $\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathcal{N} \otimes_k \mathcal{D} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathcal{N} \otimes_k \mathfrak{P}$ of the map induced by the right $\mathcal{D}$-coaction map $\mathcal{N} \longrightarrow \mathcal{N} \otimes_k \mathcal{D}$ and the map induced by the evaluation map $\mathcal{D} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathfrak{P}$. The functor of contratensor product of comodules and contramodules over a coalgebra $\mathcal{D}$ is right exact.

For any right $\mathcal{D}$-comodule $\mathcal{N}$ and a k-vector space V there is a natural isomorphism of k-vector spaces
$$\mathcal{N} \odot_{\mathcal{D}} \mathrm{Hom}_k(\mathcal{D}, V) \simeq \mathcal{N} \otimes_k V,$$
while for any right $\mathcal{D}$-comodule $\mathcal{N}$, any left $\mathcal{D}$-contramodule $\mathfrak{P}$, and a k-vector space V there is a natural isomorphism of k-vector spaces
$$\mathrm{Hom}_k(\mathcal{N} \odot_{\mathcal{D}} \mathfrak{P}, \, V) \simeq \mathrm{Hom}^{\mathcal{D}}(\mathfrak{P}, \mathrm{Hom}_k(\mathcal{N}, V)).$$

The *cotensor product* $\mathcal{N} \square_{\mathcal{C}} \mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ [26, Sections 2.5–6] is a k-vector space constructed as the kernel of the pair of maps
$$\mathcal{N} \otimes_k \mathcal{M} \rightrightarrows \mathcal{N} \otimes_k \mathcal{C} \otimes_k \mathcal{M},$$
one of which is induced by the right $\mathcal{C}$-coaction in $\mathcal{N}$ and the other one by the left $\mathcal{C}$-coaction in $\mathcal{M}$. The functor of cotensor product of comodules over a coalgebra $\mathcal{C}$ is left exact.

For any right $\mathcal{C}$-comodule $\mathcal{N}$, left $\mathcal{C}$-comodule $\mathcal{M}$, and k-vector space V there are natural isomorphisms of k-vector spaces
$$\mathcal{N} \square_{\mathcal{C}} (\mathcal{C} \otimes_k V) \simeq \mathcal{N} \otimes_k V \quad \text{and} \quad (V \otimes_k \mathcal{C}) \square_{\mathcal{C}} \mathcal{M} \simeq V \otimes_k \mathcal{M}.$$
For any left $\mathcal{C}$-comodule $\mathcal{M}$ and any subcoalgebra $\mathcal{E} \subset \mathcal{C}$ there is a natural isomorphism of left $\mathcal{E}$-comodules
$$_{\mathcal{E}}\mathcal{M} \simeq \mathcal{E} \square_{\mathcal{C}} \mathcal{M},$$
where the left $\mathcal{E}$-comodule structure on the cotensor product is induced by the left $\mathcal{E}$-comodule structure on $\mathcal{E}$.

The k-vector space of *cohomomorphisms* $\mathrm{Cohom}_{\mathcal{D}}(\mathcal{M}, \mathfrak{P})$ from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{D}$-contramodule $\mathfrak{P}$ is a k-vector space constructed as the cokernel of the pair of maps
$$\mathrm{Hom}_k(\mathcal{D} \otimes_k \mathcal{M}, \, \mathfrak{P}) \simeq \mathrm{Hom}_k(\mathcal{M}, \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P})) \rightrightarrows \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}),$$
one of which is induced by the left $\mathcal{D}$-coaction in $\mathcal{M}$ and the other one by the left $\mathcal{D}$-contraaction in $\mathfrak{P}$. The functor of cohomomorphisms from left comodules to left contramodules over a coalgebra $\mathcal{D}$ is right exact.

For any left $\mathcal{D}$-comodule $\mathcal{M}$, left $\mathcal{D}$-contramodule $\mathfrak{P}$, and k-vector space V there are natural isomorphisms of k-vector spaces
$$\mathrm{Cohom}_{\mathcal{D}}(\mathcal{D} \otimes_k V, \mathfrak{P}) \simeq \mathrm{Hom}_k(V, \mathfrak{P}) \quad \text{and} \quad \mathrm{Cohom}_{\mathcal{D}}(\mathcal{M}, \mathrm{Hom}_k(\mathcal{D}, V)) \simeq \mathrm{Hom}_k(\mathcal{M}, V).$$

For any left $\mathcal{D}$-contramodule $\mathfrak{P}$ and any subcoalgebra $\mathcal{E} \subset \mathcal{D}$ there is a natural isomorphism of left $\mathcal{E}$-contramodules
$$^{\mathcal{E}}\mathfrak{P} \simeq \mathrm{Cohom}_{\mathcal{D}}(\mathcal{E}, \mathfrak{P}),$$

where the left $\mathcal{E}$-contramodule structure on the Cohom space is induced by the right $\mathcal{E}$-comodule structure on $\mathcal{E}$.

Lemma C.13. *For any right $\mathcal{D}$-comodule $\mathcal{N}$ and any left $\mathcal{D}$-contramodule $\mathfrak{P}$ there is a natural surjective map of k-vector spaces from the tensor product over the algebra $\mathcal{D}^*$ to the contratensor product over the coalgebra $\mathcal{D}$*

$$\mathcal{N} \otimes_{\mathcal{D}^*} \mathfrak{P} \longrightarrow \mathcal{N} \odot_{\mathcal{D}} \mathfrak{P}.$$

This map is an isomorphism, at least, whenever either
  (a) *the left $\mathcal{D}$-contramodule $\mathfrak{P}$ is finitely presented, or*
  (b) *the coalgebra $\mathcal{D}$ is left co-Noetherian.*

Proof. To construct the surjective k-linear map in question, one notices that the tensor product $\mathcal{N} \otimes_{\mathcal{D}^*} \mathfrak{P}$ is the cokernel of a natural map $\mathcal{N} \otimes_k \mathcal{D}^* \otimes_k \mathfrak{P} \longrightarrow \mathcal{N} \otimes_k \mathfrak{P}$, while the contratensor product $\mathcal{N} \square_{\mathcal{D}} \mathfrak{P}$ is the cokernel of a map $\mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}) \longrightarrow \mathcal{N} \otimes_k \mathfrak{P}$. These two maps form a commutative diagram with the natural embedding

$$\mathcal{N} \otimes_k \mathcal{D}^* \otimes_k \mathfrak{P} \longrightarrow \mathcal{N} \otimes_k \mathrm{Hom}_k(\mathcal{D}, \mathfrak{P}).$$

To prove part (a), one considers the induced map of the dual vector spaces

$$(\mathcal{N} \square_{\mathcal{D}} \mathfrak{P})^* \simeq \mathrm{Hom}^{\mathcal{D}}(\mathfrak{P}, \mathcal{N}^*) \longrightarrow \mathrm{Hom}_{\mathcal{D}^*}(\mathfrak{P}, \mathcal{N}^*) \simeq (\mathcal{N} \otimes_{\mathcal{D}^*} \mathfrak{P})^*$$

and applies Proposition C.9.

To prove part (b), notice that any right $\mathcal{D}$-comodule $\mathcal{N}$ is the union of its maximal $\mathcal{E}$-subcomodules $\mathcal{N}_{\mathcal{E}}$ over all the finite-dimensional subcoalgebras $\mathcal{E} \subset \mathcal{D}$. Since bothe the tensor and the contratensor products preserve inductive limits in their first arguments, it suffices to consider the case of a right $\mathcal{E}$-comodule $\mathcal{N} = \mathcal{N}_{\mathcal{E}}$. Then one has $\mathcal{N} \otimes_{\mathcal{D}^*} \mathfrak{P} \simeq \mathcal{N} \otimes_{\mathcal{E}^*} (\mathcal{E}^* \otimes_{\mathcal{D}^*} \mathfrak{P})$ and $\mathcal{N} \odot_{\mathcal{D}} \mathfrak{P} \simeq \mathcal{N} \otimes_{\mathcal{E}^*} {}^{\mathcal{E}}\mathfrak{P}$, so it remains to show that the natural map

$$\mathcal{E}^* \otimes_{\mathcal{D}^*} \mathfrak{P} \longrightarrow {}^{\mathcal{E}}\mathfrak{P} \simeq \mathcal{E}^* \odot_{\mathcal{D}} \mathfrak{P} \simeq \mathrm{Cohom}_{\mathcal{D}}(\mathcal{E}, \mathfrak{P})$$

is an isomorphism. For this purpose, one presents the left $\mathcal{D}$-comodule $\mathcal{E}$ as the kernel of a morphism of finitely cogenerated cofree left $\mathcal{D}$-comodules and uses the right exactness property of the functor $\mathrm{Cohom}_{\mathcal{D}}$ together with the natural isomorphism

$$\mathcal{J}^* \otimes_{\mathcal{D}^*} \mathfrak{P} = (V^* \otimes_k \mathcal{D}^*) \otimes_{\mathcal{D}^*} \mathfrak{P} \simeq \mathrm{Cohom}_{\mathcal{D}}(\mathcal{D} \otimes_k V, \mathfrak{P}) = \mathrm{Cohom}_{\mathcal{D}}(\mathcal{J}, \mathfrak{P})$$

for a finitely cogenerated cofree left $\mathcal{D}$-comodule $\mathcal{J} = \mathcal{D} \otimes_k V$ and any left $\mathcal{D}$-contramodule $\mathfrak{P}$. $\qquad\square$

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