COEQUIVALENCES AND CONTRAEQUIVALENCES OF CURVED DG-ALGEBRAS

LEONID POSITSELSKI

Abstract. We provide sufficient conditions for a morphism of CDG-coalgebras, CDG-rings, or topological CDG-algebras to induce an equivalence of the coderived or contraderived categories of CDG-modules, discrete CDG-modules, CDG-comodules, or CDG-contramodules.

Contents

Introduction

A *DG-scheme* is the result of glueing of nonpositively cohomologically graded (super)commutative DG-rings in the topology of the spectra of their degree-zero cohomology rings. So, let $R^\bullet$ be a nonpositively cohomologically graded commutative DG-ring, let $f \in H^0(R^\bullet)$ be a degree-zero cohomology class, and let $\tilde f \in R^0$ be an element representing f. Then the DG-ring of sections of the structure sheaf of the DG-spectrum of $R^\bullet$ over the open subset $\operatorname{Spec} H^0(R^\bullet)[f^{-1}] \subset \operatorname{Spec} H^0(R^\bullet)$ can be constructed as the localization $R^\bullet[\tilde f^{-1}]$ of the commutative DG-ring $R^\bullet$. However, the DG-ring $R^\bullet[\tilde f^{-1}]$ changes when a representative $\tilde f$ of the cohomology class f is changed; it is only the quasi-isomorphism class of the DG-ring $R^\bullet[\tilde f^{-1}]$ that is determined by f and does not depend on $\tilde f$. This example illustrates the importance of having a well-described equivalence relation on DG-rings.

The kind of equivalence relation required for the differential rings depends on the kind of derived category construction one intends to apply to the differential modules over them. It is well known that a morphism of DG-rings $f\colon A^\bullet \longrightarrow B^\bullet$ induces an equivalence $\mathsf{D}(A^\bullet\text{–}\mathsf{mod}) \simeq \mathsf{D}(B^\bullet\text{–}\mathsf{mod})$ of the derived categories of left DG-modules over A and B if and only if f is a quasi-isomorphism (see, e. g., [3, Section 7.2], [2, Section 3], or [5, Section 1.7]). Characterizing morphisms of DG-rings f inducing an equivalence of the coderived or contraderived categories of left DG-modules

$\mathsf{D}^{\mathsf{co}}(A^\bullet\text{--}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{co}}(B^\bullet\text{--}\mathsf{mod})$ or $\mathsf{D}^{\mathsf{ctr}}(A^\bullet\text{--}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B^\bullet\text{--}\mathsf{mod})$ (see [5, Section 3] for the definitions) is an important open question. The natural generality level for this question must be at least that of a morphism of CDG-rings $f\colon A \longrightarrow B$ [4, 5]. In fact, the results of this paper argue for the assertion that the natural generality level is that of a morphism of *topological CDG-rings*.

Let us call a morphism of CDG-rings $f\colon A \longrightarrow B$ a *left coequivalence* if the induced functor of restriction of scalars acting between the coderived categories of left CDG-modules $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(B\text{--}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(A\text{--}\mathsf{mod})$ is an equivalence of triangulated categories. Similarly, a morphism of CDG-rings $f\colon A \longrightarrow B$ is called a *left contraequivalence* if the induced functor of restriction of scalars acting between the contraderived categories of left CDG-modules $\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(B\text{--}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(A\text{--}\mathsf{mod})$ is an equivalence of triangulated categories.

Example 0.1. *Let $A^\bullet = (A^{-N} \to \cdots \to A^{-1} \to A^0)$ and $B^\bullet = (B^{-N} \to \cdots \to B^{-1} \to B^0)$ be two nonpositively cohomologically graded DG-rings that are bounded in the cohomological grading. Assume that the underlying graded rings of the DG-rings $A^\bullet$ and $B^\bullet$ are left Noetherian. Then any quasi-isomorpism of DG-rings $f\colon A^\bullet \longrightarrow B^\bullet$, i. e., any morphism of DG-rings inducing an isomorphism of the cohomology rings $H^*(A^\bullet) \simeq H^*(B^\bullet)$, is a left coequivalence.*

Proof. The following argument is essentially presumed in [1]. The Noetherianness assumption guarantees that the coderived categories $\mathsf{D}^{\mathsf{co}}(A^\bullet\text{--}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{co}}(B^\bullet\text{--}\mathsf{mod})$ are compactly generated by their full triangulated subcategories that can be identified with the absolute derived categories of finitely generated DG-modules $\mathsf{D}^{\mathsf{abs}}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$ and $\mathsf{D}^{\mathsf{abs}}(B^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$ [5, Section 3.11]. The assumption of boundedness of the cohomological grading of the DG-rings $A^\bullet$ and $B^\bullet$ implies the similar boundedness of any finitely generated DG-modules over them. In view of the assumption of nonpositivity of the cohomological grading of $A^\bullet$ and $B^\bullet$, it follows [5, proof of Theorem 3.4.1(a)] that any acyclic finitely generated DG-module over the DG-ring $A^\bullet$ or $B^\bullet$ is absolutely acyclic. In other words, the absolute derived categories of finitely generated DG-modules coincide with their conventional derived categories, $\mathsf{D}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}}) = \mathsf{D}^{\mathsf{abs}}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$ and $\mathsf{D}(B^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}}) = \mathsf{D}^{\mathsf{abs}}(B^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$.

Since the DG-submodules and quotient DG-modules of canonical filtration of finitely generated left DG-modules over $A^\bullet$ and $B^\bullet$ are finitely generated, the standard arguments show that the derived categories of finitely generated DG-modules $\mathsf{D}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$ and $\mathsf{D}(B^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}})$ are full subcategories of the derived categories $\mathsf{D}(A^\bullet\text{--}\mathsf{mod})$ and $\mathsf{D}(B^\bullet\text{--}\mathsf{mod})$. Furthermore, a DG-module over $A^\bullet$ belongs to the full subcategory $\mathsf{D}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}}) \subset \mathsf{D}(A^\bullet\text{--}\mathsf{mod})$ if and only if its cohomology module is bounded in the cohomological grading and finitely generated as a module over $H^0(A^\bullet)$. For any quasi-isomorphism of DG-rings $f\colon A^\bullet \longrightarrow B^\bullet$, the functor of restriction of scalars is an equivalence between the derived categories $\mathsf{D}(A^\bullet\text{--}\mathsf{mod}) \simeq \mathsf{D}(B^\bullet\text{--}\mathsf{mod})$, and it follows that in our assumptions it identifies the full subcategories $\mathsf{D}(A^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}}) \subset \mathsf{D}(A^\bullet\text{--}\mathsf{mod})$ and $\mathsf{D}(B^\bullet\text{--}\mathsf{mod}_{\mathsf{fg}}) \subset \mathsf{D}(B^\bullet\text{--}\mathsf{mod})$. Hence we can conclude that the restriction of scalars is also an equivalence of the coderived categories $\mathsf{D}^{\mathsf{co}}(A^\bullet\text{--}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{co}}(B^\bullet\text{--}\mathsf{mod})$. $\square$

The above argument is uncomplicated, but it is not quite what one would like to have, as the reduction of questions about coderived categories to those about the conventional derived categories that is being used here is only possible for non-positively cohomologically graded (or simply connected positively cohomologically graded) DG-rings, which do not at all exhaust the classes of DG-rings we are interested in. One would like to have, at least, a sufficient condition of coequivalence that would be applicable to (C)DG-rings such as the de Rham complex, and which would include the above Example as a particular case.

In this paper, we obtain such a sufficient condition. We start with a simpler case of a morphism of CDG-coalgebras $f\colon C \longrightarrow D$, where a sufficient condition for co/contra/weak equivalence is formulated in terms of a pair of increasing filtrations on the CDG-coalgebras C and D compatible with the morphism f. Our result in this direction is a far-reaching generalization of the assertion of [5, Theorem 4.8], obtained by a completely different method. Then we pass to morphisms of topological CDG-rings or CDG-algebras $f\colon A \longrightarrow B$, where sufficient conditions for co- and contraequivalence are formulated in terms of a pair of decreasing filtrations on A and B. Both our old and new approaches to proving weak equivalence of CDG-coalgebras are being transfered to the realm of topological CDG-algebras.

I am grateful to Wendy Lowen for stimulating questions about CDG-algebras and to Vadim Vologodsky for stressing the importance of conditions for equivalences of derived categories of the second kind.

1. Weak Equivalences of CDG-Coalgebras

We refer to [5, Section 4.1] for the definitions of CDG-coalgebras, CDG-comodules, and CDG-contramodules; to [5, Section 4.2] for the definitions of absolutely acyclic and coacyclic CDG-comodules, absolutely acyclic and contraacyclic CDG-contramodules and the coderived and contraderived categories; and to [5, Section 4.8] for the definitions of CDG-bicomodules and coacyclic CDG-bicomodules.

Let $f\colon C \longrightarrow D$ be a morphism of CDG-coalgebras. We recall from [5, Section 4.8] that the functor of (co)restriction of scalars $\mathbb{I}R_f\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ is left adjoint to the derived functor of (co)extension of scalars $\mathbb{R}E_f\colon \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, while the functor of (contra)restriction of scalars $\mathbb{I}R^f\colon \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra})$ is right adjoint to the functor of (contra)extension of scalars $\mathbb{L}E^f\colon \mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.

Furthermore, according to [5, Section 5.4], the natural equivalences of categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ and $\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra})$ transform the functor of coextension of scalars $\mathbb{R}E_f$ into the functor of contraextension of scalars $\mathbb{L}E^f$. Therefore, if any one of the functors $\mathbb{I}R_f$, $\mathbb{R}E_f$, $\mathbb{L}E^f$, or $\mathbb{I}R^f$ is an equivalence of triangulated categories, then so are all of them.

Besides, the coderived category of left CDG-comodules over any CDG-coalgebra E is compactly generated, and its full subcategory of compact objects is the idempotent closure of the absolute derived category $\mathsf{D}^{\mathsf{abs}}(E\text{–}\mathsf{comod}_{\mathsf{fd}})$ of finite-dimensional left CDG-comodules over E [5, Sections 3.11, 4.6, and 5.5]. So the functor of corestriction of scalars is an equivalence of the coderived categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ if and only if its restriction to finite-dimensional CDG-comodules induces an equivalence between the idempotent closures of the absolute derived categories $\mathsf{D}^{\mathsf{abs}}(C\text{–}\mathsf{comod}_{\mathsf{fd}})$ and $\mathsf{D}^{\mathsf{abs}}(D\text{–}\mathsf{comod}_{\mathsf{fd}})$ of finite-dimensional left CDG-comodules over the CDG-coalgebras C and D.

Since the functor of the passage to the dual vector space $M \longmapsto M^*$ provides an anti-equivalence $\mathsf{D}^{\mathsf{abs}}(E\text{–}\mathsf{comod}_{\mathsf{fd}})^{\mathsf{op}} \simeq \mathsf{D}^{\mathsf{abs}}(\mathsf{comod}_{\mathsf{fd}}\text{–}E)$ between the absolute derived categories of finite-dimensional left and right CDG-comodules over any CDG-coalgebra E, we conclude that the corestriction of scalars with respect to a morphism of CDG-coalgebras $f\colon C \longrightarrow D$ is an equivalence between the coderived categories of left CDG-comodules $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ if and only if it is an equivalence between the coderived categories of right CDG-comodules $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}D)$. If this is the case, we will call the morphism $f\colon C \longrightarrow D$ a *weak equivalence* of CDG-coalgebras.

A morphism of DG-coalgebras is called a *weak equivalence* if, viewed as a morphism of CDG-coalgebras, it is a weak equivalence in the sense of the above definition. Weak equivalences, in the sense of the above definition, whose sources and targets are *conilpotent* CDG- or DG-coalgebras form a part of model category structures on the categories of conilpotent CDG- and DG-coalgebras [5, Section 9.3].

Let $F_0C \subset F_1C \subset F_2C \subset \cdots$, $C = \bigcup_n F_nC$, be an exhaustive increasing filtration of a CDG-coalgebra $C = (C,d,h)$ compatible with the comultiplication and the differential d on C. We recall from [5, Section 4.8] that the associated graded coalgebra $\mathrm{gr}^F C$ has a natural CDG-coalgebra structure with the differential induced by the differential d on the coalgebra C and the curvature linear function obtained by restricting the curvature linear function h of the CDG-coalgebra C to the subcoalgebra $F_0C \subset C$. In particular, F_0C is a CDG-coalgebra that can be viewed as a CDG-subcoalgebra simultaneously in C and in $\mathrm{gr}^F C$, and at the same time as a quotient CDG-coalgebra of $\mathrm{gr}^F C$. The components $F_nC/F_{n-1}C$ of the associated graded coalgebra $\mathrm{gr}^F C$ acquire natural structures of CDG-bicomodules over F_0C.

Now let C and D be two CDG-coalgebras endowed with increasing filtrations F as above, and let $f\colon C \longrightarrow D$ be a morphism of CDG-coalgebras preserving the filtrations. Then the restriction of f provides a morphism of CDG-coalgebras $F_0C \longrightarrow F_0D$, and there is also the induced morphism of associated graded CDG-coalgebras $\mathrm{gr}^F C \longrightarrow \mathrm{gr}^F D$. The maps of the associated graded components $F_nC/F_{n-1}C \longrightarrow F_nD/F_{n-1}D$ can be naturally viewed as morphisms of CDG-bicomodules over F_0D.

The following theorem (cf. [5, Theorem 4.8]) is the main result of this section.

Theorem 1.1. *Assume that the cones of the closed morphisms $F_nC/F_{n-1}C \longrightarrow F_nD/F_{n-1}D$ are coacyclic CDG-bicomodules over F_0D for all $n \geq 1$. Then the morphism of CDG-coalgebras $f\colon C \longrightarrow D$ is a weak equivalence provided that its restriction $F_0C \longrightarrow F_0D$ is a weak equivalence of CDG-coalgebras.*

Proof. The functors of corestriction of scalars acting between the coderived categories of left CDG-comodules over the CDG-coalgebras F_0C, F_0D, C, and D form a commutative square of triangulated functors; so do the functors of corestriction of scalars acting between the absolute derived categories of finite-dimensional left CDG-comodules over F_0C, F_0D, C, and D. Since we have assumed that the corestriction of scalars is an equivalence between the coderived categories of CDG-comodules over F_0C and F_0D, any finite-dimensional CDG-comodule over F_0D is isomorphic to a direct summand of a finite-dimensional CDG-comodule over F_0C as an object of the coderived category $\mathsf{D}^{\mathsf{co}}(F_0D\text{–}\mathsf{comod})$.

Furthermore, any finite-dimensional left CDG-comodule M over C admits a finite filtration by CDG-subcomodules F_nM defined as the full preimages of the CDG-subcomodules $F_nC \otimes_k M \subset C \otimes_k M$ under the coaction map $M \longrightarrow C \otimes_k M$ with the quotient CDG-comodules $F_nM/F_{n-1}M$ being CDG-comodules over F_0C; and similarly for finite-dimensional left CDG-comodules over D. Hence finite-dimensional left CDG-comodules over F_0C form a set of compact generators in both the coderived categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$. Therefore, in order to prove the assertion of theorem it suffices to check that the triangulated functor $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ induces an isomorphism of the Hom spaces

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})}(L, M[*]) \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})}(L, M[*])$$

for any two finite-dimensional left CDG-comodules L and M over F_0C.

The graded vector space of morphisms $L \longrightarrow M[*]$ in the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ can be computed as the cohomology space of the complex of morphisms of CDG-comodules from the CDG-comodule L to the CDG-comodule cobar-resolution $\widetilde{\mathrm{Cob}}(C, M)$ of the left CDG-comodule M over the CDG-coalgebra C. The left CDG-comodule $\widetilde{\mathrm{Cob}}(C, M)$ over C is constructed as the direct sum of graded left C-comodules

$$\widetilde{\mathrm{Cob}}(C, M) = C \otimes_k M \oplus C \otimes_k C \otimes_k M[-1] \oplus C \otimes_k C \otimes_k C \otimes_k M[-2] \oplus \cdots$$

endowed with the differential that is the sum of three summands, one of them defined in terms of the comultiplication on C and the coaction of C in M, the other one in terms of the differentials on C and M, and the third one in terms of the curvature linear function on C [5, proof of Theorem 4.4] (see also [4, §3]). The complex $\mathrm{Cob}^\bullet(L^*, C, M) = \mathrm{Hom}_C^\bullet(L, \widetilde{\mathrm{Cob}}(C, M))$ computing $\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})}(L, M[*])$ has the form

$$\mathrm{Cob}^\bullet(L^*, C, M) = L^* \otimes_k M \oplus L^* \otimes_k C \otimes_k M[-1] \oplus L^* \otimes_k C \otimes_k C \otimes_k M[-2] \oplus \cdots$$

with the differential consisting of the three summands as described above.

More generally, given a sequence of CDG-coalgebras E_1, …, E_m over a field k, a right CDG-comodule N over E_1, a sequence of CDG-bicomodules $K_{j-1,j}$ over E_{j-1} and E_j, and a left CDG-comodule M over E_m, one can construct the related cobar-complex $\mathrm{Cob}^\bullet(N, E_1, K_{12}, \dots, K_{m-1,m}, E_m, M)$ as the direct sum of shifted tensor products

$$\bigoplus\nolimits_{n_1,\dots,n_m\ge0} N\otimes E_1^{\otimes n_1}\otimes K_{12}\otimes\dots\otimes K_{m-1,m}\otimes E_m^{\otimes n_m}\otimes M[-n_1-\dots-n_m]$$

with the differential consisting of the three summands similar to the above.

Now we have to show that the morphism of cobar-complexes $\mathrm{Cob}^\bullet(N,C,M)\longrightarrow\mathrm{Cob}^\bullet(N,D,M)$ induced by the morphism of CDG-coalgebras $f\colon C\longrightarrow D$ is a quasi-isomorphism for any finite-dimensional right CDG-comodule N and any finite-dimensional left CDG-comodule M over the CDG-coalgebra F_0C. Let us endow both the cobar-complexes with the increasing filtrations F induced by the filtrations F on the CDG-coalgebras C and D. The associated graded complexes $\mathrm{gr}^F\mathrm{Cob}^\bullet(N,C,M)$ and $\mathrm{gr}^F\mathrm{Cob}^\bullet(N,D,M)$ are naturally identified with the cobar-complexes $\mathrm{Cob}^\bullet(N,\mathrm{gr}^F C,M)$ and $\mathrm{Cob}^\bullet(N,\mathrm{gr}^F D,M)$ of the associated graded coalgebras $\mathrm{gr}^F C$ and $\mathrm{gr}^F D$ with coefficients in the CDG-comodules N and M.

The grading components of the latter two complexes with respect to the grading by the indices of the filtrations F admit finite filtrations whose associated quotient complexes are direct sums of the shifts of the cobar-complexes of the sequences of CDG-coalgebras and CDG-(bi)comodules N, F_0C, $F_{n_1}C/F_{n_1-1}C$, F_0C, $\dots$, F_0C, $F_{n_m}C/F_{n_m-1}C$, F_0C, M, and similarly for D. In order to deduce the assertion of theorem, it remains to apply the following lemma. ☐

Lemma 1.2. *Let*

$$(N,C_1,K_{12},\dots,K_{m-1,m},C_m,M)\longrightarrow(N',D_1,L_{12},\dots,L_{m-1,m},D_m,M')$$

be a morphism of sequences of CDG-coalgebras and CDG-(bi)comodules as above, i. e., $C_j\longrightarrow D_j$ are morphisms of CDG-coalgebras, N is a right CDG-comodule over C_1, M is a left CDG-comodule over C_m, and $K_{j-1,j}$ are CDG-bicomodules over C_{j-1} and C_j, while $N\longrightarrow N'$ is a morphism of right CDG-comodules over D_1, $M\longrightarrow M'$ is a morphism of left CDG-comodules over D_m, and $K_{j-1,j}\longrightarrow L_{j-1,j}$ are morphisms of CDG-bicomodules over D_{j-1} and D_j. Suppose that the cones of closed morphisms $N\longrightarrow N'$, $M\longrightarrow M'$, and $K_{j-1,j}\longrightarrow L_{j-1,j}$ are coacyclic CDG-(bi)comodules over the corresponding CDG-coalgebras, while the morphisms of CDG-coalgebras $C_j\longrightarrow D_j$ are weak equivalences. Then the induced morphism of cobar-complexes

$$\mathrm{Cob}^\bullet(N,C_1,K_{12},\dots,K_{m-1,m},C_m,M)\longrightarrow\mathrm{Cob}^\bullet(N',D_1,L_{12},\dots,L_{m-1,m},D_m,M')$$

is a quasi-isomorphism.

Proof. One can transform the source sequence $(N,C_1,K_{12},\dots,K_{m-1,m},C_m,M)$ into the target sequence $(N',D_1,L_{12},\dots,L_{m-1,m},D_m,M')$ by replacing its elements one by one, starting from replacing the CDG-coalgebras C_j in the first sequence with the CDG-coalgebras D_j and then proceeding to replace the CDG-(bi)comodules N, M, and $K_{j-1,j}$ with the CDG-(bi)comodules N', M', and $L_{j-1,j}$, respectively. This reduces the question to the case when the source and target sequences differ in one position only, i. e., either they differ by a one CDG-coalgebra only, or only by one CDG-(bi)comodule. Since the cobar-complex of a sequence of CDG-coalgebras and CDG-(bi)comodules is obviously (co)acyclic whenever one of the CDG-(bi)comodules is coacyclic, the latter situation is clear.

To consider the former situation, assume that $N = N'$, $M = M'$, $K_{j-1,j} = L_{j-1,j}$ for all j, and $C_j = D_j$ for all j except some $j = j_0$. Here it remains to notice that the cobar-complex $\mathrm{Cob}^\bullet(N, C_1, K_{12}, \dots, C_{j_0}, \dots, K_{m-1,m}, C_m, M)$ is isomorphic to the cobar-complex $\mathrm{Cob}^\bullet(N'', C_{j_0}, M'')$ for a certain right CDG-comodule N'' and left CDG-comodule M'' over the CDG-coalgebra C_{j_0} (and similarly for D_{j_0} in place of C_{j_0}). One can denote these two CDG-comodules by

$$N'' = \mathrm{Cob}(N, C_1, K_{12}, \dots, K_{j_0-1,j_0}) \quad \text{and} \quad M'' = \mathrm{Cob}(K_{j_0,j_0+1}, \dots, K_{m-1,m}, C_m, M),$$

in the obvious sense of the notation.

So the question reduces to showing that the natural morphism of complexes $\mathrm{Cob}^\bullet(N'', C_{j_0}, M'') \longrightarrow \mathrm{Cob}^\bullet(N'', D_{j_0}, M'')$ is a quasi-isomorphism for any right CDG-comodule N'' and left CDG-comodule M'' over C_{j_0}. When the CDG-comodule N'' is finite-dimensional, these are the complexes computing morphisms $N''^* \longrightarrow M''[*]$ in the coderived categories $\mathsf{D}^{\mathsf{co}}(C_{j_0}\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{co}}(D_{j_0}\text{–}\mathsf{comod})$, so the map between them is a quasi-isomorphism by the assumption that the morphism of CDG-coalgebras $C_{j_0} \longrightarrow D_{j_0}$ is weak equivalence. The general case follows by the passage to the direct limit over finite-dimensional CDG-subcomodules of N''. $\square$

We refer to [5, Sections 4.3, 4.5, and 5.5] for the definitions of a *cosemisimple graded coalgebra* and the *maximal cosemisimiple graded subcoalgebra* $E^{\mathrm{ss}} \subset E$ of a graded coalgebra E. One can easily see that the maximal cosemisimple subcoalgebra of a nonnegatively graded coalgebra E is contained in its degree-zero component E^0.

The following example illustrates a possible way of applying Theorem 1.1. Its assertion is not difficult to prove by a direct argument generalizing [5, proof of Proposition 3.1]; however, we prefer to use it as a demonstration of our general method.

Example 1.3. *Let $C^\bullet = (C^0 \to C^1 \to C^2 \to \cdots)$ and $D^\bullet = (D^0 \to D^1 \to D^2 \to \cdots)$ be two nonnegatively cohomologically graded DG-coalgebras with the maximal cosemisimple subcoalgebras contained in the kernels of the differentials d^0, that is $d^0(C^{\mathrm{ss}}) = 0 = d^0(D^{\mathrm{ss}})$. Then any comultiplicative quasi-isomorphism $f\colon C^\bullet \longrightarrow D^\bullet$, i. e., a morphism of DG-coalgebras inducing an isomorphism $H^*(C^\bullet) \simeq H^*(D^\bullet)$ of their cohomology coalgebras, is a weak equivalence of DG-coalgebras.*

Proof. Before proceeding with the proof as such, let us have a short discussion of *canonical filtrations* of complexes (DG-algebras, DG-coalgebras) on which this proof is based. Let $\cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow E^{n+2} \longrightarrow \cdots$ be a complex of modules over an algebra R. Then a subcomplex in $E^\bullet$ with the cohomology modules isomorphic to those of $E^\bullet$ in the degrees $\le n$ and vanishing in the degrees $\ge n+1$ can be constructed either as

$$\cdots \longrightarrow E^{n-1} \longrightarrow \ker(d^n) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

or as

$$\cdots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow \mathrm{im}(d^{n+1}) \longrightarrow 0 \longrightarrow \cdots$$

Let us denote the former subcomplex by $\tau_{\le n}E^\bullet$ and the latter one by $\tau'_{\le n} \subset E^\bullet$. The two constructions are dual to each other; we will call the filtration of a complex $E^\bullet$ by its subcomplexes $\tau_{\le n}E^\bullet$ the *canonical filtration* and the filtration by the subcomplexes $\tau'_{\le n}E^\bullet$ the *cocanonical filtration* of a complex $E^\bullet$.

The difference between the two constructions may appear to be insignificant, as the quotient complexes $\tau'_{\le n}E^\bullet/\tau_{\le n}E^\bullet$ are just contractible two-term complexes. The situation is similar for a complex of comodules $E^\bullet$ over a coalgebra D. However, the difference between the two filtrations becomes essential when one passes from complexes of (co)modules to DG-algebras or DG-coalgebras.

Given a DG-algebra $A^\bullet$, its subcomplexes of canonical filtration $\tau_{\le n}A^\bullet$ form a multiplicative filtration on $A^\bullet$, i. e., for any p and $q \in \mathbb{Z}$ one has

$$\tau_{\le p}A^\bullet \cdot \tau_{\le q}A^\bullet \subset \tau_{\le p+q}A^\bullet.$$

The cocanonical filtration on $A^\bullet$ does *not* have this property, i. e., one has $\tau'_{\le p}A^\bullet \cdot \tau'_{\le q}A^\bullet \not\subset \tau'_{\le p+q}A^\bullet$ in general. Similarly, given a DG-coalgebra $C^\bullet$, its subcomplexes of cocanonical filtration $\tau'_{\le n}C^\bullet$ form a comultiplicative filtration on $C^\bullet$, that is

$$\mu(\tau'_{\le n}C^\bullet) \subset \textstyle\sum_{p+q=n} \tau'_{\le p}C^\bullet \otimes \tau'_{\le q}C^\bullet,$$

where $\mu\colon C \longrightarrow C \otimes_k C$ denotes the comultiplication map. The canonical filtration on $C^\bullet$ does *not* have this property in general.

The argument proving the desired assertion now proceeds as follows. Endow the DG-coalgebras $C^\bullet$ and $D^\bullet$ with the cocanonical filtrations $F_n = \tau'_{\le n}$. Given our assumptions on the cohomological grading of $C^\bullet$ and $D^\bullet$, these are increasing filtrations starting from $F_0C^\bullet = (C^0 \to d(C^0))$, $F_1C^\bullet = (C^0 \to C^1 \to d(C^1))$, …, and similarly for $D^\bullet$. The cones of the induced morphisms $F_nC^\bullet/F_{n-1}C^\bullet \longrightarrow F_nD^\bullet/F_{n-1}D^\bullet$ are acyclic three-term complexes with DG-bicomodule structures over $F_0D^\bullet$. By [5, Theorem 4.3.1(a)], these are coacyclic (and in fact even absolutely acyclic) DG-bicomodules. Hence, according to Theorem 1.1, in order to prove that the morphism of DG-coalgebras $C^\bullet \longrightarrow D^\bullet$ is a weak equivalence, it suffices to show that so is the morphism of DG-coalgebras $F_0C^\bullet \longrightarrow F_0D^\bullet$.

Notice that is follows from the assumptions of the example that the morphism of graded coalgebras $C \longrightarrow D$ identifies their maximal cosemisimple subcoalgebras, $C^{\mathrm{ss}} \simeq D^{\mathrm{ss}}$. Introduce the *coradical filtrations* on the grading components C^0 and D^0, i. e., the filtrations by the kernels of iterated comultiplication maps $N_nC^0 = \ker(C^0 \to (C^0/C^{\mathrm{ss}})^{\otimes n+1})$ taking values in the tensor powers of the coalgebra without counit C^0/C^{ss}, and similarly for D^0. Extend the filtrations N_n to the whole DG-coalgebras $F_0C^\bullet$ and $F_0D^\bullet$ by the rules $N_nd(C^0) = d(N_nC^0)$ and similarly for $F_0D^\bullet$.

Then, in particular, one has $N_0F_0C^\bullet = C^{\mathrm{ss}} \simeq D^{\mathrm{ss}} = N_0F_0D^\bullet$. The associated quotient complenes $N_nF_0C^\bullet/N_{n-1}F^0C^\bullet$ and $N_nF_0D^\bullet/N_{n-1}F_0D^\bullet$ are contractible two-term complexes of bicomodules over the coalgebra $N_0F_0C^\bullet \simeq N_0F_0D^\bullet$. Applying Theorem 1.1 again, we conclude that the morphism of DG-coalgebras $F_0C^\bullet \longrightarrow F_0D^\bullet$ is a weak equivalence. $\square$

2. Topological Graded Abelian Groups and Rings

3. Coequivalences of Topological CDG-Algebras

4. Co- and Contraequivalences of Topological CDG-Rings

References

[1] D. Gaitsgory. Ind-coherent sheaves. Electronic preprint `arXiv:1105.4857 [math.AG]` .

[2] V. Hinich. Homological algebra of homotopy algebras. *Communications in Algebra* **25**, #10, p. 3291–3323, 1997. `arXiv:q-alg/9702015v1` . Erratum, `arXiv:math/0309453v3 [math.QA]` .

[3] B. Keller. Deriving DG-categories. *Ann. Sci. de l'École Norm. Sup. (4)* **27**, #1, p. 63–102, 1994.

[4] L. Positselski. Nonhomogeneous quadratic duality and curvature. *Functional Analysis and its Appl.* **27**, #3, p. 197–204, 1993. `arXiv:1411.1982 [math.RA]`

[5] L. Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs of the American Math. Society* **212**, #996, 2011. vi+133 pp. `arXiv:0905.2621 [math.CT]`

[6] L. Positselski. Koszulity of cohomology = $K(\pi,1)$-ness + quasi-formality. Electronic preprint `arXiv:1507.04691 [math.KT]` .

Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127994; and

Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312, Russia; and

Mathematics Department, Technion — Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `posic@mccme.ru`