THE CONTRA VERSION OF THE CATEGORY O

LEONID POSITSELSKI

Abstract. The semicoderived category of the category O over a Tate Lie algebra (or, more precisely, over a Tate Harish-Chandra pair) is equivalent to the semicontraderived category of the contra version of the category O with the complementary or shifted central charge. This is the main result of the book [2]. In this paper we show that the forgetful functor from the contra version of the category O into the category of modules over the Lie algebra is fully faithful for many Tate Lie algebras, such as the Virasoro or the Kac–Moody. Similarly, the forgetful functor from the category of contramodules over the topological Lie algebra into the category of modules over the underlying abstract Lie algebra, or over any its dense subalgebra, is fully faithful for such Lie algebras. These assertions extend the line of known results claiming that the forgetful functors from contramodules to modules are fully faithful under certain assumptions.

Contents

Introduction

0.1. *Contramodules* is the common name for objects of the categories *dual-analogous to* the categories of comodules over coalgebras and corings, or discrete modules over topological rings or Lie algebras, or smooth modules over locally profinite groups, or torsion modules over a commutative ring with a fixed ideal. The words "dual-analogous" mean that the category of contramodules is somewhat similar to but quite different from the opposite category to the category of comodules. We refer to the overview paper [5] and the introductory exposition [7] for in-depth discussions of various classes of contramodules and contramodule categories.

In good situations, both the comodule and the contramodule categories are abelian, and furthermore, there are *covariant* triangulated equivalences connecting their ("appropriately defined", i. e., possibly "exotic") derived categories. Results of the latter kind are known under the common name of the *derived comodule-contramodule correspondence*. We refer to the introductions to the book [2] and the papers [6, 8] for substantial discussions.

The simplest result of this kind is the triangulated equivalence between the (bounded or unbounded, conventional) derived categories of the abelian category of p-primary torsion abelian groups and the abelian category of contramodules over the topological ring of p-adic integers,

$$\mathsf{D}^\star(\mathbb{Z}_p\text{–}\mathsf{discr}) \simeq \mathsf{D}^\star(\mathbb{Z}_p\text{–}\mathsf{contra}),$$

where $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$. Here p-primary torsion abelian groups are, of course, the same thing as discrete $\mathbb{Z}_p$-modules.

The most deep and important result of this kind is the triangulated equivalence between the (always unbounded) *semiderived categories* of the abelian category $\mathsf{O}_\varkappa(\mathfrak{g}, H)$ of *Tate Harish-Chandra modules* with the central charge $\varkappa$ over a *Tate Harish-Chandra pair* $(\mathfrak{g}, H)$ and the abelian category $\mathsf{O}^{\mathsf{ctr}}_{\varkappa_0+\varkappa}(\mathfrak{g}, H)$ of *Tate Harish-Chandra contramodules* with the complementary or shifted central charge $\varkappa + \varkappa_0$ over the same Tate Harish-Chandra pair [2, Corollary D.3.1],

$$\mathsf{D}^{\mathsf{sico}}(\mathsf{O}_\varkappa(\mathfrak{g}, H)) \simeq \mathsf{D}^{\mathsf{sictr}}(\mathsf{O}^{\mathsf{ctr}}_{\varkappa_0+\varkappa}(\mathfrak{g}, H)).$$

Here the "semiderived category" is a generic term for the two dual notions of the *semicoderived* and the *semicontraderived* categories, which appear on the two sides of the triangulated equivalence. The canonical central charge κ_0 is the central charge with which the Tate Lie algebra $\mathfrak{g}$ acts in the space of semi-infinite exterior forms over it. For example, for the Virasoro Lie algebra, the canonical central charge $\varkappa_0$ corresponds to $c = -26$ in the standard notation. The category of Tate Harish-Chandra contramodules with the central charge $\varkappa_0 + \varkappa$ contains, in particular, the dual vector spaces to Tate Harish-Chandra modules with the central charge $-\varkappa_0 - \varkappa$; in this sense this is called "complementary or shifted central charge".

0.2. The discrete, the smooth, and the torsion modules form full subcategories in the related categories of modules, by the definition. So do the comodules over coalgebras over fields. The contramodules, on the other hand, are module-like structures with *infinite summation operations* [5]. Thus the claim that they form a full subcategory in the related category of modules sounds somewhat unexpected. An infinite summation operation is uniquely determined by its finite aspects. Why should it be? Nevertheless, such results are known to hold in a number of situations.

The observation that weakly p-complete/Ext-p-complete abelian groups carry a uniquely defined natural structure of $\mathbb{Z}_p$-modules that is preserved by any group homomorphisms between them goes back to [1, Lemma 4.13]. The claim that the forgetful functor provides an isomorphism between the category of $\mathbb{Z}_p$-contramodules $\mathbb{Z}_p\text{–}\mathsf{contra}$ and the full subcategory of weakly p-complete abelian groups in $\mathbb{Z}\text{–}\mathsf{mod}$

was announced in [2, Remark A.3], where the definition of a contramodule over a topological associative ring also first appeared. Similarly, it was mentioned in [2, Remark A.1.1] that contramodules over the coalgebra $\mathcal{C}$ dual to the algebra of formal power series $\mathcal{C}^* = k[[x]]$ in one variable x over a field k form a full subcategory in the category of $k[x]$-modules; and a generalization to the coalgebras dual to power series in several commutative variables $\mathcal{C}^* = k[[x_1, \dots, x_m]]$ was formulated.

A more elaborated discussion of these results for the rings $k[[x]]$ and $\mathbb{Z}_p$ can be found in [5, Section 1.6]; and a generalization to the adic completions of Noetherian rings with respect to arbitrary ideals was proved in [3, Theorem B.1.1]. A version for adic completions of right Noetherian associative rings at their centrally generated ideals was obtained in [4, Theorem C.5.1]. A definitive treatment based on modern techniques can be found in [10, Examples 2.2 and 2.3].

In the most straightforward terms, the situation can be explained as follows. One says that an *x-power infinite summation operation* is defined on an abelian group P if for every sequence of elements p_0, p_1, p_2, $\ldots \in P$ an element denoted formally by $\sum_{n=0}^\infty x^n p_n \in P$ is defined in such a way that the equations of additivity

$$\sum\nolimits_{n=0}^\infty x^n(p_n + q_n) = \sum\nolimits_{n=0}^\infty x^n p_n + \sum\nolimits_{n=0}^\infty x^n q_n \qquad \forall\, p_n,\, q_n \in P,$$

contraunitality

$$\sum\nolimits_{n=0}^\infty x^n p_n = p_0 \qquad \text{if } p_1 = p_2 = p_3 = \cdots = 0,$$

and contraassociativity

$$\sum\nolimits_{i=0}^\infty x^i \sum\nolimits_{j=0}^\infty x^j p_{ij} = \sum\nolimits_{n=0}^\infty x^n \sum\nolimits_{i+j=n} p_{ij} \qquad \forall\, p_{ij} \in P, \quad i,\, j = 0,\, 1,\, 2,\, \ldots$$

are satisfied. Then it is an elementary linear algebra exercise [7, Section 3] to show that an x-power infinite summation operation on an abelian group P is uniquely determined by the additive operator $x\colon P \longrightarrow P$,

$$xp = \sum\nolimits_{n=0}^\infty x^n p_n, \qquad \text{where } p_1 = p \text{ and } p_0 = p_2 = p_3 = p_4 = \cdots = 0.$$

Furthermore, a $\mathbb{Z}[x]$-module structure on P comes from an x-power infinite summation operation if and only if

$$\text{(1)} \qquad \operatorname{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x, x^{-1}], P) = 0 = \operatorname{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z}[x, x^{-1}], P).$$

"An abelian group with an x-power infinite summation operation" is another name for a contramodule over the topological ring of formal power series $\mathbb{Z}[[x]]$; so the forgetful functor $\mathbb{Z}[[x]]\text{–}\mathsf{contra} \longrightarrow \mathbb{Z}[x]\text{–}\mathsf{mod}$ is fully faithful, and the conditions (1) describe its image. Notice that the dual condition

$$\mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}[x]} M = 0$$

describes the category of discrete/torsion $\mathbb{Z}[[x]]$-modules as a full subcategory in the category of $\mathbb{Z}[x]$-modules. In the case of several commutative variables x_1, $\ldots$, x_m, rewriting an $[x_1, \dots, x_m]$-power infinite summation operation as the composition of x_j-power infinite summations over the indices $j = 1$, $\ldots$, m allows to obtain a similar description of the abelian category of $\mathbb{Z}[[x_1,\dots,x_m]]$-contramodules as a full subcategory in the abelian category of $\mathbb{Z}[x_1,\dots,x_m]$-modules [7, Section 4].

0.3. The discussion in Section 0.2 applies to finite collections of commutative variables only. So it was a kind of breakthrough when it was shown in [9, Theorem 1.1] that the forgetful functor from the category of left contramodules over the coalgebra $\mathcal{C}$ dual to the algebra of formal power series $\mathcal{C}^* = k\{\{x_1,\dots,x_m\}\}$ in noncommutative variables x_1, …, x_m into the category of left modules over the free associative algebra $k\{x_1,\dots,x_m\}$ is fully faithful.

To give another example, let us consider contramodules over the Virasoro Lie algebra $\mathbb{V}\mathrm{ir}$. The Virasoro Lie algebra is the topological vector space $\mathbb{V}\mathrm{ir} = k((z))d/dz \oplus kC$ of vector fields with coefficients in the field of Laurent power series in one variable z over a field k of characteristic 0, extended by adding a one-dimensional vector space kC as a second direct summand. The vectors $L_n = z^{n+1}d/dz$, $n\in\mathbb{Z}$, and C form a topological basis in $\mathbb{V}\mathrm{ir}$, and the Lie bracket is defined by the formula

$$[L_i, L_j] = (j-i)L_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}C, \qquad [L_i, C] = 0, \qquad i,\, j\in\mathbb{Z},$$

where δ is the Kronecker symbol.

A $\mathbb{V}\mathrm{ir}$-*contramodule* P is a k-vector space endowed with a linear operator $C\colon P\longrightarrow P$ and an infinite summation operation assigning to every sequence of vectors p_{-N}, p_{-N+1}, p_{-N+2}, … $\in P$, $N\in\mathbb{Z}$, a vector denoted formally by $\sum_{i=-N}^{\infty} L_ip_i \in P$. The equations of agreement

$$\sum\nolimits_{i=-N}^{\infty} L_ip_i = \sum\nolimits_{i=-M}^{\infty} L_ip_i \qquad \text{when } -N<-M \text{ and } p_{-N}=\dots=p_{-M+1}=0,$$

linearity

$$\sum\nolimits_{i=-N}^{\infty} L_i(ap_i+bq_i) = a\sum\nolimits_{i=-N}^{\infty} L_ip_i + b\sum\nolimits_{i=-N}^{\infty} L_iq_i \qquad \forall\, a,\, b\in k,\ \ p_i,\, q_i\in P,$$

centrality of C

$$C\sum\nolimits_{i=-N}^{\infty} L_ip_i = \sum\nolimits_{i=-N}^{\infty} L_i(Cp_i),$$

and contra-Jacobi identity

$$\sum\nolimits_{i=-N}^{\infty} L_i\Big(\sum\nolimits_{j=-M}^{\infty} L_jp_{ij}\Big) - \sum\nolimits_{j=-M}^{\infty} L_j\Big(\sum\nolimits_{i=-N}^{\infty} L_ip_{ij}\Big)$$
$$= \sum\nolimits_{n=-N-M}^{\infty} L_n\Big(\sum\nolimits_{i+j=n}^{i\ge -N,\, j\ge -M}(j-i)p_{ij}\Big) + C\sum\nolimits_{i+j=0}^{i\ge -N,\, j\ge -M}\left(\frac{i^3-i}{12}p_{ij}\right)$$

have to be satisfied [2, Section D.2.7], [5, Section 1.7].

One of the main results of this paper claims that the forgetful functor $\mathbb{V}\mathrm{ir}$–contra $\longrightarrow$ $\mathbb{V}\mathrm{ir}$–mod from the abelian category of $\mathbb{V}\mathrm{ir}$-contramodules to the abelian category of modules over the Lie algebra $\mathbb{V}\mathrm{ir}$ is fully faithful. Moreover, so is the forgetful functor $\mathbb{V}\mathrm{ir}$–$\mathsf{contra}\longrightarrow \mathrm{Vir}$–$\mathsf{mod}$ from the category of $\mathbb{V}\mathrm{ir}$-contramodules to the category of modules over a discrete version $\mathrm{Vir} = k[z,z^{-1}]d/dz\oplus kC \subset \mathbb{V}\mathrm{ir}$ of Virasoro Lie algebra.

Acknowlegdement. The author's research is supported by the Israel Science Foundation grant # 446/15 and by the Grant Agency of the Czech Republic under the grant P201/12/G028.

1. Main Theorem for Topological Rings

We refer to [11, Sections 1.1–2 of the Introduction] and [10, Section 1] for a discussion of additive accessible monads on the category of sets. In this paper, we are interested in the monads associated with topological rings [2, Remark A.3], [3, Section 1.2], [5, Section 2.1], [11, Section 5], [10, Example 1.3(2)].

Let $\mathfrak{R}$ be a separated and complete topological associative ring with a base of neighborhoods of zero formed by open right ideals. The former two conditions mean that the natural map $\mathfrak{R} \longrightarrow \varprojlim_{\mathfrak{J}} \mathfrak{R}/\mathfrak{J}$, where the projective limit is taken over all the open right ideals $\mathfrak{J}$ in $\mathfrak{R}$, is an isomorphism of abelian groups (or right $\mathfrak{R}$-modules). For any set X, we denote by $\mathfrak{R}[[X]]$ the set $\varprojlim_{\mathfrak{J}} \mathfrak{R}/\mathfrak{J}[X]$, where, once again, the projecitive limit is taken over all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$, and the notation $A[X]$ for any abelian group A and a set X stands for the direct sum of X copies of A.

Elements of the set $\mathfrak{R}[[X]]$ are interpreted as infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with families of coefficients $r_x \in R$ converging to zero in the topology of $\mathfrak{R}$. The latter condition means that an infinite formal linear combination $\sum_{x\in X} r_x x$ belongs to $\mathfrak{R}[[X]]$ if and only if for every neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ the set of all $x \in X$ such that $r_x \notin \mathfrak{U}$ is finite.

For any map of sets $f\colon X \longrightarrow Y$, one constructs the induced map $\mathfrak{R}[[f]]\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{R}[[Y]]$ by computing infinite sums of coefficients, in the sense of the limit of finite partial sums in the topology of $\mathfrak{R}$, along the fibers of the map f. The convergence condition on the families of coefficients r_x and the separatedness and completeness condition on the topology of the ring $\mathfrak{R}$ are used here.

The covariant functor $\mathbb{T}_{\mathfrak{R}}\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ taking every set X to the set $\mathfrak{R}[[X]]$ is an (additive and accessible) monad on the category of sets. The monad unit map $\epsilon_{\mathfrak{R}}(X)\colon X \longrightarrow \mathfrak{R}[[X]]$ simply takes every element $x_0 \in X$ to the formal linear combination $\sum_{x\in X} r_x x$ with the coefficients $r_{x_0} = 1$ and $r_x = 0$ for $x \neq x_0$; while the monad multiplication map $\phi_{\mathfrak{R}}(X)\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ opens the parentheses in a formal linear combination of formal linear combinations, computing the coefficients of the resulting formal linear combination of elements of X using the multiplication in the ring $\mathfrak{R}$ and the infinite summation understood, once again, in the sense of the limit of finite partial sums in the topology of $\mathfrak{R}$.

More precisely, the map $\mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ is constructed as the projective limit of natural maps $\mathfrak{R}/\mathfrak{J}[\mathfrak{R}[[X]]] \longrightarrow \mathfrak{R}/\mathfrak{J}[X]$ over the open right ideals $\mathfrak{J} \subset \mathfrak{R}$. To construct the latter map, one has to show that for every element $\bar r \in \mathfrak{R}/\mathfrak{J}$ there is a well-defined map of left multiplication with $\bar r$ acting from $\mathfrak{R}[[X]]$ to $\mathfrak{R}/\mathfrak{J}[X]$. Here it is important that $\mathfrak{J}$ is a right ideal in $\mathfrak{R}$ and that for every element $r \in \mathfrak{R}$ there exists an open right ideal $\mathfrak{J}_r \subset \mathfrak{R}$ such that $r\mathfrak{J}_r \subset \mathfrak{J}$. The latter property follows from the continuity condition on the multiplication in a topological ring.

A *left* $\mathfrak{R}$*-contramodule* is an algebra/module over the monad $\mathbb{T}_{\mathfrak{R}}\colon X \longmapsto \mathfrak{R}[[X]]$ on the category of sets. In other words, a left $\mathfrak{R}$-contramodule $\mathfrak{P}$ is a set endowed with a map of sets, called the *contraaction map*, $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ satisfying the *contraassociativity* and *contraunitality* equations.

Specifically, the two maps $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]]$, one of them being the monad multiplication map $\phi_{\mathfrak{R}}(\mathfrak{P})$ and the other one the map $\mathfrak{R}[[\pi_{\mathfrak{P}}]]$ induced by the contraaction map $\pi_{\mathfrak{P}}$, should have equal compositions with the map $\pi_{\mathfrak{P}}$,

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P},$$

and the composition of the monad unit map $\epsilon_{\mathfrak{R}}(\mathfrak{P})\colon \mathfrak{P} \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ with the contraaction map $\pi_{\mathfrak{P}}$ should be equal to the identity map $\mathrm{id}_{\mathfrak{P}}$ of the set $\mathfrak{P}$,

$$\mathfrak{P} \longrightarrow \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}.$$

The category $\mathfrak{R}\text{–}\mathsf{contra}$ of left contramodules over a topological ring $\mathfrak{R}$ satisfying the above conditions is abelian with set-indexed direct sums and products, and a natural projective generator $\mathfrak{P} = \mathfrak{R}$. More generally, for any set X, the left $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$, with the contraaction map $\pi_{\mathfrak{R}[[X]]} = \phi_{\mathfrak{R}}(X)$, is called the *free left* $\mathfrak{R}$*-contramodule* generated by the set X. There are enough projective objects in the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$, and a left $\mathfrak{R}$-contramodule is projective if and only if it is a direct summand of a free left $\mathfrak{R}$-contramodule.

In particular, when $\mathfrak{R}$ is a discrete ring, a left $\mathfrak{R}$-contramodule is the same thing as a conventional left $\mathfrak{R}$-module, because left $\mathfrak{R}$-modules can be defined as modules over the monad $X \longmapsto \mathfrak{R}[X]$ on the category of sets. Here, as above, $\mathfrak{R}[X]$ is the set of all finite formal linear combinations of elements of X with coefficients in $\mathfrak{R}$. So, when $\mathfrak{R}$ is a topological ring, $\mathfrak{R}[X]$ is a subset in $\mathfrak{R}[[X]]$. Composing the inclusion map $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ with the contraaction map $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}] \longrightarrow \mathfrak{P}$, one recovers the *underlying left* $\mathfrak{R}$*-module structure* of a left $\mathfrak{R}$-contramodule.

The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ is exact and preserves infinite products (but not infinite direct sums). More generally, let R be an associative ring and $\theta\colon R \longrightarrow \mathfrak{R}$ be an associative ring homomorphism. Composing the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ with the functor of restriction of scalars $\mathfrak{R}\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}$ induced by θ, one obtains a forgetful functor $\Theta\colon \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$. The latter functor is also exact and preserves infinite products.

Let $\mathfrak{J} \subset \mathfrak{R}$ be a right ideal. We will say that a finite set of elements $s_1, \dots, s_m \in \mathfrak{J}$ *strongly generates* the right ideal $\mathfrak{J}$ if for every family of elements $r_x \in \mathfrak{J}$, indexed by some set X and converging to zero in the topology of $\mathfrak{R}$, there exist families of elements $r_{j,x} \in \mathfrak{R}$, $j = 1, \dots, m$, each of them indexed by the set X and converging to zero in the topology of $\mathfrak{R}$, such that $r_x = \sum_{j=1}^m s_j r_{j,x}$ for all $x \in X$. Since any finite family of elements in $\mathfrak{R}$ converges to zero in the topology of $\mathfrak{R}$, any finite set of elements of a right ideal $\mathfrak{J} \subset \mathfrak{R}$ strongly generating the ideal $\mathfrak{J}$ also generates the right ideal $\mathfrak{J}$ in the conventional sense (cf. [3, Section B.4]).

The following theorem is our main result. It is a generalization of [9, Theorem 1.1], and its proof is similar to that [9] (see Examples 2.3 for a discussion).

Theorem 1.1. *Let $\mathfrak{R}$ be a complete, separated topological associative ring, R be an associative ring, and $\theta\colon R \longrightarrow \mathfrak{R}$ be a ring homomorphism with a dense image. Assume that $\mathfrak{R}$ has a countable base of neighborhoods of zero consisting of open two-sided ideals, each of which, viewed as a right ideal, is strongly generated by a finite set of elements lying in the image of the map θ. Then the forgetful functor $\Theta\colon \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful.*

Proof. Given a set X and a complete, separated topological abelian group $\mathfrak{A}$ with a base of neighborhoods of zero formed by open subgroups $\mathfrak{U} \subset \mathfrak{A}$, denote by $\mathfrak{A}[[X]]$ the abelian group $\varprojlim_{\mathfrak{U}\subset\mathfrak{A}} \mathfrak{A}/\mathfrak{U}[X]$. Following the notation in [4, Section D.1] and [11, Sections 5–6], for any closed subgroup $\mathfrak{A} \subset \mathfrak{R}$ and any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, we denote by $\mathfrak{A} \leftthreetimes \mathfrak{P} \subset \mathfrak{P}$ the image of the composition $\mathfrak{A}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ of the natural embedding $\mathfrak{A}[[\mathfrak{P}]] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ and the contraaction map $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$. The map $\mathfrak{A}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is an abelian group homomorphism, so $\mathfrak{A} \leftthreetimes \mathfrak{P}$ is a subgroup in $\mathfrak{P}$.

As usually, for any left $\mathfrak{R}$-module P and any subgroup $A \subset \mathfrak{R}$, we denote by $AP \subset P$ the subgroup generated by the products ap, where $a \in A$ and $p \in P$. So we have $\mathfrak{A}\mathfrak{P} \subset \mathfrak{A} \leftthreetimes \mathfrak{P}$ for any closed subgroup $\mathfrak{A} \subset \mathfrak{R}$ and any left $\mathfrak{R}$-contramodule $\mathfrak{P}$. For any left ideal $I \subset \mathfrak{R}$ and any left $\mathfrak{R}$-module P, the subgroup $IP \subset P$ is a left $\mathfrak{R}$-submodule in P. For any closed left ideal $\mathfrak{I} \subset \mathfrak{R}$ and any left $\mathfrak{R}$-contramodule $\mathfrak{P}$, the subgroup $\mathfrak{I} \leftthreetimes \mathfrak{P} \subset \mathfrak{P}$ is a left $\mathfrak{R}$-subcontramodule in $\mathfrak{P}$. On the other hand, when a closed right ideal $\mathfrak{J} \subset \mathfrak{R}$ is strongly generated by a finite set of elements $s_1, \dots, s_m \in \mathfrak{J}$, one has $\mathfrak{J} \leftthreetimes \mathfrak{P} = \mathfrak{J}\mathfrak{P} = s_1\mathfrak{P} + \dots + s_m\mathfrak{P}$ for any left $\mathfrak{R}$-contramodule $\mathfrak{P}$.

Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two left $\mathfrak{R}$-contramodules, and let $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ be a left R-module morphism. The contraaction map $\pi_{\mathfrak{P}}\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is a surjective morphism of left $\mathfrak{R}$-contramodules. In order to show that f is an $\mathfrak{R}$-contramodule morphism, it suffices to check that the composition $\mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q}$ is an $\mathfrak{R}$-contramodule morphism. Hence we can replace $\mathfrak{P}$ with $\mathfrak{R}[[\mathfrak{P}]]$ and suppose that $\mathfrak{P} = \mathfrak{R}[[X]]$ is a free left $\mathfrak{R}$-contramodule generated by a set X.

Then the composition $X \longrightarrow \mathfrak{Q}$ of the natural embedding $X \longrightarrow \mathfrak{R}[[X]]$ with a left R-module morphism $f\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{Q}$ can be extended uniquely to a left $\mathfrak{R}$-contramodule morphism $f'\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{Q}$. Setting $g = f - f'$, we have a left R-module morphism $g\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{Q}$ taking the elements of the set X to zero elements in $\mathfrak{Q}$. We have to show that $g = 0$. Without loss of generality, we can assume that the left $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is generated by its subset $g(\mathfrak{R}[[X]]) \subset \mathfrak{Q}$ (otherwise, replace $\mathfrak{Q}$ with its subcontramodule generated by $g(\mathfrak{R}[[X]])$). Then we have to show that $\mathfrak{Q} = 0$.

Let $\mathfrak{I} \subset \mathfrak{R}$ be an open two-sided ideal strongly generated, as a right ideal, by a finite set of elements $s_1, \dots, s_m \in \mathfrak{I}$ belonging to the image of θ. Denote by $I \subset R$ the full preimage of the ideal $\mathfrak{I}$ with respect to ring homomorphism θ; so I is a two-sided ideal in R and $R/I \simeq \mathfrak{R}/\mathfrak{I}$ (since the image of θ is dense in $\mathfrak{R}$). Then we have $\mathfrak{I}[[X]] = \mathfrak{I} \leftthreetimes \mathfrak{R}[[X]] = I\mathfrak{R}[[X]] = s_1\mathfrak{R}[[X]] + \dots + s_m\mathfrak{R}[[X]]$ and $\mathfrak{I} \leftthreetimes \mathfrak{Q} = I\mathfrak{Q} = s_1\mathfrak{Q} + \dots + s_m\mathfrak{Q}$.

Now the induced left (R/I)-module morphism $g/I\colon \mathfrak{R}[[X]]/I\mathfrak{R}[[X]] \longrightarrow \mathfrak{Q}/I\mathfrak{Q}$ vanishes, because the left (R/I)-module $\mathfrak{R}[[X]]/I\mathfrak{R}[[X]] = \mathfrak{R}/\mathfrak{I}[X] = R/I[X]$ is generated by elements from X. So the image of the morphism g is contained in $I\mathfrak{Q}$. Since the left $\mathfrak{R}$-contramodule $\mathfrak{Q}$ is generated by $g(\mathfrak{R}[[X]])$ and $I\mathfrak{Q} = \mathfrak{I} \leftthreetimes \mathfrak{Q}$ is a left $\mathfrak{R}$-subcontramodule in $\mathfrak{Q}$, it follows that $\mathfrak{Q} = I\mathfrak{Q}$.

We have shown that $\mathfrak{Q} = \mathfrak{I} \leftthreetimes \mathfrak{Q}$ for a countable set of open two-sided ideals $\mathfrak{I} \subset \mathfrak{R}$ forming a base of neighborhoods of zero in $\mathfrak{R}$. According to the contramodule Nakayama lemma ([4, Lemma D.1.2] or [11, Lemma 6.14]), it follows that $\mathfrak{Q} = 0$. □

The following proposition explains, in a context more general than that of Theorem 1.1, how to distinguish the objects of the full subcategory $\mathfrak{R}\text{–}\mathsf{contra}$ among the objects of the ambient category $R\text{–}\mathsf{mod}$.

Proposition 1.2. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, R be an associative ring, and $\theta\colon R \longrightarrow \mathfrak{R}$ be a ring homomorphism such that the related forgetful functor $\Theta\colon \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful. Let Y be a set of the cardinality greater or equal to the cardinality of a base of neighborhoods of zero in $\mathfrak{R}$. Denote by θ_Y the natural left R-module morphism $R[Y] \longrightarrow \mathfrak{R}[[Y]]$. Then a left R-module P belongs to $\Theta(\mathfrak{R}\text{–}\mathsf{contra})$ if and only if the morphism of abelian groups $\operatorname{Hom}_R(\theta_Y, P)\colon \operatorname{Hom}_R(\mathfrak{R}[[Y]], P) \longrightarrow \operatorname{Hom}_R(R[Y], P) = P^Y$ is an isomorphism.*

In particular, in the context of Theorem 1.1, it suffices to take a countable set Y.

Proof. This is essentially the particular case of (the proof of) [10, Theorem 3.5(a)] that occurs when one considers the accessible additive monad $\mathbb{T} = \mathbb{T}_{\mathfrak{R}}$ associated with a topological ring $\mathfrak{R}$. Notice that if Z is a subset in Y and the map $\operatorname{Hom}_R(\theta_Y, P)$ is an isomorphism for a certain left R-module P, then the map $\operatorname{Hom}_R(\theta_Z, P)$ is also an isomorphism, because the left R-module morphism $\theta_Z\colon R[Z] \longrightarrow \mathfrak{R}[[Z]]$ is a direct summand of the morphism θ_Y. □

2. Examples to the Main Theorem

Below we list some examples of topological rings $\mathfrak{R}$ together with ring homomorphisms $R \longrightarrow \mathfrak{R}$ into $\mathfrak{R}$ from a discrete ring R for which one can show the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful using Theorem 1.1 (or parts of the argument from its proof), as well as a certain counterexample. Further such examples will be provided in Sections 3–4.

Remark 2.1. Let $\mathfrak{R}$ be a complete, separated topological associative ring with a countable base of neighborhoods of zero consisting of open right ideals. Let $\mathfrak{K} \subset \mathfrak{R}$ be a closed two-sided ideal and $\mathfrak{S} = \mathfrak{R}/\mathfrak{K}$ be the quotient ring endowed with the quotient topology. Then $\mathfrak{S}$ is also a complete, separated topological associative ring with a countable base of neighborhoods of zero consisting of open right ideals, and the functor of restriction of scalars $\mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ is fully faithful. Indeed, any family of elements of $\mathfrak{S}$ converging to zero in the topology of $\mathfrak{S}$ can be lifted to a family of elements of $\mathfrak{R}$ converging to zero in the topology of $\mathfrak{R}$.

Moreover, if a homomorphism of associative rings $\theta\colon R \longrightarrow \mathfrak{R}$ satisfies the assumptions of Theorem 1.1, then so does the composition $\bar\theta\colon R \longrightarrow \mathfrak{S}$ of the homomorphism θ with the natural surjective homomorphism $\mathfrak{R} \longrightarrow \mathfrak{S}$. Indeed, for any open right/two-sided ideal $\mathfrak{I} \subset \mathfrak{R}$, one can consider the open right/two-sided ideal $\mathfrak{J} = (\mathfrak{I} + \mathfrak{K})/\mathfrak{K} \subset \mathfrak{S}$. Then any family of elements of $\mathfrak{J}$ converging to zero in the topology of $\mathfrak{S}$ can be lifted to a family of elements of $\mathfrak{I}$ converging to zero in the topology of $\mathfrak{R}$, hence the image of any finite set of elements strongly generating the right ideal $\mathfrak{I} \subset \mathfrak{R}$ strongly generated the right ideal $\mathfrak{J} \subset \mathfrak{S}$.

Examples 2.2. (1) Let k be a commutative ring and $\mathfrak{R} = k\{\{x_1, \dots, x_m\}\}$ be the k-algebra of noncommutative formal Taylor power series in the variables $x_1, \dots, x_m$ with the coefficients in k, endowed with the formal power series topology (or, in other words, the $\mathfrak{I}$-adic topology for the ideal $\mathfrak{I} = (x_1, \dots, x_m) \subset \mathfrak{R}$). We observe that the ideal $\mathfrak{I}^n \subset \mathfrak{R}$ of all the formal power series with vanishing coefficients at all the noncommutative monomials of the total degree less than n in $x_1, \dots, x_m$ is strongly generated, as a left ideal in $\mathfrak{R}$, by the finite set of all the noncommutative monomials of the total degree n in $x_1, \dots, x_m$. Denoting by $R = k\{x_1, \dots, x_m\}$ the k-algebra of noncommutative polynomials in $x_1, \dots, x_m$ and by $\theta\colon R \longrightarrow \mathfrak{R}$ the natural embedding, we conclude, by applying Theorem 1.1, that the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful.

(2) More generally, let R be a quotient algebra of the algebra $k\{x_1, \dots, x_m\}$ of noncommutative polynomials in the variables $x_1, \dots, x_m$ over a commutative ring k by a two-sided ideal $K \subset k\{x_1, \dots, x_m\}$. Let $\mathfrak{R} = \varprojlim_n R/I^n$ be the adic completion of the algebra R with respect to the two-sided ideal $I = (x_1, \dots, x_m) \subset \mathfrak{R}$, endowed with the projective limit topology ($=$ I-adic topology of the left or right R-module $\mathfrak{R}$). Then $\mathfrak{R}$ is the topological quotient ring of the algebra of noncommutative formal power series $k\{\{x_1, \dots, x_m\}\}$ by the closure $\mathfrak{K}$ of the image of the ideal $K \subset k\{x_1, \dots, x_m\}$ in $k\{\{x_1, \dots, x_m\}\}$. In view of Remark 2.1, the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful.

(3) Even more generally, let $\mathfrak{R}$ be the quotient ring of the topological algebra of noncommutative formal power series $k\{\{x_1, \dots, x_m\}\}$ by a closed two-sided ideal $\mathfrak{K} \subset k\{\{x_1, \dots, x_m\}\}$, endowed with the quotient topology. Then the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow k\{x_1, \dots, x_m\}\text{–}\mathsf{mod}$ is fully faithful.

(4) One can also drop the commutatity assumption on the ring k, presuming only that the elements of k commute with the variables $x_1, \dots, x_m$ (while the variables do not commute with each other and the elements of k do not necessarily commute with each other). All the assertions of (1–3) remain valid in this setting.

Examples 2.3. (1) Let k be a field and $\mathcal{C}$ be a coassociative, counital k-coalgebra. Then the dual vector space $\mathcal{C}^\vee$ to the k-vector space $\mathcal{C}$ has a natural topological k-algebra structure. The category $\mathcal{C}\text{–}\mathsf{contra}$ of left contramodules over the coalgebra $\mathcal{C}$ is isomorphic to the category $\mathcal{C}^\vee\text{–}\mathsf{contra}$ of left contramodules over the topological algebra $\mathcal{C}^*$ [3, Section 1.10], [5, Section 2.3].

(2) In particular, when $\mathcal{C}$ is a conilpotent coalgebra over k with a finite dimensional cohomology space $H^1(\mathcal{C})$, the topological algebra $\mathcal{C}^\vee$ is a topological quotient algebra of the algebra of noncommutative formal power series $k\{\{x_1,\dots,x_m\}\}$, where $m = \dim_k H^1(\mathcal{C})$, by a closed two-sided ideal, as in Example 2.2(3). This allows to recover the result of [9, Thereom 1.1] as a particular case of our Theorem 1.1. (A change of variables may be needed in order to ensure that an arbitrary dense subalgebra $R\subset\mathcal{C}^\vee$ contains the images of the generators $x_j\in k\{\{x_1,\dots,x_m\}\}$.)

Examples 2.4. (1) Let R be an associative ring with a two-sided ideal $I\subset R$. Consider the associated graded ring $\operatorname{gr}_I R=\bigoplus_{n=0}^\infty I^n/I^{n+1}$ and the I-adic completion $\mathfrak{R}=\varprojlim_n R/I^n$ (viewed as a topological ring in the projective limit topology). Assume that the ideal $\operatorname{gr}_I I=\bigoplus_{n=1}^\infty I^n/I^{n+1}$ in the graded ring $\operatorname{gr}_I R$ is generated by a finite set of central elements $\bar s_1$, …, $\bar s_m$ of grading 1.

Choose some liftings s_1, …, $s_m\in I$ of the elements $\bar s_1$, …, $\bar s_m\in I/I^2$. Then the open two-sided ideal $\mathfrak{I}_n=\varprojlim_i I^n/I^{n+i}$ in the topological ring $\mathfrak{R}$, viewed as a right ideal, is strongly generated by the images of the monomials of degree n in the elements s_1, …, s_m. According to Theorem 1.1, it follows that the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow R\text{–}\mathsf{mod}$ is fully faithful.

(2) In particular, let R be an associative ring and $I\subset R$ be the ideal generated by a finite set of central elements s_1, …, $s_m\in R$. Let $\mathfrak{R}=\varprojlim_n R/I^n$ be the I-adic completion of the ring R, endowed with the projective limit ($=I$-adic) topology. Then the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow R\text{–}\mathsf{mod}$ is fully faithful. This result was announced in [10, Examples 2.2(2) and 2.3(3)].

Examples 2.5. (1) Let R be an associative ring and $S\subset R$ be a multiplicative subset consisting of some central elements in R. Let $\mathfrak{R}=\varprojlim_{s\in S} R/sR$ denote the S-completion of the ring R, viewed as a topological ring in the projective limit topology. Assume that the projective limit topology of $\mathfrak{R}$ coincides with the S-topology of R-module $\mathfrak{R}$, and moreover, assume that for every set X the projective limit topology of the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]=\varprojlim_{s\in S} R/sR[X]$ coincides with the S-topology of the R-module $\mathfrak{R}[[X]]$ (cf. [8, Theorem 2.3]).

The latter condition can be expressed by saying that for any X-indexed family of elements $r_x\in\mathfrak{R}$, converging to zero in the topology of $\mathfrak{R}$ and belonging to the kernel ideal of the natural ring homomorphism $\mathfrak{R}\longrightarrow R/sR$, there exists an X-indexed family of elements in $t_x\in\mathfrak{R}$, converging to zero in the topology of $\mathfrak{R}$, such that $r_x=st_x$ for all $x\in X$. In other words, it means that the kernel ideals of the ring homomorphism $\mathfrak{R}\longrightarrow R/sR$ is strongly generated by (the image in $\mathfrak{R}$ of) an element s, for every $s\in S$.

Following the proof of Theorem 1.1, we would be able to conclude that the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow R\text{–}\mathsf{mod}$ is fully faithful if we knew that, for every left $\mathfrak{R}$-contramodule $\mathfrak{Q}$, the equations $\mathfrak{Q}=s\mathfrak{Q}$ for all $s\in S$ imply $\mathfrak{Q}=0$. This condition holds whenever every S-divisible left R-module is S-h-divisible (see the discussion in [8, Section 1] and the references therein).

To sum up, the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful whenever the S-completion of the free left R-module $R[X]$ is S-complete for every set X and all the S-divisible left R-modules are S-h-disible.

(2) In particular, for any countable multiplicative subset S consisting of some central elements in an associative ring R, the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful (see [8, Proposition 2.2]).

3. Topological Lie Algebras

4. Conilpotent Coenveloping Coalgebras

5. Semialgebras and Semicontramodules

6. Contratensor Product

References

[1] U. Jannsen. Continuous étale cohomology. *Mathematische Annalen* **280**, #2, p. 207–245, 1988.

[2] L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. `arXiv:0708.3398 [math.CT]`

[3] L. Positselski. Weakly curved A_∞-algebras over a topological local ring. Electronic preprint `arXiv:1202.2697 [math.CT]`.

[4] L. Positselski. Contraherent cosheaves. Electronic preprint `arXiv:1209.2995 [math.CT]`.

[5] L. Positselski. Contramodules. Electronic preprint `arXiv:1503.00991 [math.CT]`.

[6] L. Positselski. Dedualizing complexes and MGM duality. *Journ. of Pure and Appl. Algebra* **220**, #12, p. 3866–3909, 2016. `arXiv:1503.05523 [math.CT]`

[7] L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. Electronic preprint `arXiv:1605.03934 [math.CT]`.

[8] L. Positselski. Triangulated Matlis equivalence. Electronic preprint `arXiv:1605.08018 [math.CT]`. To appear in *Journ. of Algebra and its Appl.*

[9] L. Positselski. Smooth duality and co-contra correspondence. Electronic preprint `arXiv:1609.04597 [math.CT]`.

[10] L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint `arXiv:1705.04960 [math.CT]`.

[11] L. Positselski, J. Rosický. Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories. *Journ. of Algebra* **483**, p. 83–128, 2017. `arXiv:1512.08119 [math.CT]`

Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa 31905, Israel; and

Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312; and

Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051, Russia; and

Charles University, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail address: `positselski@yandex.ru`