ARTIN–TATE MOTIVIC SHEAVES WITH FINITE COEFFICIENTS OVER AN ALGEBRAIC VARIETY

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Abstract. We propose a construction of a tensor exact category $\mathcal{F}_X^m$ of Artin–Tate motivic sheaves with finite coefficients $\mathbb{Z}/m$ over an algebraic variety X (over a field K of characteristic prime to m) in terms of étale sheaves of $\mathbb{Z}/m$-modules over X. Among the objects of $\mathcal{F}_X^m$, in addition to the Tate motives $\mathbb{Z}/m(j)$, there are the cohomological relative motives with compact support $\mathcal{M}_{cc}^m(Y/X)$ of varieties Y quasi-finite over X. Exact functors of inverse image with respect to morphisms of algebraic varieties and direct image with compact supports with respect to quasi-finite morphisms of varieties $Y \longrightarrow X$ act on the exact categories $\mathcal{F}_X^m$. Assuming the existence of triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ over algebraic varities X over K and a weak version of the "six operations" in these categories, we identify $\mathcal{F}_X^m$ with the exact subcategory in $\mathcal{DM}(X, \mathbb{Z}/m)$ consisting of all the iterated extensions of the Tate twists $\mathcal{M}_{cc}^m(Y/X)(j)$ of the motives $\mathcal{M}_{cc}^m(Y/X)$. An isomorphism of the $\mathbb{Z}/m$-modules Ext between the Tate motives $\mathbb{Z}/m(j)$ in the exact category $\mathcal{F}_X^m$ with the motivic cohomology modules predicted by the Beilinson–Lichtenbaum étale descent conjecture (recently proven by Voevodsky, Rost, et al.) holds for smooth varieties X over K if and only if the similar isomorphism holds for Artin–Tate motives over fields containing K. When K contains a primitive m-root of unity, the latter condition is equivalent to a certain Koszulity hypothesis, as shown in our previous paper [14].

Contents

Introduction

The motivic cohomology of an algebraic variety X over a field K with coefficients in a ring $k = \mathbb{Z},\ \mathbb{Z}/m,\ \mathbb{Q}, \dots$ can be defined [19, 20, 21] as

$$H^i_{\mathcal{M}}(X, k(j)) = \mathrm{Hom}_{\mathcal{DM}(X,k)}(k, k(j)[i]) = \mathrm{Hom}_{\mathcal{DM}(K,k)}(\mathcal{M}^k_h(X), k(j)[i]),$$

where $\mathcal{DM}(K, k)$ is the derived category of mixed motives over K with coefficients in k and $\mathcal{M}^k_h(X) \in \mathcal{DM}(K, k)$ is the homological (covariant) motive of X over K, while $\mathcal{DM}(X, k)$ is the derived category of motives (motivic sheaves) over X with coefficients in k. The motivic cohomology localize in the Zariski topology, so one has

$$H^i_{\mathcal{M}}(X, k(j)) = \mathbb{H}^i_{Zar}(X, \underline{k(j)}),$$

where $\underline{k(j)}$ are certain complexes of sheaves of k-modules on the big Zariski site of varieties over K (defined naturally up to a quasi-isomorphism). Furthermore, one has $\underline{k(j)} = k \otimes^{\mathbb{L}}_{\mathbb{Z}} \underline{\mathbb{Z}(j)}$. Here $\mathbb{H}_{Zar}$ denotes the hypercohomology of complexes of sheaves in the Zariski topology and $\otimes^{\mathbb{L}}_{\mathbb{Z}}$ is the notation for the left derived functor of tensor product of sheaves over $\mathbb{Z}$.

Now assume that the variety X is smooth and the field K is perfect of characteristic not dividing a positive integer m. Then the motivic cohomology of X with coefficients in $\mathbb{Z}/m$ can be computed by the formula

$$H^i_{\mathcal{M}}(X, \mathbb{Z}/m(j)) = \mathbb{H}^i_{Zar}(X, \tau_{\leqslant j}\mathbb{R}\pi_*\mu_m^{\otimes j}), \tag{1}$$

where $\pi\colon \acute{E}t \longrightarrow Zar$ is the natural map between the big étale and Zariski sites of varieties over K (the direction of the map of sites being opposite to that of the functor between the categories of "open sets"), $\mathbb{R}\pi_*$ denotes the right derived functor of direct image of sheaves, $\tau_{\leqslant j}$ are the canonical truncations of complexes of Zariski sheaves, and the cyclotomic étale sheaves $\mu_m^{\otimes j}$ are the tensor powers over $\mathbb{Z}/m$ of the étale sheaf μ_m of m-roots of unity. In fact, the formula (1) is a combination of two assertions: the Beilinson–Lichtenbaum modified étale descent rule

$$\underline{\mathbb{Z}(j)} = \tau_{\leqslant j}\mathbb{R}\pi_*\pi^*\underline{\mathbb{Z}(j)} = \tau_{\leqslant j+1}\mathbb{R}\pi_*\pi^*\underline{\mathbb{Z}(j)}, \tag{2}$$

and (a version of) the Suslin rigidity theorem

$$\pi^*\underline{\mathbb{Z}/m(j)} = \mu_m^{\otimes j}, \qquad j \geqslant 0. \tag{3}$$

The isomorphism (3) goes back to A. Suslin's paper [16]; its proof in the form stated above can be found in [11, 7.20 and 10.3]. The rules (1–2) were conjectured by A. Beilinson [1, 5.10.D(v-vi)] and S. Lichtenbaum [10]. According to [17, 6], the formula (1) follows from the Milnor–Bloch–Kato conjecture, the long work on the proof of which was recently finished by Voevodsky, Rost, et al. [22]. One can replace the Zariski topology with the Nisnevich topology in these results [11, 13.9 and 22.2].

For singular varieties X, the formula (1) no longer holds, as one can see in the following simple example. Let X be the affine line (say, over the field K of complex numbers) with two different points glued together. Then one has $\mathcal{M}^{\mathbb{Z}}_h(X) = \mathbb{Z} \oplus \mathbb{Z}[1]$, so $H^1_{\mathcal{M}}(X, \mathbb{Z}/m(0)) = \mathbb{Z}/m$, while $\tau_{\leqslant 0}\mathbb{R}\pi_*\mathbb{Z}/m = \mathbb{Z}/m$ and $H^1_{Zar}(X, \mathbb{Z}/m) = 0$. In fact, the restriction of $\pi^*\underline{\mathbb{Z}/m(0)}$ to the small étale site of X is still isomorphic to

$\mathbb{Z}/m$, since our first motivic cohomology class of X with coefficients in $\mathbb{Z}/m$ dies in étale covers. So it must be the modified étale descent rule (2) for the Zariski topology that breaks down in this case.

For the above singular curve X, the formula (1) can still be saved by replacing the Zariski topology with the Nisnevich topology, as one has $H^1_{Nis}(X, \mathbb{Z}/m) = \mathbb{Z}/m$. However, for more complicated singularities the Nisnevich topology is not enough either, and the *cdh* topology is needed [11, Theorem 14.20]. E. g., if Y is a normal surface with a point singularity such that the exceptional fiber of its resolution is a self-intersecting projective line, then one has $H^2_{\mathcal{M}}(Y, \mathbb{Z}/m(0)) = \mathbb{Z}/m$, while $\tau_{\leqslant 0}\mathbb{R}\rho_*\mathbb{Z}/m = \mathbb{Z}/m$ and $H^2_{Nis}(Y, \mathbb{Z}/m) = 0$ (ρ being the natural map $\acute{E}t \longrightarrow Nis$ between the big étale and Nisnevich sites) [11, Exercise 12.32]. Now it is the formula (3) that breaks down.

The formula (1) suggests that it might be possible to construct the derived category of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ on a variety X, or at least some parts of this category, in terms of the étale topology of X. The aim of this paper is to suggest such a construction for the triangulated category $\mathcal{DMAT}(X, \mathbb{Z}/m)$ of *Artin–Tate motivic sheaves* over X. This is defined as the full triangulated tensor subcategory of $\mathcal{DM}(X, \mathbb{Z}/m)$ generated by Tate motives $\mathbb{Z}/m(j)$ and the compactly supported relative cohomological motives of varieties Y quasi-finite over X. In fact, we construct a $\mathbb{Z}/m$-linear exact category of filtered constructible étale sheaves of $\mathbb{Z}/m$-modules $\mathcal{F}_X^m$ over X, whose derived category $\mathcal{D}^b(\mathcal{F}_X^m)$ is similar to $\mathcal{DMAT}(X, \mathbb{Z}/m)$ "insofar as no complicated singularities get involved".

We also establish some functoriality properties of the exact categories $\mathcal{F}_X^m$ with respect to morphisms of varieties $f\colon Y \longrightarrow X$. Namely, we construct exact functors of inverse image $f^*\colon \mathcal{F}_X^m \longrightarrow \mathcal{F}_Y^m$ for all morphisms f and exact functors of direct image with compact supports $f_!\colon \mathcal{F}_Y^m \longrightarrow \mathcal{F}_X^m$ for quasi-finite morphisms f. The functors f^* and $f_!$ are adjoint to each other from different sides depending on whether f is finite or étale. For each quasi-finite morphism of varieties $Y \longrightarrow X$ we define the relative cohomological motive with compact supports $\mathcal{M}^m_{cc}(Y/X) \in \mathcal{E}_X^m$ of Y over X. Here $\mathcal{E}_X^m \subset \mathcal{F}_X^m$ is the full exact subcategory of *Artin motivic sheaves.* For any quasi-finite morphism of smooth varieties $Y \longrightarrow X$ we have the relative homological motive $\mathcal{M}^m_h(Y/X) \in \mathcal{D}^b(\mathcal{F}_X^m)$.

Let us first discuss the case when $X = \operatorname{Spec} L$ is the spectrum of a field. In this case, the Tate twists of the motives of the spectra of finite separable extensions of L generate (using iterated extensions) an exact subcategory $\mathcal{MAT}(L, \mathbb{Z}/m) \subset \mathcal{DMAT}(L, \mathbb{Z}/m)$ which was computed in [14] in terms of the absolute Galois group G_L of the field L. The category $\mathcal{MAT}(L, \mathbb{Z}/m)$ is equivalent to the exact category of finitely filtered discrete G_L-modules over $\mathbb{Z}/m$ whose successive quotient modules are cyclotomically twisted (finitely generated) permutational modules.

The triangulated category $\mathcal{DMAT}(L, \mathbb{Z}/m)$ is equivalent to the derived category $\mathcal{D}^b\mathcal{MAT}(L, \mathbb{Z}/m)$ of the exact category $\mathcal{MAT}(L, \mathbb{Z}/m)$ if and only if the natural maps from the $\mathbb{Z}/m$-modules of Yoneda Ext between the objects of the exact category to the modules of higher Hom between the same objects in the triangulated category are isomorphisms. The latter property can be called the $K(\pi,1)$-*conjecture for Artin–Tate motives over L with coefficients* $\mathbb{Z}/m$. In the case when L contains a primitive m-root of unity, this conjecture has been interpreted in [14] as a certain Koszulity hypothesis for the "big graded ring" of diagonal Hom in $\mathcal{DMAT}(L,\mathbb{Z}/m)$. Another name for the $K(\pi,1)$-conjecture is the *silly filtration conjecture*.

This paper purports to explain how to "globalize" the $K(\pi,1)$-conjecture to smooth varieties, in the particular case of Artin–Tate motives with finite coefficients. (Note that we do not know how to globalize the more conventional $K(\pi,1)$-conjecture for Tate motives.) We proceed in two steps. First of all, we construct natural maps from the $\mathbb{Z}/m$-modules $\mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$ to the "naive" motivic cohomology modules $\mathbb{H}^i_{Nis}(X,\tau_{\leqslant j}\mathbb{R}\rho_*\mu_m^{\otimes j})$, and show that these are isomorphisms (for X and all varieties étale over X) if and only if the $K(\pi,1)$-conjecture holds for Artin–Tate motives over the residue fields of the scheme points of X.

Secondly, we assume the existence of reasonably well-behaved triangulated categories of motivic sheaves $\mathcal{DM}(X,\mathbb{Z}/m)$ over algebraic varieties X over K and identify the exact category $\mathcal{F}_X^m$ with the full subcategory $\mathcal{MAT}(X,\mathbb{Z}/m)\subset\mathcal{DM}(X,\mathbb{Z}/m)$ generated, using iterated extensions, by the Tate twists $\mathcal{M}^m_{cc}(Y/X)(j)$ of the compactly supported relative cohomological motives of varieties Y quasi-finite over X, with its induced exact category structure. Note that the construction of this fully faithful functor and equivalence of exact categories does not yet depend on the $K(\pi,1)$-conjectures of any kind.

The $\mathbb{Z}/m$-modules of higher Hom in $\mathcal{DM}(X,\mathbb{Z}/m)$ between objects of $\mathcal{MAT}(X,\mathbb{Z}/m)$ may differ from the modules Ext computed in $\mathcal{F}_X^m=\mathcal{MAT}(X,\mathbb{Z}/m)$, but, assuming the $K(\pi,1)$-conjecture for Artin–Tate motives over fields, this only happens for singularities-related reasons. In particular, in the mentioned assumptions, the groups Ext in $\mathcal{F}_X^m$ coincide with groups of higher Hom in $\mathcal{DMAT}(X,\mathbb{Z}/m)$ when X is a curve, as singularities of curves can be resolved by finite morphisms.

In other words, the triangulated subcategory $\mathcal{DMAT}(X,\mathbb{Z}/m)\subset\mathcal{DM}(X,\mathbb{Z}/m)$ is equivalent to the derived category $\mathcal{D}^b\mathcal{MAT}(X,\mathbb{Z}/m)$ when X is a curve. Let us emphasize that the latter assertion is certainly *not* true for surfaces (not even for smooth surfaces over algebraically closed fields, as the category $\mathcal{MAT}(X,\mathbb{Z}/m)$ for such a surface X contains objects related to surfaces Y with bad enough singularities mapping finitely onto X). However, one has $\mathrm{Ext}^i_{\mathcal{F}_X^m}(M,\mathcal{M}^m_{cc}(Y/X)(j))\simeq\mathrm{Hom}_{\mathcal{DMAT}(X,\mathbb{Z}/m)}(M,\mathcal{M}^m_{cc}(Y/X)(j)[i])$ for any smooth variety Y finite over a variety X and any object $M\in\mathcal{F}_X^m$.

We also prove the basic properties of the relative homological motives $\mathcal{M}^m_h(Y/X)\in\mathcal{D}^b(\mathcal{F}_X^m)$ of smooth varieties Y quasi-finite over a fixed smooth variety X using some of the conventional assumptions about the triangulated categories of motivic sheaves $\mathcal{DM}(X,\mathbb{Z}/m)$. In particular, the group $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{F}_X^m)}(\mathcal{M}^m_h(Y/X),\mathbb{Z}/m(j)[i])$ is identified with the group $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{F}_Y^m)}(\mathbb{Z}/m,\mathbb{Z}/m(j)[i])$ and the motivic cohomology group $\mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m,\mathbb{Z}/m(j)[i])$ (assuming, as above, the $K(\pi,1)$-conjecture for Artin–Tate motives over fields).

Thus our category $\mathcal{F}_X^m$ is proposed as a solution to the problem of "constructing explicitly the category of 'fine étale $\mathbb{Z}/l^n$-sheaves'" posed in [1, 5.10.D(vi)].

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1. Exact Category of Artin–Tate Motivic Sheaves

All schemes in this paper are presumed to be separated. We fix an integer $m \geqslant 2$, and a perfect field K of characteristic not dividing m. By an (*algebraic*) *variety* over K we mean a scheme of finite type over $\operatorname{Spec} K$, which is not distinguished from its maximal reduced closed subscheme. For the purposes of notation and terminology related to the dimensions, all smooth varieties are presumed to be equidimensional.

Recall that for any Noetherian scheme X the category $\acute{E}t_X^{m,\infty}$ of étale sheaves of $\mathbb{Z}/m$-modules over X is a locally Noetherian Grothendieck abelian category. In other words, $\acute{E}t_X^{m,\infty}$ is equivalent to the category of ind-objects in the abelian category $\acute{E}t_X^m$ of constructible étale sheaves of $\mathbb{Z}/m$-modules over X. Here an étale sheaf is called *constructible* if it is generated by a finite set of its sections (over some étale schemes of finite type over X). This is equivalent to the existence of a finite stratification of X by locally closed subschemes, in restriction to which the sheaf is locally constant (lisse) with finitely generated stalks. An étale sheaf of $\mathbb{Z}/m$-modules is constructible if and only if it is a Noetherian object in the category of étale sheaves [4, Arcata IV.3].

For any algebraic variety X over K, consider the following exact category $\mathcal{E}_X^m$. The objects of $\mathcal{E}_X^m$ are constructible étale sheaves of $\mathbb{Z}/m$-modules N over X such that for any scheme point $x \in X$ with the residue field K_x the stalk N_x of N over x, considered as a discrete module over the absolute Galois group G_{K_x}, is a (finitely generated) permutational module with coefficients in $\mathbb{Z}/m$. In other words, the $\mathbb{Z}/m$-module N_x must admit a system of free generators that is preserved as a set (permuted) by the action of G_{K_x}. As an additive category, $\mathcal{E}_X^m$ is a full subcategory in $\acute{E}t_X^m$. The exact triples in $\mathcal{E}_X^m$ are the short exact sequences of étale sheaves from $\mathcal{E}_X^m$ for which the related short sequences of stalks at x are *split* short exact sequences of G_{K_x}-modules over $\mathbb{Z}/m$ for all the scheme points $x \in X$.

The category $\mathcal{E}_X^m$ is suggested as our candidate for the role of the *exact category of Artin motivic sheaves* over X. The larger exact category $\mathcal{F}_X^m$ of *Artin–Tate motivic sheaves* is constructed on the basis of $\mathcal{E}_X^m$ and $\acute{E}t_X^m$ in the following way.

The objects of $\mathcal{F}_X^m$ are filtered étale sheaves of $\mathbb{Z}/m$-modules (N, F) over X with a finite decreasing filtration by étale subsheaves $F^jN,\ j \in \mathbb{Z},\ F^jN = N$ for $j \ll 0$ and 0 for $j \gg 0$. The successive quotient sheaves $\mathrm{gr}_F^j N = F^jN/F^{j+1}N$ must be isomorphic to the tensor products over $\mathbb{Z}/m$ of étale sheaves from $\mathcal{E}_X^m$ with the cyclotomic étale sheaves $\mu_m^{\otimes j}$. The latter are the inverse images to X of the étale sheaves over $\operatorname{Spec} K$ corresponding to the cyclotomic representations of the Galois group G_K. So one must have $\mathrm{gr}_F^j N \otimes_{\mathbb{Z}/m} \mu_m^{\otimes -j} \in \mathcal{E}_X^m$. The morphisms in $\mathcal{F}_X^m$ are the filtration-preserving morphisms of étale sheaves of $\mathbb{Z}/m$-modules. The exact triples in $\mathcal{F}_X^m$ are the pairs of morphisms with zero composition whose successive quotients with respect to the filtration F are exact triples in $\mathcal{E}_X^m$ twisted with $\mu_m^{\otimes j}$. In other words, it is required that the successive quotients form short exact sequences of étale sheaves whose stalks over every scheme point $x \in X$ are split exact triples of G_{K_x}-modules.

The exact categories $\mathcal{E}_X$ and $\mathcal{F}_X$ have natural structures of associative, commutative, and unital tensor categories with exact functors of tensor product. These are given by the tensor products of étale sheaves over $\mathbb{Z}/m$ and the tensor products of filtrations. The *Tate motive* $\mathbb{Z}/m(j) \in \mathcal{F}_X^m$ is the étale sheaf $\mu_m^{\otimes j}$ over X, placed in the filtration component j, so that $F^j\mathbb{Z}/m(j) = \mathbb{Z}/m(j)$ and $F^{j+1}\mathbb{Z}/m(j) = 0$. The objects $\mathbb{Z}/m(j)$ are invertible in the tensor category F_X; there are natural isomorphisms $\mathbb{Z}/m(i) \otimes \mathbb{Z}/m(j) = \mathbb{Z}/m(i+j)$. For any object $N \in \mathcal{F}_X^m$ the tensor product $N \otimes \mathbb{Z}/m(j)$ is denoted by $N(j)$ and called the *Tate twist* of N by j. The category $\mathcal{E}_X^m$ is naturally a full exact tensor subcategory in $\mathcal{F}_X^m$ consisting of all the objects concentrated in the filtration component 0.

Given an integer n dividing m, and any variety X over K, there is a natural tensor exact functor $\mathcal{F}_X^m \longrightarrow \mathcal{F}_X^n$, assigning to a filtered étale sheaf of $\mathbb{Z}/m$-modules (M, F) the étale sheaf of $\mathbb{Z}/n$-modules $M/n \simeq (m/n)M$ with the induced filtration.

2. Inverse Image and Direct Image with Compact Supports

Let $f\colon Y \longrightarrow X$ be a morphism of algebraic varieties over K. Then the inverse image functor $f^*\colon \acute{E}t_X^m \longrightarrow \acute{E}t_Y^m$ takes $\mathcal{E}_X^m$ to $\mathcal{E}_Y^m$ and is exact as a functor between these exact categories (and as a functor between the abelian categories $\acute{E}t_X^m$ and $\acute{E}t_Y^m$, too). The functor f^* is also a tensor functor taking μ_m to μ_m, and in particular commuting with the cyclotomic twists. Hence there is the induced exact functor $f^*\colon \mathcal{F}_X^m \longrightarrow \mathcal{F}_Y^m$. It takes the Tate motives $\mathbb{Z}/m(j)$ over X to the similar Tate motives $\mathbb{Z}/m(j)$ over Y and commutes with the tensor products in $\mathcal{F}_X$ and $\mathcal{F}_Y$.

Let $f\colon Y \longrightarrow X$ be a quasi-finite morphism of varieties over K. Then the functor of direct image with compact supports $f_!\colon \acute{E}t_Y^m \longrightarrow \acute{E}t_X^m$ is exact. For any étale sheaf of $\mathbb{Z}/m$-modules N over Y the stalk of the sheaf $f_!N$ over a scheme point $x \in X$ is isomorphic to the direct sum of the direct images of the stalks of N over the points $y \in Y$ such that $f(y) = x$ with respect to the finite morphisms of spectra of fields $y \to x$ [4, Arcata IV.5]. On the level of discrete modules over the absolute Galois groups of the fields K_x and K_y, the direct image is the functor of induction from the open subgroup $G_{K_y} \subset G_{K_x}$.

Hence the functor $f_!$ takes $\mathcal{E}_Y^m$ to $\mathcal{E}_X^m$ and induces an exact functor between these exact categories. According to the projection formula, the functor $f_!$ also commutes with the twists by (locally constant constructible) sheaves lifted from X, and in particular from $\operatorname{Spec} K$. Thus there is the induced exact functor $f_!\colon \mathcal{F}_Y^m \longrightarrow \mathcal{F}_X^m$, commuting with the Tate twists.

For any étale morphism of algebraic varieties $f\colon Y \longrightarrow X$, the functor $f_!\colon \mathcal{F}_Y^m \longrightarrow \mathcal{F}_X^m$ is left adjoint to the functor $f^*\colon \mathcal{F}_X^m \longrightarrow \mathcal{F}_Y^m$. For any finite morphism $f\colon Y \longrightarrow X$, the functor $f_!\colon \mathcal{F}_Y^m \longrightarrow \mathcal{F}_X^m$ is right adjoint to the functor $f^*\colon \mathcal{F}_X^m \longrightarrow \mathcal{F}_Y^m$. These adjunctions are induced by the similar adjunctions of functors between the categories $\acute{E}t_X^m$ and $\acute{E}t_Y^m$.

For any quasi-finite morphism $Y \longrightarrow X$, the functors $f_!$ and f^* between the exact categories $\mathcal{F}_X^m$ and $\mathcal{F}_Y^m$ satisfy the projection formula with respect to the tensor products over $\mathbb{Z}/m$. These functors also commute with each other in any base change situation in which they are all defined.

Lemma 2.1. *For any pair of adjoint exact functors between exact categories $\mathcal{E}' \longrightarrow \mathcal{E}''$ and $\mathcal{E}'' \longrightarrow \mathcal{E}'$ the induced pair of triangulated functors between the derived categories $\mathcal{D}^b(\mathcal{E}')$ and $\mathcal{D}^b(\mathcal{E}'')$ are naturally adjoint. The same assertion holds for unbounded derived categories.*

Proof. This follows from a general result about adjunctions and quotients for triangulated categories. For any pair of adjoint functors between triangulated categories $\mathcal{H}'$ and $\mathcal{H}''$ taking their triangulated subcategories $\mathcal{A}'$ and $\mathcal{A}''$ into each other, the induced functors between the quotient categories $\mathcal{H}'/\mathcal{A}'$ and $\mathcal{H}''/\mathcal{A}''$ are adjoint. ☐

3. Relative Motives

For any quasi-finite morphism $f\colon Y \longrightarrow X$ of varieties over K, the *cohomological relative motive of Y over X with compact supports*, defined as $\mathcal{M}_{cc}^m(Y/X) = f_!(\mathbb{Z}/m)$, is an object of the exact subcategory $\mathcal{E}_X^m$ in $\mathcal{F}_X^m$. The relative motive $\mathcal{M}_{cc}^m(Y/X) \in \mathcal{E}_X^m$ is contravariantly functorial with respect to finite morphisms of varieties Y quasi-finite over X and covariantly functorial with respect to étale[1] morphisms of varieties Y quasi-finite over X. For any quasi-finite morphism $Y \longrightarrow X$ and any closed subvariety $Z \subset Y$ there is a natural short exact sequence

$$(4) \qquad 0 \longrightarrow \mathcal{M}_{cc}^m((Y\backslash Z)/X) \longrightarrow \mathcal{M}_{cc}^m(Y/X) \longrightarrow \mathcal{M}_{cc}^m(Z/X) \longrightarrow 0$$

in $\mathcal{E}_X^m \subset \mathcal{F}_X^m$.

[1]In fact, the motive $\mathcal{M}_{cc}^m(Y/X) \in \mathcal{E}_X^m$ is even covariantly functorial with respect to flat morphisms $g\colon Z \longrightarrow Y$ of varieties quasi-finite over X, since for any flat quasi-finite morphism g there is a natural map $g_!\,\mathbb{Z}/m \longrightarrow \mathbb{Z}/m$ of étale sheaves over Y. However, when the schemes Y and Z are not necessarily reduced, this map depends on their non-reduced structures (i. e., it is *not* determined by the induced morphism of the maximal reduced closed subschemes). When g is a finite flat morphism, the composition $\mathcal{M}_{cc}^m(Y/X) \longrightarrow \mathcal{M}_{cc}^m(Z/X) \longrightarrow \mathcal{M}_{cc}^m(Y/X)$ is the multiplication with the degree of the morphism g, defined as the rank of the locally free sheaf $g_*\mathcal{O}_Z$ over Y.

For any quasi-finite morphism of smooth varieties $f\colon Y \longrightarrow X$, the *homological relative motive of Y over X* is the object

$$\mathcal{M}_h^m(Y/X) = \mathcal{M}_{cc}^m(Y/X)(\dim Y/X)[2\dim Y/X]$$

in the derived category $\mathcal{D}^b(\mathcal{F}_X^m)$, where $\dim Y/X = \dim Y - \dim X$ is the relative dimension. For any quasi-finite morphism of smooth varieties $Y \longrightarrow X$ and a smooth closed subvariety $Z \subset Y$ there is a natural distinguished triangle

$$(5)\quad \mathcal{M}_h^m((Y\backslash Z)/X) \longrightarrow \mathcal{M}_h^m(Y/X) \longrightarrow \mathcal{M}_h^m(Z/X)(c)[2c] \longrightarrow \mathcal{M}_h^m((Y\backslash Z)/X)[1]$$

in $\mathcal{D}^b(\mathcal{F}_X^m)$, where $c = -\dim Z/Y$ is the codimension of Z in Y.

If $Y' \longrightarrow X$ and $Y'' \longrightarrow X$ are two quasi-finite morphisms of algebraic varieties and $Y = Y' \times_X Y''$, then $\mathcal{M}_{cc}^m(Y) = \mathcal{M}_{cc}^m(Y') \otimes \mathcal{M}_{cc}^m(Y'')$ in $\mathcal{E}_X^m \subset \mathcal{F}_X^m$. If $Y' \longrightarrow X$ and $Y'' \longrightarrow X$ are two quasi-finite morphisms of smooth varieties with transversal singularities, i. e., the variety $Y = Y' \times_X Y''$ is smooth and $\dim Y/X = \dim Y'/X + \dim Y''/X$, then $\mathcal{M}_h^m(Y) = \mathcal{M}_h^m(Y') \otimes \mathcal{M}_h^m(Y'')$ in $\mathcal{D}^b(\mathcal{F}_X^m)$.

The basic results justifying the definition of the relative homological motive $\mathcal{M}_h^m(Y/X)$ will be obtained below in Section 11.

Lemma 3.1. *For any algebraic variety X, the exact category $\mathcal{E}_X^m$ is generated by the objects $\mathcal{M}_{cc}^m(Y/X)$, where Y runs over all varieties finite and étale over smooth locally closed subvarieties of X, using the (iterated) operation of passage to an extension. The exact category $\mathcal{F}_X^m$ is generated by the objects $\mathcal{M}_{cc}^m(Y/X)(j)$, with Y as above and $j \in \mathbb{Z}$, in the same way.*

Proof. Given an object $N \in \mathcal{E}_X^m$, consider a stratification of X by smooth locally closed subvarieties such that the restriction of N to each stratum is a lisse étale sheaf. Then notice that whenever for a lisse étale sheaf M of $\mathbb{Z}/m$-modules on a smooth connected variety U the corresponding module over the absolute Galois group of the generic point of U is a permutational module with coefficients in $\mathbb{Z}/m$, the étale sheaf M is the direct image of $\mathbb{Z}/m$ from a finite étale morphism into U. This is so because the maps of étale fundamental groups induced by open embeddings of connected smooth varieties are surjective (since an open subvariety of a connected smooth covering variety is connected). Now it remains to use the exact sequences of étale sheaves related to the extension by zero from open subvarieties. The second assertion follows from the first one in the obvious way. □

Lemma 3.2. *The exact category $\mathcal{E}_X^m$ is generated by the objects $\mathcal{M}_{cc}^m(Y/X)$, where Y runs over all varieties étale over X, using the operations of passage to the cokernel of an admissible monomorphism and (iterated) extension. Furthermore, any object of $\mathcal{E}_X^m$ is the target of an admissible epimorphism whose source is the object $\mathcal{M}_{cc}^m(Y/X)$, where Y is a variety étale over X.*

Proof. One deduces the first assertion from Lemma 3.1 by showing that any étale morphism into a locally closed subvariety $Z \subset X$ factors through an étale morphism into X, perhaps after Z is replaced by its dense open subvariety.

Passing to the local ring of a generic point of Z in X, it suffices to check that any étale morphism to the closed point of a local scheme can be extended to an étale morphism to the whole local scheme. This can be done by choosing a primitive element in a separable field extension of the residue field and lifting the coefficients of its irreducible equation to the local ring in an arbitrary way. Then it remains to use the exact triple (4) and the Noetherian induction.

To prove the second assertion, one can identify the absolute Galois group of the residue field of a scheme point of X with the étale fundamental group of the spectum of the Henselization of its local ring and present the latter as a filtered projective limit of varieties étale over X. $\square$

Lemma 3.3. *The exact category $\mathcal{E}_X^m$ is generated by the objects $\mathcal{M}_{cc}^m(Y/X)$, where Y runs over all normal varieties finite over X, using the operations of passage to the kernel of an admissible epimorphism and (iterated) extension.*

Proof. Use Lemma 3.1 and the fact that any quasi-finite morphism of algebraic varieties is the composition of an open embedding and a finite morphism (Grothendieck's form of Zariski's main theorem [8, Théorème 8.12.6]) together with the fact that the normalization is a finite morphism. $\square$

Lemmas 3.2–3.3 allow to compute (at least, in principle) the $\mathbb{Z}/m$-modules Ext in the exact categories $\mathcal{E}_X^m$ and $\mathcal{F}_X^m$ in terms of the $\mathbb{Z}/m$-modules Ext between the Tate motives $\mathbb{Z}/m(j)$ in the categories $\mathcal{E}_Y^m$ and $\mathcal{F}_Y^m$ over some other varieties Y, using the adjunctions of the functors $f_!$ and f_*. The problem is that the varieties Y may turn out to be singular, even if the original variety X was smooth. On the other hand, it always suffices to know $\operatorname{Ext}^*_{\mathcal{F}_Y^m}(\mathbb{Z}/m, \mathbb{Z}/m(j))$ for normal varieties Y. These observations will be used in the proofs in Sections 9–10.

4. Nisnevich Topology: Distinguished Pairs and Points

The aim of this section and the next one is to prepare ground for the proof of the motivic cohomology comparison theorems in Section 6.

It is explained in Appendix that one can construct a complex of $\mathbb{Z}/m$-modules $C^\bullet_{\mathcal{E}}(M, N)$ with good functorial properties computing the modules $\operatorname{Ext}^*_{\mathcal{E}}(M, N)$ for any two objects M and N in a $\mathbb{Z}/m$-linear exact category $\mathcal{E}$. In this section we establish some properties of the complexes computing Ext in the exact categories $\mathcal{F}_X^m$ with respect to the Nisnevich topology on the category of algebraic varieties X.

Let X be a variety over K and let M and N be two objects in the exact category $\mathcal{F}_X^m$. To any morphism of varieties $f\colon Y \longrightarrow X$ we assign the complex of $\mathbb{Z}/m$-modules $C^\bullet_{M,N}(Y) = C^\bullet_{\mathcal{F}_Y^m}(f^*M, f^*N)$. The map $Y \longmapsto C^\bullet_{M,N}(Y)$ is a complex of presheaves on the category of algebraic varieties mapping into X.

A *distinguished pair of morphisms* (for the Nisnevich topology) [11, Definition 12.5] is a pair of morphisms of varieties $U \longrightarrow Y$ and $Z \longrightarrow Y$ such that $U \longrightarrow Y$ is an open embedding and $Z \longrightarrow Y$ is an étale morphism which is an isomorphism over $Y \backslash U$, i.e., $(Y \backslash U) \times_Y Z \simeq Y \backslash U$.

Lemma 4.1. *Let $\upsilon\colon U \longrightarrow Y$ and $g\colon Z \longrightarrow Y$ be a distinguished pair of morphisms of varieties mapping into X. Then the total complex of the bicomplex with three rows*
$$C^\bullet_{M,N}(Y) \longrightarrow C^\bullet_{M,N}(U) \oplus C^\bullet_{M,N}(Z) \longrightarrow C^\bullet_{M,N}(U \times_Y Z)$$
is acyclic.

Proof. For any étale morphism $g\colon Z \longrightarrow Y$ of varieties mapping into X, the composition of the natural morphisms of complexes
$$C^\bullet_{\mathcal{F}^m_Y}(g_!\, g^*f^*M,\ f^*N) \longrightarrow C^\bullet_{\mathcal{F}^m_Z}(g^*g_!\, g^*f^*M,\ g^*f^*N) \longrightarrow C^\bullet_{\mathcal{F}^m_Z}(g^*f^*M,\ g^*f^*N)$$
is a quasi-isomorphism. Indeed, passing to the cohomology turns this composition into the adjunction isomorphism for Ext, related to the pair of adjoint exact functors $g_!$ and g^* between the exact categories $\mathcal{F}^m_Y$ and $\mathcal{F}^m_Z$ and the pair of objects $g^*f^*M \in \mathcal{F}^m_Z$ and $f^*N \in \mathcal{F}^m_Y$ (see Lemma 2.1). Hence the bicomplex we are interested in is row-wise quasi-isomorphic to the bicomplex
$$\begin{aligned} C^\bullet_{\mathcal{F}^m_Y}(f^*M,\ f^*N) \longrightarrow C^\bullet_{\mathcal{F}^m_Y}&(\upsilon_!\, \upsilon^*f^*M,\ f^*N) \\ &\oplus C^\bullet_{\mathcal{F}^m_Y}(g_!\, g^*f^*M,\ f^*N) \longrightarrow C^\bullet_{\mathcal{F}^m_Y}(h_!\, h^*f^*M,\ f^*N), \end{aligned}$$
where h denotes the morphism $U \times_Y Z \longrightarrow Y$. Exactness of the total complex of the latter bicomplex follows from exactness of the short sequence
$$0 \longrightarrow h_!\, h^*f^*M \longrightarrow \upsilon_!\, \upsilon^*f^*M \oplus g_!\, g^*f^*M \longrightarrow f^*M \longrightarrow 0$$
in the exact category $\mathcal{F}^m_Y$, which is easy to check. $\square$

Let H be the spectrum of the Henselization of the local ring of a scheme point of an algebraic variety over K. Just as for varieties over K, one can define the exact category of étale sheaves $\mathcal{E}^m_H$ and the exact category of filtered étale sheaves $\mathcal{F}^m_H$ for the scheme H. For any morphism h from H to a variety Y over K there is an exact functor of inverse image $h^*\colon \mathcal{F}^m_Y \longrightarrow \mathcal{F}^m_H$.

Let M and N be objects of the category $\mathcal{F}^m_X$ for some variety X over K. Given a morphism $h\colon H \longrightarrow X$, we denote by $C^\bullet_{M,N}(H) = C^\bullet_{\mathcal{F}^m_H}(h^*M, h^*N)$ the corresponding complex computing $\mathrm{Ext}_{\mathcal{F}^m_H}(h^*M, h^*N)$.

Lemma 4.2. *Let y be a scheme point of an algebraic variety Y mapping into X, and let $h_y\colon H_y \longrightarrow Y$ be the related morphism into Y from the Henselian local scheme H_y corresponding to this point. Then the natural map of complexes of $\mathbb{Z}/m$-modules*
$$(h_y^*\, C^\bullet_{M,N})(H_y) \longrightarrow C^\bullet_{M,N}(H_y),$$
where $h_y^*\, C^\bullet_{M,N}$ denotes the inverse image of the complex of presheaves $C^\bullet_{M,N}$ under the functor between the categories of étale morphisms into H_y and Y related to the morphism of schemes h_y, is a quasi-isomorphism.

Proof. The complex $(h_y^*\, C^\bullet_{M,N})(H_y)$ is the filtered inductive limit of the complexes $C^\bullet_{M,N}(Z)$ over all varieties Z, étale over Y and endowed with a lifting $H_y \longrightarrow Z$ of the morphism $H_y \longrightarrow Y$. The scheme H_y is the projective limit in the category of schemes of the filtered projective system of varieties Z. Moreover, the abelian category $\acute{E}t^m_{H_y}$ of constructible étale sheaves of $\mathbb{Z}/m$-modules over H_y is equivalent to the inductive limit of the abelian categories $\acute{E}t^m_Z$ (with respect to the functors of inverse image of étale sheaves).

Furthermore, on a filtered inductive limit of exact categories there is a natural exact category structure. The exact categories $\mathcal{E}^m_{H_y}$ and $\mathcal{F}^m_{H_y}$ are equivalent to the inductive limits of the exact categories $\mathcal{E}^m_Z$ and $\mathcal{F}^m_Z$, respectively. Finally, the filtered inductive limits of exact categories commute with the passage to the Ext groups. $\square$

Lemma 4.3. *Let H be the spectrum of the Henselization of the local ring of a scheme point of an algebraic variety over K, and let $\iota\colon \eta \longrightarrow H$ be the closed point of H. Let M and N be objects of the category $\mathcal{F}^m_H$. Assume that the successive quotient objects $\mathrm{gr}^j_F M$ of the object M with respect to the filtration F are locally constant étale sheaves on H. Then the natural map*

$$C^\bullet_{\mathcal{F}_H}(M, N) \longrightarrow C^\bullet_{\mathcal{F}_\eta}(\iota^*M,\ \iota^*N),$$

where $\iota^*\colon \mathcal{F}^m_H \longrightarrow \mathcal{F}^m_\eta$ denotes the inverse image with respect to ι, is a quasi-isomorphism of complexes of $\mathbb{Z}/m$-modules.

Proof. The restriction of the functor ι^* to the full exact subcategory of $\mathcal{F}^m_H$ consisting of the objects with locally constant successive quotients is an equivalence of exact categories, since the étale fundamental groups of H and η coincide. The embedding of exact categories $\mathcal{F}^m_\eta \longrightarrow \mathcal{F}^m_H$ inverse to this restriction is left adjoint to the functor ι^*.

Essentially, this is so because finite étale covers of H are cofinal among all étale covers and any morphism from a scheme finite and étale over η to a scheme étale over H forming a commutative diagram with ι factors through a scheme finite and étale over H. One can consider the site $\mathfrak{G}$ of finite étale morphisms into H (with the obvious topology). Then the étale site of H maps to the site $\mathfrak{G}$ in the obvious way (the direction of the site map being opposite to that of the functor).

The category of constructible sheaves over $\mathfrak{G}$ is equivalent to the category of locally constant constructible étale sheaves over H. The inverse image functor for the above map of sites is the embedding of the locally constant constructible étale sheaves into arbitrary constructible étale sheaves over H, while the direct image is identified with the functor ι^*. So the former is left adjoint to the latter. This adjunction of exact functors between the abelian categories $\acute{E}t^m_H$ and $\acute{E}t^m_\eta$ induces the desired adjunction of exact functors between the exact categories $\mathcal{F}^m_H$ and $\mathcal{F}^m_\eta$.

It remains to apply Lemma 2.1 in order to deduce the desired isomorphism of the Ext modules. $\square$

Notice that the assertions of the above three Lemmas are equally applicable to the complexes computing the $\mathbb{Z}/m$-modules Ext in the abelian categories of constructible étale sheaves $\acute{E}t^m_Y$, $\acute{E}t^m_H$, etc. in place of the exact categories $\mathcal{F}^m_Y$, $\mathcal{F}^m_H$, etc., and the complexes of presheaves formed from these complexes of $\mathbb{Z}/m$-modules.

5. Hypercohomology and Derived Direct Image

In this section we prove several technical lemmas. Given a presheaf P on the category of varieties étale over a given variety X or on the category of all varieties over K, we denote by P_{Nis} the sheafification of P in the Nisnevich topology. For a complex of presheaves $P^\bullet$, the similar notation $P^\bullet_{Nis}$ stands for the complex of Nisnevich sheaves obtained by sheafifying every term of the complex $P^\bullet$. Given a complex of Nisnevich sheaves $S^\bullet$ with the cohomology sheaves bounded from below, $\mathbb{H}^i_{Nis}(X, S^\bullet)$ denotes the Nisnevich hypercohomology of X with coefficients in $S^\bullet$.

Lemma 5.1. *Let $P^\bullet$ be a complex of presheaves of $\mathbb{Z}/m$-modules on the category of varieties étale over a fixed variety X over K with the cohomology presheaves $H^iP^\bullet$ bounded from below. Then there is a natural map $H^iP^\bullet(X) \longrightarrow \mathbb{H}^i_{Nis}(X, P^\bullet_{Nis})$ from the cohomology modules $H^iP^\bullet(X)$ of the complex of sections $P^\bullet(X)$ to the hypercohomology modules $\mathbb{H}^i_{Nis}(X, P^\bullet_{Nis})$.*

Proof. Let $PreSh$ denote the abelian category of presheaves of $\mathbb{Z}/m$-modules on the category of varieties étale over X, and let Sh_{Nis} be the category of Nisnevich sheaves of $\mathbb{Z}/m$-modules on the same category/site. Then there is the functor $Nis\colon \mathcal{D}^+(PreSh) \longrightarrow \mathcal{D}^+(Sh_{Nis})$ induced by the (exact) Nisnevich sheafification functor $PreSh \longrightarrow Sh_{Nis}$. The functor Nis has a right adjoint functor $J\colon \mathcal{D}^+(Sh_{Nis}) \longrightarrow \mathcal{D}^+(PreSh)$ constructed as follows. Given a bounded below complex of sheaves $S^\bullet$, choose its bounded below injective resolution $I^\bullet$ in Sh_{Nis} and consider it as a complex of presheaves; then $J(S^\bullet) = I^\bullet$.

Now for any complex of presheaves $P^\bullet$ there is the adjunction morphism $P^\bullet \longrightarrow J(P^\bullet_{Nis})$ in $\mathcal{D}^+(PreSh)$. The induced morphism of complexes of sections $P^\bullet(X) \longrightarrow J(P^\bullet_{Nis})(X)$ represents the desired cohomology map $H^*P^\bullet(X) \longrightarrow \mathbb{H}^*_{Nis}(X, P^\bullet_{Nis})$. $\square$

Lemma 5.2. *Let $P^\bullet$ be a complex of presheaves of $\mathbb{Z}/m$-modules on the category of varieties étale over X with the cohomology presheaves $H^iP^\bullet$ bounded from below. Assume that for any distinguished pair $U \longrightarrow Y$ and $Z \longrightarrow Y$ of morphisms of varieties étale over X the total complex of the bicomplex with three rows*

$$P^\bullet(Y) \longrightarrow P^\bullet(U) \oplus P^\bullet(Z) \longrightarrow P^\bullet(U \times_Y Z)$$

is acyclic. Then the natural map $H^iP^\bullet(X) \longrightarrow \mathbb{H}^i_{Nis}(X, P^\bullet_{Nis})$ is an isomorphism.

Proof. We use the notation from the proof of Lemma 5.1. It suffices to show that the morphism $P^\bullet \longrightarrow J(P^\bullet_{Nis})$ is a quasi-isomorphism of complexes of presheaves whenever the complex of presheaves $P^\bullet$ satisfies the condition of Lemma.

The total complex of the bicomplex with three rows entering into this condition is the complex of morphisms from the complex of presheaves $(\mathbb{Z}/m)_{U\times_Y Z} \longrightarrow (\mathbb{Z}/m)_U \oplus (\mathbb{Z}/m)_Z \longrightarrow (\mathbb{Z}/m)_Y$ into $P^\bullet$, where $(\mathbb{Z}/m)_Y$ is the presheaf of $\mathbb{Z}/m$-modules freely generated by the presheaf of sets represented by Y. The presheaves $(\mathbb{Z}/m)_Y$ are projective objects, so this total complex computes also the Hom in the derived category $\mathcal{D}^+(PreSh)$. The sheafification functor sends the above three-term complex of presheaves to an acyclic complex of Nisnevich sheaves. Hence the complex $J(P^\bullet_{Nis})$, being a complex of injective sheaves, also satisfies the same condition.

So does the cone $Q^\bullet$ of the morphism of complexes of presheaves $P^\bullet \longrightarrow J(P^\bullet_{Nis})$. Besides, the complex of presheaves $Q^\bullet$ is annihilated by the sheafification functor. From these two properties, we will deduce that $Q^\bullet$ is acyclic. Clearly, it suffices to assume that $Q^i = 0$ for $i < 0$ and prove that the differential $d\colon Q^0 \longrightarrow Q^1$ is injective.

Suppose $s \in Q^0(Y)$ is a section annihilated by the morphism $Q^0 \longrightarrow Q^1$. Then s must be also annihilated by the map $Q^0(Y) \longrightarrow Q^0_{Nis}(Y)$, since otherwise the morphism of sheaves $Q^0_{Nis} \longrightarrow Q^1_{Nis}$ wouldn't be injective and the complex of sheaves $Q^\bullet_{Nis}$ wouldn't be acyclic.

Consider a Nisnevich cover $V_\alpha \longrightarrow Y$ of the variety Y such that the section s vanishes in the restriction to V_α. Choose a stratification of Y by connected locally closed subvarieties, over each of which V_α has a section. Assume that $s \neq 0$. Let us throw away closed strata from this stratification one by one until we find an open subvariety $Y' \subset Y$ such that $s|_{Y'}$ is still nonzero, but there is a closed stratum $W \subset Y$ such that $s|_U = 0$, where $U = Y' \backslash W$. There exists $\beta \in \{\alpha\}$ such that the full preimage $W \times_Y V_\beta$ of W in V_β contains a connected component whose morphism to W is an isomorphism. Let $Z \subset Y' \times_Y V_\beta$ denote the complement to all the other connected components of this full preimage. Then the morphisms $U \longrightarrow Y'$ and $Z \longrightarrow Y'$ form a distinguished pair.

Since $s|_U = 0 = s|_Z$ and $d(s) = 0$, while $s|_{Y'} \neq 0$ in $Q^0(Y')$ and $Q^{-1} = 0$, the section $s|_{Y'}$ represents a nonzero cohomology class in the total complex of the bicomplex associated with this distinguished pair and the complex of presheaves $Q^\bullet$. This contradiction proves that $s = 0$ and the complex $Q^\bullet$ is acyclic. $\square$

Now let $\mathcal{U}$ be a Grothendieck universe set such that $K \in \mathcal{U}$. Given a variety X over K, let $\acute{E}t_X^{m,\mathcal{U}}$ denote the abelian category of étale sheaves of $\mathbb{Z}/m$-modules over X belonging to $\mathcal{U}$; so $\acute{E}t_X^m \subset \acute{E}t_X^{m,\mathcal{U}} \subset \acute{E}t_X^{m,\infty}$.

Let $\rho\colon \acute{E}t \longrightarrow Nis$ denote the natural map between the (big) étale and Nisnevich sites of algebraic varieties over K. Let $P \in \mathcal{U}$ be an étale sheaf of $\mathbb{Z}/m$-modules on the site of all varieties over K. The restriction of P to the (small) étale site of étale varieties over a given variety X defines an object of $\acute{E}t_X^{m,\mathcal{U}}$, which we will denote also by P. Consider the presheaf of complexes of $\mathbb{Z}/m$-modules $C^\bullet_P(X) = C^\bullet_{\acute{E}t_X^{m,\mathcal{U}}}(\mathbb{Z}/m, P)$ on the category of varieties over K.

Lemma 5.3. *The complex of Nisnevich sheaves $(C^\bullet_P)_{Nis}$ on the site of algebraic varieties over K represents the derived direct image $\mathbb{R}\rho_*(P)$ of the étale sheaf P with respect to the map of sites $\rho\colon \acute{E}t \longrightarrow Nis$.*

Proof. Applying the canonical truncation, we can assume that the complex of presheaves $C^\bullet_P$ is concentrated in nonnegative cohomological degrees. Furthermore, a section of an étale sheaf P over a variety X can be viewed as a morphism $\mathbb{Z}/m \longrightarrow P$ in $\acute{E}t_X^{m,\mathcal{U}}$, and to such a morphism one can naturally assign a cocycle of degree 0 in the complex $C_P^\bullet(X)$. Hence there is a natural morphism $P \longrightarrow C_P^\bullet$ of complexes of presheaves on the category of varieties over K.

Now let $I^\bullet$ be an injective resolution of the étale sheaf P. Then there is the bicomplex of presheaves $C_{I^\bullet}^\bullet$ on the category of varieties over K. The complexes of presheaves $I^\bullet$ and $C_P^\bullet$ map naturally into the total complex of the bicomplex $C_{I^\bullet}^\bullet$. The map from the former is a quasi-isomorphism of complexes of presheaves, since $\mathrm{Ext}^i_{\acute{E}t_X^{m,\mathcal{U}}}(\mathbb{Z}/m, I^j) = 0$ for $i > 0$. The map from the latter is also a quasi-isomorphism due to the property of the complexes $C_P^\bullet$ with respect to short exact sequences of the objects $P \in \acute{E}t_X^{m,\mathcal{U}}$ (see item (5) in Appendix).

The Nisnevich sheafification transforms quasi-isomorphisms of complexes of presheaves into quasi-isomorphisms of complexes of Nisnevich sheaves. It also does not change the complex of étale sheaves $I^\bullet$. Thus the complexes $(C_P^\bullet)_{Nis}$ and $I^\bullet$ are connected by a natural chain of quasi-isomorphisms of complexes of Nisnevich sheaves. By the definition, the complex $I^\bullet$, considered as a complex of Nisnevich sheaves, computes $\mathbb{R}\rho_*(P)$. □

6. Comparison Theorem

The following two theorems constitute the first main result of this paper.

Theorem 6.1. *For any algebraic variety X over a field K of characteristic not dividing m, there are natural maps of $\mathbb{Z}/m$-modules*

$$\theta_X^{m,i,j}\colon \mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(j)) \longrightarrow \mathbb{H}^i_{Nis}(X, \tau_{\leqslant j}\mathbb{R}\rho_*\mu_m^{\otimes j}). \tag{6}$$

The compositions of these maps with the Nisnevich hypercohomology maps induced by the morphisms $\tau_{\leqslant j}\mathbb{R}\rho_*\mu_m^{\otimes j} \longrightarrow \mathbb{R}\rho_*\mu_m^{\otimes j}$ coincide with the maps $\mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(j)) \longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$ induced by the exact forgetful functor $\mathcal{F}_X^m \longrightarrow \acute{E}t_X^m$.

The maps (6) for $X = \operatorname{Spec} L$, where L is a field of characteristic not dividing m, were discussed in the paper [14]. In this case, the right hand side of (6) is isomorphic to the Galois cohomology group $H^i(G_L, \mu_m^{\otimes j})$ when $i \leqslant j$ and vanishes for $i > j$. The left hand side of (6) for $X = \operatorname{Spec} L$ can be also easily seen to vanish for $i > j$ [14, Theorem 6.1(1)]. The maps $\theta_{\operatorname{Spec} L}^{m,i,j}$ are simply induced by the forgetful functor from the exact category $\mathcal{F}_{\operatorname{Spec} L}^m$ of filtered G_L-modules with a restriction on the successive quotients to the abelian category of arbitrary (discrete) G_L-modules over $\mathbb{Z}/m$.

Assuming the Beilinson–Lichtenbaum conjecture for the field L, these maps can be also described as being induced by the embedding of the exact category $\mathcal{F}_{\operatorname{Spec} L}^m \simeq \mathcal{MAT}(L, \mathbb{Z}/m)$ into the triangulated category $\mathcal{DM}(L, \mathbb{Z}/m)$ [14, Theorem 3.1(1)]. The assertion that the maps $\theta_{\operatorname{Spec} L}^{m,i,j}$ are isomorphisms is (a particular case of) what was called the *silly filtration* or *$K(\pi, 1)$-conjecture for Artin–Tate motives over L with coefficients in $\mathbb{Z}/m$* in [14]. Specifically, it is [14, Conjecture 9.2] for the field denoted by K in [14] being our field L and the field denoted by M being its separable closure (see [14, Sections 9.3 and 9.9] for some further details).

When the field L contains a primitive m-root of unity, this conjecture is equivalent to the Koszul property of the big graded ring of the diagonal Ext between the Artin–Tate motives [14, Proposition 8.1, Theorem 9.1, and Section 9.5]. More precisely, the maps $\theta^{m,i,j}_{\operatorname{Spec} L}$ are isomorphisms for all (finite, separable) algebraic extensions L of a given field $K \ni \sqrt[m]{1}$ if and only if the big graded ring A [14, formula (9.4)] describing the Artin–Tate motives over K with coefficients $\mathbb{Z}/m$ is Koszul.

Theorem 6.2. *Given an integer m and an algebraic variety X, the comparison maps* (6) *are isomorphisms for X and all the varieties Y étale over X if and only if they are isomorphisms for all (the spectra of the residue fields of) the scheme points $y \in Y$.*

Proof of two Theorems. Consider the complex of presheaves $C^\bullet_{\mathbb{Z}/m,\,\mathbb{Z}/m(j)}(X) = C^\bullet_{\mathcal{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(j))$ on the category of varieties over K. There is a natural morphism from it to the complex of presheaves $C^\bullet_{\mathbb{Z}/m,\,\mu_m^{\otimes j}}(X) = C^\bullet_{\acute{E}t_X^m}(\mathbb{Z}/m, \mu_m^{\otimes j})$ induced by the forgetful functors $\mathcal{F}_X^m \longrightarrow \acute{E}t_X^m$. Consider the induced morphism of complexes of Nisnevich sheaves

$$(C^\bullet_{\mathbb{Z}/m,\,\mathbb{Z}/m(j)})_{Nis} \longrightarrow (C^\bullet_{\mathbb{Z}/m,\,\mu_m^{\otimes j}})_{Nis}. \tag{7}$$

By Lemmas 4.2–4.3, the map of the cohomology of the stalks at the Henzelization of the local ring of a scheme point $y \in Y$ induced by (7) is identified with the map $\theta_y^{m,i,j}$. In particular, the stalks of the left hand side of (7) are concentrated in the cohomological degrees $\leqslant j$, hence there is the induced morphism

$$(C^\bullet_{\mathbb{Z}/m,\,\mathbb{Z}/m(j)})_{Nis} \longrightarrow \tau_{\leqslant j}(C^\bullet_{\mathbb{Z}/m,\,\mu_m^{\otimes j}})_{Nis}. \tag{8}$$

in the derived category of Nisnevich sheaves over the site of algebraic varieties over K.

Now consider the map of Nisnevich hypercohomology of X induced by the morphism (8). By Lemmas 4.1 and 5.2, the $\mathbb{Z}/m$-modules $\mathbb{H}^i_{Nis}(X, (C^\bullet_{\mathbb{Z}/m,\,\mathbb{Z}/m(j)})_{Nis})$ are naturally isomorphic to $\operatorname{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(j))$. By Lemma 5.3, the complex of Nisnevich sheaves $(C^\bullet_{\mathbb{Z}/m,\,\mu_m^{\otimes j}})_{Nis}$ represents the derived direct image $\mathbb{R}\rho_*\mu_m^{\otimes j}$. This proves the assertions of both Theorems. $\square$

Remark 6.3. There is a simple way to make the equivalent assertions of Theorem 6.2 automatically true by replacing the exact categories $\mathcal{F}_X^m$ with somewhat larger exact categories $\mathcal{F}_X'^m$. Namely, let $\mathcal{E}_X'^m$ denote the abelian category $\acute{E}t_X^m$ endowed with the exact category structure in which a short sequence is exact if it is split exact at every scheme point of X. In other words, we just drop the condition that the Galois representations at stalks be permutational. Let $\mathcal{F}_X'^m$ be the exact category of finitely filtered objects of $\acute{E}t_X^m$ with the exact triples defined by the condition that the triples of successive quotients by the filtration must be exact in $\mathcal{E}_X'^m$. Then our proof of Theorems 6.1–6.2 applies to the categories $\mathcal{F}_X'^m$ as well as to the categories $\mathcal{F}_X^m$. Moreover, the analogues of the maps $\theta_y^{m,i,j}$ for the categories $\mathcal{F}_y'^m$ are easily seen to be isomorphisms [14, Example 8.3 and Remark 9.3]. However, Lemmas 3.1–3.3, of course, do not hold for the exact categories $\mathcal{E}'^m_X \supset \mathcal{E}^m_X$ and $\mathcal{F}'^m_X \supset \mathcal{F}^m_X$, nor is our embedding theorem (Theorem 9.1 below) applicable to them.

On the other hand, let us emphasize that it is of key importance to our Ext computation that the exact category structure on $\mathcal{F}^m_X$ or $\mathcal{F}'^m_X$ is defined by the condition of *split* exactness of the short sequences of Galois modules of stalks at scheme points. Indeed, this splitting condition is the reason why the canonical truncation appears in the right hand side of the formula (6), i. e., we obtain the modified étale descent rule of the Beilinson–Lichtenbaum type, rather than the conventional étale descent, for the Ext spaces in our exact categories (or at least certainly for the exact categories $\mathcal{F}'^m_X$). For comparison, define $\mathcal{F}''^m_X$ as the exact category of filtered objects in $\acute{E}t^m_X$ with the exact triples being the short sequences with zero composition for which the exact triples of successive quotients are exact in $\acute{E}t^m_X$. Then one has $\operatorname{Ext}^i_{\mathcal{F}''^m_X}(\mathbb{Z}/m, \mathbb{Z}/m(j)) \simeq H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$ [14, Example D.1].

Yet another way to modify our definition of the exact categories $\mathcal{E}^m_X$ and $\mathcal{F}^m_X$ is to allow the Galois representations at the stalks of the (successive quotient) sheaves to be direct summands of permutational representations rather than permutational representations as such. An argument similar to the proof of Lemma 3.1 shows that the exact categories one obtains in this way are simply the closures of the exact categories $\mathcal{E}^m_X$ and $\mathcal{F}^m_X$ with respect to adjoining the images of idempotent endomorphisms. Thus such a change in the definitions does not affect the $\mathbb{Z}/m$-modules Ext, and only leads to the necessity to mention the passage to the direct summands in the formulations of Lemmas 3.1–3.3.

Corollary 6.4. *Let X be an algebraic variety over K. Then*

(a) *for any $j \in \mathbb{Z}$, the map $\theta^{m,i,j}_X$ from (6) is an isomorphism for $i = 0$, 1, and a monomorphism for $i = 2$;*
(b) *for any $j \leqslant 2$, the map $\theta^{m,i,j}_X$ is an isomorphism for all $i \in \mathbb{Z}$;*
(c) *whenever X is normal and connected, the $\mathbb{Z}/m$-module $\operatorname{Ext}^i_{\mathcal{F}^m_X}(\mathbb{Z}/m, \mathbb{Z}/m)$ vanishes for all $i \neq 0$, and is freely generated by the identity endomorphism when $i = 0$.*

Proof. First of all, it follows from the proof of Theorem 6.2 above that its assertion holds when j is fixed and i is restricted to an interval $0 \leqslant i \leqslant n$ with a fixed $n \geqslant 0$. That is, given $j \in \mathbb{Z}$ and $n \geqslant 0$, the maps $\theta^{m,i,j}_Y$ are isomorphisms for all the varieties Y étale over X and all $0 \leqslant i \leqslant n$ if and only if the maps $\theta^{m,i,j}_y$ are isomorphisms for all the scheme points $y \in Y$ and the same i. Moreover, if this is the case, then the map $\theta^{m,n+1,j}_Y$ is a monomorphism for all Y étale over X if and only if the map $\theta^{m,n+1,j}_y$ is a monomorphism for all the scheme points $y \in Y$.

Now part (a) follows from [14, Theorem 3.1(2)]. To prove part (b), one only has to check that the map $\theta^{m,2,2}_y$ is surjective, and this so because the Galois cohomology module $H^2(G_{K_y}, \mu_m^{\otimes 2})$ is multiplicatively generated by $H^1(G_{K_y}, \mu_m)$ [12]. Finally, part (c) is [11, Exercise 12.32(1)]. ☐

7. Weak Version of Six Operations

Here we spell out the additional structures on and the properties of the triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ over algebraic varieties X over K that we will need in order to construct our embeddings $\mathcal{F}_X^m \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$. Of the six operations in the conventional formalism, we will use only three: the inverse image f^*, the direct image with compact supports $f_!$, and the tensor product $\otimes = \otimes_{\mathbb{Z}/m}$.

So, suppose that we are given the following data. For any algebraic variety X over K, there is a $\mathbb{Z}/m$-linear symmetric tensor triangulated category $\mathcal{DM}(X, \mathbb{Z}/m)$ with the unit object $\mathbb{Z}/m$ and a fixed invertible object $\mathbb{Z}/m(1) \in \mathcal{DM}(X, \mathbb{Z}/m)$. As usually, we set $M(j) = M \otimes \mathbb{Z}/m(1)^{\otimes j}$ for $M \in \mathcal{DM}(X, \mathbb{Z}/m)$. For any morphism of algebraic varieties $f\colon Y \longrightarrow X$, there is a tensor triangulated functor $f^*\colon \mathcal{DM}(X, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(Y, \mathbb{Z}/m)$ taking $\mathbb{Z}/m(1)$ to $\mathbb{Z}/m(1)$, and a triangulated functor $f_!\colon \mathcal{DM}(Y, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$. The following constraints and conditions are imposed.

(i) The assignment of the functors f^* and $f_!$ to morphisms of varieties $f\colon Y \longrightarrow X$ takes identity morphisms to identity functors, and compositions to compositions. For an étale morphism f, the functor $f_!$ is left adjoint to f^*. For a proper morphism f, the functor $f_!$ is right adjoint to f^*. When the morphism f is a universal homeomorphism, the functors $f_!$ and f^* are equivalences of triangulated categories.

(ii) In a base change situation, i. e., given morphisms of varieties $f\colon Y \longrightarrow X$ and $g\colon Z \longrightarrow X$, and $W = Z \times_X Y$ being their Cartesian product with the natural morphisms $f'\colon W \longrightarrow Z$ and $g'\colon W \longrightarrow Y$, the functor $f_!$ commutes with g^*. In other words, there is an isomorphism of functors $g^* f_! \simeq f'_! \, g'^*$. The compatibility of these base change isomorphisms with the compositions of the morphisms f and g holds. When the morphism f is étale or proper, the base change isomorphism is provided by the morphism defined in terms of the adjunction and the compatibilities with the compositions, as stated in (i).

(iii) For any variety X with an open subvariety $\upsilon\colon U \longrightarrow X$ and its closed complement $\iota\colon Z \longrightarrow X$, and any object $M \in \mathcal{DM}(X, \mathbb{Z}/m)$, there is a distinguished triangle

$$\upsilon_! \, \upsilon^* M \longrightarrow M \longrightarrow \iota_! \, \iota^* M \longrightarrow \upsilon_! \, \upsilon^* M[1].$$

in the triangulated category $\mathcal{DM}(X, \mathbb{Z}/m)$. Here the leftmost and the middle morphisms are the adjunction morphisms for the open embedding υ and the closed embedding ι. The rightmost morphism is functorial in M and compatible with the inverse image functors f^* with respect to morphisms of varieties $f\colon Y \longrightarrow X$.

(iv) For any morphism of varieties $f\colon Y \longrightarrow X$, object $M \in \mathcal{DM}(X, \mathbb{Z}/m)$, and object $N \in \mathcal{DM}(Y, \mathbb{Z}/m)$, there is an isomorphism $f_!(f^* M \otimes N) \simeq M \otimes f_! N$ in the category $\mathcal{DM}(X, \mathbb{Z}/m)$. This projection formula isomorphism is functorial in M and N, and compatible with the compositions of the morphisms f. When the morphism f is étale or proper, the projection formula isomorphism is provided by the morphism defined in terms of the adjunction and the preservation of the tensor product by the functor f^*.

Furthermore, we will need to have étale realization functors acting on our triangulated categories of motivic sheaves. These are presumed to be tensor triangulated functors $\Phi_X\colon \mathcal{DM}(X,\mathbb{Z}/m) \longrightarrow \mathcal{D}(\acute{E}t_X^{m,\infty})$ taking values in the derived categories of étale sheaves of $\mathbb{Z}/m$-modules over the varieties X. For the sake of generality, we allow the derived categories to be unbounded and the étale sheaves to be nonconstructible. The following constraints and conditions are imposed.

(v) One has $\Phi_X(\mathbb{Z}/m(1)) = \mu_m$ for all varieties X.

(vi) For any morphism of varieties $f\colon Y \longrightarrow X$, the functors Φ_X and Φ_Y form commutative diagrams with the functors f^* and $f_!$ between the triangulated categories $\mathcal{DM}(X,\mathbb{Z}/m)$ and $\mathcal{DM}(Y,\mathbb{Z}/m)$, and the similar functors between the derived categories $\mathcal{D}(\acute{E}t_X^{m,\infty})$ and $\mathcal{D}(\acute{E}t_Y^{m,\infty})$.

(vii) The functors Φ transform the constraints (i–iv) for the triangulated categories of motivic sheaves into the similar constraints for the derived categories of étale sheaves [4, Arcata IV.5].

In addition, we will have to assume that the triangulated categories $\mathcal{DM}(X,\mathbb{Z}/m)$ satisfy the following formulation of the Beilinson–Lichtenbaum conjecture (proven in [17, 6] and [22]).

(viii) For any variety X, the $\mathbb{Z}/m$-modules $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i])$ vanish for $j < 0$ and all i.

(ix) For any smooth connected variety X, the $\mathbb{Z}/m$-module $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m)$ is freely generated by the identity endomorphism, while the $\mathbb{Z}/m$-modules $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[i])$ vanish for all $i \neq 0$.

(x) For any smooth variety X, the morphisms $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$ induced by the functor Φ_X are isomorphisms for all $i \leqslant j$ and monomorphisms for $i = j+1$.

The above assumptions will suffice for the purposes of Sections 8–9, but in Sections 10–11 we will need a more precise version of the Beilinson–Lichtenbaum conjecture. Let H be the spectrum of the Henselization of the local ring of a scheme point of a smooth variety over K. Then varieties X over K endowed with scheme morphisms $H \longrightarrow X$ form a filtered category. Here is our last assumption.

(xi) The filtered inductive limit $\varinjlim_{H\to X} \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i])$ vanishes for all $i > j$ and any Henselian scheme H as above.

A notable attempt to construct the triangulated categories of motivic sheaves with the six operations formalism was undertaken by Cisinski and Déglise [2]. However, these categories do not seem to have all the properties that we need in the case of motives with finite coefficients. The new approach by the same authors [3] may be closer to what is needed.

8. Resolution of Singularities

Recall that an *abstract blow-up* of a variety X with a center $Z \subset X$ is a proper birational morphism $\widetilde{X} \longrightarrow X$ that is an isomorphism over a dense open subvariety U to which $Z = X \backslash U$ is the closed complement [19, 11]. We will say that *varieties of dimension* $\leqslant d$ *over* K *admit resolution of singularites* if for any normal variety X of dimension $\leqslant d$ over K there exists a sequence of abstract blow-ups $X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X$ with normal centers such that the map $X_n \longrightarrow X$ factors through a proper birational morphism onto X from a smooth variety.

Notice that all the conditions (i–xi) of Section 7 make sense for varieties of bounded dimension, except for the condition (ii), which presumes that the dimension can be increased. So we will say that the condition (ii) holds for varieties of dimension $\leqslant d$ if it holds whenever the varieties X and Z have dimensions $\leqslant d$ and the morphism $f\colon Y \longrightarrow X$ is quasi-finite.

Lemma 8.1. *Assume that triangulated categories of motivic sheaves* $\mathcal{DM}(X, \mathbb{Z}/m)$ *satisfying the conditions* (i–x) *of Section 7 are defined for varieties* X *of dimension* $\leqslant d$ *over* K*, and that such varieties admit resolution of singularities. Then*

(a) *for any normal connected variety* X *of dimension* $\leqslant d$ *over* K*, the* $\mathbb{Z}/m$*-module* $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[i])$ *vanishes for all* $i < 0$ *and* $i = 1$*, and is freely generated by the identity endomorphism for* $i = 0$*;*

(b) *for any variety* X *of dimension* $\leqslant d$ *over* K *and any* $j \geqslant 1$*, the morphism of* $\mathbb{Z}/m$*-modules* $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$ *induced by the étale realization functor* Φ_X *is an isomorphism for* $i \leqslant 1$ *and a monomorphism for* $i = 2$*.*

Proof. Let $p\colon \widetilde{X} \longrightarrow X$ be an abstract blow-up of a variety X with a closed center $\iota : Z \longrightarrow X$ and its open complement $\upsilon\colon U \longrightarrow X$. Let $\widetilde{Z}$ be the Cartesian product $Z \times_X \widetilde{X}$; denote the related closed embedding by $\tilde{\iota}\colon \widetilde{Z} \longrightarrow \widetilde{X}$, and the open embedding by $\tilde{\upsilon}\colon U \longrightarrow \widetilde{X}$. Then we have the distinguished triangle

$$\upsilon_! \mathbb{Z}/m \longrightarrow \mathbb{Z}/m \longrightarrow \iota_! \mathbb{Z}/m \longrightarrow \upsilon_! \mathbb{Z}/m[1] \tag{9}$$

in $\mathcal{DM}(X, \mathbb{Z}/m)$ and the distinguished triangle

$$\tilde{\upsilon}_! \mathbb{Z}/m \longrightarrow \mathbb{Z}/m \longrightarrow \tilde{\iota}_! \mathbb{Z}/m \longrightarrow \tilde{\upsilon}_! \mathbb{Z}/m[1]$$

in $\mathcal{DM}(\widetilde{X}, \mathbb{Z}/m)$. Applying the functor $p_!$ to the latter triangle, we obtain a distinguished triangle

$$\upsilon_! \mathbb{Z}/m \longrightarrow p_! \mathbb{Z}/m \longrightarrow \iota_! \pi_! \mathbb{Z}/m \longrightarrow \upsilon_! \mathbb{Z}/m[1] \tag{10}$$

in $\mathcal{DM}(X, \mathbb{Z}/m)$. There is a natural morphism from the triangle (9) to the triangle (10) acting by identity on their common first vertex. Thus the octahedron axiom applies, and we obtain two distinguished triangles with a common first vertex

$$\begin{gathered} M \longrightarrow \mathbb{Z}/m \longrightarrow p_! \mathbb{Z}/m \longrightarrow M[1] \\ M \longrightarrow \iota_! \mathbb{Z}/m \longrightarrow \iota_! \pi_! \mathbb{Z}/m \longrightarrow M[1] \end{gathered} \tag{11}$$

in $\mathcal{DM}(X,\mathbb{Z}/m)$. Twisting with (j) and applying the functor $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, -[*])$, we get the exact sequences

$$\begin{aligned}(12)\quad \cdots \longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, M(j)[i]) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \\ \longrightarrow \mathrm{Hom}_{\mathcal{DM}(\widetilde{X},\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, M(j)[i+1])\end{aligned}$$

and

$$\begin{aligned}(13)\quad \mathrm{Hom}_{\mathcal{DM}(Z,\mathbb{Z}/m)}(\mathbb{Z}/m,\ \mathbb{Z}/m(j)[i-1]) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(\widetilde{Z},\mathbb{Z}/m)}(\mathbb{Z}/m,\ \mathbb{Z}/m(j)[i-1]) \\ \longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M, \mathbb{Z}/m(j)[i]) &\longrightarrow \mathrm{Hom}_{\mathcal{DM}(Z,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \longrightarrow \cdots\end{aligned}$$

Applying to the triangles (11) the étale realization functor Φ_X, twisting the result with $\mu_m^{\otimes j}$, and passing to the étale cohomology, we obtain the exact sequences

$$\begin{aligned}(14)\quad \cdots \longrightarrow \mathbb{H}^i_{\acute{e}t}(X,\ \Phi_X(M)\otimes\mu_m^{\otimes j}) &\longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j}) \\ &\longrightarrow H^i_{\acute{e}t}(\widetilde{X}, \mu_m^{\otimes j}) \longrightarrow \mathbb{H}^{i+1}_{\acute{e}t}(X,\ \Phi_X(M)\otimes\mu_m^{\otimes j})\end{aligned}$$

and

$$\begin{aligned}(15)\quad H^{i-1}_{\acute{e}t}(Z, \mu_m^{\otimes j}) &\longrightarrow H^{i-1}_{\acute{e}t}(\widetilde{Z}, \mu_m^{\otimes j}) \\ &\longrightarrow \mathbb{H}^i_{\acute{e}t}(X,\ \Phi_X(M)\otimes\mu_m^{\otimes j}) \longrightarrow H^i_{\acute{e}t}(Z, \mu_m^{\otimes j}) \longrightarrow \cdots\end{aligned}$$

together with morphisms of exact sequences from (12) to (14) and from (13) to (15).

We will argue by induction in $\dim X$. The case of a smooth variety X is covered by the condition (ix) from Section 7 for part (a) and the condition (x) for part (b).

The assertion about the vanishing of $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[i])$ for $i < 0$ holds for any variety X. One proves this using a proper birational morphism $\widetilde{X} \longrightarrow X$ onto X from a smooth variety $\widetilde{X}$ and the exact sequences (12–13). The assertion about $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m)$ holds for any connected variety X. Once again, one proves this using an abstract blow-up of X with a smooth variety $\widetilde{X}$, using the fact that there is at least one connected component of $\widetilde{Z}$ over every component of Z.

The assertion about the vanishing of $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1])$ holds for any normal variety X. To prove this, one first shows that the morphism $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1]) \longrightarrow \mathrm{Hom}_{\mathcal{DM}(\widetilde{X},\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1])$ is injective for any abstract blow-up $\widetilde{X} \longrightarrow X$ with a normal center Z. This follows from the the exact sequences (12–13) and the fact that there is at most one connected component of $\widetilde{Z}$ over every connected component of Z (Zariski's main theorem [7, Proposition 4.3.5]). Since a composition of such abstract blow-ups factors through a smooth variety, the desired vanishing assertion follows.

To prove the assertions of part (b), consider a proper birational morphism $\widetilde{X} \longrightarrow X$ onto X from a smooth variety $\widetilde{X}$ and apply the 5-lemma to the morphisms of exact sequences (12) $\to$ (14) and (13) $\to$ (15). □

The above proof is the only argument in this paper where any kind of resolution of singularities is used. With the exception of Section 11, it is also the only argument that uses the direct image functors $f_!$ for morphisms f that are not necessarily quasi-finite. To the extent that, for a particular definition of the triangulated categories $\mathcal{DM}(X,\mathbb{Z}/m)$, the assertions of Lemma 8.1 could be established differently, neither the resolution of singularities, nor the functors $f_!$ for any but quasi-finite morphisms f would be needed for our purposes.

9. Embedding Theorem

In this section we only use the conditions (i–viii) of Section 7, together with the results of Section 8. The next theorem is the second main result of this paper.

Theorem 9.1. *Assume that the triangulated categories of motivic sheaves $\mathcal{DM}(X,\mathbb{Z}/m)$ satisfying the conditions* (i–x) *of Section* 7 *are defined for varieties X of dimension $\leqslant d$ over K, and that such varieties admit resolution of singularities.*

Then for any variety X of dimension $\leqslant d$ over K there is a natural tensor fully faithful functor $\Theta_X\colon \mathcal{F}_X^m \longrightarrow \mathcal{DM}(X,\mathbb{Z}/m)$. The image of this functor is an exact subcategory closed under extensions in the triangulated category $\mathcal{DM}(X,\mathbb{Z}/m)$ in the sense of [14, *Section* A.8]*, and the induced exact category structure coincides with the exact category structure on $\mathcal{F}_X^m$ defined in Section* 1*.*

The functors Θ form commutative diagrams with the inverse image functors f^* for morphisms f of varieties of dimension $\leqslant d$ and the direct image functors $f_!$ for quasi-finite morphisms of varieties of dimension $\leqslant d$. The composition of the embedding $\Theta_X\colon \mathcal{F}_X^m \longrightarrow \mathcal{DM}(X,\mathbb{Z}/m)$ with the étale realization functor $\Phi_X\colon \mathcal{DM}(X,\mathbb{Z}/m) \longrightarrow \mathcal{D}(\acute{E}t_X^{m,\infty})$ coincides with the composition of the exact forgetful functor $\mathcal{F}_X^m \longrightarrow \acute{E}t_X^m$ and the natural embedding $\acute{E}t_X^m \longrightarrow \mathcal{D}(\acute{E}t_X^{m,\infty})$.

Proof. Let $\mathcal{MAT}(X,\mathbb{Z}/m)$ denote the minimal full subcategory of the triangulated category $\mathcal{DM}(X,\mathbb{Z}/m)$ containing all the objects $f_!\,\mathbb{Z}/m(j)$, for quasi-finite morphisms $f\colon Y \longrightarrow X$ and $j\in\mathbb{Z}$, and closed under extensions. We will check that $\mathcal{MAT}(X,\mathbb{Z}/m)$ is an exact subcategory of $\mathcal{DM}(X,\mathbb{Z}/m)$, then refine the restriction of the étale realization functor Φ_X to the full subcategory $\mathcal{MAT}(X,\mathbb{Z}/m)$ so as to obtain a tensor exact functor $\Theta_X^{-1}\colon \mathcal{MAT}(X,\mathbb{Z}/m) \longrightarrow \mathcal{F}_X^m$, and finally show that the functor Θ_X^{-1} is an equivalence of exact categories, so it can be inverted, providing the desired fully faithful embedding Θ_X.

Let $\mathcal{MA}(X,\mathbb{Z}/m)$ denote the minimal full subcategory of $\mathcal{DM}(X,\mathbb{Z}/m)$ containing the objects $f_!\,\mathbb{Z}/m$, with f being quasi-finite morphisms into X, and closed under extensions. Clearly, the full subcategory $\mathcal{MA}(X,\mathbb{Z}/m)\subset\mathcal{DM}(X,\mathbb{Z}/m)$ is also generated, using iterated extensions, by the objects $f_!\,\mathbb{Z}/m$ with the morphisms f being finite étale morphisms onto smooth locally closed subvarieties of X.

Arguing as in the proof of Lemma 3.2, one shows that the subcategory $\mathcal{MA}(X,\mathbb{Z}/m)$ is generated by the objects $f_!\,\mathbb{Z}/m$, with f being etale morphisms into X, using the operations of passage to a cone (when such a cone belongs to $\mathcal{MA}(X,\mathbb{Z}/m)$) and iterated extension. Let us restate the latter assertion in the following more precise form, which we will need later.

Lemma 9.2. *For any variety X of dimension $\leqslant d$, the full subcategory $\mathcal{MA}(X, \mathbb{Z}/m) \subset \mathcal{DM}(X, \mathbb{Z}/m)$ is generated, using iterated extensions, by the objects $h_! \mathbb{Z}/m$, where $h \colon Z \longrightarrow X$ is the restriction of an étale morphism $f \colon Y \longrightarrow X$ to a smooth closed subvariety $Z \subset Y$.*

Proof. See the proof of Lemma 3.2. $\square$

Similarly, one shows in the same way as in the proof of Lemma 3.3 that the subcategory $\mathcal{MA}(X, \mathbb{Z}/m)$ is generated by the objects $g_! \mathbb{Z}/m$, with g being finite morphisms into X from normal varieties, using the operation of passage to a cocone (when such a cocone belongs to $\mathcal{MA}(X, \mathbb{Z}/m)$) and iterated extension.

Lemma 9.3. *For any variety X of dimension $\leqslant d$, and*

(a) *for all M, $N \in \mathcal{MA}(X, \mathbb{Z}/m)$, any $i \in \mathbb{Z}$, and $j < 0$,*

(b) *for all M, $N \in \mathcal{MA}(X, \mathbb{Z}/m)$, any $j \in \mathbb{Z}$, and $i < 0$*

one has $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M, N(j)[i]) = 0$.

Proof. As explained above, it suffices to consider the case when $M = f_! \mathbb{Z}/m$ and $N = g_! \mathbb{Z}/m(j)$, the morphism $f \colon Y \longrightarrow X$ being étale and the morphism $g \colon Z \longrightarrow X$ being finite. Using the adjunction properties of the inverse and direct images for étale and finite morphisms together with the base change, we conclude that

$$\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(f_! \mathbb{Z}/m, g_! \mathbb{Z}/m(j)[i]) \simeq \mathrm{Hom}_{\mathcal{DM}(Y \times_X Z, \mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]).$$

The latter group vanishes for $j < 0$ by the condition (viii) from Section 7 and for $i < 0$ by Lemma 8.1(a-b). $\square$

It follows from Lemma 9.3(b) that $\mathcal{MA}(X, \mathbb{Z}/m) \subset \mathcal{MAT}(X, \mathbb{Z}/m)$ are two exact subcategories of $\mathcal{DM}(X, \mathbb{Z}/m)$ (see [14, Section A.8]). They are also tensor subcategories, as $f_! \mathbb{Z}/m \otimes g_! \mathbb{Z}/m \simeq h_! \mathbb{Z}/m$ in $\mathcal{DM}(X, \mathbb{Z}/m)$ for any quasi-finite morphisms $f \colon Y \longrightarrow X$ and $g \colon Z \longrightarrow X$ with the Cartesian product $h \colon Y \times_X Z \longrightarrow X$ (as one can see from the projection formula and the base change). The tensor products in these exact categories are exact functors.

According to [14, Section 3.1], it follows from Lemma 9.3(a) that there is a natural finite decreasing filtration on every object M of the exact category $\mathcal{MAT}(X, \mathbb{Z}/m)$ with the successive quotient objects $\mathrm{gr}^j M \in \mathcal{MA}(X, \mathbb{Z}/m)(j)$. All morphisms in $\mathcal{MAT}(X, \mathbb{Z}/m)$ preserve these filtrations, and a short sequence with zero composition in $\mathcal{MAT}(X, \mathbb{Z}/m)$ is exact if and only if its short sequence of successive quotients is exact in every component number j. The filtration on the tensor product of any two objects of $\mathcal{MAT}(X, \mathbb{Z}/m)$ is the tensor product of their filtrations.

For any morphism $f \colon Y \longrightarrow X$ of varieties of dimension $\leqslant d$ over K, the tensor triangulated functor $f^* \colon \mathcal{DM}(X, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(Y, \mathbb{Z}/m)$ takes $\mathcal{MA}(X, \mathbb{Z}/m)$ to $\mathcal{MA}(Y, \mathbb{Z}/m)$ and $\mathcal{MAT}(X, \mathbb{Z}/m)$ to $\mathcal{MAT}(Y, \mathbb{Z}/m)$, defining tensor exact functors f^* between these exact categories. This follows from the base change property (ii) from Section 7. For any quasi-finite morphism $g \colon Z \longrightarrow X$, the triangulated functor $g_! \colon \mathcal{DM}(Z, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$ takes $\mathcal{MA}(Z, \mathbb{Z}/m)$ to $\mathcal{MA}(X, \mathbb{Z}/m)$ and $\mathcal{MAT}(Z, \mathbb{Z}/m)$ to $\mathcal{MAT}(X, \mathbb{Z}/m)$, defining exact functors $g_!$ between these exact categories. The exact functors f^* and $g_!$ satisfy the base change and the projection formula, because the triangulated functors f^* and $g_!$ do.

Clearly, the étale realization functor $\Phi_X\colon \mathcal{DM}(X,\mathbb{Z}/m) \longrightarrow \mathcal{D}(\acute{E}t_X^{m,\infty})$ takes $\mathcal{MAT}(X,\mathbb{Z}/m)$ into $\acute{E}t_X^m$, defining a tensor exact functor between these two exact categories (the second of which is actually abelian). The following lemma provides a more precise assertion.

Lemma 9.4. *The functor $\Phi_X\colon \mathcal{MAT}(X,\mathbb{Z}/m) \longrightarrow \acute{E}t_X^m$ takes the the full subcategory $\mathcal{MA}(X,\mathbb{Z}/m) \subset \mathcal{MAT}(X,\mathbb{Z}/m)$ into the full subcategory $\mathcal{E}_X^m \subset \acute{E}t_X^m$, defining a tensor exact functor $\mathcal{MA}(X,\mathbb{Z}/m) \longrightarrow \mathcal{E}_X^m$.*

Proof. Let $h\colon x \longrightarrow X$ be the embedding of (the spectrum of the residue field of) a scheme point of X. Then the functor $h^*\colon \mathcal{MA}(X,\mathbb{Z}/m) \longrightarrow \mathcal{MA}(x,\mathbb{Z}/m)$ forms a commutative diagram with the functor $h^*\colon \acute{E}t_X^m \longrightarrow \acute{E}t_x^m$ and the étale realization functors $\mathcal{MA}(X,\mathbb{Z}/m) \longrightarrow \acute{E}t_X^m$ and $\mathcal{MA}(x,\mathbb{Z}/m) \longrightarrow \acute{E}t_x^m$.

By the condition (ix) from Section 7 (or by Lemma 8.1(a)), one has $\mathrm{Hom}_{\mathcal{DM}(x,\mathbb{Z}/m)}(f_!\,\mathbb{Z}/m, g_!\,\mathbb{Z}/m[1]) = 0$ for any (quasi-)finite morphisms $f\colon y \longrightarrow x$ and $g\colon z \longrightarrow x$. Hence the exact category structure on $\mathcal{MA}(x,\mathbb{Z}/m)$ is trivial, with all exact triples being split and all objects isomorphic to direct sums of the objects $f_!\,\mathbb{Z}/m$. It follows that the functor Φ_X takes any object of $\mathcal{MA}(X,\mathbb{Z}/m)$ to an étale sheaf on X whose stalk at x is a permutational G_{K_x}-module, and any exact triple in $\mathcal{MA}(X,\mathbb{Z}/m)$ to an exact triple of étale sheaves whose stalk at x is a split exact triple. ☐

Applying the functor Φ_X to the natural filtration of an object of $\mathcal{MAT}(X,\mathbb{Z}/m)$, we obtain a finitely filtered object of $\acute{E}t_X^m$ with the successive quotients in $\mathcal{E}_X^m \otimes \mu_m^{\otimes j}$, i. e., an object of $\mathcal{F}_X^m$. This defines the desired tensor exact functor $\Theta_X^{-1}\colon \mathcal{MAT}(X,\mathbb{Z}/m) \longrightarrow \mathcal{F}_X^m$. Since the inverse and direct image functors f^* and f_* on the exact categories $\mathcal{MAT}(-,\mathbb{Z}/m)$ preserve the natural filtrations, the functors Θ^{-1} commute with the inverse and direct images.

It remains to show that the functor Θ_X^{-1} is an equivalence of exact categories. For this purpose, we will apply the result of [14, Lemma 3.2]. The functor Θ_X^{-1} takes the generating objects $f_!\,\mathbb{Z}/m(j)$ of the exact category $\mathcal{MAT}(X,\mathbb{Z}/m)$, where $f\colon Y \longrightarrow X$ are quasi-finite morphisms into X, to the generating objects $\mathcal{M}_{cc}^m(Y/X)(j)$ of the exact category $\mathcal{F}_X^m$. We have to show that the induced morphisms

$$\mathrm{Ext}^i_{\mathcal{MAT}(X,\mathbb{Z}/m)}(f_!\,\mathbb{Z}/m, g_!\,\mathbb{Z}/m(j)) \longrightarrow \mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathcal{M}_{cc}^m(Y/X), \mathcal{M}_{cc}^m(Z/X)(j))$$

are isomorphisms for $i = 0$, 1 and monomorphisms for $i = 2$, for all pairs of quasi-finite morphisms $f\colon Y \longrightarrow X$ and $g\colon Z \longrightarrow X$.

Arguing as in the beginning of this proof and using the 5-lemma, we conclude that we can assume the morphism f to be étale and the morphism g to be finite with a normal source. Using the adjunctions and the base change, and the compatibility of these with the functors Θ^{-1}, we reduce the problem to showing that the morphisms

$$\mathrm{Ext}^i_{\mathcal{MAT}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)) \longrightarrow \mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(j)) \tag{16}$$

are isomorphisms for $i = 0$, 1 and monomorphisms for $i = 2$ for all normal varieties X of dimension $\leqslant d$ over K. (Indeed, a scheme étale over a normal scheme is normal [9, Corollaire I.9.10].) We can also assume X to be connected.

Recall that for any objects M, $N \in \mathcal{MAT}(X, \mathbb{Z}/m)$, the natural map

$$\operatorname{Ext}^i_{\mathcal{MAT}(X,\mathbb{Z}/m)}(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M, N[i]) \tag{17}$$

is an isomorphism for $i = 0$, 1 and a monomorphism for $i = 2$ [14, Corollary A.17].

Both sides of (16) vanish when $j < 0$. When $j = 0$ and $i \leqslant 1$, the map (16) is an isomorphism by Corollary 6.4(c) and Lemma 8.1(a). In order to prove that $\operatorname{Ext}^2_{\mathcal{MAT}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m) = 0$, we will use the following lemma.

Lemma 9.5. *For any normal variety X of dimension $\leqslant d$, one has $\operatorname{Ext}^1_{\mathcal{MA}(X,\mathbb{Z}/m)}$ $(M, \mathbb{Z}/m) = 0$ for all objects $M \in \mathcal{MA}(X, \mathbb{Z}/m)$. In other words, $\mathbb{Z}/m$ is an injective object of the exact category $\mathcal{MA}(X, \mathbb{Z}/m)$.*

Proof. By Lemma 9.2, we can assume that $M = h_!\,\mathbb{Z}/m$, where $h\colon Z \longrightarrow X$ and Z is the closed complement to an open subvariety U in a variety Y étale over X. Let $v\colon U \longrightarrow Y$ and $f\colon Y \longrightarrow X$ denote the related morphisms; then we have the exact triple $f_!v_!\,\mathbb{Z}/m \longrightarrow f_!\,\mathbb{Z}/m \longrightarrow h_!\,\mathbb{Z}/m$ in $\mathcal{MA}(X, \mathbb{Z}/m)$. Hence the induced exact sequence

$$\begin{aligned}\operatorname{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m) &\longrightarrow \operatorname{Hom}_{\mathcal{DM}(U,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m)\\ \longrightarrow \operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(h_!\,\mathbb{Z}/m, \mathbb{Z}/m[1]) &\longrightarrow \operatorname{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1]).\end{aligned}$$

By [9, Proposition I.10.1], the connected components of Y are irreducible. Since $\operatorname{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1]) = 0$ by Lemma 8.1(a), and there is at most one connected component of U in every connected component of Y, we are done. ☐

Finally, when $j \geqslant 1$, both maps

$$\operatorname{Ext}^n_{\mathcal{F}^m_X}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \longrightarrow \mathbb{H}^i_{Nis}(X, \tau_{\leqslant j}\mathbb{R}\rho_*\mu_m^{\otimes j}) \longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$$

are isomorphisms for $i \leqslant 1$ and monomorphisms for $i = 2$ (the rightmost one obviously, and the leftmost one by Corollary 6.4(a)). So are the maps

$$\operatorname{Ext}^i_{\mathcal{MAT}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m) \longrightarrow \operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[i]) \longrightarrow H^i_{\acute{e}t}(X, \mu_m^{\otimes j})$$

(see Lemma 8.1(b)). As the diagram is commutative (the functors $\mathcal{F}^m_X \longrightarrow \acute{E}t^m_X$ and $\mathcal{MAT}(X, \mathbb{Z}/m) \longrightarrow \acute{E}t^m_X$ forming a commutative diagram with the functor Θ_X^{-1}), bijectivity of the maps (16) for $i = 0$, 1 and their injectivity for $i = 2$ follows.

We have shown that the functor Θ_X^{-1} is an equivalence of exact categories, so the desired fully faithful functor $\Theta_X\colon \mathcal{F}^m_X \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$ is constructed. The functor Θ_X commutes with the inverse and direct image functors f^* and $f_!$, since the functor Θ_X^{-1} does. The same applies to the compatibility with the étale realization functors. All the assertions having been verified, the embedding theorem is proven. ☐

10. Particular Cases and Applications

In this section we discuss conditions under which the morphisms (17) are isomorphisms, or an equivalence between the derived category of an exact category of Artin–Tate motivic sheaves and an appropriate triangulated subcategory of the triangulated category of motivic sheaves can be established.

We will identify the exact category $\mathcal{F}_X^m$ with the exact subcategory $\mathcal{MAT}(X,\mathbb{Z}/M) \subset \mathcal{DM}(X,\mathbb{Z}/m)$ using the functor Θ_X from Theorem 9.1, and in particular, use the notation $\mathcal{M}_{cc}^m(Y/X) = f_!\,\mathbb{Z}/m$ for the corresponding objects of both categories $\mathcal{F}_X^m$ and $\mathcal{DM}(X,\mathbb{Z}/m)$ (where $f\colon Y \longrightarrow X$ is a quasi-finite morphism).

Proposition 10.1. *Assume that the triangulated categories of motivic sheaves $\mathcal{DM}(X,\mathbb{Z}/m)$ satisfying the conditions* (i–xi) *of Section* 7 *are defined for varieties X of dimension $\leqslant d$ over K, and that such varieties admit resolution of singularities. Assume further that the maps $\theta^{m,i,j}_{\operatorname{Spec} L}$ from* (6) *are isomorphisms for all (the residue fields L of) the scheme points of varieties of dimension $\leqslant d$ over K. Then the natural maps*

$$\operatorname{Ext}^i_{\mathcal{F}_X^m}(M,\mathcal{M}_{cc}^m(Y/X)(j)) \longrightarrow \operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M,\mathcal{M}_{cc}^m(Y/X)(j)[i]) \tag{18}$$

are isomorphisms for all varieties X of dimension $\leqslant d$ over K, all objects $M \in \mathcal{F}_X^m$, and all smooth varieties Y finite over X.

Proof. Since the functor Θ_X is compatible with the adjunction isomorphisms between the groups of morphisms in $\mathcal{F}_X^m$ and $\mathcal{DM}(X,\mathbb{Z}/m)$, it suffices to check that the map

$$\operatorname{Ext}^i_{\mathcal{F}_Y^m}(M,\mathbb{Z}/m(j)) \longrightarrow \operatorname{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(M,\mathbb{Z}/m(j)[i])$$

is an isomorphism. For the same reason, and using Lemma 3.2, the question can be reduced to checking that the map

$$\operatorname{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m,\mathbb{Z}/m(j)) \longrightarrow \operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m,\mathbb{Z}/m(j)[i]) \tag{19}$$

is an isomorphism for a smooth variety X of dimension $\leqslant d$ over K.

Both sides of (19) vanish when $j<0$ or $i<0$ (see Lemma 9.3). When $i \leqslant j$, both sides map isomorphically to $H^i_{\acute{e}t}(X,\mu_m^{\otimes j})$ (see Theorem 6.2 and the condition (x) from Section 7) and the diagram is commutative, so (19) is an isomorphism.

The case $0 \leqslant j < i$ is dealt with using the condition (xi). We argue by induction in i for a fixed j. First we show that the map (19) is injective. Let x be an element in $\operatorname{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$ that dies in $\operatorname{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m,\mathbb{Z}/m(j)[i])$. As explained in the proof of Theorems 6.1–6.2, there exists a Nisnevich cover $U_\alpha \longrightarrow X$ such that the element x vanishes in the restriction to U_α.

Arguing as in the proof of Lemma 5.2, we can assume that there exists a distinguished pair of morphisms $\upsilon\colon U \longrightarrow X$ and $g\colon Z \longrightarrow X$ in the Nisnevich topology such that x vanishes in the restriction to U and Z. Let h denote the morphism $W = U \times_X Z \longrightarrow X$. Consider the exact triple

$$0 \longrightarrow h_!\,\mathbb{Z}/m \longrightarrow \upsilon_!\,\mathbb{Z}/m \oplus g_!\,\mathbb{Z}/m \longrightarrow \mathbb{Z}/m \longrightarrow 0$$

in $\mathcal{F}_X^m$. From the related long exact sequence of $\mathrm{Ext}^*_{\mathcal{F}_X^m}(-,\mathbb{Z}/m(j))$ we see that the element x comes from an element $w \in \mathrm{Ext}^{i-1}_{\mathcal{F}_W^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$. In the similar long exact sequence of $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(-,\mathbb{Z}/m(j)[*])$, the image of the element w in $\mathrm{Hom}_{\mathcal{DM}(W,\mathbb{Z}/m)}(\mathbb{Z}/m,\,\mathbb{Z}/m(j)[i-1])$ comes from an element $\zeta \in \mathrm{Hom}_{\mathcal{DM}(U,\mathbb{Z}/m)}(\mathbb{Z}/m,$ $\mathbb{Z}/m(j)[i-1]) \oplus \mathrm{Hom}_{\mathcal{DM}(Z,\mathbb{Z}/m)}(\mathbb{Z}/m,\,\mathbb{Z}/m(j)[i-1])$. By the induction assumption, the latter comes from an element $z \in \mathrm{Ext}^{i-1}_{\mathcal{F}_U^m}(\mathbb{Z}/m,\mathbb{Z}/m(j)) \oplus \mathrm{Ext}^{i-1}_{\mathcal{F}_Z^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$. Continuing this diagram chase, one easily concludes that $x = 0$.

Now we can check that the map (19) is surjective. Let ξ be an element in $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m,\mathbb{Z}/m(j)[i])$ which does not come from $\mathrm{Ext}^i_{\mathcal{F}_X^m}(\mathbb{Z}/m,\mathbb{Z}/m)$. By the condition (xi), there exists a Nisnevich cover $U_\alpha \longrightarrow X$ such that the element ξ vanishes in the restriction to U_α. Arguing as in the proof of Lemma 5.2 again, we can assume that there exists a distinguished pair of morphisms υ and g such that the elements $\upsilon^*\xi$ and $g^*\xi$ come from $\mathrm{Ext}^i_{\mathcal{F}_U^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$ and $\mathrm{Ext}^i_{\mathcal{F}_Z^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$, respectively. Let z be the corresponding element in the direct sum of the latter two groups. In view of the injectivity assertion that we have proven, the image of z vanishes in $\mathrm{Ext}^i_{\mathcal{F}_W^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$, so the element z comes from a certain element in $\mathrm{Ext}_{\mathcal{F}_X^m}(\mathbb{Z}/m,\mathbb{Z}/m(j))$, etc. □

Corollary 10.2. *Assume that the triangulated categories of motivic sheaves* $\mathcal{DM}(X,$ $\mathbb{Z}/m)$ *satisfying the conditions* (i–xi) *of Section* 7 *are defined for varieties* X *of dimension* $\leqslant 1$ *over* K. *Assume further that the maps* $\theta^{m,i,j}_{\mathrm{Spec}\,L}$ *from* (6) *are isomorphisms for all (the residue fields* L *of) the scheme points of varieties of dimension* $\leqslant 1$ *over* K. *Then the natural maps*

$$\mathrm{Ext}^i_{\mathcal{F}_X^m}(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M,N[i]) \tag{20}$$

(*see also* (17)) *are isomorphisms for all varieties* X *of dimension* $\leqslant 1$ *over* K *and all objects* M, $N \in \mathcal{F}_X^m$.

Proof. Notice that varieties of dimension $\leqslant 1$ over a perfect field K always admit resolution of singularities (in the sense of Section 8) by definition. By Lemma 3.3, the exact category $\mathcal{F}_X^m$ is generated, using the operations of passage to the kernel of an admissible epimorphism and iterated extension, by objects $\mathcal{M}^m_{cc}(Y/X)(j)$, where Y runs over all smooth varieties finite over X (since normal curves are smooth). So it remains to use Proposition 10.1. □

Corollary 10.3. *Assume that the triangulated categories of motivic sheaves* $\mathcal{DM}(X,$ $\mathbb{Z}/m)$ *satisfying the conditions* (i–xi) *of Section* 7 *are defined for varieties* X *of dimension* $\leqslant 1$ *over* K. *Then the natural maps*

$$\mathrm{Ext}^i_{\mathcal{F}_X^m}(M,N(j)) \longrightarrow \mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(M,N(j)[i]) \tag{21}$$

are isomorphisms for all varieties X *of dimension* $\leqslant 1$ *over* K, *all objects* M, $N \in$ $\mathcal{E}_X^m$, *and all* $j \leqslant 2$.

Proof. It suffices to notice that the proof of Proposition 10.1, and hence also of Corollary 10.2, holds for every fixed value of j. Then one applies Corollary 6.4(b). □

Notice that the assertion of the latter Corollary is certainly *not* true for surfaces. The problem arises when $j = 0$. Indeed, let Y be a normal surface with a point singularity such that the exceptional fiber Z of its resolution $\widetilde{Y}$ is a self-intersecting projective line (see Introduction). Using the exact sequences (12–13), one easily computes $\mathrm{Hom}_{\mathcal{DM}(Z,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[1]) = \mathbb{Z}/m$ and $\mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[2]) = \mathbb{Z}/m$. However, $\mathrm{Ext}^2_{\mathcal{F}^m_Y}(\mathbb{Z}/m, \mathbb{Z}/m) = 0$ by Corollary 6.4(c) or Lemma 9.5.

Furthermore, given a smooth surface X over an algebraically closed field K, one can find a normal surface Y finite over X having at least one singularity of the above type. Then $\mathrm{Ext}^2_{\mathcal{F}^m_X}(\mathbb{Z}/m, \mathcal{M}^m_{cc}(Y/X)) \simeq \mathrm{Ext}^2_{\mathcal{F}^m_Y}(\mathbb{Z}/m, \mathbb{Z}/m) = 0$ and $\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathcal{M}^m_{cc}(Y/X)[2]) \simeq \mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m[2]) \neq 0$.

In order to extend the functor Θ_X to a triangulated functor $\mathcal{D}^b(\mathcal{F}^m_X) \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$, one needs some additional structure on the triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$. E. g., it would suffice if these triangulated categories had algebraic origin.

Corollary 10.4. *Assume that the triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ satisfying the conditions* (i–x) *of Section* 7 *are defined for varieties X of dimension $\leqslant d$ over K, and that such varieties admit resolution of singularities. Assume additionally that the triangulated categories $\mathcal{DMF}(X, \mathbb{Z}/m)$ of two-step filtrations, or morphisms in the triangulated categories $\mathcal{DM}(X, \mathbb{Z}/m)$* [14, *Appendix* D] *are defined. Then*

(a) *there exist triangulated functors*

$$\widetilde{\Theta}_X \colon \mathcal{D}^b(\mathcal{F}^m_X) \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m) \tag{22}$$

extending the fully faithful functors $\Theta_X \colon \mathcal{F}^m_X \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$;

(b) *if the functors of inverse image f^* for morphisms of varieties f and direct image with compact supports $f_!$ for quasi-finite morphisms f act on the triangulated categories $\mathcal{DMF}(X, \mathbb{Z}/m)$ in a way compatible with the structure functors κ_0, κ_1, w between the triangulated categories $\mathcal{DM}(X, \mathbb{Z}/m)$ and $\mathcal{DMF}(X, \mathbb{Z}/m)$, then the triangulated functors* (22) *commute with the functors f^* and $f_!$.*

Proof. Part (a) is provided by [14, Theorem D.4], and part (b) follows from the construction in the proof of that Theorem. $\square$

Let $\mathcal{F}^m_{X;\,[0,2]}$ denote the full exact subcategory of $\mathcal{F}^m_X$ consisting of all the objects concentrated in the filtration components 0, 1, and 2 (i. e., the iterated extensions of the objects of $\mathcal{E}^m_X$, $\mathcal{E}^m_X(1)$, and $\mathcal{E}^m_X(2)$). Denote by $\mathcal{DMAT}(X, \mathbb{Z}/m) \subset \mathcal{DM}(X, \mathbb{Z}/m)$ the full triangulated subcategory generated by $\mathcal{MAT}(X, \mathbb{Z}/m)$ in $\mathcal{DM}(X, \mathbb{Z}/m)$, and by $\mathcal{DMAT}_{[0,2]}(X, \mathbb{Z}/m)$ the full triangulated subcategory generated by $\mathcal{MAT}_{[0,2]} = \mathcal{F}^m_{X;\,[0,2]}$ in $\mathcal{DM}(X, \mathbb{Z}/m)$.

Corollary 10.5. *Assume that the triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ satisfying the conditions* (i–xi) *of Section* 7 *are defined for varieties X*

of dimension $\leqslant$ 1 over K. Assume additionally that triangulated categories $\mathcal{DMF}(X, \mathbb{Z}/m)$ are defined. Then

(a) *the functors* (22) *are fully faithful in restriction to $\mathcal{D}^b(\mathcal{F}^m_{X;\,[0,2]})$ and induce equivalences of triangulated categories $\mathcal{D}^b(\mathcal{F}^m_{X;\,[0,2]}) \simeq \mathcal{DMAT}_{[0,2]}(X, \mathbb{Z}/m)$;*

(b) *if the maps $\theta^{m,i,j}_{\operatorname{Spec} L}$ from* (6) *are isomorphisms for all (the residue fields L of) the scheme points of varieties of dimension $\leqslant 1$ over K, then the functors* (22) *are fully faithful and induce equivalences of triangulated categories $\mathcal{D}^b(\mathcal{F}^m_X) \simeq \mathcal{DMAT}(X, \mathbb{Z}/m)$.*

Proof. Follows from Corollaries 10.2–10.4. □

Remark 10.6. Using Tsen's theorem together with Corollary 6.4(a), one can easily see that the maps $\theta^{m,i,j}_L$ are isomorphisms for fields L of transcendence degree 1 over an algebraically closed field K. Thus the conclusions of Corollaries 10.2 and 10.5(b) hold when the field K is algebraically closed (provided that the relevant motivic triangulated categories are defined).

One would like to extend this result, e. g., to finite fields K. A step in this direction was made in the paper [15], where the $K(\pi, 1)$-conjecture was proven for Tate motives (see [14, Sections 2 and 9.1] and [15, Section 5]) and for Artin–Tate motives related to a fixed cyclic extension of prime degree (see [14, Section 9.8] and [15, Section 6]) over a one-dimensional global field containing certain roots of unity.

11. Homological Motives

Here we discuss the properties of the relative homological motives $\mathcal{M}^m_h(Y/X) \in \mathcal{D}^b(\mathcal{F}^m_X)$, which were defined in Section 3 for quasi-finite morphisms of smooth varieties $Y \longrightarrow X$.

For this purpose we will need to have a fourth operation $f^!$ of the "six operations" formalism defined on our triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$. So we assume, in addition to the assumptions of Section 7, the following.

(xii) For any morphism $f\colon Y \longrightarrow X$ of varieties over K, the functor $f_!\colon \mathcal{DM}(Y, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$ admits a right adjoint functor $f^!\colon \mathcal{DM}(X, \mathbb{Z}/m) \longrightarrow \mathcal{DM}(Y, \mathbb{Z}/m)$. Whenever the morphism f is smooth of relative dimension n, there is a functorial isomorphism $f^!(M) \simeq f^*(M)(n)[2n]$ for all objects $M \in \mathcal{DM}(X, \mathbb{Z}/m)$.

Proposition 11.1. *Assume that the triangulated categories of motivic sheaves $\mathcal{DM}(X, \mathbb{Z}/m)$ satisfying the conditions* (i–xii) *are defined for varieties X of dimension $\leqslant d$ over K, and that such varieties admit resolution of singularities. Assume further that the maps $\theta^{m,i,j}_L$ from* (6) *are isomorphisms for all (the residue fields L of) the scheme points of varieties of dimension $\leqslant d$ over K.*

Then the relative motive $\mathcal{M}^m_h(Y/X) \in \mathcal{D}^b(\mathcal{F}^m_X)$ is covariantly functorial with respect to arbitrary morphisms of smooth varieties Y quasi-finite over a fixed smooth *variety X of dimension $\leqslant d$ over K. For any quasi-finite morphism of smooth varieties $Y \longrightarrow X$ of dimension $\leqslant d$ there is a natural isomorphism of $\mathbb{Z}/m$-modules*

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{F}_X^m)}(\mathcal{M}_h^m(Y/X), \mathbb{Z}/m(j)[i]) \simeq \mathrm{Hom}_{\mathcal{D}^b(\mathcal{F}_Y^m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i]). \tag{23}$$

Both $\mathbb{Z}/m$-modules are also naturally isomorphic to the motivic cohomology module $\mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m(j)[i])$.

Proof. No triangulated categories $\mathcal{DM}(X, \mathbb{Z}/m)$ are mentioned in the formulations of the functoriality claim and the isomorphism (23), but they are used in the following proof, so we need them in the assumptions. Notice that the motive $\mathcal{M}_h^m(Y/X) \in \mathcal{F}_X^m$ is a shift of an object of $\mathcal{F}_X^m$ (or, at worst, is naturally presented as a direct sum of such shifts), so an object of the category $\mathcal{DM}(X, \mathbb{Z}/m)$ can be associated with the motive $\mathcal{M}_h^m(Y/X)$ using the functor $\Theta_X : \mathcal{F}_X^m \longrightarrow \mathcal{DM}(X, \mathbb{Z}/m)$.

The object so obtained is naturally isomorphic to $f_!\, f^!\, \mathbb{Z}/m \in \mathcal{DM}(X, \mathbb{Z}/m)$. One can see this by applying the condition (xii) to the smooth morphisms $X \longrightarrow \operatorname{Spec} K$ and $Y \longrightarrow \operatorname{Spec} K$ together with the object $M = \mathbb{Z}/m \in \mathcal{DM}(\operatorname{Spec} K, \mathbb{Z}/m)$, and using the compatibility of the functors $f^!$ with the compositions of the morphisms f (which follows from the similar compatibility of the functors $f_!$).

Both sides of the desired isomorphism (23) are (at worst, direct sums of) certain Ext groups in the exact categories $\mathcal{F}_X^m$ and $\mathcal{F}_Y^m$, respectively. In our assumptions, Proposition 10.1 allows to identify these groups with the related Hom groups in the categories $\mathcal{DM}(X, \mathbb{Z}/m)$ and $\mathcal{DM}(Y, \mathbb{Z}/m)$. In order to deduce (23), it remains to use the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{DM}(X,\mathbb{Z}/m)}(f_!\, f^!\, \mathbb{Z}/m, \mathbb{Z}/m(j)[i]) \simeq \mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(f^!\, \mathbb{Z}/m, f^!\mathbb{Z}/m(j)[i])$$

together with the isomorphism

$$\mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(f^!\, \mathbb{Z}/m, f^!\mathbb{Z}/m(j)[i]) \simeq \mathrm{Hom}_{\mathcal{DM}(Y,\mathbb{Z}/m)}(\mathbb{Z}/m, \mathbb{Z}/m),$$

which again holds in view of the condition (xii).

In particular, given a quasi-finite morphism of smooth varieties $g : Z \longrightarrow Y$, there is a natural morphism $\mathcal{M}_h^m(Z/Y) \longrightarrow \mathbb{Z}/m$ in $\mathcal{D}^b(\mathcal{F}_Y^m)$ corresponding to the identity morphism $\mathbb{Z}/m \longrightarrow \mathbb{Z}/m$ in $\mathcal{D}^b(\mathcal{F}_Z^m)$. Applying the direct image functor $f_!$ with respect to a quasi-finite morphism of smooth varieties $f : Y \longrightarrow X$ together with the twist and the shift, we obtain the desired functoriality map

$$g_* : \mathcal{M}_h^m(Z/X) \longrightarrow \mathcal{M}_h^m(Y/X) \tag{24}$$

in $\mathcal{D}^b(\mathcal{F}_X^m)$. □

Alternatively, given a morphism $g : Z \longrightarrow Y$ of smooth varieties quasi-finite over X with $\dim Y - \dim Z \leqslant 1$, one can define the map (24) in terms of the adjunction morphism $f_!\, g_!\, g^!\, f^!\, \mathbb{Z}/m \longrightarrow f_!\, f^!\, \mathbb{Z}/m$ in $\mathcal{DM}(X, \mathbb{Z}/m)$, using the fact that $\mathcal{F}_X^m$ is a full exact subcategory closed under extensions in $\mathcal{DM}(X, \mathbb{Z}/m)$ (see (17)). The latter construction does not even depend on any $K(\pi, 1)$-conjectures.

Appendix. Complexes Computing Ext in Exact Categories

Let k be a commutative ring and $\mathcal{E}$ be a k-linear small exact category. To any two objects M and $N \in \mathcal{E}$ we would like to assign a homotopy projective complex of projective k-modules $C^\bullet_{\mathcal{E}}(M,N)$ with the following properties.

(1) To any morphism $f\colon M \longrightarrow N$ in $\mathcal{E}$ a cocycle $c(f) \in C^0_{\mathcal{E}}(M,N)$ is assigned. One has $c(f)=0$ whenever $f=0$.

(2) For any three objects L, M, $N \in \mathcal{E}$ there is a composition map $C^\bullet_{\mathcal{E}}(M,N) \otimes_k C^\bullet_{\mathcal{E}}(L,M) \longrightarrow C^\bullet_{\mathcal{E}}(L,N)$. These multiplications are associative, the elements $c(\mathrm{id}_M)$ are unit objects for them, and the composition $c(f)c(g)$ is equal to $c(fg)$ whenever f and g are composable.

In particular, it follows from (1–2) that $C^\bullet_{\mathcal{E}}(M,N)$ are contravariantly functorial in M and covariantly functorial in N. However, the maps $f \longmapsto c(f)$ are *not* necessarily compatible with the addition of morphisms f or their multiplication by constants from k. So the bifunctor $(M,N) \longmapsto C^\bullet_{\mathcal{E}}(M,N)$ may be neither biadditive nor k^*-biequivariant.

(3) For any k-linear exact functor $\gamma\colon \mathcal{E} \longrightarrow \mathcal{F}$ and all objects M, $N \in \mathcal{E}$ there are morphisms of complexes $C^\bullet_{\mathcal{E}}(M,N) \longrightarrow C^\bullet_{\mathcal{F}}(\gamma(M),\gamma(N))$, compatible with the structures in (1–2) in the obvious sense.

(4) For any M, $N \in \mathcal{E}$, there are isomorphisms between the cohomology k-modules $H^iC^\bullet_{\mathcal{E}}(M,N)$ and the Yoneda Ext k-modules $\mathrm{Ext}^i_{\mathcal{E}}(M,N)$, compatible with all the structures in (1–3).

(5) The three-term sequences of complexes $C^\bullet_{\mathcal{E}}(M,N)$ corresponding to exact triples in either of the arguments M and N, considered as bicomplexes with three rows, have acyclic total complexes.

Proposition A.1. *Complexes $C^\bullet_{\mathcal{E}}(M,N)$ with the above-listed properties exist.*

Proof. Let us start with the simple case when k is a field. Then it suffices to consider the DG-category of (bounded or unbounded) complexes over $\mathcal{E}$ and take its Drinfeld localization [5] over k with respect to the full DG-subcategory of acyclic complexes. The localization is a k-linear DG-category $D^\bullet_{\mathcal{E}}$ endowed with a natural k-linear functor into it from the category $\mathcal{E}$. The complexes of morphisms between the images of objects M, $N \in \mathcal{E}$ in the DG-category $D^\bullet_{\mathcal{E}}$ provide the desired complexes $C^\bullet_{\mathcal{E}}(M,N)$.

In the general case, there is a problem that the Drinfeld localization is only defined for DG-categories whose complexes of morphisms are homotopy flat complexes of k-modules. In order to deal with it, we will use the following lemma.

Lemma A.2. *There exists a functor P from the category of complexes of k-modules to the category of homotopy projective complexes of projective k-modules with the following properties. The functor P is endowed with natural transformations $A^\bullet \longrightarrow P(A^\bullet) \longrightarrow A^\bullet$ for any complex of k-modules $A^\bullet$. The map $P(A^\bullet) \longrightarrow A^\bullet$ is a quasi-isomorphism of complexes of k-modules. The map $A^\bullet \longrightarrow P(A^\bullet)$ is a map of graded sets taking zero elements to zero elements and commuting with the differentials (but* *not necessarily preserving either the addition of elements or their multiplication by constants from k). The composition $A^\bullet \longrightarrow P(A^\bullet) \longrightarrow A^\bullet$ is the identity map.*

Furthermore, P is a pseudotensor functor, i. e., for any complexes of k-modules $A^\bullet$ and $B^\bullet$ there is a natural morphism of complexes of k-modules $P(A^\bullet) \otimes_k P(B^\bullet) \longrightarrow P(A^\bullet \otimes_k B^\bullet)$ compatible with the associativity and (graded) commutativity constraints in the tensor category of complexes. There is a morphism of complexes of k-modules $k \longrightarrow P(k)$ that is a pseudounit for the pseudotensor functor P. So, in particular, P transforms DG-algebras over k to DG-algebras and k-linear DG-categories to DG-categories. The maps of section $A^\bullet \longrightarrow P(A^\bullet)$ and projection $P(A^\bullet) \longrightarrow A^\bullet$ are multiplicative (for homogeneous elements) with respect to this pseudotensor structure. The image of the element $1 \in k$ under the pseudounit map $k \longrightarrow P(k)$ coincides with the image of the same element under the section map $k \longrightarrow P(k)$.

Notice that an alternative way to assign to a DG-category $D^\bullet$ a DG-category $F(D^\bullet)$ with homotopy k-projective complexes of morphisms and a quasi-isomorphism $F(D^\bullet) \longrightarrow D^\bullet$ is to construct a functorial cofibrant resolution of $D^\bullet$ in the model category of DG-categories over k [18]. This is not difficult to do, and in addition one can have a natural section $D^\bullet \longrightarrow F(D^\bullet)$. However, this section will not be multiplicative. This problem is solved by the construction of the functor P, which produces from a DG-category $D^\bullet$ the DG-category $P(D^\bullet)$ whose complexes of morphisms are cofibrant as complexes of k-modules only.

Proof of Lemma. To convince ourselves that such a construction is at all possible, let us first discuss the simple case when the ring k contains a field f. Then one can set $P(A^\bullet)$ to be the (direct sum total complex of) the (reduced or nonreduced) bar-construction of $A^\bullet$ over k relative to f. The desired pseudotensor structure is given by the operation of shuffle product on the bar-complexes [13, Proposition 1.1 of Chapter 3]. The section map in this case is even additive and f-linear (but not k-linear).

In the general case, our functor P takes a k-module M to its resolution whose degree-zero term is the k-module freely generated by all the elements of M, modulo the only relation that the generator corresponding to the zero element is equal to zero. This construction is iterated to obtain the whole resolution. To a complex of k-modules $A^\bullet$, the functor P assigns the total complex of the bicomplex formed by the above-described resolutions of the terms of the complex $A^\bullet$. The total complex is constructed by taking infinite direct sums along the diagonals. The key step is to construct the pseudotensor structure on this functor P.

Formally, for any complex of k-modules $A^\bullet$ we construct the terms of the bicomplex $P_i^j(A^\bullet)$ by induction in i. The k-module $P_0^j(A^\bullet)$ is generated by the symbols $[a]$, where $a \in A^j$, with the relation $[0] = 0$. The map $\pi\colon P_0^j(A^\bullet) \longrightarrow A^\bullet$ takes $[a]$ to a. For every $i > 0$, we define the k-module $P_i^j(A^\bullet)$ together with the differential $\partial\colon P_i^j(A^\bullet) \longrightarrow P_{i-1}^j(A^\bullet)$ as follows. The k-module $P_i^j(A^\bullet)$ is generated by the symbols $\langle p\rangle$, where $p \in P_{i-1}^j(A^\bullet)$ and $\partial(p) = 0$ (if $i > 1$) or $\pi(p) = 0$ (if $i = 1$), with the relation $\langle 0\rangle = 0$. The differential ∂ is defined by the rules $\partial\langle p\rangle = p$ and $\partial[a] = 0$.

The degree $|p|$ of an element $p \in P_i^j(A^\bullet)$ is set to be equal to $j-i$ (notice that $\langle\ \rangle$ is consequently an operation of degree -1). The differential d on $P(A^\bullet)$ is defined by induction and in terms of the differential d on A by the rules $d[a]=[da]$ and $d\langle p\rangle = \langle -dp\rangle$. The total differential on $P(A^\bullet)$ is $\partial + d$. Finally, the pseudotensor structure is defined by induction by the rules

$$\langle p\rangle \times \langle q\rangle = \langle p\times\langle q\rangle - (-1)^{|p|}\langle p\rangle \times q\rangle,$$
$$\langle p\rangle \times [a] = \langle p\times [a]\rangle, \quad [a]\times\langle q\rangle = \langle (-1)^{|a|}[a]\times q\rangle,$$

and $[a]\times[b]=[a\otimes b]$. The projection map $\pi\colon P(A^\bullet)\longrightarrow A^\bullet$ is extended from $P_0^*(A^\bullet)$ to the whole of $P(A^\bullet)$ by the rule $\pi(\langle p\rangle)=0$. The section map $s\colon A^\bullet \longrightarrow P(A^\bullet)$ is defined on homogeneous elements $a\in A$ by the rule $s(a)=[a]$. The pseudounit map $e\colon k\longrightarrow P(k)$ is defined by the rule $e(1)=[1]$. $\square$

Now, given a k-linear exact category $\mathcal{E}$, we apply the pseudotensor functor P to (every complex of morphisms in) the k-linear DG-category of complexes over $\mathcal{E}$. To the k-linear DG-category so obtained we apply, in turn, the Drinfeld localization construction with respect to the full DG-subcategory consisting of the acyclic complexes over $\mathcal{E}$. The resulting DG-category $D^\bullet_{\mathcal{E}}$ comes together with a (nonadditive) functor $S\colon \mathcal{E}\longrightarrow D^\bullet_{\mathcal{E}}$, which is defined in terms of the section map for the pseudotensor functor P. Finally, we set $C^\bullet_{\mathcal{E}}(M,N) = \mathrm{Hom}_{D^\bullet_{\mathcal{E}}}(SM,SN)$. $\square$

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