COHERENT ANALOGUES OF MATRIX FACTORIZATIONS AND RELATIVE SINGULARITY CATEGORIES

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Abstract. We define the triangulated category of relative singularities of a closed subscheme in a scheme. When the closed subscheme is a Cartier divisor, we consider matrix factorizations of the related section of a line bundle, and their analogues with locally free sheaves replaced by coherent ones. The appropriate exotic derived category of coherent matrix factorizations is then identified with the triangulated category of relative singularities, while the similar exotic derived category of locally free matrix factorizations is its full subcategory. The latter category is identified with the kernel of the direct image functor corresponding to the closed embedding of the zero locus and acting between the conventional (absolute) triangulated categories of singularities. Similar results are obtained for matrix factorizations of infinite rank; and two different "large" versions of the triangulated category of relative singularities, corresponding to the approaches of Orlov and Krause, are identified in the case of a Cartier divisor. Contravariant (coherent) and covariant (quasi-coherent) versions of the Serre–Grothendieck duality theorems for matrix factorizations are established, and pull-backs and push-forwards of matrix factorizations are discussed at length. A number of general results about derived categories of the second kind for CDG-modules over quasi-coherent CDG-algebras are proven on the way. Hochschild (co)homology of matrix factorization categories are discussed in an appendix.

Contents

Introduction

A *matrix factorization* of an element w in a commutative ring R is a pair of square matrices (Φ, Ψ) of the same size, with entries from R, such that both the products $\Phi\Psi$ and $\Psi\Phi$ are equal to w times the identity matrix. In the coordinate-free language, a matrix factorization is a pair of finitely generated free R-modules M^0 and M^1

together with R-module homomorphisms $M^0 \longrightarrow M^1$ and $M^1 \longrightarrow M^0$ such that both the compositions $M^0 \longrightarrow M^1 \longrightarrow M^0$ and $M^1 \longrightarrow M^0 \longrightarrow M^1$ are equal to the multiplication with w. Matrix factorizations were introduced by Eisenbud [16] and used by Buchweitz [4] for the purposes of the study of maximal Cohen–Macaulay modules over hypersurface local rings.

Another name for this notion is "D-branes in the Landau–Ginzburg B model" (as suggested by Kontsevich) [20]; in this context, the element w is called the *potential*. One generalizes the above definition, replacing free modules with projective modules [20, 29], with locally free sheaves [31], and finally with coherent sheaves [22]. The importance of the latter generalization is emphasized in the present paper.

Being particular cases of curved DG-modules over a curved DG-ring [20, 37], matrix factorizations form a DG-category. So one can consider the corresponding category of closed degree-zero morphisms up to chain homotopy, which is a triangulated category. Generally speaking, however, the homotopy category is "too big" for most purposes, and one would like to pass from it to an appropriately defined derived category. One can use the homotopy category in lieu of the derived one when dealing with projective modules [20, 29]; for locally free matrix factorizations over a nonaffine scheme, there is an option of working with the quotient category of the homotopy category by the locally contractible objects [33, Definition 3.13]. When dealing with coherent (analogues of) matrix factorizations, having some kind of a derived category construction is apparently unavoidable.

The relevant concept of a derived category is that of the derived category of the second kind, as developed in [37, 36]. There are several versions of this notion; the appropriate one for quasi-coherent sheaves is called the *coderived category* and for coherent sheaves it is the *absolute derived category*. The absolute derived category of locally free matrix factorizations was studied in [31]; for coherent matrix factorizations over a smooth variety, it was considered in [22]. These two absolute derived categories are equivalent for regular schemes, but may be different otherwise.

The *triangulated category of singularities* of a Noetherian scheme was defined by D. Orlov in [29] as the quotient category of the bounded derived category of coherent sheaves by its full triangulated subcategory of perfect complexes, i. e., the objects locally presentable as finite complexes of locally free sheaves. This triangulated category vanishes if and only if the Noetherian scheme is regular. It was shown in [29, Theorem 3.9], under mild assumptions on an affine regular Noetherian scheme X and a potential (regular function) w on it, that the homotopy category of locally free matrix factorizations of w over X is equivalent to the triangulated category of singularities of the zero locus X_0 of w in X.

In his recent paper [31], Orlov shows that the affineness assumption on X can be dropped in this result, if one replaces the homotopy category of locally free matrix factorizations with their absolute derived category. He also considers the general case of a nonaffine singular scheme X, for which he obtains a fully faithful functor from the absolute derived category of locally free matrix factorizations over X to the triangulated category of singularities of X_0. The problem of studying the difference between these two triangulated categories was posed in the introduction to [31].

The first aim of the present paper is to provide an alternative proof of these results of Orlov for regular schemes, an alternative generalization of them to singular schemes, and a more precise version of Orlov's original generalization. We replace the triangulated category at the source of Orlov's fully faithful functor by a "larger" category (containing the original one) and the triangulated category at the target by a "smaller" category (a quotient of the original one), thereby transforming this functor into an equivalence of triangulated categories. We also describe the image of Orlov's fully faithful functor as the kernel of a certain other triangulated functor.

More precisely, we show that the absolute derived category of coherent matrix factorizations of w over X is equivalent to what we call the *triangulated category of singularities of* X_0 *relative to* X. The latter category is a certain quotient category of the triangulated category of singularities of X_0; it measures, roughly speaking, how much worse are the singularities of X_0 compared to those of X. As to the image of Orlov's fully faithful embedding, it consists precisely of those objects of the conventional (absolute) triangulated category of singularities of X_0 whose direct images vanish in the triangulated category of singularities of X.

The paper consists of three sections and two appendices. In Section 1, we prove three rather general technical assertions about derived categories of the second kind for CDG-modules over a quasi-coherent CDG-algebra with a restriction on the homological dimension. One of them, claiming that certain embeddings of DG-categories of CDG-modules induce equivalences of the derived categories of the second kind, is a generalization of [32, Theorem 3.2] based on a modification of the same argument, originally introduced for the proof of [36, Theorem 7.2.2].

The idea of the proof of the other assertion, according to which certain natural functors between derived categories of the second kind are fully faithful, is new. The third technical assertion explains when the coderived category coincides with the absolute derived category of the same class of CDG-modules: e. g., for the locally projective CDG-modules this is true.

A version of (the former two of) these results is used in Section 2 in order to extend Orlov's cokernel functor from the absolute derived category of locally free matrix factorizations to the absolute derived category of coherent ones. This extension of the cokernel functor admits a simple construction of a functor in the opposite direction, suggested in [22]. We use these constructions to obtain a new proof of Orlov's theorem, and our own generalization of it to the singular case.

When X is regular, Orlov's and our results amount to the same assertion, since the absolute derived categories of locally free and coherent matrix factorizations are equivalent by our Theorem 1.4. When X is singular, the natural functor between these two absolute derived categories is fully faithful by our Proposition 1.5, and Orlov's full-and-faithfullness theorem follows from ours by virtue of an appropriate semiorthogonality property.

We also compare a "large" version of the triangulated category of relative singularities with the coderived category of quasi-coherent matrix factorizations, strengthening some results of Polishchuk–Vaintrob [33]. A "large" version of the absolute triangulated category of singularities, defined by Orlov in [29], is identified with H. Krause's stable derived category [21] in the case of a divisor in a regular scheme. Similar results are proven in the case of a Cartier divisor in a singular scheme, where we extend Krause's theory by defining the *relative stable derived category.* For any closed subscheme of finite flat dimension in a separated Noetherian scheme, the relative stable derived category is compactly generated by its full triangulated subcategory equivalent to the triangulated category of relative singularities.

The homotopy categories of unbounded complexes of projective modules over a ring and injective quasi-coherent sheaves over a scheme were studied in the papers by Jørgensen [18] and Krause [21]; subsequently, Iyengar and Krause have constructed an equivalence between these two categories for rings with dualizing complexes [17]. These results were extended to quasi-coherent sheaves over schemes by Neeman [28] and Murfet [23], who found a way to define a replacement of the homotopy category of (nonexistent) projective sheaves in terms of the flat ones. The equivalence between these two categories is a covariant version of the Serre–Grothendieck duality [14]. It is also very similar to the derived comodule-contramodule correspondence theory, developed by the present author in [37, 36].

The Serre–Grothendieck duality for matrix factorizations in the situation of a smooth variety X (and an isolated singularity of X_0) was studied in [24]. In this paper we extend the duality to matrix factorizations over much more general schemes X, constructing an equivalence between two "large" exotic derived categories, namely, the coderived category of flat (or locally free) matrix factorizations of possibly infinite rank and the coderived category of quasi-coherent matrix factorizations. Unless X is Gorenstein, this equivalence is *not* provided by the natural functor induced by the embedding of DG-categories, but rather differs from it in that the tensor product with the dualizing complex has to be taken along the way. A contravariant Serre duality in the form of an auto-anti-equivalence of the absolute derived category of coherent matrix factorizations is also obtained.

There was some attention paid to pull-backs and push-forwards of matrix factorizations recently [33, 8, 34]. In Section 3, we approach this topic with our techniques, constructing the push-forwards of locally free matrix factorizations of infinite rank for any morphism of finite flat dimension between schemes of finite Krull dimension, and the push-forwards of locally free matrix factorizations of finite rank for any such morphism for which the induced morphism of the zero loci of w is proper. At the price of having to adjoin the images of idempotent endomorphisms, the preservation of finite rank under push-forwards is proven assuming only the support of the matrix factorization [33] to be proper over the base.

Push-forwards of quasi-coherent matrix factorizations are well-defined for any morphism of Noetherian schemes, and push-forwards of coherent matrix factorizations exist under properness assumptions similar to the above. A general study of category-theoretic and set-theoretic supports of quasi-coherent and coherent CDG-modules is undertaken in this paper in order to obtain an independent proof of the preservation of coherence under the push-forwards not based on the passage to the triangulated categories of singularities.

The compatibility with pull-backs and push-forwards in an organic part of the Serre–Grothendieck duality theory. The contravariant duality agrees with push-forwards of coherent sheaves (or matrix factorizations) with respect to proper morphisms [14], while the covariant duality transforms the conventional inverse image of flat matrix factorizations into the extraordinary inverse image of quasi-coherent ones [39]. We use the latter result in order to construct the Hartshorne–Deligne extraordinary inverse image functor, which is denoted by $f^!$ in [14] and which we denote by f^+, in the case of quasi-coherent matrix factorizations.

Appendix A contains proofs of some basic facts about flat, locally projective, and injective quasi-coherent graded modules which are occasionally used in the main body of the paper. Appendix B is essentially a complement to the paper [32], containing both some variations and improvements on the results about Hochschild (co)homology of (C)DG-categories and matrix factorizations in [32], and an alternative approach to Hochschild (co)homology of coherent matrix factorizations based on the techniques developed in the main body of this paper.

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1. Exotic Derived Categories of Quasi-Coherent CDG-Modules

1.1. **CDG-rings and CDG-modules.** A *CDG-ring* (curved differential graded ring) $B=(B,d,h)$ is defined as a graded ring $B=\bigoplus_{i\in\mathbb{Z}}B^i$ endowed with an odd derivation $d\colon B\longrightarrow B$ of degree 1 and an element $h\in B^2$ such that $d^2(b)=[h,b]$ for all $b\in B$ and $d(h)=0$. So one should have $d\colon B^i\longrightarrow B^{i+1}$ and $d(ab)=d(a)b+(-1)^{|a|}ad(b)$; the brackets $[-,-]$ denote the supercommutator $[a,b]=ab-(-1)^{|a||b|}ba$. The element h is called the *curvature element*.

A morphism of CDG-rings $B\longrightarrow A$ is a pair (f,a), with a morphism of graded rings $f\colon B\longrightarrow A$ and an element $a\in A^1$, such that $f(d_Bb)=d_Af(b)+[a,f(b)]$ for all $b\in B$ and $f(h_B)=h_A+d_Aa+a^2$. The composition of morphisms of CDG-rings is defined by the obvious rule $(f,a)\circ(g,b)=(f\circ g,\,a+f(b))$. The element a is called the *change-of-connection element*. A discussion of the origins of these definitions can be found in the paper [35], where the above terminology first appeared (see also an earlier paper [10], where the motivation was entirely different).

A *left CDG-module* $M = (M, d_M)$ over a CDG-ring B is a graded B-module endowed with an odd derivation $d_M\colon M \longrightarrow M$ compatible with the derivation d on B such that $d_M^2(m) = hm$ for all $m \in M$. Given a morphism of CDG-rings $(f, a)\colon B \longrightarrow A$ and a CDG-module (M, d) over A, the CDG-module (M, d') over B is defined by the rule $d'(m) = d(m) + am$.

Given graded left B-modules M and N, homogeneous B-module morphisms $f\colon M \longrightarrow N$ of degree n are defined as homogeneous maps supercommuting with the action of B, i. e., $f(bm) = (-1)^{n|b|}bf(m)$. When M and N are CDG-modules, the homogeneous B-module morphisms $M \longrightarrow N$ form a complex of abelian groups with the differential $d(f)(m) = d(f(m)) - (-1)^{|f|}f(d(m))$. The curvature-related terms cancel out in the computation of the square of this differential, so one has $d^2(f) = 0$. Therefore, left CDG-modules over B form a DG-category.

Two aspects of the above definitions are worth to be pointed out. First, the CDG-rings or modules have *no* cohomology modules, as their differentials do not square to zero. Second, given a CDG-ring B, there is *no* natural way to define a CDG-module structure on the free graded B-module B (though B is naturally a CDG-bimodule over itself, in the appropriate sense).

We refer the reader to [37, Section 3.1] or [36, Sections 0.4.3–0.4.5] for more detailed discussions of the above notions. We will not need to consider any gradings different from $\mathbb{Z}$-gradings in this paper, though all the general results will be equally applicable in the Γ-graded situation in the sense of [32, Section 1.1].

1.2. **Quasi-coherent CDG-algebras.** Throughout this paper, unless specified otherwise, X is separated Noetherian scheme with enough vector bundles; in other words, it is assumed that every coherent sheaf on X is the quotient sheaf of a locally free sheaf of finite rank. Note that the class of all schemes satisfying these conditions is closed under the passages to open and closed subschemes [29, Section 1.2] and contains all regular separated Noetherian schemes [15, Exercise III.6.8].

Recall the definition of a *quasi-coherent CDG-algebra* from [37, Appendix B]. A quasi-coherent CDG-algebra $\mathcal{B}$ over X is a graded quasi-coherent $\mathcal{O}_X$-algebra such that for each affine open subscheme $U \subset X$ the graded ring $\mathcal{B}(U)$ is endowed with a structure of CDG-ring, i. e., a (not necessarily $\mathcal{O}_X$-linear) odd derivation $d\colon \mathcal{B}(U) \longrightarrow \mathcal{B}(U)$ of degree 1 and an element $h \in \mathcal{B}^2(U)$. For each pair of embedded affine open subschemes $U \subset V \subset X$, an element $a_{UV} \in \mathcal{B}^1(U)$ is fixed such that the restriction morphism $\mathcal{B}(V) \longrightarrow \mathcal{B}(U)$ together with the element a_{UV} form a morphism of CDG-rings. The obvious compatibility condition is imposed for triples of embedded affine open subschemes $U \subset V \subset W \subset X$.

A *quasi-coherent left CDG-module* $\mathcal{M}$ over $\mathcal{B}$ is an $\mathcal{O}_X$-quasi-coherent (or, equivalently, $\mathcal{B}$-quasi-coherent) sheaf of graded left modules over $\mathcal{B}$ together with a family of differentials $d\colon \mathcal{M}(U) \longrightarrow \mathcal{M}(U)$ defined for all affine open subschemes $U \subset X$ such that $\mathcal{M}(U)$ is a CDG-module over $\mathcal{B}(U)$ and the appropriate compatibility condition holds with respect to the restriction morphisms of CDG-rings $\mathcal{B}(V) \longrightarrow \mathcal{B}(U)$.

Specifically, for a quasi-coherent left CDG-module $\mathcal{M}$ one should have $d(s)|_U = d(s|_U) + a_{UV}s|_U$ for any $s \in \mathcal{M}(V)$.

Quasi-coherent left CDG-modules over a quasi-coherent CDG-algebra $\mathcal{B}$ form a DG-category [37]. The complex of morphisms between CDG-modules $\mathcal{N}$ and $\mathcal{M}$ is the graded abelian group of homogeneous $\mathcal{B}$-module morphisms $f\colon \mathcal{N} \longrightarrow \mathcal{M}$ with the differential $d(f)$ defined locally as the supercommutator of f with the differentials in $\mathcal{N}(U)$ and $\mathcal{M}(U)$. We denote this DG-category by $\mathcal{B}\text{–}\mathsf{qcoh}$.

We will call a quasi-coherent graded algebra $\mathcal{B}$ over X *Noetherian* if the graded ring $\mathcal{B}(U)$ is left Noetherian for any affine open subscheme $U \subset X$. Equivalently, $\mathcal{B}$ is Noetherian if the abelian category of quasi-coherent graded left $\mathcal{B}$-modules is a locally Noetherian Grothendieck category. In this case, the full DG-subcategory in $\mathcal{B}\text{–}\mathsf{qcoh}$ formed by CDG-modules whose underlying graded $\mathcal{B}$-modules are coherent (i. e., finitely generated over $\mathcal{B}$) is denoted by $\mathcal{B}\text{–}\mathsf{coh}$.

Given a quasi-coherent graded left $\mathcal{B}$-module $\mathcal{M}$ and a quasi-coherent graded right $\mathcal{B}$-module $\mathcal{N}$, one can define their tensor product $\mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}$, which is a quasi-coherent graded $\mathcal{O}_X$-module. A quasi-coherent graded left $\mathcal{B}$-module $\mathcal{M}$ is called *flat* if the functor $- \otimes_{\mathcal{B}} \mathcal{M}$ is exact on the abelian category of quasi-coherent graded right $\mathcal{B}$-modules. Equivalently, $\mathcal{M}$ is flat if the graded left $\mathcal{B}(U)$-module $\mathcal{M}(U)$ is flat for any affine open subscheme $U \subset X$. The *flat dimension* of a quasi-coherent graded module $\mathcal{M}$ is the minimal length of its flat left resolution.

The full DG-subcategory in $\mathcal{B}\text{–}\mathsf{qcoh}$ formed by CDG-modules whose underlying graded $\mathcal{B}$-modules are flat is denoted by $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$, and the full subcategory formed by CDG-modules whose underlying graded $\mathcal{B}$-modules have finite flat dimension is denoted by $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$. The similarly defined DG-categories of coherent CDG-modules are denoted by $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$ and $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{ffd}}$.

All the above DG-categories of quasi-coherent CDG-modules (and the similar ones defined below in this paper) admit shifts and twists, and, in particular, cones. It follows that their homotopy categories $H^0(\mathcal{B}\text{–}\mathsf{qcoh})$, $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, $H^0(\mathcal{B}\text{–}\mathsf{coh})$, etc. are triangulated. Besides, to any finite complex (of objects and closed morphisms) in one of these DG-categories one can assign its total object, which is an object of (i. e., a CDG-module belonging to) the same DG-category [37, Section 1.2].

The DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}$ and $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ also admit infinite direct sums. Hence in these two DG-categories one can totalize even an unbounded complex by taking infinite direct sums along the diagonals.

The DG-category $\mathcal{B}\text{–}\mathsf{qcoh}$ also admits infinite products (which one can obtain using the coherator construction from [42, Section B.14]), but these are not well-behaved (neither exact nor local), so we will not use them.

1.3. **Derived categories of the second kind.** The nonexistence of the cohomology groups for curved structures stands in the way of the conventional definition of the derived category of CDG-modules, which therefore does not seem to make sense. The suitable class of constructions of derived categories for CDG-modules is that of the *derived categories of the second kind* [36, 37].

Let $\mathcal{B}$ be a quasi-coherent CDG-algebra over X; assume that the quasi-coherent graded algebra $\mathcal{B}$ is Noetherian. Then a coherent CDG-module over $\mathcal{B}$ is called *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category of coherent CDG-modules $H^0(\mathcal{B}\text{–}\mathsf{coh})$ containing the total CDG-modules of all the short exact sequences of coherent CDG-modules over $\mathcal{B}$ (with closed morphisms between them). The quotient category of $H^0(\mathcal{B}\text{–}\mathsf{coh})$ by the thick subcategory of absolutely acyclic CDG-modules is called the *absolute derived category* of coherent CDG-modules over $\mathcal{B}$ and denoted by $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ [37].

For any quasi-coherent CDG-algebra $\mathcal{B}$ over X, a quasi-coherent CDG-module over $\mathcal{B}$ is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category of quasi-coherent CDG-modules $H^0(\mathcal{B}\text{–}\mathsf{qcoh})$ containing the total CDG-modules of all the short exact sequences of quasi-coherent CDG-modules over $\mathcal{B}$ and closed under infinite direct sums. The quotient category of $H^0(\mathcal{B}\text{–}\mathsf{coh})$ by the thick subcategory of coacyclic CDG-modules is called the *coderived category* of quasi-coherent CDG-modules over $\mathcal{B}$ and denoted by $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ [36, 37].

Given an exact subcategory E in the abelian category of quasi-coherent graded left $\mathcal{B}$-modules, one can define the *absolute derived category of left CDG-modules over $\mathcal{B}$ with the underlying graded $\mathcal{B}$-modules belonging to* E as the quotient category of the corresponding homotopy category by its minimal thick subcategory containing the total CDG-modules of all the exact triples of CDG-modules with the underlying graded $\mathcal{B}$-modules belonging to E. The objects of the latter subcategory are called *absolutely acyclic with respect to* E (or with respect to the DG-category of CDG-modules with the underlying graded modules belonging to E) [32].

So one defines the absolute derived categories $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh_{ffd}})$ and $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh_{fl}})$ as the quotient categories of the homotopy categories $H^0(\mathcal{B}\text{–}\mathsf{coh_{ffd}})$ and $H^0(\mathcal{B}\text{–}\mathsf{coh_{fl}})$ by the thick subcategories of CDG-modules absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh_{ffd}}$ and $\mathcal{B}\text{–}\mathsf{coh_{fl}}$, respectively.

When the exact subcategory E is closed under infinite direct sums, the thick subcategory of CDG-modules *coacyclic with respect to* E is the minimal triangulated subcategory of the homotopy category CDG-modules with the underlying graded modules belonging to E, containing the total CDG-modules of all the exact triples of CDG-modules with the underlying graded modules belonging to E and closed under infinite direct sums. The quotient category by this thick subcategory is called the *coderived category of left CDG-modules over $\mathcal{B}$ with the underlying graded modules belonging to* E [36, 32].

Thus one defines the coderived category $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh_{fl}})$ as the quotient categories of the homotopy category $H^0(\mathcal{B}\text{–}\mathsf{qcoh_{fl}})$ by the thick subcategory of CDG-modules coacyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh_{fl}}$. The definition of the coderived category $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh_{ffd}})$ requires a little more care, since the class of graded modules of finite flat dimension is not in general closed under infinite direct sums. An object $\mathcal{M} \in H^0(\mathcal{B}\text{–}\mathsf{qcoh_{ffd}})$ is said to be *coacyclic with respect to* $\mathcal{B}\text{–}\mathsf{qcoh_{ffd}}$ if there exists an integer $d \geq 0$ such that $\mathcal{M}$ is coacyclic with respect to the exact category of quasi-coherent CDG-modules of flat dimension $\leq d$. The coderived category of quasi-coherent CDG-modules of finite flat dimension is, by the definition, the quotient category of $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ by the above-defined thick subcategory of coacyclic CDG-modules [32, Section 3.2].

Remark. One may wonder whether coacyclicity (absolute acyclicity) of quasi-coherent CDG-modules (of a certain class) is a local notion. One general approach to this kind of problems is to consider the Mayer–Vietoris/Čech exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \bigoplus_\alpha j_{U_\alpha *} j_{U_\alpha}^* \mathcal{M} \longrightarrow \bigoplus_{\alpha<\beta} j_{U_\alpha \cap \mathcal{U}_\beta *} j_{U_\alpha \cap U_\beta}^* \mathcal{M} \longrightarrow \cdots \longrightarrow 0$$

for a finite affine open covering U_α of X. Since the inverse and direct images with respect to affine open embeddings are exact and compatible with direct sums, they preserve coacyclicity (absolute acyclicity). Hence if the restrictions of $\mathcal{M}$ to all U_α are coacyclic (absolutely acyclic), then so is $\mathcal{M}$ itself.

Alternatively, one can base this kind of argument on the implications of the Noetherianness assumption, rather than the separatedness assumption. For this purpose, one replaces a quasi-coherent CDG-module $\mathcal{M}$ with its injective resolution (see Lemma 1.7(b)) before writing down its Čech resolution. In this approach, the covering need not be affine, as injective coacyclic objects are contractible, and direct images preserve contractibility; but it is important that the restrictions to open subschemes should preserve injectivity of quasi-coherent graded $\mathcal{B}$-modules (see [14, Theorem II.7.18] and Theorem A.3; cf. [42, Appendix B]).

When one is working with coherent CDG-modules, the Čech sequence argument is to be used in conjuction with Proposition 1.5 below. (Cf. Sections 1.10 and 3.1.)

1.4. **Finite flat dimension theorem.** The next theorem is our main technical result on which the proofs in Section 2 are based.

Though we generally prefer the coderived categories of (various classes of) infinitely generated CDG-modules over their absolute derived categories, technical considerations sometimes force us to deal with the latter (see Remark 1.5). Therefore, let $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$, and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$ denote the absolute derived categories of (flat, of finite flat dimension, or arbitrary) quasi-coherent CDG-modules over a quasi-coherent CDG-algebra $\mathcal{B}$.

Theorem. (a) *For any quasi-coherent CDG-algebra $\mathcal{B}$ over X, the functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.*

(b) *For any quasi-coherent CDG-algebra $\mathcal{B}$ over X, the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.*

(c) *For any quasi-coherent CDG-algebra $\mathcal{B}$ over X such that the underlying quasi-coherent graded algebra $\mathcal{B}$ is Noetherian, the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{ffd}})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{coh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.*

Proof. The proof follows that of [32, Theorem 3.2] (see also [36, Theorem 7.2.2]) with some modifications. We will prove part (a); the proofs of parts (b-c) are completely similar. (Alternatively, parts (b-c) can be deduced from Proposition 1.5(a-b) below.)

Given an affine open subscheme $U \subset X$ and a graded module P over the graded ring $\mathcal{B}(U)$, one can construct the freely generated CDG-module $G^+(P)$ over the CDG-ring $\mathcal{B}(U)$ in the way explained in [37, proof of Theorem 3.6]. The elements of $G^+(P)$ are formal expressions of the form $p + dq$, where p, $q \in P$. Given a quasi-coherent graded module $\mathcal{P}$ over $\mathcal{B}$, the CDG-modules $G^+(\mathcal{P}(U))$ glue together to form a quasi-coherent CDG-module $G^+(\mathcal{P})$ over $\mathcal{B}$. For any quasi-coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$, there is a bijective correspondence between morphisms of graded $\mathcal{B}$-modules $\mathcal{P} \longrightarrow \mathcal{M}$ and closed morphisms of CDG-modules $G^+(\mathcal{P}) \longrightarrow \mathcal{M}$ over $\mathcal{B}$. There is a natural short exact sequence of quasi-coherent graded $\mathcal{B}$-modules $\mathcal{P} \longrightarrow G^+(\mathcal{P}) \longrightarrow \mathcal{P}[-1]$. The quasi-coherent CDG-module $G^+(\mathcal{P})$ is naturally contractible with the contracting homotopy $t_{\mathcal{P}}$ given by the composition $G^+(\mathcal{P}) \longrightarrow \mathcal{P}[-1] \longrightarrow G^+(\mathcal{P})[-1]$.

Due to our assumption on X, for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{K}$ over X there exists a surjective morphism $\mathcal{E} \longrightarrow \mathcal{K}$ onto $\mathcal{K}$ from a direct sum $\mathcal{E}$ of locally free sheaves of finite rank on X. Hence for any quasi-coherent graded $\mathcal{B}$-module $\mathcal{M}$ there is a surjective morphism onto $\mathcal{M}$ from a flat quasi-coherent graded $\mathcal{B}$-module $\mathcal{P} = \bigoplus_n \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}_n[n]$, and for any quasi-coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ there is a surjective closed morphism onto $\mathcal{M}$ from the CDG-module $G^+(\mathcal{P}) \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$. (In fact, parts (a-b) of Theorem can be proven without the assumption of enough vector bundles on X, since there are always enough flat sheaves; see Remark 2.6 and Lemma A.1.)

Now the construction from [37, proof of Theorem 3.6] provides for any object $\mathcal{M}$ of $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ a closed morphism onto $\mathcal{M}$ from an object of $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ with the cone absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$. To obtain this morphism, one picks a finite left resolution of $\mathcal{M}$ consisting of objects from $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ with closed morphisms between them, and takes the total CDG-module of this resolution. By [37, Lemma 1.6], it follows that the triangulated category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ is equivalent to the quotient category of $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ by its intersection in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ with the thick subcategory of CDG-modules coacyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$. It only remains to show that any object of $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ that is coacyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ is coacyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}$.

Let us call a quasi-coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ *d-flat* if its underlying quasi-coherent graded $\mathcal{B}$-module $\mathcal{M}$ has flat dimension not exceeding d. A d-flat quasi-coherent CDG-module is said to be *d-coacyclic* if it is homotopy equivalent to a CDG-module obtained from the total CDG-modules of exact triples of d-flat CDG-modules using the operations of cone and infinite direct sum. Our goal is to show that any 0-flat d-coacyclic CDG-module is 0-coacyclic. For this purpose, we will prove that any $(d-1)$-flat d-coacyclic CDG-module is $(d-1)$-coacyclic; the desired assertion will then follow by induction.

It suffices to construct for any d-coacyclic CDG-module $\mathcal{M}$ a $(d-1)$-coacyclic CDG-module $\mathcal{L}$ with a $(d-1)$-coacyclic CDG-submodule $\mathcal{K}$ such that the quotient

CDG-module $\mathcal{L}/\mathcal{K}$ is isomorphic to $\mathcal{M}$. Then if $\mathcal{M}$ is $(d-1)$-flat, it would follow that both the cone of the morphism $\mathcal{K} \longrightarrow \mathcal{L}$ and the total CDG-module of the exact triple $\mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M}$ are $(d-1)$-coacyclic, so $\mathcal{M}$ also is. The construction is based on four lemmas similar to those in [32, Section 3.2].

Lemma A. *Let $\mathcal{M}$ be the total CDG-module of an exact triple of d-flat quasi-coherent CDG-modules $\mathcal{M}' \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M}'''$ over $\mathcal{B}$. Then there exists a surjective closed morphism onto $\mathcal{M}$ from a contractible 0-flat CDG-module $\mathcal{P}$ with a $(d-1)$-coacyclic kernel $\mathcal{K}$.*

Proof. Choose 0-flat quasi-coherent CDG-modules $\mathcal{P}'$ and $\mathcal{P}'''$ such that there exist surjective closed morphisms $\mathcal{P}' \longrightarrow \mathcal{M}'$ and $\mathcal{P}''' \longrightarrow \mathcal{M}''$. Then there exists a surjective morphism from the exact triple of CDG-modules $\mathcal{P}' \longrightarrow \mathcal{P}' \oplus \mathcal{P}''' \longrightarrow \mathcal{P}'''$ onto the exact triple $\mathcal{M}' \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M}'''$. The rest of the proof is similar to that in [32]. ☐

Lemma B. (a) *Let $\mathcal{K}' \subset \mathcal{L}'$ and $\mathcal{K}'' \subset \mathcal{L}''$ be $(d-1)$-coacyclic CDG-submodules in $(d-1)$-coacyclic CDG-modules, and let $\mathcal{L}'/\mathcal{K}' \longrightarrow \mathcal{L}''/\mathcal{K}''$ be a closed morphism of CDG-modules. Then there exists a $(d-1)$-coacyclic CDG-module $\mathcal{L}$ with a $(d-1)$-coacyclic CDG-submodule $\mathcal{K}$ such that $\mathcal{L}/\mathcal{K} \simeq \operatorname{cone}(\mathcal{L}'/\mathcal{K}' \to \mathcal{L}''/\mathcal{K}'')$.*

(b) *In the situation of* (a)*, assume that the morphism $\mathcal{L}'/\mathcal{K}' \longrightarrow \mathcal{L}''/\mathcal{K}''$ is injective with a d-flat cokernel $\mathcal{M}_0$. Then there exists a $(d-1)$-coacyclic CDG-module $\mathcal{L}_0$ with a $(d-1)$-coacyclic CDG-submodule $\mathcal{K}_0$ such that $\mathcal{L}_0/\mathcal{K}_0 \simeq \mathcal{M}_0$.*

Proof. The proof is similar to that in [32]. ☐

Lemma C. *For any contractible d-flat CDG-module $\mathcal{M}$ there exists a surjective closed morphism onto $\mathcal{M}$ from a contractible 0-flat CDG-module $\mathcal{L}$ with a $(d-1)$-coacyclic kernel $\mathcal{K}$.*

Proof. Let $p\colon \mathcal{P} \longrightarrow \mathcal{M}$ be a surjective morphism onto the quasi-coherent graded $\mathcal{B}$-module $\mathcal{M}$ from a flat quasi-coherent graded $\mathcal{B}$-module $\mathcal{P}$, and $\tilde{p}\colon G^+(\mathcal{P}) \longrightarrow \mathcal{M}$ be the induced surjective closed morphism of quasi-coherent CDG-modules. Let $t\colon \mathcal{M} \longrightarrow \mathcal{M}$ be a contracting homotopy for $\mathcal{M}$ and $t_\mathcal{P}\colon G^+(\mathcal{P}) \longrightarrow G^+(\mathcal{P})$ be the natural contracting homotopy for $G^+(\mathcal{P})$. Then $\tilde{u} = \tilde{p}t_\mathcal{P} - t\tilde{p}\colon G^+(\mathcal{P}) \longrightarrow \mathcal{M}$ is a closed morphism of quasi-coherent CDG-modules of degree -1. Denote by u the restriction of $\tilde{u}$ to $\mathcal{P} \subset G^+(\mathcal{P})$. There exists a surjective morphism from a flat quasi-coherent graded $\mathcal{B}$-module $\mathcal{Q}$ onto the fibered product of the morphisms $p\colon \mathcal{P} \longrightarrow \mathcal{M}$ and $u\colon \mathcal{P} \longrightarrow \mathcal{M}$. Hence we obtain a surjective morphism of quasi-coherent graded $\mathcal{B}$-modules $q\colon \mathcal{Q} \longrightarrow \mathcal{P}$ and a morphism of quasi-coherent graded $\mathcal{B}$-modules $v\colon \mathcal{Q} \longrightarrow \mathcal{P}$ of degree -1 such that $uq = pv$.

The morphism q induces a surjective closed morphism of quasi-coherent CDG-modules $\tilde{q}\colon G^+(\mathcal{Q}) \longrightarrow G^+(\mathcal{P})$. The morphism $\tilde{q}$ is homotopic to zero with the natural contracting homotopy $\tilde{q}t_\mathcal{Q} = t_\mathcal{P}\tilde{q}$. The morphism v induces a closed morphism of CDG-modules $\tilde{v}\colon G^+(\mathcal{Q}) \longrightarrow G^+(\mathcal{P})$ of degree -1. The morphism $t_\mathcal{P}\tilde{q} - \tilde{v}$ is another contracting homotopy for $\tilde{q}$. The latter homotopy forms a commutative square with the morphisms $\tilde{p}$, $\tilde{p}\tilde{q}$, and the contracting homotopy t for the CDG-module $\mathcal{M}$.

Let $\mathcal{N}$ be the kernel of the morphism $\tilde{p}\tilde{q}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{M}$ and $\mathcal{K}$ be the kernel of the morphism $\tilde{p}\colon G^+(\mathcal{P}) \longrightarrow \mathcal{M}$. Then the natural surjective closed morphism $r\colon \mathcal{N} \longrightarrow \mathcal{K}$ is homotopic to zero; the restriction of the map $t_{\mathcal{P}}\tilde{q} - \tilde{v}$ provides the contracting homotopy that we need. In addition, the kernel $G^+(\ker q)$ of the morphism r is contractible. So the cone of the morphism r is isomorphic to $\mathcal{K} \oplus \mathcal{N}[1]$, and on the other hand there is an exact triple $G^+(\ker q)[1] \longrightarrow \operatorname{cone}(r) \longrightarrow \operatorname{cone}(\mathrm{id}_{\mathcal{K}})$. Since $\mathcal{K}$ is $(d-1)$-flat and $\ker q$ is flat, this proves that $\mathcal{K}$ is $(d-1)$-coacyclic. It remains to take $\mathcal{L} = G^+(\mathcal{P})$. □

Lemma D. *Let $\mathcal{M} \longrightarrow \mathcal{M}'$ be a homotopy equivalence of d-flat CDG-modules such that $\mathcal{M}'$ is the quotient CDG-module of a $(d-1)$-coacyclic CDG-module by a $(d-1)$-coacyclic CDG-submodule. Then $\mathcal{M}$ is also such a quotient.*

Proof. The proof is similar to that in [32]. □

It is clear that the property of a CDG-module to be presentable as the cokernel of an injective closed morphism of $(d-1)$-coacyclic CDG-modules is stable under infinite direct sums. This finishes our construction and the proof of Theorem. □

Remark. The assertion of part (c) of Theorem 1.4 can be equivalently rephrased with flat modules replaced by locally projective ones. Indeed, a finitely presented module over a ring is flat if and only if it is projective.

In the infinitely generated situation of parts (a-b), flatness of quasi-coherent sheaves is different from their local projectivity (which is a stronger condition), but the assertions remain true after one replaces the former with the latter. The same applies to Proposition 1.5(a) below. Indeed, by Theorem A.2, for any quasi-coherent graded algebra $\mathcal{B}$ over an affine scheme U, projectivity of a graded module over the graded ring $\mathcal{B}(U)$ is a local notion. Taking this fact into account, our proof goes through for locally projective quasi-coherent graded modules in place of flat ones and the locally projective dimension (defined as the minimal length of a locally projective resolution) in place of the flat dimension.

When $\mathcal{B} = \mathcal{O}_X$, local projectivity of quasi-coherent modules is equivalent to local freeness [2, Corollary 4.5]. Furthermore, in this case, assuming additionally that X has finite Krull dimension, the classes of quasi-coherent sheaves of finite flat dimension and of finite locally projective dimension coincide [40, Corollaire II.3.3.2].

1.5. **Fully faithful embedding.** The next proposition is stronger than Theorem 1.4 in some respects, and is proven by an entirely different technique.

Proposition. (a) *For any quasi-coherent CDG-algebra $\mathcal{B}$ over X, the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ is fully faithful.*

Furthermore, let $\mathcal{B}$ be a quasi-coherent CDG-algebra over X such that the underlying quasi-coherent graded algebra $\mathcal{B}$ is Noetherian. Then

(b) *the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{coh}$ is fully faithful;*

(c) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{–}\mathsf{coh} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ *is fully faithful;*

(d) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{–}\mathsf{coh} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ *is fully faithful and its image forms a set of compact generators for* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$.

Proof. The proof of part (d) in the case when X is affine can be found in [37, Section 3.11] (the part concerning compact generation belongs to D. Arinkin). The proof in the general case is similar; and part (c) can be also proven in the way similar to [37, Theorem 3.11.1]. Part (b) in the affine case is easy and follows from the semiorthogonality property of CDG-modules with projective underlying graded modules and absolutely acyclic/contraacyclic CDG-modules [37, Theorem 3.5(b)], since finitely generated flat modules over a Noetherian ring are projective. A detailed proof of part (b) in the general case is given below; and the proof of part (a) (which does not automatically simplify in the affine case) is similar.

We will show that any morphism $\mathcal{E} \longrightarrow \mathcal{L}$ from a CDG-module $\mathcal{E} \in H^0(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$ to a CDG-module $\mathcal{L} \in H^0(\mathcal{B}\text{–}\mathsf{coh})$ absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh}$ can be annihilated by a morphism $\mathcal{P} \longrightarrow \mathcal{E}$ from a CDG-module $\mathcal{P} \in H^0(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$ with a cone of the morphism $\mathcal{P} \longrightarrow \mathcal{E}$ being absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$. By the definition, the CDG-module $\mathcal{L}$ is a direct summand of a CDG-module homotopy equivalent to a CDG-module obtained from the totalizations of exact triples of CDG-modules in $\mathcal{B}\text{–}\mathsf{coh}$ using the operation of passage to the cone of a closed morphism repeatedly. It suffices to consider the case when $\mathcal{L}$ itself is obtained from totalizations of exact triples using cones. We proceed by induction in the number of operations of passage to the cone in such a construction of $\mathcal{L}$.

So we assume that there is a distinguished triangle $\mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}[1]$ in $H^0(\mathcal{B}\text{–}\mathsf{coh})$ such that $\mathcal{M}$ is the total CDG-module of an exact triple of CDG-modules in $\mathcal{B}\text{–}\mathsf{coh}$, while the CDG-module $\mathcal{K}$ has the desired property with respect to morphisms into it from all CDG-modules $\mathcal{F} \in H^0(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$. If we knew that the object $\mathcal{M}$ also has the same property, it would follow that the composition $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M}$ can be annihilated by a morphism $\mathcal{F} \longrightarrow \mathcal{E}$ with $\mathcal{F} \in H^0(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$ and a cone absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$. The composition $\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}$ then factorizes through $\mathcal{K}$, and the morphism $\mathcal{F} \longrightarrow \mathcal{K}$ can be annihilated by a morphism $\mathcal{P} \longrightarrow \mathcal{F}$ with $\mathcal{P} \in H^0(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$ and a cone absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$. The composition $\mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}$ provides the desired morphism $\mathcal{P} \longrightarrow \mathcal{E}$.

Thus it remains to construct a morphism $\mathcal{F} \longrightarrow \mathcal{E}$ with the required properties annihilating a morphism $\mathcal{E} \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is the total CDG-module of an exact triple of CDG-modules $\mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$. For any graded module $\mathcal{N}$ over $\mathcal{B}$, morphisms of graded $\mathcal{B}$-modules $\mathcal{N} \longrightarrow \mathcal{M}$ of degree n are represented by triples (f, g, h), where $f\colon \mathcal{N} \longrightarrow \mathcal{U}$ is a morphism of degree $n+1$, $g\colon \mathcal{N} \longrightarrow \mathcal{V}$ is a morphism of degree n, and $h\colon \mathcal{N} \longrightarrow \mathcal{W}$ is a morphism of degree $n-1$. Denote the closed morphisms in the exact triple $\mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ by $j\colon \mathcal{U} \longrightarrow \mathcal{V}$ and $k\colon \mathcal{V} \longrightarrow \mathcal{W}$.

Lemma E. *Let* $\mathcal{N}$ *be a CDG-module over* $\mathcal{B}$ *and* $\mathcal{M}$ *be the total CDG-module of an exact triple of CDG-modules* $\mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ *as above. Then*

(a) *the differential of a morphism of graded $\mathcal{B}$-modules $\mathcal{N} \longrightarrow \mathcal{M}$ of degree n represented by a triple (f, g, h) is given by the rule $d(f, g, h) = (-df,\ -jf + dg,\ kg - dh)$;*

(b) *when (f, g, h) is a closed morphism of CDG-modules of degree n and the morphism of graded $\mathcal{B}$-modules $h\colon \mathcal{N} \longrightarrow \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t\colon \mathcal{N} \longrightarrow \mathcal{V}$ of degree $n-1$, the morphism (f, g, h) is homotopic to zero.*

Proof. The complex of morphisms in the DG-category of CDG-modules $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{M})$ is the total complex of the bicomplex of abelian groups $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$. The formula in (a) is the formula for the differential of a total complex.

Furthermore, the sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ is exact. Let $\mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ denote the cokernel of the morphisms of complexes $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V})$; then $\mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ is a subcomplex of $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ and the total complex of the bicomplex $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \longrightarrow \mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ is an acyclic subcomplex of $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{M})$. Hence any cocycle in $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{M})$ that belongs to this subcomplex is a coboundary.

To present the same argument using our letter notation for morphisms, assume that $kt = h$. Then $k(dt - g) = dh - kg = 0$, so there exists a morphism of graded $\mathcal{B}$-modules $s\colon \mathcal{N} \longrightarrow \mathcal{U}$ of degree n such that $dt - g = js$. Then $jds = -dg = -jf$, hence $ds = -f$ and $d(s, t, 0) = (f, g, h)$. □

Recall the notation $G^+(\mathcal{Q})$ for the CDG-module freely generated by a graded $\mathcal{B}$-module $\mathcal{Q}$ (see the beginning of the proof of Theorem 1.4).

Lemma F. *Let $\mathcal{M}$ be the total CDG-module of an exact triple of CDG-modules $\mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ as above, and let $\mathcal{Q}$ be a graded $\mathcal{B}$-module. Assume that a morphism of graded $\mathcal{B}$-modules $p\colon \mathcal{Q} \longrightarrow \mathcal{M}$ of degree n with the components (f, g, h) is given such that the component $h\colon \mathcal{Q} \longrightarrow \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t\colon \mathcal{Q} \longrightarrow \mathcal{V}$ of degree $n-1$. Let $\tilde{p}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{M}$ be the induced closed morphism of CDG-modules of degree n and $(\tilde{f}, \tilde{g}, \tilde{h})$ be its three components. Then the morphism of graded $\mathcal{B}$-modules $\tilde{h}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $\tilde{t}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{V}$ of degree $n-1$.*

Proof. Notice that any closed morphism of CDG-modules $G^+(\mathcal{Q}) \longrightarrow \mathcal{M}$ is homotopic to zero, since the CDG-module $G^+(\mathcal{Q})$ is contractible. The conclusion of the lemma is stronger, and we will need its full strength. The argument consists in a computation in the letter notation for morphisms.

For any CDG-module $\mathcal{N}$ over $\mathcal{B}$, morphisms of graded $\mathcal{B}$-modules $\tilde{r}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{N}$ of degree $n-1$ are uniquely determined by their restriction to $\mathcal{Q}$ and the restriction to $\mathcal{Q}$ of their differential $d\tilde{r}$, which can be arbitrary morphisms of graded $\mathcal{B}$-modules $\mathcal{Q} \longrightarrow \mathcal{N}$ of the degrees $n-1$ and n, respectively. Extend our morphism $t\colon \mathcal{Q} \longrightarrow \mathcal{V}$ to a morphism of graded $\mathcal{B}$-modules $\tilde{t}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{V}$ of degree $n-1$ such that $(d\tilde{t})|_{\mathcal{Q}} = g$. Then $k\tilde{t}|_{\mathcal{Q}} = kt = h = \tilde{h}|_{\mathcal{Q}}$ and $(d(k\tilde{t}))|_{\mathcal{Q}} = k(d\tilde{t})|_{Q} = kg = k\tilde{g}|_{\mathcal{Q}} = (d\tilde{h})|_{\mathcal{Q}}$ by Lemma E(a), hence $k\tilde{t} = \tilde{h}$. □

Now represent a closed morphism $\mathcal{E} \longrightarrow \mathcal{M}$ by a triple (f, g, h) of morphisms of the degrees 1, 0, and -1, respectively. Let $\mathcal{Q}$ be a flat coherent graded $\mathcal{B}$-module mapping surjectively onto the fibered product of the morphisms $k\colon \mathcal{V} \longrightarrow \mathcal{W}$ and $h\colon \mathcal{E} \longrightarrow \mathcal{W}$ (see the beginning of the proof of Theorem 1.4 again). Then there is a surjective morphism of graded $\mathcal{B}$-modules $q\colon \mathcal{Q} \longrightarrow \mathcal{E}$ and its composition with the morphism $h\colon \mathcal{E} \longrightarrow \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t\colon \mathcal{Q} \longrightarrow \mathcal{V}$ of degree -1. Consider the induced morphism of CDG-modules $\tilde{q}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{E}$. By Lemma F, the composition $h\tilde{q}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $\tilde{t}\colon G^+(\mathcal{Q}) \longrightarrow \mathcal{V}$ of degree -1.

Let $\mathcal{R}$ denote the kernel of the closed morphism $\tilde{q}$. Then the cone $\mathcal{F}$ of the embedding $\mathcal{R} \longrightarrow G^+(\mathcal{Q})$ maps naturally onto $\mathcal{E}$ with the cone absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$. As a graded $\mathcal{B}$-module, the CDG-module $\mathcal{F}$ is isomorphic to $G^+(\mathcal{Q}) \oplus \mathcal{R}[1]$; the composition $\mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M}$ factorizes through the direct summand $G^+(\mathcal{Q})$, where it is defined by the triple $(f\tilde{q},\ g\tilde{q},\ h\tilde{q})$. Since the morphism $h\tilde{q}$ can be lifted to $\mathcal{V}$, so can the corresponding component $\mathcal{F} \longrightarrow \mathcal{W}$ of the morphism $\mathcal{F} \longrightarrow \mathcal{M}$. Thus the latter morphism is homotopic to zero by Lemma E(b). ☐

In some cases the use of Lemma F in the above proof of part (b) can be avoided. Assume that X is a projective scheme over a Noetherian ring and the category of coherent graded $\mathcal{B}$-modules is equivalent to the category of coherent modules over some coherent (graded) $\mathcal{O}_X$-algebra $\mathcal{A}$. In this situation, one takes $\mathcal{Q}$ to be the graded $\mathcal{B}$-module corresponding to the (graded) $\mathcal{A}$-module induced from a large enough finite direct sum of (shifts of) copies of a sufficiently negative invertible $\mathcal{O}_X$-module; then there is a surjective morphism of graded $\mathcal{B}$-modules $\mathcal{Q} \longrightarrow \mathcal{E}$ and any morphism of graded $\mathcal{B}$-modules $G^+(\mathcal{Q}) \longrightarrow \mathcal{W}$ lifts to $\mathcal{V}$.

Remark. We do *not* know how to extend the proof of Proposition 1.5(a-b) to the coderived categories of quasi-coherent CDG-modules. Instead, this argument appears to be well-suited for use with the *contraderived* categories (see [37, Section 3.3] for the definition). In particular, it allows to show that the contraderived category of left CDG-modules over a CDG-ring B with a right coherent underlying graded ring is equivalent to the contraderived category of CDG-modules whose underlying graded B-modules are flat (cf. [37, paragraph after the proof of Theorem 3.8]).

This is the main reason why we sometimes find it easier to deal with the absolute derived rather than the coderived categories of infinitely generated CDG-modules (cf. Remark 2.8). On the other hand, for the coderived category of quasi-coherent CDG-modules we have the compact generation result (part (d) of Proposition), the results and arguments of Sections 1.7, 1.10, 2.5, 2.9, etc. The conditions under which these two versions of the construction of the derived category of the second kind for a given class of CDG-modules lead to the same triangulated category are discussed below in Section 1.6.

1.6. **Finite homological dimension theorem.** Let $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ denote the DG-category of quasi-coherent CDG-modules over $\mathcal{B}$ whose underlying graded $\mathcal{B}$-modules are locally projective (see Remark 1.4 and Theorem A.2). Denote by $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ the corresponding coderived and absolute derived categories.

Theorem. *The triangulated categories $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ coincide, i. e., every CDG-module over $\mathcal{B}$ that is coacyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ is also absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$.*

Proof. The reason for this assertion to be true is that the exact category of locally projective graded $\mathcal{B}$-modules has finite homological dimension [29, Lemma 1.12] and exact functors of infinite direct sums. If this exact category also had enough injectives, the simple argument from [37, Theorem 3.6(a) and Remark 3.6] would suffice to establish the desired $\mathsf{D}^{\mathsf{co}} = \mathsf{D}^{\mathsf{abs}}$ isomorphism for it (see also [36, Remark 2.1]). The lengthy argument below is designed to provide a way around the injective objects issue in this kind of proof.

Our aim is to show that for any closed morphism $\mathcal{P} \longrightarrow \mathcal{L}$ from a CDG-module $\mathcal{P} \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ to a CDG-module absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ there exists an exact sequence $0 \longrightarrow \mathcal{Q}_d \longrightarrow \mathcal{Q}_{d-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0 \longrightarrow \mathcal{P} \longrightarrow 0$ of CDG-modules and closed morphisms in $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ such that the induced morphism from the total CDG-module of $\mathcal{Q}_d \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$ to $\mathcal{L}$ is homotopic to zero. Here d is a fixed integer equal to the homological dimension of the exact category of locally projective graded $\mathcal{B}$-modules, which does not exceed the number of open subsets in an affine covering of X minus one.

Taking $\mathcal{P} = \mathcal{L}$ and the morphism $\mathcal{P} \longrightarrow \mathcal{L}$ to be the identity, we will then conclude that $\mathcal{P}$ is isomorphic to a direct summand of the total CDG-module of $\mathcal{Q}_d \longrightarrow \cdots \longrightarrow \mathcal{Q}_0 \longrightarrow \mathcal{P}$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$. Hence an object of $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ is absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ if and only if it is isomorphic to a direct summand of the total CDG-module of a $(d+2)$-term exact sequence of CDG-modules from $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ with closed morphisms between them. It will immediately follow that the class of CDG-modules absolutely acyclic with respect to $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ is closed under infinite direct sums, so it coincides with the class of coacyclic CDG-modules.

We can suppose that there exists a sequence of distinguished triangles $\mathcal{K}_{i-1} \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{K}_{i-1}[1]$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ such that $\mathcal{K}_0 = 0$, $\mathcal{K}_n = \mathcal{L}$, and $\mathcal{M}_i$ is the total CDG-module of an exact triple $\mathcal{U}_i \longrightarrow \mathcal{V}_i \longrightarrow \mathcal{W}_i$ of CDG-modules from $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ for all $1 \le i \le n$. We will start with constructing an exact sequence $0 \longrightarrow \mathcal{Q}'_n \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P} \longrightarrow 0$ with the above properties, but of the length n rather than d. Then we will use the finite homological dimension property of locally projective graded $\mathcal{B}$-modules in order to obtain the desired resolution $\mathcal{Q}_\bullet$ of a fixed length d from a resolution $\mathcal{Q}'_\bullet$.

Lemma G. *Let $\mathcal{M}$ be the total CDG-module of an exact triple $\mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ of CDG-modules from $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ and $\mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}[1]$ be a distinguished triangle in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$. Then for any CDG-module $\mathcal{P} \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ and a morphism $\mathcal{P} \longrightarrow \mathcal{L}$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ there exists an exact triple $\mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{P}$ of CDG-modules from $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ and a morphism $\mathcal{R}[1] \longrightarrow \mathcal{K}$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ such* *that the composition $\mathcal{F} \longrightarrow \mathcal{P} \longrightarrow \mathcal{L}$, where $\mathcal{F}$ is the cone of the closed morphism $\mathcal{R} \longrightarrow \mathcal{Q}$, is equal to the composition $\mathcal{F} \longrightarrow \mathcal{R}[1] \longrightarrow \mathcal{K} \longrightarrow \mathcal{L}$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$.*

Proof. The argument is based on Lemmas E–F from Section 1.5. We can assume that $\mathcal{L}$ is the cone of a closed morphism $\mathcal{M}[-1] \longrightarrow \mathcal{K}$ and fix a closed morphism $\mathcal{P} \longrightarrow \mathcal{L}$ representing the given morphism in the homotopy category. Arguing as in the proof of Proposition 1.5, we can construct a surjective closed morphism $\mathcal{Q}' \longrightarrow \mathcal{P}$ onto $\mathcal{P}$ from a CDG-module $\mathcal{Q}' \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ such that the composition $\mathcal{Q}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{W}[-1]$ lifts to a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}' \longrightarrow \mathcal{V}[-1]$. Here it suffices to apply the functor G^+ to the fibered product of the morphisms of graded $\mathcal{B}$-modules $\mathcal{P} \longrightarrow \mathcal{W}[-1]$ and $\mathcal{V}[-1] \longrightarrow \mathcal{W}[-1]$, and use Lemma F.

Then the morphism $\mathcal{Q}' \longrightarrow \mathcal{M}$ is homotopic to zero with a natural contracting homotopy (provided by the proof of Lemma E), so the morphism $\mathcal{Q}' \longrightarrow \mathcal{L}$ factorizes, up to a homotopy, as the composition of a naturally defined closed morphism $\mathcal{Q}' \longrightarrow \mathcal{K}$ and the closed morphism $\mathcal{K} \longrightarrow \mathcal{L}$. Set $\mathcal{Q}$ to be the cocone of the closed morphism $\mathcal{Q}' \longrightarrow \mathcal{K}$; then we have a surjective closed morphism $\mathcal{Q} \longrightarrow \mathcal{Q}'$ such that the composition $\mathcal{Q} \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{K}$ is homotopic to zero.

Let $\mathcal{R}$ be kernel of the morphism $\mathcal{Q} \longrightarrow \mathcal{P}$ and $\mathcal{F}$ be the cone of the morphism $\mathcal{R} \longrightarrow \mathcal{Q}$; then there is a natural closed morphism $\mathcal{F} \longrightarrow \mathcal{P}$. Using Lemma E and arguing as in the end of the proof of Proposition 1.5 again, we can conclude that the composition $\mathcal{F} \longrightarrow \mathcal{P} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M}$ is homotopic to zero. Indeed, the composition $\mathcal{F} \longrightarrow \mathcal{M} \longrightarrow \mathcal{W}[-1]$ lifts to a graded $\mathcal{B}$-module morphism $\mathcal{F} \longrightarrow \mathcal{V}[-1]$, since $\mathcal{F} \simeq \mathcal{Q} \oplus \mathcal{R}[-1]$ as a graded $\mathcal{B}$-module, the morphism $\mathcal{F} \longrightarrow \mathcal{M}$ factorizes through the projection of $\mathcal{F}$ onto $\mathcal{Q}$, and the morphism $\mathcal{Q} \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{W}[-1]$ lifts to a graded $\mathcal{B}$-module morphism $\mathcal{Q} \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{V}[-1]$ by our construction.

Notice that the contracting homotopy that we have obtained for the closed morphism $\mathcal{F} \longrightarrow \mathcal{M}$ forms a commutative diagram with the closed morphisms $\mathcal{Q} \longrightarrow \mathcal{F}$, $\mathcal{Q} \longrightarrow \mathcal{Q}'$, and the contracting homotopy that we have previously had for the closed morphism $\mathcal{Q}' \longrightarrow \mathcal{M}$ (since so do the liftings $\mathcal{F} \longrightarrow \mathcal{V}[-1]$ and $\mathcal{Q}' \longrightarrow \mathcal{V}[-1]$). This allows to factorize, up to a homotopy, the closed morphism $\mathcal{F} \longrightarrow \mathcal{L}$ as the composition of a closed morphism $\mathcal{F} \longrightarrow \mathcal{K}$ and the closed morphism $\mathcal{K} \longrightarrow \mathcal{L}$ in such a way that the morphism $\mathcal{F} \longrightarrow \mathcal{K}$ forms a commutative diagram with the closed morphisms $\mathcal{Q} \longrightarrow \mathcal{F}$, $\mathcal{Q} \longrightarrow \mathcal{Q}'$, and the closed morphism $\mathcal{Q}' \longrightarrow \mathcal{K}$ that we have previously constructed. The composition $\mathcal{Q} \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}$, being equal to the composition $\mathcal{Q} \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{K}$, is homotopic to zero; hence the morphism $\mathcal{F} \longrightarrow \mathcal{K}$ factorizes through the closed morphism $\mathcal{F} \longrightarrow \mathcal{R}[1]$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$. □

Applying Lemma G to the morphism $\mathcal{P} \longrightarrow \mathcal{L}$ and the distinguished triangle $\mathcal{K}_{n-1} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M}_n \longrightarrow \mathcal{K}_{n-1}$, we obtain an exact triple $\mathcal{R}'_0 \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P}$ and a morphism $\mathcal{R}'_0[1] \longrightarrow \mathcal{K}_{n-1}$ in $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$. Applying the same lemma again to the morphism $\mathcal{R}'_0[1] \longrightarrow \mathcal{K}_{n-1}$ and the distinguished triangle $\mathcal{K}_{n-2} \longrightarrow \mathcal{K}_{n-1} \longrightarrow \mathcal{M}_{n-1} \longrightarrow \mathcal{K}_{n-2}[1]$, we construct an exact triple $\mathcal{R}'_1 \longrightarrow \mathcal{Q}'_1 \longrightarrow \mathcal{R}'_0$ and a morphism $\mathcal{R}'_1[2] \longrightarrow \mathcal{K}_{n-2}$, etc. Finally we obtain an exact triple $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \mathcal{R}'_{n-2}$ and a morphism $\mathcal{R}'_{n-1}[n] \longrightarrow \mathcal{K}_0 = 0$.

Let us check that the natural morphism from the total CDG-module of the complex $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0$ to the CDG-module $\mathcal{L}$ is homotopic to zero. Denote this morphism by f_n. It factorizes naturally through the cone $\mathcal{F}_0$ of the closed morphism $\mathcal{R}'_0 \longrightarrow \mathcal{Q}'_0$, and the morphism $\mathcal{F}_0 \longrightarrow \mathcal{L}$ is homotopic to the composition $\mathcal{F}_0 \longrightarrow \mathcal{R}'_0[1] \longrightarrow \mathcal{K}_{n-1} \longrightarrow \mathcal{L}$. Hence, up to the homotopy, the morphism f_n factorizes through the morphism f_{n-1} from the the total CDG-module of the complex $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_1$ to $\mathcal{K}_{n-1}$ induced by the morphism $\mathcal{R}'_0[1] \longrightarrow \mathcal{K}_{n-1}$. Continuing to argue in this way, we conclude that the morphism f factorizes, up to a homotopy, through the morphism $f_0\colon \mathcal{R}'_{n-1}[n] \longrightarrow \mathcal{K}_0 = 0$.

It remains to "cut" our exact sequence of an unknown length n to a fixed size d. For this purpose, we will assume that $n > d$ and construct from our exact sequence of length n another exact sequence with the same properties, but of the length $n-1$. This part of the argument is based on the following lemma.

Lemma H. *For any CDG-module $\mathcal{M} \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$, locally projective graded $\mathcal{B}$-module $\mathcal{E}$, and a homogeneous surjective morphism of locally projective graded $\mathcal{B}$-modules $\mathcal{E} \longrightarrow \mathcal{M}$, there exist a CDG-module $\mathcal{Q} \in \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$, a surjective closed morphism of CDG-modules $\mathcal{Q} \longrightarrow \mathcal{M}$, and a homogeneous surjective morphism of locally projective graded $\mathcal{B}$-modules $\mathcal{Q} \longrightarrow \mathcal{E}$, such that the triangle $\mathcal{Q} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M}$ commutes.*

Proof. For any open subscheme $U \subset X$, one can simply define $\mathcal{Q}^i(U)$ as the abelian group of all pairs $(e' \in \mathcal{E}^{i+1}(U),\ e \in \mathcal{E}^i(U))$ such that $df(e) = f(e')$, where f denotes the morphism of graded $\mathcal{B}$-modules $\mathcal{E} \longrightarrow \mathcal{M}$ and d is the differential in $\mathcal{M}$. The action of $\mathcal{B}$ in $\mathcal{Q}$ is defined by the formula $b(e',e) = ((-1)^{|b|}be' + d(b)e,\ be)$; the differential in $\mathcal{Q}$ is given by the obvious rule $d(e',e) = (he, e')$. The morphism $\mathcal{Q} \longrightarrow \mathcal{E}$ is defined as $(e',e) \longmapsto e$; the morphism $\mathcal{Q} \longrightarrow \mathcal{M}$, given by $(e',e) \longmapsto f(e)$, obviously commutes with the differentials.

It remains to check that the graded $\mathcal{B}$-module $\mathcal{Q}$ is locally projective. This can be done by comparing the above construction with the constructions of the freely (co)generated CDG-modules $G^+(\mathcal{E})$ and $G^-(\mathcal{E})$ from [37, proof of Theorem 3.6] (see the beginning of the proof of Theorem 1.4). One can simply define $G^-(\mathcal{E})$ as being isomorphic to $G^+(\mathcal{E})[1]$. Since $\mathcal{M}$ is a CDG-module, there is a natural closed morphism of CDG-modules $\mathcal{M} \longrightarrow G^-(\mathcal{M})$. The CDG-module $\mathcal{Q}$ is the fibered product of the surjective closed morphism of CDG-modules $G^-(\mathcal{E}) \longrightarrow G^-(\mathcal{M})$ and the closed morphism $\mathcal{M} \longrightarrow G^-(\mathcal{M})$; hence the graded $\mathcal{B}$-module $\mathcal{Q}$ is locally projective. The morphism $\mathcal{Q} \longrightarrow \mathcal{E}$ is induced by the natural morphism of graded $\mathcal{B}$-modules $G^-(\mathcal{E}) \longrightarrow \mathcal{E}$. It forms a commutative diagram with the morphism $\mathcal{E} \longrightarrow \mathcal{M}$, since the composition $\mathcal{M} \longrightarrow G^-(\mathcal{M}) \longrightarrow \mathcal{M}$ is the identity morphism. □

The exact sequence of CDG-modules $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P} \longrightarrow 0$ represents a certain Yoneda Ext class of degree n between the locally projective graded $\mathcal{B}$-modules $\mathcal{P}$ and $\mathcal{R}'_{n-1}$. Since the homological dimension of the exact category of such $\mathcal{B}$-modules is equal to d and we assume that $n > d$, this Ext class has to vanish. This means that there exists an exact sequence of locally projective graded $\mathcal{B}$-modules $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{P} \longrightarrow 0$ mapping to our original exact sequence, with the maps on the rightmost and lefmost terms being the identity maps, such that the embedding of $\mathcal{B}$-modules $\mathcal{R}'_{n-1} \longrightarrow \mathcal{E}_{n-1}$ splits.

As explained in [38, proof of Lemma 4.4], one can assume the morphisms $\mathcal{E}_i \longrightarrow \mathcal{Q}'_i$ to be surjective. Applying Lemma H, we obtain a surjective closed morphism of CDG-modules $\mathcal{Q}_0 \longrightarrow \mathcal{Q}'_0$ and a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}_0 \longrightarrow \mathcal{E}_0$ forming a commutative triangle with the morphism $\mathcal{E}_0 \longrightarrow \mathcal{Q}'_0$. Applying Lemma H to the surjective morphism of fibered products $\mathcal{Q}_0 \times_{\mathcal{E}_0} \mathcal{E}_1 \longrightarrow \mathcal{Q}_0 \times_{\mathcal{Q}'_0} \mathcal{Q}'_1$, we obtain a surjective closed morphism $\mathcal{Q}_1 \longrightarrow \mathcal{Q}'_1$ and a closed morphism $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_0$ forming a commutative square with the closed morphisms $\mathcal{Q}_0 \longrightarrow \mathcal{Q}'_0$ and $\mathcal{Q}'_1 \longrightarrow \mathcal{Q}'_0$. Besides, the sequence $\mathcal{Q}_1 \longrightarrow \mathcal{Q}_0 \longrightarrow \mathcal{P}$ is exact at $\mathcal{Q}_0$. We also obtain a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}_1 \longrightarrow \mathcal{E}_1$ forming a commutative triangle with the morphisms to $\mathcal{Q}'_1$ and a commutative square with the morphisms to $\mathcal{E}_0$.

Proceeding in this way, we construct a sequence $\mathcal{Q}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0 \longrightarrow \mathcal{P} \longrightarrow 0$, which is exact at all the middle terms, maps onto the sequence $\mathcal{Q}'_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P}$ by closed morphisms, and maps into the sequence $\mathcal{E}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{P}$ so that the triangle of the maps of sequences commutes. Finally, notice that $\mathcal{E}_{n-1} \simeq \mathcal{E}_{n-2} \times_{\mathcal{Q}'_{n-2}} \mathcal{Q}'_{n-1}$, and set $\mathcal{Q}_{n-1} = \mathcal{Q}_{n-2} \times_{\mathcal{Q}'_{n-2}} \mathcal{Q}'_{n-1}$. Then the exact sequence of CDG-modules $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0 \longrightarrow \mathcal{P} \longrightarrow 0$ maps onto the exact sequence $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P} \longrightarrow 0$ by closed morphisms, and this map of exact sequences factorizes through the exact sequence of graded $\mathcal{B}$-modules $0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{E}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{P} \longrightarrow 0$. The composition of the morphism $\mathcal{Q}_{n-1} \longrightarrow \mathcal{E}_{n-1}$ with the splitting $\mathcal{E}_{n-1} \longrightarrow \mathcal{R}'_{n-1}$ of the embedding $\mathcal{R}'_{n-1} \longrightarrow \mathcal{E}_{n-1}$ provides a graded $\mathcal{B}$-module splitting $\mathcal{Q}_{n-1} \longrightarrow \mathcal{R}'_{n-1}$ of the embedding of CDG-modules $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}_{n-1}$.

Denote by $\mathcal{R}_{n-2}$ the image of the morphism of CDG-modules $\mathcal{Q}_{n-1} \longrightarrow \mathcal{Q}_{n-2}$. The morphism from the total CDG-module of the complex $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0$ to the CDG-module $\mathcal{L}$ is homotopic to zero, hence so is the morphism to $\mathcal{L}$ from the total CDG-module of the complex $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$. The latter morphism factorizes naturally through the total CDG-module of the complex $\mathcal{R}_{n-2} \longrightarrow \mathcal{Q}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$. The cone of this closed morphism between two total CDG-modules is homotopy equivalent to the total CDG-module of the exact triple $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}_{n-1} \longrightarrow \mathcal{R}_{n-2}$. Since this exact triple splits as an exact triple of graded $\mathcal{B}$-modules, its total CDG-module is contractible. Consequently, the morphism between the total CDG-modules of $\mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$ and $\mathcal{R}_{n-2} \longrightarrow \mathcal{Q}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$ is a homotopy equivalence.

It follows that the natural morphism from the total CDG-module of the resolution $\mathcal{R}_{n-2} \longrightarrow \mathcal{Q}_{n-2} \longrightarrow \cdots \longrightarrow \mathcal{Q}_0$ of the CDG-module $\mathcal{P}$ to the CDG-module $\mathcal{L}$ is homotopic to zero, and we are done. □

So far we have only considered flat coherent CDG-modules over quasi-coherent CDG-algebras $\mathcal{B}$ whose underlying quasi-coherent graded algebras are Noetherian. But the latter restriction is not necessary, as flat and locally finitely presented (or, which is equivalent, locally projective and finitely generated) quasi-coherent graded $\mathcal{B}$-modules always form an exact subcategory of flat (or locally projective) graded $\mathcal{B}$-modules. The notation $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{lp}}$ (understood in the obvious sense as the DG-category of CDG-modules over $\mathcal{B}$ with coherent and locally projective underlying graded $\mathcal{B}$-modules) is synonymous to $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}$ (see Remark 1.4).

Corollary. *The functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{lp}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{lp}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}$ *is fully faithful.*

Proof. When $\mathcal{B}$ is Noetherian, one can show that the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{lp}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}})$ is fully faithful by comparing parts (a-c) of Proposition 1.5 (with the flatness condition replaced by the local projectivity). In the general case, one proves this assertion directly, using an argument similar to the proof of Proposition 1.5(a-b). Then it remains to use the above Theorem. □

When every flat quasi-coherent graded module over $\mathcal{B}$ has finite locally projective dimension (see Remark 1.4), one has $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ by appropriate versions of Theorem 1.4. Consequently, it follows from Theorem above that $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ in this case. Thus the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ is fully faithful; when $\mathcal{B}$ is Noetherian, so is the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$.

1.7. **Gorenstein case.** Here we establish a sufficient condition for the functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ to be an equivalence of triangulated categories.

Let $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{inj}}$ denote the full DG-subcategory in $\mathcal{B}\text{–}\mathsf{qcoh}$ consisting of the CDG-modules whose underlying quasi-coherent graded $\mathcal{B}$-modules are injective. Furthermore, let $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}}$ be the full DG-subcategory in $\mathcal{B}\text{–}\mathsf{qcoh}$ consisting of the CDG-modules whose underlying quasi-coherent graded $\mathcal{B}$-modules have finite injective dimension (i. e., admit a finite right resolution by injective quasi-coherent graded $\mathcal{B}$-modules). Let $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}})$ denote the corresponding derived categories of the second kind. (The difficulty in the definition of the latter category, similar to the difficulty in the definition of $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ discussed in Section 1.3, does not actually arise, as it is clear from part (a) of the next lemma.)

Lemma. (a) *For any quasi-coherent CDG-algebra $\mathcal{B}$ over X, the natural functors* $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}})$ *are equivalences of triangulated categories.*

(b) *Let $\mathcal{B}$ be a quasi-coherent CDG-algebra over X whose underlying quasi-coherent graded algebra $\mathcal{B}$ is Noetherian. Then the functor* $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ *induced by the embedding* $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{inj}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

Proof. Part (a) is provided by [37, Theorem and Remark in Section 3.6]. Part (b) is a particular case of [37, Theorem and Remark in Section 3.7], since the class of injective quasi-coherent graded $\mathcal{B}$-modules is closed under infinite direct sums in its assumptions. (Cf. [22, Proposition 2.4].) □

Proposition. *Let $\mathcal{B}$ be a quasi-coherent CDG-algebra over X such that the quasi-coherent graded algebra $\mathcal{B}$ is Noetherian and the classes of quasi-coherent graded $\mathcal{B}$-modules of finite flat dimension and of finite injective dimension coincide. Then the functors* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ *induced by the embedding* $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ *are equivalences of triangulated categories.*

Proof. Since $\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}} = \mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}}$, the isomorphism of categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ follows from part (a) of Lemma. Applying Theorem 1.4, we obtain the isomorphism of categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$. Similarly, it suffices to compare parts (a) and (b) of Lemma in order to conclude that the functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ is an equivalence of categories, hence so are the functors $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$. (Cf. [37, Section 3.9].) ☐

1.8. **Pull-backs and push-forwards.** Let $f\colon Y \longrightarrow X$ be a morphism of separated Noetherian schemes with enough vector bundles, $\mathcal{B}_X$ be a quasi-coherent CDG-algebra over X, and $\mathcal{B}_Y$ a quasi-coherent CDG-algebra over Y. A *morphism of quasi-coherent CDG-algebras $\mathcal{B}_X \longrightarrow \mathcal{B}_Y$ compatible with the morphism $Y \longrightarrow X$* is the data of a CDG-ring morphism $\mathcal{B}_X(U) \longrightarrow \mathcal{B}_Y(V)$ for each pair of affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$. This data should satisfy the obvious compatibility condition: for any affine open subschemes $U' \subset U$ and $V' \subset V$ such that $f(V') \subset U'$, the square diagram of CDG-ring morphisms between the CDG-rings $\mathcal{B}_X(U)$, $\mathcal{B}_X(U')$, $\mathcal{B}_Y(V)$, and $\mathcal{B}_Y(V')$ must be commutative.

Let $\mathcal{B}_X \longrightarrow \mathcal{B}_Y$ be a morphism of quasi-coherent CDG-algebras compatible with a morphism of schemes $Y \longrightarrow X$. Then for any quasi-coherent left CDG-module $\mathcal{M}$ over $\mathcal{B}_X$ the quasi-coherent graded left module $f^*\mathcal{M} = \mathcal{B}_Y \otimes_{f^{-1}\mathcal{B}_X} f^{-1}\mathcal{M}$ over $\mathcal{B}_Y$ has a natural structure of quasi-coherent CDG-module over $\mathcal{B}_Y$. Similarly, for any quasi-coherent left CDG-module $\mathcal{N}$ over $\mathcal{B}_Y$ the quasi-coherent graded left module $f_*\mathcal{N}$ over $\mathcal{B}_X$ has a natural structure of quasi-coherent CDG-module over $\mathcal{B}_X$. These CDG-module structures are defined in terms of the CDG-ring morphisms $\mathcal{B}_X(U) \longrightarrow \mathcal{B}_Y(V)$. The above constructions provide the underived direct and inverse image functors, which can be viewed as triangulated functors $f^*\colon H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}) \longrightarrow H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ and $f_*\colon H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow H^0(\mathcal{B}_X\text{–}\mathsf{qcoh})$. The functor f_* is right adjoint to the functor f^*.

The derived inverse image functor $\mathbb{L}f^*$ is in general only defined on CDG-modules satisfying certain finite flat dimension conditions. Restricting the functor f^* to flat CDG-modules, we obtain a triangulated functor $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, which takes objects coacyclic with respect to $\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ to objects coacyclic with respect to $\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{fl}}$, since the inverse image preserves infinite direct sums and short exact sequences of flat quasi-coherent graded modules. Hence there is the induced triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{fl}})$. Applying Theorem 1.4(a), we construct the derived inverse image functor

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}}).$$

Restricting the functor f^* to flat coherent CDG-modules, we obtain a triangulated functor $H^0(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow H^0(\mathcal{B}_Y\text{–}\mathsf{coh}_{\mathsf{fl}})$, which induces a triangulated functor

$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{–}\mathsf{coh}_{\mathsf{fl}})$. Assuming that the quasi-coherent graded algebras $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian and applying Theorem 1.4(c), we construct the derived inverse image functor

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{–}\mathsf{coh}_{\mathsf{ffd}}).$$

When f is an affine morphism, the direct image of quasi-coherent sheaves is an exact functor (preserving also infinite direct sums), so the functor $f_*\colon H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow H^0(\mathcal{B}_X\text{–}\mathsf{qcoh})$ induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$. To construct the derived direct image functor between the coderived categories in the general case, we need to use injective resolutions.

From now on we assume that $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian; so Lemma 1.7(b) is applicable to $\mathcal{B}_Y$. Restricting the functor f_* to the full subcategory $H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \subset H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ and composing it with the localization functor $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$, we obtain the derived direct image functor

$$\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}).$$

Proposition. *The functors* $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{–}\mathsf{coh}_{\mathsf{ffd}})$ *and* $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$ *are "partially adjoint" to each other in the following sense: for any objects* $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{ffd}})$ *and* $\mathcal{N} \in \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ *there is a natural isomorphism of abelian groups*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})}(\iota_X\mathcal{M},\, \mathbb{R}f_*\mathcal{N}) \simeq \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh})}(\iota_Y\mathbb{L}f^*\mathcal{M},\, \mathcal{N}),$$

where $\iota_X\colon \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$ *and* $\iota_Y\colon \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ *are the natural fully faithful triangulated functors.*

Proof. The functors ι_X and ι_Y are fully faithful by Theorem 1.4(c) and Proposition 1.5(b, d). Using Theorem 1.4(c), let us assume that $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}})$. We can also assume that $\mathcal{N} \in H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{inj}})$.

Then the left-hand side is the (filtered) inductive limit of $\operatorname{Hom}_{H^0(\mathcal{B}_X\text{–}\mathsf{qcoh})}(\mathcal{M}'', f_*\mathcal{N})$ over all morphisms $\mathcal{M}'' \longrightarrow \mathcal{M}$ in $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh})$ with a cone coacyclic with respect to $\mathcal{B}_X\text{–}\mathsf{qcoh}$. According to the proofs of Proposition 1.5(b) and [37, Theorem 3.11.1], any morphism from $\mathcal{M}$ to an object coacyclic with respect to $\mathcal{B}_X\text{–}\mathsf{qcoh}$ factorizes through an object absolutely acyclic with respect to $\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}}$. Thus the above inductive limit coincides with the similar limit taken over all morphisms $\mathcal{M}' \longrightarrow \mathcal{M}$ in $H^0(\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}})$ with a cone absolutely acyclic with respect to $\mathcal{B}_X\text{–}\mathsf{coh}_{\mathsf{fl}}$.

By [37, Theorem 3.5(a), Remark 3.5, and Lemma 1.3], the right-hand side is isomorphic to $\operatorname{Hom}_{H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})}(f^*\mathcal{M},\mathcal{N})$ and to $\operatorname{Hom}_{H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})}(f^*\mathcal{M}',\mathcal{N})$, since the objects of $H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ are right orthogonal to any coacyclic objects in $H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})$. So the assertion follows from the adjointness of the functors f^* and f_* on the level of the homotopy categories of quasi-coherent CDG-modules. ☐

Remark. It is not immediately obvious from the above construction that the derived functor $\mathbb{R}f_*$ is compatible with the compositions, i. e., for $g\colon Z \longrightarrow Y$ and $f\colon Y \longrightarrow X$ one has $\mathbb{R}(fg)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_*$. The problem is that the direct image functor f_* does not preserve injectivity of quasi-coherent graded modules in general. When the derived direct image functors are adjoint to appropriately defined derived inverse images (see Section 1.9 below for some results of this kind), the problem reduces to checking that the derived inverse images are compatible with the compositions, which may be easier to see from our definitions.

One general approach to this problem is to replace injective quasi-coherent graded $\mathcal{B}$-modules with quasi-coherent graded $\mathcal{B}$-modules that are flabby as sheaves of graded abelian groups in our construction of the derived direct images. The class of flabby sheaves of abelian groups is closed under infinite direct sums, since the underlying topological space of the scheme is Noetherian; it is also always closed under extensions and cokernels of injective morphisms. Whenever the quasi-coherent graded algebra $\mathcal{B}$ is Noetherian, all injective quasi-coherent graded $\mathcal{B}$-modules are flabby by Theorem A.3. Therefore, the coderived category of flabby quasi-coherent CDG-modules over $\mathcal{B}$ is equivalent to the homotopy category $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ by a version of Lemma 1.7(b), hence it is also equivalent to the coderived category of all quasi-coherent CDG-modules $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ (cf. the proof of Proposition 1.7).

The direct images preserve exact triples of flabby sheaves, so derived direct images can be defined using flabby resolutions. The direct images also take flabby sheaves to flabby sheaves, hence the desired compatibility of their derived functors with the compositions of scheme morphisms follows.

Moreover, assuming additionally that the scheme has finite Krull dimension, the absolute derived category of flabby quasi-coherent CDG-modules is equivalent to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$ by a dual version of Theorem 1.4(b), as the "flabby dimension" of any quasi-coherent graded $\mathcal{B}$-module is finite. This allows to define the derived direct images on the absolute derived categories of quasi-coherent CDG-modules (another approach to this question is to use the construction from the proof of Proposition 1.9 below). Notice that all our constructions of derived inverse images are also applicable to the categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$.

Finally, let us point out that the functor $\mathbb{R}f_*$ has a right adjoint functor

$$f^!\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}).$$

Indeed, the triangulated category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ is compactly generated by Proposition 1.5(d), and the functor $\mathbb{R}f_*$ preserves infinite direct sums, since the class of injective quasi-coherent graded $\mathcal{B}_Y$-modules is closed under infinite direct sums, due to Noetherianness of $\mathcal{B}_Y$. So it remains to apply [26, Theorem 4.1].

1.9. **Morphisms of finite flat dimension.** Let $f\colon Y \longrightarrow X$ be a morphism of schemes as above, and $\mathcal{B}_X \longrightarrow \mathcal{B}_Y$ is a compatible morphism of quasi-coherent CDG-algebras. We will say that the quasi-coherent graded algebra $\mathcal{B}_Y$ *has finite flat dimension over* $\mathcal{B}_X$ if (the left derived functor of) the functor of inverse image f^* acting between the abelian categories of quasi-coherent graded modules over $\mathcal{B}_X$ and $\mathcal{B}_Y$ has finite homological dimension. Equivalently, for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ the graded right $\mathcal{B}_X(U)$-module $\mathcal{B}_Y(V)$ should have finite flat dimension.

A quasi-coherent graded $\mathcal{B}_X$-module is said to be *adjusted to* f^* if its derived inverse image under f, as an object of the derived category of the abelian category of quasi-coherent graded $\mathcal{B}_Y$-modules, coincides with the underived inverse image. Denote the DG-category of quasi-coherent CDG-modules over $\mathcal{B}_X$ whose underlying graded $\mathcal{B}_X$-modules are adjusted to f^* by $\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}}$. When $\mathcal{B}_X$ is Noetherian, let $\mathcal{B}_X\text{–}\mathsf{coh}_{f\text{–}\mathsf{adj}}$ denote the similarly defined DG-category of coherent CDG-modules. We will use our usual notation for the absolute derived and coderived categories of these DG-categories of CDG-modules.

Lemma. *Assume that the quasi-coherent graded algebra $\mathcal{B}_Y$ has finite flat dimension over $\mathcal{B}_X$. Then*

(a) *the functor* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}} \longrightarrow \mathcal{B}_X\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories;*

(b) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}} \longrightarrow \mathcal{B}_X\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories;*

(c) *if $\mathcal{B}_X$ is Noetherian, the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}_{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{–}\mathsf{coh}_{f\text{–}\mathsf{adj}} \longrightarrow \mathcal{B}_X\text{–}\mathsf{coh}$ *is an equivalence of triangulated categories.*

Proof. This is a version of Theorem 1.4, provable in the same way (cf. Corollary 2.6 below). The assertions hold, because any quasi-coherent graded $\mathcal{B}_X$-module has a finite left resolution consisting of quasi-coherent CDG-modules adjusted to f^*, and similarly for coherent CDG-modules. ☐

The functor of inverse image $f^*\colon H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}) \longrightarrow H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh})$ takes CDG-modules coacyclic with respect to $\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}}$ to CDG-modules coacyclic with respect to $\mathcal{B}_Y\text{–}\mathsf{qcoh}$, and hence induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh})$. Taking Lemma into account, we construct the derived inverse image functor

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}).$$

One shows that this functor is left adjoint to the functor $\mathbb{R}f_*$ constructed in 1.8 in the way analogous to (but simpler than) the proof of Proposition 1.8.

When $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian, we construct the derived inverse image functor

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{–}\mathsf{coh})$$

in the similar way.

Let $\mathcal{B}_X^{\mathrm{op}}$ and $\mathcal{B}_Y^{\mathrm{op}}$ denote the quasi-coherent graded algebras with the opposite multiplication to $\mathcal{B}_X$ and $\mathcal{B}_Y$.

Proposition. *When $\mathcal{B}_Y^{\mathrm{op}}$ has finite flat dimension over $\mathcal{B}_X^{\mathrm{op}}$, the derived inverse image functor* $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ *constructed in* 1.8 *has a right adjoint functor*

$$\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}).$$

Proof. Let $\{U_\alpha\}$ be a finite affine covering of Y. To any object $\mathcal{N} \in \mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$, assign the total CDG-module $\mathbb{R}_{\{U_\alpha\}}f_*\mathcal{N}$ of the finite Čech complex

$$\textstyle\bigoplus_\alpha f|_{U_\alpha *}(\mathcal{N}|_{U_\alpha}) \longrightarrow \bigoplus_{\alpha<\beta} f|_{U_\alpha\cap\mathcal{U}_\beta *}(\mathcal{N}|_{U_\alpha\cap U_\beta}) \longrightarrow \cdots$$

of CDG-modules over $\mathcal{B}_X$.

The terms of this complex belong to $\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$, since the morphism $f|_V\colon V \longrightarrow X$ is affine for any intersection V of a nonempty subset of affine open subschemes $U_\alpha \subset Y$ and the quasi-coherent graded algebra $\mathcal{B}_Y^{\mathrm{op}}$ has finite flat dimension over $\mathcal{B}_X^{\mathrm{op}}$. Hence one has $\mathbb{R}_{\{U_\alpha\}}f_*\mathcal{N} \in \mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$; it is clear that $\mathbb{R}_{\{U_\alpha\}}f_*$ is a DG-functor $\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}} \longrightarrow \mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ taking coacyclic objects to coacyclic objects. So we have the induced functor $\mathbb{R}f_*$ between the coderived categories.

It remains to obtain the adjuction isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathcal{M}, \mathbb{R}f_*\mathcal{N}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathbb{L}f^*\mathcal{M}, \mathcal{N})$$

for $\mathcal{M} \in \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$. Denote by $\mathcal{N}_+$ the total CDG-module of the finite complex

$$C^\bullet_{\{U_\alpha\}}\mathcal{N} = \big(\textstyle\bigoplus_\alpha j_{U_\alpha *}j^*_{U_\alpha}\mathcal{N} \longrightarrow \bigoplus_{\alpha<\beta} j_{U_\alpha\cap\mathcal{U}_\beta *}j^*_{U_\alpha\cap U_\beta}\mathcal{N} \longrightarrow \cdots\big)$$

of CDG-modules over $\mathcal{B}_Y$ (where $j_V\colon V \longrightarrow Y$ denotes the embedding of an affine open subscheme). Then we have $\mathbb{R}_{\{U_\alpha\}}f_*\mathcal{N} \simeq f_*\mathcal{N}_+$. There is a natural closed morphism $\mathcal{N} \longrightarrow \mathcal{N}_+$ of CDG-modules over $\mathcal{B}_Y$ with the cone coacyclic (and even absolutely acyclic) with respect to $\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$.

For any CDG-module $\mathcal{Q} \in \mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$, there is a natural map

$$\psi\colon \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathcal{M}, f_*\mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathbb{L}f^*\mathcal{M}, \mathcal{Q}).$$

Indeed, by (the proof of) Theorem 1.4(a), any morphism $\mathcal{M} \longrightarrow f_*\mathcal{Q}$ in $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ can be represented as a fraction formed by a morphism $\mathcal{M}' \longrightarrow \mathcal{M}$ in $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ with $\mathcal{M}' \in \mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ and a cone coacyclic with respect to $\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$, and a morphism $\mathcal{M}' \longrightarrow f_*\mathcal{Q}$ in $H^0(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$. To such a fraction, the map ψ assigns the related morphism $\mathbb{L}f^*\mathcal{M} = f^*\mathcal{M}' \longrightarrow \mathcal{Q}$.

For a fixed $\mathcal{M}$, the map ψ is a morphism of cohomological functors of the argument $\mathcal{Q} \in H^0(\mathcal{B}_Y\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$. Thus in order to show that it is an isomorphism for $\mathcal{Q} = \mathcal{N}_+$, it suffices to check that it is an isomorphism for $\mathcal{Q} = j_{V*}\mathcal{P}$ for every affine $V \subset Y$ and $\mathcal{P} \in \mathcal{B}_Y|_V\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$. This follows from the adjunction isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathcal{M}, f|_{V*}\mathcal{P}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y|_V\text{–}\mathsf{qcoh}_{\mathsf{ffd}})}(\mathbb{L}f|_V^*\mathcal{M}, \mathcal{P})$$

and the similar isomorphism for the embedding j_V, which hold because the functors $f|_{V*}$ and j_{V*} are exact, the morphisms $f|_V$ and j_V being affine. □

Remark. One can also use the above Čech complex approach in order to construct a version of the derived functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{–}\mathsf{qcoh})$. One can check that this construction agrees with the injective resolution construction from Section 1.8, using the fact that the restrictions of injective quasi-coherent graded $\mathcal{B}_Y$-modules to open subschemes are injective (Theorem A.3). Alternatively, in the assumption of finite flat dimension of $\mathcal{B}_Y$ over $\mathcal{B}_X$, one checks that both constructions provide functors right adjoint to $\mathbb{L}f^*$, hence they are isomorphic.

This allows to conclude that the derived functors $\mathbb{R}f_*$ acting on arbitrary quasi-coherent CDG-modules and quasi-coherent CDG-modules of finite flat dimension form a commutative diagram with the natural functors from the coderived categories of the latter to the coderived categories of the former.

1.10. **Supports of CDG-modules.** Let X be a Noetherian scheme. The *set-theoretic support* of a quasi-coherent sheaf $\mathcal{M}$ on X is the minimal closed subset $T \subset X$ such that the restriction of $\mathcal{M}$ to the open subscheme $X \setminus T$ vanishes. Given a Noetherian quasi-coherent graded algebra $\mathcal{B}$ over X and a quasi-coherent graded $\mathcal{B}$-module $\mathcal{M}$, the set-theoretic support $T = \operatorname{Supp} \mathcal{M}$ of $\mathcal{M}$ is defined similarly. It only depends on the underlying quasi-coherent $\mathcal{O}_X$-module of $\mathcal{M}$.

Let $\mathcal{B}$ be a quasi-coherent CDG-algebra over X whose underlying quasi-coherent graded algebra $\mathcal{B}$ is Noetherian. Fix a closed subset $T \subset X$. Denote by $\mathcal{B}\text{–}\mathsf{qcoh}_T$ the full DG-subcategory in $\mathcal{B}\text{–}\mathsf{qcoh}$ consisting of all the quasi-coherent CDG-modules whose underlying quasi-coherent graded $\mathcal{B}$-modules have their set-theoretic supports contained in T. The DG-category $\mathcal{B}\text{–}\mathsf{coh}_T$ of coherent CDG-modules with the set-theoretic support in T is defined similarly.

Let $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T)$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_T)$ denote the coderived and the absolute derived category of these DG-categories of CDG-modules. Finally, let $\mathcal{B}\text{–}\mathsf{qcoh}_{T,\mathsf{inj}}$ denote the DG-category of quasi-coherent CDG-modules over $\mathcal{B}$ whose underlying quasi-coherent graded modules are injective objects of the abelian category of quasi-coherent graded $\mathcal{B}$-modules with the set-theoretic support contained in T.

Proposition. (a) *The functor $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{T,\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T)$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}_{T,\mathsf{inj}} \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}_T$ is an equivalence of triangulated categories.*

(b) *The functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T)$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{coh}_T \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}_T$ is fully faithful and its image is a set of compact generators of the target category.*

(c) *The functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{qcoh}_T \longrightarrow \mathcal{B}\text{–}\mathsf{qcoh}$ is fully faithful.*

(d) *The functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ induced by the embedding of DG-categories $\mathcal{B}\text{–}\mathsf{coh}_T \longrightarrow \mathcal{B}\text{–}\mathsf{coh}$ is fully faithful.*

Proof. Part (a) is essentially a particular case of [37, Theorem and Remark in Section 3.7]. It is only important here that there are enough injective objects in the abelian category of quasi-coherent graded $\mathcal{B}$-modules supported set-theoretically in T and the class of such injective objects is closed under infinite direct sums. This is so because the abelian category in question is a locally Noetherian Grothendieck category (since X and $\mathcal{B}$ are Noetherian). Part (b) can be proven in the same way as the results of [37, Section 3.11]. Part (d) follows from parts (b-c) and Proposition 1.5(d).

Finally, part (c) follows from part (a), Lemma 1.7(b), and the fact that any injective object $\mathcal{J}$ in the category of quasi-coherent graded $\mathcal{B}$-modules supported set-theoretically in T is also an injective object in the category of arbitrary quasi-coherent graded $\mathcal{B}$-modules. The latter is essentially a reformulation of the Artin–Rees lemma.

Indeed, it suffices to check that for any coherent graded $\mathcal{B}$-module $\mathcal{M}$ and its coherent graded $\mathcal{B}$-submodule $\mathcal{N}$, any morphism of quasi-coherent graded $\mathcal{B}$-modules $\phi\colon \mathcal{N} \longrightarrow \mathcal{J}$ can be extended to $\mathcal{M}$. Let Z be a closed subscheme structure on the closed subset $T \subset X$. Then there is an integer $n \geq 0$ such that the morphism ϕ annihilates $\mathcal{I}_Z^n\mathcal{N}$ (where $\mathcal{I}_Z$ is the sheaf of ideals of the closed subscheme Z). By Lemma A.3, there exists $m \geq 0$ such that $\mathcal{I}_Z^m\mathcal{M} \cap \mathcal{N} \subset \mathcal{I}_Z^n\mathcal{N}$. Then there exists a morphism $\mathcal{M}/\mathcal{I}_Z^m\mathcal{M} \longrightarrow \mathcal{J}$ of quasi-coherent graded $\mathcal{B}$-modules supported set-theoretically in T which extends the given morphism into $\mathcal{J}$ from the quasi-coherent graded $\mathcal{B}$-submodule $\mathcal{N}/(\mathcal{I}_Z^m\mathcal{M} \cap \mathcal{N}) \subset \mathcal{M}/\mathcal{I}_Z^m\mathcal{M}$. ☐

Let $U \subset X$ denote the open subscheme $X \setminus T$.

Theorem. (a) *The functor of restriction to the open subscheme* $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(\mathcal{B}|_U\text{–}\mathsf{qcoh})$ *is the Verdier localization functor by the thick subcategory* $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T) \subset \mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$. *In particular, the kernel of the restriction functor coincides with the subcategory* $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T)$.

(b) *The functor of restriction to the open subscheme* $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^{abs}}(\mathcal{B}|_U\text{–}\mathsf{coh})$ *is the Verdier localization functor by the triangulated subcategory* $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh}_T) \subset \mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh})$. *In particular, the kernel of the restriction functor coincides with the thick envelope of (i. e., the minimal thick subcategory containing)* $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh}_T)$ *in* $\mathsf{D^{abs}}(\mathcal{B}\text{–}\mathsf{coh})$.

Proof. Let $j\colon U \longrightarrow X$ denote the natural open embedding. To prove part (a), consider the functor $\mathbb{R}j_*\colon \mathsf{D^{co}}(\mathcal{B}|_U\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ as constructed in Section 1.8. The quasi-coherent graded algebra $\mathcal{B}|_U$ being flat over $\mathcal{B}$, the functor $\mathbb{R}j_*$ is right adjoint to the restriction functor $j^*\colon \mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(\mathcal{B}|_U\text{–}\mathsf{qcoh})$. Obviously, the composition $j^*\mathbb{R}j_*$ is the identity functor. It follows that the functor j^* is a Verdier localization functor by its kernel, which is the full subcategory consisting of all the cones of the adjunction morphisms $\mathcal{M} \longrightarrow \mathbb{R}j_*j^*\mathcal{M}$, where $\mathcal{M} \in \mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$.

Represent the object $\mathcal{M}$ by a CDG-module with an injective underlying quasi-coherent graded $\mathcal{B}$-module. By Theorem A.3, the quasi-coherent graded $\mathcal{B}|_U$-module $j^*\mathcal{M}$ is then also injective, so we have $\mathbb{R}j_*j^*\mathcal{M} = j_*j^*\mathcal{M}$. Obviously, both the kernel and the cokernel of the closed morphism of CDG-modules $\mathcal{M} \longrightarrow j_*j^*\mathcal{M}$ belong to $\mathcal{B}\text{–}\mathsf{qcoh}_T$, and it follows, in view of part (c) of Proposition, that the cone also belongs to $\mathsf{D^{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_T)$.

To prove part (b), notice first that any coherent CDG-module over $\mathcal{B}|_U$ can be extended to a coherent CDG-module over $\mathcal{B}$ (because a coherent sheaf $\mathcal{K}$ on U can be extended to a coherent subsheaf of $j_*\mathcal{K}$), so the restriction functor is essentially surjective. Taking this observation into account, part (b) follows from part (a), part (b) of the above Proposition, Proposition 1.5(d), and the standard results about localization of compactly generated triangulated categories [25, Lemma 2.5 to Theorem 2.1]. ☐

Define the *category-theoretic support* $\operatorname{supp}\mathcal{M}$ of a quasi-coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ as the minimal closed subset $T \subset X$ such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is a coacyclic CDG-module over $\mathcal{B}|_U$. In other words,

$X \setminus \operatorname{supp} \mathcal{M}$ is the union of all open subschemes $V \subset X$ such that $\mathcal{M}|_V$ is a coacyclic CDG-module over $\mathcal{B}|_V$ (see Remark 1.3). Obviously, one has $\operatorname{supp} \mathcal{M} \subset \operatorname{Supp} \mathcal{M}$.

The category-theoretic support of a coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ can be equivalently defined as the minimal closed subset $T \subset X$ such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is absolutely acyclic. Indeed, any CDG-module from $\mathcal{B}|_U$–coh that is coacyclic with respect to $\mathcal{B}|_U$–qcoh is also absolutely acyclic with respect to $\mathcal{B}|_U$–coh by Proposition 1.5(d).

Corollary. (a) *For any quasi-coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ with the category-theoretic support* $\operatorname{supp} \mathcal{M}$ *contained in T, there exists a quasi-coherent CDG-module $\mathcal{M}'$ over $\mathcal{B}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{M}'$ in* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ *and the set-theoretic support* $\operatorname{Supp} \mathcal{M}'$ *is contained in T.*

(b) *For any coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ with the category-theoretic support* $\operatorname{supp} \mathcal{M}$ *contained in T, there exists a coherent CDG-module $\mathcal{M}'$ over $\mathcal{B}$ such that $\mathcal{M}$ is isomorphic to a direct summand of $\mathcal{M}'$ in* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ *and the set-theoretic support* $\operatorname{Supp} \mathcal{M}'$ *is contained in T.*

Proof. Follows immediately from Theorem. □

Remark. One can prove that the restriction functor in part (a) of Theorem is a Verdier localization functor without assuming the quasi-coherent graded algebra $\mathcal{B}$ to be Noetherian. Indeed, one can construct a right adjoint functor $\mathbb{R}j_*$ to the restriction functor j^* in the way similar to that of Proposition 1.9; then it is easy to see that $j^*\mathbb{R}j_*$ is the identity functor.

When $\mathcal{B}$ is Noetherian, the above Theorem can be generalized as follows. Let S and T be closed subsets in X; set $U = X \setminus T$. Then the restriction functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_S) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}|_U\text{–}\mathsf{qcoh}_{U\cap S})$ is the Verdier localization functor by the thick subcategory $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh}_{T\cap S})$, and the restriction functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_S) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}|_U\text{–}\mathsf{coh}_{U\cap S})$ is the Verdier localization functor by the triangulated subcategory $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{T\cap S})$. The proof is similar to the above.

It is not difficult to deduce from the latter assertions, using the result of [26, Theorem 2.1(5)], that the property of an object of $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$ to belong to the thick envelope of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ is local in X. Using the Čech exact sequence as in Remark 1.3, one can easily see that the property of an object of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$ to belong to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ is also local.

We do *not* know whether the property of an object of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh})$ or $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ to belong to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{coh}_{\mathsf{fl}})$ is local in general. In the particular case of matrix factorizations, such results will be proven in Section 3.1 using the connection with singularity categories (cf. Remarks 3.1 and 3.5).

2. Triangulated Categories of Relative Singularities

2.1. **Relative singularity category.** Recall that X denotes a separated Noetherian scheme with enough vector bundles. The *triangulated category of singularities*

$\mathsf{D}^{\mathsf{b}}_{Sing}(X)$ of the scheme X is defined [29, Section 1.2] as the quotient category of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ of coherent sheaves on X by its thick subcategory $Perf(X)$ of perfect complexes on X.

The perfect complexes, in our assumptions, can be simply defined as bounded complexes of locally free sheaves of finite rank, so $Perf(X) = \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_{\mathsf{lf}})$ is the bounded derived category of the exact category $X\text{–}\mathsf{coh}_{\mathsf{lf}}$ of locally free sheaves of finite rank on X. Equivalently, the perfect complexes are the compact objects of the unbounded derived category of quasi-coherent sheaves $\mathsf{D}(X\text{–}\mathsf{qcoh})$ on the scheme X [26, Examples 1.10–1.11 and Corollary 2.3].

Let $Z \subset X$ be a closed subscheme such that $\mathcal{O}_Z$ has finite flat dimension as an $\mathcal{O}_X$-module. In this case the derived inverse image functor $\mathbb{L}i^*$ for the closed embedding $i\colon Z \longrightarrow X$ acts on the bounded derived categories of coherent sheaves, $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$. We call the quotient category of $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ by the thick subcategory generated by the objects in the image of this functor the *triangulated category of singularities of Z relative to X* and denote it by $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$.

Note that the triangulated category of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is a quotient category of the conventional (absolute) triangulated category of singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ of the scheme Z. Indeed, the thick subcategory $Perf(Z) \subset \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ is generated by any ample family of vector bundles on Z, since any such family is a set of compact generators of the unbounded derived category of quasi-coherent sheaves $\mathsf{D}(Z\text{–}\mathsf{qcoh})$ on Z [26]; in particular, it is generated by the restrictions to Z of vector bundles from X (see also Lemma 2.8).

The functor $\mathbb{L}i^*\colon \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ induces a triangulated functor $i^\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z)$. Furthermore, since the sheaf $i_*\mathcal{O}_Z$ belongs to $Perf(X)$, the functor $i_*\colon \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ takes $Perf(Z)$ to $Perf(X)$ (cf. [29, paragraphs before Proposition 1.14]). Hence the functor i_* induces a triangulated functor $i_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ right adjoint to i°. The triangulated category $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is the quotient category of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ by the thick subcategory generated by the image of the functor i°.

When X is regular, any coherent sheaf on X has a finite resolution by locally free sheaves of finite rank. So $\mathsf{D}^{\mathsf{b}}_{Sing}(X) = 0$, hence the triangulated categories $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ coincide. The converse is also true: the structure sheaf of the reduced scheme structure on the closure of any singular point of X is not a perfect complex on X, so $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \neq 0$ when X is not regular.

Remark. Roughly speaking, the triangulated category of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ measures how much worse are the singularities of Z compared to the singularities of X in a neighborhood of Z.

The basic formal properties of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ are similar to those of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$. When the $\mathcal{O}_X$-module $\mathcal{O}_Z$ has finite flat dimension, the derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ is generated by coherent sheaves adjusted to i^*. Let $\mathsf{E}_{Z/X}$ denote the minimal full subcategory of the abelian category of coherent sheaves on Z containing the restrictions of such coherent sheaves from X and closed under extensions and the kernels of epimorphisms of sheaves. Then $\mathsf{E}_{Z/X}$ is naturally an exact category and its bounded derived category $\mathsf{D^b}(\mathsf{E}_{Z/X})$ is equivalent to the thick subcategory of $\mathsf{D^b}(Z\text{–}\mathsf{coh})$ generated by the derived restrictions of coherent sheaves from X, so $\mathsf{D^b_{\mathit{Sing}}}(Z/X) = \mathsf{D^b}(Z\text{–}\mathsf{coh})/\mathsf{D^b}(\mathsf{E}_{Z/X})$. One can define the E-homological dimension of a coherent sheaf (or bounded complex) on Z as the minimal length of a left resolution consisting of objects from $\mathsf{E}_{Z/X}$. This dimension does not depend on the choice of a resolution (in the same sense as the conventional flat dimension doesn't). The thick subcategory $\mathsf{D^b}(\mathsf{E}_{Z/X})$ consists of those objects of $\mathsf{D^b}(Z\text{–}\mathsf{coh})$ that have finite E-homological dimensions.

Unlike in the case of perfect complexes, we do not know whether the property to belong to $\mathsf{E}_{Z/X}$ or $\mathsf{D^b}(\mathsf{E}_{Z/X})$ is local, though. In the case when Z is a Cartier divisor, locality can be established using Theorem 2.7 below and Remark 1.3.

2.2. **Matrix factorizations.** Following [33], we will consider matrix factorizations of a global section of a line bundle. So let $\mathcal{L}$ be a line bundle (invertible sheaf) on X and $w \in \mathcal{L}(X)$ be a fixed section, called the *potential*.

Let $\mathcal{B} = (X, \mathcal{L}, w)$ denote the following $\mathbb{Z}$-graded quasi-coherent CDG-algebra over X. The component $\mathcal{B}^n$ is isomorphic to $\mathcal{L}^{\otimes n/2}$ for $n \in 2\mathbb{Z}$ and vanishes for $n \in 2\mathbb{Z}+1$, the multiplication in $\mathcal{B}$ being given by the natural isomorphisms $\mathcal{L}^{\otimes n/2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m/2} \longrightarrow \mathcal{L}^{\otimes (n+m)/2}$. For any affine open subscheme $U \subset X$, the differential on $\mathcal{B}(U)$ is zero, and the curvature element is $w|_U \in \mathcal{B}^2(U) = \mathcal{L}(U)$. The elements a_{UV} defining the restriction morphisms of CDG-rings $\mathcal{B}(V) \longrightarrow \mathcal{B}(U)$ all vanish.

The category of quasi-coherent $\mathbb{Z}$-graded $\mathcal{B}$-modules is equivalent to the category of quasi-coherent $\mathbb{Z}/2$-graded $\mathcal{O}_X$-modules, the equivalence assigning to a graded $\mathcal{B}$-module $\mathcal{M}$ the pair of $\mathcal{O}_X$-modules which we denote symbolically by $\mathcal{U}^0 = \mathcal{M}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} = \mathcal{M}^1$. Conversely, $\mathcal{M}^n \simeq \mathcal{U}^{n \bmod 2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n/2}$ for all $n \in \mathbb{Z}$ (the meaning of the notation in the right-hand side being the obvious one). This equivalence of abelian categories preserves all the properties of coherence, flatness, flat dimension, local projectivity/localy freeness, etc. that we have been interested in in Section 1.

Following [22], we will consider CDG-modules over $\mathcal{B} = (X, \mathcal{L}, w)$ whose underlying graded $\mathcal{B}$-modules correspond to coherent or quasi-coherent $\mathcal{O}_X$-modules, rather than just locally free sheaves (as in the conventional matrix factorizations). A quasi-coherent CDG-module over $(X, \mathcal{L}, w)$ is the same thing as a pair of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{U}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ endowed with $\mathcal{O}_X$-linear morphisms $\mathcal{U}^0 \longrightarrow \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \longrightarrow \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ such that both compositions $\mathcal{U}^0 \longrightarrow \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \longrightarrow \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \longrightarrow \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L} \longrightarrow \mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes 3/2}$ are equal to the multiplications with w.

2.3. **Exotic derived categories of matrix factorizations.** The following corollary is a restatement of the results of Section 1 in the application to the quasi-coherent CDG-algebra $\mathcal{B} = (X, \mathcal{L}, w)$. We will use the notation $(X, \mathcal{L}, w)\text{–}\mathsf{coh_{lf}}$ (instead ot the previously introduced $\mathcal{B}\text{–}\mathsf{coh_{fl}}$) for the DG-category of locally free matrix factorizations of finite rank, and the notation $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh_{lf}}$ (instead of the previously introduced $\mathcal{B}\text{–}\mathsf{qcoh_{lp}}$) for the DG-category of locally free matrix factorizations of possibly infinite rank (see Remark 1.4). The rest of our notation system for various classes of quasi-coherent CDG-modules over $\mathcal{B} = (X, \mathcal{L}, w)$ remains in use.

In addition, we also denote by $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}}$ the DG-category of quasi-coherent CDG-modules of finite locally free/locally projective dimension over $(X, \mathcal{L}, w)$ (see Remark 1.4 again). Let $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ be the corresponding derived categories of the second kind.

Corollary. (a) *The functor* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(b) *The functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(c) *The functors* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}}$ *are equivalences of triangulated categories.*

(d) *The triangulated categories* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ *coincide, as do the categories* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$. *The natural functors between these four categories form a commutative square of equivalences of triangulated categories.*

(e) *When the scheme X has finite Krull dimension, the functors* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ *are equivalences of triangulated categories. The natural functors between these four categories form a commutative square of equivalences.*

(f) *When the scheme X has finite Krull dimension, the triangulated category* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *coincides with* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *and the triangulated category* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ *coincides with* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$. *The natural functors between these four categories form a commutative square of equivalences.*

(g) *The functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(h) *The triangulated functors* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ *induced by the embeddings of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}$ *are fully faithful.*

(i) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{coh}$ *is fully faithful.*

(j) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}$ *is fully faithful.*

(k) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{coh} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}$ *is fully faithful.*

(l) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{coh} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}$ *is fully faithful and its image forms a set of compact generators for* $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$.

Proof. Parts (a-b) and (g) are particular cases of Theorem 1.4, and the proof of part (c) is similar (see Remark 1.4). Part (g) also essentially follows from Proposition 1.5(b) (and part (b) can be proven similarly). Parts (h-i) and (k-l) are particular cases of Proposition 1.5 (except for "locally free half" of part (h), which is similar to the "flat half"). Part (d) is Theorem 1.6 together with part (c). Part (j) is Corollary 1.6. Part (e) follows from parts (a-c) and Remark 1.4 (cf. the discussion in the end of Section 1.6). Part (f) follows from parts (a-b) and (d-e); alternatively, it can be proven directly in the way similar to part (d), using the fact that the exact category of flat quasi-coherent sheaves on X has finite homological dimension when the Krull dimension of X is finite. ☐

2.4. **Regular and Gorenstein scheme cases.** When the scheme X is regular or Gorenstein, the assertions of Corollary 2.3 simplify as follows.

Corollary. (a) *When the scheme* X *is Gorenstein of finite Krull dimension, the functors* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}$ *are equivalences of triangulated categories.*

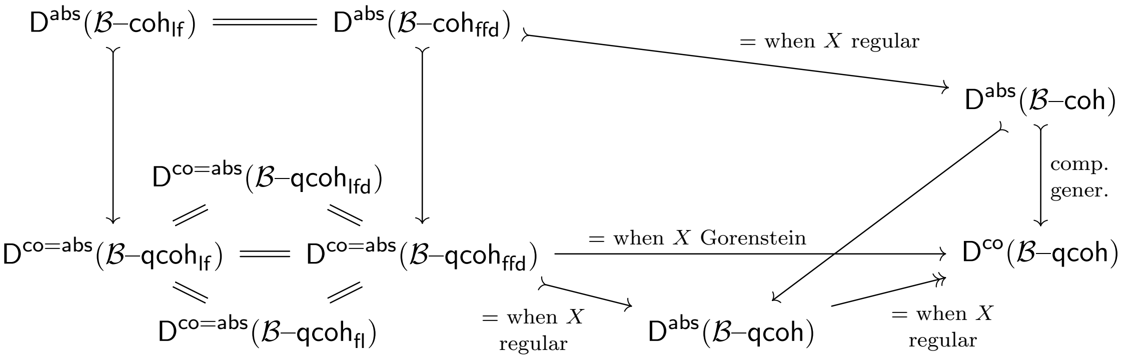
(b) *When the scheme* X *is regular of finite Krull dimension, the natural functors between the categories* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$, *and* $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ *form a commutative square of equivalences of triangulated categories.*

(c) *When the scheme* X *is regular, the natural functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ *is an equivalence of triangulated categories.*

Proof. Part (a) is a particular case of Proposition 1.7. Part (c) follows from Corollary 2.3(g), since any coherent sheaf on a regular scheme has finite flat dimension. In the assumptions of part (b), the functor $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ is an isomorphism of triangulated categories by [37, Theorem 3.6(a) and Remark 3.6], since the abelian category of quasi-coherent sheaves on a regular scheme of finite Krull dimension has finite homological dimension and enough injectives (cf. Theorem 1.6). The remaining assertions of part (b) follow from Corollary 2.3(a-b), or alternatively from part (a). ☐

Assuming that X has finite Krull dimension, the assertions of Corollaries 2.3–2.4 may be summarized by the following commutative diagram of triangulated functors.

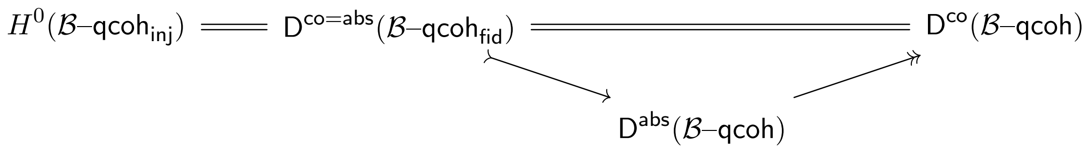
Here, as above, $\mathcal{B}$ denotes the quasi-coherent CDG-algebra $(X, \mathcal{L}, w)$:



The four categories in the left lower area are coderived categories coinciding with absolute derived categories (of the same classes of quasi-coherent CDG-modules). The five double lines between these four categories are equivalences, as is the upper left horizontal line. All the arrows going down are fully faithful functors. The image of the rightmost vertical arrow is a set of compact generators in the target category. The only arrow going up is a Verdier localization functor.

Nothing is claimed about the long horizontal arrow in the right lower area of the diagram in general; but when X is Gorenstein, this functor is an equivalence of categories. When X is regular, all the arrows going right are equivalences of categories (so the whole diagram reduces to one triangulated category with infinite direct sums, containing a full triangulated subcategory of compact generators).

Recall also that, by Lemma 1.7, for any X we have a commutative diagram of triangulated functors



with equivalences of categories in the upper line. The fully faithful embedding $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fid}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$, which in the Gorenstein case (of finite Krull dimension) coincides with the embedding $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh})$, is always right adjoint to the localization functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{–}\mathsf{qcoh})$.

Remark. When X is an affine Noetherian scheme of finite Krull dimension, the embeddings of DG-categories $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lp}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ induce equivalences $H^0(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{B}\text{–}\mathsf{qcoh})$ between the homotopy category of (locally) projective matrix factorizations of infinite rank, the absolute derived category of flat matrix factorizations, and the contraderived category of arbitrary quasi-coherent matrix factorizations (see [37, Section 3.8]; cf. Remark 1.5).

2.5. **Serre–Grothendieck duality.** The aim of this section is to show that the somewhat misterious long horizontal arrow in the above large diagram is actually a functor between two equivalent triangulated categories, for a rather wide class of schemes X. The functor $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ in the above diagram, which is induced by the embedding of DG-categories $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}$, is *not* the equivalence that we have in mind, however (unless the scheme is Gorenstein). Instead, the equivalence between the categories $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ is constructed using a dualizing complex on X [14, Section V.2].

Before recalling the definition of a dualizing complex, let us discuss the notion of the *quasi-coherent internal Hom.* Given quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ over X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{N})$ is defined by the isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(-\otimes_{\mathcal{O}_X}\mathcal{M},\ \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(-,\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{N}))$ of functors from the category of quasi-coherent sheaves to the category of abelian groups. Equivalently, the quasi-coherent sheaf $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{N})$ can be obtained by applying the coherator functor [42, Sections B.12–B.14] to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$. Whenever $\mathcal{M}$ is a coherent sheaf, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ of $\mathcal{O}_X$-module internal Hom is quasi-coherent, and $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{N}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$.

Notice that the construction of the sheaf $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{N})$ is *not* local in general, i. e., it does not commute with the restrictions of quasi-coherent sheaves to open subschemes; when the sheaf $\mathcal{M}$ is coherent, it does.

Lemma. (a) *For any injective quasi-coherent sheaf $\mathcal{J}$ over a separated Noetherian scheme X, the functor $\mathcal{M} \longmapsto \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M},\mathcal{J})$ is exact.*

(b) *For any flat quasi-coherent sheaf $\mathcal{F}$ and injective quasi-coherent sheaf $\mathcal{J}$ over X, the quasi-coherent sheaves $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{J}$ and $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{F},\mathcal{J})$ are injective.*

(c) *For any injective quasi-coherent sheaves $\mathcal{J}'$ and $\mathcal{J}$ over X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{J}',\mathcal{J})$ is flat.*

Proof. The second assertion of part (b) is obvious from the universal property defining $\mathcal{H}om_{X\text{–}\mathsf{qc}}$. To prove the first one, notice that injectivity of quasi-coherent sheaves over a Noetherian scheme is a local property ([14, Lemma II.7.16 and Theorem II.7.18] or Theorem A.3), a flat quasi-coherent sheaf over an affine scheme is a filtered inductive limit of locally free sheaves of finite rank [3, No. 1.5–6], and injectivity of modules over a Noetherian ring is preserved by filtered inductive limits.

The proof of parts (a) and (c) follows the argument in [23, Lemma 8.7]. Choose a finite affine covering U_α of the scheme X and consider the morphism $\mathcal{J} \longrightarrow \bigoplus_\alpha j_{U_\alpha *}j_{U_\alpha}^*\mathcal{J}$. Being an embedding of injective quasi-coherent sheaves, it splits, so $\mathcal{J}$ is a direct summand of the direct sum of $j_{U_\alpha *}j_{U_\alpha}^*\mathcal{J}$. Hence it suffices to prove both assertions in the case when $\mathcal{J} = j_{V*}\mathcal{J}''$, where $\mathcal{J}''$ is an injective quasi-coherent sheaf on an affine open subscheme $V \subset X$.

Now we have $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, j_{V*}\mathcal{J}'') \simeq j_{V*}\,\mathcal{H}om_{V\text{–}\mathsf{qc}}(j_V^*\mathcal{M},\mathcal{J}'')$. Since $V \longrightarrow X$ is a flat affine morphism, the functor j_{V*} is exact and preserves flatness of quasi-coherent sheaves. This proves part (a), and reduces part (c) to the case of an affine scheme $X = V$. Then it remains to apply [5, Proposition VI.5.3]. ☐

For our purposes, a *dualizing complex* $\mathcal{D}_X^\bullet$ on X is a finite complex of injective quasi-coherent sheaves such that the cohomology sheaves of $\mathcal{D}_X^\bullet$ are coherent and for any coherent sheaf $\mathcal{M}$ over X the natural morphism of finite complexes of quasi-coherent sheaves $\mathcal{M} \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}^\bullet_X), \mathcal{D}^\bullet_X)$ is a quasi-isomorphism. Note that it follows from the former two conditions on $\mathcal{D}^\bullet_X$ that the complex $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}^\bullet_X)$ has coherent cohomology sheaves. This makes the conditions imposed on $\mathcal{D}^\bullet_X$ actually local in X, so the restriction $\mathcal{D}^\bullet_U = \mathcal{D}^\bullet_X|_U$ of the complex of sheaves $\mathcal{D}^\bullet_X$ to an open subscheme $U \subset X$ is a dualizing complex on U.

Given a quasi-coherent CDG-algebra $\mathcal{B}$ over X, a quasi-coherent left CDG-module $\mathcal{M}$ over $\mathcal{B}$, and a complex of quasi-coherent sheaves $\mathcal{F}^\bullet$ on X, one can consider the complexes of quasi-coherent left CDG-modules $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$ and $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{F}^\bullet, \mathcal{M})$ over $\mathcal{B}$. Taking their totalizations (formed, if necessary, by taking infinite direct sums along the diagonals), one constructs two triangulated functors $H^0(\mathcal{B}\text{–}\mathsf{qcoh}) \longrightarrow H^0(\mathcal{B}\text{–}\mathsf{qcoh})$ depending on a complex $\mathcal{F}^\bullet$. Given a right CDG-module $\mathcal{N}$ over $\mathcal{B}$ (see [37, Sections 3.1 and B.1]), one can similarly construct a complex of quasi-coherent left CDG-modules $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{N}, \mathcal{F}^\bullet)$ over $\mathcal{B}$, obtaining a triangulated functor from the homotopy category of right CDG-modules $H^0(\mathsf{qcoh}\text{–}\mathcal{B})$ to $H^0(\mathcal{B}\text{–}\mathsf{qcoh})$.

In the particular case of matrix factorizations, we conclude that the covariant functors $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} -$ and $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{F}^\bullet, -)$ take quasi-coherent matrix factorizations of a potential $w \in \mathcal{L}(X)$ to (complexes of) quasi-coherent matrix factorizations of w, while the contravariant functor $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-, \mathcal{F}^\bullet)$ transforms quasi-coherent matrix factorizations of the opposite potential $-w \in \mathcal{L}(X)$ into (complexes of) quasi-coherent matrix factorizations of w. Of course, the quasi-coherent CDG-algebras $(X, \mathcal{L}, w)$ and $(X, \mathcal{L}, -w)$ over a scheme X are naturally isomorphic, but we prefer to keep the distinction between the two.

The next proposition provides the matrix factorization version of the conventional (contravariant) Serre–Grothendieck duality for bounded complexes of coherent sheaves. We assume that X is a separated Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet_X$. Recall that any such scheme has finite Krull dimension [14, Corollary V.7.2].

Proposition. *The triangulated functor* $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-, \mathcal{D}^\bullet_X)\colon H^0((X, \mathcal{L}, -w)\text{–}\mathsf{qcoh}) \longrightarrow H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ *induces a well-defined triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w)\text{–}\mathsf{qcoh} \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ *taking the full triangulated subcategory* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w)\text{–}\mathsf{qcoh})$ *into the full subcategory* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$*. The composition of the duality functors* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$ *is the identity functor.*

Proof. The functor $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-, \mathcal{D}^\bullet_X)$ preserves absolute acyclicity, because $\mathcal{D}^\bullet_X$ is a complex of injective quasi-coherent sheaves, so part (a) of Lemma applies. Given a coherent matrix factorization $\mathcal{M}$, the finite complex of matrix factorizations $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-, \mathcal{D}^\bullet_X)$ has coherent cohomology matrix factorizations, so one can use its canonical truncations in order to prove by induction that its totalization belongs to the triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$.

Finally, for any quasi-coherent matrix factorization $\mathcal{M}$ consider the bicomplex of matrix factorizations $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}^\bullet_X), \mathcal{D}^\bullet_X)$ and take its totalization in the two directions where it is a complex, obtaining a complex of matrix factorizations. Then there is a natural morphism of finite complexes of matrix factorizations

$\mathcal{M} \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}_X^\bullet), \mathcal{D}_X^\bullet)$, which is a quasi-isomorphism of complexes of matrix factorizations when $\mathcal{M}$ is coherent. The induced closed morphism of the total matrix factorizations is an isomorphism in $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$, since the totalization of a finite acyclic complex of matrix factorizations is absolutely acyclic. It remains to use the fact that the functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ is fully faithful (see Corollary 2.3(k)) again. □

The next result is our covariant Serre–Grothendieck duality theorem for matrix factorizations. It is the matrix factorization analogue of the similar results for complexes of projective and injective modules [17, Theorem 4.2] and sheaves [23, Theorem 8.4]. It also strongly resembles the *derived comodule-contramodule correspondence* theory (see [37, Theorem 5.2], [36, Corollaries 5.4 and 6.3]; cf. Remark 2.4 above). Notice that our proof is more akin to the arguments in [37, 36] than those of [17, 23] in that we give a direct proof of the covariant duality independent of both the contravariant duality and any descriptions of the compact objects in the categories to be compared.

Theorem. *The functors* $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -\colon H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ *and* $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet, -)\colon H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \longrightarrow H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *induce mutually inverse equivalences between the coderived categories* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ *and* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$.

Proof. Recall that $H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ by Lemma 1.7(b) and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ by Corollary 2.3(f) (though we will reprove the latter fact rather than use it in the following argument; see also Remark 2.6 below and Lemma A.1). The functor $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -\colon H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ obviously takes matrix factorizations coacyclic with respect to $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}$ to matrix factorizations coacyclic with respect to $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}}$, which are all contractible. It remains to check that the induced functors are mutually inverse.

Let $\mathcal{E}$ be a matrix factorization from $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}$. As in the previous proof, consider the bicomplex of matrix factorizations $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$ and take its total complex of matrix factorizations. Then there is a natural morphism $\mathcal{E} \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$ of finite complexes of matrix factorizations from $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}$. To prove that the induced morphism of the total matrix factorizations is an isomorphism in $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$, we once again use the fact that the totalization of a finite acyclic complex of matrix factorizations is absolutely acyclic. So it suffices to check that for any flat quasi-coherent sheaf $\mathcal{F}$ over X the natural morphism $\mathcal{F} \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism of complexes of flat quasi-coherent sheaves. This will be done below.

Similarly, let $\mathcal{M}$ be a matrix factorization from $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{inj}}$. Consider the morphism of finite complexes of injective matrix factorizations $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}} (\mathcal{D}_X^\bullet, \mathcal{M}) \longrightarrow \mathcal{M}$. To prove that the cone of the induced morphism of the total matrix factorizations is contractible, it suffices to check that for any injective quasi-coherent sheaf $\mathcal{J}$ over X the natural morphism of complexes of injective sheaves $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$ is a quasi-isomorphism.

Let ${}'\mathcal{D}_X^\bullet$ denote a finite complex of coherent sheaves over X endowed with a quasi-isomorphism ${}'\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet$. Then the morphism $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet,\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}) \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet,\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism for any flat quasi-coherent sheaf $\mathcal{F}$. The construction of the composition $\mathcal{F} \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet,\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}) \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet,\, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is local in X, so it suffices to check that the composition is a quasi-isomorphism when X is affine. Then, using the passage to the filtered inductive limit, we may assume that $\mathcal{F}$ is locally free of finite rank, and further that $\mathcal{F} = \mathcal{O}_X$. It remains to recall that the morphism $\mathcal{O}_X \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet)$ is a quasi-isomorphism by the definition of $\mathcal{D}_X^\bullet$.

Let ${}''\mathcal{D}_X^\bullet$ be a bounded above complex of flat quasi-coherent sheaves mapping quasi-isomorphically to ${}'\mathcal{D}_X^\bullet$. Then for any injective quasi-coherent sheaf $\mathcal{J}$ over X there are quasi-isomorphisms ${}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J})$ and ${}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow {}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet, \mathcal{J})$ forming a commutative diagram with the evaluation morphisms into $\mathcal{J}$. Hence it remains to check that the morphism ${}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$ is a quasi-isomorphism, which is a local question. Assume futher that ${}''\mathcal{D}_X^\bullet$ is a bounded above complex of locally free sheaves of finite rank. Then there is a natural isomorphism of complexes of sheaves ${}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}({}'\mathcal{D}_X^\bullet, \mathcal{J}) \simeq \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{H}om_{X\text{–}\mathsf{qc}}({}''\mathcal{D}_X^\bullet, {}'\mathcal{D}_X^\bullet), \mathcal{J})$. The related morphism $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{H}om_{X\text{–}\mathsf{qc}}({}''\mathcal{D}_X^\bullet, {}'\mathcal{D}_X^\bullet), \mathcal{J}) \longrightarrow \mathcal{J}$ is induced by the natural morphism of complexes $\mathcal{O}_X \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}({}''\mathcal{D}_X^\bullet, {}'\mathcal{D}_X^\bullet)$. The latter is again a quasi-isomorphism essentially by the definition of $\mathcal{D}_X^\bullet$. ☐

From this point on we resume assuming that X has enough vector bundles.

Notice that the equivalence functor $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ that we have constructed takes the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ into the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$. This is so because the dualizing complex $\mathcal{D}_X^\bullet$ has bounded coherent cohomology sheaves.

Now we will use the above Proposition and Theorem in order to construct compact generators of the triangulated category $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ (cf. [18, 28]).

Consider the abelian category $Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh})$ of coherent matrix factorizations of $-w$ and closed morphisms of degree 0 between them, and its exact subcategory of locally free matrix factorizations of finite rank $Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}})$. The natural functor between the bounded above derived categories of our abelian category and its exact subcategory $\mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh})$ is an equivalence of triangulated categories.

The vector bundle duality functor $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-,\mathcal{O}_X)\colon Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}})^{\mathsf{op}} \longrightarrow Z^0((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})$ induces a triangulated functor $\mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}})^{\mathsf{op}} \longrightarrow \mathsf{D}^+(Z^0(X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})$ taking bounded above complexes to bounded below ones. Here D^{op} denotes the opposite category to a category D.

Let $\mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}))$ denote the bounded below derived category of the exact category of locally free matrix factorizations of possibly infinite rank. Since the bounded below acyclic complexes over any exact category with infinite direct sums are coacyclic [36, Lemma 2.1], there is a well-defined, triangulated totalization functor

$\mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$. Consider the composition

$$\begin{aligned} Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh})^{\mathsf{op}} &\longrightarrow \mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh}))^{\mathsf{op}} \\ &\simeq \mathsf{D}^-(Z^0(X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}}))^{\mathsf{op}} \longrightarrow \mathsf{D}^+(Z^0(X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})) \longrightarrow \\ &\qquad \mathsf{D}^+(Z^0(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}), \end{aligned}$$

where two of the functors are the duality and the totalization discussed above, while the other two are the natural embedding and the functor induced by such.

One easily checks that this composition takes cones of closed morphisms in $Z^0((X,\mathcal{L},-w)\text{–}\mathsf{coh})$ to cocones in $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$, hence induces a triangulated functor $H^0((X,\mathcal{L},-w)\text{–}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$. Similarly, the above composition takes the totalizations of short exact sequences in $(X,\mathcal{L},-w)\text{–}\mathsf{coh}$ to objects corresponding to the totalizations of short exact sequences in $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}$; one checks this by considering a left locally free resolution of a short exact sequence of coherent matrix factorizations. Thus we obtain a triangulated functor

$$\Omega\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{–}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}).$$

Corollary. *The functor Ω is fully faithful, and its image forms a set of compact generators in $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$. The following diagram of triangulated functors is commutative:*

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{–}\mathsf{coh}_{\mathsf{lf}})^{\mathsf{op}} & \xrightarrow{\ \upsilon^{\mathsf{op}}\ } & \mathsf{D}^{\mathsf{abs}}(X,\mathcal{L},-w)\text{–}\mathsf{coh})^{\mathsf{op}} \\
\Big\| {\scriptstyle \mathcal{H}om_{X\text{-}\mathsf{qc}}(-,\mathcal{O}_X)} & & \Big\| {\scriptstyle \mathcal{H}om_{X\text{-}\mathsf{qc}}(-,\mathcal{D}^\bullet_X)} \\
\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) & \xrightarrow{\ \mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}-\ } & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \\
\downarrow {\scriptstyle \kappa} & \swarrow {\scriptstyle \Omega} & \downarrow {\scriptstyle \gamma \ \text{comp. gener.}} \\
\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) & \overset{\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}-}{\underset{\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X,-)}{=\!=\!=\!=}} & \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})
\end{array}$$

Here υ, κ, and γ denote the fully faithful functors induced by the natural embeddings of DG-categories of CDG-modules. The two upper vertical lines are the natural contravariant dualities (anti-equivalences) on the (absolute derived) categories of locally free matrix factorizations of finite rank and coherent matrix factorizations. The lower horizontal line is the equivalence of categories from Theorem, and the middle horizontal arrow is the fully faithful functor discussed after the proof of Theorem.

The above diagram is to be compared with the following subdiagram of the large diagram in the end of Section 2.4:

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) & \xrightarrow{\ \upsilon\ } & \mathsf{D}^{\mathsf{abs}}(X,\mathcal{L},w)\text{–}\mathsf{coh}) \\
\downarrow {\scriptstyle \kappa} & & \downarrow {\scriptstyle \gamma \ \text{comp. gener.}} \\
\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) & \xrightarrow{\ \lambda\ } & \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})
\end{array}$$

Here λ denotes the triangulated functor induced by the embedding of DG-categories of CDG-modules $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{lf}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}$.

Notice that it is clear from these two diagrams that the functor λ is an equivalence of triangulated categories whenever the functor v is. Indeed, if v is an equivalence of categories, then the image of κ is a set of compact generators in the target category, and λ is an infinite direct sum-preserving triangulated functor identifying triangulated subcategories of compact generators, hence an equivalence. In this case, the functor $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -$ becomes an auto-equivalence of the triangulated category $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ and restricts to an auto-equivalence of its full subcategory of compact generators $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$.

Proof of Corollary. The assertions in the first sentence follow from the second one, as we know γ to be fully faithful and its image to be a set of compact generators by Corollary 2.3(l). The commutativity of both squares and the upper left triangle is clear. To check commutativity of the lower right triangle, consider a coherent matrix factorization $\mathcal{M}$ of the potential $-w$; let $\mathcal{E}_\bullet$ be its left resolution in the abelian category $Z^0((X, \mathcal{L}, -w)\text{–}\mathsf{coh})$ whose terms $\mathcal{E}_n$ belong to $Z^0((X, \mathcal{L}, -w)\text{–}\mathsf{coh}_{\mathsf{lf}})$. Then the finite complex of injective matrix factorizations $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}_X^\bullet)$ maps quasi-isomorphically to the bounded below complex of injective matrix factorizations $\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{D}_X^\bullet) \simeq \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{O}_X)$, so the cone of the corresponding morphism of the total matrix factorizations is coacyclic (and in fact contractible). $\square$

2.6. **w-flat matrix factorizations.** From now on we will assume that for any affine open subscheme $U \subset X$ the element $w|_U$ is not a zero divisor in the $\mathcal{O}(U)$-module $\mathcal{L}(U)$; in other words, the morphism of sheaves $w\colon \mathcal{O}_X \longrightarrow \mathcal{L}$ is injective.

The following results will be used in the proof of the main theorem and its analogues below. Let us call a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ *w-flat* if the map $w\colon \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Notice that any submodule of a w-flat module is w-flat, so the "w-flat dimension" of a quasi-coherent sheaf over X never exceeds 1.

Denote by $(X, \mathcal{L}, w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}$ the DG-category of coherent CDG-modules over $(X, \mathcal{L}, w)$ with w-flat underlying graded $\mathcal{O}_X$-modules and by $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}$ the similar DG-category of quasi-coherent CDG-modules. Let $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}$, and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}})$ denote the corresponding derived categories of the second kind.

Furthermore, denote by $(X, \mathcal{L}, w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ the DG-category of coherent CDG-modules over $(X, \mathcal{L}, w)$ whose underlying graded $\mathcal{O}_X$-modules are both w-flat and of finite flat dimension, and by $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}}$ the DG-category of w-flat quasi-coherent CDG-modules of finite locally free dimension. Let $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}})$, $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}})$, and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}})$ denote the corresponding exotic derived categories.

Corollary. (a) *The functor* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}} \longrightarrow (X, \mathcal{L}, w)\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

(b) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

(c) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{coh}$ *is an equivalence of triangulated categories.*

(d) *The functor* $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}}$ *is an equivalence of triangulated categories.*

(e) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{lfd}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lfd}}$ *is an equivalence of triangulated categories.*

(f) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}} \longrightarrow (X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

Proof. The proofs are analogous to those of Corollary 2.3(a-c) and (g) (except that no induction in d is needed, as it suffices to consider the case $d=1$). Parts (d), (e), (f) are analogous to parts (a), (b), (c), respectively. Parts (b-c) and (e-f) can be also proven in the way similar to Corollary 2.3(h-i). ☐

Remark. The assertions of parts (a-b) hold under somewhat weaker assumptions than above: namely, one does not need to assume the existence of enough vector bundles on X. And one can make parts (d-e) hold without vector bundles by replacing the finite locally free dimension condition in their formulation with the finite flat dimension condition. The reason is that there are enough flat sheaves on any reasonable scheme (see Lemma A.1).

In fact, even part (c) does not depend on the existence of vector bundles, since a surjective morphism onto a given coherent sheaf $\mathcal{M}$ from a w-flat coherent sheaf can be easily constructed, e. g., by starting from a surjective morphism onto $\mathcal{M}$ from a flat quasi-coherent sheaf $\mathcal{F}$ and picking a large enough coherent subsheaf in $\mathcal{F}$. Accordingly, one does not need vector bundles to prove the equivalence of categories in the lower horizontal line in Theorem 2.7 below and the other two equivalences in Theorem 2.8. Replacing locally free sheaves with flat ones in the relevant definitions and assuming the Krull dimension to be finite, one can have the whole of Proposition 2.8 hold without vector bundles as well.

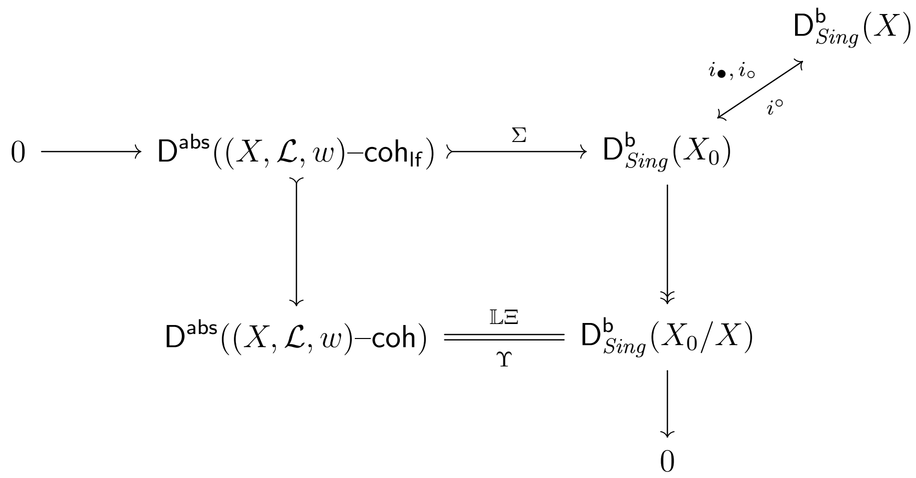
Another alternative is to use *very flat* quasi-coherent sheaves, which there are always enough of and which always form a category of finite homological dimension on a quasi-compact semi-separated scheme [39, Section 4.1]. Similarly, the existence of vector bundles is not needed for the validity of Theorem 1.4(a-b), Proposition 1.5(a, c-d), all the assertions of Sections 1.7 and 1.10, Corollaries 2.3(a-b, f, k-l) and 2.4(a-b), Proposition 2.5 and Theorem 2.5, and some other results.

2.7. **Main theorem.** Let $X_0 \subset X$ be the closed subscheme defined locally by the equation $w = 0$, and $i\colon X_0 \longrightarrow X$ be the natural closed embedding. The next theorem is the main result of this paper.

Theorem. *There is a natural equivalence of triangulated categories*

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

Together with the functor $\Sigma\colon \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *constructed in* [31], *this equivalence forms the following diagram of triangulated functors*



where the upper horizontal arrow Σ *is fully faithful, the left vertical arrow is fully faithful, the right vertical arrow is a Verdier localization functor, and the lower horizonal line* $\mathbb{L}\Xi = \Upsilon^{-1}$ *is an equivalence of categories. The square is commutative; the three diagonal arrows* $i_\bullet$, i°, $i_\circ$ *(the middle one pointing down and the two other ones pointing up) are adjoint.*

Furthermore, the image of the functor Σ *is precisely the full subcategory of objects annihilated by the functor* $i_\circ$, *or equivalently, by the functor* $i_\bullet$. *In other words, the image of* Σ *is equal both to the left and to the right orthogonal complements to the thick subcategory generated by the image of the functor* i°, *i. e., an object* $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *is isomorphic to* $\Sigma(\mathcal{M})$ *for some* $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}})$ *if and only if for every* $\mathcal{E} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ *one has* $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}(i^\circ\mathcal{E}, \mathcal{F}) = 0$, *or equivalently, for every* $\mathcal{E} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ *one has* $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}(\mathcal{F}, i^\circ\mathcal{E})$.

The thick subcategory generated by the image of the functor i° is the kernel of the right vertical arrow. So the upper horizontal arrow and the right vertical arrow are included into "exact sequences" of triangulated categories (as marked by the zeroes at the ends; there is no exactness at the uppermost rightmost end).

When X is a regular scheme, the functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$ $\text{–}\mathsf{coh})$ is an equivalence of categories by Corollary 2.4(c), and so is the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ (as explained in Section 2.1). Hence it follows that the functor Σ is an equivalence of categories, too. Thus we recover the result of Orlov [31, Theorem 3.5] claiming the equivalence of triangulated categories $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ for a regular X.

Proof of the lower horizontal equivalence. To obtain the equivalence of triangulated categories in the lower horizontal line, we will construct triangulated functors in both directions, and then check that they are mutually inverse. Given a bounded complex of coherent sheaves $\mathcal{F}^\bullet$ over X_0, consider the CDG-module $\Upsilon(\mathcal{F}^\bullet)$ over $(X,\mathcal{L},w)$ with the underlying coherent graded module given by the rule

$$\Upsilon^n(\mathcal{F}^\bullet) = \bigoplus\nolimits_{m\in\mathbb{Z}} i_*\mathcal{F}^{n-2m}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes m}$$

and the differential induced by the differential on $\mathcal{F}^\bullet$. Since $d^2=0$ on $\mathcal{F}^\bullet$ and w acts by zero in $i_*\mathcal{F}^j$, this is a CDG-module. It is clear that Υ is a well-defined triangulated functor $\mathsf{D^b}(X_0\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$, since the derived category of bounded complexes over an abelian category coincides with their absolute derived category.

Let us check that Υ annihilates the image of the functor $\mathbb{L}i^*$. It suffices to consider a w-flat coherent sheaf $\mathcal{E}$ on X and check that $\Upsilon(\operatorname{coker} w) = 0$, where $w\colon \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes -1}\longrightarrow\mathcal{E}$. Indeed, $\Upsilon(\operatorname{coker} w)$ is the cokernel of the injective morphism of contractible coherent CDG-modules $\mathcal{N}\longrightarrow\mathcal{M}$, where $\mathcal{N}^{2n+1}=\mathcal{M}^{2n+1}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{N}^{2n}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n-1}$, while $\mathcal{M}^{2n}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ for $n\in\mathbb{Z}$.

This provides the desired triangulated functor

$$\Upsilon\colon\ \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \longrightarrow \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}).$$

The functor in the opposite direction is a version of Orlov's cokernel functor, but in our situation it has to be constructed as a derived functor, since the functor of cokernel of an arbitrary morphism is not exact. Recall the equivalence of triangulated categories $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ from Corollary 2.6(c).

Define the functor $\Xi\colon Z^0((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ from the category of w-flat coherent CDG-modules over $(X,\mathcal{L},w)$ and closed morphisms of degree 0 between them to the triangulated category of relative singularities by the rule

$$\Xi(\mathcal{M}) = \operatorname{coker}(d\colon \mathcal{M}^{-1}\to\mathcal{M}^0) = \operatorname{coker}(i^*d\colon\ i^*\mathcal{M}^{-1}\to i^*\mathcal{M}^0),$$

where the former cokernel, which is by definition a coherent sheaf on X annihilated by w, is considered as a coherent sheaf on X_0. One can immediately see that the functor Ξ transforms morphisms homotopic to zero into morphisms factorizable through the restrictions to X_0 of w-flat coherent sheaves on X. Hence the functor Ξ factorizes through the homotopy category $H^0((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}})$.

It is explained in [33, Lemma 3.12] (see also Lemma 3.5 below) that the functor Ξ is triangulated and in [31, Proposition 3.2] that the functor Ξ factorizes through $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}})$. The latter assertion can be also deduced by considering the complex (1.3) from [33]. Indeed, the complex $i^*\mathcal{M}$ corresponding to the total CDG-module $\mathcal{M}$ of an exact triple in $\mathcal{B}\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}$ is the total complex of an exact triple of complexes in the exact category $\mathsf{E}_{X_0/X}$ from Remark 2.1, hence the complex $i^*\mathcal{M}$ is exact with respect to $\mathsf{E}_{X_0/X}$ and the cokernels of its differentials belong to this exact subcategory in the abelian category of coherent sheaves over X_0. So we obtain the triangulated functor

$$\Xi\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X),$$

and consequently, the left derived functor

$$\mathbb{L}\Xi\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

Let us check that the two functors Υ and $\mathbb{L}\Xi$ are mutually inverse. For any w-flat coherent CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$, there is a natural surjective closed morphism of CDG-modules $\phi\colon \mathcal{M} \longrightarrow \Upsilon\Xi(\mathcal{M})$ with a contractible kernel. Clearly, $\phi\colon \mathrm{Id} \longrightarrow \Upsilon\mathbb{L}\Xi$ is an (iso)morphism of functors.

Conversely, any object of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ can be represented by a coherent sheaf on X_0, and any morphism in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is isomorphic to a morphism coming from the abelian category of such coherent sheaves. Indeed, the bounded above derived category $\mathsf{D}^-(X_0\text{–}\mathsf{coh})$ of coherent sheaves over X_0 is equivalent to the bounded above derived category $\mathsf{D}^-(X_0\text{–}\mathsf{coh}_{\mathsf{lf}})$ of locally free sheaves; using a truncation far enough to the left, one can represent any object or morphism in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ by a long enough shift of a coherent sheaf or a morphism of coherent sheaves. Now for any coherent sheaf $\mathcal{F}$ on X_0 there is a natural distinguished triangle $\mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[1] \longrightarrow \mathbb{L}i^*i_*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[2]$ in $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{coh})$, which provides a natural isomorphism $\mathcal{F}\simeq\mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[2]$ in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$.

Let $\mathcal{F}$ be a coherent sheaf on X_0; pick a vector bundle $\mathcal{E}$ on X together with a surjective morphism $\mathcal{E}\longrightarrow i_*\mathcal{F}$ with the kernel $\mathcal{E}'$. Then the CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{M}^{2n}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{2n-1}=\mathcal{E}'\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ maps surjectively onto $\Upsilon(\mathcal{F})$ with a contractible kernel, and $\mathbb{L}\Xi\Upsilon(\mathcal{F})=\Xi(\mathcal{M})=\mathcal{F}$ (cf. [22, Lemma 2.18]). Denote the isomorphism we have constructed by $\psi\colon \mathbb{L}\Xi\Upsilon(\mathcal{F})\longrightarrow\mathcal{F}$. The composition $\Upsilon\psi\circ\phi\Upsilon\colon \Upsilon(\mathcal{F})\longrightarrow\Upsilon\mathbb{L}\Xi\Upsilon(\mathcal{F})\longrightarrow\Upsilon(\mathcal{F})$ is clearly the identity morphism. It is obvious that ψ commutes with any morphisms of coherent sheaves $\mathcal{F}$ on X_0, but checking that it commutes with all morphisms, or all isomorphisms, in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is a little delicate.

Notice that $\Upsilon\psi$ is an (iso)morphism of functors since $\phi\Upsilon$ is, and consequently $\mathbb{L}\Xi\Upsilon\psi$ is an (iso)morphism of functors. Thus it remains to check that the functor $\mathbb{L}\Xi\Upsilon$ is faithful, i. e., does not annihilate any morphisms. Indeed, any morphism in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is isomorphic to a morphism coming from the abelian category of coherent sheaves on X_0, and the functor $\mathbb{L}\Xi\Upsilon$ transforms such morphisms into isomorphic ones. The construction of the equivalence of categories in the lower horizontal line is finished. One still has to check that the isomorphisms ϕ commute with the isomorphisms $\Upsilon\Xi(\mathcal{M}[1])\simeq\Upsilon\Xi(\mathcal{M})[1]$, but this is straightforward.

Alternatively, one can use w-flat coherent sheaves on X or objects of the exact category $\mathsf{E}_{X_0/X}$ of coherent sheaves on X_0 (as applicable) instead of the locally free sheaves everywhere in the above argument. □

Proof of "exactness" in the upper line. We start with a discussion of the three adjoint functors in the right upper corner. The functor $i_\circ$ right adjoint to the functor $i^\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(X)\longrightarrow\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ was constructed in Section 2.1.

To construct the left adjoint functor to i°, notice that the right derived functor of subsheaf with the scheme-theoretic support in the closed subscheme $\mathbb{R}i^!\colon \mathsf{D^b}(X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^b}(X_0\text{–}\mathsf{coh})$ only differs from the functor $\mathbb{L}i^*$ by a shift and a twist, $\mathbb{R}i^!\mathcal{E}^\bullet \simeq \mathbb{L}i^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_0}} \mathcal{L}|_{X_0}[-1]$. One can check this first for w-flat coherent sheaves $\mathcal{E}$, when both objects to be identified are shifts of sheaves, so it suffices to compare their direct images under i, which are both computed by the same two-term complex $\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$; then replace a complex $\mathcal{E}^\bullet$ with a finite complex of w-flat coherent sheaves (for a general result of this kind, see [26, Theorem 5.4]).

Hence the functor $\mathbb{R}i^!$ takes $\mathit{Perf}(X)$ to $\mathit{Perf}(X_0)$ and induces a triangulated functor $i^\bullet\colon \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$ right adjoint to $i_\circ$. It follows that the functor $i_\bullet(\mathcal{F}) = i_\circ(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}[-1]$ is left adjoint to the functor i°.

To prove the vanishing of the composition of functors in the upper line and the orthogonality assertions, notice that

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)}(i^\circ\mathcal{E}, \Sigma\mathcal{M}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)}(\mathcal{E}, i_\circ\Sigma\mathcal{M})$$

and $i_*\Sigma(\mathcal{M}) = \mathrm{coker}(\mathcal{M}^{-1} \to \mathcal{M}^0) \in \mathit{Perf}(X)$ for any $\mathcal{M} \in \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$, since the morphism $\mathcal{M}^{-1} \longrightarrow \mathcal{M}^0$ of locally free sheaves on X is injective. Similarly,

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)}(\Sigma\mathcal{M}, i^\circ\mathcal{E}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)}(i_\bullet\Sigma\mathcal{M}, \mathcal{E})$$

and $i_\bullet\Sigma(\mathcal{M}) = i_\circ\Sigma(\mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{L}[-1] = 0$ in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)$.

Obviously, our derived cokernel functor $\mathbb{L}\Xi$ makes a commutative diagram with the cokernel functor Σ from [31]. The left vertical arrow is fully faithful by Corollary 2.3(i). The assertion that the upper horizontal arrow is fully faithful is due to Orlov [31, Theorem 3.4]. We have just obtained a new proof of it with our methods. Indeed, it follows from the orthogonality that the functor $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0/X)$ induces isomorphisms on the groups of morphisms between any two objects one of which comes from $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$. Conversely, Orlov's theorem together with the orthogonality argument and the equivalence of categories in the lower horizontal line imply that the left vertical arrow is fully faithful.

Now assume that $i_\circ\mathcal{F} = 0$ for some $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$. Clearly, there exists $n \geq 0$ and a coherent sheaf $\mathcal{K}$ on X_0 such that $\mathcal{F} \simeq \mathcal{K}[n]$ in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$. Then $i_*\mathcal{K}$ is a perfect complex, i. e., a coherent sheaf of finite flat dimension on X. Let us view it as an object of $(X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}}$, i. e., consider the CDG-module $\mathcal{N}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{N}^{2n} = i_*\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ and $\mathcal{N}^{2n+1} = 0$.

The construction of the cokernel functor Σ can be straightforwardly extended to w-flat coherent matrix factorizations of finite flat dimension, providing a triangulated functor $\widetilde{\Sigma}\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$. The functor $\widetilde{\Sigma}$ is well-defined, since one has $i^*\mathcal{M} \in \mathit{Perf}(X_0)$ for any w-flat coherent sheaf $\mathcal{M}$ of finite flat dimension on X. Using the equivalence of triangulated categories $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}) \simeq \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}})$ from Corollary 2.6(f), one constructs the derived functor $\mathbb{L}\widetilde{\Sigma}\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}} \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$ in the same way as it was done above for the derived functor $\mathbb{L}\Xi$. Since the functor

$\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}}) \longrightarrow \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}})$ is an equivalence of categories by Corollary 2.3(g), the (essential) images of the functors Σ and $\mathbb{L}\widetilde{\Sigma}$ coincide.

Let us check that $\mathbb{L}\widetilde{\Sigma}(\mathcal{N}) \simeq \mathcal{K}$ as an object of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. We argue as above, picking a vector bundle $\mathcal{E}$ on X together with a surjective morphism $\mathcal{E} \longrightarrow i_*\mathcal{K}$ with the kernel $\mathcal{E}'$. Then the CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{M}^{2n} = \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{2n-1} = \mathcal{E}'\otimes_{O_X}\mathcal{L}^{\otimes n}$ maps surjectively onto $\mathcal{N}$ with a contractible kernel. Hence the object $\mathcal{M}\in(X,\mathcal{L},w)\text{–}\mathsf{coh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ is isomorphic to $\mathcal{N}$ in $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}})$, and we have $\mathbb{L}\widetilde{\Sigma}(\mathcal{N}) = \widetilde{\Sigma}(\mathcal{M}) = \mathcal{K}$. Therefore, the object $\mathcal{K}\in\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ belongs to the (essential) image of the functor Σ, and it follows that so does the object $\mathcal{F}\simeq\mathcal{K}[n]$.

One can strengthen the above argument so as to obtain a construction of the (partial) inverse functor Δ to the functor Σ similar to the above construction of the functor Υ inverse to the functor $\mathbb{L}\Xi$. Consider the full subcategory $\mathsf{F}_{X_0/X}\subset X_0\text{–}\mathsf{coh}$ in the abelian category of coherent sheaves on X_0 consisting of all the sheaves $\mathcal{F}$ such that the sheaf $i_*\mathcal{F}$ has finite flat dimension (i. e., is a perfect complex) on X. The category $\mathsf{F}_{X_0/X}$ contains all the locally free sheaves on X_0 and is closed under the kernels of surjections, the cokernels of embeddings, and the extensions.

Hence $\mathsf{F}_{X_0/X}$ is an exact subcategory in $X_0\text{–}\mathsf{coh}$. The natural functor $\mathsf{D^b}(\mathsf{F}_{X_0/X}) \longrightarrow \mathsf{D^b}(X_0\text{–}\mathsf{coh})$ is fully faithful; its image coincides with the kernel of the composition of the direct image and Verdier localization functors $\mathsf{D^b}(X_0\text{–}\mathsf{coh}) \longrightarrow \mathsf{D^b}(X\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X)$. Accordingly, the quotient category $\mathsf{D^b}(\mathsf{F}_{X_0/X})/\mathsf{D^b}(X_0\text{–}\mathsf{coh_{lf}})$ is identified with the kernel of the direct image functor $i_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)\longrightarrow\mathsf{D}^{\mathsf{b}}_{Sing}(X)$.

Now the functor

$$\Delta\colon \mathsf{D^b}(\mathsf{F}_{X_0/X})/\mathsf{D^b}(X_0\text{–}\mathsf{coh_{lf}}) \longrightarrow \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{ffd}})$$

is constructed in the way similar to the construction of the functor Υ, by taking the direct image from X_0 to X and applying the periodicity summation. That is

$$\Delta^n(\mathcal{F}^\bullet) = \textstyle\bigoplus_{m\in\mathbb{Z}} i_*\mathcal{F}^{n-2m}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes m}$$

for any $\mathcal{F}^\bullet\in\mathsf{D^b}(\mathsf{F}_{X_0/X})$. One checks that the functor Δ is inverse to the functor $\mathbb{L}\widetilde{\Sigma}$, the latter being viewed as a functor taking values in the triangulated subcategory $\mathsf{D^b}(\mathsf{F}_{X_0/X})/\mathsf{D^b}(X_0\text{–}\mathsf{coh_{lf}})\subset\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$, in the same way as it was done above for the functors Υ and $\mathbb{L}\Xi$. This provides yet another proof of the fact that the functor Σ is fully faithful, together with another proof of our description of its image. It is also obvious from the constructions that the functor Δ makes a commutative diagram with the functor Υ. □

2.8. **Infinite matrix factorizations.** Following [29, paragraphs after Remark 1.9], one can define a "large" version of the triangulated category of singularities $\mathsf{D}'_{Sing}(X)$ of a scheme X as the quotient category of the bounded derived category of quasi-coherent sheaves $\mathsf{D^b}(X\text{–}\mathsf{qcoh})$ by the thick subcategory $\mathsf{D^b}(X\text{–}\mathsf{qcoh_{lf}})$ of bounded complexes of locally free sheaves (of infinite rank). When X has finite Krull dimension, the latter subcategory coincides with the thick subcategory $\mathsf{D^b}(X\text{–}\mathsf{qcoh_{fl}})$ of bounded complexes of flat sheaves (see Remark 1.4).

Similarly, let $Z \subset X$ be a closed subscheme such that $\mathcal{O}_Z$ has finite flat dimension as an $\mathcal{O}_X$-module. Let us define a "large" triangulated category of relative singularities $\mathsf{D}'_{Sing}(Z/X)$ as the quotient category of $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$ by the minimal thick subcategory containing the image of the functor $\mathbb{L}i^*\colon \mathsf{D^b}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^b}(Z\text{–}\mathsf{qcoh})$ and closed under those infinite direct sums that exist in $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$. The quotient category of $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$ by the minimal thick subcategory containing $\mathbb{L}i^*\mathsf{D^b}(X\text{–}\mathsf{qcoh})$ (without the direct sum closure) will be also of interest to us; let us denote it by $\mathsf{D}''_{Sing}(Z/X)$.

Lemma. *The triangulated categories* $\mathsf{D}'_{Sing}(Z/X)$ *and* $\mathsf{D}''_{Sing}(Z/X)$ *are quotient categories of* $\mathsf{D}'_{Sing}(Z)$. *When the scheme* X *is regular of finite Krull dimension, these three triangulated categories coincide.*

Proof. To prove the first assertion, let us show that any locally free sheaf on Z, considered as an object of $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$, is a direct summand of a bounded complex whose terms are direct sums of locally free sheaves of finite rank restricted from X. Indeed, pick a finite left resolution of a given locally free sheaf on Z with the middle terms as above, long enough compared to the number of open subsets in an affine covering of Z. Then the corresponding Ext class between the cohomology sheaves at the rightmost and leftmost terms has to vanish in view of the Mayer–Vietoris sequence for Ext groups between quasi-coherent sheaves [29, Lemma 1.12]. Hence the rightmost term is a direct summand of the complex formed by the middle terms.

The second assertion holds for the categories $\mathsf{D}''_{Sing}(Z/X)$ and $\mathsf{D}'_{Sing}(Z)$, since any quasi-coherent sheaf on a regular scheme of finite Krull dimension has a finite left resolution consisting of locally free sheaves. To identify these two categories with $\mathsf{D}'_{Sing}(Z/X)$, one needs to know that the subcategory of bounded complexes of locally free sheaves on Z is closed under those infinite direct sums that exist in $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$. The latter is true for any Noetherian scheme Z of finite Krull dimension with enough vector bundles, since the finitistic projective dimension of a commutative ring of finite Krull dimension is finite [40, Théorème II.3.2.6]. ☐

Now let $\mathcal{L}$ be a line bundle on X, $w \in \mathcal{L}(X)$ be a global section corresponding to an injective morphism of sheaves $\mathcal{O}_X \longrightarrow \mathcal{L}$, and $X_0 \subset X$ be the locus of $w = 0$.

Proposition. *There is a natural equivalence of triangulated categories*

$$\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \simeq \mathsf{D}''_{Sing}(X_0/X).$$

Together with the infinite-rank version $\Sigma'\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{lf}}) \longrightarrow \mathsf{D}'_{Sing}(X_0)$ *of Orlov's cokernel functor* Σ *from* [31], *this equivalence forms the following diagram of* *triangulated functors*

$$\begin{array}{ccccccc}
 & & & & & & \mathsf{D}'_{Sing}(X) \\
 & & & & & {\scriptstyle i_\bullet,\,i_\circ}\nearrow\ \swarrow{\scriptstyle i^\circ} & \\
0 & \longrightarrow & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) & \overset{\Sigma'}{\rightarrowtail} & \mathsf{D}'_{Sing}(X_0) & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) & = & \mathsf{D}''_{Sing}(X_0/X) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

where the upper horizontal arrow Σ' is fully faithful, the left vertical arrow is fully faithful, the right vertical arrow is the Verdier localization functor by the thick subcategory generated by the image of the diagonal down arrow i°, and the lower horizonal line is an equivalence of categories. The square is commutative; the three diagonal arrows $i_\bullet$, i°, $i_\circ$ are adjoint.

Furthermore, the image of the functor Σ' is precisely the full subcategory of objects annihilated by the functor $i_\circ$, or equivalently, by the functor $i_\bullet$. In other words, the image of Σ' is equal both to the left and to the right orthogonal complements to (the thick subcategory generated by) the image of the functor i°.

Proof. The proof is completely similar to that of Theorem 2.7. It uses Corollaries 2.6(b), 2.3(h), 2.6(e), and 2.3(c). Alternatively, one can prove that the functor Σ' is fully faithful in the same way as it was done for the functor Σ in [31, Theorem 3.4], and deduce the assertion that the left vertical arrow is fully faithful from the orthogonality.

Note that one check in a straightforward way that the functor Σ' annihilates the objects coacyclic with respect to $(X,\mathcal{L},w)$–$\mathsf{qcoh}_{\mathsf{lf}}$. This provides another proof of Corollary 2.3(d), working in the assumption that w is a local nonzero-divisor. □

The functors Σ and Σ' together with the direct image functors $i_\circ$ form the commutative diagram of an embedding of "exact sequences" of triangulated functors

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) & \overset{\Sigma}{\rightarrowtail} & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) & \xrightarrow{i_\circ} & \mathsf{D}^{\mathsf{b}}_{Sing}(X) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) & \overset{\Sigma'}{\rightarrowtail} & \mathsf{D}'_{Sing}(X_0) & \xrightarrow{i_\circ} & \mathsf{D}'_{Sing}(X)
\end{array}$$

The leftmost vertical arrow is fully faithful by Corollary 2.3(j). The other two vertical arrows are fully faithful by Orlov's theorem [29, Proposition 1.13] claiming that the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}'_{Sing}(X)$ is fully faithful for any separated Noetherian scheme X with enough vector bundles. The leftmost nontrivial terms in both lines are the kernels of the rightmost arrows by Theorem 2.7 and Proposition above.

Theorem. *There is a natural equivalence of triangulated categories*

$$\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \simeq \mathsf{D}'_{Sing}(X_0/X)$$

forming a commutative diagram of triangulated functors

$$\begin{array}{ccccccc} & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) & = & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) & \\ \text{comp.} & \downarrow & & \downarrow & \text{comp.} \\ \text{gener.} & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) & = & \mathsf{D}''_{Sing}(X_0/X) & \text{gener.} \\ & \downarrow & & \downarrow & \\ & \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) & = & \mathsf{D}'_{Sing}(X_0/X) & \end{array}$$

with the equivalences of categories from Theorem 2.7 and the above Proposition. The upper vertical arrows are fully faithful, the lower ones are Verdier localization functors, and the vertical compositions are fully faithful. The categories in the lower line admit arbitrary direct sums, and the images of the vertical compositions are sets of compact generators in the target categories.

Proof. The construction of the desired equivalence of categories is very similar to the construction of the equivalence of categories in Theorem 2.7 and the Proposition. Using Corollary 2.6(a), one defines the infinite-rank version of the functor $\mathbb{L}\Xi$, then shows that the obvious infinite-rank version of the functor Υ is inverse to it. Notice that the functor $\Xi\colon Z^0((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$ preserves infinite direct sums and the functor $\Upsilon\colon \mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ preserves those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$, so the functors $\Xi\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$ and $\Upsilon\colon \mathsf{D}'_{Sing}(X_0/X) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ are well-defined.

The upper left vertical arrow is fully faithful by Corollary 2.3(k); it follows that the upper right vertical arrow is fully faithful, too. The assertions about the vertical compositions are proved similarly. The category $\mathsf{D}'_{Sing}(X_0/X)$ admits arbitrary direct sums, since the category $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ does. By Corollary 2.3(l), the left vertical composition is fully faithful and its image is a set of compact generators in the target, so the right vertical composition has the same properties. ☐

The following square diagram of triangulated functors is commutative:

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) & \xrightarrow{\ \Sigma'\ } & \mathsf{D}'_{Sing}(X_0) \\ \downarrow & & \downarrow \\ \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) & = & \mathsf{D}'_{Sing}(X_0/X) \end{array}$$

The upper horizontal arrow Σ' is fully faithful; the right vertical arrow is a Verdier localization functor. The lower line is an equivalence of triangulated categories. Nothing is claimed about the left vertical arrow in general.

When the scheme X is Gorenstein of finite Krull dimension, the left vertical arrow is an equivalence of categories by Corollary 2.4(a). When X is also regular, the right vertical arrow is an equivalence of categories by the above Lemma. So Σ' is an equivalence of categories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \simeq \mathsf{D}'_{Sing}(X_0)$ and we have obtained a strengthened version of [33, Theorem 4.2] (in the scheme case).

Remark. It is well-known that the Verdier localization functor of a triangulated category with infinite direct sums by a thick subcategory closed under infinite direct sums preserves infinite direct sums [27, Lemma 3.2.10]. This result is not applicable to the localization functors $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(X)$ and $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(Z/X)$, as the category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$ does not admit arbitrary infinite direct sums.

Using the equivalence of categories from the above Theorem and the observation that the functor Υ preserves infinite direct sums, one can show that the localization functor $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$ takes those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$ into direct sums in the triangulated category of relative singularities $\mathsf{D}'_{Sing}(X_0/X)$ of the zero locus of w in X. However, there is *no* obvious reason why the localization functor $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(X_0)$ should take those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$ into direct sums in the absolute triangulated category of singularities $\mathsf{D}'_{Sing}(X_0)$.

That is the problem one encounters attempting to prove that the kernel of the localization functor $\mathsf{D}'_{Sing}(X_0) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$ is semiorthogonal to the image of the functor Σ'.

2.9. **Stable derived category.** Following Krause [21], we define the *stable derived category* of a Noetherian scheme X as the homotopy category of acyclic unbounded complexes of injective quasi-coherent sheaves on X. As explained below, this is another (and in some respects better) "large" version of the triangulated category of singularities of X; for this reason, we denote it by $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$.

In view of Lemma 1.7 (see also [36, Remark 5.4]), one can equivalently define $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$ as the quotient category of the homotopy category of acyclic complexes of quasi-coherent sheaves over X by the thick subcategory of coacyclic complexes, or as the full subcategory of acyclic complexes in the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ of (complexes of) quasi-coherent sheaves over X. It is the latter definition that will be used in the sequel.

Clearly, the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$ has arbitrary infinite direct sums. In [21, Corollary 5.4], Krause constructs a fully faithful functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X)$ and proves that its image is a set of compact generators of the target category.

Theorem. *For any separated Noetherian scheme Z with enough vector bundles, there is a natural triangulated functor $\mathsf{D}'_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ forming a commutative diagram with the natural functors from $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ into both these categories. The composition $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ preserves those infinite direct sums* *that exist in* $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})$. *When* $Z = X_0$ *is a divisor in a regular separated Noetherian scheme of finite Krull dimension, the functor* $\mathsf{D}'_{Sing}(X_0) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$ *is an equivalence of triangulated categories.*

Proof. The construction of the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ in [21] is given in terms of the Verdier localization functor $Q\colon \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Z\text{–}\mathsf{qcoh})$ by the triangulated subcategory $\mathsf{D}^{\mathsf{st}}_{Sing}(Z) \subset \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ and its adjoint functors on both sides, which exist according to [21, Corollary 4.3]. The proof of our Theorem is based on explicit constructions of the restrictions of these adjoint functors to some subcategories of bounded complexes in $\mathsf{D}(Z\text{–}\mathsf{qcoh})$.

It is well known that the Verdier localization functor $H^0(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Z\text{–}\mathsf{qcoh})$ from the homotopy category of (complexes of) quasi-coherent sheaves on Z to their derived category has a right adjoint functor $\mathsf{D}(Z\text{–}\mathsf{qcoh}) \longrightarrow H^0(Z\text{–}\mathsf{qcoh})$. The objects in the image of this functor are called *homotopy injective complexes* of quasi-coherent sheaves on Z. The composition $\mathsf{D}(Z\text{–}\mathsf{qcoh}) \longrightarrow H^0(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ provides the functor $Q_\rho\colon \mathsf{D}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ right adjoint to Q. In particular, any bounded below complex in $\mathsf{D}(Z\text{–}\mathsf{qcoh})$ has a bounded below injective resolution and any bounded below complex of injectives is homotopy injective. Furthermore, any bounded below acyclic complex is coacyclic [36, Lemma 2.1]. It follows that any bounded below complex from $\mathsf{D}^+(Z\text{–}\mathsf{qcoh})$, considered as an object of $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$, represents its own image under the functor Q_ρ.

On the other hand, any bounded above complex from $\mathsf{D}(Z\text{–}\mathsf{qcoh})$ has a locally free left resolution defined uniquely up to a quasi-isomorphism of complexes in the exact category of locally free sheaves, i. e., there is an equivalence of bounded above derived categories $\mathsf{D}^-(Z\text{–}\mathsf{qcoh_{lf}}) \simeq \mathsf{D}^-(Z\text{–}\mathsf{qcoh})$. Since the exact category $Z\text{–}\mathsf{qcoh_{lf}}$ has finite homological dimension, any acyclic complex in it is coacyclic (and even absolutely acyclic [36, Remark 2.1]), so there are natural functors $\mathsf{D}^-(Z\text{–}\mathsf{qcoh_{lf}}) \longrightarrow \mathsf{D}(Z\text{–}\mathsf{qcoh_{lf}}) \simeq \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh_{lf}}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$.

Lemma. *The composition of the embedding* $\mathsf{D}^-(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Z\text{–}\mathsf{qcoh})$ *with the functor* $Q_\lambda\colon \mathsf{D}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *left adjoint to* Q *is isomorphic to the functor* $\mathsf{D}^-(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *constructed above.*

Proof. We have to show that $\mathrm{Hom}_{\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{E}^\bullet) = 0$ for any bounded above complex of locally free sheaves $\mathcal{L}^\bullet$ and any acyclic complex $\mathcal{E}^\bullet$ of quasi-coherent sheaves on Z. Let us check that any morphism $\mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ in $H^0(Z\text{–}\mathsf{qcoh})$ factorizes through a coacyclic complex of quasi-coherent sheaves. Clearly, we can assume that the complex $\mathcal{E}^\bullet$ is bounded above, too. Let $\mathcal{K}^\bullet$ be the cocone of a closed morphism of complexes $\mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$; then $\mathcal{K}^\bullet$ is bounded above and the composition $\mathcal{K}^\bullet \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ is homotopic to zero. Pick a bounded above complex of locally free sheaves $\mathcal{F}^\bullet$ together with a quasi-isomorphism $\mathcal{F}^\bullet \longrightarrow \mathcal{K}^\bullet$. Then the cone of the composition $\mathcal{F}^\bullet \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathcal{L}^\bullet$, being a bounded above acyclic complex of locally free sheaves, is coacyclic. Since the composition $\mathcal{F}^\bullet \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ is homotopic to zero, the morphism $\mathcal{L}^\bullet \longrightarrow \mathcal{E}^\bullet$ factorizes, up to homotopy, through this cone. □

Now we can describe the action of the functor $I_\lambda\colon \mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z\text{–}\mathsf{qcoh})$ left adjoint to the embedding $\mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh})$ on bounded above complexes in $\mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh})$. If $\mathcal{K}^\bullet$ is a bounded above complex of quasi-coherent sheaves and $\mathcal{F}^\bullet$ is its locally free left resolution, then the cone of the closed morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{K}^\bullet$ represents the object $I_\lambda(\mathcal{K}^\bullet) \in \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z\text{–}\mathsf{qcoh})$. In view of the above Lemma, this cone is functorial and does not depend on the choice of $\mathcal{F}^\bullet$ for the usual semiorthogonality reasons.

The embedding of compact generators $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$ is constructed in [21] as the functor induced by the restriction of the composition $I_\lambda \circ Q_\rho\colon \mathsf{D}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$ to the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}) \subset \mathsf{D}(Z\text{–}\mathsf{qcoh})$. Let us explain why this is so. By Proposition 1.5(d) (cf. [21, Proposition 2.3 and Remark 3.8]), the natural functor $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh})$ is fully faithful and its image is a set of compact generators in the target. This is the image of $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}) \subset \mathsf{D}(Z\text{–}\mathsf{qcoh})$ under the functor Q_ρ, as constructed above. It is clear from the above construction of the functor Q_λ that it preserves compactness (and in fact coincides with the functor Q_ρ on perfect complexes in $\mathsf{D}(Z\text{–}\mathsf{qcoh})$ [21, Lemma 5.2]). Since the functors Q_λ and I_λ, being left adjoints, preserve infinite direct sums, and I_λ is a Verdier localization functor by the image of Q_λ, it follows that the image of any set of compact generators of $\mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh})$ under I_λ is a set of compact generators of $\mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$ [26, Theorem 2.1(4)].

In order to define the desired functor $\mathsf{D}'_{\mathit{Sing}}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$, restrict the same composition $I_\lambda \circ Q_\rho$ to the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}) \subset \mathsf{D}(Z\text{–}\mathsf{qcoh})$. According to the above, this restriction assigns to any bounded complex of quasi-coherent sheaves $\mathcal{K}^\bullet$ on Z the cone of a morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{K}^\bullet$ into it from its locally free left resolution $\mathcal{F}^\bullet$. Clearly, the functor $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$ that we have constructed preserves those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ and annihilates the triangulated subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$. So we have the induced functor $\mathsf{D}'_{\mathit{Sing}}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$, and the first two assertions of Theorem are proven.

To prove the last assertion, we use the results of Section 2.8. Assume that $Z = X_0$ is the zero locus of a section $w \in \mathcal{L}(X)$ of a line bundle on X; as usually, $w\colon \mathcal{O}_X \longrightarrow \mathcal{L}$ has to be an injective morphism of sheaves. Then by Theorem 2.8 and Lemma 2.8, the category $\mathsf{D}'_{\mathit{Sing}}(Z)$ admits infinite direct sums and the image of the fully faithful functor $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0) \longrightarrow \mathsf{D}'_{\mathit{Sing}}(X_0)$ is a set of compact generators in the target. Furthermore, it follows from the proof of Theorem 2.8 that any object of $\mathsf{D}'_{\mathit{Sing}}(X_0)$ can be represented by a quasi-coherent sheaf on X_0 and the direct sum of an infinite family of such objects is represented by the direct sums of such sheaves (see Remark 2.8). Thus the functor $\mathsf{D}'_{\mathit{Sing}}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{\mathit{Sing}}(Z)$, being an infinite direct sum-preserving triangulated functor identifying triangulated subcategories of compact generators, is an equivalence of triangulated categories. □

We keep the assumptions of Theorem and the notation of the last paragraph of its proof, i. e., X is a regular separated Noetherian scheme of finite Krull dimension with enough vector bundles and $X_0 \subset X$ is the divisor of zeroes of a locally nonzero-dividing section $w \in \mathcal{L}(X)$. The closed embedding $X_0 \longrightarrow X$ is denoted by i.

Corollary. *The functor* $\Lambda\colon \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{lf}}) \simeq \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$ *assigning to a locally free (or just w-flat) quasi-coherent matrix factorization* $\mathcal{M}$ *the acyclic complex of locally free (or quasi-coherent) sheaves* $i^*\mathcal{M}$ *on* X_0 *is an equivalence of triangulated categories.*

Proof. Given a w-flat matrix factorization $\mathcal{M}$, the complex of sheaves $i^*\mathcal{M}$ on X_0 is acyclic by [33, Lemma 1.5]. Clearly, the assignment $\mathcal{M}\longmapsto i^*\mathcal{M}$ defines a triangulated functor $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$.

To prove that this functor is an equivalence of categories, it suffices to identify it, up to a shift, with the composition of the equivalences $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{lf}}) \longrightarrow \mathsf{D}'_{Sing}(X_0) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$. Here one simply notices that for any $\mathcal{M}\in \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{lf}})$ the complex $i^*\mathcal{M}$ is isomorphic in $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$ to its canonical truncation $\tau_{\le 1}i^*\mathcal{M}$, and the latter complex is the cocone of the morphism into $\Sigma(\mathcal{M})$ from one of its left locally free resolutions. So the functor Λ is identified with $\Sigma[-1]$. ☐

2.10. **Relative stable derived category.** The goal of this section is to generalize the results of the previous one to the case of a singular Noetherian scheme X. The relative version of stable derived category, defined for a closed embedding of finite flat dimension $i\colon Z\longrightarrow X$, is equivalent to the categories $\mathsf{D}'_{Sing}(X_0/X)$ and $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ in the case of the Cartier divisor $Z=X_0$ corresponding to a locally nonzero-dividing section w of a line bundle $\mathcal{L}$ on X.

Let X be a separated Noetherian scheme of finite Krull dimension and $i\colon Z\longrightarrow X$ be a closed embedding of schemes such that $i_*\mathcal{O}_Z$ has a finite flat dimension as $\mathcal{O}_X$-module. According to Section 1.9, there is a left derived inverse image functor $\mathbb{L}i^*\colon \mathsf{D^{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$. This functor forms a commutative diagram with the similar functor $\mathbb{L}i^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Z\text{–}\mathsf{qcoh})$, and consequently, takes acyclic complexes in $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ to acyclic complexes in $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$.

Proposition. *The following four triangulated categories are naturally equivalent:*

(a) *the full subcategory in* $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *consisting of all the objects annihilated by the direct image functor* $i_*\colon \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$*;*

(b) *the quotient category of the homotopy category of complexes over* Z–qcoh *whose direct images are coacyclic complexes over* X–qcoh *by the thick subcategory of coacyclic complexes over* Z–qcoh*;*

(c) *the quotient category of* $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *by its minimal triangulated subcategory, containing the objects in* $\mathbb{L}i^*\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ *and closed under infinite direct sums;*

(d) *the quotient category of the full subcategory of acyclic complexes in* $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ *by its minimal triangulated subcategory, containing the left derived inverse images of acyclic complexes in* $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ *and closed under infinite direct sums.*

Proof. The equivalence of (a) and (b) is obvious. To show that the natural functor from the category (c) to the category (d) is an equivalence, notice that the minimal triangulated subcategory containing flat quasi-coherent sheaves and closed under infinite direct sums together with the triangulated subcategory of acyclic complexes form a semiorthogonal decomposition of $\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$, and similarly for Z [39, Corollary A.4.7]. Since flat quasi-coherent sheaves on Z belong to the thick subcategory in $\mathsf{D^b}(Z\text{–}\mathsf{qcoh}) \subset \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ generated by the inverse images of flat quasi-coherent sheaves from X (see the proof of Lemma 2.8), the assertion follows.

Finally, the functor $\mathbb{L}i^*$ preserves infinite direct sums and compactness of objects, since its right adjoint functor i_* preserves infinite direct sums. Hence the minimal triangulated subcategory in $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$ containing $\mathbb{L}i^*\mathsf{D^{co}}(X\text{–}\mathsf{qcoh})$ and closed under infinite direct sums is compactly generated by some objects which are compact in $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$. By Brown representability, the quotient category in (c) is equivalent to the right orthogonal complement to this triangulated subcategory, which is the kernel category in (a). ☐

We call any of the equivalent triangulated categories in Proposition 2.10 the *relative stable derived category of Z over X* and denote it by $\mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$. In particular, defining the relative stable derived category by the construction (c), we have natural triangulated functors $\mathsf{D^b}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$. Clearly, the composition $\mathsf{D^b}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$ factorizes through the relative singularity category $\mathsf{D}'_{Sing}(Z/X)$, providing a natural functor $\mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$.

Lemma. *The composition of triangulated functors $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \longrightarrow \mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$ is fully faithful and its image forms a set of compact generators for the triangulated category $\mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$.*

Proof. By Proposition 1.5(d), the full triangulated subcategory $\mathsf{D^{abs}}(Z\text{–}\mathsf{coh})$ compactly generates the triangulated category $\mathsf{D^{co}}(Z\text{–}\mathsf{qcoh})$, and similarly for X. In view of the construction (c) and the argument in the proof of Proposition, the assertion follows from [25, Theorem 2.1]. ☐

Now let $\mathcal{L}$ be a line bundle on X, let $w \in \mathcal{L}(X)$ be a locally nonzero-dividing section of $\mathcal{L}$, and let $i\colon X_0 \longrightarrow X$ be closed embedding of the zero locus of w. Defining the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ by the construction (d), let $\mathbb{L}\Lambda\colon \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ be the triangulated functor assigning to a w-flat quasi-coherent matrix factorization $\mathcal{M}$ the acyclic complex $i^*\mathcal{M}$ over X_0–qcoh.

Since any bounded below acyclic complex over X_0–qcoh is coacyclic, and any any bounded above complex belongs to the minimal triangulated subcategory in $\mathsf{D^{co}}(X_0\text{–}\mathsf{qcoh})$ generated by its terms and closed under infinite direct sums, the following diagram of triangulated functors is commutative (cf. Corollary 2.9)

$$\begin{array}{ccc} \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) & \xrightarrow{\ \mathbb{L}\Xi[-1]\ } & \mathsf{D}'_{Sing}(X_0/X) \\ & {\scriptstyle \mathbb{L}\Lambda}\searrow \qquad \swarrow & \\ & \mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X) & \end{array}$$

Theorem. *For any locally nonzero-dividing section w of a line bundle $\mathcal{L}$ on a separated Noetherian scheme X of finite Krull dimension, all the three functors on the above diagram are equivalences of triangulated categories.*

Proof. The functor $\mathbb{L}\Xi$ is an equivalence by Theorem 2.8. To show that the functor $\mathbb{L}\Lambda$ is an equivalence, let us check that it identifies compact generators. By Proposition 1.5(d), the category $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ is compactly generated by its full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$, while according to Lemma the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ is compactly generated by its full triangulated subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$. The restriction of the functor $\mathbb{L}\Lambda$ being an equivalence between these two subcategories (in view of commutativity of the diagram and) by Theorem 2.7, it follows that the functor $\mathbb{L}\Lambda$ itself is an equivalence, too. $\square$

3. Supports, Pull-Backs, and Push-Forwards

3.1. **Locality of local freeness.** The aim of this section is to show that the property of an object of $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ or $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ to be a direct summand of an object from $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})$ is local in a separated Noetherian scheme X with a dualizing complex and enough vector bundles, assuming that the potential $w\in\mathcal{L}(X)$ is not locally zero-dividing. The author is grateful to A. Efimov for emphasizing the importance of this problem.

Let Z be a Noetherian scheme of finite Krull dimension with enough vector bundles. Recall that the natural functor $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)\longrightarrow\mathsf{D}'_{Sing}(Z)$ is fully faithful [29, Proposition 1.13] (cf. Section 2.8).

Proposition. *Let $Z=U\cup V$ be a covering by two open subschemes. Then any object of $\mathsf{D}'_{Sing}(Z)$ whose restrictions to U and V belong to the full subcategories $\mathsf{D}^{\mathsf{b}}_{Sing}(U)\subset\mathsf{D}'_{Sing}(U)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(V)\subset\mathsf{D}'_{Sing}(V)$, respectively, is a direct summand of an object belonging to the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)\subset\mathsf{D}'_{Sing}(Z)$.*

Proof. Consider the bounded derived category of quasi-coherent sheaves $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ on Z and two full triangulated subcategories $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ in it. Clearly, the intersection $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})\cap\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ coincides with the full subcategory of perfect complexes $\mathit{Perf}(Z)=\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_{\mathsf{lf}})\subset\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$.

Lemma. *Any morphism from an object of the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ into an object of the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})\subset\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ factorizes through an object belonging to $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_{\mathsf{fl}})$.*

Proof. See the proof of [29, Proposition 1.13]. $\square$

It follows from Lemma (by the way of the octahedron axiom) that any object $\mathcal{K}^\bullet$ of the full triangulated subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ generated by $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ can be included in a distinguished triangle $\mathcal{F}^\bullet\longrightarrow\mathcal{K}^\bullet\longrightarrow\mathcal{M}^\bullet\longrightarrow\mathcal{F}^\bullet[1]$ with $\mathcal{F}^\bullet\in\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{M}^\bullet\in\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$. Besides, the natural functor $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})/\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_{\mathsf{lf}})\longrightarrow\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ is fully faithful.

To prove Proposition, one has to show that any object $\mathcal{K}^\bullet\in\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ whose restrictions to U and V belong to the subcategories $\mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ and $\mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{qcoh}_{\mathsf{fl\text{-}c}})$, respectively, is a direct summand of an object from $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}\subset\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$.

According to the above, there exist two objects $\mathcal{F}_U^\bullet \in \mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{F}_V^\bullet \in \mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and two morphisms $\mathcal{F}_U^\bullet \longrightarrow \mathcal{K}^\bullet|_U$ and $\mathcal{F}_V^\bullet \longrightarrow \mathcal{K}^\bullet|_V$ whose cones belong to $\mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{coh})$, respectively.

Set $W = U \cap V \subset Z$; then the restrictions of $\mathcal{F}_U^\bullet$ and $\mathcal{F}_V^\bullet$ to W are isomorphic in $\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh})$, and consequently, in $\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{qcoh}_{\mathsf{fl}})/\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh}_{\mathsf{lf}})$, too. Notice that the category $\mathit{Perf}(W) = \mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh}_{\mathsf{lf}})$ is idempotent complete, and therefore, a thick subcategory in $\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{qcoh}_{\mathsf{fl}})$. It follows that there exists a finite complex of flat quasi-coherent sheaves $\mathcal{F}_W^\bullet$ on W together with two morphisms $\mathcal{F}_U^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet$ and $\mathcal{F}_V^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet$ whose cones are perfect complexes. Denote the cocones of these morphisms by $\mathcal{G}_W^\bullet$ and $\mathcal{H}_W^\bullet$.

For any object A of a triangulated category D, let us denote by $'A$ the object $A \oplus A[1]$. For any triangulated subcategory $\mathsf{C} \subset \mathsf{D}$, whenever an object $A \in \mathsf{D}$ is a direct summand of an object from C, the object $'A$ belongs to C, as $A \oplus B \in \mathsf{C}$ implies $A \oplus A[1] \in \mathsf{C}$ in view of the distinguished triangle $A \oplus B \longrightarrow A \oplus B \longrightarrow A \oplus A[1] \longrightarrow A[1] \oplus B[1]$ [41, Theorem 2.1].

By the Thomason–Trobaugh theorem [42, Section 5], the objects $'\mathcal{G}_W^\bullet$ and $'\mathcal{H}_W^\bullet$ can be extended to perfect complexes on U and V, respectively. Moreover, these extensions $\mathcal{G}_U^\bullet \in \mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{coh}_{\mathsf{lf}})$ and $\mathcal{H}_V^\bullet \in \mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{coh}_{\mathsf{lf}})$ can be chosen in such a way that the morphisms $'\mathcal{G}_W^\bullet \longrightarrow {}'\mathcal{F}_U^\bullet|_W$ and $'\mathcal{H}_W^\bullet \longrightarrow {}'\mathcal{F}_V^\bullet|_W$ would be extendable to morphisms $\mathcal{G}_U^\bullet \longrightarrow {}'\mathcal{F}_U^\bullet$ and $\mathcal{H}_V^\bullet \longrightarrow {}'\mathcal{F}_V^\bullet$ [26, Theorem 2.1(4-5)].

Furthermore, the objects $'\mathcal{G}_U^\bullet$ and $'\mathcal{H}_V^\bullet$ can be extended to perfect complexes $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ on the whole scheme Z so that the compositions of morphisms $'\mathcal{G}_U^\bullet \longrightarrow {}''\mathcal{F}_U^\bullet \longrightarrow {}''\mathcal{K}^\bullet|_U$ and $'\mathcal{H}_V^\bullet \longrightarrow {}''\mathcal{F}_V^\bullet \longrightarrow {}''\mathcal{K}^\bullet|_V$ would be extendable to morphisms $\mathcal{G}^\bullet \longrightarrow {}''\mathcal{K}^\bullet$ and $\mathcal{H}^\bullet \longrightarrow {}''\mathcal{K}^\bullet$. Denote by $\mathcal{K}_{(1)}^\bullet$ a cone of the morphism $\mathcal{G}^\bullet \oplus \mathcal{H}^\bullet \longrightarrow {}''\mathcal{K}^\bullet$, by $\mathcal{F}_{U,(1)}^\bullet$ a cone of the morphism $'\mathcal{G}_U^\bullet \longrightarrow {}''\mathcal{F}_U^\bullet$, and by $\mathcal{F}_{V,(1)}^\bullet$ a cone of the morphism $'\mathcal{H}_V^\bullet \longrightarrow {}''\mathcal{F}_V^\bullet$. We have come back to the original situation with an object $\mathcal{K}_{(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$, two objects $\mathcal{F}_{U,(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{F}_{V,(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, and two morphisms $\mathcal{F}_{U,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet|_U$ and $\mathcal{F}_{V,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet|_V$ whose cones belong to $\mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{coh})$, respectively. In addition, the objects $\mathcal{F}_{U,(1)}^\bullet|_W$ and $\mathcal{F}_{V,(1)}^\bullet|_W$ are now isomorphic in $\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{qcoh}_{\mathsf{fl}})$.

The construction does not guarantee commutativity of the diagram formed by the isomorphism $\mathcal{F}_{U,(1)}^\bullet|_W = \mathcal{F}_{W,(1)}^\bullet \simeq \mathcal{F}_{V,(1)}^\bullet|_W$ and the restrictions of the morphisms $\mathcal{F}_{U,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet$ and $\mathcal{F}_{V,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet$ to W. However, the original choice of the morphisms $\mathcal{F}_U^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet$ and $\mathcal{F}_V^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet$ makes this diagram commute in the quotient category $\mathsf{D}^b(W\text{–}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh})$. Hence the difference of two morphisms $\mathcal{F}_{W,(1)}^\bullet \rightrightarrows \mathcal{K}_{(1)}^\bullet|_W$ factorizes through a bounded complex of coherent sheaves on W, and consequently (according to Lemma) also through a perfect complex on W. Denote the latter by $\mathcal{E}^\bullet \in \mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh}_{\mathsf{lf}})$.

Now let $j\colon U \longrightarrow Z$, $k\colon V \longrightarrow Z$, and $h\colon W \longrightarrow Z$ denote the natural open embeddings. Consider the square diagram formed by the morphisms $\mathbb{R}j_*\mathcal{F}_{U,(1)}^\bullet \oplus \mathbb{R}k_*\mathcal{F}_{V,(1)}^\bullet \longrightarrow \mathbb{R}h_*\mathcal{F}_{U,(1)}^\bullet|_W$ and $\mathbb{R}j_*\mathcal{K}_{(1)}^\bullet|_U \oplus \mathbb{R}k_*\mathcal{K}_{(1)}^\bullet|_V \longrightarrow \mathbb{R}h_*\mathcal{K}_{(1)}^\bullet|_W$. According to the above, this diagram is not necessarily commutative; but it can be made commutative by adding the new direct summand $\mathbb{R}h_*\mathcal{E}^\bullet$ to the term $\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V$ with the morphism $\mathbb{R}h_*\mathcal{E}^\bullet \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W$ induced by the morphism $\mathcal{E}^\bullet \longrightarrow \mathcal{K}^\bullet_{(1)}|_W$ and the morphism $\mathbb{R}j_*\mathcal{F}^\bullet_{U,(1)} \oplus \mathbb{R}k_*\mathcal{F}^\bullet_{V,(1)}$ equal to zero on the first direct summand and induced by the morphism $\mathcal{F}^\bullet_{V,(1)}|_W \simeq \mathcal{F}^\bullet_{W,(1)} \longrightarrow \mathcal{E}^\bullet$ on the second one.

Let $\mathcal{F}^\bullet$ denote a cocone of the morphism $\mathbb{R}j_*\mathcal{F}^\bullet_{U,(1)} \oplus \mathbb{R}k_*\mathcal{F}^\bullet_{V,(1)} \longrightarrow \mathbb{R}h_*\mathcal{F}^\bullet_{U,(1)}|_W$ and $\mathcal{L}^\bullet$ denote a cocone of the morphism $\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V \oplus \mathbb{R}h_*\mathcal{E}^\bullet \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W$. Then the commutative square can be extended to a morphism of distinguished triangles, so we obtain a morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{L}^\bullet$. Since $\mathcal{K}^\bullet_{(1)}$ is a cocone of the morphism $\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W$, there is also a distinguished triangle $\mathcal{K}^\bullet_{(1)} \longrightarrow \mathcal{L}^\bullet \longrightarrow \mathbb{R}h_*\mathcal{E} \longrightarrow \mathcal{K}^\bullet_{(1)}[1]$.

Notice that the complexes $\mathcal{F}^\bullet$ and $\mathbb{R}h_*\mathcal{E}^\bullet$ belong to $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ (since the class of bounded complexes of flat quasi-coherent sheaves is preserved by the derived direct images with respect to flat morphisms of Noetherian schemes; cf. Proposition 1.9). Furthermore, the complex $\mathbb{R}h_*\mathcal{E}^\bullet$ is perfect over W. Restricting to W our morphism of distinguished triangules, and recalling that cones of the morphisms $\mathcal{F}^\bullet_{U,(1)} \longrightarrow \mathcal{K}^\bullet_{(1)}|_U$ and $\mathcal{F}^\bullet_{V,(1)} \longrightarrow \mathcal{K}^\bullet_{(1)}|_V$ are coherent complexes over U and V, one easily concludes that a cone of the morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{L}^\bullet$ is a coherent complex over W.

Denote this cone temporarily by $\mathcal{K}^\bullet_{(2)}$. Clearly, in order to show that the original complex $\mathcal{K}^\bullet$ is a direct summand of an object from $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ (which is our goal) it suffices to check that so is the complex $\mathcal{K}^\bullet_{(2)}$. It also follows from the constructions that the restrictions of the complex $\mathcal{K}^\bullet_{(2)}$ to U and V belong to $\mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ and $\mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{qcoh}_{\mathsf{fl\text{-}c}})$, respectively. Dropping the lower index and redenoting $\mathcal{K}^\bullet_{(2)}$ simply by $\mathcal{K}^\bullet$, we are coming back to the situation in the beginning of the proof with the new knowledge that $\mathcal{K}^\bullet$ may be assumed to be a coherent complex over W.

The next fragment of our proof is based on the localization theory for coderived categories of quasi-coherent sheaves on Noetherian schemes (similar to the Thomason–Trobaugh–Neeman theory for the conventional derived categories, the difference being that arbitrary bounded complexes of coherent sheaves play the role of perfect complexes). What we need is a particular case of the theory developed in Section 1.10 (corresponding to the choice of the quasi-coherent CDG-algebra $\mathcal{O}_Z$ over Z).

Specifically, it follows from Proposition 1.5(d) and Theorem 1.10 together with [26, Theorem 2.1(4-5)] that any morphism from an object of $\mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{coh})$ into a restriction to W of an object $\mathcal{K}^\bullet$ from $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$ (or even from $\mathsf{D}^{\mathsf{co}}(Z\text{–}\mathsf{qcoh})$) can be extended to a morphism to $\mathcal{K}^\bullet$ from an object of $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$. Applying this assertion to the identity morphism $\mathcal{K}^\bullet|_W \longrightarrow \mathcal{K}^\bullet|_W$ in the above situation, we obtain a morphism $\mathcal{M}^\bullet \longrightarrow \mathcal{K}^\bullet$ into $\mathcal{K}^\bullet$ from a coherent complex $\mathcal{M}^\bullet$ over Z that is a quasi-isomorphism over W. Passing to a cone of this morphism, we may assume $\mathcal{K}^\bullet$ to be acyclic over W.

By Corollary 1.10, such a complex $\mathcal{K}^\bullet$ is quasi-isomorphic to a (bounded) complex of quasi-coherent sheaves on Z whose terms are concentrated set-theoretically in the complement $Z \setminus W$. The latter is a disjout union of two nonintersecting closed subsets in Z, namely, the complements $S = Z \setminus U$ and $T = Z \setminus V$. Now the complex $\mathcal{K}^\bullet$ decomposes into a direct sum of two complexes with the set-theoretic supports inside S and T, respectively.

One can consider the two direct summands separately. We have to show that any bounded complex of quasi-coherent sheaves $\mathcal{K}^\bullet$ on Z, which is supported set-theoretically in T and whose restriction to U belongs to $\mathsf{D^b}(U\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$, itself belongs to $\mathsf{D^b}(Z\text{–}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$. Arguing as in the beginning of this proof, we have an object $\mathcal{G}^\bullet \in \mathsf{D^b}(U\text{–}\mathsf{qcoh_{fl}})$ together with a morphism $\mathcal{G}^\bullet \longrightarrow \mathcal{K}^\bullet|_U$ whose cone belongs to $\mathsf{D^b}(U\text{–}\mathsf{coh})$. The restriction $\mathcal{G}^\bullet|_W$ then belongs to both $\mathsf{D^b}(W\text{–}\mathsf{qcoh_{fl}})$ and $\mathsf{D^b}(W\text{–}\mathsf{coh})$, and is, therefore, a perfect complex on W.

Again by the Thomason–Trobaugh theorem, the object $'\mathcal{G}^\bullet|_W$ can be extended to a perfect complex $\mathcal{H}^\bullet$ on V. A cocone of the morphism $\mathbb{R}j_*{}'\mathcal{G}^\bullet \oplus \mathbb{R}k_*\mathcal{H}^\bullet \longrightarrow \mathbb{R}h_*{}'\mathcal{G}^\bullet|_W$ provides an object $\mathcal{F}^\bullet \in \mathsf{D^b}(Z\text{–}\mathsf{qcoh_{fl}})$ isomorphic to $'\mathcal{G}^\bullet$ over U and to $\mathcal{H}^\bullet$ over V. Now the morphism $'\mathcal{G}^\bullet \longrightarrow {}'\mathcal{K}^\bullet|_U$ over U extends uniquely to a morphism $\mathcal{F}^\bullet \longrightarrow {}'\mathcal{K}^\bullet$ over Z, since the set-theoretic support of $'\mathcal{K}^\bullet$ is contained in a closed subset lying inside U. A cone of the morphism $\mathcal{F}^\bullet \longrightarrow {}'\mathcal{K}^\bullet$ is a coherent complex on Z, since it is so in restrictions to U and V. Proposition is proven. □

Now let X be a separated Noetherian scheme of finite Krull dimension with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a locally nonzero-dividing potential. Let $X_0 \subset X$ be the zero locus of w.

Corollary. *Let $X = U \cap V$ be a covering by two open subschemes. Then any object of $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ whose restrictions to U and V belong to the full triangulated subcategories $\mathsf{D^{abs}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{coh_{lf}}) \subset \mathsf{D^{co}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{qcoh_{fl}})$ and $\mathsf{D^{abs}}((V,\mathcal{L}|_V,w|_V)\text{–}\mathsf{coh_{lf}}) \subset \mathsf{D^{co}}((V,\mathcal{L}|_V,w|_V)\text{–}\mathsf{qcoh_{fl}})$, respectively, is a direct summand of an object from the full triangulated subcategory $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{fl}}) \subset \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$.*

Proof. By Proposition 2.8, the category $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ is a full triangulated subcategory of the triangulated category $\mathsf{D}'_{Sing}(X_0)$. The (essential) intersection of the full subcategories $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ is the triangulated category $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$.

Indeed, an object of $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ belongs to $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$ if and only if the object $i_\circ\mathcal{F}$ vanishes in $\mathsf{D}^{\mathsf{b}}_{Sing}(X)$ (Theorem 2.7); an object $\mathcal{F} \in \mathsf{D}'_{Sing}(X_0)$ belongs to $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ if and only if the object $i_\circ\mathcal{F}$ vanishes in $\mathsf{D}'_{Sing}(X)$ (Proposition 2.8); and the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}'_{Sing}(X)$ is fully faithful.

Moreover, the (essential) intersection of $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ with the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ is the thick envelope of $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$ in $\mathsf{D}'_{Sing}(X_0)$. Indeed, let $\mathcal{M}$ be an object of the intersection; then $\mathcal{M} \oplus \mathcal{M}[1]$ belongs to both $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$, hence also to $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$, and consequently $\mathcal{M}$ belongs to the thick envelope of $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})$.

Now let $\mathcal{K}$ be our object of $\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}})$; it can be also viewed as an object of $\mathsf{D}'_{Sing}(X_0)$. If its restrictions to U and V belong to $\mathsf{D^{abs}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{coh_{lf}})$ and

$\mathsf{D^{abs}}((V,\mathcal{L}|_V,w|_V)\text{–}\mathsf{coh_{lf}})$, they also belong to $\mathsf{D}^{\mathsf{b}}_{Sing}(U_0)\subset \mathsf{D}'_{Sing}(U_0)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(V_0)\subset \mathsf{D}'_{Sing}(V_0)$ (where we set $U_0=U\cap X_0$ and $V_0=V\cap X_0$).

Applying Proposition, we can conclude that $\mathcal{K}$ belongs to the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$. The assertion of Corollary follows from the above. ☐

Assume additionally that the scheme X admits a dualizing complex $\mathcal{D}^\bullet_X$.

Theorem. *Let $X=U\cap V$ be a covering by two open subschemes. Then any object of $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ whose restrictions to U and V belong to the thick envelopes of the triangulated subcategories $\mathsf{D^{abs}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{coh_{lf}})\subset\mathsf{D^{abs}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{coh})$ and $\mathsf{D^{abs}}((V,\mathcal{L}|_V,w|_V)\text{–}\mathsf{coh_{lf}})\subset\mathsf{D^{abs}}((V,\mathcal{L}_V,w|_V)\text{–}\mathsf{coh})$ itself belongs to the thick envelope of the triangulated subcategory $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{fl}})\subset\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$.*

Proof. The argument is based on the Serre–Grothendieck duality theory for matrix factorizations as developed in Section 2.5, which allows to reduce the question to the result of Corollary. Specifically, let $\mathcal{M}$ be our coherent matrix factorization over X. Replacing, if necessary, $\mathcal{M}$ with $\mathcal{M}\oplus\mathcal{M}[1]$, we may assume the restrictions of $\mathcal{M}$ to U and V to be isomorphic to locally free matrix factorizations of finite rank.

Let us apply the construction of functor $\Omega\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})^{\mathsf{op}}\longrightarrow\mathsf{D^{co}}((X,\mathcal{L},-w)\text{–}\mathsf{qcoh_{fl}})$ from Section 2.5 to the matrix factorization $\mathcal{M}$. That is, we pick a left resolution of $\mathcal{M}$ by locally free matrix factorizations of finite rank, dualize by applying $\mathrm{Hom}_{X\text{–}\mathsf{qc}}(-,\mathcal{O}_X)$, and totalize using infinite direct sums. By Corollary 2.5, the functor Ω is fully faithful; it also identifies $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})^{\mathsf{op}}$ with $\mathsf{D^{abs}}((X,\mathcal{L},-w)\text{–}\mathsf{coh_{lf}})$. Hence it suffices to check that the matrix factorization $\Omega(\mathcal{M})$ belongs to the thick envelope of $\mathsf{D^{abs}}((X,\mathcal{L},-w)\text{–}\mathsf{coh_{lf}})$ in $\mathsf{D^{co}}((X,\mathcal{L},-w)\text{–}\mathsf{qcoh_{lf}})$. But we know as much from the above Corollary. ☐

Remark. As it was pointed out to the author by A. Efimov, one would like to have a version of the Thomason–Trobaugh localization theory for locally free matrix factorizations of finite rank. A key question is whether the restriction functor $\mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh_{lf}})\longrightarrow\mathsf{D^{abs}}((U,\mathcal{L}|_U,w|_U)\text{–}\mathsf{coh_{lf}})$ for an open subscheme $U\subset X$ is the composition of a Verdier localization functor with a fully faithful embedding adjoining some direct summands only. The lack of a workable notion of the conventional derived category (as opposed to the coderived category) for quasi-coherent matrix factorizations stands in the way of a direct extension of the Thomason–Trobaugh–Neeman arguments to the matrix factorization case. (Cf. Theorem 1.10 and Remark 3.5.)

3.2. **Supports.** This section paves the ground for the results about preservation of finite rank or coherence by the push-forwards of matrix factorizations with proper supports, which will be proven in Sections 3.4–3.5.

Let X be a separated Noetherian scheme and $T\subset X$ be a Zariski closed subset. Denote by $X\text{–}\mathsf{coh}_T$ the abelian category of coherent sheaves on X with the set-theoretic support in T; and similarly for quasi-coherent sheaves.

It is a well-known fact (essentially, a reformulation of the Artin–Rees lemma) that the embedding of abelian categories $X\text{–}\mathsf{qcoh}_T\longrightarrow X\text{–}\mathsf{qcoh}$ takes injectives to injectives. It follows that the functor $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ is fully faithful. Clearly, its image is a thick subcategory and the corresponding quotient category can be naturally identified with $\mathsf{D}^{\mathsf{b}}(U\text{–}\mathsf{coh})$, where $U = X \setminus T$ (cf. Section 1.10).

Assume additionally that X has enough vector bundles. Let $\mathit{Perf}_T(X) \subset \mathit{Perf}(X)$ denote the full subcategory of perfect complexes with the cohomology sheaves set-theoretically supported in T. By the above result, $\mathit{Perf}_T(X)$ can be considered as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_T)$. According to [30, Lemma 2.6], the functor $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_T)/\mathit{Perf}_T(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)$ induced by the embedding $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ is fully faithful. We denote the source (or the image) category of this functor by $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X, T)$.

By [6, Theorem 1.3], the restriction functor $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(U)$ is the Verdier localization functor by the triangulated subcategory $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X, T)$. In particular, the kernel of the restriction functor coincides with the thick envelope of (i. e., the minimal thick subcategory containing) $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X, T)$ in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)$.

Now we are going to establish the similar results for the triangulated categories of relative singularities. Let $i\colon Z \longrightarrow X$ be a closed subscheme such that $i_*\mathcal{O}_Z \in \mathit{Perf}(X)$, and let $\mathit{Perf}(Z/X) = \mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$ (see Remark 2.1) denote the thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ generated by $\mathbb{L}i^*\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$. Let $T \subset Z$ be a Zariski closed subset; put $U = X \setminus T$ and $V = Z \setminus T$. We denote by $\mathit{Perf}_T(Z/X)$ the full subcategory of all objects of $\mathit{Perf}(Z/X)$ with the cohomology sheaves set-theoretically supported in T. Consider it as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_T)$, and denote by $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X, T)$ the quotient category $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_T)/\mathit{Perf}_T(Z/X)$.

Lemma. (a) *The functor* $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X, T) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X)$ *induced by the embedding* $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ *is fully faithful.*

(b) *The restriction functor* $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(V/U)$ *is the Verdier localization functor by the triangulated subcategory* $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X, T)$*. In particular, the kernel of the restriction functor coincides with the thick envelope of* $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X, T)$ *in* $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(Z/X)$*.*

Proof. The proof of part (a) is similar to that of [30, Lemma 2.6]. One only needs to notice that the tensor product of an object of $\mathit{Perf}(Z/X)$ with an object of $\mathit{Perf}(Z)$ belongs to $\mathit{Perf}(Z/X)$. This follows from the fact that $\mathit{Perf}(Z)$ as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{coh})$ is generated by the restrictions of vector bundles from X (see Section 2.1). Part (b) is true, since the thick subcategory $\mathit{Perf}(V/U) \subset \mathsf{D}^{\mathsf{b}}(V\text{–}\mathsf{coh})$ is generated by the image of the restriction functor $\mathit{Perf}(Z/X) \longrightarrow \mathit{Perf}(V/U)$, which is because any coherent sheaf on U can be extended to a coherent sheaf on X. ☐

Let $\mathcal{L}$ be a line bundle over X and $w \in \mathcal{L}(X)$ be a section; set $X_0 = \{w = 0\} \subset X$. The definitions of the set-theoretic and category-theoretic supports $\operatorname{Supp} \mathcal{M}$ and $\operatorname{supp} \mathcal{M}$ of a coherent matrix factorization $\mathcal{M} \in (X, \mathcal{L}, w)\text{–}\mathsf{coh}$ were given (in a greater generality of coherent CDG-modules) in Section 1.10.

Given a locally free matrix factorization of finite rank $\mathcal{M} \in (X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{lf}}$, define the *(category-theoretic) support* $\operatorname{supp} \mathcal{M} \subset X$ as the minimal closed subset $T \subset X$

such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is absolutely acyclic with respect to $(U, \mathcal{L}|_U, w|_U)$–$\mathsf{coh_{lf}}$. By Corollary 2.3(i), the definitions of category-theoretic supports of coherent matrix factorizations and of locally free matrix factorizations of finite rank agree when they are both applicable.

Equivalently, for a locally free matrix factorization $\mathcal{M}$ of finite rank over X, the open subscheme $X \setminus \operatorname{supp} \mathcal{M}$ is the union of all affine open subschemes $U \subset X$ such that the matrix factorization $\mathcal{M}|_U$ is contractible (see Remark 1.3). For any coherent matrix factorization $\mathcal{M}$ one has $\operatorname{supp} \mathcal{M} \subset X_0$, since any matrix factorization of an invertible potential is contractible (cf. [33, Section 5]).

Let $T \subset X$ be a closed subset. Denote by $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}})$ (respectively, $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh}))$ the quotient category of the homotopy category of locally free matrix factorizations of finite rank (resp., coherent matrix factorizations) supported category-theoretically inside T by the thick subcategory of matrix factorizations absolutely acyclic with respect to $(X, \mathcal{L}, w)$–$\mathsf{coh_{lf}}$ (resp., $(X, \mathcal{L}, w)$–coh). Clearly, the functors $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}})$ and $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh})$ are fully faithful [33].

By the definition, the thick subcategories $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}})$ and $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh})$ only depend on the intersection $X_0 \cap T$ (rather than the whole of T). Equivalently, they can be defined as the full subcategories of objects annihilated by the restriction functors $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)$–$\mathsf{coh_{lf}})$ and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)$–$\mathsf{coh})$, where $U = X \setminus T$.

As in Section 1.10, we denote by $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T)$ the absolute derived category of coherent matrix factorizations with the set-theoretic support in T. The functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh})$ is fully faithful by Proposition 1.10(d). By Corollary 1.10(b), the full subcategory $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh})$ is the thick envelope of the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T)$.

Now assume that $w\colon \mathcal{O}_X \longrightarrow \mathcal{L}$ is an injective morphism of sheaves.

Proposition. (a) *The equivalence of categories* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ *identifies the triangulated subcategory* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T)$ *with the triangulated subcategory* $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X, X_0 \cap T)$. *In particular, the former triangulated subcategory only depends on the intersection* $X_0 \cap T$.

(b) *The full preimage of the thick envelope of the triangulated subcategory* $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0, X_0 \cap T) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *under the fully faithful functor* $\Sigma\colon \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *coincides with the triangulated subcategory* $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}})$.

Proof. Part (b) follows from the fact that the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0,\ X_0 \cap T)$ is the kernel of the restriction functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0 \setminus T)$, the similar fact for $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)$–$\mathsf{coh_{lf}})$, and the compatibility of the functors Σ with the restrictions to open subschemes, together with their full-and-faithfulness.

To prove part (a), notice first that the functor Υ obviously takes $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,\ X_0 \cap T)$ into $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T)$. Let us check that the functor $\mathbb{L}\Xi$ takes $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh}_T)$ into $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,\ X_0 \cap T)$. Let $\mathcal{M}$ be a coherent matrix factorization supported set-theoretically in T. Present $\mathcal{M}$ as the cokernel of an injective morphism of w-flat coherent matrix factorizations $\mathcal{K} \longrightarrow \mathcal{N}$. The functor $\mathbb{L}\Xi$ being triangulated, the object $\mathbb{L}\Xi(\mathcal{M}) \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is isomorphic to the cone of the morphism $\Xi(\mathcal{K}) \longrightarrow \Xi(\mathcal{N})$ (cf. Lemma 3.5). The morphism $\Xi(\mathcal{K}) \longrightarrow \Xi(\mathcal{N})$ of coherent sheaves on X_0 is an isomorphism outside T, so its kernel and cokernel are supported in $X_0 \cap T$. Thus the cone is quasi-isomorphic to a two-term complex of coherent sheaves on X_0 with the terms supported set-theoretically in $X_0 \cap T$. $\square$

3.3. **Pull-backs and push-forwards in singularity categories.** Let $f\colon Y \longrightarrow X$ be a morphism of separated Noetherian schemes with enough vector bundles. The morphism f is said to have *finite flat dimension* if the derived inverse image functor $\mathbb{L}f^*\colon \mathsf{D}^-(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$ to $\mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{qcoh})$.

In this case, the functor $\mathbb{L}f^*$ induces the inverse image functors on the triangulated categories of singularities

$$f^\circ\colon \mathsf{D}'_{Sing}(X) \longrightarrow \mathsf{D}'_{Sing}(Y)$$
$$f^\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Y).$$

Under the same assumption of finite flat dimension, the derived direct image functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ to $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}_{\mathsf{fl}})$, as one can see by computing $\mathbb{R}f_*$ in terms of an affine covering of Y in the spirit of the proof of Proposition 1.9. When the scheme X has finite Krull dimension, one has $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}_{\mathsf{lf}})$, so the functor $\mathbb{R}f_*$ induces the direct image functor

$$f_\circ\colon \mathsf{D}'_{Sing}(Y) \longrightarrow \mathsf{D}'_{Sing}(X),$$

which is right adjoint to f°.

Whenever the morphism f is proper of finite type and has finite flat dimension, the functor $\mathbb{R}f_*$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{coh})$ to $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ [12, Théorème 3.2.1] and induces the direct image functor

$$f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X),$$

which is right adjoint to f° [29, paragraphs before Proposition 1.14]. More generally, for a morphism f of finite flat dimension and any closed subset $T \subset Y$ such that (a closed subscheme structure on) T is proper of finite type over X, the functor $\mathbb{R}f_*$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{coh}_T)$ to $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ and induces the direct image functor

$$f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y, T) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X).$$

Indeed, the intersection of $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$ in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$ is equal to $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh}_{\mathsf{lf}})$, as any complex of finite flat dimension with bounded coherent cohomology is easily seen to be perfect.

Let $Z \subset X$ and $W \subset Y$ be closed subschemes such that $\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module, $\mathcal{O}_W$ is a perfect $\mathcal{O}_Y$-module, and $f(W) \subset Z$. Assume that both morphisms $f\colon Y \longrightarrow X$ and $f|_W\colon W \longrightarrow Z$ have finite flat dimensions. Then the derived inverse image functor $\mathbb{L}f|_W^*\colon \mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(W\text{–}\mathsf{qcoh})$ induces the inverse image functors on the triangulated categories of relative singularities

$$f^\circ\colon \mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}'_{Sing}(W/Y)$$
$$f^\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y).$$

Now let $Z \subset X$ be a closed subscheme; set $W = Z \times_X Y$. Denote the closed embeddings $Z \longrightarrow X$ and $W \longrightarrow Y$ by i and i', respectively; let also f' denote the morphism $f|_W\colon W \longrightarrow Z$. Assume that W coincides with the derived product of Z and Y over X, i. e., $\mathbb{L}f^*i_*\mathcal{O}_Z = i'_*\mathcal{O}_W$. Assume further that $i_*\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module; then also $i'_*\mathcal{O}_W$ is a perfect $\mathcal{O}_Y$-module.

For any $\mathcal{M} \in \mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{qcoh})$ there is a natural morphism $\phi_{\mathcal{M}}\colon \mathbb{L}i^*\mathbb{R}f_*\mathcal{M} \longrightarrow \mathbb{R}f'_*\mathbb{L}i'^*\mathcal{M}$ in $\mathsf{D}^{\mathsf{b}}(Z\text{–}\mathsf{qcoh})$. Using the projection formula for tensor products with perfect complexes, one easily checks that the morphism $i_*\phi_{\mathcal{M}}$ is an isomorphism. Hence so is the morphism $\phi_{\mathcal{M}}$, since the functor i_* does not annihilate any objects of the derived category. Hence we obtain the induced functor of direct image

$$f_\circ\colon \mathsf{D}'_{Sing}(W/Y) \longrightarrow \mathsf{D}'_{Sing}(Z/X).$$

When the morphism f is proper of finite type, there is also the induced functor

$$f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X).$$

Assume additionally that the morphism f has finite flat dimension; then so does the morphism f'. In this case the functor $f_\circ\colon \mathsf{D}'_{Sing}(W/Y) \longrightarrow \mathsf{D}'_{Sing}(Z/X)$ is right adjoint to the functor $f^\circ\colon \mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}'_{Sing}(W/Y)$. When the morphism f is proper of finite type, the functor $f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is right adjoint to the functor $f^\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y)$.

Remark. In the case when Z is a Cartier divisor in X, we will construct the functor $f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ under somewhat weaker assumptions below in Section 3.4. Namely, it will suffice that the morphism $f'\colon W \longrightarrow Z$ be proper of finite type, while the morphism $f\colon Y \longrightarrow Z$ need not be. A generalization to the case of proper support will also be obtained.

3.4. **Push-forwards of matrix factorizations.** Let $f\colon Y \longrightarrow X$ be a morphism of separated Noetherian schemes with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a section.

Set $\mathcal{B}_X = (X, \mathcal{L}, w)$ and $\mathcal{B}_Y = (Y, f^*\mathcal{L}, f^*w)$; then there is a natural morphism of CDG-algebras $\mathcal{B}_X \longrightarrow \mathcal{B}_Y$ compatible with the morphism of schemes $f\colon Y \longrightarrow X$. Therefore, according to Section 1.8, there are the derived inverse image functors

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$$
$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}_{\mathsf{ffd}})$$

and the derived direct image functor

$$\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X, \mathcal{L}, w)\text{–}\mathsf{qcoh}).$$

The latter two functors are “partially adjoint” to each other.

Given a triangulated category D, we denote by $\overline{\mathsf{D}}$ its idempotent completion. By [1, Section 1], the category $\overline{\mathsf{D}}$ has a natural structure of triangulated category.

Lemma. *For any closed subset $T \subset Y$ such that (for a closed subscheme structure on T) the morphism $f|_T \colon T \longrightarrow X$ is proper of finite type, the functor $\mathbb{R}f_*$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}_T) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh})$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$, thus definining a triangulated functor of direct image*

$$\mathbb{R}f_* \colon \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}).$$

Consequently, there is the triangulated functor

$$\overline{\mathbb{R}f_*} \colon \overline{\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})}.$$

Proof. We will use the construction of the functor $\mathbb{R}f_* \colon \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ similar to the one in the proof of Proposition 1.9 (see Remark 1.9). According to this construction, given a matrix factorization $\mathcal{M} \in (Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh}$, the object $\mathbb{R}f_*\mathcal{M} \in \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ is represented by the total matrix factorization $\mathbb{R}_{\{U_\alpha\}}f_*\mathcal{M}$ of the finite Čech complex $f_*C^\bullet_{\{U_\alpha\}}\mathcal{M}$ of matrix factorizations on X. The derived functor of direct image of complexes of quasi-coherent sheaves $\mathbb{R}f_* \colon \mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$ can be constructed in the same way.

By [12, Théorème 3.2.1], the latter functor takes $\mathsf{D}^{\mathsf{b}}(Y\text{–}\mathsf{coh}_T)$ into $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{coh})$. Hence the cohomology matrix factorizations of the finite complex of matrix factorizations $f_*C^\bullet_{\{U_\alpha\}}\mathcal{M}$ belong to $(X, \mathcal{L}, w)\text{–}\mathsf{coh}$ when the matrix factorization $\mathcal{M}$ belongs to $(Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}_T$. It follows that the object $\mathbb{R}f_*\mathcal{M}$ belongs to $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$ $\subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{–}\mathsf{qcoh})$ in this case.

To prove the last assertion, it remains to apply Corollary 1.10(b). ☐

Now assume that both morphisms of sheaves $w \colon \mathcal{O}_X \longrightarrow \mathcal{L}$ and $f^*w \colon \mathcal{O}_Y \longrightarrow f^*\mathcal{L}$ are injective. Let $X_0 \subset X$ and $Y_0 \subset Y$ denote the closed subschemes defined locally by the equations $w = 0$ and $f^*w = 0$, respectively. In this setting, we will compare the constructions of direct image functors for matrix factorizations and for the triangulated categories of relative singularities, and prove the assertions of Lemma in a different way. Recall that in Section 3.3 we have constructed the functor of direct image $f_\circ \colon \mathsf{D}'_{Sing}(Y_0/Y) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$.

Proposition. (a) *Whenever the morphism $f_0 = f|_{Y_0} \colon Y_0 \longrightarrow X_0$ is proper of finite type, the functor $\mathbb{R}f_*$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh})$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)$ $\text{–}\mathsf{qcoh})$, thus defining a triangulated functor*

$$\mathbb{R}f_* \colon \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}).$$

(b) *For any closed subset $T \subset Y_0$ such that (for a closed subscheme structure on T) the morphism $f_0|_T \colon T \longrightarrow X_0$ is proper of finite type, the functor $f_\circ$ takes the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T) \subset \mathsf{D}'_{Sing}(Y_0/Y)$ into the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \subset$*

$\mathsf{D}'_{Sing}(X_0/X)$, *thus defining a triangulated functor*

$$f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

(c) *The equivalences of categories* $\mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_T) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,T)$ *from Proposition* 3.2(a) *and* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ *from Theorem* 2.7 *transform the direct image functor* $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ *from Lemma into the direct image functor* $f_\circ$ *from part* (b).

Proof. Part (a) follows from Lemma and Proposition 3.2(a), or alternatively, from part (b) and the proof of part (c) below. In part (b), the fact of key importance is that the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$ is fully faithful (by Theorem 2.8). The functor $f_\circ$ takes $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)$ into $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$, because the functor $\mathbb{R}f_{0*}\colon \mathsf{D}^{\mathsf{b}}(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(Y_0\text{–}\mathsf{coh})$ into $\mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{coh})$ [12]. To prove part (c), we will check that the equivalences of categories from Theorem 2.8 transform the functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ into the functor $f_\circ\colon \mathsf{D}'_{Sing}(Y_0/Y) \longrightarrow \mathsf{D}'_{Sing}(X_0/X)$. (Together with part (b) and Proposition 3.2(a), this will also provide another proof of Lemma.)

For this purpose, extend the functor $\Upsilon_Y\colon \mathsf{D}^{\mathsf{b}}(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$ to a functor $\widetilde{\Upsilon}_Y\colon \mathsf{D}^+(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$ in the obvious way (taking infinite direct sums of quasi-coherent sheaves in the construction of the matrix factorization $\widetilde{\Upsilon}_Y(\mathcal{F}^\bullet)$). The functor $\widetilde{\Upsilon}_Y$ is well-defined, since any bounded below acyclic complex of quasi-coherent sheaves is coacyclic [36, Lemma 2.1]. Furthermore, the functor $\widetilde{\Upsilon}_Y$ can be presented as the composition of the "periodicity summation" functor $\mathsf{D}^+(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y_0,i'^*f^*\mathcal{L},0)\text{–}\mathsf{qcoh})$ taking values in the coderived category of quasi-coherent matrix factorizations of the zero potential on Y_0, and the functor of direct image $i'_*\colon \mathsf{D}^{\mathsf{co}}((Y_0,i'^*f^*\mathcal{L},0)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$ with respect to the closed embedding i'.

The functors $\mathbb{R}f_{0*}\colon \mathsf{D}^+(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(X_0\text{–}\mathsf{qcoh})$ and $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ form a commutative diagram with the functors $\widetilde{\Upsilon}_X$ and $\widetilde{\Upsilon}_Y$. Indeed, the "periodicity summations" of bounded below complexes of quasi-coherent sheaves on Y_0 and X_0, taking injective resolutions to injective resolutions, obviously commute with the derived direct images with respect to f', as the direct image preserves infinite direct sums. Furthermore, the derived direct images of quasi-coherent matrix factorizations are compatible with the compositions of morphisms of schemes (see Remark 1.8), hence also commute with each other. It follows that the functors $\mathbb{R}f_*$ and $f_\circ$ agree as they should. (Alternatively, one can prove this in the way similar to the proof of Proposition 3.5 below.) ☐

3.5. **Push-forwards for morphisms of finite flat dimension.** Let $f\colon Y \longrightarrow X$ be a morphism of finite flat dimension between separated Noetherian schemes with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a section. As in Section 3.4, we have a natural morphism of CDG-algebras $\mathcal{B}_X = (X,\mathcal{L},w) \longrightarrow \mathcal{B}_Y = (Y,f^*\mathcal{L},f^*w)$ compatible with the morphism of schemes $Y \longrightarrow X$.

The quasi-coherent graded algebra $\mathcal{B}_Y$ has finite flat dimension over $\mathcal{B}_X$. Therefore, according to Section 1.9, there are derived inverse image functors

$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$$
$$\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}),$$

the former of which is left adjoint to the functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ from Section 3.4.

Furthermore, according to Proposition 1.9, there is a derived direct image functor

$$\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$$

which is right adjoint to the functor $\mathbb{L}f^*\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ from Section 3.4.

Now assume that X and Y have finite Krull dimensions. Recall that the natural triangulated functors $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ are fully faithful by Corollary 2.3(e) and (j).

As in the second half of Section 3.4, assume that both morphisms of sheaves $w\colon \mathcal{O}_X \longrightarrow \mathcal{L}$ and $f^*w\colon \mathcal{O}_Y \longrightarrow f^*\mathcal{L}$ are injective, and denote by $f_0\colon Y_0 \longrightarrow X_0$ the induced morphism between the zero loci schemes of f^*w and w. Since the morphism f has finite flat dimension, so does the morphism f_0.

Proposition. (a) *Whenever the morphism f_0 is proper of finite type, the functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$. Besides, the functor $f_{0\circ}\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0)$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. Both restrictions define the same triangulated functor*

$$\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}).$$

(b) *For any closed subset $T \subset Y_0$ such that (for a closed subscheme structure on T) the morphism $f_0|_T\colon T \longrightarrow X_0$ is proper of finite type, the functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ into the thick envelope of the full subcategory $\overline{\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})} \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$. Besides, the triangulated functor $\overline{f_{0\circ}}\colon \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0,T)} \longrightarrow \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0,T)}$ into the thick envelope of the full subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}$. Both restrictions define the same triangulated functor*

$$\overline{\mathbb{R}f_*}\colon \overline{\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh}_{\mathsf{lf}})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})}.$$

Proof. Both categories $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ are full triangulated subcategories of the triangulated category $\mathsf{D}'_{Sing}(X_0)$ (see Proposition 2.8 and [29, Proposition 1.13]). According to the proof of Corollary 3.1, the intersection of $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ with (the thick envelope of) $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ (is the thick envelope of) the subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}'_{Sing}(X_0)$.

Thus it suffices to show that the direct image functor $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ agrees with the direct image functor $f_{0\circ}\colon \mathsf{D}'_{Sing}(Y_0) \longrightarrow \mathsf{D}'_{Sing}(X_0)$. The latter assertion does not depend on any properness assumptions.

Recall that the derived functor $\mathbb{R}f_*$ was constructed in the proof of Proposition 1.9 in terms of the Čech complex whose terms are direct sums of the CDG-modules $f|_{V*}\mathcal{M}|_V$, where $\mathcal{M} \in \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{ffd}})$ and $V \subset Y$. The derived direct image $\mathbb{R}f_{0*}\colon \mathsf{D}^{\mathsf{b}}(Y_0\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{–}\mathsf{qcoh})$ can be constructed in the similar way; moreover, one can use for this purpose the restriction to Y_0 of an affine open covering U_α of the scheme Y.

We will make use of the flat dimension analogue of Corollary 2.6(d). Let $\widetilde{\Sigma}'_X$ and $\widetilde{\Sigma}'_Y$ denote the obvious extensions of the functors Σ' from $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}$ to the category of w-flat matrix factorizations of finite flat dimension $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ and from $(Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{lf}}$ to $(Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{f^*w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ (see the proofs of Proposition 2.8 and Theorem 2.7). Notice that the direct image functors $f|_{V*}$ take f^*w-flat sheaves to w-flat sheaves and $(V,f^*\mathcal{L}|_V,f^*w|_V)\text{–}\mathsf{qcoh}_{f^*w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ to $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$.

Let $\mathcal{N}$ be a matrix factorization from $(Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{f^*w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$. Since the open subschemes V are presumed to be affine, there are natural isomorphisms $\widetilde{\Sigma}'_X(f|_{V*}\mathcal{N}|_V) \simeq f_0|_{V\cap Y_0}{}_*\widetilde{\Sigma}'_Y(\mathcal{N})|_{V\cap Y_0}$ of quasi-coherent sheaves on X_0. Now it remains to use the next lemma. ☐

Lemma. *Let $\mathcal{M}^{-n} \longrightarrow \cdots \longrightarrow \mathcal{M}^N$ be a finite complex of matrix factorizations from $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}\cap\mathsf{ffd}}$ and $\mathcal{M}$ be its totalization. Then the complex $\widetilde{\Sigma}'(\mathcal{M}^{-n}) \longrightarrow \cdots \longrightarrow \widetilde{\Sigma}'(\mathcal{M}^N)$ and the quasi-coherent sheaf $\widetilde{\Sigma}'(\mathcal{M})$ on X_0 represent naturally isomorphic objects in the triangulated category of singularities $\mathsf{D}'_{Sing}(X_0)$. The same applies to a finite complex of matrix factorizations from $(X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{w\text{–}\mathsf{fl}}$, the functor Ξ, and the triangulated category of relative singularities $\mathsf{D}''_{Sing}(X_0/X)$.*

Proof. For each $-n \le p \le N$, the restriction of the matrix factorization $\mathcal{M}^p$ to the closed subscheme $X_0 \subset X$ is an unbounded complex of quasi-coherent sheaves $i^*\mathcal{M}^{p,\bullet}$. By [33, Lemma 1.5], this complex is acyclic.

The complex $\widetilde{\Sigma}'(\mathcal{M}^{-n}) \longrightarrow \cdots \longrightarrow \widetilde{\Sigma}'(\mathcal{M}^N)$ of quasi-coherent sheaves on X_0 is quasi-isomorphic to the total complex of the bicomplex $\mathcal{K}^{\bullet,\bullet}$ with the terms $\mathcal{K}^{p,0} = i^*\mathcal{M}^{p,0}$, $\mathcal{K}^{p,-1} = i^*\mathcal{M}^{p,-1}$, $\mathcal{K}^{p,-2} = \ker(i^*\mathcal{M}^{p,-1} \to i^*\mathcal{M}^{p,0})$, and $\mathcal{K}^{p,q} = 0$ for $q \ne 0$, -1, -2. Similarly, the quasi-coherent sheaf $\widetilde{\Sigma}'(\mathcal{M})$ on X_0 is quasi-isomorphic to the total complex of the bicomplex $\mathcal{E}^{\bullet,\bullet}$ with the terms $\mathcal{E}^{p,p} = i^*\mathcal{M}^{p,p}$, $\mathcal{E}^{p,p-1} = i^*\mathcal{M}^{p,p-1}$, $\mathcal{E}^{p,p-2} = \ker(i^*\mathcal{M}^{p,p-1} \to i^*\mathcal{M}^{p,p})$, and $\mathcal{E}^{p,q} = 0$ for $q - p \ne 0$, -1, -2.

We can assume that N, $n \ge 0$. Consider the bicomplex $\mathcal{F}^{\bullet,\bullet}$ with the terms $\mathcal{F}^{p,q} = i^*\mathcal{M}^{p,q}$ for $-n-1 \le q \le N$, $\mathcal{F}^{p,-n-2} = \ker(i^*\mathcal{M}^{p,-n-1} \to i^*\mathcal{M}^{p,-n})$, and $\mathcal{F}^{p,q} = 0$ for $q < -n-2$ or $q > N$. Then there are natural surjective morphisms of bicomplexes $\mathcal{F}^{\bullet,\bullet} \longrightarrow \mathcal{K}^{\bullet,\bullet}$ and $\mathcal{F}^{\bullet,\bullet} \longrightarrow \mathcal{E}^{\bullet,\bullet}$. The kernels of both morphisms are the direct sums of a finite bicomplex of quasi-coherent sheaves of finite flat dimension on X_0 and a finite bicomplex of quasi-coherent sheaves on X_0 with acyclic columns. Thus both morphisms become isomorphisms in $\mathsf{D}'_{Sing}(X_0)$. $\square$

Remark. One would like to have a theory of set-theoretic supports for locally free matrix factorizations of finite rank that would allow to prove the above Proposition in the way similar to the proof of Lemma 3.4. However, we do not know how to do this. In particular, we do *not* know whether every locally free matrix factorization of finite rank with the category-theoretic support in T is isomorphic in the absolute derived category to a direct summand of an object represented by a coherent matrix factorization of finite flat dimension with the set-theoretic support in T (cf. Corollary 1.10).

Another alternative approach to proving Proposition would be to show that the intersection of the full subcategories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{lf}})$ in the absolute derived category $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ coincides with the full subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh}_{\mathsf{lf}})$. We do *not* know whether this is true.

3.6. **Duality and push-forwards.** In the following two sections we discuss the compatibility properties of the derived direct and inverse image functors for matrix factorizations with the Serre–Grothendieck duality functors from Section 2.5.

Let X be a separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$, and let $f\colon Y \longrightarrow X$ be a separated morphism of finite type. As usually, we set $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$, where f^+ is the functor denoted by $f^!$ in [14] (right adjoint to $\mathbb{R}f_*$ for proper morphisms f and left adjoint to $\mathbb{R}f_*$ for open embeddings f) (see [26, Example 4.2] and [14, Remark before Proposition V.8.5 and Deligne's Appendix]). This formula defines the dualizing complex $\mathcal{D}_Y^\bullet$ up to a natural quasi-isomorphism only, and we presume this derived category object (as well as $\mathcal{D}_X^\bullet$) to be represented by a finite complex of injective quasi-coherent sheaves.

Proposition. *Let $T \subset Y_0$ be a closed subset such that that (for some closed subscheme structure on T) the morphism $f|_T\colon T \longrightarrow X_0$ is proper. Then the derived direct image functor $\overline{\mathbb{R}f_*}\colon \overline{\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})}$ and the similar functor for the potential $-w$ form a commutative diagram with the Serre duality functors $\mathcal{H}om_{X\text{–}\mathsf{qc}}(-,\mathcal{D}_X^\bullet)\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{–}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{coh})$ and $\mathcal{H}om_{Y\text{–}\mathsf{qc}}(-,\mathcal{D}_Y^\bullet)\colon \mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},-f^*w)\text{–}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{coh})$.*

Two proofs of Proposition are given below. One of them is based on the theory of set-theoretic supports of coherent CDG-modules developed in Section 1.10 and the arguments similar to the proof of Lemma 3.4. It does not depend on the assumption about w and f^*w being local nonzero-divisors and does not mention the zero loci. The other proof is based on the passage to the triangulated categories of relative singularities and uses Proposition 3.4(c).

First proof. First of all, the duality functor $\mathcal{H}om_{Y\text{–}\mathsf{qc}}(-,\mathcal{D}_Y^\bullet)\colon \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},-f^*w)$ $\text{–}\mathsf{qcoh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$ obviously takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y,$

$f^*\mathcal{L}, -f^*w)$–$\mathsf{coh}_T)^{\mathsf{op}}$ into $\mathsf{D^{abs}}((Y, f^*\mathcal{L}, f^*w)$–$\mathsf{coh}_T)$ and vice versa. Furthermore, for any quasi-coherent sheaf $\mathcal{K}$ on Y denote by $\Gamma_T\mathcal{K} \subset \mathcal{K}$ the maximal quasi-coherent subsheaf with the set-theoretic support in T. Then for any matrix factorization $\mathcal{M} \in \mathsf{D^{abs}}((Y, f^*\mathcal{L}, -f^*w)$–$\mathsf{coh}_T)$ the natural morphism $\mathcal{H}om_{Y\text{–}\mathsf{qc}}(\mathcal{M}, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{Y\text{–}\mathsf{qc}}(\mathcal{M}, \mathcal{D}_Y^\bullet)$ is an isomorphism in $\mathsf{D^{abs}}((Y, f^*\mathcal{L}, f^*w)$–$\mathsf{coh}_T)$.

As in the proof of Lemma 3.4, we will use the construction of the functor $\mathbb{R}f_*\colon \mathsf{D^{abs}}((Y, f^*\mathcal{L}, f^*w)$–$\mathsf{qcoh}) \longrightarrow \mathsf{D^{abs}}((X, \mathcal{L}, w)$–$\mathsf{qcoh})$ similar to the one from the proof of Proposition 1.9 (see Remarks 1.8 and 1.9). Let $\{U_\alpha\}$ and $\{V_\beta\}$ be two affine open coverings of the scheme Y. For any matrix factorization $\mathcal{N} \in (Y, f^*\mathcal{L}, -f^*w)$–$\mathsf{qcoh}$, there is a natural morphism of bicomplexes of matrix factorizations $f_*C^\bullet_{\{U_\alpha\}}\mathcal{H}om_{Y\text{–}\mathsf{qc}}(\mathcal{N}, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(f_*\mathcal{N}, f_*C^\bullet_{\{U_\alpha\}}\Gamma_T\mathcal{D}_Y^\bullet)$. Passing to the total complexes and taking the composition with the adjunction morphism $f_*C^\bullet_{\{U_\alpha\}}\Gamma_T\mathcal{D}_Y^\bullet = \mathbb{R}f_*(\Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{D}_X^\bullet$, we obtain a natural morphism of complexes of matrix factorizations $f_*C^\bullet_{\{U_\alpha\}}\mathcal{H}om_{Y\text{–}\mathsf{qc}}(\mathcal{N}, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(f_*\mathcal{N}, \Gamma_T\mathcal{D}_X^\bullet)$ (cf. [26, beginning of Section 6]).

Substituting $\mathcal{N} = C^\bullet_{\{V_\beta\}}\mathcal{M}$ for some $\mathcal{M} \in (Y, f^*\mathcal{L}, -f^*w)$–$\mathsf{qcoh}$, we get a natural morphism of bicomplexes of matrix factorizations $f_*C^\bullet_{\{U_\alpha\}}\mathcal{H}om_{Y\text{–}\mathsf{qc}}(C^\bullet_{\{V_\beta\}}\mathcal{M}, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{–}\mathsf{qc}}(f_*C^\bullet_{\{V_\beta\}}\mathcal{M}, \mathcal{D}_X^\bullet)$. When $\mathcal{M}$ is a coherent matrix factorization supported set-theoretically in T, the induced morphism of the total complexes is a quasi-isomorphism of complexes of matrix factorizations by the conventional Serre–Grothendieck duality theorem for bounded derived categories of coherent sheaves and proper morphisms of schemes (see [14, Theorem VII.3.3] or [26, Section 6]). Hence the induced morphism of the total matrix factorizations is an isomorphism in $\mathsf{D^{abs}}((X, \mathcal{L}, w)$–$\mathsf{qcoh})$, and consequently also in $\mathsf{D^{abs}}((X, \mathcal{L}, w)$–$\mathsf{coh})$. □

Second proof. Assume that w and f^*w are locally nonzero-dividing sections of the respective line bundles. Let $i\colon X_0 \longrightarrow X$ be the zero locus of w and $i'\colon Y_0 \longrightarrow Y$ be the zero locus of f^*w. As above, we set $\mathcal{D}_{X_0}^\bullet = \mathbb{R}i^!\mathcal{D}_X^\bullet$ and $\mathcal{D}_{Y_0}^\bullet = \mathbb{R}i'^!\mathcal{D}_Y^\bullet$ [14, Proposition V.2.4], and presume all these dualizing complexes to be finite complexes of injective quasi-coherent sheaves.

The duality functor $\mathcal{H}om_{Y\text{–}\mathsf{qc}}(-, \mathcal{D}_Y^\bullet)\colon \mathsf{D^{abs}}((Y, f^*\mathcal{L}, -f^*w)$–$\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D^{abs}}((Y, f^*\mathcal{L}, f^*w)$–$\mathsf{coh})$ is compatible with the restrictions to the open subscheme $Y \setminus T$ and therefore identifies the full subcategories $\mathsf{D}_T^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)$–$\mathsf{coh})^{\mathsf{op}}$ and $\mathsf{D}_T^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)$–$\mathsf{coh})$. To prove the proposition, we will define the Serre duality functors on the triangulated categories of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X/X_0)$, then check that the equivalences of triangulated categories $\mathbb{L}\Xi = \Upsilon^{-1}$ commute with the dualities, and finally reduce to the conventional Serre–Grothendieck duality theorem for bounded complexes of coherent sheaves.

The duality functor $\mathcal{H}om_{X_0\text{–}\mathsf{qc}}(-, \mathcal{D}_{X_0}^\bullet)\colon \mathsf{D^b}(X_0\text{–}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D^b}(X_0\text{–}\mathsf{coh})$ takes objects of the form $\mathbb{L}i^*\mathcal{K}^\bullet$, where $\mathcal{K}^\bullet \in \mathsf{D^b}(X\text{–}\mathsf{coh})$, to similar objects. Indeed, one has $\mathcal{H}om_{X_0\text{–}\mathsf{qc}}(\mathbb{L}i^*\mathcal{K}^\bullet, \mathcal{D}_{X_0}^\bullet) \simeq \mathbb{R}i^!\,\mathcal{H}om_{X\text{–}\mathsf{qc}}(\mathcal{K}^\bullet, \mathcal{D}_X^\bullet)$ [14, Proposition V.8.5] and $\mathbb{R}i^! \simeq$

$\mathcal{L}|_{X_0}[-1]\otimes_{\mathcal{O}_{X_0}}\mathbb{L}i^*$ (see the proof of Theorem 2.7). Therefore, we have the induced duality functor $\mathcal{H}om_{X_0\text{–qc}}(-,\mathcal{D}^\bullet_{X_0})\colon \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)^{\mathsf{op}}\longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$. Similarly, the duality functor $\mathcal{H}om_{Y_0\text{–qc}}(-,\mathcal{D}^\bullet_{Y_0})\colon \mathsf{D}^{\mathsf{b}}(Y_0\text{–coh})^{\mathsf{op}}\longrightarrow \mathsf{D}^{\mathsf{b}}(Y_0\text{–coh})$ takes the full subcategory $\mathsf{D}^{\mathsf{b}}(Y_0\text{–coh}_T)^{\mathsf{op}}$ into $\mathsf{D}^{\mathsf{b}}(Y_0\text{–coh}_T)$ and $Perf_T(Y_0/Y)^{\mathsf{op}}$ into $Perf_T(Y_0/Y)$. Hence the induced duality functor $\mathcal{H}om_{Y_0\text{–qc}}(-,\mathcal{D}^\bullet_{Y_0})\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)^{\mathsf{op}}\longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)$.

Checking that the equivalence of categories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–coh})\simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ commutes with the dualities is easily done using the functor Υ. It suffices to notice the functorial quasi-isomorphism $\mathcal{H}om_{X\text{–qc}}(i_*\mathcal{F}^\bullet,\mathcal{D}^\bullet_X)\simeq i_*\mathcal{H}om_{X_0\text{–qc}}(\mathcal{F}^\bullet,\mathcal{D}^\bullet_{X_0})$ for any complex $\mathcal{F}^\bullet\in \mathsf{D}^{\mathsf{b}}(X_0\text{–coh})$ [14, Theorem III.6.7]. The same applies to the equivalence of categories $\overline{\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–coh})}\simeq \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)}$. Furthermore, by Proposition 3.4(c), the equivalences of categories $\mathbb{L}\Xi=\Upsilon^{-1}$ transform the derived direct image functor $\overline{\mathbb{R}f_*}\colon \overline{\mathsf{D}^{\mathsf{abs}}_T((Y,f^*\mathcal{L},f^*w)\text{–coh})}\longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–coh})}$ into (the idempotent closure of) the direct image functor $f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)\longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$.

Finally, the direct image functor $f_\circ\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y,T)\longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ commutes with the Serre duality functors, since so do the derived direct image functors $\mathbb{R}f|_{T*}\colon \mathsf{D}^{\mathsf{b}}(\widetilde{T}\text{–coh})\longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{–coh})$ for all the closed subscheme structures $\widetilde{T}\subset Y_0$ on the closed subset T and the similar functors related to the closed embeddings $\widetilde{T}'\longrightarrow \widetilde{T}''$ of various such subscheme structures into each other. This is the conventional Serre–Grothendieck duality theorem for proper morphisms of schemes. ☐

3.7. **Duality and pull-backs.** Let X be a separated Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet_X$ and $f\colon Y\longrightarrow X$ be a separated morphism of finite type; set $\mathcal{D}^\bullet_Y=f^+\mathcal{D}^\bullet_X$. Let $\mathcal{L}$ be a line bundle on X and $w\in\mathcal{L}(X)$ be a section.

Let us first suppose that the morphism f is smooth of relative dimension n. Then the functor $f^+\colon \mathsf{D}^+(X\text{–qcoh})\longrightarrow \mathsf{D}^+(Y\text{–qcoh})$ is naturally isomorphic to $\omega_{Y/X}[n]\otimes_{\mathcal{O}_Y}f^*$, where $\omega_{Y/X}$ is the line bundle of relative top forms.

In particular, $\mathcal{D}^\bullet_Y\simeq\omega_{Y/X}[n]\otimes_{\mathcal{O}_Y}f^*\mathcal{D}^\bullet_X$ (where $f^*\mathcal{D}^\bullet_X$ is also presumed to have been replaced by a complex of injectives). Then it is clear that the equivalences of categories $\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}-\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–qcoh}_{\mathsf{fl}})\longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–qcoh})$ and $f^*\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_Y}-\colon \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–qcoh}_{\mathsf{fl}})\longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–qcoh})$ from Section 2.5 transform the inverse image functor for flat matrix factorizations $f^*\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–qcoh}_{\mathsf{fl}})\longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–qcoh}_{\mathsf{fl}})$ into the (underived, as the morphism f is flat) inverse image functor for quasi-coherent matrix factorizations $f^*\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–qcoh})\longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–qcoh})$.

Furthermore, for any quasi-coherent matrix factorization $\mathcal{M}$ on X there is a natural morphism of finite complexes of matrix factorizations $f^*\mathcal{H}om_{X\text{–qc}}(\mathcal{M},\mathcal{D}^\bullet_X)\longrightarrow \mathcal{H}om_{Y\text{–qc}}(f^*\mathcal{M},f^*\mathcal{D}^\bullet_X)$ on Y. When $\mathcal{M}$ is a coherent matrix factorization, this is a quasi-isomorphism of complexes of matrix factorizations (since the similar assertion holds for coherent sheaves [14, Proposition II.5.8]), so the related morphism of total matrix factorizations has an absolutely acyclic cone. Thus the anti-equivalences of categories $\mathcal{H}om_{X\text{–qc}}(-,\mathcal{D}^\bullet_X)\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{–coh})^{\mathsf{op}}\longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–coh})$ and $\mathcal{H}om_{Y\text{–qc}}(-,f^*\mathcal{D}^\bullet_X)\colon \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},-f^*w)\text{–coh})^{\mathsf{op}}\longrightarrow \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–coh})$ form a commutative diagram with the inverse image functors f^* for coherent matrix factorizations.

Now suppose that f is a proper morphism of finite type. The following theorem describes the compability property of the covariant Serre–Grothendieck duality with the inverse images of matrix factorizations (cf. [39, Theorem 5.15.3], where the similar result is proven for complexes of quasi-coherent sheaves).

Theorem. *The equivalences of categories* $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}}) \longrightarrow \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ *and* $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} -\colon \mathsf{D^{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh_{fl}}) \longrightarrow \mathsf{D^{co}}((Y,f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh})$ *transform the inverse image functor* $f^*\colon \mathsf{D^{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh_{fl}}) \longrightarrow \mathsf{D^{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh_{fl}})$ *into the functor* $f^!\colon \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}((Y,f^*\mathcal{L}, f^*w)\text{–}\mathsf{qcoh})$ *right adjoint to the direct image functor* $\mathbb{R}f_*\colon \mathsf{D^{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$.

Proof. For any quasi-coherent matrix factorization $\mathcal{N}$ on Y and any flat quasi-coherent matrix factorization $\mathcal{E}$ on X we have to construct an isomorphism

$$\mathrm{Hom}_{\mathsf{D^{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})}(\mathbb{R}f_*\mathcal{N},\ \mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}\mathcal{E}) \simeq \mathrm{Hom}_{\mathsf{D^{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})}(\mathcal{N},\ \mathcal{D}_Y^\bullet\otimes_{\mathcal{O}_Y} f^*\mathcal{E}).$$

The composition $\mathrm{Hom}_Y(\mathcal{N},\ \mathcal{D}_Y^\bullet\otimes_{\mathcal{O}_Y} f^*\mathcal{E}) \longrightarrow \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ \mathbb{R}f_*(\mathcal{D}_Y^\bullet\otimes_{\mathcal{O}_Y} f^*\mathcal{E})) \simeq \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ f_*\mathcal{D}_Y^\bullet\otimes_{\mathcal{O}_X}\mathcal{E}) \longrightarrow \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ \mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}\mathcal{E})$ provides a morphism from the right-hand to the left-hand side. Here all the Hom functors are taken in the coderived categories of quasi-coherent matrix factorizations on Y and X; the middle isomorphism holds since $\mathcal{D}_Y^\bullet\otimes_{\mathcal{O}_Y} f^*\mathcal{E}$ is an injective matrix factorization on Y (so the derived direct image can be computed for it by applying the underived direct image functor f_* termwise) and by the projection formula; the last morphism is induced by the adjunction $f_*\mathcal{D}_Y^\bullet \longrightarrow \mathcal{D}_X^\bullet$.

Furthermore, on both sides of the desired isomorphism we have injective matrix factorizations in the second arguments of the Hom functors; hence the Hom can be computed in the homotopy category of matrix factorizations instead of the coderived category in both cases. Finally, one can assume $\mathcal{N}$ to be an injective matrix factorization, too, and compute $\mathbb{R}f_*\mathcal{N} = f_*\mathcal{N}$ termwise (alternatively, one could use the Čech construction). Similarly, the tensor products in the second arguments are totalizations of termwise tensor products.

Now one can fix the components involved for both matrix factorizations $\mathcal{N}$ and $\mathcal{E}$, obtaining a morphism of finite complexes of abelian groups of the same kind as above, but related to (one-term) complexes of quasi-coherent sheaves rather than matrix factorizations. The latter is an isomorphism by [39, Theorem 5.15.3]. It remains to notice that the totalization of an acyclic finite complex of (unbounded) complexes of abelian groups is acyclic. ☐

The next corollary is a matrix factorization version of the main result of Deligne's appendix to [14] (see also [39, Section 5.16]).

Corollary. *For any morphism of finite type between separated Noetherian schemes with dualizing complexes* $f\colon Y \longrightarrow X$*, a line bundle* $\mathcal{L}$ *on* X*, and a section*

$w \in \mathcal{L}(X)$, *one can define a triangulated functor* $f^+\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w))\text{–}\mathsf{qcoh})$ *in such a way that*

(i) *for an open embedding* f, *one has* $f^+ = f^*$, *and more generally, for a smooth morphism* f *of relative dimension* n *one has* $f^+ = \omega_{Y/X}[n] \otimes_{\mathcal{O}_Y} f^*$;

(ii) *for a proper morphism* f, *the functor* $f^+ = f^!$ *is right adjoint to* $\mathbb{R}f_*$;

(iii) *the construction is compatible with the compositions of the morphisms* f.

Proof. It suffices to define $f^+\colon \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w))\text{–}\mathsf{qcoh})$ as the functor corresponding to the inverse image of flat quasi-coherent matrix factorizations $f^*\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}})$ under the identifications of categories $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -\colon \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{–}\mathsf{qcoh})$ and $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} -\colon \mathsf{D}^{\mathsf{abs}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{–}\mathsf{qcoh})$, where $\mathcal{D}_X^\bullet$ is any dualizing complex on X and $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$. $\square$

Appendix A. Quasi-Coherent Graded Modules

A.1. **Flat quasi-coherent sheaves.** I am grateful to A. Neeman for suggesting that a result of the following kind can be proven without much difficulty.

Lemma. *On any quasi-compact semi-separated scheme, any quasi-coherent sheaf is the quotient sheaf of a flat quasi-coherent sheaf.*

Proof. Let X be our scheme. Assume that a quasi-coherent sheaf $\mathcal{M}$ over X is flat over an open subscheme $V \subset X$; given an affine open subscheme $U \subset X$, we will construct a surjective morphism $\mathcal{N} \longrightarrow \mathcal{M}$ onto $\mathcal{M}$ from a quasi-coherent sheaf $\mathcal{N}$ over X that is flat over $U \cup V$. Let j denote the embedding $U \longrightarrow X$. There exists a surjective morphism onto $j^*\mathcal{M}$ from a flat quasi-coherent sheaf $\mathcal{F}$ over U; let $\mathcal{K}$ denote the kernel of this morphism of sheaves.

The morphism $j\colon U \longrightarrow X$ being affine and flat, the functor j_* is exact and preserves flatness. Consider the pull-back of the exact triple $j_*\mathcal{K} \longrightarrow j_*\mathcal{F} \longrightarrow j_*j^*\mathcal{M}$ with respect to the morphism $\mathcal{M} \longrightarrow j_*j^*\mathcal{M}$; denote the middle term of the resulting exact triple by $\mathcal{N}$. One has $\mathcal{N}|_U = \mathcal{F}|_U$, so $\mathcal{N}$ is flat over U. Furthermore, the sheaf $j^*\mathcal{M}$ is flat over $V \cap U$, hence so is the sheaf $\mathcal{K}$. The embedding $U \cap V \longrightarrow V$ is an affine flat morphism, so the sheaf $j_*\mathcal{K}$ is flat over V. From the exact triple $j_*\mathcal{K} \longrightarrow \mathcal{N} \longrightarrow \mathcal{M}$ we conclude that $\mathcal{N}$ is flat over V. $\square$

It follows immediately that any quasi-coherent graded module over a quasi-coherent graded algebra $\mathcal{B}$ over X is a quotient module of a flat quasi-coherent graded module.

A.2. **Locally projective quasi-coherent graded modules.** The following result is essentially due to Raynaud and Gruson [40] (for a discussion, see [7, Section 2]); here we just briefly explain how to deduce the formulation that interests us from their assertions.

Theorem. *Let X be an affine scheme and $\{U_\alpha\}$ be its finite affine covering. Let $\mathcal{B}$ be a quasi-coherent graded algebra over X and $\mathcal{P}$ be a quasi-coherent graded module over $\mathcal{B}$. Then the graded $\mathcal{B}(X)$-module $\mathcal{P}(X)$ is projective if and only if the graded $\mathcal{B}(U_\alpha)$-module $\mathcal{P}(U_\alpha)$ is projective for every α.*

Proof. First of all, a graded module P over a graded ring B is projective if and only if it is projective as an ungraded module. Indeed, if P is graded projective, then it is a homogeneneous direct summand of a free graded module, hence P is also ungraded projective. Conversely, pick a homogeneous (of degree 0) surjective homomorphism $F \longrightarrow P$ onto a given graded module P from a free graded module F. If P is ungraded projective, this homomorphism has a (perhaps nonhomogeneous) section s, and the homogeneous component of s of degree 0 provides a homogeneous section. Hence it suffices to consider ungraded modules over an ungraded quasi-coherent algebra $\mathcal{B}$.

It is clear that if $\mathcal{P}(X)$ is a projective $\mathcal{B}(X)$-module, then $\mathcal{P}(V)$ is a projective $\mathcal{B}(V)$-module for any affine open subscheme $V \subset X$. Conversely, assume that the $\mathcal{B}(U_\alpha)$-module $\mathcal{P}(U_\alpha)$ is projective for every α. Then by the result of [19] the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ are direct sums of countably generated modules, and it follows easily that so is the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ (essentially, since a connected graph with an at most countable set of edges at each vertex has a countable number of vertices). Hence we can assume the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ to be countably generated.

Besides, the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ are flat, hence so is the $\mathcal{B}(X)$-module $\mathcal{P}(X)$. By [40, Corollaire II.2.2.2], it remains to show that the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ satisfies the Mittag-Leffler condition; this can be easily deduced from the similar property of the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ using the formulation of this condition given in Proposition II.2.1.4(iii) or Propositions II.2.1.4(ii) and II.2.1.1(i) of [40] (cf. Sections II.2.5 and II.3.1 of the same paper). ☐

A.3. **Injective quasi-coherent graded modules.** The following result is a noncommutative generalization of a theorem of Hartshorne [14, Theorem II.7.18] about injective quasi-coherent sheaves on Noetherian schemes. Our proof method, based on the Artin–Rees lemma, is different from the one in [14].

Theorem. *Let $\mathcal{B}$ be a Noetherian quasi-coherent graded algebra over a Noetherian scheme X. Then any injective object in the category of quasi-coherent graded left modules over $\mathcal{B}$ is also an injective object of the category of arbitrary sheaves of graded $\mathcal{B}$-modules over X.*

Consequently, the restriction $\mathcal{J}|_U$ of an injective quasi-coherent graded module $\mathcal{J}$ over $\mathcal{B}$ to an open subscheme $U \subset X$ is an injective quasi-coherent graded module over $\mathcal{B}|_U$. Conversely, if U_α is an open covering of X and the quasi-coherent graded $\mathcal{B}|_{U_\alpha}$-modules $\mathcal{J}|_{U_\alpha}$ are injective, then a quasi-coherent graded $\mathcal{B}$-module $\mathcal{J}$

is injective. Besides, the underlying sheaf of graded abelian groups of any injective quasi-coherent graded $\mathcal{B}$-module $\mathcal{J}$ is flabby.

Proof. First of all, notice that the abelian category $\mathcal{B}$–qcoh of quasi-coherent graded modules over $\mathcal{B}$ is a locally Noetherian Grothendieck category with coherent graded modules forming the subcategory of Noetherian generators [15, Exercise II.5.15]; so in particular $\mathcal{B}$–qcoh has enough injectives and the assertions of Theorem are not vacuous. The category of sheaves of graded $\mathcal{B}$-modules $\mathcal{B}$–mod has similar properties, with the extensions by zero of the restrictions of $\mathcal{B}$ to (small) open subschemes of X forming a set of Noetherian generators [14, Theorem II.7.8].

Secondly, let us check that the main result in the first paragraph implies the assertions in the second one. Indeed, injective sheaves of graded $\mathcal{B}$-modules have all the properties we are interested in. They remain injective after being restricted to an open subscheme, since the extension by zero from an open subscheme is an exact functor. They are flabby, since given two open subschemes $U \subset V \subset X$ and j_U, j_V being their identity embeddings U, $V \longrightarrow X$, the morphism of sheaves of graded $\mathcal{B}$-modules $j_{U!}\mathcal{B}|_U \longrightarrow j_{V!}\mathcal{B}|_V$ is injective. And their property is local [14, Lemma II.7.16], because sheaves of graded $\mathcal{B}$-modules supported inside one of the subschemes U_α form a set of generators of the category $\mathcal{B}$–mod.

Now let $\mathcal{J}$ be an injective quasi-coherent graded module over $\mathcal{B}$. To prove the main assertion, we have to show that for any open subscheme $U \subset X$ and a subsheaf of graded $\mathcal{B}$-modules $\mathcal{G} \subset j_{U!}\mathcal{B}|_U$, any homogeneous morphism of sheaves of graded $\mathcal{B}$-modules $\mathcal{G} \longrightarrow \mathcal{J}$ can be extended to a similar morphism $j_{U!}\mathcal{B}|_U \longrightarrow \mathcal{J}$. Indeed, $\mathcal{G}$ is a subsheaf of graded $\mathcal{B}$-modules in the coherent graded $\mathcal{B}$-module $\mathcal{B}$, hence according to the following proposition there exists a quasi-coherent graded $\mathcal{B}$-module $\mathcal{G} \subset \mathcal{F} \subset \mathcal{B}$ such that the morphism $\mathcal{G} \longrightarrow \mathcal{J}$ can be extended to a homogeneous morphism of quasi-coherent graded $\mathcal{B}$-modules $\mathcal{F} \longrightarrow \mathcal{J}$.

Since $\mathcal{J}$ is injective in $\mathcal{B}$–qcoh, the latter morphism can in turn be extended to a similar morphism $\mathcal{B} \longrightarrow \mathcal{J}$. Restricting to $j_{U!}\mathcal{B}|_U$, we obtain the desired morphism of sheaves of graded $\mathcal{B}$-modules $j_{U!}\mathcal{B}|_U \longrightarrow \mathcal{J}$. □

Proposition. *In the assumptions of Theorem, let $\mathcal{E}$ be a coherent graded left $\mathcal{B}$-module, $\mathcal{G} \subset \mathcal{E}$ be a subsheaf of graded $\mathcal{B}$-modules, $\mathcal{M}$ be a quasi-coherent graded $\mathcal{B}$-module, and $\phi\colon \mathcal{G} \longrightarrow \mathcal{M}$ be a morphism of sheaves of graded $\mathcal{B}$-modules. Then there exists a coherent graded $\mathcal{B}$-module $\mathcal{G} \subset \mathcal{F} \subset \mathcal{E}$ such that the morphism ϕ can be extended to $\mathcal{F}$.*

Proof. Before proving Proposition, let us reformulate its conclusion in follows. In the same setting, there exists a quasi-coherent graded $\mathcal{B}$-module $\mathcal{K}$ together with an injective morphism $\mathcal{M} \longrightarrow \mathcal{K}$ and a morphism $\mathcal{E} \longrightarrow \mathcal{K}$ forming a commutative diagram with the embedding $\mathcal{G} \longrightarrow \mathcal{E}$ and the morphism $\phi\colon \mathcal{G} \longrightarrow \mathcal{M}$. Indeed, if a coherent $\mathcal{B}$-module $\mathcal{F}$ exists, one can take $\mathcal{K}$ to be the fibered coproduct of $\mathcal{E}$ and $\mathcal{M}$ over $\mathcal{F}$; conversely, if a quasi-coherent $\mathcal{B}$-module $\mathcal{K}$ exists, one can take $\mathcal{F}$ to be the full preimage of $\mathcal{M} \subset \mathcal{K}$ under the morphism $\mathcal{E} \longrightarrow \mathcal{K}$. Notice also that one can always replace $\mathcal{M}$ with its sufficiently big coherent graded $\mathcal{B}$-submodule.

Now let us state the version of Artin–Rees lemma that we will use.

Lemma. *In the assumptions of Theorem, let $\mathcal{M}$ be a coherent graded $\mathcal{B}$-module, $\mathcal{N} \subset \mathcal{M}$ a coherent graded $\mathcal{B}$-submodule, and $Z \subset X$ a closed subscheme with the sheaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$. Then for any $n \geq 0$ there exists $m \geq 0$ such that the intersection $\mathcal{I}_Z^m\mathcal{M} \cap \mathcal{N}$ is contained in $\mathcal{I}_Z^n\mathcal{N}$.*

Proof. Clearly, the question is local, so it suffices to consider the case of an affine scheme X. Then (the graded version of) the Artin–Rees lemma for ideals generated by central elements in noncommutative Noetherian rings [11, Theorem 13.3] applies. □

Being a Noetherian object, the sheaf of graded $\mathcal{B}$-modules $\mathcal{G}$ is generated by a finite number of homogeneous sections $s_n \in \mathcal{G}(U_n)$, where $U_n \subset X$ are some open subschemes. If all of these subschemes coincide with X, the sheaf $\mathcal{G}$, being a subsheaf of a coherent sheaf generated by global sections, is itself coherent, so there is nothing to prove. In the general case, we will argue by induction in the number of open subschemes U_n that are not equal to X.

Let $U = U_1 \subsetneq X$ be one such open subscheme, and $T = X \setminus U$ be its closed complement. We can assume that $\mathcal{M}$ is a coherent graded $\mathcal{B}$-module. Let $\mathcal{N}$ denote its maximal coherent graded $\mathcal{B}$-submodule supported set-theoretically in T. Applying Lemma to $\mathcal{N} \subset \mathcal{M}$, we conclude that there is a closed subscheme structure $i\colon Z \longrightarrow X$ on T such that the morphism $\mathcal{N} \longrightarrow i_*i^*\mathcal{M}$ is injective. Consequently, so is the morphism $\mathcal{M} \longrightarrow i_*i^*\mathcal{M} \oplus j_*j^*\mathcal{M}$, where j denotes the open embedding $U \longrightarrow X$.

Let us show that there is a thicker closed subscheme structure $i'\colon Z' \longrightarrow X$ on T such that the kernel of the morphism of sheaves $i'_*i'^*\mathcal{G} \longrightarrow i'_*i'^*\mathcal{E}$ is contained in the kernel of the morphism of sheaves $i'_*i'^*\mathcal{G} \longrightarrow i_*i^*\mathcal{G}$. Indeed, there exists a finite collection of subsheaves of graded $\mathcal{B}$-modules in $\mathcal{G}$, each of them an extension by zero of a coherent graded $\mathcal{B}|_V$-module from some open subscheme $V \subset X$, such that the the stalk of $\mathcal{G}$ at each point of X coincides with the stalk of one of these subsheaves. So the assertion reduces to the case when $\mathcal{G}$ is a coherent graded $\mathcal{B}$-submodule in $\mathcal{E}$, when it is an equivalent reformulation of Lemma.

Let $\mathcal{H} \subset i'^*\mathcal{E}$ denote the image of the morphism of sheaves of graded $i'^*\mathcal{B}$-modules $i'^*\mathcal{G} \longrightarrow i'^*\mathcal{E}$ over the scheme Z'. Let $\iota\colon Z \longrightarrow Z'$ be the natural closed embedding. Then, according to the above, the morphism of sheaves of graded $i'^*\mathcal{B}$-modules $i'^*\mathcal{G} \longrightarrow \iota_*i^*\mathcal{G}$ induces a morphism $\mathcal{H} \longrightarrow \iota_*i^*\mathcal{G}$.

The sheaf of graded $i'^*\mathcal{B}$-modules $\mathcal{H}$ is generated by the images of the restrictions of the sections s_n, $n \geq 2$, to the closed subschemes $Z' \cap U_n \subset U_n$. Hence the induction assumption is applicable to $\mathcal{H}$, and we can conclude that there exists a quasi-coherent graded $i'^*\mathcal{B}$-module $\mathcal{K}$ on the scheme Z' together with an injective morphism $\iota_*i^*\mathcal{M} \longrightarrow \mathcal{K}$ and a morphism $i'^*\mathcal{E} \longrightarrow \mathcal{K}$ forming a commutative diagram with the embedding $\mathcal{H} \longrightarrow i'^*\mathcal{E}$ and the composition $\mathcal{H} \longrightarrow \iota_*i^*\mathcal{G} \longrightarrow \iota_*i^*\mathcal{M}$.

Similarly, the sheaf of graded $\mathcal{B}|_U$-modules $j^*\mathcal{G}$ is generated by the restrictions of the sections s_n to the open subschemes $U_1 \cap U_n \subset U_n$, among which the (restriction of) the section s_1 is a global section over $U = U_1$. Hence the induction assumption is applicable to $j^*\mathcal{G}$, and there exists a quasi-coherent graded $\mathcal{B}|_U$-module $\mathcal{L}$ together with an injective morphism $j^*\mathcal{M} \longrightarrow \mathcal{L}$ and a morphism $j^*\mathcal{E} \longrightarrow \mathcal{L}$ forming a commutative diagram with the embedding $j^*\mathcal{G} \longrightarrow j^*\mathcal{E}$ and the morphism $j^*\mathcal{G} \longrightarrow j^*\mathcal{M}$.

Now the injective morphism $\mathcal{M} \longrightarrow i'_*\mathcal{K} \oplus j_*\mathcal{L}$ (whose first component is the composition $\mathcal{M} \longrightarrow i_*i^*\mathcal{M} \simeq i'_*\iota_*i^*\mathcal{M} \longrightarrow i'_*\mathcal{K}$) and the morphism $\mathcal{E} \longrightarrow i'_*\mathcal{K} \oplus j_*\mathcal{L}$ provide the desired commutative diagram of morphisms of sheaves of graded $\mathcal{B}$-modules over X. □

Appendix B. Hochschild (Co)Homology of Matrix Factorizations

This appendix complements the paper [32] in two ways. Section B.1 contains some modifications and improvements of the main results of [32] generally, and as applied to locally free matrix factorizations of finite rank in particular. The main thrust consists in replacing the finite homological dimension conditions in [32] with the Noetherianness conditions to the (limited) extent possible.

Section B.2, on the other hand, presents an elementary approach to computation of Hochschild (co)homology of coherent matrix factorizations, entirely unrelated to that in [32] and not based on any notion of Hochschild (co)homology of the second kind, but rather on the Serre–Grothendieck duality theory.

B.1. **Locally free matrix factorizations of finite rank.** In Sections B.1.1–B.1.4, we start with a bit of categorical nonsense, following the lines of [32, Sections 3.3–3.5], but with the additional coherence/Noetherianness conditions imposed from the very beginning. We use the notation from [32] rather than that of the main body of this paper. Then in Section B.1.5 we turn to locally free matrix factorizations of finite rank over certain possibly singular, affine algebraic varieties. The final Section B.1.6 presents an improvement over the discussion of matrix factorizations over smooth affine varieties in [32, Section 4.8]. An example of application of our techniques to nonaffine varieties can be found in the preprint [9].

B.1.1. *Coherent and Noetherian CDG-categories.* Let $(\Gamma, \sigma, \mathbf{1})$ be a grading group data [32, Section 1.1] and $B^\#$ be a small Γ-graded preadditive category [38, Section A.1]. Both left and right Γ-graded $B^\#$-modules form abelian categories.

A Γ-graded $B^\#$-module is said to be *finitely generated* (respectively, *finitely presented*) if it is a quotient module of a finitely generated free Γ-graded $B^\#$-module [32, Section 1.5] (respectively, the cokernel of a morphism of finitely generated free Γ-graded $B^\#$-modules).

A Γ-graded preadditive category $B^\#$ is called *left Noetherian* if any submodule of a finitely generated Γ-graded left $B^\#$-module is finitely generated, or equivalently, if the abelian category of Γ-graded left $B^\#$-modules is locally Noetherian. A Γ-graded preadditive category $B^\#$ is called *left coherent* if any submodule of a finitely presented Γ-graded left $B^\#$-module is finitely presented.

Let B be a small (Γ-graded) CDG-category [32, Section 1.2] and $B^\#$ be its underlying Γ-graded preadditive category. Following [32], we denote the DG-categories of left and right CDG-modules over B by $B\text{-mod}^{\text{cdg}}$ and $\text{mod}^{\text{cdg}}\text{-}B$. The DG-subcategories of left CDG-modules whose underlying Γ-graded $B^\#$-modules are flat or injective are denoted by $B\text{-mod}^{\text{cdg}}_{\text{fl}}$ and $B\text{-mod}^{\text{cdg}}_{\text{inj}} \subset B\text{-mod}^{\text{cdg}}$. Similarly, the DG-subcategories of left and right CDG-modules over B whose underlying Γ-graded $B^\#$-modules are projective and finitely generated are denoted by $B\text{-mod}^{\text{cdg}}_{\text{fgp}}$ and $\text{mod}^{\text{cdg}}_{\text{fgp}}\text{-}B$.

Assuming that the Γ-graded category $B^\#$ is left Noetherian, the DG-subcategory of left CDG-modules whose underlying Γ-graded $B^\#$-modules are finitely generated is denoted by $B\text{-mod}^{\text{cdg}}_{\text{fg}} \subset B\text{-mod}^{\text{cdg}}$. Assuming that the Γ-graded category $B^\#$ is right coherent, the DG-subcategory of right CDG-modules whose underlying Γ-graded $B^\#$-modules are finitely presented is denoted by $\text{mod}^{\text{cdg}}_{\text{fp}}\text{-}B$.

The coderived and contraderived categories of left CDG-modules over B are denoted by $\mathrm{D}^{\text{co}}(B\text{-mod}^{\text{cdg}})$ and $\mathrm{D}^{\text{ctr}}(B\text{-mod}^{\text{cdg}})$, respectively [32, Section 3.2]. Assuming that the Γ-graded category $B^\#$ is right coherent, the class of flat Γ-graded left B-modules [32, Section 2.2] is closed under infinite products, so the contraderived category $\mathrm{D}^{\text{ctr}}(B\text{-mod}^{\text{cdg}}_{\text{fl}})$ is well-defined. The homotopy category of the DG-category $B\text{-mod}^{\text{cdg}}_{\text{inj}}$ is denoted, as usually, by $H^0(B\text{-mod}^{\text{cdg}}_{\text{inj}})$.

In the respective assumptions of left Noetherianness or right coherence of the Γ-graded category $B^\#$, the absolute derived categories of CDG-modules with finitely generated or finitely presented underlying Γ-graded $B^\#$-modules are denoted by $\mathrm{D}^{\text{abs}}(B\text{-mod}^{\text{cdg}}_{\text{fg}})$ and $\mathrm{D}^{\text{abs}}(\text{mod}^{\text{cdg}}_{\text{fp}}\text{-}B)$, respectively.

B.1.2. *Derived functors of the second kind.* Let k be a commutative ring and B be a small k-linear CDG-category. Assume that the Γ-graded category $B^\#$ is left Noetherian. Let L and M be left CDG-modules over B; suppose that the Γ-graded left $B^\#$-module $L^\#$ underlying the CDG-module L over B is finitely generated.

As in [32, Sections 2.1-2], we denote by $Z^0(B\text{-mod}^{\text{cdg}})$ and $Z^0(\text{mod}^{\text{cdg}}\text{-}B)$ the abelian categories of left and right CDG-modules over B. Let $Z^0(B\text{-mod}^{\text{cdg}}_{\text{fg}}) \subset Z^0(B\text{-mod}^{\text{cdg}})$ and $H^0(B\text{-mod}^{\text{cdg}}_{\text{fg}}) \subset H^0(B\text{-mod}^{\text{cdg}})$ denote the abelian and homotopy categories of left CDG-modules over B with finitely generated underlying Γ-graded $B^\#$-modules, and $Z^0(\text{mod}^{\text{cdg}}_{\text{fp}}\text{-}B) \subset Z^0(\text{mod}^{\text{cdg}}\text{-}B)$ and $H^0(\text{mod}^{\text{cdg}}_{\text{fp}}\text{-}B) \subset H^0(\text{mod}^{\text{cdg}}\text{-}B)$ be the similar categories of right CDG-modules with finitely presented underlying Γ-graded modules.

Let $J^\bullet$ be a right resolution of M in $Z^0(B\text{-mod}^{\text{cdg}})$ such that the Γ-graded left $B^\#$-modules $J^{i\#}$ are injective, and let J be the total CDG-module of the complex of CDG-modules $J^\bullet$ constructed by taking infinite direct sums along the diagonals. Then the complex $\mathrm{Tot}^\oplus \mathrm{Hom}^B(L, J^\bullet)$ computing $\mathrm{Ext}^{I\!I}_B(L, M)$ [32, Section 2.2] is isomorphic to the complex $\mathrm{Hom}^B(L, J)$ [32, formula (6)], which computes the k-modules of morphisms from L into $M[*]$ in the coderived category $\mathrm{D}^{\text{co}}(B\text{-mod}^{\text{cdg}})$ [37, Theorems 3.5(a) and 3.7]. Thus,

$$H^* \mathrm{Ext}^{I\!I}_B(L, M) \simeq \mathrm{Hom}_{\mathrm{D}^{\text{co}}(B\text{-mod}^{\text{cdg}})}(L, M[*]).$$

Just as in [32, Section 3.3], one can lift this isomorphism from the level of cohomology modules to that of the derived category $\mathrm{D}(k\text{-mod})$ in the following way. Consider the functor

$$\mathrm{Hom}^B\colon H^0(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}} \times H^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod})$$

and restrict it to the full subcategory $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{inj}})$ in the second argument. This restriction factorizes through the coderived category $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ in the first argument. Taking into account [37, Theorem 3.7], we obtain a right derived functor

$$\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}} \times \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}).$$

Restricting to the full subcategory $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})^{\mathrm{op}} \subset \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}}$ [37, Theorem 3.11.1] in the first argument, we have the derived functor

$$\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})^{\mathrm{op}} \times \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}). \tag{1}$$

The composition of this functor with the localization functors $Z^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \longrightarrow \mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ and $Z^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ agrees with the derived functor $\mathrm{Ext}_B^{\mathit{II}}$ where the former is defined.

Now assume that the Γ-graded category $B^\#$ is right coherent. Consider the functor [32, formula (5)]

$$\otimes_B\colon H^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B) \times H^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod})$$

and restrict it to the Cartesian product of full subcategories $H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \times H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fl}}) \subset H^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B) \times H^0(B\text{-mod}^{\mathrm{cdg}})$. Since the tensor product with a finitely presented Γ-graded right $B^\#$-module commutes with infinite products of Γ-graded left $B^\#$-modules, this restriction factorizes through the contraderived category $\mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fl}})$ in the second argument. Clearly, it also factorizes through the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ in the first argument.

By Remark 1.5 of the main body of this paper (see also [39, Proposition A.3.1(b)]), the natural functor $\mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fl}}) \longrightarrow \mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}})$ is an equivalence of triangulated categories. Hence we obtain a left derived functor

$$\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \times \mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}). \tag{2}$$

Up to composing with the localization functors $Z^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \longrightarrow \mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ and $Z^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}})$, this functor agrees with the derived functor $\mathrm{Tor}^{B,\mathit{II}}$ from [32, Section 2.2] where the former is defined.

Indeed, let N be an object of $Z^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$. Let $P_\bullet$ be a left resolution of an object $M \in Z^0(B\text{-mod}^{\mathrm{cdg}})$ by left CDG-modules over B with flat underlying Γ-graded $B^\#$-modules, and let P be the total CDG-module of the complex $P_\bullet$ constructed by taking infinite products along the diagonals. Then the complex $\mathrm{Tot}^\sqcap(N \otimes_B P_\bullet)$ computing $\mathrm{Tor}^{B,\mathit{II}}(N, M)$ is isomorphic to the complex $N \otimes_B P$ computing the derived functor (2) on the objects N and M.

B.1.3. *Comparison of the two theories.* Let C be a small k-linear (Γ-graded) DG-category. The above constructions applicable to CDG-categories and CDG-modules over them can be also applied to DG-categories and DG-modules as a particular case. Following [32], we denote the DG-categories of left and right DG-modules over C by $C\text{-mod}^{\text{dg}}$ and $\text{mod}^{\text{dg}}\text{-}C$, and generally use the upper index "dg" instead of "cdg" in the notation related to DG-modules.

As in [32, Sections 2.1, 3.1 and 3.4], we denote by $H^0(C\text{-mod}^{\text{dg}})_{\text{inj}}$ and $H^0(C\text{-mod}^{\text{dg}})_{\text{fl}}$ the homotopy categories of h-injective and h-flat left DG-modules over C. The notation $H^0(C\text{-mod}^{\text{dg}}_{\text{inj}})_{\text{inj}}$ and $H^0(C\text{-mod}^{\text{dg}}_{\text{fl}})_{\text{fl}}$ stands for the full triangulated subcategories in $H^0(C\text{-mod}^{\text{dg}})$ formed by h-injective DG-modules over C whose underlying Γ-graded $C^\#$-modules are injective, or h-flat DG-modules whose underlying Γ-graded $C^\#$-modules are flat, respectively. Finally, let $H^0(C\text{-mod}^{\text{dg}}_{\text{fgp}})_{\text{prj}} \subset H^0(C\text{-mod}^{\text{dg}})$ and $H^0(\text{mod}^{\text{dg}}_{\text{fgp}}\text{-}C)_{\text{fl}} \subset H^0(\text{mod}^{\text{dg}}\text{-}C)$ denote the full triangulated subcategories of h-projective left and h-flat right DG-modules whose underlying Γ-graded $C^\#$-modules are projective and finitely generated.

Assume that the Γ-graded category $C^\#$ is left Noetherian. Let L be an object of $Z^0(C\text{-mod}^{\text{dg}}_{\text{fg}})$. Given a left DG-module M over C, pick its injective resolution $J^\bullet$ in the exact category $Z^0(C\text{-mod}^{\text{dg}})$ [32, Section 2.1]. Let $\text{Tot}^\oplus(J^\bullet) \longrightarrow \text{Tot}^\sqcap(J^\bullet)$ be the natural closed morphism between the total DG-modules of the complex $J^\bullet$ constructed by taking infinite direct sums and infinite products along the diagonals. Then the induced morphism of complexes of k-modules

$$\text{Hom}^C(L,\ \text{Tot}^\oplus(J^\bullet)) \longrightarrow \text{Hom}^C(L,\ \text{Tot}^\sqcap(J^\bullet))$$

represents the comparison morphism $\text{Ext}^{II}_C(L,M) \longrightarrow \text{Ext}_C(L,M)$ [32, formula (10)] in $\mathsf{D}(k\text{-mod})$ between the two kinds of Ext objects for the DG-modules L and M.

Similarly, assume that the Γ-graded category $C^\#$ is right coherent. Let N be an object of $Z^0(\text{mod}^{\text{dg}}_{\text{fp}}\text{-}C)$. Given a left DG-module M over C, pick its projective resolution $P_\bullet$ in the exact category $Z^0(C\text{-mod}^{\text{dg}})$. Let $\text{Tot}^\oplus(P_\bullet) \longrightarrow \text{Tot}^\sqcap(P_\bullet)$ be the natural closed morphism between the total DG-modules of the complex $P_\bullet$ constructed by taking infinite direct sums and infinite products along the diagonals. Then the induced morphism of complexes of k-modules

$$N \otimes_C \text{Tot}^\oplus(P_\bullet) \longrightarrow N \otimes_C \text{Tot}^\sqcap(P_\bullet)$$

represents the comparison morphism $\text{Tor}^C(N,M) \longrightarrow \text{Tor}^{C,II}(N,M)$ [32, formula (9)] in $\text{D}(k\text{-mod})$ between the two kinds of Tor objects for the DG-modules N and M.

Proposition A. *Assume that the Γ-graded category $C^\#$ is left Noetherian. Let L be a left DG-module over C whose underlying Γ-graded left $C^\#$-module is finitely generated, and let M be a left DG-module over C. Then the natural morphism $\text{Ext}^{II}_C(L,M) \longrightarrow \text{Ext}_C(L,M)$ is an isomorphism provided that either*

(i) *the object $M \in \text{D}^{\text{co}}(C\text{-mod}^{\text{dg}})$ belongs to the image of the fully faithful functor $H^0(C\text{-mod}^{\text{dg}}_{\text{inj}})_{\text{inj}} \longrightarrow \text{D}^{\text{co}}(C\text{-mod}^{\text{dg}})$; or*

(ii) *the object* $L \in \mathrm{D^{abs}}(C\text{-mod}^{\mathrm{dg}})$ *belongs to the image of the fully faithful functor* $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}})_{\mathrm{prj}} \longrightarrow \mathrm{D^{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$.

Proof. Let $J^\bullet$ be an injective resolution of the DG-module M in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$. Then the natural morphism $M \longrightarrow \mathrm{Tot}^{\oplus}(J^\bullet)$ is always an isomorphism in $\mathrm{D^{co}}(C\text{-mod}^{\mathrm{dg}})$ [37, proof of Theorem 3.7], while the morphism $M \longrightarrow \mathrm{Tot}^{\sqcap}(J^\bullet)$ is an isomorphism in the conventional derived category $\mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ [37, proofs of Theorems 1.4-5]. Furthermore, one has $\mathrm{Tot}^{\oplus}(J^\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})$ and $\mathrm{Tot}^{\sqcap}(J^\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$.

Part (i): the functor is fully faithful by [37, Theorem 3.5(a) and Lemma 1.3]. According to formula (1) from Section B.1.2 and [32, Section 3.1], both kinds of Ext involved are well-defined as functors of the argument $M \in \mathrm{D^{co}}(C\text{-mod}^{\mathrm{dg}})$. Hence one can assume $M \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$. Then both morphisms $M \longrightarrow \mathrm{Tot}^{\oplus}(J^\bullet)$ and $M \longrightarrow \mathrm{Tot}^{\sqcap}(J^\bullet)$ are homotopy equivalences by semiorthogonality, hence so is the morphism $\mathrm{Tot}^{\oplus}(J^\bullet) \longrightarrow \mathrm{Tot}^{\sqcap}(J^\bullet)$ and the assertion follows.

Part (ii): in view of the first paragraph of this proof, a cone K of the morphism $\mathrm{Tot}^{\oplus}(J^\bullet) \longrightarrow \mathrm{Tot}^{\sqcap}(J^\bullet)$ in $H^0(C\text{-mod}^{\mathrm{dg}})$ is an acyclic DG-module over C whose underlying Γ-graded $C^{\#}$-module is injective. Hence the complex of morphisms $\mathrm{Hom}^C(-,K)$ is a well-defined functor $\mathsf{D^{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})^{\mathrm{op}} \longrightarrow \mathrm{D}(k\text{-mod})$ annihilating $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}})_{\mathrm{prj}}$. ☐

Proposition B. *Assume that the* Γ*-graded category* $C^{\#}$ *is right coherent. Let* N *be a right DG-module over* C *whose underlying* Γ*-graded right* $C^{\#}$*-module is finitely presented, and let* M *be a left DG-module over* C*. Then the natural morphism* $\mathrm{Tor}^C(N,M) \longrightarrow \mathrm{Tor}^{C,\mathit{II}}(N,M)$ *is an isomorphism provided that either*

(i) *there is a closed morphism* $P \longrightarrow M$ *into* M *from a DG-module* $P \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$ *with a cone contraacyclic with respect to* $C\text{-mod}^{\mathrm{dg}}$ *or completely acyclic with respect to* $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$ *(see* [32, *Sections* 3.2 *and* 4.7]*); or*

(ii) *the object* $N \in \mathrm{D^{abs}}(\mathrm{mod}^{\mathrm{dg}}\text{-}C)$ *belongs to the image of the fully faithful functor* $H^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fgp}}\text{-}C)_{\mathrm{fl}} \longrightarrow \mathrm{D^{abs}}(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C)$.

Proof. Let $P_\bullet$ be a projective resolution of the DG-module M in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$. Then the natural morphism $\mathrm{Tot}^{\sqcap}(P_\bullet) \longrightarrow M$ is always an isomorphism in $\mathsf{D^{ctr}}(C\text{-mod}^{\mathrm{dg}})$ [37, proof of Theorem 3.8], while the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \longrightarrow M$ is an isomorphism in $\mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ [37, proof of Theorem 1.4]. Furthermore, one has $\mathrm{Tot}^{\sqcap}(P_\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})$ and $\mathrm{Tot}^{\oplus}(P_\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$.

Part (i): acyclic DG-modules in the second argument are annihilated by the functor Tor^C by [32, Section 3.1], while contraacyclic DG-modules in the second argument are annihilated by the functor $\mathrm{Tor}^{C,\mathit{II}}(N,-)$ according to the formula (2). The latter also applies to DG-modules competely acyclic with respect to $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$, since the functor of tensor product with a finitely presented DG-module preserves infinite direct sums and products. So one can replace M with P and assume that $M \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$.

Then a cone of the morphism $\mathrm{Tot}^{\sqcap}(P_\bullet) \longrightarrow M$ is contraacyclic with respect to $C\text{-mod}^{\mathrm{dg}}$ with a flat underlying Γ-graded $C^\#$-module, hence also contraacyclic with respect to $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$. On the other hand, a cone of the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \longrightarrow M$ is acyclic and h-flat. It follows that the functor $N \otimes_C -$ transforms both these morphisms, and therefore also the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \longrightarrow \mathrm{Tot}^{\sqcap}(P_\bullet)$, into quasi-isomorphisms of complexes of k-modules.

Part (ii): a cone K of the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \longrightarrow \mathrm{Tot}^{\sqcap}(P_\bullet)$ in $H^0(C\text{-mod}^{\mathrm{dg}})$ is an acyclic DG-module over C whose underlying Γ-graded $C^\#$-module is flat. Hence the tensor product $- \otimes_C K$ is a well-defined functor $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C) \longrightarrow \mathrm{D}(k\text{-mod})$ annihilating $H^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fgp}}\text{-}C)_{\mathrm{fl}}$. □

In particular, assuming that the category $C^\#$ is left Noetherian, the natural morphism $\mathrm{Ext}^{II}_C(L, M) \longrightarrow \mathrm{Ext}_C(L, M)$ is an isomorphism for all $L \in C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}}$ and $M \in C\text{-mod}^{\mathrm{dg}}$ provided that the Verdier localization functor $\mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}}) \longrightarrow \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ is an equivalence of triangulated categories. Assuming that the category $C^\#$ is right coherent, the natural morphism $\mathrm{Tor}^C(N, M) \longrightarrow \mathrm{Tor}^{C,II}(N, M)$ is an isomorphism for all $N \in \mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C$ and $M \in C\text{-mod}^{\mathrm{dg}}$ provided that the Verdier localization functor $\mathrm{D}^{\mathrm{ctr}}(C\text{-mod}^{\mathrm{dg}}) \longrightarrow \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ is an equivalence of categories, or alternatively, that any acyclic DG-module from $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$ is completely acyclic with respect to $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$.

B.1.4. *Comparison for the DG-category of CDG-modules.* Let B be a k-linear CDG-category and $C = \mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$ be the DG-category of right CDG-modules over B whose underlying Γ-graded $B^\#$-modules are projective and finitely generated. The DG-categories of (left or right) CDG-modules over B and DG-modules over C are naturally equivalent [32, Sections 1.5 and 2.6] (as are the categories of Γ-graded modules over $B^\#$ and $C^\#$). Following [32, Section 3.5], we denote by M_C the DG-module over C corresponding to a CDG-module M over B.

Let $k^\vee$ be an injective cogenerator of the abelian category of k-modules. Introduce the notation $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{prj}} \subset B\text{-mod}^{\mathrm{cdg}}$ for the DG-category of left CDG-modules over B with projective underlying Γ-graded $B^\#$-modules. The results below in this section are to be compared with those from [32, Sections 3.5 and 4.7].

Proposition A. *Assume that the Γ-graded category $B^\#$ is left Noetherian. Let L be a left CDG-module over B whose underlying Γ-graded left $B^\#$-module $L^\#$ is finitely generated, and let M be a left CDG-module over B. Then the natural morphism $\mathrm{Ext}^{II}_C(L_C, M_C) \longrightarrow \mathrm{Ext}_C(L_C, M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ containing the objects $\mathrm{Hom}_k(F, k^\vee)$ for all $F \in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and closed under infinite products; or*

(ii) *the object L belongs to the minimal thick subcategory of $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ containing the image of $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$.*

Proof. Part (i): the equivalence of categories $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \simeq \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ makes the embedding functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \longrightarrow \mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ right adjoint to the localization functor $\mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}}) \longrightarrow \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$. It follows that the functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \longrightarrow \mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ preserves infinite products (also, all infinite products exist in the coderived category, since it is compactly generated [37, Theorem 3.11.2]). Since the category $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$ is the minimal triangulated subcategory of $H^0(C\text{-mod}^{\mathrm{dg}})$ containing the objects $\mathrm{Hom}_k(F_C, k^\vee)$ and closed under infinite products [37, Theorem 1.5], the assertion follows from Proposition B.1.3.A(i).

Part (ii): the equivalence of absolute derived categories $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \simeq \mathrm{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$ takes objects of the full subcategory $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}}) \subset \mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ to representable (and, consequently, perfect and h-projective) DG-modules in $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}}) \subset \mathrm{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$, so it remains to apply Proposition B.1.3.A(ii). ☐

Proposition B. *Assume that the Γ-graded category $B^\#$ is right coherent. Let N be a right CDG-module over B whose underlying Γ-graded right $B^\#$-module $N^\#$ is finitely presented, and let M be a left CDG-module over B. Then the natural morphism $\mathrm{Tor}^C(N_C, M_C) \longrightarrow \mathrm{Tor}^{C,II}(N_C, M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{prj}}) \subset \mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}})$ containing the image of $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and closed under infinite direct sums; or*

(ii) *the object N belongs to the minimal thick subcategory of $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ containing the image of $H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$.*

Proof. Similar to that of Proposition A and based on Proposition B.1.3.B. ☐

Now assume that the commutative ring k has finite weak homological dimension and all the Γ-graded k-modules of morphisms in the category $B^\#$ are flat. Clearly, the DG-categories of left and right CDG-modules over the CDG-category $B \otimes_k B^{\mathrm{op}}$ are naturally equivalent, as are the DG-categories of left and right DG-modules over the DG-category $C \otimes_k C^{\mathrm{op}}$. The DG-category of CDG-modules over $B \otimes_k B^{\mathrm{op}}$ is also naturally equivalent to the DG-category of DG-modules over $C \otimes_k C^{\mathrm{op}}$ [32, Section 2.6]. As above, we denote by M_C the DG-module over $C \otimes_k C^{\mathrm{op}}$ corresponding to a CDG-module M over $B \otimes_k B^{\mathrm{op}}$.

To any left CDG-module G and right CDG-module F over B one can assign the left CDG-module $G \otimes_k F$ and the right CDG-module $F \otimes_k G$ (corresponding to each other under the above equivalence) over the CDG-category $B \otimes_k B^{\mathrm{op}}$. There are also the natural *diagonal* CDG-module B over $B \otimes_k B^{\mathrm{op}}$ and DG-module C over $C \otimes_k C^{\mathrm{op}}$ [32, Section 2.4]; these also correspond to each other with respect to the above equivalence of DG-categories.

For any DG-module M_C over $C \otimes_k C^{\mathrm{op}}$, we are interested in the comparison morphisms between the two kinds of Hochschild cohomology $HH^{II,*}(C, M_C) \longrightarrow HH^*(C, M_C)$ and homology $HH_*(C, M_C) \longrightarrow HH^{II}_*(C, M_C)$ [32, formula (23)].

Proposition C. *Assume that the Γ-graded category $B^{\#}\otimes_k B^{\#\mathrm{op}}$ is Noetherian and the diagonal Γ-graded module $B^{\#}$ over it is finitely generated. Let M be a CDG-module over $B\otimes_k B^{\mathrm{op}}$. Then the natural morphism $HH^{II,*}(C,M_C)\longrightarrow HH^*(C,M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of $\mathrm{D^{co}}(B\otimes_k B^{\mathrm{op}}\text{-}\mathrm{mod^{cdg}})$ containing the CDG-modules $Hom_k(F\otimes_k G,\,k^\vee)$ for all $F\in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G\in H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and closed under infinite products; or*

(ii) *the diagonal CDG-module B over $B\otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of $\mathrm{D^{abs}}(B\otimes_k B^{\mathrm{op}}\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ containing the CDG-modules $G\otimes_k F$ for all $F\in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G\in H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$.*

Proposition D. *Assume that the Γ-graded category $B^{\#}\otimes_k B^{\#\mathrm{op}}$ is coherent and the diagonal Γ-graded module $B^{\#}$ over it is finitely presented. Let M be a CDG-module over $B\otimes_k B^{\mathrm{op}}$. Then the natural morphism $HH_*(C,M_C)\longrightarrow HH^{II}_*(C,M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of $H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{prj}})\subset \mathrm{D^{ctr}}(B\otimes_k B^{\mathrm{op}}\text{-}\mathrm{mod^{cdg}})$ containing the CDG-modules $G\otimes_k F$ for all $F\in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G\in H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and closed under infinite direct sums; or*

(ii) *the diagonal CDG-module B over $B\otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of $\mathrm{D^{abs}}(B\otimes_k B^{\mathrm{op}}\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ containing the CDG-modules $G\otimes_k F$ for all $F\in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G\in H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$.*

Proofs of Propositions C-D. Similar to the proofs of Propositions A-B. ☐

In particular, assume that the Γ-graded category $B^{\#}\otimes_k B^{\#\mathrm{op}}$ is Noetherian and the diagonal Γ-graded module $B^{\#}$ over it is finitely generated. Suppose that the diagonal CDG-module B over $B\otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of $\mathrm{D^{abs}}(B\otimes_k B^{\mathrm{op}}\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ containing the CDG-modules $G\otimes_k F$ for all $F\in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G\in H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$. Then, according to [32, formulas (44-45) in Section 2.6] and parts (ii) of Propositions C-D, there are natural isomorphisms

$$HH^*(C,M_C)\simeq HH^{II,*}(C,M_C)\simeq HH^{II,*}(B,M) \tag{3}$$

$$HH_*(C,M_C)\simeq HH^{II}_*(C,M_C)\simeq HH^{II}_*(B,M) \tag{4}$$

for any CDG-module M over $B\otimes_k B^{\mathrm{op}}$. Specializing to the case of the diagonal CDG-module $M=B$ and DG-module $M_C=C$, we obtain

$$HH^*(C)\simeq HH^{II,*}(C)\simeq HH^{II,*}(B)\quad\text{and}\quad HH_*(C)\simeq HH^{II}_*(C)\simeq HH^{II}_*(B). \tag{5}$$

B.1.5. *Locally free matrix factorizations.* Let k be a regular commutative Noetherian ring of finite Krull dimension and X be an affine scheme of finite type over $\operatorname{Spec} k$. Let $w\in\mathcal{O}(X)$ be a global regular function on X. Consider the $\mathbb{Z}/2$-graded CDG-algebra B over k with $B^0=\mathcal{O}(X)$, $B^1=0$, $d=0$, and $h=-w\in B^0$. We will find it convenient to denote the CDG-algebra B simply by $(X,h)=(X,-w)$ (cf. Section 2.2 of the main body of this paper).

Then $C = \mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$ is the $\mathbb{Z}/2$-graded DG-category of locally free matrix factorizations of finite rank of the potential w on X. Furthermore, one has $B \otimes_k B^{\mathrm{op}} = (X \times_k X,\ w_2 - w_1)$, where $w_i = p_i^* w \in \mathcal{O}(X \times_k X)$, $i = 1, 2$, and $p_i \colon X \times_k X \longrightarrow X$ denote the coordinate projections. Let $\Delta \colon X \longrightarrow X \times_k X$ be the diagonal embedding and $\Delta_* \mathcal{O}_X$ be the corresponding coherent sheaf on $X \times_k X$.

Consider the coherent matrix factorization of the potential $w_2 - w_1$ on $X \times X$ whose even-degree component is the sheaf $\Delta_* \mathcal{O}_X$, while the odd-degree component vanishes. We will denote this "diagonal" matrix factorization simply by $\Delta_* \mathcal{O}_X \in H^0((X \times_k X,\ w_2 - w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$. Applying the machinery of the previous sections leads to the following result (cf. [32, Sections 4.8–4.10]).

Corollary. *Suppose that the diagonal matrix factorization $\Delta_* \mathcal{O}_X$ belongs to the minimal thick subcategory of $\mathsf{D}^{\mathrm{abs}}((X \times_k X,\ w_2 - w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ containing the external tensor products of locally free matrix factorizations of finite rank $p_1^* G \otimes_k p_2^* F$ for all $G \in H^0((X, -w)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and $F \in H^0((X, w)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$. Then the natural isomorphisms* (5) *hold for the CDG-algebra $B = (X, w)$ and the DG-category of locally free matrix factorizations $C = \mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$.* □

Notice that the condition under which the conclusion of Corollary has been proven is a rather strong one, particularly when X is not assumed to be a regular scheme. Then it is not even clear when or why the diagonal matrix factorization $\Delta_* \mathcal{O}_X$ should belong to the thick envelope of the full triangulated subcategory of locally free matrix factorizations $H^0((X \times_k X,\ w_2 - w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}) \subset \mathsf{D}^{\mathrm{abs}}((X \times_k X,\ w_2 - w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ on $X \times_k X$, let alone to the thick subcategory generated by external tensor products of locally free matrix factorizations from the two copies of X.

B.1.6. *Smooth stratifications.* The author learned the following definition from A. Efimov. A scheme X of finite type over a field k is said to admit a *smooth stratification* if it can be presented as a disjoint union of its locally closed subsets $X = \coprod_\alpha S_\alpha$ so that each S_α, when endowed with the structure of a reduced locally closed subscheme in X, becomes a smooth scheme over k. In particular, every scheme of finite type over a perfect field k admits a smooth stratification, as any regular scheme of finite type over a perfect field is smooth over it [13, Corollaires 17.15.2 and 17.15.13]. Notice that a scheme of finite type over a field admits a smooth stratification if and only if its maximal reduced closed subscheme does.

The definition of a *regular stratification* of a Noetherian scheme is similar, except that the strata S_α are only required to be regular schemes in their reduced locally closed subscheme structures. Any scheme of finite type over a field admits a regular stratification.

Let X be a smooth affine scheme over a field k and $w \in \mathcal{O}(X)$ be a regular function on X. Set $X_0 = \{w = 0\} \subset X$ to be the zero locus of w. The following result is a slight generalization of [32, Corollary 4.8.A] based on the above definitions.

Corollary. *Assume that there exists a closed subscheme $Z \subset X$ such that $w \colon X \setminus Z \longrightarrow \mathbb{A}^1_k$ is a smooth morphism, $w|_Z = 0$, and the scheme Z admits* *a smooth stratification over k. Then the conditions of Corollary* B.1.5 *are satisfied, so its conclusions apply.*

Proof. According to the argument in [32, Section 4.8], it suffices to show that the bounded derived category of coherent sheaves on $Z \times Z$ is generated by external tensor products of coherent sheaves on the two Cartesian factors. This is a particular case of the following lemma. □

Lemma. *Let Z' and Z'' be schemes of finite type over a field k. Assume that the scheme Z' admits a smooth stratification. Then the bounded derived category of coherent sheaves $\mathsf{D^b}((Z' \times Z'')\text{–}\mathsf{coh})$ on the Cartesian product $Z' \times_k Z''$ coincides with its minimal thick subcategory containing the external tensor products $\mathcal{K}' \otimes_k \mathcal{K}''$ of coherent sheaves on $\mathcal{K}'$ on Z' and $\mathcal{K}''$ on Z''.*

Proof. One proceeds by induction in the total number of strata in a smooth stratification of Z' and a regular stratification of Z''. Clearly, one can replace Z' and Z'' with their maximal reduced closed subschemes. Now if S_{α_0} is an open stratum in Z' and T_{β_0} is an open stratum in Z'', then S_{α_0} is smooth as an open subscheme in Z' and T_{β_0} is regular as an open subscheme in Z'', while the induction assumption applies to $(Z' \setminus S_{\alpha_0}) \times_k Z''$ and $Z' \times_k (Z'' \setminus T_{\beta_0})$. The scheme $S_{\alpha_0} \times_k T_{\beta_0}$ is regular, since it is smooth over a regular scheme. The rest of the argument is based on [30, Proposition 2.7] and follows the lines of [22, proof of Theorem 3.7]. □

B.2. **Coherent matrix factorizations.** In this section we return to the notation system typical for the main body of this paper. The notion of a critical value of a regular function on a singular variety is defined in Section B.2.1. In Section B.2.2 we show that the external tensor product of coherent matrix factorizations is a fully faithful functor between the absolute derived categories, and provide a sufficient condition for the pretriangulated extension of its DG-category version to be a quasi-equivalence. The Hochschild cohomology of the DG-category corresponding to the absolute derived category of coherent matrix factorizations of a potential having no critical values but zero is computed in Section B.2.3.

B.2.1. *Noncritical functions.* Let k be a field and X be a scheme of finite type over $\operatorname{Spec} k$. Let $f \in \mathcal{O}(X)$ be a global regular function on X; assume that $f\colon X \longrightarrow \mathbb{A}^1_k$ is a flat morphism from X to the affine line (when k is algebraically closed, this means that the function $f - c$ is a local nonzero-divisor on X for every $c \in k$).

Let Y be a scheme of finite type over $\operatorname{Spec} k$ and $g \in \mathcal{O}(Y)$ be a global regular function. Let $p_1\colon X \times_k Y \longrightarrow X$ and $p_2\colon X \times_k Y \longrightarrow Y$ be the natural projections. Consider the regular function $f_1 - g_2 = p_1^* f - p_2^* g$ on $X \times_k Y$. The corresponding morphism $f_1 - g_2\colon X \times_k Y \longrightarrow \mathbb{A}^1_k$ is flat as the composition of two flat morphisms $X \times_k Y \longrightarrow \mathbb{A}^1_k \times_k Y \longrightarrow \mathbb{A}^1_k$ (the former morphism being flat since the morphism $f\colon X \longrightarrow \mathbb{A}^1_k$ is and the latter one because the polynomial $x - g$ does not divide zero in $B[x]$ for any commutative ring B and element $g \in B$). In particular, the function $f_1 - g_2$ is a local nonzero-divisor on $X \times_k Y$.

The function $f \in \mathcal{O}(X)$ is said to be *noncritical* (or to *have no critical values*) if for any regular function $g \in \mathcal{O}(Y)$ on a scheme Y of finite type over $\operatorname{Spec} k$ the triangulated category $\mathsf{D}^{\mathsf{abs}}((X \times_k Y, \mathcal{O}, f_1 - g_2)\text{–}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(\{f_1 - g_2 = 0\}/X \times_k Y)$ of relative singularities of the zero locus of the function $f_1 - g_2$ on $X \times_k Y$ (see Theorem 2.7 for the underlying equivalence of categories) vanishes. According to Remark 1.3 and Theorem 1.10(b), this condition is local in both X and Y.

Therefore, given a scheme X of finite type over k and a regular function $f \in \mathcal{O}(X)$, there is a unique maximal open subscheme $X'_f \subset X$ where the function f is noncritical. We will see below that the open subscheme X'_f is always dense in X if the morphism $f\colon X \longrightarrow \mathbb{A}^1_k$ is flat and the field k has zero characteristic.

Similarly, there is a unique maximal open subscheme $\mathbb{A}^1_{k,f} \subset \mathbb{A}^1_k$ such that the restriction of f to its full preimage in X is noncritical. The scheme $\mathbb{A}^1_{k,f}$ is always nonempty if the field k has zero characteristic. The points in the complement $\mathbb{A}^1_k \setminus \mathbb{A}^1_{k,f}$ are called the *critical values* of f. In particular, one says that f *has no critical values but zero* if the restriction of f to $f^{-1}(\mathbb{A}^1_k \setminus \{0\}) \subset X$ is noncritical.

Remark. It would be interesting to have a geometric characterization of noncriticality of functions on singular schemes. For example, how does our definition of noncriticality relate to the condition that the differential of f at every closed point $x \in X$ be a nonzero element of the Zariski cotangent space $T^*_x X$? We do *not* know this; cf. the smooth stratification approach below.

Lemma. *Let $X = \coprod_\alpha S_\alpha$ be a scheme of finite type over* $\operatorname{Spec} k$ *presented as a disjoint union of its locally closed subsets, endowed with their reduced locally closed subscheme structures. Let $\mathcal{L}$ be a line bundle on X and $w \in \mathcal{L}(X)$ be its global section. In this setting, if the absolute derived categories* $\mathsf{D}^{\mathsf{abs}}((S_\alpha, \mathcal{L}|_{S_\alpha}, w|_{S_\alpha})\text{–}\mathsf{coh})$ *vanish for all α, then so does the absolute derived category* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh})$.

Proof. Proceeding by induction in the number of strata in the stratification S_α, it suffices to consider the case when there are only two of them, namely, a closed subset $S \subset X$ and its open complement $X \setminus S$. One can also replace X with its maximal reduced closed subscheme. Then the desired assertion follows from Theorem 1.10(b), since the triangulated category $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_S)$ is generated by the image of the natural functor $\mathsf{D}^{\mathsf{abs}}((S, \mathcal{L}|_S, w|_S)\text{–}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{–}\mathsf{coh}_S)$. □

Proposition. *Let X be a scheme of finite type over* $\operatorname{Spec} k$ *and $f \in \mathcal{O}(X)$ be a regular function on X. Let $X = \coprod_\alpha S_\alpha$ be a smooth stratification of the scheme X over k (see Section* B.1.6*). Assume that the morphism of schemes $f\colon X \longrightarrow \mathbb{A}^1_k$ is flat and the morphisms of schemes $f|_{S_\alpha}\colon S_\alpha \longrightarrow \mathbb{A}^1_k$ are smooth for all α. Then the function f is noncritical on X.*

Proof. Let Y be a scheme of finite type over $\operatorname{Spec} k$ and $g \in \mathcal{O}(Y)$ be a regular function. We have to show that the absolute derived category $\mathsf{D}^{\mathsf{abs}}((X \times_k Y, \mathcal{O}, f_1 - g_2)\text{–}\mathsf{coh})$ vanishes. Choosing a stratification of Y by regular locally closed subschemes and applying Lemma, one can assume that X is smooth over k and Y is regular. Then the scheme $X \times_k Y$ is also regular, the derivative of the function $f_1 - g_2 \in \mathcal{O}(X \times_k Y)$, viewed as an element of the Zariski cotangent space, does not vanish at any points where the function itself does (and, in a sense, at any other points, too), and it follows that the zero locus of $f_1 - g_2$ in $X \times_k Y$ is also a regular scheme.

Notice that, defining noncriticality in terms of the absolute derived categories of coherent matrix factorizations, one does not need the flatness assuption on f for this argument to be applicable. $\square$

It follows from Proposition that, for any scheme of finite type X with a smooth stratification $X = \coprod_\alpha S_\alpha$ over $\operatorname{Spec} k$ and any regular function $f \in \mathcal{O}(X)$ corresponding to a flat morphism $f\colon X \longrightarrow \mathbb{A}^1_k$, the set of critical values of the function f on X is contained in the union of the sets of critical values of the functions $f|_{S_\alpha}$. In particular, if the characteristic of k is zero, then all these sets are finite.

B.2.2. *External tensor products.* Let X' and X'' be schemes of finite type over a field k, and let $w' \in \mathcal{O}(X')$ and $w'' \in \mathcal{O}(X'')$ be regular functions. Let $X' \times_k X''$ be the Cartesian product, p_1 and p_2 be its natural projections onto the factors X' and X'', and $w'_1 + w''_2 = p_1^* w' + p_2^* w''$ be the related regular function on $X' \times_k X''$. Then there is the external tensor product functor

$$(6)\quad \otimes_k\colon \mathsf{D^{co}}((X', \mathcal{O}, w')\text{–}\mathsf{qcoh}) \times \mathsf{D^{co}}((X'', \mathcal{O}, w'')\text{–}\mathsf{qcoh}) \\ \longrightarrow \mathsf{D^{co}}((X' \times_k X'',\ \mathcal{O},\ w'_1 + w''_2)\text{–}\mathsf{qcoh}),$$

which restricts to the similar functor

$$(7)\quad \otimes_k\colon \mathsf{D^{abs}}((X', \mathcal{O}, w')\text{–}\mathsf{coh}) \times \mathsf{D^{abs}}((X'', \mathcal{O}, w'')\text{–}\mathsf{coh}) \\ \longrightarrow \mathsf{D^{abs}}((X' \times_k X'',\ \mathcal{O},\ w'_1 + w''_2)\text{–}\mathsf{coh})$$

on coherent matrix factorizations.

Proposition. *Let $\mathcal{K}'$ and $\mathcal{M}'$ be coherent matrix factorizations of the potential w' on the scheme X', and let $\mathcal{K}''$ and $\mathcal{M}''$ be coherent matrix factorizations of the potential w'' on the scheme X''. Then the natural map of $\mathbb{Z}/2$-graded k-vector spaces of morphisms*

$$(8)\quad \operatorname{Hom}_{\mathsf{D^{abs}}((X',\mathcal{O},w')\text{–}\mathsf{coh})}(\mathcal{K}', \mathcal{M}'[*]) \otimes_k \operatorname{Hom}_{\mathsf{D^{abs}}((X'',\mathcal{O},w'')\text{–}\mathsf{coh})}(\mathcal{K}'', \mathcal{M}''[*]) \\ \longrightarrow \operatorname{Hom}_{\mathsf{D^{abs}}((X'\times_k X'',\, \mathcal{O},\, w'_1+w''_2)\text{–}\mathsf{coh})}(\mathcal{K}' \otimes_k \mathcal{K}'',\ \mathcal{M}' \otimes_k \mathcal{M}''[*])$$

induced by the additive functor of two arguments (7) *is an isomorphism.*

Proof. By Proposition 1.5(d), it suffices to show that the natural map

$$(9)\quad \operatorname{Hom}_{\mathsf{D^{co}}((X',\mathcal{O},w')\text{–}\mathsf{qcoh})}(\mathcal{K}', \mathcal{M}'[*]) \otimes_k \operatorname{Hom}_{\mathsf{D^{co}}((X'',\mathcal{O},w'')\text{–}\mathsf{qcoh})}(\mathcal{K}'', \mathcal{M}''[*]) \\ \longrightarrow \operatorname{Hom}_{\mathsf{D^{co}}((X'\times_k X'',\, \mathcal{O},\, w'_1+w''_2)\text{–}\mathsf{qcoh})}(\mathcal{K}' \otimes_k \mathcal{K}'',\ \mathcal{M}' \otimes_k \mathcal{M}''[*])$$

induced by the functor (6) is an isomorphism for any coherent matrix factorizations $\mathcal{K}'$, $\mathcal{K}''$ and quasi-coherent matrix factorizations $\mathcal{M}'$, $\mathcal{M}''$ of the potentials w' and w''. One easily checks that the desired assertion holds for the Hom spaces in the homotopy categories of matrix factorizations (since it holds for morphisms between the external tensor products of coherent and quasi-coherent sheaves).

Furthermore, one can assume the quasi-coherent matrix factorizations $\mathcal{M}'$ and $\mathcal{M}''$ to be injective. Then the Hom spaces in the left-hand side of the map (9) coincide with the similar Hom spaces computed in the homotopy categories of matrix factorizations. Let $\mathcal{I}^\bullet$ be a right resolution of $\mathcal{M}' \otimes_k \mathcal{M}''$ in the abelian category of quasi-coherent matrix factorizations (and closed morphisms between them) consisting of injective matrix factorizations, and let $\mathcal{J}$ be the total matrix factorization of the complex $\mathcal{I}^\bullet$ constructed by taking infinite direct sums along the diagonals. Then the k-vector spaces of morphisms from $\mathcal{K}' \otimes_k \mathcal{K}''$ into $\mathcal{J}$ in the homotopy category of matrix factorizations are isomorphic to the right-hand side of (9) [37, Theorem 3.7].

It remains to show that the spaces of morphisms from $\mathcal{K}' \otimes_k \mathcal{K}''$ to $\mathcal{M}' \otimes_k \mathcal{M}''$ in the homotopy category of matrix factorizations are isomorphic to the similar spaces of morphisms from $\mathcal{K}' \otimes_k \mathcal{K}''$ to $\mathcal{J}$. Indeed, taking the termwise Hom from $\mathcal{K}' \otimes_k \mathcal{K}''$ preserves exactness of the sequence $0 \longrightarrow \mathcal{M}' \otimes_k \mathcal{M}'' \longrightarrow \mathcal{I}^\bullet$, since the higher Ext spaces from the components of $\mathcal{K}' \otimes_k \mathcal{K}''$ into those of $\mathcal{M}' \otimes_k \mathcal{M}''$ in the abelian category of coherent sheaves on $X' \otimes_k X''$ vanish. The latter assertion can be checked for affine schemes X', X'' using projective resolutions and then globally for the cohomology of quasi-coherent sheaves using, e. g., the Čech approach. $\square$

Theorem. *Assume that the morphisms of schemes $w' \colon X' \longrightarrow \mathbb{A}^1_k$ and $w'' \colon X'' \longrightarrow \mathbb{A}^1_k$ are flat. Suppose that there exist closed subschemes $Z' \subset X'$ and $Z'' \subset X''$ such that $w'|_{Z'} = 0 = w''_{Z''}$, the functions w' and w'' are noncritical on $X' \setminus Z'$ and $X'' \setminus Z''$, and the scheme Z' admits a smooth stratification over k. Then the absolute derived category $\mathsf{D}^{\mathsf{abs}}((X' \times_k X'', \mathcal{O}, w'_1 + w''_2)\text{–}\mathsf{coh})$ coincides with its minimal thick subcategory containing the image of the functor* (7).

Proof. By the definition of noncriticality, one has $\mathsf{D}^{\mathsf{abs}}((((X' \setminus Z') \times_k X''), \mathcal{O}, w'_1 + w''_2)\text{–}\mathsf{coh}) = 0 = \mathsf{D}^{\mathsf{abs}}(((X' \times_k (X'' \setminus Z'')), \mathcal{O}, w'_1 + w''_2)\text{–}\mathsf{coh})$. Therefore, any coherent matrix factorization of the potential $w'_1 + w''_2$ on $X' \times_k X''$ has its category-theoretic support inside $Z' \times_k Z''$, and is consequently isomorphic in $\mathsf{D}^{\mathsf{abs}}((X' \times_k X'', \mathcal{O}, w'_1 + w''_2)\text{–}\mathsf{coh})$ to a direct summand of an object represented by a coherent matrix factorization supported set-theoretically inside $Z' \times_k Z''$ (see Corollary 1.10(b)). It follows that the triangulated category $\mathsf{D}^{\mathsf{abs}}((X' \times_k X'', \mathcal{O}, w'_1 + w''_2)\text{–}\mathsf{coh})$ is generated by the direct images of coherent matrix factorizations of the zero potential from the closed embedding $Z' \times_k Z'' \longrightarrow X' \times_k X''$.

Furthermore, let X'_0, X''_0, and Y_0 denote the zero loci of the functions w', w'', and $w'_1 + w''_2$ on X', X'', and $X' \times_k X''$, respectively. Denote the natural closed embeddings by $i' \colon X'_0 \longrightarrow X'$, $i'' \colon X''_0 \longrightarrow X''$, $\iota \colon X'_0 \times X''_0 \longrightarrow Y_0$, and $h \colon Y_0 \longrightarrow Y$. The external tensor product functor (cf. [32, Lemma 4.8.B])

$$\otimes_k \colon \mathsf{D}^{\mathsf{b}}_{Sing}(X'_0/X') \times \mathsf{D}^{\mathsf{b}}_{Sing}(X''_0/X'') \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/(X' \times_k X'')) \qquad (10)$$

is well-defined, since for any bounded complexes of coherent sheaves $\mathcal{F}^\bullet$ on X' and $\mathcal{K}^\bullet$ on X''_0 one has $\iota_*(\mathbb{L}i'^*\mathcal{F}^\bullet \otimes_k \mathcal{K}^\bullet) \simeq \mathbb{L}h^*((\mathrm{id}_{X'} \times i'')_*(\mathcal{F}^\bullet \otimes_k \mathcal{K}^\bullet))$. Indeed, the square diagram of closed embeddings

$$\begin{array}{ccc} X'_0 \times_k X''_0 & \longrightarrow & X' \times_k X''_0 \\ \big\downarrow & & \big\downarrow \\ Y_0 & \longrightarrow & X' \times X'' \end{array}$$

is Cartesian and the higher derived tensor products related to the construction of this Cartesian product all vanish.

The functor $\Upsilon\colon \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/(X' \times_k X'')) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X' \times_k X'',\ \mathcal{O},\ w'_1 + w''_2)$ (see Section 2.7) and the similar functors for the potentials w' and w'' on X' and X'' transform the external product functor (7) into the external tensor product functor (10). By the assumption, one has $Z' \subset X'_0$ and $Z'' \subset X''_0$. It remains to apply Lemma B.1.6 in order to finish the proof of the theorem. □

B.2.3. *Hochschild cohomology.*

B.2.4. *Cotensor product.*

B.2.5. *Hochschild homology.*

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