TWO EXERCISES IN DERIVED KOSZUL DUALITY

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Introduction

The goal of this paper is to demonstrate how the derived Koszul duality theory as developed in [8] can be applied in some familiar settings of the homological theory of associative and commutative rings. In particular, we show how to use conilpotent coalgebras in place of positively graded algebras or local rings (see also [9]).

The two topics that we discuss are vaguely related by the point of view and the methods applied, but there is not direct connection between them. The first one is the Artin–Schelter Gorenstein property. The Artin–Schelter regularity axioms [1] provide a way to study noncommutative deformations or analogues of the graded algebras of commutative polynomials in several variables in intrinsic terms. The two of three axioms that have homological nature define the class of algebras that we call *AS Gorenstein*. The main result about these algebras [5, Corollary D] claims that a positively graded algebra is AS Gorenstein if and only if its Yoneda Ext-algebra is Frobenius. In this note we generalize this theorem from positively graded algebras to ungraded conilpotent coalgebras, and further to coalgebras with a finite-dimensional maximal cosemisimple subcoalgebra.

Our second exercise concerns the "relative" properties of morphisms of positively graded commutative algebras, or more generally, ungraded commutative conilpotent coalgebras. The latter can be viewed as dual analogues of commutative local rings. Here we generalize and strengthen (the commutative case of) the duality principle for morphisms of Koszul algebras as developed in [6, Sections 7–8].

The class of morphisms that is in the focus of our attention is that of Golod maps [4, 2], but our main result is applicable to any morphism of conilpotent commutative DG-coalgebras $C \longrightarrow D$. It claims that the homotopy kernel of the Koszul dual morphism of Lie DG-algebras $\mathrm{Lie}(C) \longrightarrow \mathrm{Lie}(D)$, concidered as a Lie DG-algebra, is Koszul dual to the derived cotensor product $k \mathbin{\square}_D^{\mathbb{R}} C$, considered as a commutative DG-coalgebra (where k denotes the ground field). In the particular case when a morphism of conilpotent coalgebras is replaced with a morphism of locally finite-dimensional positively graded algebras $S \longrightarrow T$, one deals with the commutative DG-algebra of the derived tensor product $k \otimes_S^{\mathbb{L}} T$ in place of the cotensor product.

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1. Recollection on Koszul Duality

The following brief exposition roughly corresponds to [8, parts of Sections 1–2 and 4–6]. The main difference is that we restrict ourselves to the augmented conilpotent duality. In particular, no curved structures are mentioned in this paper.

1.1. **DG-modules.** Here the reader can find an incomplete reminder of the definitions and results from [3] or [8, Section 1].

1.1.1. *DG-algebras.* We work over a fixed ground field k. A *DG-algebra* is an associative k-algebra A with a unit element 1, graded by the integers and endowed with an odd derivation d of degree 1 with a zero square. In other words, $d(A^i) \subset A^{i+1}$ and $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for all a, $b \in A$, where $|a| \in \mathbb{Z}$ here, as in the sequel, denotes the degree of a homogeneous element of a graded vector space, i. e., $a \in A^{|a|}$.

A *left DG-module* M over A is a unital graded module over the associative algebra A, endowed with an odd derivation $d = d_M$ of degree 1 compatible with the derivation on A, i. e., $d_M(am) = d(a)m + (-1)^{|a|}ad_M(m)$ for all $a \in A$ and $m \in M$, such that $d_M^2 = 0$. *Right DG-modules* N over A are defined in the similar way, except that the signs in the Leibniz rule have the form $d_N(na) = d_N(n)a + (-1)^{|n|}nd(a)$.

Given an integer i and a left DG-module M over A, the shifted DG-module $M[i] = [i]M$ is defined by the rules $M[i]^j = M^{i+j}$, $d([i]m) = (-1)^i[i](d(m))$, and $a([i]m) = (-1)^{i|a|}(am)$. The first two formulas are applicable to, and define the shift of, any complex of k-vector spaces. Similarly, for a right DG-module N over A one sets $N[i]^j = N^{i+j}$, $d([i]n) = (-1)^i[i](d(n))$, and $([i]n)a = [i](na)$.

1.1.2. *Tensor operations.* Let L and M be left DG-modules over a DG-algebra A. Then the complex of morphisms $\mathrm{Hom}_A(L, M)$ from L to M over A is a complex of k-vector spaces whose term $\mathrm{Hom}_A^i(L, M)$ is the vector space of graded A-module morphisms $L \longrightarrow M[i]$. The differential on $\mathrm{Hom}_A(L, M)$ is given by the usual formula $d(f)(l) = d(f(l)) - (-1)^{|f|} f(d(l))$. The definition of the complex $\mathrm{Hom}_A(R, N)$ for two right DG-modules R and N over A is similar.

Let N be a right DG-module and M be a left DG-module over A. Then the tensor product $N \otimes_A M$ is a complex of k-vector spaces whose underlying graded k-vector space is the tensor product of graded modules N and M over a graded algebra A. The differential on $N \otimes_A M$ is given by the usual formula $d(n \otimes m) = d(n) \otimes m + (-1)^{|n|} n \otimes d(m)$.

1.1.3. *Derived categories.* Let A be a DG-algebra. The category $\mathsf{DG}(A\text{–}\mathsf{mod})$ whose objects are all left DG-modules over A and the morphisms from a DG-module L to a DG-module M are defined as the complex $\mathrm{Hom}_A(L, M)$ is a DG-category with a zero object, shifts, twists, infinite direct sums, and infinite products [8, Section 1.2]; in particular, it is a DG-category with cones. So its homotopy category $\mathsf{Hot}(A\text{–}\mathsf{mod}) = H^0\mathsf{DG}(A\text{–}\mathsf{mod})$ is a triangulated category with infinite direct sums and products. The same applies to the the DG-category $\mathsf{DG}(\mathsf{mod}\text{–}A)$ of right DG-modules over A and its homotopy category $\mathsf{Hot}(\mathsf{mod}\text{–}A)$.

A DG-module M is said to be *acyclic* if it is acyclic as a complex of vector spaces, i. e., its cohomology module $H(M)$ vanishes. A (closed) morphism of DG-modules is a *quasi-isomorphism* if it is a quasi-isomorphism as a morphism of complexes of vector spaces, i. e., a cohomology isomorphism. The *derived category* $\mathsf{D}(A\text{–}\mathsf{mod})$ of left DG-modules over A is obtained by inverting quasi-isomorphisms in the homotopy category, i. e., it is the quotient category of $\mathsf{Hot}(A\text{–}\mathsf{mod})$ by thick subcategory of acyclic DG-modules $\mathsf{Acycl}(A\text{–}\mathsf{mod})$. The derived category $\mathsf{D}(\mathsf{mod}\text{–}A)$ of right DG-modules over A is defined in the similar way.

1.1.4. *Homotopy adjusted objects.* A left DG-module P over a DG-algebra A is called *homotopy projective* (h-projective for brevity) if the complex $\mathrm{Hom}_A(P, M) = 0$ is acyclic for any acyclic left DG-module M over A. Analogously, a DG-module J over A is called *h-injective* if the complex $\mathrm{Hom}_A(M, J)$ is acyclic for any acyclic DG-module M. The full subcategories of h-projective and h-injective DG-modules in $\mathsf{Hot}(A\text{–}\mathsf{mod})$ are denoted by $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}}$ and $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}}$, respectively.

Theorem. (a) *The subcategory* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}}$ *is the minimal full triangulated subcategory of* $\mathsf{Hot}(A\text{–}\mathsf{mod})$*, containing the DG-module A over A and closed under infinite direct sums. The composition of functors* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}} \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

(b) *The subcategory* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}}$ *is the minimal full triangulated subcategory of* $\mathsf{Hot}(A\text{–}\mathsf{mod})$*, containing the DG-module* $\mathrm{Hom}_k(A, k)$ *over A and closed under infinite products. The composition of functors* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}} \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

Sketch of proof. Part (a): it is clear that the full subcategory $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}}$ contains the DG-module A and is closed under shifts, cones, and infinite direct sums. To prove both assertions, it remains to construct for any DG-module M over A a quasi-isomorphism $P \longrightarrow M$ of DG-modules over A such that P is homotopy equivalent to a DG-module obtained from the DG-module A by the above three operations. For this purpose, one considers the bar-resolution $\cdots \longrightarrow A\otimes_k A\otimes_k \otimes_k M \longrightarrow A\otimes_k M$ of the DG-module M and sets P to be the total DG-module of this complex of DG-modules over A, constructed by taking infinite direct sums along the diagonals of the bicomplex. The proof of part (b) is similar, except that one considers the cobar-resolution $\mathrm{Hom}_k(A,M) \longrightarrow \mathrm{Hom}_k(A,\mathrm{Hom}_k(A,M)) \longrightarrow \cdots$ and totalizes it by taking infinite products. See [8, proofs of Theorems 1.4–1.5] for further details. $\square$

1.1.5. *Derived functors.* Let $k\text{–}\mathsf{vect}$ denote the abelian category of k-vector spaces and $\mathsf{D}(k\text{–}\mathsf{vect})$ be its derived category. Let A be a DG-algebra over k. The right derived functor

$$\mathrm{Ext}_A\colon \mathsf{D}(A\text{–}\mathsf{mod})^{\mathsf{op}}\times\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{vect})$$

of the functor Hom_A can be simply defined as $\mathrm{Ext}_A(L,M)=\mathrm{Hom}_{\mathsf{D}(A\text{–}\mathsf{mod})}(L,M[*])$.

To compute the graded vector space $\mathrm{Ext}_A(L,M)$, it suffices to either replace the DG-module L with an h-projective DG-module P over A endowed with a quasi-isomorphism $P\longrightarrow L$, or replace the DG-module M with an h-injective DG-module J over A endowed with a quasi-isomorphism $M\longrightarrow J$. Then all three complexes $\mathrm{Hom}_A(P,M)$, $\mathrm{Hom}_A(L,J)$, and $\mathrm{Hom}_A(P,J)$ represent the object $\mathrm{Ext}_A(L,M)\in \mathsf{D}(k\text{–}\mathsf{mod})$ [8, Lemma 1.3].

The definition of the left derived functor

$$\mathrm{Tor}^A\colon \mathsf{D}(\mathsf{mod}\text{–}A)\times\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{vect})$$

of the functor

$$\otimes_A\colon \mathsf{Hot}(\mathsf{mod}\text{–}A)\times\mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{vect})$$

requires just a little more care. Restrict the functor $\otimes_A$ to the full subcategory $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{prj}}$ of h-projective right DG-modules over A in the first argument. Then this restriction factorizes through the derived category $\mathsf{D}(A\text{–}\mathsf{mod})$ in the second argument, providing the desired functor Tor^A. Indeed, it is easy to see that the tensor product $Q\otimes_A M$ is acyclic whenever a DG-module Q over A is h-projective and the DG-module M is acyclic. The same derived functor can be obtained by restricting the functor $\otimes_A$ to the full subcategory $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}}$ in the second argument.

More generally, one can compute the derived functor $\mathrm{Tor}^A(N,M)$ by replacing one of the DG-modules N or M with a *homotopy flat* (h-flat) DG-module quasi-isomorphic to it [8, Section 1.6]. A right DG-module F over A is said to be h-flat if the tensor product $F\otimes_A M$ is acyclic for any acyclic left DG-module M over A.

1.1.6. *Pull-backs and push-forwards.* Let $f\colon A\longrightarrow B$ be a morphism of DG-algebras over k. Then any (left or right) DG-module N over B can be also considered as a DG-module over A; let us denote the DG-module so obtained by R_fN. The functor

$R_f\colon \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ has a left adjoint functor E_f and a right adjoint functor E^f. Given a left DG-module M over A, the DG-module E_fM over B is constructed as $E_fM = B \otimes_A M$; while the DG-module E^fM over B is obtained as $E^fM = \mathrm{Hom}_A(B, M)$.

The functor R_f obviously preserves acyclicity of DG-modules, so it induces a functor on the derived categories

$$R_f\colon \mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod}).$$

The functor E_f has a left derived functor

$$\mathbb{L}E_f\colon \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod}),$$

which can be constructed by restricting the functor E_f to the full subcategory $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{prj}} \subset \mathsf{Hot}(A\text{–}\mathsf{mod})$, or, more generally, to the full subcategory of h-flat DG-modules. The functor $\mathbb{L}E_f\colon \mathsf{D}(\mathsf{mod}\text{–}A) \longrightarrow \mathsf{D}(\mathsf{mod}\text{–}B)$ is defined in the similar way. The functor E^f has a right derived functor

$$\mathbb{R}E^f\colon \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod}),$$

which can be constructed by restricting the functor E^f to the full subcategory of h-injective DG-modules $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}} \subset \mathsf{Hot}(A\text{–}\mathsf{mod})$.

The functors $\mathbb{L}E_f$ and $\mathbb{R}E^f$ are adjoint to the functor $R_f\colon \mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ from the left and from the right, respectively. For any right DG-module N over B and left DG-module M over A there is a natural isomorphism

$$\mathrm{Tor}^A(R_fN, M) \simeq \mathrm{Tor}^B(N,\, \mathbb{L}E_fM)$$

in $\mathsf{D}(k\text{–}\mathsf{mod})$.

The functors R_f, $\mathbb{L}E_f$, $\mathbb{R}E^f$ are equivalences of triangulated categories if and only if the morphism of DG-algebras f is a quasi-isomorphism.

1.2. **DG-comodules and DG-contramodules.** Here is a very incomplete sketch of [8, Sections 2 and 4–5].

1.2.1. *DG-coalgebras.* A *DG-coalgebra* C over a field k is a complex of k-vector spaces endowed with morphisms of complexes $C \longrightarrow C \otimes_k C$ and $C \longrightarrow k$, called the *comultiplication* and the *counit* maps, which define a structure of a graded coaccosiative coalgebra with counit on the underlying graded k-vector space of C. In other words, the comultiplication map must be coassociative and counital.

A *left DG-comodule* M over a DG-coalgebra C is a complex of vector spaces endowed with a morphism of complexes $M \longrightarrow C \otimes_k M$, called the *left coaction* map, which satisfies the coassociative and counital coaction equations as a map of graded vector spaces. Analogously, a *right DG-comodule* N over C is a complex of vector spaces endowed with a morphism of complexes $N \longrightarrow N \otimes_k C$, called the *right coaction* map, satisfying the coassociativity and counitality equations.

A *left DG-contramodule* P over a DG-coalgebra C is a complex of vector spaces endowed with a morphism of complexes $\mathrm{Hom}_k(C, P) \longrightarrow P$, called the *left contraaction* map, which satisfies the *contraassociativity* and counitality equations. Here

$\mathrm{Hom}_k(C,P)$ denotes the graded vector space of homogeneous k-linear maps $C \longrightarrow P$, endowed with the differential by the usual rule. The contraassociativity means that the two maps $\mathrm{Hom}_k(C\otimes_k C,\, P) \simeq \mathrm{Hom}_k(C, \mathrm{Hom}_k(C,P)) \rightrightarrows \mathrm{Hom}_k(C,P)$ induced by the comultiplication and contraaction maps have equal compositions with the contraaction map $\mathrm{Hom}_k(C,P) \longrightarrow P$. The presumed isomorphism $\mathrm{Hom}_k(U\otimes_k V,\, W) \simeq \mathrm{Hom}_k(V, \mathrm{Hom}_k(U,W))$ for any graded k-vector spaces U, V, W is given by the formula $g'(v)(u) = (-1)^{|u||v|}g''(u\otimes v)$. The counitality means that the composition of the map $P \longrightarrow \mathrm{Hom}_k(C,P)$ induced by the counit map with the contraaction map is the identity endomorphism of P.

1.2.2. *Tensor operations.* Let L and M be left DG-comodules over a DG-coalgebra C. Then the complex of morphisms $\mathrm{Hom}_C(L,M)$ from L to M over C is a complex of k-vector spaces whose term $\mathrm{Hom}_C^i(L,M)$ is the vector space of homogeneous k-linear maps $f\colon L \longrightarrow M$ of degree i such that the coaction maps of L and M form a commutative diagram with the map f and the map $f_*\colon C\otimes_k L \longrightarrow C\otimes_k M$ given by the formula $f_*(c\otimes x) = (-1)^{i|c|}c\otimes f(x)$. The differential d on the complex $\mathrm{Hom}_C(L,M)$ is defined by the usual formula $d(f)(x) = d(f(x)) - (-1)^{|f|}f(d(x))$.

The complex of morphisms $\mathrm{Hom}_C(R,N)$ between right DG-comodules R and N over C is defined in the similar way, except that the map $f_*\colon R\otimes_k C \longrightarrow N\otimes_k C$ given by the formula $f_*(x\otimes c) = f_*(x)\otimes c$ is used.

Let P and Q be left DG-contramodules over a DG-coalgebra C. Then the complex of morphisms $\mathrm{Hom}^C(P,Q)$ from P to Q over C is a complex of vector spaces whose term $\mathrm{Hom}^{C,i}(P,Q)$ is the vector space of homogeneous k-linear maps $f\colon P \longrightarrow Q$ of degree i such that the contraaction maps of P and Q form a commutative diagram with the map f and the map $f_*\colon \mathrm{Hom}_k(C,P) \longrightarrow \mathrm{Hom}_k(C,Q)$ given by the rule $f_*(g) = f\circ g$. The differential d on the complex $\mathrm{Hom}^C(P,Q)$ is defined by the usual formula.

For any right DG-comodule N over C and complex of k-vector spaces V, the complex $\mathrm{Hom}_k(N,V)$ has a natural structure of left DG-contramodule over C. The contraaction map $\mathrm{Hom}_k(C, \mathrm{Hom}_k(N,V)) \simeq \mathrm{Hom}_k(N\otimes_k C,\, V) \longrightarrow \mathrm{Hom}_k(N,V)$ is induced by the coaction map $N \longrightarrow N\otimes_k C$.

The *contratensor product* $N \odot_C P$ of a right DG-comodule N and a left DG-contramodule P over C is a complex of vector spaces constructed as the cokernel of the pair of morphisms $N\otimes_k \mathrm{Hom}_k(C,P) \rightrightarrows N\otimes_k P$, one of which is induced by the C-contraaction in P and the other one is the composition $N\otimes_k \mathrm{Hom}_k(C,P) \longrightarrow N\otimes_k C\otimes_k \mathrm{Hom}_k(C,P) \longrightarrow N\otimes_k P$ of the map induced by the C-coaction in N and the evaluation map. For any complex of k-vector spaces V, there is a natural isomorphism $\mathrm{Hom}_k(N\odot_C P,\, V) \simeq \mathrm{Hom}^C(P, \mathrm{Hom}_k(N,V))$.

The *cotensor product* $N \,\square_C\, M$ of a right DG-comodule N and a left DG-comodule M over C is a complex of vector spaces constructed as the kernel of the pair of morphisms $N\otimes_k M \rightrightarrows N\otimes_k C\otimes_k M$, one of which is induced by the coaction of C in N and the other one by the coaction of C in M. The *complex of cohomomorphisms* $\mathrm{Cohom}_C(M,P)$ from a left DG-comodule M to a right DG-contramodule P over C is a complex of vector spaces constructed as the cokernel of the pair of morphisms

$\mathrm{Hom}_k(C\otimes_k M, P)\simeq \mathrm{Hom}_k(M,\mathrm{Hom}_k(C,P)) \rightrightarrows \mathrm{Hom}_k(M,P)$, one of which is induced by the C-coaction in M and the other one by the C-contraaction in P.

1.2.3. *Derived categories of the second kind.* Let C be a DG-coalgebra. Then the categories of all left DG-comodules, right DG-comodules, and left DG-contramodules over C, with the complexes of morphisms defined as above, are DG-categories with zero objects, shifts, twists, infinite direct sums, and infinite products. So their homotopy categories $\mathsf{Hot}(C\text{–}\mathsf{comod})$, $\mathsf{Hot}(\text{–}\mathsf{comod}C)$, and $\mathsf{Hot}(C\text{–}\mathsf{contra})$ are triangulated categories with infinite direct sums and products.

Given a short exact sequence $K'\longrightarrow K\longrightarrow K''$ of DG-comodules (resp., DG-contramodules) over C and closed morphisms between them, one can form its total DG-comodule (DG-contramodule). The full subcategory $\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})\subset \mathsf{Hot}(C\text{–}\mathsf{comod})$ of *coacyclic* left DG-comodules over C is defined as the minimal full triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{comod})$, containing the total DG-comodules of exact triples of left DG-comodules and closed under infinite direct sums. The full subcategory $\mathsf{Acycl}^{\mathsf{co}}(\mathsf{comod}\text{–}C)\subset\mathsf{Hot}(\mathsf{comod}\text{–}C)$ of coacyclic right DG-comodules over C is defined in the similar way. The *coderived categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$ of left and right DG-comodules over C are the quotient categories of the homotopy categories $\mathsf{Hot}(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}(\mathsf{comod}\text{–}C)$ by their thick subcategories $\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $\mathsf{Acycl}^{\mathsf{ctr}}(\mathsf{comod}\text{–}C)$.

The full subcategory $\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})\subset\mathsf{Hot}(C\text{–}\mathsf{contra})$ of *contraacyclic* left DG-contramodules over C is defined as the minimal full triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{contra})$, containing the total DG-contramodules of exact triples of left DG-contramodules and closed under infinite products. The *contraderived category* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ of left DG-contramodules over C is the quotient category of the homotopy category of DG-contramodules $\mathsf{Hot}(C\text{–}\mathsf{contra})$ by its thick subcategory of contraacyclic DG-contramodules $\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.

Any coacyclic DG-comodule or contraacyclic DG-contramodule is acyclic in the usual sense (has zero cohomology comodule or contramodule), by the converse is not true [8, Examples 3.3]. The reason one prefers to use infinite direct sums of comodules and infinite products of contramodules, but not the other way, is that these are the better behaved operations (i. e., exact and compatible with the forgetful functors to graded vector spaces, which the other two operations aren't).

1.2.4. *Graded adjusted objects.* Given a DG-coalgebra C, let us denote by $C^{\#}$ its underlying graded coalgebra, i. e., the result of forgetting the differential on C. Similarly, given a (left or right) DG-comodule M or DG-contramodule P, we denote by $M^{\#}$ or $P^{\#}$ its underlying graded C-comodule or C-contramodule.

Denote by $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$ the full triangulated subcategory of the homotopy category of left DG-comodules over C formed by the DG-comodules whose underlying graded $C^{\#}$-comodules are injective. Analogously, denote by $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{prj}})$ the full triangulated subcategory of the homotopy category of left DG-contramodules over C formed by the DG-contramodules whose underlying graded $C^{\#}$-contramodules are projective. We refer to [8, Sections 2.1-2] for a discussion of cofree/injective graded comodules and free/projective graded contramodules.

Theorem. (a) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *is an equivalence of triangulated categories.*

(b) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{prj}}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *is an equivalence of triangulated categories.*

Proof. Part (a): it is easy to see that the complex $\mathrm{Hom}_C(L, J)$ is acyclic for any DG-modules $L \in \mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $J \in \mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$. So it remains to construct for any left DG-comodule M over C a closed morphism of DG-comodules $M \longrightarrow J$, where $J \in \mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$, with a coacyclic cone. For this purpose, consider the cobar-resolution $C \otimes_k M \longrightarrow C \otimes_k C \otimes_k M \longrightarrow \cdots$ and set J to be the total DG-comodule of this complex of DG-comodules, constructed by taking infinite direct sums along the diagonals. The proof of part (b) is similar, except that one considers the bar-resolution $\cdots \longrightarrow \mathrm{Hom}_k(C, \mathrm{Hom}_k(C, P)) \longrightarrow \mathrm{Hom}_k(C, P)$ and totalizes it by taking infinite products. □

Notice the difference between the constructions of resolutions in Theorems 1.1.4 and 1.2.4: while in the former one totalizes a left resolution using direct sums and the right one using direct products, in the latter case it is the other way.

1.2.5. *Derived functors.* Let C be a DG-coalgebra over k. The right derived functor

$$\mathrm{Ext}_C\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{mod})$$

of the functor Hom_C can be simply defined as $\mathrm{Ext}_C(L, M) = \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})}(L, M[*])$. To compute the graded vector space $\mathrm{Ext}_C(L, M)$, it suffices to replace the DG-comodule M with a DG-comodule J endowed with a closed morphism of DG-comodules $M \longrightarrow J$ such that the graded $C^\#$-comodule $J^\#$ is injective and the cone of the morphism $M \longrightarrow J$ is a coacyclic DG-comodule. Then the complex $\mathrm{Hom}_C(L, J)$ represents the object $\mathrm{Ext}_C(L, M) \in \mathsf{D}(k\text{–}\mathsf{mod})$.

Similarly, the right derived functor

$$\mathrm{Ext}^C\colon \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{mod})$$

of the functor Hom^C can be defined as $\mathrm{Ext}^C(P, Q) = \mathrm{Hom}^{\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})}(P, Q[*])$. To compute the graded vector space $\mathrm{Ext}^C(P, Q)$, it suffices to replace the DG-contramodule P with a DG-contramodule F endowed with a closed morphism of DG-contramodules $F \longrightarrow J$ such that the graded $C^\#$-contramodule $F^\#$ is projective and the cone of the morphism $F \longrightarrow P$ is a contraacyclic DG-contramodule. Then the complex $\mathrm{Hom}^C(F, Q)$ represents the object $\mathrm{Ext}^C(P, Q) \in \mathsf{D}(k\text{–}\mathsf{mod})$.

To construct the left derived functor

$$\mathrm{Ctrtor}^C\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \times \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{mod})$$

of the functor

$$\odot_C\colon \mathsf{Hot}(\mathsf{comod}\text{–}C) \times \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{mod}),$$

one restricts the functor $\odot_C$ to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra_{prj}})$ of DG-contramodules whose underlying graded $C^{\#}$-contramodules are projective in the first argument. Then this restriction factorizes through the coderived category $\mathsf{D^{co}}(\mathsf{comod}\text{–}C)$, providing the desired derived functor. Indeed, the contratensor product with a projective graded $C^{\#}$-contramodule sends exact triples of graded $C^{\#}$-comodules to exact triples of vector spaces, hence the contratensor product of a coacyclic DG-comodule with a DG-contramodule from $\mathsf{Hot}(C\text{–}\mathsf{contra_{prj}})$ is an acyclic complex.

To construct the right derived functor

$$\operatorname{Cotor}^C\colon \mathsf{D^{co}}(\mathsf{comod}\text{–}C)\times\mathsf{D^{co}}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{D}(k\text{–}\mathsf{mod})$$

of the functor

$$\square_C\colon \mathsf{Hot}(\mathsf{comod}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{D}(k\text{–}\mathsf{mod}),$$

one restricts the functor $\square_C$ to the full subcategory $\mathsf{Hot}(\mathsf{comod_{inj}}\text{–}C)$ of DG-comodules whose underlying graded $C^{\#}$-comodules are injective in the first argument. Then this restriction factorizes through the coderived category $\mathsf{D^{co}}(C\text{–}\mathsf{comod})$ in the second argument, providing the desired derived functor. The same derived functor can be obtained by restricting the functor $\square_C$ to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})$ in the second argument. To check the latter assertion, one can restrict the functor $\square_C$ to the Cartesian product $\mathsf{Hot}(\mathsf{comod_{inj}}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})$.

To construct the left derived functor

$$\operatorname{Coext}_C\colon \mathsf{D^{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{D^{ctr}}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{D}(k\text{–}\mathsf{mod})$$

of the functor

$$\operatorname{Cohom}_C\colon \mathsf{Hot}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{Hot}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{D}(k\text{–}\mathsf{mod}),$$

one restricts the functor Cohom_C either to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})^{\mathsf{op}}$ in the first argument, or to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra_{prj}})$ in the second argument. Both restrictions factorize through the Cartesian product of the coderived and contraderived categories, inducing the same derived functor.

1.2.6. *Comodule-contramodule correspondence.* Let C be a DG-coalgebra. For any left DG-contramodule P over C, define the left DG-comodule $\Phi_C(P)$ as the contratensor product $C\odot_C P$ endowed with the comodule structure induced by the left C-comodule structure on C. For any left DG-comodule M over C, define the left DG-contramodule $\Psi_C(M)$ as the comodule Hom complex $\operatorname{Hom}_C(C,M)$ endowed with the contramodule structure induced by the right C-comodule structure on C. The functor

$$\Phi_C\colon \mathsf{Hot}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{Hot}(C\text{–}\mathsf{comod})$$

is left adjoint to the functor

$$\Psi_C\colon \mathsf{Hot}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{Hot}(C\text{–}\mathsf{contra}).$$

The similarly defined functors $\Phi_{C^\#}$ and $\Psi_{C^\#}$ between the abelian categories of graded left $C^{\#}$-comodules and graded left $C^{\#}$-contramodules define an equivalence between the additive categories of injective graded comodules and projective graded contramodules (as one can check by considering cofree comodules and free contramodules). Therefore, restricting the functor Φ_C to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{prj}})$ and the functor Ψ_C to the full subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$, one obtains derived functors $\mathbb{L}\Phi_C$ and $\mathbb{R}\Psi_C$, which are mutually inverse equivalences between the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.

1.2.7. *Pull-backs and push-forwards.* Let $g\colon C \longrightarrow D$ be a morphism of DG-coalgebras over k. Then any (left or right) DG-comodule M over C can be also considered as a DG-comodule over D; let us denote the DG-comodule so obtained by R_gM. Similarly, any DG-contramodule P over C can be also considered as a DG-contramodule over D; we will denote this DG-contramodule by R^gP.

The functor $R_g\colon \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(D\text{–}\mathsf{comod})$ has a right adjoint functor E_g. Given a left DG-comodule N over D, the DG-comodule E_gN can be constructed as $E_gN = C\,\square_D\,N$. Analogously, the functor $R^g\colon \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(D\text{–}\mathsf{contra})$ has a left adjoint functor E^g. For any DG-contramodule Q over D, the DG-contramodule E^gQ is constructed as $E^gQ = \mathrm{Cohom}_D(C, N)$.

The functors R_g and R^g obviously preserve coacyclicity of DG-comodules and contraacyclicity of DG-contramodules, so they induce functors on the exotic derived categories

$$R_g\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$$

and

$$R^g\colon \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra}).$$

The functor E_g has a right derived functor

$$\mathbb{R}E_g\colon \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}),$$

which can be constructed by restricting the functor E_g to the full subcategory $\mathsf{Hot}(D\text{–}\mathsf{comod}_{\mathsf{inj}})$. The functor $\mathbb{R}E_g\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}D) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$ is defined in the similar way. The functor E^g has a left derived functor

$$\mathbb{L}E^g\colon \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{contra}),$$

which can be contructed by restricting the functor E^g to the full subcategory $\mathsf{Hot}(D\text{–}\mathsf{contra}_{\mathsf{prj}})$.

The equivalences of categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ and $\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ identify the functor $\mathbb{R}E_g$ with the functor $\mathbb{L}E^g$. Taking these identifications into account, the functor R_g is left adjoint to the functor $\mathbb{R}E_g = \mathbb{L}E^g$, while the functor R^g is right adjoint to the functor $\mathbb{R}E_g = \mathbb{L}E^g$.

For any right DG-comodule M over C and any left DG-contramodule Q over D there is a natural isomorphism

$$\mathrm{Ctrtor}^C(M,\, \mathbb{L}E^gQ) \simeq \mathrm{Ctrtor}^D(R_gM, Q).$$

For any right DG-comodule N over D and left DG-comodule M over C there is a natural isomorphism

$$\mathrm{Cotor}^C(\mathbb{R}E_gN, M) \simeq \mathrm{Cotor}^D(N, R_gM).$$

For any left DG-comodule M over C and left DG-contramodule Q over D, there is a natural isomorphism

$$\mathrm{Coext}_C(M,\ \mathbb{L}E^g Q) \simeq \mathrm{Coext}_D(R_g M, Q).$$

For any left DG-comodule N over D and left DG-contramodule P over C, there is a natural isomorphism

$$\mathrm{Coext}_C(\mathbb{R}E_g N, P) \simeq \mathrm{Coext}_D(N,\ \mathbb{L}E^g P).$$

1.3. **Koszul duality.** This subsection corresponds to parts of [8, Section 6].

1.3.1. *(Co)bar-constructions.* Let A be a DG-algebra over k endowed with an augmentation, i. e., a morphism of DG-algebras $A \longrightarrow k$ with the kernel A_+. Denote by $\mathrm{Bar}(A)$ the tensor coalgebra $\bigoplus_{n=0}^\infty A_+[1]^{\otimes n}$ cogenerated by the graded vector space $A_+[1]$. Coderivations of the graded coalgebra $\mathrm{Bar}(A)$ are uniquely determined by their compositions with the projection $\mathrm{Bar}(A) \longrightarrow A_+$. Let d_{Bar} be the odd coderivation of degree 1 on $\mathrm{Bar}(A)$ whose composition with this projection is given by the rules $a_1\otimes\cdots\otimes a_n \longmapsto 0$ for $n=0$ or $n\ge 3$, $a_1\otimes a_2 \longmapsto (-1)^{|a_1|+1}a_1a_2$, and $a\longmapsto -d_A(a)$ for a, a_1, $a_2\in A_+$. One can readily check that $d_{\mathrm{Bar}}^2=0$, so the differential d_{Bar} defines a structure of DG-coalgebra on the graded coalgebra $\mathrm{Bar}(A)$. This DG-coalgebra is called the *bar-construction* of an augmented DG-algebra A.

Analogously, let C be a DG-coalgebra over k endowed with a coaugmentation, i. e., a morphism of DG-coalgebras $k \longrightarrow C$ with the cokernel C_+. Denote by $\mathrm{Cob}(C)$ the tensor algebra $\bigoplus_{n=0}^\infty C_+[-1]^{\otimes n}$ of the graded vector space $C_+[-1]$. Let d_{Cob} be the odd derivation of degree 1 on $\mathrm{Cob}(C)$ given by the rule $d_{\mathrm{Cob}}(c)=(-1)^{|c_{(1)}|+1}c_{(1)}\otimes c_{(2)} - d_C(c)$ for $c\in C_+$, where $c\longmapsto c_{(1)}\otimes c_{(2)}$ is the notation for the comultiplication map. One can check that $d_{\mathrm{Cob}}^2=0$, so the differential d_{Cob} defines a structure of DG-algebra on the graded algebra $\mathrm{Cob}(C)$. This DG-algebra is called the *cobar-construction* of a coagumented DG-coalgebra C. For any DG-coalgebra C, the DG-algebra $\mathrm{Cob}(C)$ is naturally augmented; and similarly, for any DG-algebra A, the DG-coalgebra $\mathrm{Bar}(A)$ is naturally coaugmented.

1.3.2. *Twisting cochains.* A graded coalgebra without counit D is called *conilpotent* if for any element $x\in D$ there exists an integer $n\ge 0$ such that x is annihilated by the iterated comultiplication map $D\longrightarrow D^{\otimes n+1}$. A coaugmented DG-coalgebra C is said to be conilpotent if the graded coalgebra $C_+=C/k$ is conilpotent. The bar-construction $\mathrm{Bar}(A)$ of any augmented DG-algebra A is a conilpotent DG-coalgebra.

Let A be an augmented DG-algebra and C be a coaugmented DG-coalgebra over a field k. Introduce a DG-algebra structure on the graded vector space of homologeneous k-linear maps $\mathrm{Hom}_k(C,A)$ by the rules $(fg)(c) = (-1)^{|g||c_1|}f(c_1)g(c_2)$ and $d(f)(c)=d_A(f(c))-(-1)^{|f|}f(d_C(c))$. A *twisting cochain* $\tau\colon C\longrightarrow A$ is a homogeneous k-linear map of degree 1 annihilating the coaugmentation $k\longrightarrow C$ and the augmentation $A\longrightarrow k$, and satisfying the Maurer–Cartan equation $\tau^2+d(\tau)=0$.

For any coaugmented DG-coalgebra C, the composition $C\longrightarrow C_+\longrightarrow \mathrm{Cob}(C)$ is a twising cochain for C and $\mathrm{Cob}(C)$. For any augmented DG-algebra A, minus the composition $\mathrm{Bar}(A)\longrightarrow A_+\longrightarrow A$ is a twisting cochain for $\mathrm{Bar}(A)$ and A. We will denote these twisting cochains by τ_C and τ_A, respectively. The functor $C \longmapsto \mathrm{Cob}(C)$, considered as a functor from the category of conilpotent DG-coalgebras to the category of augmented DG-algebras, is left adjoint to the functor $A \longmapsto \mathrm{Bar}(A)$. For any conilpotent DG-coalgebra C and augmented DG-algebra A, one constructs a bijective correspondence between the set of all DG-algebra morphisms $\mathrm{Cob}(C) \longrightarrow A$ and the set of all DG-coalgebra morphisms $C \longrightarrow \mathrm{Bar}(A)$ by identifying both sets with the set of all twisting cochains $\tau\colon C \longrightarrow A$.

A twisting cochain $\tau\colon C \longrightarrow A$ between a conilpotent DG-coalgebra C and an augmented DG-algebra A is said to be *acyclic* if the related morphism of DG-algebras $\mathrm{Cob}(C) \longrightarrow A$ is a quasi-isomorphism. This is *not* equivalent to the morphism of DG-coalgebras $C \longrightarrow \mathrm{Bar}(A)$ being a quasi-isomorphism; the latter condition is weaker than the former. This is so because the "right" equivalence relation on (conilpotent) DG-coalgebras is more delicate than the conventional quasi-isomorphism; see 1.3.4 below. In fact, the functor Cob, unlike the functor Bar, does not preserve quasi-isomorphisms.

1.3.3. *Triality for modules.* Let $\tau\colon C \longrightarrow A$ be a twisting cochain for a DG-coalgebra C and a DG-algebra A. Then for any left DG-module M over A there is a natural structure $C\otimes^\tau M$ of left DG-comodule over C on the tensor product $C\otimes_k M$, obtained by adding to the conventional differential of the tensor product the twisting term $c\otimes m \longmapsto (-1)^{|c_{(1)}|}c_{(1)}\otimes\tau(c_{(2)})m$. The coaction of C in $C\otimes^\tau M$ is simply induced by the left coaction of C in itself. For any right DG-module M over A, the natural right DG-comodule structure $M\otimes^\tau C$ on the tensor product $M\otimes_k C$ is obtained by adding to the conventional tensor product differential the twisting term $m\otimes c \longmapsto (-1)^{|m|+1}m\tau(c_{(1)})\otimes c_{(2)}$.

Analogously, for any left DG-comodule N over C, there is a natural structure $A\otimes^\tau N$ of left DG-module over A on the tensor product $A\otimes_k M$, obtained by adding to the tensor product differential the twisting term $a\otimes n \longmapsto (-1)^{|a|+1}a\tau(n_{(-1)})\otimes n_{(0)}$, where $n \longmapsto n_{(-1)}\otimes n_{(0)}$ is the notation for the left coaction map $N \longrightarrow C\otimes_k N$. The action of A in $A\otimes^\tau N$ is simply induced by the left action of A in itself. For any right DG-comodule N over C, the natural right DG-module structure $N\otimes^\tau A$ on the tensor product $N\otimes_k A$ is obtained by adding to the tensor product differential the twisting term $n\otimes a \longmapsto (-1)^{|n_{(0)}|}n_{(0)}\otimes\tau(n_{(1)})a$, where $n \longmapsto n_{(0)}\otimes n_{(1)}$ is the notation for the right coaction map.

Furthermore, for any left DG-contramodule Q over C there is a natural structure $\mathrm{Hom}^\tau(A,Q)$ of left DG-module over A on the graded vector space of homogeneous k-linear maps $\mathrm{Hom}_k(A,Q)$, obtained by adding to the conventional differential on Hom the twisting term $f \longmapsto (a \longmapsto \pi(c \mapsto (-1)^{|f|+1+|c||a|}f(\tau(c)a)))$, where $f \in \mathrm{Hom}_k(A,Q)$ and π denotes the contraaction map $\mathrm{Hom}_k(C,Q) \longrightarrow Q$. The left action of A in $\mathrm{Hom}^\tau(A,Q)$ is induced by the right action of A in itself in the usual way. For any left DG-module P over A there is a natural structure $\mathrm{Hom}^\tau(C,P)$ of left DG-contramodule over C on the graded Hom space $\mathrm{Hom}_k(C,P)$, obtained by adding to the Hom differential the twisting term $f \longmapsto (c \longmapsto (-1)^{|f||c_{(1)}|}\tau(c_{(1)})f(c_{(2)}))$. The left contraaction of C in $\mathrm{Hom}^\tau(C,P)$ is induced by the right coaction of C in itself.

Theorem. *Let $\tau\colon C \longrightarrow A$ be a twisting cochain for a conilpotent DG-coalgebra C and an augmented DG-algebra A. Then the following assertions hold.*

(a) *The triangulated functor $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ defined by the rule $N \longmapsto A \otimes^\tau N$ transforms arbitrary DG-comodules over C into h-projective DG-modules over A and coacyclic DG-comodules over C into contractible DG-modules over A. The triangulated functor $\mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod})$ defined by the rule $M \longmapsto C \otimes^\tau M$ transforms arbitrary DG-modules over A into DG-comodules over C with injective underlying graded $C^\#$-comodules, and acyclic DG-modules over A into contractible DG-comodules over C. The former functor is left adjoint to the latter one.*

(b) *The triangulated functor $\mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ defined by the rule $Q \longmapsto \mathrm{Hom}^\tau(A, Q)$ transforms arbitrary DG-contramodules over C into h-injective DG-modules over A and contraacyclic DG-contramodules over C into contractible DG-modules over A. The triangulated functor $\mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra})$ defined by the rule $P \longmapsto \mathrm{Hom}^\tau(C, P)$ transforms arbitrary DG-modules over A into DG-contramodules over C with projective underlying graded $C^\#$-contramodules, and acyclic DG-modules over A into contractible DG-contramodules over C. The former functor is right adjoint to the latter one.*

(c) *Let $\tau\colon C \longrightarrow A$ be an acyclic twisting cochain. Then the functors between the (exotic) derived categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$, and $\mathsf{D}(A\text{–}\mathsf{mod})$ induced by the above four functors between the homotopy categories, together with the derived functors $\mathbb{L}\Phi_C$ and $\mathbb{R}\Psi_C$, form a commutative triangle of mutually inverse equivalences of triangulated categories.*

Proof. The proof can be found in [8, Sections 6.3–6.5]. In each of the parts (a) and (b), the second and the third assertions are equivalent by adjunction, and so are the first and the fourth ones. The former ones do not depend on the conilpotency assumption on C, while the latter ones do. ☐

Corollary. *A twisting cochain $\tau\colon C \longrightarrow A$ is acyclic if and only if the cone of the natural morphism of left DG-comodules $k \longrightarrow C \otimes^\tau A$ is a coacyclic DG-comodule over C, or equivalently, the cone of the natural morphism of right DG-comodules $k \longrightarrow A \otimes^\tau C$ is a coacyclic DG-comodule over C.*

Proof. The morphism of DG-comodules $k \longrightarrow C \otimes^\tau A$ is induced by the coaugmentation of C and the unit of A; it is the adjunction morphism $k \longrightarrow C \otimes^\tau A \otimes^\tau k$. Thus if τ is acyclic, the cone of this morphism is coacyclic by the assertion (c) of Theorem. In particular, the cone of the morphism $k \longrightarrow \mathrm{Cob}(C) \otimes^{\tau_C} C$ is a coacyclic DG-comodule over C. Hence if the cone of the morphism $k \longrightarrow C \otimes^\tau A$ is coacyclic, then so is the cone of the morphism $\mathrm{Cob}(C) \otimes^{\tau_C} C \longrightarrow A \otimes^\tau C$. The latter cone is also injective as a graded $C^\#$-comodule; by Theorem 1.2.4, it follows that this cone is contractible. Applying the functor $- \mathbin{\square}_C k$ to it, we conclude that the cone of the morphism $\mathrm{Cob}(C) \longrightarrow A$ is a contractible complex of vector spaces. To prove the assertion related to the morphism $k \longrightarrow A \otimes^\tau C$, one can pass to the opposite DG-algebra and DG-coalgebra. ☐

1.3.4. *Duality for algebras.* An *admissible filtration* of a coaugmented DG-coalgebra C over a field k is an increasing filtration $F_0C \subset F_1C \subset F_2C \subset \cdots$ by homogeneous subspaces, preserved by the differential and compatible with the comultiplication, such that $C = \bigcup_j F_jC$ and $F_0C = k$ is the image of the coaugmentation map. A coaugmented DG-coalgebra is C conilpotent if and only if there exists an admissible filtration F on it.

A morphism of conilpotent DG-coalgebras $f\colon C \longrightarrow D$ over k is said to be a *filtered quasi-isomorphism* if there exist admissible filtrations F on C and D such that $f(F_jC) \subset F_jD$ and the induced map of the associated graded DG-coalgebras $\mathrm{gr}^F f\colon \mathrm{gr}^F C \longrightarrow \mathrm{gr}^F D$ is a quasi-isomorphism (i. e., the map $\mathrm{gr}^F_j C \longrightarrow \mathrm{gr}^F_j D$ is a quasi-isomorphism of complexes of vector spaces for each j).

Theorem. (a) *The functor $A \longmapsto \mathrm{Bar}(A)$ transforms quasi-isomorphisms of augmented DG-algebras into filtered quasi-isomorphisms of conilpotent DG-coalgebras. The functor $C \longmapsto \mathrm{Cob}(C)$ transforms filtered quasi-isomorphisms of conilpotent DG-coalgebras into quasi-isomorphisms of augmented DG-algebras.*

(b) *The functors between the localization of the category of augmented DG-algebras by the class of quasi-isomorphisms and the localization of the category of conilpotent DG-coalgebras by the class of filtered quasi-isomorphisms induced by the adjoint functors* Bar *and* Cob *are mutually inverse equivalences of categories.*

Proof. See [8, Section 6.10]. $\square$

1.3.5. *Pull-backs and push-forwards.* Let $f\colon A \longrightarrow B$ be a morphism of augmented DG-algebras and $g\colon C \longrightarrow D$ be a morphism of conilpotent DG-coalgebras. Let $\tau'\colon C \longrightarrow A$ and $\tau''\colon D \longrightarrow B$ be two acyclic twisting cochains forming a commutative square with the morphisms f and g.

Then the equivalences of categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra}) \simeq \mathsf{D}(B\text{–}\mathsf{mod})$ induced by τ' and τ'' transform the restriction-of-scalars functor R_f into the derived coextension-of-scalars functors $\mathbb{R}E_g$ and $\mathbb{L}E^g$. They also transform the derived extension-of-scalars functor $\mathbb{L}E_f$ into the corestriction-of-scalars functor R_g and the derived extension-of-scalars functor $\mathbb{R}E^f$ into the corestriction-of-scalars functor R^g.

2. Artin–Schelter Gorenstein Coalgebras

2.1. **Homological dimension.** Let C be an (ungraded) coalgebra over k. Then the homological dimensions of the abelian categories of left C-comodules, right C-comodules, and left C-contramodules are all equal to each other. They are also equal to the homological dimensions of the derived functor Cotor^C (defined in terms of injective resolutions of either argument), the derived functor Ctrtor^C (defined in terms of a projective resolution of the second argument), and and the derived functor Coext^C (defined in terms of an injective resolution of the first argument or a projective resolution of the second argument); see 1.2.5.

To prove these assertions, one defines *coflat* left C-comodules M by the condition that the functor $-\square_C M$ be exact and *coprojective* left C-comodules M by the condition that the functor $\mathrm{Cohom}_C(M,-)$ be exact. One also defines *coinjective* left C-contramodules P by the condition that the functor $\mathrm{Cohom}_C(-,P)$ be exact and *contraflat* left C-contramodules P by the condition that the functor $-\odot_C P$ be exact.

One shows [7, Section 0.2.9] that a comodule is coflat if and only if it is coprojective and if and only if it is injective (this follows easily from the fact that any comodule is the union of its finite-dimensional subcomodules). Furthermore, a contramodule is contraflat if and only if it is coinjective and if and only if it is projective (these results are a little harder to establish). All the above assertions follow from this.

The common homological dimension of the above abelian categories and derived functors is called the *homological dimension* of the coalgebra C.

Let C be a coalgebra of finite homological dimension, considered as a DG-coalgebra concentrated in degree 0 with a zero differential. Then the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ coincides with the conventional (unbounded) derived category $\mathsf{D}(C\text{–}\mathsf{comod})$ of the abelian category of left C-comodules, and similarly, the contraderived category $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ coincides with the conventional (unbounded) derived category $\mathsf{D}(C\text{–}\mathsf{contra})$ of the abelian category of left C-contramodules. Indeed, one easily checks that any acyclic complex of injective C-comodules is contractible, and so is any acyclic complex of projective C-contramodules. In view of Theorem 1.2.4, it follows that any acyclic complex of C-comodules is coacyclic and any acyclic complex of C-contramodules is contraacyclic.

2.2. **Artin–Schelter Gorenstein coalgebras.** Let C be an (ungraded) coalgebra over a field k; we will consider it also as a DG-coalgebra concentrated in degree 0 with a zero differential. Assume that C is coaugmented and conilpotent; then there exists a unique left C-comodule structure on the one-dimensional vector space k (namely, the trivial comodule structure defined in terms of the coaugmentation). By [7, Corollary A.2], the trivial C-contramodule structure on k (defined in terms of the coaugmentation) is also the unique left C-contramodule structure on k.

Definition-Proposition. *A conilpotent coalgebra C of is called* AS Gorenstein of dimension d *if it has finite homological dimension and one of the following conditions, which are equivalent for conilpotent coalgebras of finite homological dimension, holds:*

(a) *the object $\Psi_C(k)\in\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ is (noncanonically) isomorphic to $k[-d]$;*
(b) *the object $\Phi_C(k)\in\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ is (noncanonically) isomorphic to $k[d]$;*
(c) *the graded k-vector space $\mathrm{Ext}_C(C,k)$ is one-dimensional and concentrated in the degree d;*
(d) *the graded k-vector space $\mathrm{Ctrtor}^C(C,k)$ is one-dimensional and concentrated in the degree $-d$.*

An AS Gorenstein coalgebra C of dimension d has homological dimension d, and the map $\mathrm{Ext}^d_C(C,k)\longrightarrow\mathrm{Ext}^d_C(k,k)$ induced by the coaugmentation morphism $k\longrightarrow C$ is an isomorphism.

Proof. The conditions (a) and (b) are equivalent by the comodule-contramodule correspondence theorem as stated in 1.2.6. The condition (a) is equivalent to (c) and (b) is equivalent to (d) for the reasons explained in 2.1.

To prove the last two assertions, notice that the homological dimension of a conilpotent coalgebra C is equal to the supremum of n for which $\operatorname{Ext}_C^n(k,k) \simeq \operatorname{Cotor}^{C,n}(k,k)$ is nonzero. Indeed, any C-comodule is the union of its finite-dimensional subcomodules, the functors $\square_C$ and Cotor^C commute with the filtered direct limits, and any finite-dimensional C-comodule is an iterated extension of copies of the trivial comodule k. If D is the homological dimension of C, then the map $\operatorname{Ext}_C^D(C,k) \longrightarrow \operatorname{Ext}_C^D(k,k)$ is surjective. It follows that if C is AS Gorenstein of dimension d, then then $d = D$ and the above map is an isomorphism. □

Notice that with any AS Gorenstein conilpotent coalgebra C one can canonically associate a one-dimensional vector space $I_C = \operatorname{Ext}_C^d(C,k) \simeq \operatorname{Ext}_C^d(k,k) \simeq \operatorname{Cotor}^{C,d}(k,k) \simeq \operatorname{Ext}^{C,d}(k,k)$, which is naturally dual to the one-dimensional vector space $\operatorname{Ctrtor}_d^C(C,k) \simeq \operatorname{Coext}_{C,d}(k,k)$.

2.3. **Frobenius Ext-algebras.** A finite-dimensional positively graded algebra $F = F^0 \oplus F^1 \oplus \dots \oplus F^d$ over a field $k = F^0$ is said to be *Frobenius* if its top nonzero component F^d is one-dimensional and all the multiplication pairings $F^i \otimes F^{d-i} \longrightarrow F^d$ taking values in this component are nondegenerate.

Theorem. *Let C be a conilpotent coalgebra over k. Then the following conditions are equivalent.*

(a) *The coalgebra C is AS Gorenstein of dimension d.*
(b) *The coalgebra C^{op} opposite to C is AS Gorenstein of dimension d.*
(c) *The Ext-algebra $\operatorname{Ext}_C^*(k,k)$ computed in the abelian category of left comodules over C is Frobenius with the top component in degree d.*
(d) *The Ext-algebra $\operatorname{Ext}_{C^{\mathrm{op}}}^*(k,k)$ computed in the abelian category of right comodules over C is Frobenius with the top component in degree d.*
(e) *The Ext-algebra $\operatorname{Ext}^{C,*}(k,k)$ computed in the abelian category of left contramodules over C is Frobenius with the top component in degree d.*

If these conditions hold, the algebras $\operatorname{Ext}_C^*(k,k)$ and $\operatorname{Ext}^{C,*}(k,k)$ are naturally isomorphic and the algebra $\operatorname{Ext}_{C^{\mathrm{op}}}^*(k,k)$ is anti-isomorphic to them.

Proof. (a) $\Longleftrightarrow$ (c): Consider the cobar-construction $\operatorname{Cob}(C)$; it is an augmented DG-algebra with the cohomology of the augmentation ideal concentrated in the grading between 1 and d. Hence it has a quasi-isomorphic augmented quotient DG-algebra A living entirely in the grading $\le d$. By Theorem 1.3.3, there is a natural equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ sending the cofree DG-comodule C to the trivial DG-module k and the trivial DG-comodule k to the free DG-module A. Consequently, there is a natural isomorphism $\operatorname{Ext}_C(C,k) \simeq \operatorname{Ext}_A(k,A)$.

Furthermore, the Koszul duality transforms the morphism $\operatorname{Ext}_C^d(C,k) \longrightarrow \operatorname{Ext}_C^d(k,k)$ into the morphism $\operatorname{Ext}_A^d(k,A) \longrightarrow H^d(A)$ assigning to a morphism

$k \longrightarrow A[d]$ in $\mathsf{D}(A\text{–}\mathsf{mod})$ the image of $1 \in k$ under the induced cohomology map. Conversely, for any class in $H^d(A)$ there is a morphism of DG-modules $k \longrightarrow A[d]$ taking 1 to a cocycle representing that class. Consider the DG-module $A^* = \mathrm{Hom}_k(A, k)$ over A. One has $\mathrm{Hom}_{\mathsf{D}(A\text{–}\mathsf{mod})}(A[d], A^*) = H^{-d}(A^*) = \mathrm{Hom}_k(H^d(A), k)$. There is a morphism $f\colon A[d] \longrightarrow A^*$ in $\mathsf{D}(A\text{–}\mathsf{mod})$ corresponding to a nonzero linear function $H^d(A) \longrightarrow k$. If C is AS Gorenstein, the morphism f induces an isomorphism $\mathrm{Ext}_A(k, A[d]) \longrightarrow \mathrm{Ext}_A(k, A^*) = k$.

Let Z be the cone of f; then $\mathrm{Ext}_A(k, Z) = 0$. Since the augmentation ideal of A is positively graded and bounded from above, the free DG-module A over A is an iterated extension of direct sums of shifts of the DG-module k. It follows that $\mathrm{Ext}_A(A, Z) = 0$, hence Z is acyclic and f is an isomorphism in $\mathsf{D}(A\text{–}\mathsf{mod})$. Therefore, the multiplication pairing $H^i(A) \otimes H^{d-i}(A) \longrightarrow H^d(A) \simeq k$ induces isomorphisms $H^i(A) \longrightarrow H^{d-i}(A)^*$. Since this holds for all i, it follows that the vector spaces $H^i(A)$ are finite-dimensional and the algebra $H^*(A)$ is Frobenius.

If the algebra $\mathrm{Ext}_C^*(k, k)$ is Frobenius, the morphism f is a quasi-isomorphism, so it induces an isomorphism $\mathrm{Ext}_A(k, A[d]) \longrightarrow \mathrm{Ext}_A(k, A^*) = k$. The equivalence (b) $\Longleftrightarrow$ (d) is obtained from the equivalence (a) $\Longleftrightarrow$ (c) by replacing C with C^{op}.

(c) $\Longleftrightarrow$ (d) $\Longleftrightarrow$(e): The functor $M \longmapsto \mathrm{Hom}_k(M, k)$ is an anti-equivalence between the categories of finite-dimensional left C-comodules and finite-dimensional right C-comodules. The embedding of the abelian categories of finite-dimensional comodules into the categories of arbitrary comodules preserves Ext, so the graded algebras $\mathrm{Ext}_C^*(k, k)$ and $\mathrm{Ext}_{C^{\mathrm{op}}}^*(k, k)$ are anti-isomorphic for any coalgebra C.

Furthermore, there is the contravariant exact functor $N \longmapsto \mathrm{Hom}_k(N, k)$ acting from the category of right C-comodules to the category of left C-contramodules. This functor transforms injective comodules into projective contramodules, cobar-resolutions into bar-resolutions, and for any right comodule N and finite-dimensional right comodule R identifies $\mathrm{Ext}^{C,i}(N^*, R^*)$ with the double dual vector space to $\mathrm{Ext}_C^i(R, N)$. Hence the algebra $\mathrm{Ext}^{C,*}(k, k)$ is (locally) finite-dimensional if and only if the algebra $\mathrm{Ext}_{C^{\mathrm{op}}}^*(k, k)$ is, and whenever they are (locally) finite-dimensional, they are naturally anti-isomorphic. ☐

Corollary. *Let C be a conilpotent AS Gorenstein coalgebra. Then for any object $M \in \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) = \mathsf{D}(C\text{–}\mathsf{comod})$ the pairing*

$$\mathrm{Ext}_C^i(k, M) \otimes_k \mathrm{Ext}_C^{d-i}(M, k) \longrightarrow \mathrm{Ext}_C^d(k, k) = I_C$$

induces an isomorphism

$$\mathrm{Ext}_C^{d-i}(M, k) \simeq \mathrm{Hom}_k(\mathrm{Ext}_C^i(k, M),\ I_C), \qquad i \in \mathbb{Z}.$$

Proof. The pairing provides a natural map from the left to the right hand side. Both sides of this map are contravariant triangulated/cohomological functors transforming infinite direct sums of the objects M into infinite products of graded vector spaces. Thus the assertion follows from the fact that any object of $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ can be obtained from the C-comodule k using the operations of shift, cone and infinite direct sum [8, Section 5.5]. ☐

Graded case
Nilpotent Lie algebra example

3. Koszul Duality over Semisimple Base Algebra

Equivalence of tensor categories of modules and comodules
(Co)bar-constructions composed with this equivalence

4. Nonconilpotent Artin-Schelter Gorenstein Coalgebras

Generalization of the conilpotent version, assuming only (?) that the cosemisimple part is finite-dimensional

5. Lie–Commutative Duality

Lie-Com version of the theory from "recollection" section

6. Homomorphisms of DG-(Co)algebras

Noncommutative DG version of the duality principle for homomorphisms
Commutative refinement of the same

7. Lie Algebra Kernel

Tor between k and the quotient enveloping algebra over the big is enveloping algebra is the commutative dual to the kernel enveloping algebra

8. Generalized Golod Homomorphisms

Free Lie algebras of cohomology and Massey products

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