Unfinished preliminary version. Comments appreciated

NONHOMOGENEOUS KOSZUL DUALITY AND $\mathcal{D}$–Ω

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The main goal of this paper is to establish a general algebraic framework, sufficient, e. g., for proving the equivalence of an appropriately defined derived category of DG-modules over the de Rham complex Ω_X with the derived category of $\mathcal{D}_X$-modules—all of this over an arbitrary smooth algebraic variety X (of any characteristic, actually). The main references are my old paper *Nonhomogeneous Quadratic Duality and Curvature* published in Functional Analysis and its Applications vol. **27**, #3, 1993 and Bernhard Keller's *Deriving DG-categories*, Annales Scientifiques de l'Ecole Normale Superieure vol. **27**, 1994. The plan of what follows is here:

(1) Quadratic duality and Koszul algebras over base rings
(2) Nonhomogeneous quadratic rings; duality with CDG-algebras
(3) Coalgebras over rings
(4) Quasi-differential structures
(5) The Poincare–Birkhoff–Witt theorem over a base ring
(6) Derived categories of the first and the second kind
(7) Koszul duality between the derived category of modules and that of quasi-differential comodules
(8) General Koszul duality

1. Quadratic Duality and Koszul Algebras over Base Rings

1.1. Quadratic duality. A graded ring $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is called *quadratic* if it is generated by A_1 over A_0 with relations in degree 2 only. This means, in other words, that if one considers the graded ring freely generated by A_0-bimodule A_1, i. e., the graded ring T_{A_0,A_1} with components

$$A_1 \otimes_{A_0} A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1,$$

then the ring A is a quotient ring of T_{A_0,A_1} by an ideal generated by a certain sub-bimodule R_A in $A_1 \otimes_{A_0} A_1$.

We will always restrict ourselves to quadratic rings A which satisfy some flatness, or even projectivity conditions as either left or right modules over A_0. When we finally get to Koszul rings, it will be simply that A is either left, or right flat (or projective) module over A_0. For general quadratic rings, it is sensible to use somewhat weaker conditions, namely as follows.

A quadratic ring A is called 2-left projectively finite, if both left A_0-modules A_1 and A_2 are projective and finitely generated. A quadratic ring A is called 3-left projectively finite, if the same applies to A_1, A_2, and A_3. Further conditions of this kind are not very sensible to consider for general quadratic rings.

Proposition 1. *There is a duality (anti-equivalence) between the categories of 2-left projectively finite quadratic rings A and 2-right projectively finite quadratic rings B. This duality sends 3-left projectively finite quadratic rings to 3-right projectively finite quadratic rings. The duality functor is defined by the formulas* $B_0 = A_0$, $B_1 = \mathrm{Hom}_{A_0\text{-left}}(A_1, A_0)$, $A_1 = \mathrm{Hom}_{B_0\text{-right}}(B_1, B_0)$, $B_2 = \mathrm{Hom}_{A_0\text{-left}}(R_A, A_0)$, $R_B = \mathrm{Hom}_{A_0\text{-left}}(A_2, A_0)$, *and vise versa.*

The proof of Proposition is based on the following lemma about commutation of "bimodule duality" with bimodule tensor product:

Lemma 2. *Let R, S, T be rings, M be an (R, S)-bimodule and N be an (S, T)-bimodule. Suppose that N is projective and finitely generated as a left S-module. Then there is a natural isomorphism of (T, R)-bimodules*

$$\mathrm{Hom}_{S\text{-left}}(N, S) \otimes_S \mathrm{Hom}_{R\text{-left}}(M, R) \longrightarrow \mathrm{Hom}_{R\text{-left}}(M \otimes_S N, R).$$

Analogously, if M is projective and finitely generated as a right S-module, then there is a natural isomorphism of (T, R)-bimodules

$$\mathrm{Hom}_{T\text{-right}}(N, T) \otimes_S \mathrm{Hom}_{S\text{-right}}(M, S) \longrightarrow \mathrm{Hom}_{T\text{-right}}(M \otimes_S N, T)$$

Proof of Lemma. The first map is defined by the formula $(f\otimes g)(m\otimes n) = g(mf(n))$. In order to prove that this is an isomorphism, one should forget about the T-module structures (which aren't used in the construction of the map) and then reduce the question to the trivial case $N = S$. The second map is given by $(f \otimes g)(m \otimes n) = f(g(m)n)$; the rest of proof is analogous. □

The only difference between this duality result and the familiar commutative one is that noncommutative factors switch their order under duality; symbolically writing, we have $(M \otimes N)^* = N^* \otimes M^*$.

Proof of Proposition. Here it remains to notice that for any 2-left projectively finite quadratic ring A the left A_0-module R_A is projective and finitely generated, and indeed all modules in the exact triple

$$0 \longrightarrow R_A \longrightarrow A_1 \otimes_{A_0} A_1 \longrightarrow A_2 \longrightarrow 0$$

are left-projective and finitely generated over A_0. So after applying $\mathrm{Hom}_{A_0\text{-left}}(-, A_0)$ we also get an exact triple.

To prove that 3-left projectively finite rings correspond to 3-right projectively finite ones, put $S_A = R_A \otimes_{A_0} A_1 \cap A_1 \otimes_{A_0} R_A$ and consider the following exact quadruple:

$$0 \longrightarrow S_A \longrightarrow R_A \otimes_{A_0} A_1 \oplus A_1 \otimes_{A_0} R_A \longrightarrow A_1 \otimes_{A_0} A_1 \otimes_{A_0} A_1 \longrightarrow A_3 \longrightarrow 0.$$

We see that S_A is a projective and finitely generated left A_0-module for any 3-left projectively finite quadratic algebra A, and after dualization one has

$$\mathrm{Hom}_{A_0\text{-left}}(S_A, A_0) \simeq B_1 \otimes_{B_0} B_2 \oplus B_2 \otimes_{B_0} B_1 / \operatorname{im} B_1 \otimes_{B_0} B_1 \otimes_{B_0} B_1 \simeq B_3. \quad \square$$

The analogous arguments in higher degrees don't work for general quadratic algebras, because the corresponding longer complexes don't have to be exact anymore. In fact, they are exact if and only if the quadratic algebra is Koszul.

1.2. **Koszul algebras.** In the standard nowadays expositions on Koszul algebras (like A. Beilinson, V. Ginzburg, and V. Schechtman's *Koszul duality* or A. Beilinson, V. Ginzburg, and W. Soergel's *Koszul duality in representation theory*) the authors restrict themselves to the case when A_0 is a semi-simple algebra over a field. In fact, the basic results about Koszul algebras can be proven for an arbitrary base ring, provided that an appropriate flatness condition is imposed on A.

Definition. A graded ring $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is called *Koszul* if it flat as either a left or a right A_0-module and one has $\operatorname{Tor}^A_{i,j}(A_0, A_0) = 0$ for all $i \neq j$. Here A_0 is endowed with left and right A-module structures via the augmentation map and the second grading j on the Tor spaces comes from the grading of the ring A. A Koszul ring A is called left/right (finitely) projectively Koszul, if it is a projective (locally finitely generated) left/right A_0-module.

Notice that if A is a ring with subring K such that A/K is a flat left K-module and L is a left A-module flat over K, then the reduced bar-construction

$$\cdots \longrightarrow A \otimes_K A/K \otimes_K A/K \otimes_K L \longrightarrow A \otimes_K A/K \otimes_K L \longrightarrow A \otimes_K L \longrightarrow L$$

is a flat resolution of the A-module L, so for any right A-module R the reduced bar-complex

$$\cdots \longrightarrow R \otimes_K A/K \otimes_K A/K \otimes_K L \longrightarrow R \otimes_K A/K \otimes_K L \longrightarrow R \otimes_K L \longrightarrow 0$$

computes $\operatorname{Tor}^A_*(R, L)$.

Theorem 3. *Assume that a graded ring A is flat as a either left or a right A_0-module. Then A is Koszul if and only if it is quadratic and for each grading n the lattice in the tensor product of n copies of A_1 over A_0, generated by $n-1$ submodules*

$$A_1^{\otimes i-1} \otimes_{A_0} R_A \otimes_{A_0} A_1^{\otimes n-i-1} = A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1 \otimes_{A_0} R_A \otimes_{A_0} A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1,$$

is distributive. That is, for any three submodules X, Y, Z that can be obtained from the generating submodules by applying the operations of sum and intersection one should have $(X + Y) \cap Z = X \cap Z + Y \cap Z$.

Moreover, if A is a left finitely-projective Koszul ring, then the quadratic dual ring B to A is right finitely-projective Koszul, and vise versa.

This theorem can be proven in a way very similar to the classical proofs of Backelin's distributivity theorem for Koszul algebras over a field. The only difference is that it is more convenient to use the definition of Koszulity in terms of the bar-complex computing the Tor (rather than the Koszul complex) in this case. A sketch of proof is given in the next subsections.

1.3. **Low-dimensional homology.** It is, actually, very easy to see that the A_0-bimodules $\operatorname{Tor}^A_{1,n}(A_0, A_0)$ and $\operatorname{Tor}^A_{2,n}(A_0, A_0)$ computed via the reduced bar-complex are isomorphic to the bimodules of generators and relations of A in degree n. It suffices, e. g., to consider the homomorphism of graded rings $B \longrightarrow A$, where B coincides with A up to degree $n-1$ and has neither generators nor relations in degree n and higher. In particular, it follows that any Koszul ring is quadratic.

1.4. **Bar-complexes and lattices.** Let W be an abelian group and $R^{(1)}, \ldots, R^{(n-1)} \subset W$ be its subgroups. The *bar-complex* $B^*(W; R^{(1)}, \ldots, R^{(n-1)})$ is defined by the formulas

$$B^k = \bigoplus\nolimits_{1 \le i_1 < \cdots < i_{k-1} \le n-1} W/(R^{(i_1)} + \cdots + R^{(i_{k-1})}); \qquad k = 1, \ldots, n$$

$$d(a)(i_1, \ldots, i_{k-1}) = \sum\nolimits_{j=1}^{k-1} (-1)^j a(i_1, \ldots, \hat{i}_j, \ldots, i_{k-1}).$$

The *cobar-complex* $B_*(W; R^{(1)}, \ldots, R^{(n-1)})$ is defined as

$$B_k = \bigoplus\nolimits_{1 \le i_1 < \cdots < i_{k-1} \le n-1} R^{(i_1)} \cap \cdots \cap R^{(i_{k-1})}; \qquad k = 1, \ldots, n$$

$$d(v_{i_1, \ldots, i_{k-1}}) = \sum\nolimits_{1 \le s \le k;\, i_0 = 0;\, i_k = n}^{i_{s-1} < j < i_s} (-1)^s (v_{i_1, \ldots, i_{k-1}})_{i_1, \ldots, i_{s-1}, j, i_s, \ldots, i_{k-1}}.$$

Proposition 4. *The following conditions are equivalent:*

(a) *The subgroups $R^{(1)}, \ldots, R^{(n-1)}$ generate a distributive lattice of subgroups of the group W;*

(b) *For any $1 \le j_1 < \ldots < j_{m-1} \le n-1$ one has*

$$H^k(B^*(W; R^{(j_1)}, \ldots, R^{(j_{m-1})})) = 0 \qquad \text{for } k \ge 2.$$

(c) *For any $1 \le j_1 < \ldots < j_{m-1} \le n-1$ one has*

$$H_k(B_*(W; R^{(j_1)}, \ldots, R^{(j_{m-1})})) = 0 \qquad \text{for } k \ge 2.$$

Notice that for $k = 1$ one has $H^1(B^*(W; R^{(1)}, \ldots, R^{(n-1)})) = R^{(1)} \cap \cdots \cap R^{(n-1)}$ and $H_1(B_*(W; R^{(1)}, \ldots, R^{(n-1)})) = W/(R^{(1)} + \cdots + R^{(n-1)})$.

1.5. **Criterion of Musti–Buttafuoco.** The following very useful result is known since the 1950's.

Lemma 5 ([MB]). *Let L be a modular lattice generated by its elements $x_1, \ldots, x_{n-1}$, and assume that all the sublattices generated by proper subsets of the x_i's in L are distributive. If for any $2 \le s \le n-2$ one has*

$$\begin{aligned} &(x_1 \cap \cdots \cap x_s) \cup x_{s+1} \cup \cdots \cup x_{n-1} \\ &= (x_1 \cup x_{s+1} \cup \cdots \cup x_{n-1}) \cap (x_2 \cup x_{s+1} \cup \cdots \cup x_{n-1}) \cap \cdots \cap (x_s \cup x_{s+1} \cup \cdots \cup x_{n-1}), \end{aligned}$$

then L is distributive. □

Lemma 6. *Let L be a modular lattice generated by $x_1, \ldots, x_{n-1}$ such that all the sublattices generated by proper subsets of the x_i's are distributive. Then if the sublattice generated by $x_1 \cup x_{n-1}$, $x_2 \cup x_{n-1}$, $\ldots$, $x_{n-2} \cup x_{n-1}$ is distributive and one has*

$$(x_1 \cap x_2 \cap \cdots \cap x_{n-2}) \cup x_{n-1} = (x_1 \cup x_{n-1}) \cap (x_2 \cup x_{n-1}) \cap \cdots \cap (x_{n-2} \cup x_{n-1}),$$

the lattice L is distributive.

Proof. The condition for $s = n-2$ in Lemma 5 coincides with the equation in Lemma 6. For $s < n-2$, the corresponding conditions in Lemma 5 can be expressed as distributivity equations depending on $x_1 \cup x_{n-1}$, $x_2 \cup x_{n-1}$, …, $x_{n-2} \cup x_{n-1}$ only using distributivity of proper subsets of x_1, …, x_{n-1}. □

Proof of Proposition. We will prove (a) $\iff$ (b); the dual property (a) $\iff$ (c) is proven in the analogous way. There is an exact triple of bar-complexes

$$0 \longrightarrow B^*(W/R^{(n-1)}; R^{(1)} \bmod R^{(n-1)}, \dots, R^{(n-2)} \bmod R^{(n-1)})[-1] \\ \longrightarrow B^*(W; R^{(1)}, \dots, R^{(n-1)}) \longrightarrow B^*(W; R^{(1)}, \dots, R^{(n-2)}) \longrightarrow 0.$$

The third complex is exact by the assumptions of the proposition and the hypothesis of induction in n. It follows easily that the second complex is exact at the required terms iff the first complex is exact as required and the connecting homomorphism

$$R^{(1)} \cap \dots \cap R^{(n-2)} \longrightarrow (R^{(1)} + R^{(n-1)}) \cap \dots \cap (R^{(n-2)} + R^{(n-1)})/R^{(n-1)}$$

is surjective. Using induction in n again, we conclude that exactness of the first complex is equivalent to distributivity of the lattice generated by $R^{(1)} + R^{(n-1)}$, …, $R^{(n-2)} + R^{(n-1)}$. Now it remains to apply Lemma 6 for the modular lattice of abelian subgroups in the abelian group W and its appropriate sublattices. □

1.6. **Distributivity and flatness.** Let K be a ring, W a left K-module, and $R^{(1)}$, …, $R^{(n-1)} \subset W$ be a collection of submodules generating a distributive lattice of submodules of W.

Proposition 7. *The following conditions are equivalent:*

(a) *Any subquotient module X/Y in the lattice of submodules generated by $R^{(1)}$, …, $R^{(n-1)}$ in W is a flat (projective) K-module;*

(b) *For any $1 \le j_1 < \dots < j_{m-1} \le n-1$, $m = 1, \dots, n$ the quotient module $W/(R^{(j_1)} + \dots + R^{(j_{m-1})})$ is a flat (projective) K-module.*

Proof. (a) $\implies$ (b) is obvious; let us prove (b) $\implies$ (a). Let us argue by induction in n. Since the lattice is distributive, any subquotient module can be presented as a successive extention of subquotient modules of the form

$$\bigcap\nolimits_{j \in J} R^{(j)} \Big/ \bigcap\nolimits_{j \in J} R^{(j)} \cap \sum\nolimits_{k \notin J} R^{(k)}, \qquad \text{where } J \subset \{1, \dots, n-1\}.$$

Whenever the inclusion $J \subset \{1, \dots, n-1\}$ is proper, this subquotient module can be presented as a submodule in the smaller lattice generated by the submodules $R^{(j)}$ modulo $\sum_{k \notin J} R^{(k)}$ in the quotient module $W/\sum_{k \notin J} R^{(k)}$. It follows from the induction hypothesis that all the submodules in this smaller lattice are flat (projective). It remains to show flatness (projectivity) of the only submodule $R^{(1)} \cap \dots \cap R^{(n-1)}$. Consider the exact triple

$$0 \longrightarrow R^{(1)} \cap \dots \cap R^{(n-1)} \longrightarrow W \longrightarrow W/R^{(1)} \cap \dots \cap R^{(n-1)} \longrightarrow 0;$$

the quotient module in this exact triple can be presented as a successive extention of subquotient modules whose flatness (projectivity) we have proven already. Since in the exact sequence the middle term and the quotient module are flat (projective), the submodule is also flat (projective). Alternatively, one could consider the bar-complex $B^*(W; R^{(1)}, \dots, R^{(n-1)})$ here. $\square$

2. Nonhomogeneous Quadratic Rings; Duality with CDG-algebras

2.1. **Nonhomogeneous quadratic rings.** Let A be a (3-left projectively finite) quadratic ring and B be the dual (3-right projectively finite quadratic) ring. Suppose that we are given a filtered algebra $\tilde{A}$ with an increasing filtration $F_0\tilde{A} \subset F_1\tilde{A} \subset F_2\tilde{A} \subset \cdots$ such that the associated graded ring $\operatorname{gr}_F \tilde{A}$ is identified with A. We will call such filtered rings *nonhomogeneous quadratic 3-left projectively finite rings*.

Which additional structure on the dual ring B corresponds to the data of such a filtered ring $\tilde{A}$? The following key example illustrates the situation very well.

Let M be a smooth affine algebraic variety (or a smooth real manifold) and $\mathcal{E}$ be a vector bundle over it. Let $\mathcal{D}_{M,\mathcal{E}}$ be the ring of differential operators acting on the sections of $\mathcal{E}$. It has a natural filtration by the order of a differential operator. The associated graded ring $\operatorname{gr}_F \mathcal{D}_{M,\mathcal{E}}$ is isomorphic to the ring of sections $A = \operatorname{End}(\mathcal{E}) \otimes_{\mathcal{O}(M)} \operatorname{Sym}^*(\mathcal{T}_M)$ of the graded vector bundle of associative algebras which is the tensor product of the endomorphisms of $\mathcal{E}$ and the symmetric algebra of the tangent bundle to M. The ring A is projective locally finitely generated and quadratic over the base ring $A_0 = \operatorname{End}(\mathcal{E})$.

The dual ring B is isomorphic to $\operatorname{End}(\mathcal{E}) \otimes_{\mathcal{O}(M)} \bigwedge^* \mathcal{T}^*(M)$, i. e., the ring of exterior forms on M with values in $\operatorname{End}(\mathcal{E})$. The induced additional structure on B will be defined as a set of data up to a certain equivalence relation. Namely, let us choose a connection ∇ in the vector bundle $\mathcal{E}$. Then we have the curvature element $\omega \in B_2$ and the de Rham differential d_∇ acting in B so that $(d_\nabla)^2(x) = [\omega, x]$ and $d_\nabla(\omega) = 0$. When we change the connection, the operator d_∇ and the element ω are replaced according to the well-known rules: if $\nabla' = \nabla + \alpha$, then $d_{\nabla'}(x) = d_\nabla(x) + [\alpha, x]$ and $\omega' = \omega + d_\nabla(\alpha) + \alpha^2$.

This is the situation which we generalize in the following definition.

2.2. **CDG-algebras.** A *CDG-ring* is a graded ring $B = \bigoplus_i B^i$ endowed with an odd derivation $d : B^i \to B^{i+1}$ and a "curvature element" $\omega \in B^2$ such that $d^2(x) = [\omega, x]$ for all $x \in B$ and $d(\omega) = 0$. A morphism of CDG-rings $B \longrightarrow C$ is a pair (f, α), where $f : B \longrightarrow C$ is a morphism of graded rings and α is an element in C^1 such that $f(d_B(x)) = d_C(f(x)) + [\alpha, f(x)]$ (the supercommutator) for all $x \in B$ and $f(\omega_B) = \omega_C + d_C(\alpha) + \alpha^2$. Composition of morphisms is defined by the rule $(g, \beta)(f, \alpha) = (gf, \beta + g(\alpha))$. Identity morphisms are $\mathrm{id}_{(B,d,\omega)} = (\mathrm{id}_B, 0)$.

The *category of CDG-rings* is hereby defined. Note that the natural functor from the category of DG-rings to the category of CDG-rings is faithful, but not fully faithful. In other words, two DG-algebras may be isomorphic in the category of CDG-algebras without being isomorphic as DG-algebras. Moreover, two CDG-algebras of the form $(B, d + [\alpha, \cdot], \omega + d(\alpha) + \alpha^2)$ and (B, d, ω) are always naturally isomorphic; the isomorphism being given by (id_B, α).

2.3. **Duality functor.** There is a fully faithful duality functor (that is, a contravariant embedding of full subcategory) from the category of all nonhomogeneous quadratic 3-left projectively finite rings to the category of CDG-rings whose underlying graded ring is 3-right projectively finite quadratic.

This functor is constructed as follows. For each nonhomogeneous 3-left projective quadratic ring let us choose a complementary left A_0-submodule V to the submodule $A_0 = F_0A^{\sim}$ in the left A_0-module $F_1A^{\sim}$. This can be done, because the quotient module $A_1 = F_1A^{\sim}/F_0A^{\sim}$ is projective. I should emphasize that the subset V will be only a left A_0-submodule in $A^{\sim}$, but not in general a right A_0-submodule. Of course, the choice of V is an arbitrary one, but we will see that changing this choice leads to a canonically isomorphic CDG-algebra, so the duality functor is well-defined (up to a uniquely defined isomorphism of functors, to be precise).

Since V projects isomorphically into $A_1 = F_1A^{\sim}/F_0A^{\sim}$, it is endowed with a structure of (A_0, A_0)-bimodule. The map $V \longrightarrow A^{\sim}$ is only a morphism of left A_0-modules, however. The right actions of A_0 on V and $A^{\sim}$ are compatible modulo $F_0A^{\sim}$ only. Put $q(v, k) = \mu(v, k) - vk$, where $\mu(v, k)$ is the product in $A^{\sim}$ and vk denotes the right action of A_0 on V. This defines a map $q\colon V \otimes_{\mathbb{Z}} A_0 \longrightarrow A_0$.

Let $\tilde{R}$ be the full preimage of R_A in $A_1 \otimes_{\mathbb{Z}} A_1$ (recall that R_A is a submodule in the tensor product of the same modules over A_0). Equivalently (using the identification of V with A_1), $\tilde{R}$ is the full preimage of $F_1A^{\sim}$ under the multiplication map $\mu\colon V \otimes_{\mathbb{Z}} V \longrightarrow A^{\sim}$. Let us split the map $\mu\colon \tilde{R} \longrightarrow F_1A^{\sim}$ into two components $(e, -h)$ according to the direct sum decomposition $F_1A^{\sim} = V + F_0A^{\sim}$, so that $e\colon \tilde{R} \longrightarrow V$, $h\colon \tilde{R} \longrightarrow A_0$, and $\mu = e - h$.

The differentials $d_0\colon B_0 \longrightarrow B_1$ and $d_1\colon B_1 \longrightarrow B_2$ are defined in terms of the maps q and e by the formulas

$$<v, d_0(k)> = q(v, k), \qquad <r, d_1(b)> = <e(\tilde{r}), b> - q(<\tilde{r}, b>),$$

where $< \, , \, >$ denote the pairings of V with B_1 and of R with B_2, $\tilde{r}$ is any lifting of r in $\tilde{R}$ (which has to be chosen for the summands in the right hand side to make sense, even though the left hand side doesn't depend on it), and $<\tilde{r}, b>$ is an element of $V \otimes_{\mathbb{Z}} A_0$ obtained by coupling the second component of $\tilde{r}$ with $b \in B_1$. Note that it's important to include the q-term in the second formula, though you wouldn't find it anywhere in duality over a field. The map h actually factors through R and defines the curvature element $\omega \in B_2$.

Finally, to a morphism of nonhomogeneous quadratic rings $f\colon A'^{\sim} \longrightarrow A''^{\sim}$ with chosen complementary submodules $V' \subset F_1A'^{\sim}$ and $V'' \subset F_1A''^{\sim}$ one assigns a morphism of dual CDG-rings $(g, \alpha)\colon (B'', d'', \omega'') \longrightarrow (B', d', \omega')$ defined as follows. The morphism of quadratic rings $g\colon B' \longrightarrow B''$ is the quadratic dual map to the associated graded morphism $\operatorname{gr} f\colon A' \longrightarrow A''$, while the map $\alpha \in B'_1 = \operatorname{Hom}_{A_0\text{-left}}(A'_1, A_0)$ is the composition of F_1f with the projection $F_1A''^{\sim} \longrightarrow F_0A''^{\sim} = A_0$ along V''. In particular, for a given nonhomogeneous quadratic ring A changing the splitting of $F_1A^{\sim}$ by the rule $V'' = \{v' + \alpha(v') \mid v' \in V'\}$ leads to a natural isomorphism of CDG-algebras $(\mathrm{id}_B, \alpha)\colon (B, d'', \omega'') \longrightarrow (B, d', \omega')$.

This is the end of the construction of the dual quadratic CDG-algebra, though one has to make quite some computations in order to see that everything is well-defined and compatible. In particular, the 3-left projectivity will be actually used in form of the duality between S_A (where some self-consistency equations on the defining relations of $A^\sim$ live) and B_3 (where the equations $d^2(x) = [\omega, x]$, $d(\omega) = 0$ are verified). One cannot but feel a sense of mystery about the formulas that make this strange and beautiful duality work.

2.4. **Augmented duality.** A map of left A_0-modules $a\colon A^\sim \longrightarrow A_0$ is called a *left augmentation* of the ring $A^\sim$ over its subring A_0 if its restriction to A_0 is the identity map and its kernel is a left ideal in $A^\sim$. There is a bijective correspondence between left augmentation maps $a\colon A^\sim \longrightarrow A_0$ and structures of left $A^\sim$-module on A_0 compatible with the obvious structure of left A_0-module on A_0.

The above duality construction sends nonhomogeneous quadratic algebras endowed with a left augmentation maps to quadratic DG-algebras. More precisely, it suffices to take for V the kernel of the restriction of a to $F_1A^\sim$ to costruct a DG-ring out of a left-augmented nonhomogeneous quadratic ring. Consequently, there is fully faithful duality functor from the category of 3-left projectively finite left-augmented nonhomogeneus quadratic rings into the category of nonnegatively graded DG-rings with differential increasing the grading by 1 whose underlying graded ring is a 3-right projectively finite quadratic ring.

2.5. **CDG-modules.** A left CDG-module over a CDG-ring (B, d_B, ω) is a graded left B-module M endowed with a d_B-derivation d_M such that $d_M^2(m) = \omega m$ for all $m \in M$. A right CDG-module over (B, d_B, ω) is a graded right B-module N with a d_B-derivation d_N such that $d_N^2(n) + n\omega = 0$ for all n. It is easy to check that the natural differential on the tensor product $N \otimes_B M$ of a left CDG-module and a right CDG-module has zero square. Likewise, for any two left CDG-modules L and M, the natural differential on the graded Hom group $\mathrm{Hom}_B(L, M)$ has zero square. Hence the category of all left CDG-modules over a given CDG-ring is naturally a *DG-category*, i. e., a category whose Hom spaces are complexes. As we will see below in section 6, this observation allows one to define for the category of CDG-modules its derived category *of the second kind*, even though the more familiar definition of derived category cannot be applied in this case.

3. Coalgebras over Rings

The only weak aspect of the duality theory of section 2 is that it is based on the bimodule duality operation $\mathrm{Hom}_{A_0}(-, A_0)$. Experience teaches that it is better to avoid taking the dual vector space whenever at all possible. In particular, we will prefer to consider the coalgebra of polyvector fields, rather then the algebra of differential forms, as the dual object to the ring of differential operators.

3.1. **Coalgebras and comodules.** Let K be a ring. Throughout this paper, by a *coalgebra over* K we mean an (K,K)-bimodule C endowed with comultiplication and counit maps $\Delta\colon C \longrightarrow C\otimes_K C$ and $\varepsilon\colon C\longrightarrow K$, which are supposed to be homomorphisms of (K,K)-bimodules satisfying the usual coassociativity and counit axioms. A left *comodule* P over a coalgebra C is a left K-module endowed with a coaction map $\Delta_P\colon P\longrightarrow C\otimes_K P$ compatible with the comultiplication and counit of C in the usual way. The category of left comodules over C is abelian provided that C is a flat right K-module (but not otherwise). In general one can consider the category of comodules as an exact category whose exact triples split over K.

The category of right comodules over C is defined analogously. For a left comodule P and a right comodule Q over C, one defines the *cotensor product* $Q\odot_C P$ as an abelian group obtained as the kernel of the homomorphism

$$\Delta_Q\otimes\mathrm{id}-\mathrm{id}\otimes\Delta_P:\ Q\otimes_K P\longrightarrow Q\otimes_K C\otimes_K P.$$

This is a functor of two arguments which is in general neither left- nor right-exact. It becomes right-exact, however, when considered as a functor on the exact categories of C-comodules with K-split exact triples.

Therefore, it makes sense to consider the left derived functor of cotensor product relative to these exact category structures. It is easy to check that the reduced cobar-resolution

$$P\longrightarrow C\otimes_K P\longrightarrow C\otimes_K C_+\otimes_K P\longrightarrow C\otimes_K C_+\otimes_K C_+\otimes_K P\longrightarrow\cdots$$

(where $C_+=\ker\varepsilon\colon C\longrightarrow K$) is split over K and consists of injective comodules relative to K. So we define the derived functors $\mathrm{Cotor}^i_{C/K}(Q,P)$ as the cohomology of reduced cobar-complex

$$0\longrightarrow Q\otimes_K P\longrightarrow Q\otimes_K C_+\otimes_K P\longrightarrow Q\otimes_K C_+\otimes_K C_+\otimes_K P\longrightarrow\cdots$$

When C is either left or right flat module over K, the functors Cotor between K-flat C-comodules can be considered as the derived functors of cotensor product on the exact categories of K-flat C-comodules with respect to arbitrary (or only K-split, it doesn't matter) exact sequences. We will always have to assume that at least one of these flatness conditions is satisfied for our coalgebras. When C is both left *and* right flat K-module, the functors Cotor can be computed in the abelian categories of comodules over C.

3.2. **Quadratic and Koszul coalgebras.** A nonnegatively graded coalgebra $C=C_0\oplus C_1\oplus C_2\oplus\cdots$ is called quadratic if the map $\varepsilon\colon C_0\longrightarrow K$ is an isomorphism, the map $\Delta_2\colon C_2\longrightarrow C_1\otimes_K C_1$ is an embedding, and coalgebra C is the universal final object in the category of all coalgebras with given C_1, C_2, and Δ_2. One can see that such a universal final object always exists. Indeed, define C_n by induction for $n\geq 3$ as the kernel of the reduced cobar-complex map $(C_+\otimes C_+)_n\longrightarrow(C_+\otimes C_+\otimes C_+)_n$, where $C_+=\ker\varepsilon$.

It follows that the categories of quadratic coalgebras and quadratic algebras are equivalent to each other: a quadratic coalgebra C corresponds to a quadratic ring A with $A_0=K$, $A_1=C_1$, and $R_A=C_2$ as a submodule in $A_1\otimes_{A_0}A_1=C_1\otimes_K C_1$.

We will denote this correspondence as $A = C^!$ and $C = A^?$. It is easy to check that $C^! \simeq \bigoplus_i \mathrm{Cotor}^{i,i}_{C/K}(K, K)$ for any quadratic coalgebra C and $A^? \simeq \bigoplus_i \mathrm{Tor}^A_{i,i}(A_0, A_0)$ for any Koszul ring A.

Definition. Let $C = C_0 \oplus C_1 \oplus C_2 \oplus \cdots$ be a graded coalgebra over a ring K such that the map $\varepsilon\colon C_0 \longrightarrow K$ is an isomorphism and either all the left, or all the right K-modules $\mathrm{Cotor}^{i,i}_{C/K}(K, K)$ are flat. Then the coalgebra C is called *Koszul* if $\mathrm{Cotor}^{i,j}_{C/K}(K, K) = 0$ for all $i \neq j$.

Theorem 8. *Let C be a graded coalgebra for which the map $\varepsilon\colon C_0 \longrightarrow K$ is an isomorphism and all the left K-modules $\mathrm{Cotor}^{i,i}_{C/K}(K, K)$ are flat. Then C is Koszul if and only if it is quadratic and the dual quadratic ring $C^!$ is Koszul.*

In this case, the graded components C_i of the coalgebra C are also flat left K-modules and the iterated comultiplication maps

$$\Delta^{(n-1)}\colon C_n \longrightarrow \bigcap\nolimits_{i=1}^{n-1} C_1^{\otimes_K i-1} \otimes_K C_2 \otimes_K C_1^{\otimes_K n-i-1} \subset C_1^{\otimes_K n}$$

are isomorphisms for all $n \geq 3$.

Proof is similar to the proof of Theorem 3. One proves that coalgebra Koszulity is equivalent to quadraticity plus distributivity under the imposed flatness conditions. $\square$

4. Quasi-Differential Structures

Having defined Koszul coalgebras, we would like to construct a duality theory connecting nonhomogeneous quadratic rings with a kind of differential quadratic coalgebras. In particular, to the ring of differential operators on a manifold this duality should assign the coalgebra of polyvector fields. The problem remains, however, that there is no de Rham differential on polyvector fields. We offer a solution to this problem here by introducing quasi-differential structures on rings and coalgebras.

4.1. **Quasi-differential graded rings.** Let (B, d, ω) be a CDG-ring. Let us define a new ring $B^{\sim}$ by adding to B a new generator δ of degree 1 with relations $[\delta, b] = d(b)$ for all $b \in B$ and $\delta^2 = \omega$. There is a natural derivation $\partial = \partial/\partial\delta$ of degree -1 on $B^{\sim}$ for which one has $\ker \partial/\partial\delta = \operatorname{im} \partial/\partial\delta = B \subset B^{\sim}$.

Definition. A quasi-differential ring $B^{\sim}$ is a graded ring endowed with a derivation ∂ of degree -1 such that $\ker\partial = \operatorname{im}\partial$. A morphism of quasi-differential rings is a ring homomorphism $B'^{\sim} \longrightarrow B''^{\sim}$ compatible with the derivations ∂' and ∂''. Given a graded ring B, the data of a quasi-differential ring $(B^{\sim}, \partial)$ with $\ker\partial = B$ will be sometimes called a *quasi-differential structure* on the ring B.

A left/right quasi-differential module over a quasi-differential ring $(B^{\sim}, \partial)$ is just a graded left/right $B^{\sim}$-module M. A $B^{\sim}$-module M is called contractible if it admits a ∂-derivation ∂_M. For a pair of left $B^{\sim}$-modules L and M, the complex $\mathrm{Hom}_{(B^{\sim},\partial)}(L, M)$ is defined as follows: its elements of degree n are homogeneous morphisms of $(\ker\partial)$-modules $L \longrightarrow M$ of degree n; the differential is defined by taking the commutator of our morphisms with any element $\delta \in B^{\sim}$ such that $\partial(\delta) = 1$.

This defines the *DG-category of quasi-differential modules.* In particular, closed morphisms in this DG-category are just morphisms of $B^{\sim}$-modules.

Proposition 9. *The categories of CDG-rings and of quasi-differential graded rings are equivalent to each other. The equivalence construction assigns a quasi-differential ring* $(B^{\sim}, \partial)$ *to a CDG-ring* (B, d, ω) *as described in the beginning of this subsection. To a morphism of CDG-rings* (f, α) *from* (B', d', ω') *to* (B'', d'', ω'') *there corresponds a morphism* $f^{\sim}\colon B'^{\sim} \longrightarrow B''^{\sim}$ *given by the formulas* $f^{\sim}(b) = f(b)$ *for* $b \in B'$ *and* $f^{\sim}(\delta') = \delta'' + \alpha$*. The inverse functor is defined as follows: for each quasi-differential ring* $B^{\sim}$*, choose an element* $\delta \in B^{\sim}_1$ *for which* $\partial(\delta) = 1$*; then set* $B = \ker\partial$*,* $d(b) = [\delta, b]$ *for all* $b \in B$*, and* $\omega = \delta^2$*. For any CDG-ring* B *and the corresponding quasi-differential ring* $B^{\sim}$*, the DG-categories of CDG-modules over* B *and quasi-differential modules over* $B^{\sim}$ *are naturally isomorphic.* □

Note that the ring $B^{\sim}$ is quadratic for any quadratic CDG-ring B, but the converse does not hold in general.

Proposition 10. *Let* $(B^{\sim}, \partial)$ *be a quasi-differential graded ring with* $\ker\partial = B$*. Suppose that* $B^{\sim}$ *is a quadratic ring. Then the ring* B *is quadratic if and only if it has no generators of degree 2 and no relations of degree 3.* □

4.2. **Quasi-differential coalgebras.** A graded coalgebra $C^{\sim}$ over a ring K is called *quasi-differential coalgebra* if it is endowed with a K-linear coderivation map $\partial\colon C^{\sim} \longrightarrow C^{\sim}$ of degree 1 such that $\ker\partial = \operatorname{im}\partial$, or equivalently, $\operatorname{im}\partial = \operatorname{coker}\partial$. The *underlying graded coalgebra* of the quasi-differential coalgebra $C^{\sim}$ is defined as $C = \operatorname{coker}\partial$. A morphism of quasi-differential coalgebras is just a homomorphism of coalgebras over K compatible with the derivations ∂.

References

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