PSEUDO-DUALIZING COMPLEXES AND PSEUDO-DERIVED CATEGORIES

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Abstract. The definition of a pseudo-dualizing complex is obtained from that of a dualizing complex by dropping the injective dimension condition, while retaining the finite generatedness and homothety isomorphism conditions. In several settings, such as those of a pair of associative rings, a pair of coassociative coalgebras, or a commutative ring with a weakly proregular ideal, we show that the datum of a pseudo-dualizing complex induces a triangulated equivalence between a pseudo-coderived category and a pseudo-contraderived category. The latter terms mean triangulated categories standing "in between" the conventional derived category and the coderived or the contraderived category. The constructions of these triangulated categories use appropriate versions of the Auslander and Bass classes of objects in the abelian categories involved. The constructions of derived functors providing the triangulated equivalences are based on a generalization of a technique developed in our previous paper [15].

Contents

Introduction

0.1. According to the philosophy elaborated in the introduction to [15], the choice of a dualizing complex induces a triangulated equivalence between the coderived category of (co)modules and the contraderived category of (contra)modules, while in order to construct an equivalence between the conventional derived categories of (co)modules and (contra)modules one needs a dedualizing complex. In particular, an associative ring A is a dedualizing complex of bimodules over itself, while a coassociaitve coalgebra $\mathcal{C}$ over a field k is a dualizing complex of bicomodules over itself. The former assertion refers to the identity equivalence

$$\mathsf{D}(A\text{–}\mathsf{mod}) = \mathsf{D}(A\text{–}\mathsf{mod}), \tag{1}$$

while the latter one points to the natural triangulated equivalence between the coderived category of comodules and the contraderived category of contramodules

$$\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{C}\text{–}\mathsf{contra}), \tag{2}$$

known as the *derived comodule-contramodule correspondence* [11, Sections 0.2.6–7 and 5.4], [12, Sections 4.4 and 5.2].

Given a left coherent ring A and a right coherent ring B, the choice of a dualizing complex of A-B-bimodules $D^\bullet$ induces a triangulated equivalence between the coderived and the contraderived category [7, Theorem 4.8], [16, Theorem 4.5]

$$\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}). \tag{3}$$

Given a left cocoherent coalgebra $\mathcal{C}$ and a right cocoherent coalgebra $\mathcal{D}$ over a field k, the choice of a dedualizing complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathcal{B}^\bullet$ induces a triangulated equivalence between the conventional derived categories of comodules and contramodules [19, Theorem 2.6]

$$\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}(\mathcal{D}\text{–}\mathsf{contra}). \tag{4}$$

0.2. The equivalences (1–4) of Section 0.1 are the "pure types". The more complicated and interesting triangulated equivalences of the "broadly understood co-contra correspondence" kind are obtained by mixing these pure types, or maybe rather building these elementary blocks on top of one another.

In particular, let R be a commutative ring and $I \subset R$ be an ideal. An R-module M is said to be *I-torsion* if

$$R[s^{-1}] \otimes_R M = 0 \qquad \text{for all } s \in I.$$

Clearly, it suffices to check this condition for a set of generators $\{s_j\}$ of the ideal I. An R-module P is said to be an *I-contramodule* if

$$\operatorname{Hom}_R(R[s^{-1}], P) = 0 = \operatorname{Ext}^1_R(R[s^{-1}], P) \qquad \text{for all } s \in I.$$

Once again, it suffices to check these conditions for a set of generators $\{s_j\}$ of the ideal I [17, Theorem 5.1]. The full subcategory of I-torsion R-modules $R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}} \subset R\text{–}\mathsf{mod}$ is an abelian category with infinite direct sums and products; the embedding functor $R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}} \longrightarrow R\text{–}\mathsf{mod}$ is exact and preserves infinite direct sums. Similarly, the full subcategory of I-contramodule R-modules $R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}} \subset R\text{–}\mathsf{mod}$ is an abelian category with infinite direct sums and products; the embedding functor $R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$ is exact and preserves infinite products.

The fully faithful exact embedding functor $R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}} \longrightarrow R\text{–}\mathsf{mod}$ has a right adjoint functor $\Gamma_I\colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}$ (assigning to any R-module its maximal I-torsion submodule). Assume for simplicity that R is a Noetherian ring; then the right derived functor $\mathbb{R}^*\Gamma_I$ has finite homological dimension (not exceeding the minimal number of generators of the ideal I). So it acts between the bounded derived categories

$$\mathbb{R}\Gamma_I\colon \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}).$$

A *dedualizing complex* for the ring R with the ideal $I \subset R$ can be produced by applying the derived functor $\mathbb{R}\Gamma_I$ to the R-module R, while a *dualizing complex* for the ring R with the ideal I can be obtained by applying the functor $\mathbb{R}\Gamma_I$ to a dualizing complex $D_R^\bullet$ for the ring R,

$$B^\bullet = \mathbb{R}\Gamma_I(R) \quad \text{and} \quad D^\bullet = \mathbb{R}\Gamma_I(D_R^\bullet).$$

Using a dedualizing complex $B^\bullet$, one can construct a triangulated equivalence between the conventional derived categories of the abelian categories of I-torsion and I-contramodule R-modules

$$\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \simeq \mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}). \tag{5}$$

This result can be generalized to the so-called *weakly proregular* finitely generated ideals I in not necessarily Noetherian commutative rings R [15, Corollary 3.5 or Theorem 5.10].

Using a dualizing complex $D^\bullet$, one can construct a triangulated equvalence between the coderived category of I-torsion R-modules and the contraderived category of I-contramodule R-modules [13, Theorem C.1.4] (see also [13, Theorem C.5.10])

$$\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \simeq \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}). \tag{6}$$

This result can be generalized from affine formal Noetherian schemes to ind-affine ind-Noetherian or ind-coherent ind-schemes with dualizing complexes [13, Theorem D.2.7] (see also [15, Remark 4.10]).

Informally, one can view the I-adic completion of a ring R as "a ring in the direction of R/I and a coalgebra in the transversal direction of R relative to R/I". In this sense, one can say that (the formulation of) the triangulated equivalence (5) is obtained by building (4) on top of (1), while (the idea of) the triangulated equivalence (6) is the result of bulding (2) on top of (3).

0.3. A number of other triangulated equivalences appearing in the present author's work can be described as mixtures of some of the equivalences (1–4). In particular, the equivalence between the coderived category of comodules and the contraderived category of contramodules over a pair of corings over associative rings in [13, Corollaries B.4.6 and B.4.10] is another way of building (2) on top of (3).

The equivalence between the conventional derived categories of semimodules and semicontramodules in [19, Theorem 3.3] is obtained by building (1) on top of (4). The equivalence between the semicoderived and the semicontraderived categories of modules in [16, Theorem 5.6] is the result of building (1) on top of (3).

The most deep and difficult in this series of triangulated equivalences is the *derived semimodule-semicontramodule correspondence* of [11, Section 0.3.7] (see the proof in a greater generality in [11, Section 6.3]). The application of this triangulated equivalence to the categories O and $\mathsf{O}^{\mathsf{ctr}}$ over Tate Lie algebras in [11, Corollary D.3.1] is of particular importance. This is the main result of the book [11]. It can be understood as obtainable by building (1) on top of (2).

Note that all the expressions like "can be obtained by" or "is the result of" above refer, at best, to the *formulations* of the mentioned theorems, rather than to their *proofs*. For example, the derived semimodule-semicontramodule correspondence, even in the generality of [11, Section 0.3.7], is a difficult theorem. There is no way to deduce it from the easy (2) and the trivial (1). The formulations of (2) and (1) serve as an inspiration and the guiding heuristics for arriving to the formulation of the derived semimodule-semicontramodule correspondence. Subsequently, one has to develop appropriate techniques leading to a proof.

0.4. More generally, beyond building things on top of one another, one may wish to develop notions providing a kind of "smooth interpolation" between various concepts. In particular, the notion of a discrete module over a topological ring can be viewed as interpolating between those of a module over a ring and a comodule over a coalgebra over a field, while the notion of a contramodule over a topological ring (see [11, Remark A.3] or [20]) interpolates between those of a module over a ring and a contramodule over a coalgebra over a field.

The notion of a *pseudo-dualizing complex* (known as a "semi-dualizing complex" in the literature) interpolates between those of a dualizing and a dedualizing complex. Similarly, the notions of a *pseudo-coderived* and a *pseudo-contraderived* category interpolate between those of the conventional derived category and the co- or contraderived category. The aim of this paper is to construct the related interpolations between the triangulated equivalences (1) and (3), or between the triangulated equivalences (2) and (4), or between the triangulated equivalences (5) and (6).

0.5. Three specific situations are considered separately in this paper. Firstly, let A and B be associative rings. A *pseudo-dualizing complex* $L^\bullet$ for the rings A and B is a finite complex of A-B-bimodules satisfying the following two conditions:

- (ii) the homothety maps $A \longrightarrow \mathrm{Hom}_{\mathsf{D^b}(\mathsf{mod}-B)}(L^\bullet, L^\bullet[*])$ and $B^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathsf{D^b}(A-\mathsf{mod})}(L^\bullet, L^\bullet[*])$ are isomorphisms of graded rings;
- (iii) as a complex of left A-modules, $L^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective A-modules, and similarly, as a complex of right B-modules, $L^\bullet$ is quasi-isomorphic to a bounded above complex of finitely generated projective B-modules.

This definition is obtained by dropping the fp-injectivity (or finite injective dimension) condition (i) from the definition of a *dualizing* complex of A-B-bimodules $D^\bullet$ in [16, Section 4] (see also the two related definitions in [16, Section 3]), removing the coherence conditions on the rings A and B, and rewriting the finite presentability condition (iii) accordingly.

For example, when the rings A and B coincide, the one-term complex $L^\bullet = A = B$ becomes the simplest example of a pseudo-dualizing complex. This is what can be called a *dedualizing complex* in this context. More generally, a "dedualizing complex of A-B-bimodules" is the same thing as a "(two-sided) tilting complex" $T^\bullet$ in the sense of Rickard's derived Morita theory [22, 23].

What in our terminology would be called "pseudo-dualizing complexes of modules over commutative Noetherian rings" were studied in the paper [2] and the references therein under some other names, such as "semi-dualizing complexes". What the authors call "semidualizing bimodules" for pairs of associative rings were considered in the paper [6]. We use this other terminology of our own in this paper, because in the context of the present author's work the prefix "semi" means something related but different and more narrow (as in [11] and [16, Sections 5–6]).

The main result of this paper in the context of pairs of associative rings A and B provides the following commutative diagram of triangulated functors associated with a pseudo-dualizing complex of A-B-bimodules $L^\bullet$:

$$
\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}_{\prime}^{L^\bullet}(A\text{–}\mathsf{mod}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}_{\prime\prime}^{L^\bullet}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(A\text{–}\mathsf{mod}) & \mathsf{D}(B\text{–}\mathsf{mod})
\end{tikzcd}
\tag{7}
$$

Here the vertical arrows are Verdier quotient functors, while the horizontal double lines are triangulated equivalences.

Thus $\mathsf{D}_{\prime}^{L^\bullet}(A\text{–}\mathsf{mod})$ and $\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod})$ are certain intermediate triangulated categories between the coderived category of left A-modules $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ and their conventional derived category $\mathsf{D}(A\text{–}\mathsf{mod})$. Similarly, $\mathsf{D}_{\prime\prime}^{L^\bullet}(B\text{–}\mathsf{mod})$ and $\mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod})$ are certain intermediate triangulated categories between the contraderived category of left B-modules $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ and their conventional derived category $\mathsf{D}(B\text{–}\mathsf{mod})$. These intermediate triangulated quotient categories depend on, and are determined by, the choice of a pseudo-dualizing complex $L^\bullet$ for a pair of associative rings A and B.

The triangulated category $\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod})$ is called the *lower pseudo-coderived category* of left A-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod})$ is called the *lower pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}_{\prime}^{L^\bullet}(A\text{–}\mathsf{mod})$ is called the *upper pseudo-coderived category* of left A-modules corresponding to $L^\bullet$. The triangulated category $\mathsf{D}_{\prime\prime}^{L^\bullet}(B\text{–}\mathsf{mod})$ is called the *upper pseudo-contraderived category* of left B-modules corresponding to $L^\bullet$. The choice of a pseudo-dualizing complex $L^\bullet$ also induces triangulated equivalences $\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod})$ and $\mathsf{D}_{\prime}^{L^\bullet}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}_{\prime\prime}^{L^\bullet}(A\text{–}\mathsf{mod})$ forming the commutative diagram (7).

In particular, when $L^\bullet = D^\bullet$ is a dualizing complex, i. e., the condition (i) of [16, Section 4] is satisfied, assuming additionally that all fp-injective left A-modules have finite injective dimensions, one has $\mathsf{D}_{\prime}^{L^\bullet}(A\text{–}\mathsf{mod}) = \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}_{\prime\prime}^{L^\bullet}(B\text{–}\mathsf{mod}) =$

$\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$, that is the upper two vertical arrows in the diagram (7) are isomorphisms of triangulated categories. The upper triangulated equivalence in the diagram (7) coincides with the one provided by [16, Theorem 4.5] in this case.

When $L^\bullet = A = B$, one has $\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod}) = \mathsf{D}(A\text{–}\mathsf{mod})$ and $\mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod}) = \mathsf{D}(B\text{–}\mathsf{mod})$, that is the lower two vertical arrows in the diagram (7) are isomorphisms of triangulated categories. The lower triangulated equivalence in the diagram (7) is just the identity isomorphism $\mathsf{D}(A\text{–}\mathsf{mod}) = \mathsf{D}(B\text{–}\mathsf{mod})$ is this case. More generally, the lower triangulated equivalence in the diagram (7) corresponding to a tilting complex $L^\bullet = T^\bullet$ recovers Rickard's derived Morita equivalence [22, Theorem 6.4], [23, Theorem 3.3].

0.6. A delicate point is that when $A = B = R$ is, e. g., a Gorenstein Noetherian commutative ring of finite Krull dimension, the ring R itself can be chosen as a dualizing complex of R-R-bimodules. So we are in both of the above-described situations at the same time. Still, the derived category of R-modules $\mathsf{D}(R\text{–}\mathsf{mod})$, the coderived category $\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod})$, and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod})$ are three quite different quotient categories of the homotopy category of (complexes of) R-modules $\mathsf{Hot}(R\text{–}\mathsf{mod})$. In this case, the commutative diagram (7) takes the form

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(R\text{–}\mathsf{mod}) \arrow[r, equal] & \mathsf{D}(R\text{–}\mathsf{mod})
\end{tikzcd}$$

More precisely, the two Verdier quotient functors $\mathsf{Hot}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod})$ and $\mathsf{Hot}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod})$ both factorize naturally through the Verdier quotient functor $\mathsf{Hot}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(R\text{–}\mathsf{mod})$ from the homotopy category onto the absolute derived category of R-modules $\mathsf{D}^{\mathsf{abs}}(R\text{–}\mathsf{mod})$. But the two resulting Verdier quotient functors $\mathsf{D}^{\mathsf{abs}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{abs}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod})$ do *not* form a commutative triangle with the equivalence $\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod})$. Rather, they are the two adjoint functors on the two sides to the fully faithful embedding of a certain (one and the same) triangulated subcategory in $\mathsf{D}^{\mathsf{abs}}(R\text{–}\mathsf{mod})$ [12, proof of Theorem 3.9].

This example shows that one cannot hope to have a procedure recovering the conventional derived category $\mathsf{D}(A\text{–}\mathsf{mod}) = \mathsf{D}(B\text{–}\mathsf{mod})$ from the dedualizing complex $L^\bullet = A = B$, and at the same time recovering the coderived category $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ from a dualizing complex $L^\bullet = D^\bullet$. Thus the distinction between the lower and and the upper pseudo-co/contraderived category constructions is in some sense inevitable.

0.7. Secondly, let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over a fixed field k. A *pseudo-dualizing complex* $\mathcal{L}^\bullet$ for the coalgebras $\mathcal{C}$ and $\mathcal{D}$ is a finite complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules satisfying the following two conditions:

(ii) the homothety maps $\mathcal{C}^* \longrightarrow \operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{comod}-\mathcal{D})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ and $\mathcal{D}^{*\mathrm{op}} \longrightarrow \operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathcal{C}-\mathsf{comod})}(\mathcal{L}^\bullet, \mathcal{L}^\bullet[*])$ are isomorphisms of graded rings;

(iii) as a complex of left $\mathcal{C}$-comodules, $\mathcal{L}^\bullet$ is quasi-isomophic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{C}$-comodules, and similarly, as a complex of right $\mathcal{D}$-comodules, $\mathcal{L}^\bullet$ is quasi-isomorphic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{D}$-comodules.

This definition is obtained by dropping the finite projective and contraflat dimension condition (i) from the definition of a *dedualizing* complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathcal{B}^\bullet$ in [19, Section 2], removing the cocoherence conditions on the coalgebras, and rewriting the finite copresentability condition (iii) accordingly. Here the quasi-finite cogeneratedness is a natural weakening of the finite cogeneratedness condition on comodules, having the advantage of being Morita-invariant [28].

For example, when the coalgebras $\mathcal{C}$ and $\mathcal{D}$ coincide, the one-term complex $\mathcal{L}^\bullet = \mathcal{C} = \mathcal{D}$ becomes the simplest example of a pseudo-dualizing complex. This is what can be called a *dualizing complex* in this context.

The main result of this paper in the setting of pairs of coalgebras $\mathcal{C}$ and $\mathcal{D}$ provides the following diagram of triangulated functors associated with a pseudo-dualizing complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathcal{L}^\bullet$:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(\mathcal{D}\text{–}\mathsf{contra}) \arrow[d, two heads] \\
\mathsf{D}_{\prime}^{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}_{\prime\prime}^{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra}) \arrow[d, two heads] \\
\mathsf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}''_{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra}) \arrow[d, two heads] \\
\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod}) & \mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})
\end{tikzcd} \tag{8}$$

Here, as above, the vertical arrows are Verdier quotient functors, while the horizontal double lines are triangulated equivalences.

Thus $\mathsf{D}_{\prime}^{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod})$ and $\mathsf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod})$ are certain intermediate triangulated categories between the coderived category of left $\mathcal{C}$-comodules $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ and their conventional derived category $\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod})$. Similarly, $\mathsf{D}_{\prime\prime}^{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ and $\mathsf{D}''_{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ are certain intermediate triangulated categories between the contraderived category of left $\mathcal{D}$-contramodules $\mathsf{D}^{\mathsf{ctr}}(\mathcal{D}\text{–}\mathsf{contra})$ and their conventional derived category $\mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})$. These intermediate triangulated quotient categories depend on, and are determined by, the choice of a pseudo-dualizing complex $\mathcal{L}^\bullet$.

The triangulated category $\mathsf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod})$ is called the *lower pseudo-coderived category* of left $\mathcal{C}$-comodules corresponding to the pseudo-dualizing complex $\mathcal{L}^\bullet$. The triangulated category $\mathsf{D}''_{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ is called the *lower pseudo-contraderived category* of left $\mathcal{D}$-contramodules corresponding to the pseudo-dualizing complex $\mathcal{L}^\bullet$. The triangulated category $\mathsf{D}_{\prime}^{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod})$ is called the *upper pseudo-coderived category* of left $\mathcal{C}$-comodules corresponding to $\mathcal{L}^\bullet$. The triangulated category $\mathsf{D}_{\prime\prime}^{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ is called the *upper pseudo-contraderived category* of left $\mathcal{D}$-contramodules corresponding to $\mathcal{L}^\bullet$. The choice of a pseudo-dualizing complex $\mathcal{L}^\bullet$ also induces triangulated equivalences $\mathsf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}''_{L^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ and $\mathsf{D}_{\prime}^{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) \simeq \mathsf{D}_{\prime\prime}^{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra})$ forming the commutative diagram (8).

In particular, when $L^\bullet = \mathcal{B}^\bullet$ is a dedualizing complex, i. e., the condition (i) of [19, Section 2] is satisfied, one has $\mathsf{D}'_{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) = \mathsf{D}(\mathcal{C}\text{–}\mathsf{comod})$ and $\mathsf{D}''_{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra}) = \mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})$, that is the lower two vertical arrows in the diagram (8) are isomorphisms of triangulated categories. The lower triangulated equivalence in the diagram (8) coincides with the one provided by [19, Theorem 2.6] in this case.

When $\mathcal{L}^\bullet = \mathcal{C} = \mathcal{D}$, one has $\mathsf{D}_{\prime}^{\mathcal{L}^\bullet}(\mathcal{C}\text{–}\mathsf{comod}) = \mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ and $\mathsf{D}_{\prime\prime}^{\mathcal{L}^\bullet}(\mathcal{D}\text{–}\mathsf{contra}) = \mathsf{D}^{\mathsf{ctr}}(\mathcal{D}\text{–}\mathsf{contra})$, that is the upper two vertical arrows in the diagram (8) are isomorphisms of triangulated categories. The upper triangulated equivalence in the diagram (8) is the *derived comodule-contramodule correspondence* of [11, Sections 0.2.6–0.2.7] and [12, Section 5.2] in this case.

0.8. Thirdly, let R be a commutative ring and $I \subset R$ be a finitely generated ideal. Let $\mathfrak{R} = \varprojlim_n R/I^n$ denote the I-adic completion of the ring R. Assume that the ideal I in R is *weakly proregular* in the sense of [24, 10]. A *pseudo-dualizing complex* $L^\bullet$ for the ideal $I \subset R$ is a finite complex of I-torsion R-modules satisfying the following two conditions:

- (ii) the homothety map $\mathfrak{R} \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}_{I\text{-tors}})}(L^\bullet, L^\bullet[*])$ is an isomorphism of graded rings;
- (iii) for any finite complex of finitely generated projective R-modules $K^\bullet$ with I-torsion cohomology modules, the complex of R-modules $\mathrm{Hom}_R(K^\bullet, L^\bullet)$ is quasi-isomorphic to a bounded above complex of finitely generated projective R-modules.

This definition is obtained by dropping the finite projective and contraflat dimension condition (i) and weakening the finiteness condition (iii) in the definition of a *dedualizing* complex of I-torsion R-modules $B^\bullet$ in [15, Section 5].

Notice that in the assumptions of [15, Section 4] the definition of a dedualizing complex of I-torsion R-modules given there also becomes a particular case of the above definition, as the condition (iii) in [15, Section 4] implies our condition (iii). Furthermore, our definition of a pseudo-dualizing complex of I-torsion R-modules can be also obtained by weakening the conditions in the definition of a *dualizing complex* of I-torsion R-modules in [13, Section C.1] (cf. the definition of a dualizing complex over a pair of pro-coherent topological rings in [13, Section D.2]).

The main result of this paper in the context of ideals I in commutative rings R provides provides the following diagram of triangulated functors associated with a pseudo-dualizing complex of I-torsion R-modules $L^\bullet$:

$$
\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}) \arrow[d, two heads] \\
\mathsf{D}_{\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}_{\prime\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}) \arrow[d, two heads] \\
\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}) \arrow[d, two heads] \\
\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) & \mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})
\end{tikzcd} \tag{9}
$$

Here, once again, the vertical arrows are Verdier quotient functors, while the horizontal double lines are triangulated equivalences.

Thus $\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$ is a certain intermediate triangulated category between the coderived category of I-torsion R-modules $\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$ and their conventional derived category $\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$. Similarly, $\mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ is a certain intermediate triangulated category between the contraderived category of I-contramodule R-modules $\mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ and their conventional derived category $\mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$. These intermediate triangulated quotient categories depend on, and are determined by, the choice of a pseudo-dualizing complex $L^\bullet$.

The triangulated category $\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$ is called the *lower pseudo-coderived category* of I-torsion R-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ is called the *lower pseudo-contraderived category* of I-contramodule R-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}_{\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$ is called the *upper pseudo-coderived category* of I-torsion R-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The triangulated category $\mathsf{D}_{\prime\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ is called the *upper pseudo-contraderived category* of I-contramodule R-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The choice of a pseudo-dualizing complex $L^\bullet$ also induces triangulated equivalences $\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \simeq \mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ and $\mathsf{D}_{\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) \simeq \mathsf{D}_{\prime\prime}^{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$ forming the commutative diagram (9).

In particular, when $L^\bullet = B^\bullet$ is a dedualizing complex, i. e., the conditions (i) and (iii) of [15, Section 5] or the conditions (i) and (iii) of [15, Section 4] are satisfied, one has $\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}}) = \mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{tors}})$ and $\mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}}) = \mathsf{D}(R\text{–}\mathsf{mod}_{I\text{-}\mathsf{ctra}})$; in other words, the lower two vertical arrows in the diagram (9) are isomorphisms of triangulated categories. The lower triangulated equivalence in the diagram (9) coincides with the one provided by [15, Theorem 5.10] or [15, Theorem 4.9] in this case.

When $L^\bullet = D^\bullet$ is a dualizing complex, i. e., the appropriate generalization of the definition in [13, Section C.1] or the appropriate particular case of the definition in [13, Section D.2] is applicable, one has $\mathsf{D}'_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-tors}}) = \mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}_{I\text{-tors}})$ and $\mathsf{D}''_{L^\bullet}(R\text{–}\mathsf{mod}_{I\text{-ctra}}) = \mathsf{D}^{\mathsf{ctr}}(R\text{–}\mathsf{mod}_{I\text{-ctra}})$; in other words, the upper two vertical arrows in the diagram (9) are isomorphisms of triangulated categories. The upper triangulated equivalence in the diagram (9) is the related generalization of the triangulated equivalence in [13, Theorem C.1.4] or a particular case of the triangulated equivalence in [13, Theorem D.2.7] in this case.

0.9. Before we finish this introduction, let us have a brief general discussion of *pseudo-derived categories*. This is a generic term for the pseudo-coderived and pseudo-contraderived categories. The following constructions of such triangulated categories were introduced for the purposes of the forthcoming paper [21].

Let A be an exact category (in Quillen's sense). We will say that a full subcategory $\mathsf{E} \subset \mathsf{A}$ is *coresolving* if E is closed under extensions and the passages to the cokernels of admissible monomorphisms in E, and every object of A is the source of an admissible monomorphism into an object of E. This definition slightly differs from that in [25, Section 2] in that we do not require E to be closed under direct summands (cf. [13, Section A.3]). Obviously, any coresolving subcategory E inherits an exact category structure from the ambient exact category A.

Let A be an exact category in which the functors of infinite direct sum are everywhere defined and exact. We refer to [11, Section 2.1], [13, Section A.1], or [15, Appendix A] for the definition of the *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathsf{A})$. A triangulated category D' is called a *pseudo-coderived* category of A if triangulated Verdier quotient functors $\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \longrightarrow \mathsf{D}' \longrightarrow \mathsf{D}(\mathsf{A})$ are given forming a commutative triangle with the canonical Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \longrightarrow \mathsf{D}(\mathsf{A})$ between the coderived and the conventional unbounded derived category of the exact category A.

Let $\mathsf{E} \subset \mathsf{A}$ be a coresolving subcategory closed under infinite direct sums. According to the dual version of [13, Proposition A.3.1(b)] (formulated explicitly in [16, Proposition 2.1]), the triangulated functor between the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{A})$ induced by the direct sum-preserving embedding of exact categories $\mathsf{E} \longrightarrow \mathsf{A}$ is an equivalence of triangulated categories. From the commutative diagram of triangulated functors

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}^{\mathsf{co}}(\mathsf{A}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}) \arrow[r] & \mathsf{D}(\mathsf{A})
\end{tikzcd}$$

one can see that the lower horizontal arrow is a Verdier quotient functor. Thus $\mathsf{D}' = \mathsf{D}(\mathsf{E})$ is a pseudo-coderived category of A.

Furthermore, let $\mathsf{E}_{\prime} \subset \mathsf{E}' \subset \mathsf{A}$ be two embedded coresolving subcategories, both closed under infinite direct sums in A. Then the canonical Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \longrightarrow \mathsf{D}(\mathsf{A})$ decomposes into a sequence of Verdier quotient functors

$$\mathsf{D}^{\mathsf{co}}(\mathsf{A}) \longrightarrow \mathsf{D}(\mathsf{E}_{\prime}) \longrightarrow \mathsf{D}(\mathsf{E}') \longrightarrow \mathsf{D}(\mathsf{A}).$$

In other words, when the full subcategory $\mathsf{E} \subset \mathsf{A}$ is expanded, the related preudo-coderived category $\mathsf{D}(\mathsf{E})$ gets deflated.

Notice that, as a coresolving subcategory closed under infinite direct sums $\mathsf{E} \subset \mathsf{A}$ varies, its conventional derived category behaves in quite different ways depending on the boundedness conditions. The functor $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{\prime}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E}')$ induced by the embedding $\mathsf{E}_{\prime} \longrightarrow \mathsf{E}'$ is fully faithful, the functor $\mathsf{D}^{+}(\mathsf{E}_{\prime}) \longrightarrow \mathsf{D}^{+}(\mathsf{E}')$ is a triangulated equivalence (by the assertion dual to [13, Proposition A.3.1(a)]), and the functor $\mathsf{D}(\mathsf{E}_{\prime}) \longrightarrow \mathsf{D}(\mathsf{E}')$ is a Verdier quotient functor.

Let B be another exact category. We will say that a full subcategory $\mathsf{F} \subset \mathsf{B}$ is *resolving* if F is closed under extensions and the passages to the kernels of admissible epimorphisms, an every object of A is the target of an admissible epimorphism from an object of F. Obviously, a resolving subcategory F inherits an exact category structure from the ambient exact category B.

Let B be an exact category in which the functors of infinite product are everywhere defined and exact. The definition of the *contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B})$ can be found in [11, Section 4.1], [13, Section A.1], or [15, Appendix A]. A triangulated category D'' is called a *pseudo-contraderived* category of B if Verdier quotient functors $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \longrightarrow \mathsf{D}'' \longrightarrow \mathsf{D}(\mathsf{B})$ are given forming a commutative triangle with the canonical Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \longrightarrow \mathsf{D}(\mathsf{B})$ between the contraderived and the convenional unbounded derived categories of the exact category B.

Let $\mathsf{F} \subset \mathsf{B}$ be a resolving subcategory closed under infinite products. According to [13, Proposition A.3.1(b)], the triangulated functor between the contraderived categories $\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{B})$ induced by the product-preserving embedding of exact categories $\mathsf{F} \longrightarrow \mathsf{B}$ is an equivalence of triangulated categories. From the commutative diagram of triangulated functors

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{F}) \arrow[r] & \mathsf{D}(\mathsf{B})
\end{tikzcd}$$

one can see that the lower horizontal arrow is a Verdier quotient functor. Thus $\mathsf{D}'' = \mathsf{D}(\mathsf{F})$ is a pseudo-contraderived category of B.

Let $\mathsf{F}_{\prime\prime} \subset \mathsf{F}'' \subset \mathsf{B}$ be two embedded resolving subcategories, both closed under infinite products in F. Then the canonical Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \longrightarrow \mathsf{D}(\mathsf{B})$ decomposes into a sequence of Verdier quotient functors

$$\mathsf{D}^{\mathsf{ctr}}(\mathsf{B}) \longrightarrow \mathsf{D}(\mathsf{F}_{\prime\prime}) \longrightarrow \mathsf{D}(\mathsf{F}'') \longrightarrow \mathsf{D}(\mathsf{B}).$$

In other words, when the full subcategory $\mathsf{F} \subset \mathsf{B}$ is expanded, the related pseudo-contraderived category $\mathsf{D}(\mathsf{F})$ gets deflated.

Once again, we notice that, as a resolving subcategory closed under infinite products $\mathsf{F} \subset \mathsf{B}$ varies, the behavior of its conventional derived category depends on the boundedness conditions. The functor $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{\prime\prime}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{F}'')$ is fully faithful, the functor $\mathsf{D}^{-}(\mathsf{F}_{\prime\prime}) \longrightarrow \mathsf{D}^{-}(\mathsf{F}'')$ is a triangulated equivalence [13, Proposition A.3.1(a)], and the functor $\mathsf{D}(\mathsf{F}_{\prime\prime}) \longrightarrow \mathsf{D}(\mathsf{F}'')$ is a Verdier quotient functor.

0.10. It remains to say a few words about where do the exact subcategories $\mathsf{E}_{\prime} \subset \mathsf{E}' \subset \mathsf{A}$ and $\mathsf{F}_{\prime\prime} \subset \mathsf{F}'' \subset \mathsf{B}$ come from in the three settings of Sections 0.5–0.8 (in the respective abelian categories A and B). The larger subcategories E' and F'' are our versions of what are called the *Auslander and Bass classes* in the literature [2, 5, 6]. Specifically, F'' is the Auslander class and E' is the Bass class.

The two full subcategories $\mathsf{E}_{\prime}$ and $\mathsf{F}_{\prime\prime}$ are certain natural smaller classes. One can say, in some approximate sense, that E' and F'' are the *maximal corresponding classes*, while $\mathsf{E}_{\prime}$ and $\mathsf{F}_{\prime\prime}$ are the *minimal corresponding classes* in the categories A and B.

More precisely, there is a natural single way to define the full subcategories $\mathsf{E}' \subset \mathsf{A}$ and $\mathsf{F}'' \subset \mathsf{B}$ when the pseudo-dualizing complex $L^\bullet$ or $\mathcal{L}^\bullet$ is a one-term complex. In the general case, we have two sequences of embedded subcategories $\mathsf{E}_{d_1} \subset \mathsf{E}_{d_1+1} \subset \mathsf{E}_{d_1+2} \subset \cdots \subset \mathsf{A}$ and $\mathsf{E}_{d_1} \subset \mathsf{F}_{d_1+1} \subset \mathsf{F}_{d_1+2} \subset \cdots \subset \mathsf{B}$ indexed by the large enough integers. All the subcategories E_{l_1} with varying index $l_1 = d_1, d_1+1, d_1+2, \ldots$ are "the same up to finite homological dimension", and so are all the subcategories F_{l_1}. Hence the triangulated functors $\mathsf{D}(\mathsf{E}_{l_1}) \longrightarrow \mathsf{D}(\mathsf{E}_{l_1+1})$ and $\mathsf{D}(\mathsf{F}_{l_1}) \longrightarrow \mathsf{D}(\mathsf{F}_{l_1+1})$ induced by the exact embeddings $\mathsf{E}_{l_1} \longrightarrow \mathsf{E}_{l_1+1}$ and $\mathsf{F}_{l_1} \longrightarrow \mathsf{F}_{l_1+1}$ are triangulated equivalences, so the pseudo-derived categories $\mathsf{D}'_{L^\bullet}(\mathsf{A}) = \mathsf{D}(\mathsf{E}_{l_1})$ and $\mathsf{D}''_{L^\bullet}(\mathsf{B}) = \mathsf{D}(\mathsf{F}_{l_1})$ do not depend on the choice of the number l_1.

The idea of the construction of the triangulated equivalence between the two lower pseudo-derived categories is that the functor $\mathsf{D}'_{L^\bullet}(\mathsf{A}) \longrightarrow \mathsf{D}''_{L^\bullet}(\mathsf{B})$ should be a version of $\mathbb{R}\operatorname{Hom}(L^\bullet, -)$, while the inverse functor $\mathsf{D}''_{L^\bullet}(\mathsf{B}) \longrightarrow \mathsf{D}'_{L^\bullet}(\mathsf{A})$ is a version of derived tensor product $L^\bullet \otimes^{\mathbb{L}} -$. The full subcategories $\mathsf{E}_{l_1} \subset \mathsf{A}$ and $\mathsf{F}_{l_1} \subset \mathsf{B}$ are defined by the conditions of uniform boundedness of cohomology of such Hom and tensor product complexes (hence dependence on a fixed bound l_1) and the composition of the two operations leading back to the original object.

The point is that the two functors $\mathbb{R}\operatorname{Hom}(L^\bullet, -)$ and $L^\bullet \otimes^{\mathbb{L}} -$ are mutually inverse when viewed as acting between the pseudo-derived categories $\mathsf{D}(\mathsf{E})$ and $\mathsf{D}(\mathsf{F})$, but objects of the pseudo-derived categories are complexes viewed up to a more delicate equivalence relation than in the conventional derived categories $\mathsf{D}(\mathsf{A})$ and $\mathsf{D}(\mathsf{B})$. When this subtlety is ignored, the two functors cease to be mutually inverse, generally speaking, and such mutual inverseness needs to be enforced as an additional adjustness restriction on the objects one is working with.

Similarly, there is a natural single way to define the full subcategories $\mathsf{E}_{\prime} \subset \mathsf{A}$ and $\mathsf{F}_{\prime\prime} \subset \mathsf{F}$ when the pseudo-dualizing complex $L^\bullet$ or $\mathcal{L}^\bullet$ is a one-term complex. In the general case, we have two sequences of embedded subcategories $\mathsf{E}^{d_2} \supset \mathsf{E}^{d_2+1} \supset$

$\mathsf{E}^{d_2+2} \supset \cdots$ in A and $\mathsf{F}^{d_2} \supset \mathsf{F}^{d_2+1} \supset \mathsf{F}^{d_2+2} \subset \cdots$ in B, indexed by large enough integers. As above, all the subcategories E^{l_2} with varying $l_2 = d_2$, d_2+1, d_2+2, … are "the same up to finite homological dimension", and so are all the subcategories F^{l_2}. Hence the triangulated functors $\mathsf{D}(\mathsf{E}^{l_2+1}) \longrightarrow \mathsf{D}(\mathsf{E}^{l_2})$ and $\mathsf{D}(\mathsf{F}^{l_2+1}) \longrightarrow \mathsf{D}(\mathsf{F}^{l_2})$ induced by the exact embeddings $\mathsf{E}^{l_2+1} \longrightarrow \mathsf{E}^{l_2}$ and $\mathsf{F}^{l_2+1} \longrightarrow \mathsf{F}^{l_2}$ are triangulated equivalences, so the pseudo-derived categories $\mathsf{D}^{L^\bullet}_{\prime}(\mathsf{A}) = \mathsf{D}(\mathsf{E}^{l_2})$ and $\mathsf{D}^{L^\bullet}_{\prime\prime}(\mathsf{B}) = \mathsf{D}(\mathsf{F}^{l_2})$ do not depend on the choice of the number l_2.

The triangulated equivalence between the two upper pseudo-derived categories is also provided by some versions of derived functors $\mathbb{R}\operatorname{Hom}(L^\bullet, -)$ and $L^\bullet \otimes^{\mathbb{L}} -$. The full subcategories $\mathsf{E}^{l_2} \subset \mathsf{A}$ and $\mathsf{F}^{l_2} \subset \mathsf{B}$ are produced by a kind of generation process. One starts from declaring that all the injectives in A belong to E^{l_2} and all the projectives in B belong to F^{l_2}. Then one proceeds with generating further objects of F^{l_2} by applying $\mathbb{R}\operatorname{Hom}(L^\bullet, -)$ to objects of E^{l_2}, and further objects of E^{l_2} by applying $L^\bullet \otimes^{\mathbb{L}} -$ to objects of F^{l_2}. One needs to resolve the complexes so obtained to produce objects of the abelian module categories, and the number l_2 indicates the length of the resolutions used. More objects are added to E^{l_2} and F^{l_2} to make these full subcategories closed under the operations mentioned above.

We refer to the main body of the paper for further details.

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1. Pairs of Associative Rings

1.1. **Strongly finitely presented modules.** Let A be an associative ring. We denote by $A\text{–}\mathsf{mod}$ the abelian category of left A-modules and by $\mathsf{mod}\text{–}A$ the abelian category of right A-modules. An A-module is said to be *strongly finitely presented* if it has a projective resolution consisting of finitely generated projective A-modules.

Lemma 1.1.1. *Let $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ be a short exact sequence of A-modules. Then whenever two of the three modules K, L, M are strongly finitely presented, so is the third one.*

Proof. If $P_\bullet \longrightarrow K$ and $R_\bullet \longrightarrow M$ are projective resolutions of the A-modules K and M, then there is a projective resolution $Q_\bullet \longrightarrow L$ of the A-module L with the terms $Q_i \simeq P_i \oplus R_i$. If $P_\bullet \longrightarrow K$ and $Q_\bullet \longrightarrow L$ are projective resolutions of the A-modules K and L, then there exists a morphism of complexes of A-modules $P_\bullet \longrightarrow Q_\bullet$ inducing the given morphism $K \longrightarrow L$ on the homology modules. The cone $R_\bullet$ of the morphism of complexes $P_\bullet \longrightarrow Q_\bullet$ is a projective resulution of the A-module M with the terms $R_i \simeq Q_i \oplus P_{i-1}$.

If $Q_\bullet \longrightarrow L$ and $R_\bullet \longrightarrow M$ are projective resolutions of the A-modules L and M, then there exists a morphism of complexes of A-modules $Q_\bullet \longrightarrow R_\bullet$ inducing the given morphism $L \longrightarrow M$ on the homology modules. The cocone $P'_\bullet$ of the morphism of complexes $Q_\bullet \longrightarrow R_\bullet$ is a bounded above complex of R-modules with the terms $P'_i = Q_i \oplus R_{i+1}$ and the only nonzero cohomology module $H_0(P'_\bullet) \simeq K$. Still, the complex $P'_\bullet$ is not yet literally a projective resolution of K, as its term $P'_{-1} \simeq R_0$ does not vanish. Setting $P_{-1} = 0$, $P_0 = \ker(P'_0 \to P'_{-1})$, and $P'_i = P_i$ for $i \geqslant 2$, one obtains a subcomplex $P_\bullet \subset P'_\bullet$ with a termwise split embedding $P_\bullet \longrightarrow P'_\bullet$ such that $P_\bullet$ is a projective resolution of the R-module K. $\square$

Abusing terminology, we will say that a bounded above complex of A-modules $M^\bullet$ is *strongly finitely presented* if it is quasi-isomorphic to a bounded above complex of finitely generated projective A-modules. Clearly, the class of all strongly finitely presented complexes is closed under shifts and cones in $\mathsf{D}^-(A\text{–}\mathsf{mod})$.

Lemma 1.1.2. (a) *Any bounded above complex of strongly finitely presented A-modules is strongly finitely presented.*

(b) *Let $M^\bullet$ be a complex of A-modules concentrated in the cohomological degrees $\leqslant n$, where n is a fixed integer. Then $M^\bullet$ is strongly finitely presented if and only if it is quasi-isomorphic to a complex of finitely generated projective A-modules concentrated in the cohomological degrees $\leqslant n$.*

(c) *Let $M^\bullet$ be a finite complex of A-modules concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$. Then $M^\bullet$ is strongly finitely presented if and only if it is quasi-isomorphic to a complex of A-modules $R^\bullet$ concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$ such that the A-modules R^m are finitely generated and projective for all $n_1 + 1 \leqslant m \leqslant n_2$, while the A-module R^{n_1} is strongly finitely presented.* $\square$

Let A and B be associative rings. A left A-module J is said to be *sfp-injective* if $\operatorname{Ext}^1_A(M, J) = 0$ for all strongly finitely presented left A-modules M, or equivalently, $\operatorname{Ext}^n_A(M, J) = 0$ for all strongly finitely presented left A-modules M and all $n > 0$. A left B-module P is said to be *sfp-flat* if $\operatorname{Tor}^B_1(N, P) = 0$ for all strongly finitely presented right B-modules N, or equivalently, $\operatorname{Tor}^B_n(N, P) = 0$ for all strongly finitely presented right B-modules N and all $n > 0$.

Lemma 1.1.3. (a) *The class of all sfp-injective left A-modules is closed under extensions, the cokernels of injective morphisms, filtered inductive limits, infinite direct sums, and infinite products.*

(b) *The class of all sfp-flat left B-modules is closed under extensions, the kernels of surjective morphisms, filtered inductive limits, infinite direct sums, and infinite products.* $\square$

Examples 1.1.4. (1) The following construction using strongly finitely presented modules provides some examples of pseudo-coderived categories of modules over an associative ring in the sense of Section 0.9 of the Introduction. Let A be an associative ring and S be a set of strongly finitely presented left A-modules. Denote by $\mathsf{E} \subset \mathsf{A} = A\text{–}\mathsf{mod}$ the full subcategory formed by all the left A-modules E such that $\operatorname{Ext}^i_A(S, E) = 0$ for all $S \in \mathsf{S}$ and all $i > 0$. Then the full subcategory $\mathsf{E} \subset A\text{–}\mathsf{mod}$ is a coresolving subcategory closed under infinite direct sums (and products). So the induced triangulated functor between the two coderived categories

$\mathsf{D^{co}}(\mathsf{E}) \longrightarrow \mathsf{D^{co}}(A\text{–}\mathsf{mod})$ is a triangulated equivalence by the dual version of [13, Proposition A.3.1(b)] (cf. [16, Proposition 2.1]). Thus the derived category $\mathsf{D}(\mathsf{E})$ of the exact category E is a pseudo-coderived category of the abelian category $A\text{–}\mathsf{mod}$, that is an intermediate quotient category between the coderived category $\mathsf{D^{co}}(A\text{–}\mathsf{mod})$ and the derived category $\mathsf{D}(A\text{–}\mathsf{mod})$, as explained in Section 0.9.

(2) In particular, if $\mathsf{S} = \varnothing$, then one has $\mathsf{E} = A\text{–}\mathsf{mod}$. On the other hand, if S is the set of all strongly finitely presented left A-modules, then the full subcategory $\mathsf{E} \subset A\text{–}\mathsf{mod}$ consists of all the sfp-injective modules. When the ring A is left coherent, all the finitely presented left A-modules are strongly finitely presented, and objects of the class E are called *fp-injective* left A-modules. In this case, the derived category $\mathsf{D}(\mathsf{E})$ of the exact category E is equivalent to the homotopy $\mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}})$ of the additive category of injective left A-modules [26, Theorem 6.12].

(3) More generally, for any associative ring A, the category $\mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}})$ can be called the *coderived category in the sense of Becker* [1] of the category of left A-modules. A complex of left A-modules $X^\bullet$ is called *coacyclic in the sense of Becker* if the complex of abelian groups $\mathrm{Hom}_A(X^\bullet, J^\bullet)$ is acyclic for any complex of injective left A-modules $J^\bullet$. According to [1, Proposition 1.3.6(2)], the full subcategories of complexes of injective modules and coacyclic complexes in the sense of Becker form a semiorthogonal decomposition of the homotopy category of left A-modules $\mathsf{Hot}(A\text{–}\mathsf{mod})$. According to [12, Theorem 3.5(a)], any coacyclic complex of left A-modules in the sense of [11, Section 2.1], [15, Appendix A] is also coacyclic in the sense of Becker. Thus $\mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}})$ occurs as an intermediate triangulated quotient category between $\mathsf{D^{co}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}(A\text{–}\mathsf{mod})$. So the coderived category in the sense of Becker is a pseudo-coderived category in our sense.

We do *not* know whether the Verdier quotient functor $\mathsf{D^{co}}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}})$ is a triangulated equivalence (or, which is the same, the natural fully faithful triangulated functor $\mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}}) \longrightarrow \mathsf{D^{co}}(A\text{–}\mathsf{mod})$ is a triangulated equivalence) for an arbitrary associative ring A. Partial results in this direction are provided by [12, Theorem 3.7] and [16, Theorem 2.4] (see also Proposition 1.6.1 below).

Examples 1.1.5. (1) The following dual version of Example 1.1.4(1) provides some examples of pseudo-contraderived categories of modules. Let B be an associative ring and S be a set of strongly finitely presented right B-modules. Denote by $\mathsf{F} \subset \mathsf{B} = B\text{–}\mathsf{mod}$ the full subcategory formed by all the left B-modules F such that $\mathrm{Tor}_i^B(S,F) = 0$ for all $S \in \mathsf{S}$ and $i > 0$. Then the full subcategory $\mathsf{F} \subset B\text{–}\mathsf{mod}$ is a resolving subcategory closed under infinite products (and direct sums). So the induced triangulated functor between the two contraderived categories $\mathsf{D^{ctr}}(\mathsf{F}) \longrightarrow \mathsf{D^{ctr}}(B\text{–}\mathsf{mod})$ is a triangulated equivalence by [13, Proposition A.3.1(b)]. Thus the derived category $\mathsf{D}(\mathsf{F})$ of the exact category F is a pseudo-contraderived category of the abelian category $B\text{–}\mathsf{mod}$, that is an intermediate quotient category between the contraderived category $\mathsf{D^{ctr}}(B\text{–}\mathsf{mod})$ and the derived category $\mathsf{D}(B\text{–}\mathsf{mod})$, as it was explained in Section 0.9 of the Introduction.

(2) In particular, if $\mathsf{S} = \varnothing$, then one has $\mathsf{F} = B\text{–}\mathsf{mod}$. On the other hand, if S is the set of all strongly finitely presented right B-modules, then the full subcategory $\mathsf{F} \subset B\text{–}\mathsf{mod}$ consists of all the sfp-flat modules. When the ring B is right coherent, all the sfp-flat left B-modules are flat and F is the full subcategory of all flat left B-modules. For any associative ring B, the derived category of exact category of flat left B-modules is equivalent to the homotopy category $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ of the additive category of projective left B-modules [9, Proposition 8.1 and Theorem 8.6].

(3) For any associative ring B, the category $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ can be called the *contraderived category in the sense of Becker* of the category of left B-modules. A complex of left B-modules $Y^\bullet$ is called *contraacyclic in the sense of Becker* if the complex of abelian groups $\mathrm{Hom}_B(P^\bullet, Y^\bullet)$ is acyclic for any complex of projective left B-modules $P^\bullet$. According to [1, Proposition 1.3.6(1)], the full subcategories of contraacyclic complexes in the sense of Becker and complexes of projective modules form a semiorthogonal decomposition of the homotopy category of left B-modules $\mathsf{Hot}(B\text{–}\mathsf{mod})$. According to [12, Theorem 3.5(b)], any contraacyclic complex of left B-modules in the sense of [11, Section 4.1], [15, Appendix A] is also contraacyclic in the sense of Becker. Thus $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ occurs as an intermediate triangulated quotient category between $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ and $\mathsf{D}(B\text{–}\mathsf{mod})$. So the contraderived category in the sense of Becker is a pseudo-contraderived category in our sense.

We do *not* know whether the Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ is a triangulated equivalence (or, which is the same, the natural fully faithful triangulated functor $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ is a triangulated equivalence) for an arbitrary associative ring B. A partial result in this direction is provided by [12, Theorem 3.8] (cf. [16, Theorem 4.4]; see also Proposition 1.6.2 below).

1.2. **Auslander and Bass classes.** We recall the definition of a pseudo-dualizing complex of bimodules from Section 0.5 of the Introduction. Let A and B be associative rings.

A *pseudo-dualizing complex* $L^\bullet$ for the rings A and B is a finite complex of A-B-bimodules satisfying the following two conditions:

- (ii) the homothety maps $A \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\text{–}B)}(L^\bullet, L^\bullet[*])$ and $B^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})}(L^\bullet, L^\bullet[*])$ are isomorphisms of graded rings;
- (iii) the complex $L^\bullet$ is strongly finitely presented as a complex of left A-modules and as a complex of right B-modules.

Here the condition (iii) refers to the definition of a strongly finitely presented complex of modules in Section 1.1. The complex $L^\bullet$ is viewed as an object of the bounded derived category of A-B-bimodules $\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod}\text{–}B)$.

We will use the following simplified notation: given two complexes of left A-modules $M^\bullet$ and $N^\bullet$, we denote by $\mathrm{Ext}^n_A(M^\bullet, N^\bullet)$ the groups $H^n\mathbb{R}\mathrm{Hom}_A(M^\bullet, N^\bullet) = \mathrm{Hom}_{\mathsf{D}(A\text{–}\mathsf{mod})}(M^\bullet, N^\bullet[n])$. Given a complex of right B-modules $N^\bullet$ and a complex of left B-modules $M^\bullet$, we denote by $\mathrm{Tor}^B_n(N^\bullet, M^\bullet)$ the groups $H^{-n}(N^\bullet \otimes^{\mathbb{L}}_B M^\bullet)$.

The tensor product functor $L^\bullet \otimes_B -\colon \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ acting between the unbounded homotopy categories of left B-modules and left A-modules is left adjoint to the functor $\mathrm{Hom}_A(L^\bullet,-)\colon \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod})$. Using homotopy flat and homotopy injective resolutions of the second arguments, one constructs the derived functors $L^\bullet \otimes_B^{\mathbb{L}} -\colon \mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ and $\mathbb{R}\mathrm{Hom}_A(L^\bullet,-)\colon \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ acting between the (conventional) unbounded derived categories of left A-modules and left B-modules. As always with the left and right derived functors (e. g., in the sense of Deligne [3, 1.2.1–2]), the functor $L^\bullet \otimes_B^{\mathbb{L}} -$ is left adjoint to the functor $\mathbb{R}\mathrm{Hom}_A(L^\bullet,-)$ [11, Lemma 8.3].

Suppose that the finite complex $L^\bullet$ is situated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$. Then one has $\mathrm{Ext}_A^n(L^\bullet,J) = 0$ for all $n > d_1$ and all sfp-injective left A-modules J. Similarly, one has $\mathrm{Tor}_n^B(L^\bullet,P) = 0$ for all $n > d_1$ and all sfp-flat left B-modules P. Choose an integer $l_1 \geqslant d_1$ and consider the following full subcategories in the abelian categories of left A-modules and left B-modules:

- $\mathsf{E}_{l_1} = \mathsf{E}_{l_1}(L^\bullet) \subset A\text{–}\mathsf{mod}$ is the full subcategory consisting of all the A-modules E such that $\mathrm{Ext}_A^n(L^\bullet,E) = 0$ for all $n > l_1$ and the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_A(L^\bullet,E) \longrightarrow E$ is an isomorphism in $\mathsf{D}^-(A\text{–}\mathsf{mod})$;
- $\mathsf{F}_{l_1} = \mathsf{F}_{l_1}(L^\bullet) \subset B\text{–}\mathsf{mod}$ is the full subcategory consisting of all the B-modules F such that $\mathrm{Tor}_n^B(L^\bullet,F) = 0$ for all $n > l_1$ and the adjunction morphism $F \longrightarrow \mathbb{R}\mathrm{Hom}_A(L^\bullet,\, L^\bullet \otimes_B^{\mathbb{L}} F)$ is an isomorphism in $\mathsf{D}^+(B\text{–}\mathsf{mod})$.

Clearly, for any $l_1'' \geqslant l_1' \geqslant d_1$, one has $\mathsf{E}_{l_1'} \subset \mathsf{E}_{l_1''} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}_{l_1'} \subset \mathsf{F}_{l_1''} \subset B\text{–}\mathsf{mod}$. The category F_{l_1} is our version of what is called the *Auslander class* in [2, 5, 6], while the category E_{l_1} is our version of the *Bass class*.

Lemma 1.2.1. (a) *The full subcategory $\mathsf{E}_{l_1} \subset A\text{–}\mathsf{mod}$ is closed under the cokernels of injective morphisms, extensions, and direct summands.*

(b) *The full subcategory $\mathsf{F}_{l_1} \subset B\text{–}\mathsf{mod}$ is closed under the kernels of surjective morphisms, extensions, and direct summands.* ☐

Lemma 1.2.2. (a) *The full subcategory $\mathsf{E}_{l_1} \subset A\text{–}\mathsf{mod}$ contains all the injective left A-modules.*

(b) *The full subcategory $\mathsf{F}_{l_1} \subset B\text{–}\mathsf{mod}$ contains all the flat left B-modules.*

Proof. Part (a): let $'L^\bullet$ be a bounded above complex of finitely generated projective right B-modules endowed with a quasi-isomorphism of complexes of right B-modules $'L^\bullet \longrightarrow L^\bullet$. Then the complex $'L^\bullet \otimes_B \mathrm{Hom}_A(L^\bullet,J)$ computes $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_A(L^\bullet,J)$ as an object of the derived category of abelian groups for any injective left A-module J. Now we have an isomorphism of complexes of abelian groups $'L^\bullet \otimes_B \mathrm{Hom}_A(L^\bullet,J) \simeq \mathrm{Hom}_A(\mathrm{Hom}_{B^{\mathrm{op}}}('L^\bullet,L^\bullet),J)$ and a quasi-isomorphism of complexes of left A-modules $A \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}('L^\bullet,L^\bullet)$, implying that the natural morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_A(L^\bullet,J) \longrightarrow J$ is an isomorphism in the derived category of abelian groups, hence also in the derived category of left A-modules.

Part (b): let $''L^\bullet$ be a bounded above complex of finitely generated projective right A-modules endowed with a quasi-isomorphism of complexes of right A-modules $''L^\bullet \longrightarrow L^\bullet$. Then the complex $\mathrm{Hom}_A(''L^\bullet,\, L^\bullet \otimes_B P)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet,\ L^\bullet\otimes_B^{\mathbb{L}} P)$ for any flat left B-module P. Now we have an isomorphism of complexes of abelian groups $\operatorname{Hom}_A(''L^\bullet,\ L^\bullet\otimes_B P)\simeq \operatorname{Hom}_A(''L^\bullet,L^\bullet)\otimes_B P$ and a quasi-isomorphism of complexes of right B-modules $B\longrightarrow \operatorname{Hom}_A(''L^\bullet,L^\bullet)$. $\square$

If $L^\bullet$ is finite complex of A-B-modules that are strongly finitely presented as left A-modules and as right B-modules, then the class E_{l_1} contains also all the sfp-injective left A-modules and the class F_{l_1} contains all the sfp-flat left B-modules.

Lemma 1.2.3. (a) *The full subcategory $\mathsf{E}_{l_1}\subset A\text{–}\mathsf{mod}$ is closed under infinite direct sums and products.*

(b) *The full subcategory $\mathsf{F}_{l_1}\subset B\text{–}\mathsf{mod}$ is closed under infinite direct sums and products.*

Proof. The functor $\mathbb{R}\operatorname{Hom}_A(L^\bullet,-)\colon \mathsf{D}(A\text{–}\mathsf{mod})\longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ preserves infinite direct sums of uniformly bounded below families of complexes and infinite products of arbitrary families of complexes. The functor $L^\bullet\otimes_B^{\mathbb{L}}-\colon \mathsf{D}(B\text{–}\mathsf{mod})\longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ preserves infinite products of uniformly bounded above families of complexes and infinite direct sums of arbitrary families of complexes. These observations imply both the assertions (a) and (b). $\square$

The full subcategories $\mathsf{E}_{l_1}\subset A\text{–}\mathsf{mod}$ and $\mathsf{F}_{l_1}\subset B\text{–}\mathsf{mod}$ inherit exact category structures from the abelian categories $A\text{–}\mathsf{mod}$ and $B\text{–}\mathsf{mod}$. It follows from Lemma 1.2.1 or 1.2.2 that the induced triangulated functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1})\longrightarrow \mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{l_1})\longrightarrow \mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$ are fully faithful. The following lemma describes their essential images.

Lemma 1.2.4. (a) *Let $M^\bullet$ be a complex of left A-modules concentrated in the cohomological degrees $-n_1\leqslant m\leqslant n_2$. Then $M^\bullet$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-n_1\leqslant m\leqslant n_2$ with the terms belonging to the full subcategory $\mathsf{E}_{l_1}\subset A\text{–}\mathsf{mod}$ if and only if $\operatorname{Ext}_A^n(L^\bullet,M^\bullet)=0$ for $n>n_2+l_1$ and the adjunction morphism $L^\bullet\otimes_B^{\mathbb{L}}\mathbb{R}\operatorname{Hom}_A(L^\bullet,M^\bullet)\longrightarrow M^\bullet$ is an isomorphism in $\mathsf{D}^-(A\text{–}\mathsf{mod})$.*

(b) *Let $N^\bullet$ be a complex of left B-modules concentrated in the cohomological degrees $-n_1\leqslant m\leqslant n_2$. Then $N^\bullet$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-n_1\leqslant m\leqslant n_2$ with the terms belonging to the full subcategory $\mathsf{F}_{l_1}\subset B\text{–}\mathsf{mod}$ if and only if $\operatorname{Tor}_n^B(L^\bullet,N^\bullet)=0$ for $n>n_1+l_1$ and the adjunction morphism $N^\bullet\longrightarrow \mathbb{R}\operatorname{Hom}_A(L^\bullet,\ L^\bullet\otimes_B^{\mathbb{L}}N^\bullet)$ is an isomorphism in $\mathsf{D}^+(B\text{–}\mathsf{mod})$.*

Proof. Standard argument based on Lemmas 1.2.1–1.2.2. $\square$

Thus the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1})\subset \mathsf{D}(A\text{–}\mathsf{mod})$ consists of all the complexes of left A-modules $M^\bullet$ with bounded cohomology such that the complex $\mathbb{R}\operatorname{Hom}_A(L^\bullet,M^\bullet)$ also has bounded cohomology and the adjunction morphism $L^\bullet\otimes_B^{\mathbb{L}}\mathbb{R}\operatorname{Hom}_A(L^\bullet,M^\bullet)\longrightarrow M^\bullet$ is an isomorphism. Similarly, the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{l_1})\subset \mathsf{D}(A\text{–}\mathsf{mod})$ consists of all the complexes of left B-modules $N^\bullet$ with bounded cohomology such that the complex $L^\bullet\otimes_B^{\mathbb{L}}N^\bullet$ also has bounded cohomology and the adjunction morphism $N^\bullet\longrightarrow \mathbb{R}\operatorname{Hom}_A(L^\bullet,\ L^\bullet\otimes_B^{\mathbb{L}}N^\bullet)$ is an isomorphism.

These full subcategories are usually called the *derived Bass* and *Auslander classes.* As any pair of adjoint functors restricts to an equivalence between the full subcategories of all objects whose adjunction morphisms are isomorphisms, the functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet,-)$ and $L^\bullet\otimes_B^{\mathbb{L}}-$ restrict to a triangulated equivalence

$$\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1})\simeq\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{l_1}). \tag{10}$$

Lemma 1.2.5. (a) *For any A-module $E\in\mathsf{E}_{l_1}$, the object $\mathbb{R}\operatorname{Hom}_A(L^\bullet,E)\in\mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$ can be represented by a complex of B-modules concentrated in the cohomological degrees $-d_2\leqslant m\leqslant l_1$ with the terms belonging to F_{l_1}.*

(b) *For any B-module $F\in\mathsf{F}_{l_1}$, the object $L^\bullet\otimes_B^{\mathbb{L}}F\in\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$ can be represented by a complex of A-modules concentrated in the cohomological degrees $-l_1\leqslant m\leqslant d_2$ with the terms belonging to E_{l_1}.*

Proof. Follows from Lemma 1.2.4. □

We refer to [25, Section 2] and [13, Section A.5] for discussions of the *coresolution dimension* of objects of an exact category A with respect to its coresolving subcategory E and the *resolution dimension* of objects of an exact category B with respect to its resolving subcategory F (called the *right* E*-homological dimension* and the *left* F*-homological dimension* in [13]).

Lemma 1.2.6. (a) *For any integers $l_1''\geqslant l_l'\geqslant d_1$, the full subcategory $\mathsf{E}_{l_1''}\subset A\text{–}\mathsf{mod}$ consists precisely of all the left A-modules whose $\mathsf{E}_{l_1'}$-coresolution dimension does not exceed $l_1''-l_1'$.*

(b) *For any integers $l_1''\geqslant l_1'\geqslant d_1$, the full subcategory $\mathsf{F}_{l_1''}\subset B\text{–}\mathsf{mod}$ consists precisely of all the left B-modules whose $\mathsf{F}_{l_1'}$-resolution dimension does not exceed $l_1''-l_1'$.* □

Remark 1.2.7. In particular, it follows from Lemmas 1.2.2 and 1.2.6 that, for any $n\geqslant 0$, all the left A-modules of injective dimension not exceeding n belong to E_{d_1+n} and all the left B-modules of flat dimension not exceeding n belong to F_{d_1+n}.

We refer to [13, Section A.1] or [15, Appendix A] for the definitions of exotic derived categories occuring in the next proposition.

Proposition 1.2.8. *For any $l_1''\geqslant l_1'\geqslant d_1$ and any conventional or exotic derived category symbol $\star=\mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs, the exact embedding functors $\mathsf{E}_{l_1'}\longrightarrow\mathsf{E}_{l_1''}$ and $\mathsf{F}_{l_1'}\longrightarrow\mathsf{F}_{l_1''}$ induce triangulated equivalences*

$$\mathsf{D}^\star(\mathsf{E}_{l_1'})\simeq\mathsf{D}^\star(\mathsf{E}_{l_1''})\quad\textit{and}\quad\mathsf{D}^\star(\mathsf{F}_{l_1'})\simeq\mathsf{D}^\star(\mathsf{F}_{l_1''}).$$

Proof. In view of [13, Proposition A.5.6], the assertions follow from Lemma 1.2.6. □

In particular, the unbounded derived category $\mathsf{D}(\mathsf{E}_{l_1})$ is the same for all $l_1\geqslant d_1$ and the unbounded derived category $\mathsf{D}(\mathsf{F}_{l_1})$ is the same for all $l_1\geqslant d_1$.

As it was explained in Section 0.9 of the Introduction, it follows from Lemmas 1.2.1–1.2.3 by virtue of [13, Proposition A.3.1(b)] that the natural Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})\longrightarrow\mathsf{D}(A\text{–}\mathsf{mod})$ factorizes into two Verdier quotient functors $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{E}_{l_1}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$, and the natural Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ factorizes into two Verdier quotient functors $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{F}_{l_1}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$. In other words, the triangulated category $\mathsf{D}(\mathsf{E}_{l_1})$ is a pseudo-coderived category of left A-modules and the triangulated category $\mathsf{D}(\mathsf{F}_{l_1})$ is a pseudo-contraderived category of left B-modules.

These are called the *lower pseudo-coderived category* of left A-modules and the *lower pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The notation is

$$\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{E}_{l_1}) \quad \text{and} \quad \mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{F}_{l_1}).$$

The next theorem provides, in particular, a triangulated equivalence between the lower pseudo-coderived and the lower pseudo-contraderived category,

$$\mathsf{D}'_{L^\bullet}(A\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{E}_{l_1}) \simeq \mathsf{D}(\mathsf{F}_{l_1}) = \mathsf{D}''_{L^\bullet}(B\text{–}\mathsf{mod}).$$

Theorem 1.2.9. *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, *or* abs, *there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{E}_{l_1}) \simeq \mathsf{D}^\star(\mathsf{F}_{l_1})$ *provided by (appropriately defined) mutually inverse functors* $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ *and* $L^\bullet \otimes_B^{\mathbb{L}} -$.

Proof. This is a particular case of Theorem 1.3.2 below. ☐

1.3. **Abstract corresponding classes.** More generally, suppose that we are given two full subcategories $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ satisfying the following conditions (for some fixed integers l_1 and l_2):

- (I) the full subcategory $\mathsf{E} \subset A\text{–}\mathsf{mod}$ is closed under extensions and the cokernels of injective morphisms, and contains all the injective left A-modules;
- (II) the full subcategory $\mathsf{F} \subset B\text{–}\mathsf{mod}$ is closed under extensions and the kernels of surjective morphisms, and contains all the projective left B-modules;
- (III) for any A-module $E \in \mathsf{E}$, the object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^+(B\text{–}\mathsf{mod})$ can be represented by a complex of B-modules concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1$ with the terms belonging to F;
- (IV) for any B-module $F \in \mathsf{F}$, the object $L^\bullet \otimes_B^{\mathbb{L}} F \in \mathsf{D}^-(A\text{–}\mathsf{mod})$ can be represented by a complex of A-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant l_2$ with the terms belonging to E.

One can see from the conditions (I) and (III), or (II) and (IV), that $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$ if $H^{-d_1}(L^\bullet) \neq 0 \neq H^{d_2}(L^\bullet)$. According to Lemmas 1.2.1, 1.2.2, and 1.2.5, the two classes $\mathsf{E} = \mathsf{E}_{l_1}$ and $\mathsf{F} = \mathsf{F}_{l_1}$ satisfy the conditions (I–IV) with $l_2 = d_2$.

The following lemma, providing a kind of converse implication, can be obtained as a byproduct of the proof of Theorem 1.3.2 below (based on the arguments in the appendix). It is somewhat counterintuitive, claiming that the adjunction isomorphism conditions on the modules in the classes E and F, which were necessary in the context of the previous Section 1.2, follow from the conditions (I–IV) in our present context. So we prefer to present a separate explicit proof.

Lemma 1.3.1. (a) *For any A-module $E \in \mathsf{E}$, the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \longrightarrow E$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$.*

(b) *For any B-module $F \in \mathsf{F}$, the adjunction morphism $F \longrightarrow \mathbb{R}\operatorname{Hom}_A(L^\bullet,$ $L^\bullet \otimes_B^{\mathbb{L}} F)$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$.*

Proof. We will prove part (a); the proof of part (b) is similar. Specifically, let $0 \longrightarrow E \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots$ be an exact sequence of left A-modules with $E \in \mathsf{E}$ and $K^i \in \mathsf{E}$ for all $i \geqslant 0$. Suppose that the adjunction morphisms $L^\bullet \otimes_B^{\mathbb{L}}$ $\mathbb{R}\operatorname{Hom}_A(L^\bullet, K^i) \longrightarrow K^i$ are isomorphisms in $\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$ for all $i \geqslant 0$. We will show that the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \longrightarrow E$ is also an isomorphism in this case. As injective left A-modules belong to E by the condition (I), the desired assertion will then follow from Lemma 1.2.2(a).

Let Z^i denote the kernel of the differential $K^i \longrightarrow K^{i+1}$; in particular, $Z^0 = E$. The key observation is that, according to the condition (I), one has $Z^i \in \mathsf{E}$ for all $i \geqslant 0$. For every $i \geqslant 0$, choose a coresolution of the short exact sequence $0 \longrightarrow Z^i \longrightarrow K^i \longrightarrow Z^{i+1}$ by short exact sequences of injective left A-modules $0 \longrightarrow Y^{i,j} \longrightarrow J^{i,j} \longrightarrow Y^{i+1,j} \longrightarrow 0$, $j \geqslant 0$. Applying the functor $\operatorname{Hom}_A(L^\bullet, -)$ to the complexes of left A-modules $J^{i,\bullet}$ and $Y^{i,\bullet}$, we obtain short exact sequences of complexes of left B-modules $0 \longrightarrow G^{i,\bullet} \longrightarrow F^{i,\bullet} \longrightarrow G^{i+1,\bullet} \longrightarrow 0$, where $G^{i,\bullet} = \operatorname{Hom}_A(L^\bullet, Y^{i,\bullet})$ and $F^{i,\bullet} = \operatorname{Hom}_A(L^\bullet, J^{i,\bullet})$. According the condition (III), each complex $G^{i,\bullet}$ and $F^{i,\bullet}$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1$ with the terms belonging to F.

For every $i \geqslant 0$, choose complexes projective left B-modules $Q^{i,\bullet}$ and $P^{i,\bullet}$, concentrated in the cohomological degrees $\leqslant l_1$ and endowed with quasi-isomorphisms of complexes of left B-modules $Q^{i,\bullet} \longrightarrow G^{i,\bullet}$ and $P^{i,\bullet} \longrightarrow F^{i,\bullet}$ so that there are short exact sequences of complexes of left B-modules $0 \longrightarrow Q^{i,\bullet} \longrightarrow P^{i,\bullet} \longrightarrow Q^{i+1,\bullet} \longrightarrow 0$ and the whole diagram is commutative. Applying the functor $L^\bullet \otimes_B -$ to the complexes of lef B-modules $P^{i,\bullet}$ and $Q^{i,\bullet}$, we obtain short exact sequences of complexes of left A-modules $0 \longrightarrow N^{i,\bullet} \longrightarrow M^{i,\bullet} \longrightarrow N^{i+1,\bullet} \longrightarrow 0$, where $N^{i,\bullet} = L^\bullet \otimes_B Q^{i,\bullet}$ and $M^{i,\bullet} = L^\bullet \otimes_B P^{i,\bullet}$. It follows from the condition (IV) that each complex $M^{i,\bullet}$ and $N^{i,\bullet}$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$. In particular, the cohomology modules of the complexes $M^{i,\bullet}$ and $N^{i,\bullet}$ are concentrated in the degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$.

Applying the functors of two-sided canonical truncation $\tau_{\geqslant -l_1-l_2}\tau_{\leqslant l_1+l_2}$ to the complexes $M^{i,\bullet}$ and $N^{i,\bullet}$, we obtain short exact sequences $0 \longrightarrow {}'N^{i,\bullet} \longrightarrow {}'M^{i,\bullet} \longrightarrow {}'N^{i+1,\bullet} \longrightarrow 0$ of complexes whose terms are concentrated in the cohomological degrees $-l_1 - l_2 \leqslant m \leqslant l_1 + l_2$. Similarly, applying the functors of canonical truncation $\tau_{\leqslant l_1+l_2}$ to the complexes $J^{i,\bullet}$ and $Y^{i,\bullet}$, we obtain short exact sequences $0 \longrightarrow {}'Y^{i,\bullet} \longrightarrow {}'J^{i,\bullet} \longrightarrow {}'Y^{i+1,\bullet} \longrightarrow 0$ of complexes whose terms are concentrated in the cohomological degrees $0 \leqslant m \leqslant l_1 + l_2$. Now we have two morphisms of bicomplexes ${}'M^{i,j} \longrightarrow {}'J^{i,j}$ and $K^i \longrightarrow {}'J^{i,j}$, which are both quasi-isomorphisms of finite complexes along the grading j at every fixed degree i, by assumption. Furthermore, we have two morphisms of bicomplexes ${}'N^{0,j} \longrightarrow {}'M^{i,j}$ and ${}'Y^{0,j} \longrightarrow {}'J^{i,j}$, which are both quasi-isomorphisms along the grading i at every fixed degree j, by construction. We also have a quasi-isomorphism $E \longrightarrow {}'Y^{0,\bullet}$.

Passing to the total complexes, we see that the morphism of complexes $'N^{0,\bullet} \longrightarrow {}'Y^{0,\bullet}$ is a quasi-isomorphism, because so are the morphisms $'N^{0,\bullet} \longrightarrow {}'M^{\bullet,\bullet}$, $'M^{\bullet,\bullet} \longrightarrow {}'J^{\bullet,\bullet}$, and $'Y^{0,\bullet} \longrightarrow {}'J^{\bullet,\bullet}$ in a commutative square. This proves that the adjunction morphism $L^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \longrightarrow E$ is an isomorphism in the derived category. □

Assuming that $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$, it is now clear from the conditions (III–IV) and Lemma 1.3.1 that one has $\mathsf{E} \subset \mathsf{E}_{l_1}$ and $\mathsf{F} \subset \mathsf{F}_{l_1}$ for any two classes of objects $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ satisfying (I–IV). Furthermore, it follows from the conditions (I–II) that the triangulated functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$ induced by the exact embeddings $\mathsf{E} \longrightarrow A\text{–}\mathsf{mod}$ and $\mathsf{F} \longrightarrow B\text{–}\mathsf{mod}$ are fully faithful. Hence so are the triangulated functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E}_{l_1})$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{F}_{l_1})$. In view of the conditions (III–IV), we can conclude that equivalence (10) restricts to a triangulated equivalence

$$\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{F}). \tag{11}$$

The following theorem is the main result of Section 1.

Theorem 1.3.2. *Let $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ be a pair of full subcategories of modules satisfying the conditions (I–IV) for a pseudo-dualizing complex of A-B-bimodules $L^\bullet$. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs, there is a triangulated equivalence $\mathsf{D}^\star(\mathsf{E}) \simeq \mathsf{D}^\star(\mathsf{F})$ provided by (appropriately defined) mutually inverse functors $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ and $L^\bullet \otimes_B^{\mathbb{L}} -$.*

Here, in the case $\star = \mathsf{co}$ it is assumed that the full subcategories $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ are closed under infinite direct sums, while in the case $\star = \mathsf{ctr}$ it is assumed that these two full subcategories are closed under infinite products.

Proof. This is a straightforward application of the results of the appendix. In the context of the appendix, set

$$\mathsf{A} = A\text{–}\mathsf{mod} \supset \mathsf{E} \supset \mathsf{J} = A\text{–}\mathsf{mod}_{\mathsf{inj}}$$
$$\mathsf{B} = B\text{–}\mathsf{mod} \supset \mathsf{F} \supset \mathsf{P} = B\text{–}\mathsf{mod}_{\mathsf{proj}}.$$

Consider the adjoint pair of DG-functors

$$\Psi = \operatorname{Hom}_A(L^\bullet, -)\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$$
$$\Phi = L^\bullet \otimes_B -\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A}).$$

Then the conditions of Sections A.1 and A.3 are satisfied, so the constructions of Sections A.2–A.3 provide the derived functors $\mathbb{R}\Psi$ and $\mathbb{L}\Phi$. The arguments in Section A.4 show that these two derived functors are naturally adjoint to each other, and the first assertion of Theorem A.5 explains how to deduce the claim that they are mutually inverse triangulated equivalences from the triangulated equivalence (11).

Alternatively, applying the second assertion of Theorem A.5 together with Lemma 1.2.2 allows to reprove the triangulated equivalence (11) rather than use it, thus obtaining another and more “conceptual” proof of Lemma 1.3.1. □

Now suppose that we have two pairs of full subcategories $\mathsf{E}_{\prime} \subset \mathsf{E}' \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}_{\prime\prime} \subset \mathsf{F}'' \subset B\text{–}\mathsf{mod}$ such that both the pairs $(\mathsf{E}_{\prime}, \mathsf{F}_{\prime\prime})$ and $(\mathsf{E}', \mathsf{F}'')$ satisfy the conditions (I–IV), and both the full subcategories $\mathsf{E}_{\prime}$ and E' are closed under infinite direct sums in $A\text{–}\mathsf{mod}$, while both the full subcategories $\mathsf{F}_{\prime\prime}$ and F'' are closed under infinite products in $B\text{–}\mathsf{mod}$. Then, in view of the discussion in Section 0.9 of the Introduction and Theorem 1.3.2 (for $\star = \varnothing$), we have a diagram of triangulated functors

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}_{\prime}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}_{\prime\prime}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{E}') \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}'') \arrow[d, two heads] \\
\mathsf{D}(A\text{–}\mathsf{mod}) & \mathsf{D}(B\text{–}\mathsf{mod})
\end{tikzcd} \tag{12}$$

The vertical arrows are Verdier quotient functors, so both the triangulated categories $\mathsf{D}(\mathsf{E}_{\prime})$ and $\mathsf{D}(\mathsf{E}')$ are preudo-coderived categories of left A-modules, and both the triangulated categories $\mathsf{D}(\mathsf{F}_{\prime\prime})$ and $\mathsf{D}(\mathsf{F}'')$ are pseudo-contraderived categories of left B-modules. The horizontal double lines are triangulated equivalences. The inner square in the diagram (12) is commutative, as one can see from the construction of the derived functors in Theorem 1.3.2.

1.4. **Minimal corresponding classes.** Let A and B be associative rings, and $L^\bullet$ be a pseudo-dualizing complex of A-B-bimodules.

Proposition 1.4.1. *Fix $l_1 = d_1$ and $l_2 \geqslant d_2$. Then there exists a unique minimal pair of full subcategories $\mathsf{E}^{l_2} = \mathsf{E}^{l_2}(L^\bullet) \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}^{l_2} = \mathsf{F}^{l_2}(L^\bullet) \subset B\text{–}\mathsf{mod}$ satisfying the conditions (I–IV) together with the additional requirements that E^{l_2} is closed under infinite direct sums in $A\text{–}\mathsf{mod}$ and F^{l_2} is closed under infinite products in $B\text{–}\mathsf{mod}$. For any pair of full subcategories $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ satisfying the conditions (I–IV) such that E is closed under infinite direct sums in $A\text{–}\mathsf{mod}$ and F is closed under infinite products in $B\text{–}\mathsf{mod}$ one has $\mathsf{E}^{l_2} \subset \mathsf{E}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}$.*

Proof. The full subcategories $\mathsf{E}^{l_2} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}^{l_2} \subset B\text{–}\mathsf{mod}$ are constructed simultaneously by a kind of generation process. By construction, for any full subcategories $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ as above we will have $\mathsf{E}^{l_2} \subset \mathsf{E}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}$. In particular, the pair of full subcategories $\mathsf{E} = \mathsf{E}_{d_1}$ and $\mathsf{F} = \mathsf{F}_{d_1}$ satisfies all the mentioned conditions, so we will have $\mathsf{E}^{l_2} \subset \mathsf{E}_{d_1}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}_{d_1}$.

Firstly, all the injective left A-modules belong to E^{l_2} and all the projective left B-modules belong to F^{l_2}, as dictated by the conditions (I–II). Secondly, let E be an A-module belonging to E^{l_2}. Then $E \in \mathsf{E}_{d_1}$, so the derived category object

$\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$ has cohomology modules concentrated in the degrees $-d_2 \leqslant m \leqslant d_1$. Pick a complex of left B-modules $F^\bullet$ representing $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E)$ such that $F^\bullet$ is concentrated in the degrees $-l_2 \leqslant m \leqslant d_1$ and the B-modules F^m are projective for all $-l_2 + 1 \leqslant m \leqslant d_1$. According to [13, Corollary A.5.2], we have $F^{-l_2} \in \mathsf{F}$. So we say that the B-module F^{-l_2} belongs to F^{l_2}. Similarly, let F be a B-module belonging to F^{l_2}. Then $F \in \mathsf{F}_{d_1}$, so the derived category object $L^\bullet \otimes_B^{\mathbb{L}} F$ has cohomology modules concentrated in the degrees $-d_1 \leqslant m \leqslant d_2$. Pick a complex of left A-modules $E^\bullet$ representing $L^\bullet \otimes_B^{\mathbb{L}} F$ such that $E^\bullet$ is concentrated in the degrees $-d_1 \leqslant m \leqslant l_2$ and the A-modules E^m are injective for all $-d_1 \leqslant m \leqslant l_2 - 1$. According to the dual version of [13, Corollary A.5.2], we have $E^{l_2} \in \mathsf{E}$. So we say that the A-module E^{l_2} belongs to E^{l_2}.

Thirdly and finally, we add to E^{l_2} all the extensions, cokernels of injective morphisms, and infinite direct sums of its objects, and similarly add to F^{l_2} all the extensions, kernels of surjective morphisms, and infinite products of its objects. Then the second and third steps are repeated in transfinite iterations, as it may be necessary, until all the modules that can be obtained in this way have been added and the full subcategories of all such modules $\mathsf{E}^{l_2} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}^{l_2} \subset B\text{–}\mathsf{mod}$ have been formed. □

Remark 1.4.2. It is clear from the construction in the proof of Proposition 1.4.1 that for any two values of the parameters $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$, and any two full subcategories $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ satisfying the conditions (I-IV) with the parameters l_1 and l_2 such that E is closed under infinite direct sums in $A\text{–}\mathsf{mod}$ and F is closed under infinite products in $B\text{–}\mathsf{mod}$, one has $\mathsf{E}^{l_2} \subset \mathsf{E}$ and $\mathsf{F}^{l_2} \subset \mathsf{F}$.

Notice that the conditions (III–IV) become weaker as the parameter l_2 increases. It follows that one has $\mathsf{E}^{l_2} \supset \mathsf{E}^{l_2+1}$ and $\mathsf{F}^{l_2} \supset \mathsf{F}^{l_2+1}$ for all $l_2 \geqslant d_2$. So the inclusion relations between our classes of modules have the form

$$\cdots \subset \mathsf{E}^{d_2+2} \subset \mathsf{E}^{d_2+1} \subset \mathsf{E}^{d_2} \subset \mathsf{E}_{d_1} \subset \mathsf{E}_{d_1+1} \subset \mathsf{E}_{d_1+2} \subset \cdots \subset A\text{–}\mathsf{mod}$$
$$\cdots \subset \mathsf{F}^{d_2+2} \subset \mathsf{F}^{d_2+1} \subset \mathsf{F}^{d_2} \subset \mathsf{F}_{d_1} \subset \mathsf{F}_{d_1+1} \subset \mathsf{F}_{d_1+2} \subset \cdots \subset B\text{–}\mathsf{mod}$$

Lemma 1.4.3. *Let $n \geqslant 0$ and $l_1 \geqslant d_1$, $l_2 \geqslant d_2 + n$ be some integers. Let $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ be a pair of full subcategories satisfying the conditons (I–IV) with the parameters l_1 and l_2. Denote by $\mathsf{E}(n) \subset A\text{–}\mathsf{mod}$ the full subcategory of all left A-modules of E-coresolution dimension not exceeding n and by $\mathsf{F}(n) \subset B\text{–}\mathsf{mod}$ the full subcategory of all left B-modules of F-resolution dimension not exceeding n. Then the two classes of modules $\mathsf{E}(n)$ and $\mathsf{F}(n)$ satisfy the conditions (I–IV) with the parameters $l_1 + n$ and $l_2 - n$.*

Proof. According to [25, Proposition 2.3(2)] or [13, Lemma A.5.4(a-b)] (and the assertions dual to these), the full subcategories $\mathsf{E}(n) \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}(n) \subset B\text{–}\mathsf{mod}$ satisfy the conditions (I–II). Using [13, Corollary A.5.5(b)], one shows that for any A-module $E \in \mathsf{E}(n)$ the derived category object $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E) \in \mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$ can be represented by a complex concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1+n$ with the terms belonging to F. Moreover, one has $\operatorname{Ext}_A^m(L^\bullet, E) = 0$ for all $m < -d_2$.

It follows that $\mathbb{R}\operatorname{Hom}_A(L^\bullet, E)$ can be also represented by a complex concentrated in the cohomological degrees $-l_2 + n \leqslant m \leqslant l_1 + n$ with the terms belonging to $\mathsf{F}(n)$. Similarly one can show that for any B-module $F \in \mathsf{F}(n)$ the derived category object $L^\bullet \otimes_B^{\mathbb{L}} F \in \mathsf{D}^b(A\text{–}\mathsf{mod})$ can be represented by a complex concentrated in the cohomological degrees $-l_1 - n \leqslant m \leqslant l_2$ with the terms belonging to E. Moreover, one has $\operatorname{Tor}^B_{-m}(L^\bullet, F) = 0$ for all $m > d_2$. It follows that $L^\bullet \otimes_B^{\mathbb{L}} F$ can be also represented by a complex concentrated in the cohomological degrees $-l_1 - n \leqslant m \leqslant l_2 - n$ with the terms belonging to $\mathsf{E}(n)$. This proves the conditions (III–IV). ☐

Proposition 1.4.4. *For any $l_2'' \geqslant l_2' \geqslant d_2$ and any conventional or exotic derived category symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs, the exact embedding functors $\mathsf{E}^{l_2''} \longrightarrow \mathsf{E}^{l_2'}$ and $\mathsf{F}^{l_2''} \longrightarrow \mathsf{F}^{l_2'}$ induce triangulated equivalences*

$$\mathsf{D}^\star(\mathsf{E}^{l_2''}) \simeq \mathsf{D}^\star(\mathsf{E}^{l_2'}) \quad \textit{and} \quad \mathsf{D}^\star(\mathsf{F}^{l_2''}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2'}).$$

The same exact embeddings also induce triangulated equivalences

$$\mathsf{D}^{\mathsf{co}}(\mathsf{E}^{l_2''}) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{E}^{l_2'}) \quad \textit{and} \quad \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}^{l_2''}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}^{l_2'}).$$

Proof. As in Proposition 1.2.8, we check that the $\mathsf{E}^{l_2''}$-coresolution dimension of any object of $\mathsf{E}^{l_2'}$ does not exceed $l_2'' - l_2'$ and the $\mathsf{F}^{l_2''}$-resolution dimension of any object of $\mathsf{F}^{l_2'}$ does not exceed $l_2'' - l_2'$. Indeed, according to Lemma 1.2.6, the pair of full subcategories $\mathsf{E}^{l_2''}(l_2'' - l_2') \subset A\text{–}\mathsf{mod}$ and $\mathsf{F}^{l_2''}(l_2'' - l_2') \subset B\text{–}\mathsf{mod}$ satisfies the conditions (I–IV) with the parameters $l_1 = d_1 + l_2'' - l_2'$ and $l_2 = l_2'$. Furthermore, since infinite direct sums are exact and the full subcategory $\mathsf{E}^{l_2''}$ is closed under infinite direct sums in $A\text{–}\mathsf{mod}$, so is the full subcategory $\mathsf{E}^{l_2''}(l_2'' - l_2')$. Since infinite products are exact and the full subcategory $\mathsf{F}^{l_2''}$ is closed under infinite products in $B\text{–}\mathsf{mod}$, so is the full subcategory $\mathsf{F}^{l_2''}(l_2'' - l_2')$. It follows that $\mathsf{E}^{l_2'} \subset \mathsf{E}^{l_2''}(l_2'' - l_2')$ and $\mathsf{F}^{l_2'} \subset \mathsf{F}^{l_2''}(l_2'' - l_2')$. ☐

In particular, the unbounded derived category $\mathsf{D}(\mathsf{E}^{l_2})$ is the same for all $l_2 \geqslant d_2$ and the unbounded derived category $\mathsf{D}(\mathsf{F}^{l_2})$ is the same for all $l_2 \geqslant d_2$.

As it was explained in Section 0.9 of the Introduction, it follows from the condition (I) together with the condition that E^{l_2} is closed under infinite direct sums in $A\text{–}\mathsf{mod}$ that the natural Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ factorizes into two Verdier quotient functors $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{E}^{l_2}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$. Similarly, it follows from the condition (II) together with the condition that F^{l_2} is closed under infinite products in $B\text{–}\mathsf{mod}$ that the natural Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ factorizes into two Verdier quotient functors $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{F}^{l_2}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$. In other words, the triangulated category $\mathsf{D}(\mathsf{E}^{l_2})$ is a pseudo-coderived category of left A-modules and the triangulated category $\mathsf{D}(\mathsf{F}^{l_2})$ is a pseudo-contraderived category of left B-modules.

These are called the *upper pseudo-coderived category* of left A-modules and the *upper pseudo-contraderived category* of left B-modules corresponding to the pseudo-dualizing complex $L^\bullet$. The notation is

$$\mathsf{D}^{L^\bullet}_{\prime}(A\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{E}^{l_2}) \quad \text{and} \quad \mathsf{D}^{L^\bullet}_{\prime\prime}(B\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{F}^{l_2}).$$

The next theorem provides, in particular, a triangulated equivalence between the upper pseudo-coderived and the upper pseudo-contraderived category,

$$\mathsf{D}^{L^\bullet}_{\prime}(A\text{–}\mathsf{mod}) = \mathsf{D}(\mathsf{E}^{l_2}) \simeq \mathsf{D}(\mathsf{F}^{l_2}) = \mathsf{D}^{L^\bullet}_{\prime\prime}(B\text{–}\mathsf{mod}).$$

Theorem 1.4.5. *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}{+}$, $\mathsf{abs}{-}$, *or* abs, *there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{E}^{l_2}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2})$ *provided by (appropriately defined) mutually inverse functors* $\mathbb{R}\operatorname{Hom}_A(L^\bullet, -)$ *and* $L^\bullet \otimes_B^{\mathbb{L}} -$.

Proof. This is another particular case of Theorem 1.3.2. ☐

Substituting $\mathsf{E}' = \mathsf{E}_{l_1}$, $\mathsf{E}_{\prime} = \mathsf{E}^{l_2}$, $\mathsf{F}'' = \mathsf{F}_{l_1}$, and $\mathsf{F}_{\prime\prime} = \mathsf{F}^{l_2}$ (for some $l_1 \geqslant d_1$ and $l_2 \geqslant d_2$) into the commutative diagram of triangulated functors (12) from Section 1.3, one obtains the commutative diagram of triangulated functors (7) promised in Section 0.5 of the Introduction.

1.5. **Dedualizing complexes.** Let A and B be associative rings. A *dedualizing complex* of A-B-bimodules $L^\bullet = T^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 1.2) satisfying the following additional condition:

(i) As a complex of left A-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of projective A-modules, and as a complex of right B-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of projective B-modules.

Taken together, the conditions (i) and (iii) mean that, as a complex of left A-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of finitely generated projective A-modules, and as a complex of right B-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of finitely generated projective B-modules. In other words, $T^\bullet$ is a perfect complex of left A-modules and a perfect complex of right B-modules.

This definition of a dedualizing complex is slightly less general than that of a *tilting complex* in the sense of [23, Theorem 1.1] and slightly more general than that of a *two-sided tilting complex* in the sense of [23, Definition 3.4].

Let $L^\bullet = T^\bullet$ be a dedualizing complex of A-B-bimodules. We refer to the beginning of Section 1.2 for the discussion of the pair of adjoint derived functors $\mathbb{R}\operatorname{Hom}_A(T^\bullet, -)\colon \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ and $T^\bullet \otimes_B^{\mathbb{L}} -\colon \mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$.

Proposition 1.5.1. *The derived functors* $\mathbb{R}\operatorname{Hom}_A(T^\bullet, -)$ *and* $T^\bullet \otimes_B^{\mathbb{L}} -$ *are mutually inverse triangulated equivalences between the conventional unbounded derived categories* $\mathsf{D}(A\text{–}\mathsf{mod})$ *and* $\mathsf{D}(B\text{–}\mathsf{mod})$.

Proof. We have to show that the adjunction morphisms are isomorphisms. Let $J^\bullet$ be a homotopy injective complex of left A-modules. Then the complex of left B-modules $\operatorname{Hom}_A(T^\bullet, J^\bullet)$ represents the derived category object $\mathbb{R}\operatorname{Hom}_A(T^\bullet, J^\bullet) \in \mathsf{D}(B\text{–}\mathsf{mod})$. Let ${}'T^\bullet$ be a finite complex of finitely generated projective right B-modules endowed with a quasi-isomorphism of complexes of right B-modules ${}'T^\bullet \longrightarrow T^\bullet$. Then the adjunction morphism $T^\bullet \otimes_B^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_A(T^\bullet, J^\bullet) \longrightarrow J^\bullet$ is represented, as a morphism in the derived category of abelian groups, by the morphism of complexes ${}'T^\bullet \otimes_B \operatorname{Hom}_A(T^\bullet, J^\bullet) \longrightarrow J^\bullet$. Now the complex of abelian groups ${}'T^\bullet \otimes_B \operatorname{Hom}_A(T^\bullet, J^\bullet)$ is naturally isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_{B^{\mathrm{op}}}({}'T^\bullet, T^\bullet), J^\bullet)$, and the morphism of complexes of left A-modules $A \longrightarrow \mathrm{Hom}_{B^{\mathrm{op}}}({}'T^\bullet, T^\bullet)$ is a quasi-isomorphism by the condition (ii).

Similarly, let $P^\bullet$ be a homotopy flat complex of left B-modules. Then the complex of left A-modules $T^\bullet \otimes_B P^\bullet$ represents the derived category object $T^\bullet \otimes_B^{\mathbb{L}} P^\bullet \in \mathsf{D}(A\text{–}\mathsf{mod})$. Let ${}''T^\bullet$ be a finite complex of finitely generated projective left A-modules endowed with a quasi-isomorphism of complexes of left A-modules ${}''T^\bullet \longrightarrow T^\bullet$. Then the adjunction morphism $P^\bullet \longrightarrow \mathbb{R}\mathrm{Hom}_A(T^\bullet,\, T^\bullet \otimes_B^{\mathbb{L}} P^\bullet)$ is represented, as a morphism in the derived category of abelian groups, by the morphism of complexes $P^\bullet \longrightarrow \mathrm{Hom}_A({}''T^\bullet,\, T^\bullet \otimes_B P^\bullet)$. Now the complex of abelian groups $\mathrm{Hom}_A({}''T^\bullet, T^\bullet \otimes_B P^\bullet)$ is naturally isomorphic to $\mathrm{Hom}_A({}''T^\bullet, T^\bullet) \otimes_B P^\bullet$, and the morphism of complexes of right B-modules $B \longrightarrow \mathrm{Hom}_A({}''T^\bullet, T^\bullet)$ is a quasi-isomorphism by the condition (ii). $\square$

In particular, it follows that the derived Bass and Auslander classes associated with a dedualizing complex $L^\bullet = T^\bullet$ (as discussed in Section 1.2) coincide with the whole bounded derived categories $\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$, and the triangulated equivalence (10) takes the form $\mathsf{D}^{\mathsf{b}}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{b}}(B\text{–}\mathsf{mod})$.

Now let us choose the parameter l_1 in such a way that $T^\bullet$ is quasi-isomorphic to a complex of (finitely generated) projective left A-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant d_2$ and to a complex of (finitely generated) projective right B-modules concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant d_2$. Then we have $\mathsf{E}_{l_1}(T^\bullet) = A\text{–}\mathsf{mod}$ and $\mathsf{F}_{l_1}(T^\bullet) = B\text{–}\mathsf{mod}$.

Corollary 1.5.2. *For any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs, there is a triangulated equivalence $\mathsf{D}^\star(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^\star(B\text{–}\mathsf{mod})$ provided by (appropriately defined) mutually inverse functors $\mathbb{R}\mathrm{Hom}_A(T^\bullet, -)$ and $T^\bullet \otimes_B^{\mathbb{L}} -$.*

Proof. This is a particular case of Theorem 1.2.9. $\square$

1.6. **Dualizing complexes.** Let A and B be associative rings. Our aim is to work out a generalization of the results of [16, Sections 2 and 4] falling in line with the exposition in the present Section 1.

Firstly we return to the discussion of sfp-injective and sfp-flat modules started in Section 1.1. Denote the full subcategory of sfp-injective left A-modules by $A\text{–}\mathsf{mod}_{\mathsf{sfpin}} \subset A\text{–}\mathsf{mod}$ and the full subcategory of sfp-flat left B-modules by $B\text{–}\mathsf{mod}_{\mathsf{sfpfl}} \subset B\text{–}\mathsf{mod}$. It is clear from Lemma 1.1.3 that the categories $A\text{–}\mathsf{mod}_{\mathsf{sfpin}}$ and $B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}$ have exact category structures inherited from the abelian categories $A\text{–}\mathsf{mod}$ and $B\text{–}\mathsf{mod}$.

Proposition 1.6.1. (a) *The triangulated functor $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}_{\mathsf{sfpin}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ induced by the embedding of exact categories $A\text{–}\mathsf{mod}_{\mathsf{sfpin}} \longrightarrow A\text{–}\mathsf{mod}$ is an equivalence of triangulated categories.*

(b) *If all sfp-injective left A-modules have finite injective dimensions, then the triangulated functor $\mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ induced by the embedding of additive/exact categories $A\text{–}\mathsf{mod}_{\mathsf{inj}} \longrightarrow A\text{–}\mathsf{mod}$ is an equivalence of triangulated categories.*

Proof. Part (a) is but an application of the assertion dual to [13, Proposition A.3.1(b)] (cf. [16, Theorem 2.2]). Part (b) was proved in [12, Section 3.7] (for a more general argument, one can use the assertion dual to [13, Corollary A.6.2]). In fact, the assumption in part (b) can be weakened by requiring only that fp-injective left A-modules have finite injective dimensions, as infinite direct sums of fp-injective left A-modules are fp-injective over an arbitrary ring (cf. [16, Theorem 2.4]). ☐

Proposition 1.6.2. (a) *The triangulated functor* $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *induced by the embedding of exact categories* $B\text{–}\mathsf{mod}_{\mathsf{sfpfl}} \longrightarrow B\text{–}\mathsf{mod}$ *is an equivalence of triangulated categories.*

(b) *If all sfp-flat left B-modules have finite projective dimensions, then the triangulated functor* $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *induced by the embedding of additive/exact categories* $B\text{–}\mathsf{mod}_{\mathsf{proj}} \longrightarrow B\text{–}\mathsf{mod}$ *is an equivalence of triangulated categories.*

Proof. Part (a) is but an application of [13, Proposition A.3.1(b)] (cf. [16, Theorem 4.4]). Part (b) was proved in [12, Section 3.8] (for a more general argument, see [13, Corollary A.6.2]). ☐

The following lemma is a version of [16, Lemma 4.1] applicable to arbitrary rings.

Lemma 1.6.3. (a) *Let P be a flat left B-module and K be an A-sfp-injective A-B-bimodule. Then the tensor product $K \otimes_B P$ is an sfp-injective left A-module.*

(b) *Let J be an injective left A-module and K be a B-sfp-injective A-B-bimodule. Then the left B-module $\operatorname{Hom}_A(K, J)$ is sfp-flat.*

Proof. This is a particular case of the next Lemma 1.6.4. ☐

Lemma 1.6.4. (a) *Let $P^\bullet$ be a complex of flat left B-modules concentrated in the cohomological degrees $-n \leqslant m \leqslant 0$ and $K^\bullet$ be a complex of A-B-bimodules which, as a complex of left A-modules, is quasi-isomorphic to a complex of sfp-injective A-modules concentrated in the cohomological degrees $-d \leqslant m \leqslant l$. Then the tensor product $K^\bullet \otimes_B P^\bullet$ is a complex of left A-modules quasi-isomorphic to a complex of sfp-injective left A-modules concentrated in the cohomological degrees $-n-d \leqslant m \leqslant l$.*

(b) *Let $J^\bullet$ be a complex of injective left A-modules concentrated in the cohomological degrees $0 \leqslant m \leqslant n$ and $K^\bullet$ be a complex of A-B-bimodules which, as a complex of right B-modules, is quasi-isomorphic to a complex of sfp-injective right B-modules concentrated in the cohomological degrees $-d \leqslant m \leqslant l$. The the complex of left B-modules $\operatorname{Hom}_A(K^\bullet, J^\bullet)$ is quasi-isomorphic to a complex of sfp-flat B-modules concentrated in the cohomological degrees $-l \leqslant m \leqslant n+d$.*

Proof. Part (a): clearly, the tensor product $K^\bullet \otimes_B P^\bullet$ is quasi-isomorphic to a complex of left A-modules concentrated in the cohomological degrees $-n-d \leqslant m \leqslant l$; the nontrivial aspect is to show that there is such a complex with sfp-injective tems. Equivalently, this means that $\operatorname{Ext}^i_A(M,\ K^\bullet \otimes_B P^\bullet) = 0$ for all strongly finitely presented left A-modules M and all $i > l$. Indeed, let $R^\bullet$ be a resolution of M by finitely generated projective left A-modules. Without loss of generality, we can assume that

$K^\bullet$ is a finite complex of A-B-bimodules. Then the complex $\mathrm{Hom}_A(R^\bullet,\ K^\bullet \otimes_B P^\bullet)$ is isomorphic to $\mathrm{Hom}_A(R^\bullet, K^\bullet) \otimes_B P^\bullet$ and the cohomology modules of the complex $\mathrm{Hom}_A(R^\bullet, K^\bullet)$ are concentrated in the degrees $-d \leqslant m \leqslant l$.

Part (b): clearly, the complex $\mathrm{Hom}_A(K^\bullet, J^\bullet)$ is quasi-isomorphic to a complex of left B-modules concentrated in the cohomological degrees $-l \leqslant m \leqslant n + d$; we have to show that there is such a complex with sfp-flat terms. Equivalently, this means that $\mathrm{Tor}_i^B(N, \mathrm{Hom}_A(K^\bullet, J^\bullet)) = 0$ for all strongly finitely presented right B-modules N and all $i > l$. Indeed, let $Q^\bullet$ be a resolution of N by finitely generated projective right B-modules. Without loss of generality, we can assume that $K^\bullet$ is a finite complex of A-B-bimodules. Then the complex $Q^\bullet \otimes_B \mathrm{Hom}_A(K^\bullet, J^\bullet)$ is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_{B^{\mathrm{op}}}(Q^\bullet, K^\bullet), J^\bullet)$ and the cohomology modules of the complex $\mathrm{Hom}_{B^{\mathrm{op}}}(Q^\bullet, K^\bullet)$ are concentrated in the degrees $-d \leqslant m \leqslant l$. ☐

A *dualizing complex* of A-B-bimodules $L^\bullet = D^\bullet$ is a pseudo-dualizing complex (according to the definition in Section 1.2) satisfying the following additional condition:

(i) As a complex of left A-modules, $D^\bullet$ is quasi-isomorphic to a finite complex of sfp-injective A-modules, and as a complex of right B-modules, $T^\bullet$ is quasi-isomorphic to a finite complex of sfp-injective B-modules.

This definition of a dualizing complex is a version of the definition of a *weak dualizing complex* in [16, Section 3] (see also the definition of a dualizing complex in [16, Section 4]) extended from the case of coherent rings to arbitrary rings A and B. Still, in order to prove the results below, we will have to impose some homological dimension conditions on the rings A and B, bringing our definition of a dualizing complex even closer to the definition of a weak dualizing complex in [16].

Specifically, we will have to assume that all sfp-injective left A-modules have finite injective dimensions. This assumption always holds when the ring A is left coherent and there exists an integer $n \geqslant 0$ such that every left ideal in A is generated by at most $\aleph_n$ elements [16, Proposition 2.3].

We will also have to assume that all sfp-flat left B-modules have finite projective dimensions. For a right coherent ring B, this would simply mean that all flat left B-modules have finite projective dimensions. The class of rings satisfying the latter condition was discussed, under the name of "left n-perfect rings", in the paper [5]. We refer to [16, Proposition 4.3], the discussions in [7, Section 3] and [12, Section 3.8], and the references therein, for further sufficient conditions.

Let us choose the parameter l_2 in such a way that $D^\bullet$ is quasi-isomorphic to a complex of sfp-injective left A-modules concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant l_2$ and to a compex of sfp-injective right B-modules concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant l_2$.

Proposition 1.6.5. *Let A and B be associative rings such that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions. Let $L^\bullet = D^\bullet$ be a dualizing complex of A-B-bimodules, and let the parameter l_2 be chosen as stated above. Then the related minimal corresponding* *classes* $\mathsf{E}^{l_2} = \mathsf{E}^{l_2}(D^\bullet)$ *and* $\mathsf{F}^{l_2} = \mathsf{F}^{l_2}(D^\bullet)$ *are contained in the classes of sfp-injective* A*-modules and spf-flat* B*-modules modules,* $\mathsf{E}^{l_2} \subset A\text{–}\mathsf{mod}_{\mathsf{sfpin}}$ *and* $\mathsf{F}^{l_2} \subset B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}$*.*

Moreover, let $n \geqslant 0$ *be an integer such that the injective dimensions of sfp-injective left* A*-modules do not exceed* n *and the projective dimensions of sfp-flat left* B*-modules do not exceed* n*. Then the classes of modules* $\mathsf{E} = A\text{–}\mathsf{mod}_{\mathsf{sfpin}}$ *and* $\mathsf{F} = B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}$ *satisfy the conditions (I–IV) with the parameters* $l_1 = n + d_1$ *and* l_2*.*

Proof. The second assertion holds, as the conditions (I–II) are satisfied by Lemma 1.1.3 and the conditions (III–IV) hold by Lemma 1.6.4. The first assertion follows from the second one together with Lemma 1.1.3. ☐

Let $B\text{–}\mathsf{mod}_{\mathsf{flat}} \subset B\text{–}\mathsf{mod}$ denote the full subcategory of flat left B-modules. It inherits the exact category structure of the abelian category $B\text{–}\mathsf{mod}$.

Corollary 1.6.6. *Let* A *and* B *be associative rings such that all sfp-injective left* A*-modules have finite injective dimensions and all sfp-flat left* B*-modules have finite projective dimensions. Let* $L^\bullet = D^\bullet$ *be a dualizing complex of* A*-*B*-bimodules, and let the parameter* l_2 *be chosen as above. Then there is a triangulated equivalence* $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *provided by (appropriately defined) mutually inverse functors* $\mathbb{R}\operatorname{Hom}_A(D^\bullet, -)$ *and* $D^\bullet \otimes_B^{\mathbb{L}} -$*.*

Furthermore, there is a chain of triangulated equivalences

$$\begin{gathered}\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{abs}=\varnothing}(A\text{–}\mathsf{mod}_{\mathsf{sfpin}}) \simeq \mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{E}^{l_2}) \simeq \\ \mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}}) \simeq \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \\ \simeq \mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{F}^{l_2}) \simeq \mathsf{D}^{\mathsf{abs}=\varnothing}(B\text{–}\mathsf{mod}_{\mathsf{flat}}) \simeq \mathsf{D}^{\mathsf{abs}=\varnothing}(B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}),\end{gathered}$$

where the notation $\mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{C})$ *is a shorthand for an identity isomorphism of triangulated categories* $\mathsf{D}^{\mathsf{abs}}(\mathsf{C}) = \mathsf{D}(\mathsf{C})$ *between the absolute derived category and the conventional derived category of an exact category* C*. Moreover, for any symbol* $\star = \mathsf{b}$*,* $+$*,* $-$*, or* $\varnothing$*, there are triangulated equivalences*

$$\begin{gathered}\mathsf{D}^\star(A\text{–}\mathsf{mod}_{\mathsf{sfpin}}) \simeq \mathsf{D}^\star(\mathsf{E}^{l_2}) \\ \simeq \mathsf{Hot}^\star(A\text{–}\mathsf{mod}_{\mathsf{inj}}) \simeq \mathsf{Hot}^\star(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \\ \simeq \mathsf{D}^\star(\mathsf{F}^{l_2}) \simeq \mathsf{D}^\star(B\text{–}\mathsf{mod}_{\mathsf{flat}}) \simeq \mathsf{D}^\star(B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}).\end{gathered}$$

Proof. The exact categories $A\text{–}\mathsf{mod}_{\mathsf{sfpin}}$ and $B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}$ have finite homological dimensions by assumption. Hence so do their full subcategories E^{l_2}, F^{l_2}, and $B\text{–}\mathsf{mod}_{\mathsf{flat}}$ satisfying the condition (I) or (II). It follows easily (see, e. g., [11, Remark 2.1] and [13, Proposition A.5.6]) that a complex in any one of these exact categories is acyclic if and only if it is absolutely acyclic, and that their (conventional or absolute) derived categories are equivalent to the homotopy categories of complexes of injective or projective objects. The same, of course, applies to the coderived and/or contraderived categories of those of these exact categories that happen to be closed under the infinite direct sums or infinite products in their respective abelian module categories. The same also applies to the bounded versions of the conventional or absolute derived categories and bounded versions of the homotopy categories.

Propositions 1.6.1 and 1.6.2 provide the equivalences with the coderived category $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ or the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$. Thus we have shown in all the cases that the mentioned triangulated categories of complexes of A-modules are equivalent to each other and the mentioned triangulated categories of complexes of B-modules are equivalent to each other. It remains to construct the equivalences connecting complexes of A-modules with complexes of B-modules.

Specifically, the equivalence $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ can be obtained in the same way as in [16, Theorem 4.5], using the equivalence $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \simeq \mathsf{Hot}(A\text{–}\mathsf{mod}_{\mathsf{inj}})$ in order to construct the derived functor $\mathbb{R}\operatorname{Hom}_A(D^\bullet, -)$ and the equivalence $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod}_{\mathsf{flat}})$ or $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \simeq \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ in order to construct the derived functor $D^\bullet \otimes_B^{\mathbb{L}} -$. More generally, the equivalence $\mathsf{D}^\star(\mathsf{E}^{l_2}) \simeq \mathsf{D}^\star(\mathsf{F}^{l_2})$ can be produced as a particular case of Theorem 1.4.5. $\square$

1.7. **Base change.** The aim of this section and the next one is to formulate a generalization of the definitions and results of [16, Section 5] that would fit naturally in our present context. Our exposition is informed by that in [2, Section 5].

Let $A \longrightarrow R$ and $B \longrightarrow S$ be two homomorphisms of associative rings. Let $\mathsf{E} \subset A\text{–}\mathsf{mod}$ be a full subcategory satisfying the condition (I), and let $\mathsf{F} \subset B\text{–}\mathsf{mod}$ be a full subcategory satisfying the condition (II). We denote by $\mathsf{G} = \mathsf{G}_{\mathsf{E}} \subset R\text{–}\mathsf{mod}$ the full subcategory formed by all the left R-modules whose underlying left A-modules belong to E, and by $\mathsf{H} = \mathsf{H}_{\mathsf{F}} \subset S\text{–}\mathsf{mod}$ the full subcategory formed by all the left S-modules whose underlying left B-modules belong to F.

Lemma 1.7.1. (a) *The full subcategory $\mathsf{G}_{\mathsf{E}} \subset R\text{–}\mathsf{mod}$ satisfies the condition (I) if and only if the underlying A-modules of all the injective left R-modules belong to E.*

(b) *The full subcategory $\mathsf{H}_{\mathsf{F}} \subset S\text{–}\mathsf{mod}$ satisfies the condition (II) if and only if the underlying B-modules of all the projective left S-modules belong to F.* $\square$

Assume further that the equivalent conditions of Lemma 1.7.1(a) and (b) hold, and additionally that the full subcategory $\mathsf{E} \subset A\text{–}\mathsf{mod}$ is closed under infinite direct sums and the full subcategory $\mathsf{F} \subset B\text{–}\mathsf{mod}$ is closed under infinite products. Then we get two commutative diagrams of triangulated functors, where the vertical arrows are Verdier quotient functors described in Section 0.9 of the Introduction, and the horizontal arrows are the forgetful functors:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}) \arrow[r] \arrow[d, two heads] & \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{G}_{\mathsf{E}}) \arrow[r] \arrow[d, two heads] & \mathsf{D}(\mathsf{E}) \arrow[d, two heads] \\
\mathsf{D}(R\text{–}\mathsf{mod}) \arrow[r] & \mathsf{D}(A\text{–}\mathsf{mod})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{D}^{\mathsf{ctr}}(S\text{–}\mathsf{mod}) \arrow[r] \arrow[d, two heads] & \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \arrow[d, two heads] \\
\mathsf{D}(\mathsf{H}_{\mathsf{F}}) \arrow[r] \arrow[d, two heads] & \mathsf{D}(\mathsf{F}) \arrow[d, two heads] \\
\mathsf{D}(S\text{–}\mathsf{mod}) \arrow[r] & \mathsf{D}(B\text{–}\mathsf{mod})
\end{tikzcd}$$

We recall that a triangulated functor is called *conservative* if it reflects isomorphisms, or equivalently, takes nonzero objects to nonzero objects. For example, the forgetful functors $\mathsf{D}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ and $\mathsf{D}(S\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ are conservative, while the forgetful functors $\mathsf{D}^{\mathsf{co}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$ and $\mathsf{D}^{\mathsf{ctr}}(S\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ are *not*, in general.

Lemma 1.7.2. *The forgetful functors $\mathsf{D}(\mathsf{G}_\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$ and $\mathsf{D}(\mathsf{H}_\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{F})$ are conservative.* □

One can say that a complex of left A-modules is E-*pseudo-coacyclic* if its image under the Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{E})$ vanishes. All coacyclic complexes are pseudo-coacyclic, and all pseudo-coacyclic complexes are acyclic.

Similarly, one can say that a complex of left B-modules is F-*pseudo-contraacyclic* if its image under the Verdier quotient functor $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{F})$ vanishes. All contraacyclic complexes are pseudo-contraacyclic, and all pseudo-contraacyclic complexes are acyclic.

Lemma 1.7.3. (a) *Let $\mathsf{E} \subset A\text{–}\mathsf{mod}$ be a full subcategory satisfying the condition (I), closed under infinite direct sums, and containing the underlying A-modules of injective left R-modules. Then a complex of left R-modules is G_E-pseudo-coacyclic if and only if it is E-pseudo-coacyclic as a complex of left A-modules.*

(b) *Let $\mathsf{F} \subset B\text{–}\mathsf{mod}$ be a full subcategory satisfying the condition (II), closed under infinite products, and containing the underlying B-modules of projective left S-modules. Then a complex of left S-modules is H_F-pseudo-contraacyclic if and only if it is F-pseudo-contraacyclic as a complex of left B-modules.*

Proof. This is a restatement of Lemma 1.7.2. □

The terminology in the following definition follows that in [16, Section 5], where "relative dualizing complexes" are discussed. In [2, Section 5], a related phenomenon is called "base change".

A *relative pseudo-dualizing complex* for a pair of associative ring homomorphisms $A \longrightarrow R$ and $B \longrightarrow S$ is a set of data consisting of a pseudo-dualizing complex of A-B-bimodules $L^\bullet$, a pseudo-dualizing complex of R-S-bimodules $U^\bullet$, and a morphism of complexes of A-B-bimodules $L^\bullet \longrightarrow U^\bullet$ satisfying the following condition:

(iv) the induced morphism $R \otimes_A^{\mathbb{L}} L^\bullet \longrightarrow U^\bullet$ is an isomorphism in the derived category of left R-modules $\mathsf{D}^-(R\text{–}\mathsf{mod})$, and the induced morphism $L^\bullet \otimes_B^{\mathbb{L}} S \longrightarrow U^\bullet$ is an isomorphism in the derived category of right S-modules $\mathsf{D}^-(\mathsf{mod}\text{–}S)$.

Notice that the condition (iii) in the definition of a pseudo-dualizing complex in Section 1.2 holds for the complex $U^\bullet$ whenever it holds for the complex $L^\bullet$ and the above condition (iv) is satisfied. The following result, which is our version of [2, Theorem 5.1], explains what happens with the condition (ii). We will assume that the complex $L^\bullet$ is concentrated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$ and the complex $U^\bullet$ is concentrated in the cohomological degrees $-t_1 \leqslant m \leqslant t_2$. Let $L^{\bullet\,\mathrm{op}}$ denote the complex $L^\bullet$ viewed as a complex of B^{op}-A^{op}-bimodules.

Proposition 1.7.4. *Let $L^\bullet$ be a pseudo-dualizing complex of A-B-bimodules, $U^\bullet$ be a finite complex of R-S-bimodules, and $L^\bullet \longrightarrow U^\bullet$ be a morphism of complexes of A-B-bimodules satisfying the condition (iv). Then $U^\bullet$ is a pseudo-dualizing complex of R-S-bimodules if and only if there exists an integer $l_1 \geqslant d_1$ such that the right A-module R belongs to the class $\mathsf{F}_{l_1}(L^{\bullet\,\mathrm{op}}) \subset A^{\mathrm{op}}\text{–}\mathsf{mod}$ and the left B-module S belongs to the class $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{–}\mathsf{mod}$.*

Proof. The key observation is that the natural isomorphism $\mathbb{R}\operatorname{Hom}_R(U^\bullet, U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_R(R \otimes_A^{\mathbb{L}} L^\bullet,\ U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_A(L^\bullet, U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_A(L^\bullet,\ L^\bullet \otimes_B^{\mathbb{L}} S)$ identifies the homothety morphism $S^{\mathrm{op}} \longrightarrow \mathbb{R}\operatorname{Hom}_R(U^\bullet, U^\bullet)$ with the adjunction morphism $S \longrightarrow \mathbb{R}\operatorname{Hom}_A(L^\bullet, L^\bullet \otimes_B^{\mathbb{L}} S)$. Similarly, the natural isomorphism $\mathbb{R}\operatorname{Hom}_{S^{\mathrm{op}}}(U^\bullet, U^\bullet) \simeq \mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(L^\bullet,\ R \otimes_A^{\mathbb{L}} L^\bullet)$ identifies the homothety morphism $R \longrightarrow \mathbb{R}\operatorname{Hom}_{S^{\mathrm{op}}}(U^\bullet, U^\bullet)$ with the adjunction morphism $R \longrightarrow \mathbb{R}\operatorname{Hom}_{B^{\mathrm{op}}}(L^\bullet,\ R \otimes_A^{\mathbb{L}} L^\bullet)$. It remains to say that one can take any integer l_1 such that $l_1 \geqslant d_1$ and $l_1 \geqslant t_1$. ☐

The next proposition is our version of [2, Proposition 5.3].

Proposition 1.7.5. *Let $L^\bullet \longrightarrow U^\bullet$ be a relative pseudo-dualizing complex for a pair of ring homomorphisms $A \longrightarrow R$ and $B \longrightarrow S$. Let l_1 be an integer such that $l_1 \geqslant d_1$ and $l_1 \geqslant t_1$. Then*

(a) *a left R-module belongs to the full subcategory $\mathsf{E}_{l_1}(U^\bullet) \subset R\text{–}\mathsf{mod}$ if and only if its underlying A-module belongs to the full subcategory $\mathsf{E}_{l_1}(L^\bullet) \subset A\text{–}\mathsf{mod}$;*

(b) *a left S-module belongs to the full subcategory $\mathsf{F}_{l_1}(U^\bullet) \subset S\text{–}\mathsf{mod}$ if and only if its underlying B-module belongs to the full subcategory $\mathsf{F}_{l_1}(L^\bullet) \subset B\text{–}\mathsf{mod}$.*

Proof. The assertions follow from the commutative diagrams of the pairs of adjoint functors and the forgetful functors

$$\begin{tikzcd} \mathsf{D}(R\text{–}\mathsf{mod}) \arrow[r, "\mathbb{R}\operatorname{Hom}_R(U^\bullet{,}-)"] \arrow[d] & \mathsf{D}(S\text{–}\mathsf{mod}) \arrow[d] & \mathsf{D}(R\text{–}\mathsf{mod}) \arrow[d] & \mathsf{D}(S\text{–}\mathsf{mod}) \arrow[l, "U^\bullet\otimes_S^{\mathbb{L}}-"'] \arrow[d] \\ \mathsf{D}(A\text{–}\mathsf{mod}) \arrow[r, "\mathbb{R}\operatorname{Hom}_A(L^\bullet{,}-)"] & \mathsf{D}(B\text{–}\mathsf{mod}) & \mathsf{D}(A\text{–}\mathsf{mod}) & \mathsf{D}(B\text{–}\mathsf{mod}) \arrow[l, "L^\bullet\otimes_B^{\mathbb{L}}-"'] \end{tikzcd}$$

together with the compatibility of the adjunctions with the forgetful functors and conservativity of the forgetful functors. ☐

Proposition 1.7.6. *Let $L^\bullet \longrightarrow U^\bullet$ be a relative pseudo-dualizing complex for a pair of ring homomorphisms $A \longrightarrow R$ and $B \longrightarrow S$, and let $\mathsf{E} \subset A\text{–}\mathsf{mod}$ and $\mathsf{F} \subset B\text{–}\mathsf{mod}$ be a pair of full subcategories satisfying the conditions (I–IV) with respect to the pseudo-dualizing complex $L^\bullet$ with some parameters l_1 and l_2 such that $l_1 \geqslant d_1$, $l_1 \geqslant t_1$, $l_2 \geqslant d_2$, and $l_2 \geqslant t_2$. Suppose that the underlying A-modules of all the injective left R-modules belong to E and the underlying B-modules of all the projective left S-modules belong to F. Then the pair of full subcategories $\mathsf{G}_{\mathsf{E}} \subset R\text{–}\mathsf{mod}$ and $\mathsf{H}_{\mathsf{F}} \subset S\text{–}\mathsf{mod}$ satisfies the conditions (I–IV) with respect to the pseudo-dualizing complex $U^\bullet$ with the same parameters l_1 and l_2.*

Proof. The conditions (I–II) hold by Lemma 1.7.1, and the conditions (III–IV) are easy to check using the standard properties of the (co)resolution dimensions [13, Corollary A.5.2]. □

Corollary 1.7.7. *In the context and assumptions of Proposition* 1.7.6*, for any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, *or* abs, *there is a triangulated equivalence* $\mathsf{D}^\star(\mathsf{G_E}) \simeq \mathsf{D}^\star(\mathsf{H_F})$ *provided by (appropriately defined) mutually inverse functors* $\mathbb{R}\operatorname{Hom}_R(U^\bullet, -)$ *and* $U^\bullet \otimes_S^{\mathbb{L}} -$.

Here, in the case $\star = \mathsf{co}$ *it is assumed that the full subcategories* $\mathsf{E} \subset A\text{–}\mathsf{mod}$ *and* $\mathsf{F} \subset B\text{–}\mathsf{mod}$ *are closed under infinite direct sums, while in the case* $\star = \mathsf{ctr}$ *it is assumed that these two full subcategories are closed under infinite products.*

Proof. This is a particular case of Theorem 1.3.2. □

1.8. **Relative dualizing complexes.** Let A be an associative ring. The *sfp-injective dimension* of an A-module is the minimal length of its coresolution by sfp-injective A-modules. The sfp-injective dimension of a left A-module E is equal to the supremum of all the integers $n \geqslant 0$ for which there exists a strongly finitely presented left A-module M such that $\operatorname{Ext}_A^n(M, E) \neq 0$. The *sfp-flat dimension* of an A-module is the minimal length of its resolution by sfp-flat A-modules. The sfp-flat dimension of a left A-module F is equal to the supremum of all the integers $n \geqslant 0$ for which there exists a strongly finitely presented right A-module N such that $\operatorname{Tor}_n^A(N, F) \neq 0$.

Lemma 1.8.1. *The sfp-flat dimension of a right A-module G is equal to the sfp-injective dimension of the left A-module* $\operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. □

Let $A \longrightarrow R$ and $B \longrightarrow S$ be homomorphisms of associative rings.

Lemma 1.8.2. (a) *The supremum of sfp-injective dimensions of the underlying left A-modules of injective left R-modules is equal to the sfp-flat dimension of the right A-module R.*

(b) *The supremum of sfp-flat dimensions of the underlying left B-modules of projective left S-modules is equal to the sfp-flat dimension of the left B-module S.* □

Assume that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions, as in Section 1.6. Fix an integer $n \geqslant 0$, and set $\mathsf{E} = A\text{–}\mathsf{mod}_{\mathsf{sfpin}}(n) \subset A\text{–}\mathsf{mod}$ to be the full subcategory of all left A-modules whose sfp-injective dimension does not exceed n. Similarly, set $\mathsf{F} = B\text{–}\mathsf{mod}_{\mathsf{sfpfl}}(n) \subset B\text{–}\mathsf{mod}$ to be the full subcategory of all left B-modules whose sfp-flat dimension does not exceed n.

Proposition 1.8.3. (a) *The embedding of exact/abelian categories* $\mathsf{E} \longrightarrow A\text{–}\mathsf{mod}$ *induces an equivalence of triangulated categories* $\mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{E}) \simeq \mathsf{D}^{\mathsf{co}}(A\text{–}\mathsf{mod})$.

(b) *The embedding of exact/abelian categories* $\mathsf{F} \longrightarrow B\text{–}\mathsf{mod}$ *induces an equivalence of triangulated categories* $\mathsf{D}^{\mathsf{abs}=\varnothing}(\mathsf{F}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$.

Proof. Follows from [11, Remark 2.1], Propositions 1.6.1–1.6.2, and [13, Proposition A.5.6] (cf. the proof of Corollary 1.6.6). □

In other words, in the terminology of Section 1.7, one can say that the class of E-pseudo-coacyclic complexes coincides with that of coacyclic complexes of left A-modules, while the class of F-pseudo-contraacyclic complexes coincides with that of contraacyclic complex of left B-modules.

The following definitions were given in the beginning of [16, Section 5]. The *R/A-semicoderived category* of left R-modules $\mathsf{D}^{\mathsf{sico}}_A(R\text{–}\mathsf{mod})$ is defined as the quotient category of the homotopy category of complexes of left R-modules $\mathsf{Hot}(R\text{–}\mathsf{mod})$ by its thick subcategory of complexes of R-modules *that are coacyclic as complexes of A-modules*. Similarly, the *S/B-semicontraderived category* of left S-modules $\mathsf{D}^{\mathsf{sictr}}_B(S\text{–}\mathsf{mod})$ is defined as the quotient category of the homotopy category of complexes of left S-modules $\mathsf{Hot}(S\text{–}\mathsf{mod})$ by its thick subcategory of complexes of S-modules *that are contraacyclic as complexes of B-modules.*

As in Section 1.7, we denote by $\mathsf{G}_\mathsf{E} \subset R\text{–}\mathsf{mod}$ the full subcategory of all left R-modules whose underlying A-modules belong to E, and by $\mathsf{H}_\mathsf{F} \subset S\text{–}\mathsf{mod}$ the full subcategory of all left S-modules whose underlying B-modules belong to F. The next proposition is our version of [16, Theorems 5.1 and 5.2].

Proposition 1.8.4. (a) *Assume that all sfp-injective left A-modules have finite injective dimensions and the sfp-flat dimension of the right A-module R does not exceed n. Then the embedding of exact/abelian categories $\mathsf{G}_\mathsf{E} \longrightarrow R\text{–}\mathsf{mod}$ induces an equivalence of triangulated categories $\mathsf{D}(\mathsf{G}_\mathsf{E}) \simeq \mathsf{D}^{\mathsf{sico}}_A(R\text{–}\mathsf{mod})$.*

(b) *Assume that all sfp-flat left B-modules have finite projective dimensions and the sfp-flat dimension of the left B-module S does not exceed n. Then the embedding of exact/abelian categories $\mathsf{H}_\mathsf{F} \longrightarrow S\text{–}\mathsf{mod}$ induces an equivalence of triangulated categories $\mathsf{D}(\mathsf{H}_\mathsf{F}) \simeq \mathsf{D}^{\mathsf{sictr}}_B(S\text{–}\mathsf{mod})$.*

Proof. The assumptions of Lemma 1.7.3(a) or (b) hold by Lemma 1.8.2, so its conclusion is applicable; and it remains to recall Proposition 1.8.3. ☐

So, in the assumptions of Proposition 1.8.4, the R/A-semicoderived category of left R-modules is a pseudo-coderived category of left R-modules and the S/B-semicontraderived category of left S-modules is a pseudo-contraderived category of left S-modules, in the sense of Section 0.9 of the Introduction.

A *relative dualizing complex* for a pair of associative ring homomorphisms $A \longrightarrow R$ and $B \longrightarrow S$ is a relative pseudo-dualizing complex $L^\bullet \longrightarrow U^\bullet$ in the sense of the definition in Section 1.7 such that $L^\bullet = D^\bullet$ is a dualizing complex of A-B-bimodules in the sense of the definition in Section 1.6. In other words, the condition (i) of Section 1.6 and the conditions (ii-iii) of Section 1.2 have to be satisfied for $D^\bullet$, the condition (ii) of Section 1.2 has to be satisfied for $U^\bullet$, and the condition (iv) of Section 1.7 has to be satisfied for the morphism $D^\bullet \longrightarrow U^\bullet$.

Notice that, in the assumption of finiteness of flat dimensions of the right A-module R and the left B-module S, the condition (ii) for the complex $U^\bullet$ follows from the similar condition for the complex $L^\bullet$ together with the condition (iv), by Proposition 1.7.4 and Remark 1.2.7.

The following corollary is our generalization of [16, Theorem 5.6].

Corollary 1.8.5. *Let A and B be associative rings such that all sfp-injective left A-modules have finite injective dimensions and all sfp-flat left B-modules have finite projective dimensions. Let $A \longrightarrow R$ and $B \longrightarrow S$ be associative ring homomorphisms such that the ring R is a right A-module of finite flat dimension and the ring S is a left B-module of finite flat dimension. Let $D^\bullet \longrightarrow U^\bullet$ be a relative dualizing complex for $A \longrightarrow R$ and $B \longrightarrow S$. Then there is a triangulated equivalence $\mathsf{D}^{\mathsf{sico}}_A(R\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{sictr}}_B(S\text{–}\mathsf{mod})$ provided by mutually inverse functors $\mathbb{R}\operatorname{Hom}_R(U^\bullet,-)$ and $U^\bullet \otimes_S^{\mathbb{L}} -$.*

Proof. Combine Corollary 1.7.7 (for $\star = \varnothing$) with Proposition 1.8.4. □

2. Pairs of Coassociative Coalgebras

We refer to the classical book [27], the introductory section and appendix [11, Section 0.2 and Appendix A], the memoir [12], the overview [14], the paper [19], and the references therein for a general discussion of coassociative coalgebras over fields and module objects (comodules and contramodules) over them.

2.1. **Strongly quasi-finitely copresented comodules.** Let k be a fixed ground field and $\mathcal{C}$ be a coassociative coalgebra (with counit) over k. We denote by $\mathcal{C}\text{–}\mathsf{comod}$ and $\mathsf{comod}\text{–}\mathcal{C}$ the abelian categories of left and right $\mathcal{C}$-comodules. The abelian category of left $\mathcal{C}$-contramodules is denoted by $\mathcal{C}\text{–}\mathsf{contra}$.

For any two left $\mathcal{C}$-comodules $\mathcal{M}$ and $\mathcal{N}$, we denote by $\operatorname{Hom}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ the k-vector space of left $\mathcal{C}$-comodule morphisms $\mathcal{M} \longrightarrow \mathcal{N}$. For any two left $\mathcal{C}$-contramodules $\mathfrak{S}$ and $\mathfrak{T}$, we denote by $\operatorname{Hom}^{\mathcal{C}}(\mathfrak{S},\mathfrak{T})$ the k-vector space of left $\mathcal{C}$-contramodule morphisms $\mathfrak{S} \longrightarrow \mathfrak{T}$. The coalgebra opposite to $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$; so a right $\mathcal{C}$-comodule is the same thing as a left $\mathcal{C}^{\mathrm{op}}$-comodule.

We recall that for any right $\mathcal{C}$-comodule $\mathcal{N}$ and k-vector space V the vector space $\operatorname{Hom}_k(\mathcal{N},V)$ has a natural left $\mathcal{C}$-contramodule structure [14, Sections 1.1–2]. We refer to [19, Section 1], [14, Section 3.1], [12, Section 2.2], or [11, Sections 0.2.6 and 5.1.1–2] for the definition and discussion of the functor of *contratensor product* $\mathcal{N} \odot_{\mathcal{C}} \mathfrak{T}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-contramodule $\mathfrak{T}$.

The construction of the *cotensor product* $\mathcal{N} \mathbin{\square}_{\mathcal{C}} \mathcal{M}$ of a right $\mathcal{C}$-comodule $\mathcal{N}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$ goes back at least to the paper [8, Section 2]. The dual-analogous construction involving contramodules is the vector space of *cohomomorphisms* $\operatorname{Cohom}_{\mathcal{C}}(\mathcal{M},\mathfrak{T})$ from a left $\mathcal{C}$-comodule $\mathcal{M}$ to a left $\mathcal{C}$-contramodule $\mathfrak{T}$. We refer to [19, Section 1], [14, Sections 2.5–6], or [11, Sections 0.2.1, 0.2.4, 1.2.1, and 3.2.1] for the definitions and discussion of these constructions.

Given a subcoalgebra $\mathcal{B} \subset \mathcal{C}$ and a left $\mathcal{C}$-comodule $\mathcal{M}$, let ${}_{\mathcal{B}}\mathcal{M} \subset \mathcal{M}$ denote the maximal $\mathcal{C}$-subcomodule in $\mathcal{M}$ whose $\mathcal{C}$-comodule structure comes from a $\mathcal{B}$-comodule structure. The $\mathcal{B}$-comodule ${}_{\mathcal{B}}\mathcal{M}$ can be computed as the full preimage of the subcomodule $\mathcal{B} \otimes_k \mathcal{M} \subset \mathcal{C} \otimes_k \mathcal{M}$ under the coaction map $\mathcal{M} \longrightarrow \mathcal{C} \otimes_k \mathcal{M}$, or as the cotensor product $\mathcal{B} \mathbin{\square}_{\mathcal{C}} \mathcal{M}$ [19, Section 1].

Lemma 2.1.1. *Let $\mathcal{C}$ be a coassociative coalgebra over k and $\mathcal{M}$ be a left $\mathcal{C}$-comodule. Then the following two conditions are equivalent:*

- *for any finite-dimensional subcoalgebra $\mathcal{B} \subset \mathcal{C}$, the k-vector space ${}_{\mathcal{B}}\mathcal{M}$ is finite-dimensional;*
- *for any cosimple subcoalgebra $\mathcal{A} \subset \mathcal{C}$, the k-vector space ${}_{\mathcal{A}}\mathcal{M}$ is finite-dimensional;*
- *for any finite-dimensional left $\mathcal{C}$-comodule $\mathcal{K}$, the k-vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K}, \mathcal{M})$ is finite-dimensional.*
- *for any irreducible left $\mathcal{C}$-comodule $\mathcal{I}$, the k-vector space $\mathrm{Hom}_{\mathcal{C}}(\mathcal{I}, \mathcal{M})$ is finite-dimensional.*

Proof. The equivalence between the first two conditions follows from a dual form of Nakayama's lemma for finite-dimensional algebras. The rest of the equivalences are easy. We refer to [27, Section 2] for the background material (in particular, one should keep in mind that $\mathcal{C}$ is the union of its finite-dimensional subcoalgeras, all $\mathcal{C}$-comodules are the unions of their finite-dimensional subcomodules, and all finite-dimensional $\mathcal{C}$-comodules are comodules over finite-dimensional subcoagebras of $\mathcal{C}$; so irreducible left $\mathcal{C}$-comodules correspond bijectively to cosimple subcoalgebras in $\mathcal{C}$). $\square$

We will say that a left $\mathcal{C}$-comodule $\mathcal{M}$ is *quasi-finitely cogenerated* if it satisfies the equivalent conditions of Lemma 2.1.1. (Such comodules were called "quasi-finite" in [28].) Any finitely cogenerated $\mathcal{C}$-comodule (in the sense of [28] or [19]) is quasi-finitely cogenerated, while the cofree left $\mathcal{C}$-comodule $\mathcal{C} \otimes_k V$ cogenerated by an infinite-dimensional k-vector space V is not quasi-finitely cogenerated when $\mathcal{C} \neq 0$.

One can see from [19, Lemma 1.2(e)] that the classes of finitely cogenerated and quasi-finitely cogenerated left $\mathcal{C}$-comodules coincide if and only if the maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}}$ of the coalgebra $\mathcal{C}$ is finite-dimensional. Unlike the finite cogeneratedness condition, the quasi-finite cogeneratedness condition on comodules is *Morita invariant*, i. e., it is preserved by equivalences of the categories of comodules $\mathcal{C}\text{–}\mathsf{comod} \simeq \mathcal{D}\text{–}\mathsf{comod}$ over different coalgebras $\mathcal{C}$ and $\mathcal{D}$ [28].

Lemma 2.1.2. (a) *The class of all quasi-finitely cogenerated left $\mathcal{C}$-comodules is closed under extensions and the passages to arbitrary subcomodules.*

(b) *Any quasi-finitely cogenerated $\mathcal{C}$-comodule is a subcomodule of a quasi-finitely cogenerated injective $\mathcal{C}$-comodule.* $\square$

Proof. To prove part (a), notice that the functor $\mathcal{M} \longmapsto {}_{\mathcal{B}}\mathcal{M}$ is left exact for any subcoalgebra $\mathcal{B} \subset \mathcal{C}$). In part (b), it suffices to say that the injective envelope of a quasi-finitely cogenerated comodule is quasi-finitely cogenerated. $\square$

A $\mathcal{C}$-comodule $\mathcal{M}$ is said to be *quasi-finitely copresented* if it is isomorphic to the kernel of a morphism of quasi-finitely cogenerated injective $\mathcal{C}$-comodules. Any finitely copresented $\mathcal{C}$-comodule in the sense of [19, Section 1] is quasi-finitely copresented. Any quasi-finitely copresented $\mathcal{C}$-comodule is quasi-finitely cogenerated.

Lemma 2.1.3. (a) *The kernel of a morphism from a quasi-finitely copresented $\mathcal{C}$-comodule to a quasi-finitely cogenerated one is quasi-finitely copresented.*

(b) *The class of quasi-finitely copresented $\mathcal{C}$-comodules is closed under extensions.*

(c) *The cokernel of an injective morphism from a quasi-finitely copresented $\mathcal{C}$-comodule to a quasi-finitely cogenerated one is quasi-finitely cogenerated.*

Proof. Follows from Lemma 2.1.2 (cf. the proof of [19, Lemma 1.8(a)]). $\square$

Given a subcoalgebra $\mathcal{B} \subset \mathcal{C}$ and a left $\mathcal{C}$-contramodule $\mathfrak{T}$, we denote by ${}^{\mathcal{B}}\mathfrak{T}$ the maximal quotient contramodule of $\mathfrak{T}$ whose $\mathcal{C}$-contramodule structure comes from a $\mathcal{B}$-contramodule structure. The $\mathcal{B}$-contramodule ${}^{\mathcal{B}}\mathfrak{T}$ can be computed as the cokernel of the composition $\mathrm{Hom}_k(\mathcal{C}/\mathcal{B}, \mathfrak{T}) \longrightarrow \mathfrak{T}$ of the natural embedding $\mathrm{Hom}_k(\mathcal{C}/\mathcal{B}, \mathfrak{T}) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \mathfrak{T})$ with the contraaction map $\mathrm{Hom}_k(\mathcal{C}, \mathfrak{T}) \longrightarrow \mathfrak{T}$, or as the space of cohomomorphisms ${}^{\mathcal{B}}\mathfrak{T} = \mathrm{Cohom}_{\mathcal{C}}(\mathcal{B}, \mathfrak{T})$ [19, Section 1].

Lemma 2.1.4. *Let $\mathcal{C}$ be a coassociative coalgebra over k and $\mathfrak{T}$ be a left $\mathcal{C}$-contramodule. Then the following two conditions are equivalent:*

- *for any finite-dimensional subcoalgebra $\mathcal{B} \subset \mathcal{C}$, the k-vector space ${}^{\mathcal{B}}\mathfrak{T}$ is finite-dimensional;*
- *for any cosimple subcoalgebra $\mathcal{A} \subset \mathcal{C}$, the k-vector space ${}^{\mathcal{A}}\mathfrak{T}$ is finite-dimensional;*
- *for any finite-dimensional left $\mathcal{C}$-contramodule $\mathfrak{K}$, the k-vector space* $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{T}, \mathfrak{K})$ *is finite-dimensional;*
- *for any irreducible left $\mathcal{C}$-contramodule $\mathfrak{I}$, the k-vector space* $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{T}, \mathfrak{I})$ *is finite-dimensional.*

Proof. The equivalence of the first two conditions follows from Nakayama's lemma for finite-dimensional algebras. The rest of the equivalences are only slightly more complicated than in Lemma 2.1.1. We refer to [11, Appendix A] and [14, Section 1] for the background material (in particular, one has to be careful in that *not* every $\mathcal{C}$-contramodule embeds into the projective limit of its finite-dimensional quotient contramodules; nevertheless, all irreducible $\mathcal{C}$-contramodules are finite-dimensional; furthermore, *not* every finite-dimensional $\mathcal{C}$-contramodule is a contramodule over a finite-dimensional subcoalgebra in $\mathcal{C}$, generally speaking; but every irreducible $\mathcal{C}$-contramodule is; so irreducible left $\mathcal{C}$-contramodules still correspond bijectively to cosimple subcoalgebras in $\mathcal{C}$). $\square$

We will say that a left $\mathcal{C}$-contramodule $\mathfrak{T}$ is *quasi-finitely generated* if it satisfies the equivalent conditions of Lemma 2.1.4. According to [19, Lemma 1.5(a) and the proof of Lemma 1.5(b)], any finitely generated $\mathcal{C}$-contramodule is quasi-finitely generated, while the free left $\mathcal{C}$-contramodule $\mathrm{Hom}_k(\mathcal{C}, V)$ generated by an infinite-dimensional vector space V is not quasi-finitely generated when $\mathcal{C} \neq 0$.

One can see from [19, Lemma 1.5(c,e)] that the classes of finitely generated and quasi-finitely generated left $\mathcal{C}$-contramodules coincide if and only if the maximal cosemisimple subcoalgebra $\mathcal{C}^{\mathrm{ss}}$ of the coalgebra $\mathcal{C}$ is finite-dimensional. Unlike the finite generatedness condition, the quasi-finite generatedness condition on contramodules is Morita invariant, i. e., it is preserved by equivalences of the categories of contramodules $\mathcal{C}$–$\mathsf{contra} \simeq \mathcal{D}$–$\mathsf{contra}$ over different coalgebras $\mathcal{C}$ and $\mathcal{D}$ (see [11, Section 7.5.3] for a discussion).

Lemma 2.1.5. (a) *The class of quasi-finitely generated left $\mathcal{C}$-contramodules is closed under extensions and the passages to arbitrary quotient contramodules.*

(b) *Any quasi-finitely generated left $\mathcal{C}$-contramodule is a quotient contramodule of a quasi-finitely generated projective $\mathcal{C}$-contramodule.*

Proof. To prove part (a), notice that the functor $\mathfrak{T} \longmapsto {}^{\mathcal{B}}\mathfrak{T}$ is right exact for any subcoalgebra $\mathcal{B} \subset \mathcal{C}$. The proof of part (b) is based on the arguments in the first half of the proof of [11, Lemma A.3]. The key step is to construct for any left $\mathcal{C}^{\mathrm{ss}}$-contramodule $\mathfrak{K}$ a projective left $\mathcal{C}$-contramodule $\mathfrak{P}$ such that the left $\mathcal{C}^{\mathrm{ss}}$-contramodule ${}^{\mathcal{C}^{\mathrm{ss}}}\mathfrak{P}$ is isomorphic to $\mathfrak{K}$. Then one applies [19, Lemma A.2.1]. $\square$

A $\mathcal{C}$-contramodule $\mathfrak{T}$ is said to be *quasi-finitely presented* if it is isomorphic to the cokernel of a morphism of quasi-finitely presented projective $\mathcal{C}$-contramodules. Any finitely presented contramodule in the sense of [19, Section 1] is quasi-finitely presented. Any quasi-finitely presented $\mathcal{C}$-contramodule is quasi-finitely generated.

Lemma 2.1.6. (a) *The cokernel of a morphism of from a quasi-finitely generated $\mathcal{C}$-contramodule to a quasi-finitely presented one is quasi-finitely presented.*

(b) *The class of quasi-finitely presented $\mathcal{C}$-contramodules is closed under extensions.*

(c) *The kernel of a surjective morphism from a quasi-finitely generated $\mathcal{C}$-contramodule to a quasi-finitely presented one is quasi-finitely generated.*

Proof. Follows from Lemma 2.1.5 (cf. [16, Lemma 1.1] and [19, Lemma 1.8(b)]). $\square$

The following proposition is our version of [19, Proposition 1.9].

Proposition 2.1.7. (a) *The functor $\mathcal{N} \longmapsto \mathcal{N}^* = \mathrm{Hom}_k(\mathcal{N}, k)$ restricts to an anti-equivalence between the additive category of quasi-finitely copresented right $\mathcal{C}$-comodules and the additive category of quasi-finitely presented left $\mathcal{C}$-contramodules.*

(b) *For any right $\mathcal{C}$-comodule $\mathcal{M}$, any quasi-finitely cogenerated right $\mathcal{C}$-comodule $\mathcal{N}$, and any k-vector space V, the natural k-linear map $\mathrm{Hom}_k(\mathcal{M},\, \mathcal{N} \otimes_k V) \longrightarrow \mathrm{Hom}_k(\mathcal{N}^*, \mathrm{Hom}_k(\mathcal{M}, V))$ restricts to an isomorphism of the Hom spaces in the categories of right $\mathcal{C}$-comodules and left $\mathcal{C}$-contramodules*

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M},\, \mathcal{N} \otimes_k V) \simeq \mathrm{Hom}^{\mathcal{C}}(\mathcal{N}^*, \mathrm{Hom}_k(\mathcal{M}, V)).$$

(c) *For any right $\mathcal{C}$-comodule $\mathcal{M}$, any quasi-finitely cogenerated right $\mathcal{C}$-comodule $\mathcal{N}$, and any k-vector space V, the natural k-linear map $(\mathcal{M} \otimes_k \mathcal{N}^*) \otimes_k V \longrightarrow \mathcal{M} \otimes_k \mathrm{Hom}_k(\mathcal{N}, V)$ induces an isomorphism of the (contra)tensor product spaces*

$$(\mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*) \otimes_k V \simeq \mathcal{M} \odot_{\mathcal{C}} \mathrm{Hom}_k(\mathcal{N}^*, V).$$

Proof. Part (b): for a right $\mathcal{C}$-comodule $\mathcal{M}$ and a subcoalgebra $\mathcal{B} \subset \mathcal{C}$, we denote the maximal subcomodule of $\mathcal{M}$ whose $\mathcal{C}$-comodule structure comes from a $\mathcal{B}$-comodule structure by $\mathcal{M}_{\mathcal{B}}$. Then for any k-vector space V we have ${}^{\mathcal{B}}\mathrm{Hom}_k(\mathcal{M}, V)$

$= \mathrm{Hom}_k(\mathcal{M}_{\mathcal{B}}, V)$. Since any right $\mathcal{C}$-comodule $\mathcal{M}$ is the union of its subcomodules $\mathcal{M}_{\mathcal{A}}$ over the finite-dimensional subcoalgebras $\mathcal{A} \subset \mathcal{C}$, it follows that

$$\mathrm{Hom}_k(\mathcal{M}, V) = \varprojlim\nolimits_{\mathcal{A}} {}^{\mathcal{A}}\mathrm{Hom}_k(\mathcal{M}, V).$$

Therefore,

$$\begin{aligned}\mathrm{Hom}^{\mathcal{C}}(\mathcal{N}^*, \mathrm{Hom}_k(\mathcal{M}, V)) &= \varprojlim\nolimits_{\mathcal{A}} \mathrm{Hom}^{\mathcal{C}}(\mathcal{N}^*, {}^{\mathcal{A}}\mathrm{Hom}_k(\mathcal{M}, V)) \\ = \varprojlim\nolimits_{\mathcal{A}} \mathrm{Hom}^{\mathcal{C}}({}^{\mathcal{A}}(\mathcal{N}^*), {}^{\mathcal{A}}\mathrm{Hom}_k(\mathcal{M}, V)) &= \varprojlim\nolimits_{\mathcal{A}} \mathrm{Hom}^{\mathcal{A}}((\mathcal{N}_{\mathcal{A}})^*, \mathrm{Hom}_k(\mathcal{M}_{\mathcal{A}}, V)) \\ &\simeq \varprojlim\nolimits_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(\mathcal{M}_{\mathcal{A}},\ \mathcal{N}_{\mathcal{A}} \otimes_k V) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(\mathcal{M},\ \mathcal{N} \otimes_k V)\end{aligned}$$

because the right $\mathcal{A}$-comodule $\mathcal{N}_{\mathcal{A}}$ is finite-dimensional.

To prove part (a), we notice from the computations above that the left $\mathcal{C}$-contramodule $\mathcal{N}^*$ is quasi-finitely generated if and only if a right $\mathcal{C}$-comodule $\mathcal{N}$ is quasi-finitely cogenerated. Furthermore, substituting $V = k$ into the assertion (b), we see that the dualization functor $\mathcal{N} \longmapsto \mathcal{N}^*\colon \mathsf{comod}\text{–}\mathcal{C} \longrightarrow \mathcal{C}\text{–}\mathsf{contra}$ is fully faithful on the full subcategory of quasi-finitely cogenerated comodules in $\mathsf{comod}\text{–}\mathcal{C}$. It remains to prove the essential surjectivity.

As the left $\mathcal{C}$-contramodule $\mathcal{J}^*$ is projective for any injective right $\mathcal{C}$-comodule $\mathcal{J}$, the dualization functor takes quasi-finitely cogenerated injective right $\mathcal{C}$-comodules to quasi-finitely generated projective left $\mathcal{C}$-contramodules. A projective left $\mathcal{C}$-contramodule $\mathfrak{P}$ is uniquely determined, up to isomorphism, by the left $\mathcal{C}^{\mathrm{ss}}$-contramodule ${}^{\mathcal{C}^{\mathrm{ss}}}\mathfrak{P}$, and all the quasi-finitely generated $\mathcal{C}^{\mathrm{ss}}$-contramodules belong to the image of the dualization functor; therefore so do all the quasi-finitely generated projective $\mathcal{C}$-contramodules (cf. the proofs of Lemmas 2.1.2(b) and 2.1.5(b)).

Finally, any quasi-finitely presented $\mathcal{C}$-contramodule is the cokernel of a morphism of quasi-finitely generated projective $\mathcal{C}$-contramodules, this morphism comes from a morphism of quasi-finitely cogenerated injective $\mathcal{C}$-comodules, the kernel of the latter morphism is a quasi-finitely copresented $\mathcal{C}$-comodule, and the dualization functor takes the kernels to the cokernels.

Part (c): For any right $\mathcal{C}$-comodule $\mathcal{M}$ and left $\mathcal{C}$-contramodule $\mathfrak{T}$, one has $\mathcal{M} \odot_{\mathcal{C}} \mathfrak{T} = (\varinjlim_{\mathcal{A}} \mathcal{M}_{\mathcal{A}}) \odot_{\mathcal{C}} \mathfrak{T} = \varinjlim_{\mathcal{A}}(\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \mathfrak{T}) = \varinjlim_{\mathcal{A}}(\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{A}} {}^{\mathcal{A}}\mathfrak{T})$, where the inductive limit is taken over all the finite-dimensional subcoalgebras $\mathcal{A} \subset \mathcal{C}$. In particular,

$$\begin{aligned}\mathcal{M} \odot_{\mathcal{C}} \mathrm{Hom}_k(\mathcal{N}, V) &= \varinjlim\nolimits_{\mathcal{A}}(\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \mathrm{Hom}_k(\mathcal{N}_{\mathcal{A}}, V)) \\ &\simeq \varinjlim\nolimits_{\mathcal{A}}((\mathcal{M}_{\mathcal{A}} \odot_{\mathcal{C}} \mathcal{N}_{\mathcal{A}}^*) \otimes_k V) = (\mathcal{M} \odot_{\mathcal{C}} \mathcal{N}^*) \otimes_k V,\end{aligned}$$

because $\mathcal{N}_{\mathcal{A}}$ is finite-dimensional. $\square$

A $\mathcal{C}$-comodule is said to be *strongly quasi-finitely copresented* if it has an injective coresolution consisting of quasi-finitely cogenerated injective $\mathcal{C}$-comodules. Similarly one could define "strongly quasi-finitely presented contramodules"; and the following two lemmas have their obvious dual-analogous contramodule versions.

Lemma 2.1.8. *Let $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow 0$ be a short exact sequence of $\mathcal{C}$-comodules. Then whenever two of the three comodules $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ are strongly quasi-finitely copresented, so is the third one.*

Proof. Dual to the proof of Lemma 1.1.1. □

Abusing terminology, we will say that a bounded below complex of $\mathcal{C}$-comodules is *strongly quasi-finitely copresented* if it is quasi-isomorphic to a bounded below complex of quasi-finitely cogenerated injective $\mathcal{C}$-comodules. Clearly, the class of all strongly quasi-finitely copresented complexes is closed under shifts and cones in $\mathsf{D}^+(\mathcal{C}\text{–}\mathsf{comod})$.

Lemma 2.1.9. (a) *Any bounded below complex of strongly quasi-finitely copresented $\mathcal{C}$-comodules is strongly quasi-finitely copresented.*

(b) *Let $\mathcal{M}^\bullet$ be a complex of $\mathcal{C}$-comodules concentrated in the cohomological degrees $\geqslant n$, where n is a fixed integer. Then $\mathcal{M}^\bullet$ is strongly quasi-finitely copresented if and only if is is quasi-isomorphic to a complex of strongly quasi-finitely copresented $\mathcal{C}$-comodules concentrated in the cohomological degrees $\geqslant n$.*

(c) *Let $\mathcal{M}^\bullet$ be a finite complex of $\mathcal{C}$-comodules concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$. Then $\mathcal{M}^\bullet$ is strongly quasi-finitely copresented if and only if it is quasi-isomorphic to a complex of $\mathcal{C}$-comodules $\mathcal{K}^\bullet$ concentrated in the cohomological degrees $n_1 \leqslant m \leqslant n_2$ such that the $\mathcal{C}$-comodules $\mathcal{K}^m$ are quasi-finitely cogenerated and injective for all $n_1 \leqslant m \leqslant n_2 - 1$, while the $\mathcal{C}$-comodule $\mathcal{K}^{n_2}$ is strongly quasi-finitely copresented.* □

Lemmas 2.1.8–2.1.9 are very similar to the respective results from Section 1.1. The following examples of pseudo-derived categories of comodules and contramodules, however, are quite different from Examples 1.1.4–1.1.5 in that the finiteness conditions (like the ones discussed above in this section) play essentially no role in them.

Given a left $\mathcal{C}$-comodule $\mathcal{M}$ and a right $\mathcal{C}$-comodule $\mathcal{N}$, the k-vector spaces $\mathrm{Cotor}_i^{\mathcal{C}}(\mathcal{N},\mathcal{M})$, $i = 0, -1, -2, \ldots$ are defined as the right derived functors of the left exact functor of cotensor product $\mathcal{N} \,\square_{\mathcal{C}}\, \mathcal{M}$, constructed by replacing any one or both of the comodules $\mathcal{N}$ and $\mathcal{M}$ by its injective coresolution, taking the cotensor product and computing the cohomology [11, Sections 0.2.2 and 1.2.2], [12, Section 4.7].

Examples 2.1.10. (1) Let $\mathcal{C}$ be a coassociative coalgebra and S be a class of right $\mathcal{C}$-comodules. Denote by $\mathsf{E} \subset \mathsf{A} = \mathcal{C}\text{–}\mathsf{comod}$ the full subcategory formed by all the left $\mathcal{C}$-comodules $\mathcal{E}$ such that $\mathrm{Cotor}_i^{\mathcal{C}}(\mathcal{S},\mathcal{E}) = 0$ for all $\mathcal{S} \in \mathsf{S}$ and all $i < 0$. Then the full subcategory $\mathsf{E} \subset \mathcal{C}\text{–}\mathsf{comod}$ is a coresolving subcategory closed under infinite direct sums. Thus the derived category $\mathsf{D}(\mathsf{E})$ of the exact category E is a pseudo-coderived category of the abelian category $\mathcal{C}\text{–}\mathsf{comod}$, that is an intermediate quotient category between the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ and the derived category $\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod})$, as explained in Section 0.9 of the Introduction.

(2) In particular, if $\mathsf{S} = \varnothing$, then one has $\mathsf{E} = \mathcal{C}\text{–}\mathsf{comod}$. On the other hand, if S is the class of all right $\mathcal{C}$-comodules, then $\mathsf{E} = \mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}}$ is the full subcategory of all injective left $\mathcal{C}$-comodules. In fact, it suffices to take S to be the set of all finite-dimensional right $\mathcal{C}$-comodules, or just irreducible right $\mathcal{C}$-comodules, to force $\mathsf{E} = \mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}}$ [14, Lemma 3.1(a)]. In this case, the derived category $\mathsf{D}(\mathsf{E}) = \mathsf{Hot}(\mathsf{E})$ of the (split) exact category E is equivalent to the coderived category of left $\mathcal{C}$-comodules $\mathsf{D}^{\mathsf{co}}(\mathcal{C}\text{–}\mathsf{comod})$ [11, Theorem 5.4(a) or 5.5(a)], [12, Theorem 4.4(c)].

Given a left $\mathcal{C}$-comodule $\mathcal{M}$ and a left $\mathcal{C}$-contramodule $\mathfrak{T}$, the k-vector spaces $\mathrm{Coext}^i_{\mathcal{C}}(\mathcal{M},\mathfrak{T})$, $i=0,\,-1,\,-2,\,\dots$ are defined as the left derived functors of the right exact functor of cohomomorphisms $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{M},\mathfrak{T})$, constructed by replacing either the comodule argument $\mathcal{M}$ by its injective coresolution, or the contramodule argument $\mathfrak{T}$ by its projective resolution, or both, taking the $\mathrm{Cohom}_{\mathcal{C}}$ and computing the homology [11, Sections 0.2.5 and 3.2.2], [12, Section 4.7].

Examples 2.1.11. (1) Let $\mathcal{D}$ be a coassociative coalgebra over k and S be a class of left $\mathcal{D}$-comodules. Denote by $\mathsf{F}\subset\mathsf{B}=\mathcal{D}\text{–}\mathsf{contra}$ the full subcategory formed by all the left $\mathcal{D}$-contramodules $\mathfrak{F}$ such that $\mathrm{Coext}^i_{\mathcal{D}}(\mathcal{S},\mathfrak{F})=0$ for all $\mathcal{S}\in\mathsf{S}$ and all $i<0$. Then the full subcategory $\mathsf{F}\subset\mathcal{D}\text{–}\mathsf{contra}$ is a resolving subcategory closed under infinite products. Thus the derived category $\mathsf{D}(\mathsf{F})$ of the exact category F is a pseudo-contraderived category of the abelian category $\mathcal{D}\text{–}\mathsf{contra}$, that is an intermediate quotient category between the contraderived category $\mathsf{D}^{\mathsf{ctr}}(\mathcal{D}\text{–}\mathsf{contra})$ and the derived category $\mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})$, as explained in Section 0.9.

(2) In particular, if $\mathsf{S}=\varnothing$, then one has $\mathsf{F}=\mathcal{D}\text{–}\mathsf{contra}$. On the other hand, if S is the class of all left $\mathcal{D}$-comodules, then $\mathsf{F}=\mathcal{D}\text{–}\mathsf{contra}_{\mathsf{proj}}$ is the full subcategory of all projective left $\mathcal{D}$-contramodules [14, Lemma 3.1(b)]. In fact, it suffices to take $\mathcal{S}$ to be the set of all finite-dimensional left $\mathcal{D}$-comodules, or just irreducible left $\mathcal{D}$-comodules, to force $\mathsf{F}=\mathcal{D}\text{–}\mathsf{contra}_{\mathsf{proj}}$ [11, Lemma A.3]. In this case, the derived category $\mathsf{D}(\mathsf{F})=\mathsf{Hot}(\mathsf{F})$ of the (split) exact category F is equivalent to the contraderived category of left $\mathcal{D}$-contramodules $\mathsf{D}^{\mathsf{ctr}}(\mathcal{D}\text{–}\mathsf{contra})$ [11, Theorem 5.4(b) or 5.5(b)], [12, Theorem 4.4(d)], [13, Corollary A.6.2].

Looking on Examples 2.1.10(1)–2.1.11(1), it appears that coresolving subcategories closed under infinite direct sums may be more common in comodules than in modules, and resolving subcategories closed under infinite products may be more common in contramodules than in modules.

2.2. **Auslander and Bass classes.** We recall the definition of a pseudo-dualizing complex of bicomodules from Section 0.7. Let $\mathcal{C}$ and $\mathcal{D}$ be coassociative coalgebras over a field k.

A *pseudo-dualizing complex* $\mathcal{L}^\bullet$ for the coalgebras $\mathcal{C}$ and $\mathcal{D}$ is a finite complex of $\mathcal{C}$-$\mathcal{D}$-bicomodules satisfying the following two condition:

- (ii) the homothety maps $\mathcal{C}^*\longrightarrow\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{comod}\text{–}\mathcal{D})}(\mathcal{L}^\bullet,\mathcal{L}^\bullet[*])$ and $\mathcal{D}^{*\mathrm{op}}\longrightarrow\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–}\mathsf{comod})}(\mathcal{L}^\bullet,\mathcal{L}^\bullet[*])$ are isomorphisms of graded rings;
- (iii) the complex $\mathcal{L}^\bullet$ is strongly quasi-finitely copresented as a complex of left $\mathcal{C}$-comodules and as a complex of right $\mathcal{D}$-comodules.

Here the condition (iii) refers to the definition of a strongly quasi-finitely copresented complex of comodules in Section 2.1. The complex $\mathcal{L}^\bullet$ is viewed as an object of the bounded derived category of $\mathcal{C}$-$\mathcal{D}$-bicomodules $\mathsf{D}^{\mathsf{b}}(\mathcal{C}\text{–}\mathsf{comod}\text{–}\mathcal{D})$.

Given a $\mathcal{C}$-$\mathcal{D}$-bicomodule $\mathcal{K}$, the functor of contratensor product $\mathcal{K}\odot_{\mathcal{D}}-\colon\mathcal{D}\text{–}\mathsf{contra}\longrightarrow\mathcal{C}\text{–}\mathsf{comod}$ is left adjoint to the functor of comodule homomorphisms $\mathrm{Hom}_{\mathcal{C}}(\mathcal{K},-)\colon\mathcal{C}\text{–}\mathsf{comod}\longrightarrow\mathcal{D}\text{–}\mathsf{contra}$. Hence, in particular, the functor of contratensor product of complexes $\mathcal{L}^\bullet \odot_{\mathcal{D}} -\colon \mathsf{Hot}(\mathcal{D}\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(\mathcal{C}\text{–}\mathsf{comod})$ is left adjoint to the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)\colon \mathsf{Hot}(\mathcal{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(\mathcal{D}\text{–}\mathsf{contra})$.

We will use the existence theorem of homotopy injective resolutions of complexes of comodules and homotopy projective resolutions of complexes of contramodules [12, Theorem 2.4] in order to work with the conventional unbounded derived categories of comodules and contramodules $\mathsf{D}(\mathcal{C}\text{–}\mathsf{comod})$ and $\mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})$. Using the homotopy projective and homotopy injective resolutions of the second arguments, one constructs the derived functors $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -\colon \mathsf{D}(\mathcal{D}\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(\mathcal{C}\text{–}\mathsf{comod})$ and $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)\colon \mathsf{D}(\mathcal{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(\mathcal{D}\text{–}\mathsf{contra})$. For the same reasons as in Section 1.2, the left derived functor $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} -$ is left adjoint to the right derived functor $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, -)$.

As in Section 1.2, we will use the following simplified notation. Given two complexes of left $\mathcal{C}$-comodules $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$, we denote by $\mathrm{Ext}_{\mathcal{C}}^n(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ the vector spaces $H^n\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{C}\text{–}\mathsf{comod})}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)$ of cohomology of the complex $\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}^\bullet, \mathcal{J}^\bullet)$, where $\mathcal{N}^\bullet \longrightarrow \mathcal{J}^\bullet$ is a quasi-isomorphism of complexes of left $\mathcal{C}$-comodules and $\mathcal{J}^\bullet$ is a homotopy injective complex of left $\mathcal{C}$-comodules. Given a complex of right $\mathcal{D}$-comodules $\mathcal{N}^\bullet$ and a complex of left $\mathcal{D}$-contramodules $\mathfrak{T}^\bullet$, we denote by $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{N}^\bullet, \mathfrak{T}^\bullet)$ the vector spaces $H^{-n}(\mathcal{N}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{T}^\bullet)$ of cohomology of the complex $\mathcal{N}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{P}^\bullet = \mathcal{N}^\bullet \odot_{\mathcal{D}} \mathfrak{P}^\bullet$, where $\mathfrak{P}^\bullet \longrightarrow \mathfrak{T}^\bullet$ is a quasi-isomorphism of complexes of left $\mathcal{D}$-contramodules and $\mathfrak{P}^\bullet$ is a homotopy projective complex of left $\mathcal{D}$-contramodules.

Suppose that the finite complex $\mathcal{L}^\bullet$ is situated in the cohomological degrees $-d_1 \leqslant m \leqslant d_2$. Then one has $\mathrm{Ext}_{\mathcal{C}}^n(\mathcal{L}^\bullet, \mathcal{J}) = 0$ for all $n > d_1$ and all injective left $\mathcal{C}$-comodules $\mathcal{J}$. Similarly, one has $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{L}^\bullet, \mathfrak{P}) = 0$ for all $n > d_1$ and all projective left $\mathcal{D}$-contramodules $\mathfrak{P}$. Choose an integer $l_1 \geqslant d_1$ and consider the following full subcategories in the abelian categories of left $\mathcal{C}$-comodules and $\mathcal{D}$-contramodules:

- $\mathsf{E}_{l_1} = \mathsf{E}_{l_1}(\mathcal{L}^\bullet) \subset \mathcal{C}\text{–}\mathsf{comod}$ is the full subcategory consisting of all the $\mathcal{C}$-comodules $\mathcal{E}$ such that $\mathrm{Ext}_{\mathcal{C}}^n(\mathcal{L}^\bullet, \mathcal{E}) = 0$ for all $n > l_1$ and the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \longrightarrow \mathcal{E}$ is an isomorphism in $\mathsf{D}^-(\mathcal{C}\text{–}\mathsf{comod})$;
- $\mathsf{F}_{l_1} = \mathsf{F}_{l_1}(\mathcal{L}^\bullet) \subset \mathcal{D}\text{–}\mathsf{contra}$ is the full subcategory consisting of all the $\mathcal{D}$-contramodules $\mathfrak{F}$ such that $\mathrm{Ctrtor}_n^{\mathcal{D}}(\mathcal{L}^\bullet, \mathfrak{F}) = 0$ for all $n > l_1$ and the adjunction morphism $\mathfrak{F} \longrightarrow \mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F})$ is an isomorphism in $\mathsf{D}^+(\mathcal{D}\text{–}\mathsf{contra})$.

Clearly, for any $l_1'' \geqslant l_1' \geqslant d_1$, one has $\mathsf{E}_{l_1'} \subset \mathsf{E}_{l_1''} \subset \mathcal{C}\text{–}\mathsf{comod}$ and $\mathsf{F}_{l_1'} \subset \mathsf{F}_{l_1''} \subset \mathcal{D}\text{–}\mathsf{contra}$. The category F_{l_1} can be called the *Auslander class of contramodules* corresponding to a pseudo-dualizing complex $\mathcal{L}^\bullet$, while the category E_{l_1} is the *Bass class of comodules.*

Lemma 2.2.1. (a) *The full subcategory* $\mathsf{E}_{l_1} \subset \mathcal{C}\text{–}\mathsf{comod}$ *is closed under the cokernels of injective morphisms, extensions, and direct summands.*

(b) *The full subcategory* $\mathsf{F}_{l_1} \subset \mathcal{D}\text{–}\mathsf{contra}$ *is closed under the kernels of surjective morphisms, extensions, and direct summands.* □

The formulation of the next lemma is similar to that of Lemma 1.2.2, but the proof is quite different. Rather, it resembles the related arguments in the proofs of [19, Theorem 2.6] and [15, Theorem 4.9].

Lemma 2.2.2. (a) *The full subcategory* $\mathsf{E}_{l_1} \subset \mathcal{C}\text{–}\mathsf{comod}$ *contains all the injective left* $\mathcal{C}$*-comodules.*

(b) *The full subcategory* $\mathsf{F}_{l_1} \subset \mathcal{D}\text{–}\mathsf{contra}$ *contains all the projective left* $\mathcal{D}$*-contramodules.*

Proof. Part (a): we have to check that for any injective left $\mathcal{C}$-comodule $\mathcal{E}$ the adjunction morphism $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) = \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \longrightarrow \mathcal{E}$ is a quasi-isomorphism. It suffices to consider the case of a cofree left $\mathcal{C}$-comodule $\mathcal{E} = \mathcal{C} \otimes_k V$, where V is a k-vector space. Then one has $\operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet, \mathcal{E}) \simeq \operatorname{Hom}_k(\mathcal{L}^\bullet, V)$.

According to the condition (iii), there exists a bounded below complex of quasi-finitely cogenerated injective right $\mathcal{D}$-comodules $\mathcal{J}^\bullet$ endowed with a quasi-isomorphism of complexes of right $\mathcal{D}$-comodules $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$. Then we have a quasi-isomorphism of complexes of left $\mathcal{D}$-contramodules $\operatorname{Hom}_k(\mathcal{J}^\bullet, V) \longrightarrow \operatorname{Hom}_k(\mathcal{L}^\bullet, V)$, and $\operatorname{Hom}_k(\mathcal{J}^\bullet, V)$ is a bounded above complex of projective left $\mathcal{D}$-contramodules. Hence $\mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \operatorname{Hom}_k(\mathcal{L}^\bullet, V) = \mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{J}^\bullet, V)$, and it remains to show that the morphism of complexes of left $\mathcal{C}$-comodules

$$\mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{J}^\bullet, V) \longrightarrow \mathcal{C} \otimes_k V \tag{13}$$

is a quasi-isomorphism. The morphism (13) is constructed in terms of the morphism of complexes of right $\mathcal{D}$-comodules $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$ and the left $\mathcal{C}$-coaction in the complex $\mathcal{L}^\bullet$.

In particular, substituting $V = k$ into (13), we have a morphism of complexes of left $\mathcal{C}$-comodules

$$\mathcal{L}^\bullet \odot_{\mathcal{D}} \mathcal{J}^{\bullet *} \longrightarrow \mathcal{C}. \tag{14}$$

Passing to the dual vector spaces in (14), we obtain a map $\mathcal{C}^* \longrightarrow \operatorname{Hom}^{\mathcal{D}}(\mathcal{J}^{\bullet *}, \mathcal{L}^{\bullet *})$, which is equal to the composition of the homothety map $\mathcal{C}^* \longrightarrow \operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet) = \mathbb{R}\operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{L}^\bullet)$ with the dualization map $\operatorname{Hom}_{\mathcal{D}}(\mathcal{L}^\bullet, \mathcal{J}^\bullet) \longrightarrow \operatorname{Hom}^{\mathcal{D}}(\mathcal{J}^{\bullet *}, \mathcal{L}^{\bullet *})$.

As the homothety map is a quasi-isomorphism by the condition (ii) and the dualization map is an isomorphism of complexes by Proposition 2.1.7(b) (because $\mathcal{J}^\bullet$ is a complex of quasi-finitely cogenerated right $\mathcal{D}$-comodules, while $\mathcal{L}^\bullet$ is a finite complex), it follows that passing to the dual vector spaces in (14) produces a quasi-isomorphism. Hence the map (14) is a quasi-isomorphism, too. Finally, by Proposition 2.1.7(c) the natural map $(\mathcal{L}^\bullet \odot_{\mathcal{D}} \mathcal{J}^{\bullet *}) \otimes_k V \longrightarrow \mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{J}^\bullet, V)$ is an isomorphism of complexes. Therefore, the map (13) is also a quasi-isomorphism.

Part (b): we have to check that for any projective left $\mathcal{D}$-contramodule $\mathfrak{F}$ the adjunction morphism $\mathfrak{F} \longrightarrow \mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet,\ \mathcal{L}^\bullet \odot_{\mathcal{D}}^{\mathbb{L}} \mathfrak{F}) = \mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet,\ \mathcal{L}^\bullet \odot_{\mathcal{D}} \mathfrak{F})$ is a quasi-isomorphism. It suffices to consider the case of a free left $\mathcal{D}$-contramodule $\mathfrak{F} = \operatorname{Hom}_k(\mathcal{D}, V)$, where V is a k-vector space. Then one has $\mathcal{L}^\bullet \odot_{\mathcal{D}} \operatorname{Hom}_k(\mathcal{D}, V) \simeq \mathcal{L}^\bullet \otimes_k V$.

According to the condition (iii), there exists a bounded below complex of quasi-finitely cogenerated injective left $\mathcal{C}$-comodules $\mathcal{J}^\bullet$ endowed with a quasi-isomorphism of complexes of left $\mathcal{C}$-comodules $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$. Then $\mathbb{R}\operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet,\ \mathcal{L}^\bullet \otimes_k V) = \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet,\ \mathcal{J}^\bullet \otimes_k V)$, and it remains to show that the morphism of complexes of left $\mathcal{D}$-contramodules

$$\operatorname{Hom}_k(\mathcal{D}, V) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(\mathcal{L}^\bullet,\ \mathcal{J}^\bullet \otimes_k V) \tag{15}$$

is a quasi-isomorphism. The morphism (13) is constructed in terms of the morphism of complexes of left $\mathcal{C}$-comodules $\mathcal{L}^\bullet \longrightarrow \mathcal{J}^\bullet$ and the right $\mathcal{D}$-coaction in the complex $\mathcal{L}^\bullet$.

In the same way as in the proof of part (a), one deduces from the condition (ii) using Proposition 2.1.7(b) that the natural morphism of complexes of right $\mathcal{D}$-comodules

$$\mathcal{L}^\bullet \odot_{\mathcal{C}^{\mathrm{op}}} \mathcal{J}^{\bullet *} \longrightarrow \mathcal{D} \tag{16}$$

is a quasi-isomorphism. Applying the functor $\operatorname{Hom}_k(-, V)$ to (16), we see that the natural map

$$\operatorname{Hom}_k(\mathcal{D}, V) \longrightarrow \operatorname{Hom}_k(\mathcal{L}^\bullet \odot_{\mathcal{C}^{\mathrm{op}}} \mathcal{J}^{\bullet *},\ V) \simeq \operatorname{Hom}^{\mathcal{C}^{\mathrm{op}}}(\mathcal{J}^{\bullet *}, \operatorname{Hom}_k(\mathcal{L}^\bullet, V)) \tag{17}$$

is a quasi-isomorphism, too. It remains to use Proposition 2.1.7(b) again in order to identify the right-hand sides of (15) and (17). ☐

3. Ideals in Commutative Rings

Appendix. Derived Functors of Finite Homological Dimension II

The aim of this appendix is to work out a generalization of the constructions of [15, Appendix B] that is needed for the purposes of the present paper. We use an idea borrowed from [4, Appendix A] in order to simplify and clarify the exposition.

A.1. **Posing the problem.** First we need to recall some notation from [15]. Given an additive category A, we denote by $\mathsf{C}^+(\mathsf{A})$ the category of bounded below complexes in A, viewed either as a DG-category (with complexes of morphisms), or simply as an additive category, with closed morphisms of degree 0. When A is an exact category, the full subcategory $\mathsf{C}^{\geqslant 0}(\mathsf{A}) \subset \mathsf{C}^+(\mathsf{A})$ of nonnegatively cohomologically graded complexes in A and closed morphisms of degree 0 between them has a natural exact category structure, with termwise exact short exact sequences of complexes.

Let E be an exact category and $\mathsf{J} \subset \mathsf{E}$ be a coresolving subcategory (in the sense of Section 0.9), endowed with the exact category structure inherited from E. As it was pointed out in [15], a closed morphism in $\mathsf{C}^+(\mathsf{J})$ is a quasi-isomorphism of complexes in J if and only if it is a quasi-isomorphism of complexes in E. A short sequence in $\mathsf{C}^{\geqslant 0}(\mathsf{J})$ is exact in $\mathsf{C}^{\geqslant 0}(\mathsf{J})$ if and only if it is exact in $\mathsf{C}^{\geqslant 0}(\mathsf{E})$.

Modifying slightly the notation in [15], we denote by ${}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J})$ the full subcategory in the exact category $\mathsf{C}^{\geqslant 0}(\mathsf{J})$ consisting of all the complexes $0 \longrightarrow J^0 \longrightarrow J^1 \longrightarrow J^2 \longrightarrow \cdots$ in J for which there exists an object $E \in \mathsf{E}$ together with a morphism $E \longrightarrow J^0$ such that the sequence $0 \longrightarrow E \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$ is exact in E. By the definition, one has ${}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J}) = \mathsf{C}^{\geqslant 0}(\mathsf{J}) \cap {}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{E}) \subset \mathsf{C}^{\geqslant 0}(\mathsf{E})$. The full subcategory ${}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J})$ is closed under extensions and the cokernels of admissible monomorphisms in $\mathsf{C}^{\geqslant 0}(\mathsf{J})$; so it inherits an exact category structure.

Let B be another exact category and $\mathsf{F} \subset \mathsf{B}$ be a resolving subcategory. We will suppose that the additive category B contains the images of idempotent endomorphisms of its objects. Let $-l_2 \leqslant l_1$ be two integers. Denote by $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})$ the exact category $\mathsf{C}^{\geqslant 0}(\mathsf{B})[l_2] \subset \mathsf{C}^{+}(\mathsf{B})$ of complexes in B concentrated in the cohomological degrees $\geqslant -l_2$, and by $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1} \subset \mathsf{C}^{\geqslant -l_2}(\mathsf{B})$ the full subcategory consisting of all complexes $0 \longrightarrow B^{-l_2} \longrightarrow \cdots \longrightarrow B^{l_1} \longrightarrow \cdots$ such that the sequence $B^{l_1} \longrightarrow B^{l_1+1} \longrightarrow B^{l_1+2} \longrightarrow \cdots$ is exact in B. Furthermore, let $\mathsf{C}_{\mathsf{F}}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1} \subset \mathsf{C}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}$ be the full subcategory of all complexes that are isomorphic in the derived category $\mathsf{D}(\mathsf{B})$ to complexes of the form $0 \longrightarrow F^{-l_2} \longrightarrow \cdots \longrightarrow F^{l_1} \longrightarrow 0$, with the terms belonging to F and concentrated in the cohomological degrees $-l_2 \leqslant m \leqslant l_1$.

For example, one has $\mathsf{C}^{\geqslant 0}(\mathsf{B})^{\leqslant 0} = {}_{\mathsf{B}}\mathsf{C}^{\geqslant 0}(\mathsf{B})$. The full subcategory $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}$ is closed under extensions and the cokernels of admissible monomorphisms in the exact category $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})$, while (essentially by [25, Proposition 2.3(2)] or [13, Lemma A.5.4(a-b)]) the full subcategory $\mathsf{C}_{\mathsf{F}}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}$ is closed under extensions and the kernels of admissible epimorphisms in $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}$. So the full subcategory $\mathsf{C}_{\mathsf{F}}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}$ inherits an exact category structure from $\mathsf{C}^{\geqslant -l_2}(\mathsf{B})$.

Suppose that we are given a DG-functor $\Psi\colon \mathsf{C}^{+}(\mathsf{J}) \longrightarrow \mathsf{C}^{+}(\mathsf{B})$ taking acyclic complexes in the exact category J to acyclic complexes in the exact category B. Suppose further that the restriction of Ψ to the subcategory ${}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J}) \subset \mathsf{C}^{+}(\mathsf{J})$ is an exact functor between exact categories

$$\Psi\colon {}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J}) \longrightarrow \mathsf{C}_{\mathsf{F}}^{\geqslant -l_2}(\mathsf{B})^{\leqslant l_1}. \tag{18}$$

Our aim is to construct the right derived functor

$$\mathbb{R}\Psi\colon \mathsf{D}^{\star}(\mathsf{E}) \longrightarrow \mathsf{D}^{\star}(\mathsf{F}) \tag{19}$$

acting between any bounded or unbounded, conventional or absolute derived categories $\mathsf{D}^{\star}$ with the symbols $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs.

Under certain conditions, one can also have the derived functor $\mathbb{R}\Psi$ acting between the coderived or contraderived categories, $\star = \mathsf{co}$ or ctr, of the exact categories E and F. When the exact categories E and B have exact functors of infinite product, the full subcategories $\mathsf{J} \subset \mathsf{E}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under infinite products, and the functor Ψ preserves infinite products, there will be the derived functor $\mathbb{R}\Psi$ acting between the contraderived categories, $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{ctr}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{F})$.

When the exact categories E and B have exact functors of infinite direct sum, the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under infinite direct sums, and for any family of complexes $J^\bullet_\alpha \in \mathsf{C}^{\geqslant 0}(\mathsf{J})$ and a complex $I^\bullet \in \mathsf{C}^{\geqslant 0}(\mathsf{J})$ endowed with a quasi-isomorphism $\bigoplus_\alpha J^\bullet_\alpha \longrightarrow I^\bullet$ of complexes in the exact category E, the induced morphism

$$\bigoplus\nolimits_\alpha \Psi(J^\bullet_\alpha) \longrightarrow \Psi(I^\bullet)$$

is a quasi-isomorphism of complexes in the exact category B, there will be the derived functor $\mathbb{R}\Psi$ acting between the coderived categories, $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{F})$.

The construction of the derived functor $\mathbb{R}\Psi$ in [15, Appendix B] is the particular case of the construction below corresponding to the situation with $\mathsf{F} = \mathsf{B}$.

A.2. **The construction of derived functor.** The following construction of the derived functor (19) is based on a version of the result of [4, Proposition A.3].

Since the DG-functor $\Psi\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$ preserves quasi-isomorphisms, it induces a triangulated functor

$$\Psi\colon \mathsf{D}^+(\mathsf{J}) \longrightarrow \mathsf{D}^+(\mathsf{B}).$$

Taking into account the triangulated equivalence $\mathsf{D}^+(\mathsf{J}) \simeq \mathsf{D}^+(\mathsf{E})$ (provided by the dual version of [13, Proposition A.3.1(a)]), we obtain the derived functor

$$\mathbb{R}\Psi\colon \mathsf{D}^+(\mathsf{E}) \longrightarrow \mathsf{D}^+(\mathsf{B}).$$

Now our assumptions on Ψ imply that the functor $\mathbb{R}\Psi$ takes the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \subset \mathsf{D}^+(\mathsf{E})$ into the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \subset \mathsf{D}^{\mathsf{b}}(\mathsf{B}) \subset \mathsf{D}^+(\mathsf{B})$; hence the triangulated functor

$$\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{F}). \tag{20}$$

For any exact category A, we denote by $\mathsf{C}(\mathsf{A})$ the exact category of unbounded complexes in A, with termwise exact short exact sequences of complexes. In order to construct the derived functor $\mathbb{R}\Psi$ for the derived categories with the symbols other than $\star = \mathsf{b}$, we are going to substitute into (20) the exact category $\mathsf{C}(\mathsf{E})$ in place of E and the exact category $\mathsf{C}(\mathsf{F})$ in place of F.

For any category Γ and DG-category DG, there is a DG-category whose objects are all the functors $\Gamma \longrightarrow \mathsf{DG}$ taking morphisms in Γ to closed morphisms of degree 0 in DG, and whose complexes of morphisms are constructed as the complexes of morphisms of functors. We denote this DG-category by DG^Γ. So diagrams of any fixed shape in a given DG-category form a DG-category. Given a DG-functor $F\colon {}'\mathsf{DG} \longrightarrow {}''\mathsf{DG}$, there is the induced DG-functor between the categories of diagrams $F^\Gamma\colon {}'\mathsf{DG}^\Gamma \longrightarrow {}''\mathsf{DG}^\Gamma$. In particular, the DG-category of complexes $\mathsf{C}(\mathsf{DG})$ in a given DG-category DG can be constructed a full DG-subcategory of the DG-category of diagrams of the corresponding shape in DG.

Applying this construction to the DG-functor Ψ and restricting to the full DG-subcategories of uniformly bounded bicomplexes, we obtain a DG-functor

$$\Psi_{\mathsf{C}}\colon \mathsf{C}^+(\mathsf{C}(\mathsf{J})) \longrightarrow \mathsf{C}^+(\mathsf{C}(\mathsf{B})).$$

Here the categories of unbounded complexes $\mathsf{C}(\mathsf{J})$ and $\mathsf{C}(\mathsf{B})$ are simply viewed as additive/exact categories of complexes and closed morphisms of degree 0 between them. The DG-structures come from the differentials raising the degree in which the bicomplexes are bounded below.

The functor Ψ_{C} takes acyclic complexes in the exact category $\mathsf{C}(\mathsf{J})$ to acyclic complexes in the exact category $\mathsf{C}(\mathsf{B})$. In view of the standard properties of the resolution dimension [13, Corollary A.5.2], the functor Ψ_{C} takes the full subcategory ${}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geqslant 0}(\mathsf{C}(\mathsf{J})) \subset \mathsf{C}^{+}(\mathsf{C}(\mathsf{J}))$ into the full subcategory $\mathsf{C}^{\geqslant -l_2}_{\mathsf{C}(\mathsf{F})}(\mathsf{C}(\mathsf{B}))^{\leqslant l_1} \subset \mathsf{C}^{+}(\mathsf{C}(\mathsf{B}))$,

$$\Psi_{\mathsf{C}}\colon {}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geqslant 0}(\mathsf{C}(\mathsf{J})) \longrightarrow \mathsf{C}^{\geqslant -l_2}_{\mathsf{C}(\mathsf{F})}(\mathsf{C}(\mathsf{B}))^{\leqslant l_1}.$$

Finally, the functor Ψ_{C} is exact in restriction to the exact category ${}_{\mathsf{C}(\mathsf{E})}\mathsf{C}^{\geqslant 0}(\mathsf{C}(\mathsf{J}))$, since the functor Ψ is exact in restriction to the exact category ${}_{\mathsf{E}}\mathsf{C}^{\geqslant 0}(\mathsf{J})$.

Applying the construction of the derived functor (20) to the DG-functor Ψ_{C} in place of Ψ, we obtain a triangulated functor

$$\mathbb{R}\Psi_{\mathsf{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{F})). \tag{21}$$

Similarly one can construct the derived functors $\mathbb{R}\Psi_{\mathsf{C}^{\leqslant 0}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{\leqslant 0}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{\leqslant 0}(\mathsf{F}))$ and $\mathbb{R}\Psi_{\mathsf{C}^{\geqslant 0}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{\geqslant 0}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{\geqslant 0}(\mathsf{F}))$ acting between the bounded derived categories of the exact categories of nonpositively or nonnegatively cohomologically graded complexes. Shifting and passing to the direct limits of fully faithful embeddings, one can obtain the derived functors $\mathbb{R}\Psi_{\mathsf{C}^-}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{-}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{-}(\mathsf{F}))$ and $\mathbb{R}\Psi_{\mathsf{C}^+}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{+}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}^{-}(\mathsf{F}))$ acting between the bounded derived categories of the exact categories of bounded above or bounded below complexes, etc.

In order to pass from (21) to (19) with $\star = \mathsf{abs}$, we will apply the following version of [4, Proposition A.3(2)]. Clearly, for any exact category A the totalization of bounded complexes of complexes in A is a triangulated functor

$$\mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{A})) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{A}). \tag{22}$$

Proposition A.1. *For any exact category* A*, the totalization functor* (22) *is a Verdier quotient functor. Its kernel is the thick subcategory generated by the contractible complexes in* A*, viewed as objects of* $\mathsf{C}(\mathsf{A})$*.*

Proof. Denote by $\mathsf{A}_{\mathsf{spl}}$ the additive category A endowed with the split exact category structure (i. e., all the short exact sequences are split). Following [4], one first checks the assertion of proposition for the exact category $\mathsf{A}_{\mathsf{spl}}$.

In this case, $\mathsf{C}(\mathsf{A}_{\mathsf{spl}})$ is a Frobenius exact category whose projective-injective objects are the contractible complexes, and $\mathsf{D}^{\mathsf{abs}}(\mathsf{A}_{\mathsf{spl}}) = \mathsf{Hot}(\mathsf{A}_{\mathsf{spl}})$ is the stable category of the Frobenius exact category $\mathsf{C}(\mathsf{A}_{\mathsf{spl}})$. The quotient category of the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{A}_{\mathsf{spl}}))$ by the bounded homotopy category of complexes of projective-injective objects in $\mathsf{C}(\mathsf{A}_{\mathsf{spl}})$ is just another construction of the stable category of a Frobenius exact category, and the totalization functor is the inverse equivalence to the comparison functor between the two constructions of the stable category.

Then, in order to pass from the functor (22) for the exact category $\mathsf{A}_{\mathsf{spl}}$ to the similar functor for the exact category A, one takes the quotient category by the acyclic bounded complexes of complexes on the left-hand side, transforming $\mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{A}_{\mathsf{spl}}))$ into

$\mathsf{D^b}(\mathsf{C}(\mathsf{A}))$, and the quotient category by the totalizations of such bicomplexes on the right-hand side, transforming $\mathsf{Hot}(\mathsf{A})$ into $\mathsf{D^{abs}}(\mathsf{A})$. □

It remains to notice that the contractible complexes in A are the direct summands of the cones of identity endomorphisms of complexes in A, and the functor (21) obviously takes the cones of identity endomorphisms of complexes in E (viewed as objects of $\mathsf{C}(\mathsf{E})$) to bicomplexes whose totalizations are contractible complexes in F. This provides the desired derived functor (19) for $\star = \mathsf{abs}$.

In order to pass from (21) to (19) with $\star = \varnothing$, the following corollary of Proposition A.1 can be applied. Consider the totalization functor

$$\mathsf{D^b}(\mathsf{C}(\mathsf{A})) \longrightarrow \mathsf{D}(\mathsf{A}). \tag{23}$$

Corollary A.2. *For any exact category* A*, the totalization functor* (23) *is a Verdier quotient functor. Its kernel is the thick subcategory generated by the acyclic complexes in* A*, viewed as objects of* $\mathsf{C}(\mathsf{A})$*.* □

Using the condition that the functor (18) takes short exact sequences to short exact sequences together with [15, Lemma B.2(e)], one shows that the functor (21) takes acyclic complexes in E (viewed as objects of $\mathsf{C}(\mathsf{E})$) to bicomplexes with acyclic totalizations. This provides the derived functor (19) for $\star = \varnothing$.

To construct the derived functors $\mathbb{R}\Psi$ acting between the bounded above and bounded below versions of the conventional and absolute derived categories (with $\star = +$, $-$, $\mathsf{abs}+$, or $\mathsf{abs}-$), one can notice that the functors $\mathbb{R}\Psi$ for $\star = \varnothing$ or abs take bounded above/below complexes to (objects representable by) bounded above/below complexes, and use the fact that the embedding functors from the bounded above/below conventional/absolute derived categories into the unbounded ones are fully faithful [13, Lemma A.1.1]. Alternatively, one can repeat the above arguments with the categories of unbounded complexes $\mathsf{C}(\mathsf{A})$ replaced with the bounded above/below ones $\mathsf{C}^-(\mathsf{A})$ or $\mathsf{C}^+(\mathsf{A})$. The derived functor $\mathbb{R}\Psi$ with $\star = \mathsf{b}$ constructed in such a way agrees with the functor (20).

To construct the derived functor $\mathbb{R}\Psi$ acting between the coderived or contraderived categories (under the respective assumptions in Section A.1), one considers the derived functor $\mathbb{R}\Psi$ for $\star = \mathsf{abs}$, and checks that the kernel of the composition $\mathsf{C}(\mathsf{E}) \longrightarrow \mathsf{D^{abs}}(\mathsf{E}) \longrightarrow \mathsf{D^{abs}}(\mathsf{F}) \longrightarrow \mathsf{D^{co}}(\mathsf{F})$ or $\mathsf{C}(\mathsf{E}) \longrightarrow \mathsf{D^{abs}}(\mathsf{E}) \longrightarrow \mathsf{D^{abs}}(\mathsf{F}) \longrightarrow \mathsf{D^{ctr}}(\mathsf{F})$ is closed under the infinite direct sums or infinite products, respectively. The facts that the kernels of the additive functors $\mathsf{C}(\mathsf{F}) \longrightarrow \mathsf{D^{co/ctr}}(\mathsf{F})$ are closed under the infinite direct sums/products and the total complex of a finite acyclic complex of unbounded complexes in F is absolutely acyclic need to be used.

A.3. **The dual setting.** The notation $\mathsf{C}^{\leqslant 0}(\mathsf{B}) \subset \mathsf{C}^-(\mathsf{B})$ for an additive or exact category B has the similar or dual meaning to the one in Section A.1.

Let F be an exact category and $\mathsf{P} \subset \mathsf{F}$ be a resolving subcategory, endowed with the inherited exact category structure. A closed morphism in $\mathsf{C}^-(\mathsf{P})$ is a quasi-isomorphism of complexes in P if and only if it is a quasi-isomorphism of complexes in F. A short sequence in $\mathsf{C}^{\leqslant 0}(\mathsf{P})$ is exact in $\mathsf{C}^{\leqslant 0}(\mathsf{P})$ if and only if it is exact in $\mathsf{C}^{\leqslant 0}(\mathsf{F})$.

Following the notation in Section A.1, denote by ${}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{P})$ the full subcategory in the exact category $\mathsf{C}^{\leqslant 0}(\mathsf{P})$ consisting of all the complexes $\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0$ in P for which there exists an object $F \in \mathsf{F}$ together with a morphism $P^0 \longrightarrow F$ such that the sequence $\cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow F \longrightarrow 0$ is exact in F. By the definition, one has ${}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{P}) = \mathsf{C}^{\leqslant 0}(\mathsf{P}) \cap {}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{F}) \subset \mathsf{C}^{\leqslant 0}(\mathsf{F})$. The full subcategory ${}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{P})$ is closed under extensions and the kernels of admissible epimorphisms in $\mathsf{C}^{\leqslant 0}(\mathsf{P})$; so it inherits an exact category structure.

Let A be another exact category and $\mathsf{E} \subset \mathsf{A}$ be a coresolving subcategory. Suppose that the additive category A contains the images of idempotent endomorphisms of its objects. Let $-l_1 \leqslant l_2$ be two integers. Denote by $\mathsf{C}^{\leqslant l_2}(\mathsf{A})$ the exact category $\mathsf{C}^{\leqslant 0}(\mathsf{A})[-l_2] \subset \mathsf{C}^-(\mathsf{A})$ of complexes in A concentrated in the cohomological degrees $\leqslant l_2$, and by $\mathsf{C}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1} \subset \mathsf{C}^{\leqslant l_2}(\mathsf{A})$ the full subcategory consisting of all complexes $\cdots \longrightarrow A^{-l_1} \longrightarrow \cdots \longrightarrow A^{l_2} \longrightarrow 0$ such that the sequence $\cdots \longrightarrow A^{-l_1-2} \longrightarrow A^{-l_1-1} \longrightarrow A^{-l_1}$ is exact in A. Furthermore, let $\mathsf{C}_{\mathsf{E}}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1} \subset \mathsf{C}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}$ be the full subcategory of all complexes that are isomorphic in the derived category $\mathsf{D}(\mathsf{A})$ to complexes of the form $0 \longrightarrow E^{-l_1} \longrightarrow \cdots \longrightarrow E^{l_2} \longrightarrow 0$, with the terms belonging to E and concentrated in the cohomological degrees $-l_1 \leqslant m \leqslant l_2$.

For example, one has $\mathsf{C}^{\leqslant 0}(\mathsf{A})^{\geqslant 0} = {}_{\mathsf{A}}\mathsf{C}^{\leqslant 0}(\mathsf{A})$. The full subcategory $\mathsf{C}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}$ is closed under extensions and the kernels of admissible epimorphisms in the exact category $\mathsf{C}^{\geqslant l_2}(\mathsf{A})$, while the full subcategory $\mathsf{C}_{\mathsf{E}}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}$ is closed under extension and the cokernels of admissible monomorphisms in $\mathsf{C}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}$. So the full subcategory $\mathsf{C}_{\mathsf{E}}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}$ inherits an exact category structure from $\mathsf{C}^{\geqslant l_2}(\mathsf{A})$.

Suppose that we are given a DG-functor $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$ taking acyclic complexes in the exact category P to acyclic complexes in the exact category A. Suppose further that the restriction of Φ to the subcategory ${}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{P}) \subset \mathsf{C}^-(\mathsf{P})$ is an exact functor between exact categories

$$ {}_{\mathsf{F}}\mathsf{C}^{\leqslant 0}(\mathsf{P}) \longrightarrow \mathsf{C}_{\mathsf{E}}^{\leqslant l_2}(\mathsf{A})^{\geqslant -l_1}. \tag{24} $$

Then the construction dual to that in Section A.2 provides the left derived functor

$$ \mathbb{L}\Phi\colon \mathsf{D}^\star(\mathsf{F}) \longrightarrow \mathsf{D}^\star(\mathsf{E}) \tag{25} $$

acting between any bounded or unbounded, conventional or absolute derived categories $\mathsf{D}^\star$ with the symbols $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs.

Under certain conditions, one can also have the derived functor $\mathbb{L}\Phi$ acting between the coderived or contraderived categories. When the exact categories F and A have exact functors of infinite direct sum, the full subcategories $\mathsf{P} \subset \mathsf{F}$ and $\mathsf{E} \subset \mathsf{A}$ are closed under infinite direct sums, and the functor Φ preserves infinite direct sums, there is the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$.

When the exact categories F and A have exact functors of infinite product, the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under infinite products, and for any family of complexes $P_\alpha^\bullet \in \mathsf{C}^{\leqslant 0}(\mathsf{P})$ and a complex $Q^\bullet \in \mathsf{C}^{\leqslant 0}(\mathsf{P})$ endowed with a quasi-isomorphism $Q^\bullet \longrightarrow \prod_\alpha P_\alpha^\bullet$ of complexes in the exact category F, the induced morphism

$$ \Phi(Q^\bullet) \longrightarrow \prod\nolimits_\alpha \Phi(P_\alpha^\bullet) $$

is a quasi-isomorphism of complexes in the exact category A, there is the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$.

Let us spell out the major steps of the construction of the derived functor (25). Since the DG-functor $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$ preserves quasi-isomorphisms, it induces a triangulated functor $\Phi\colon \mathsf{D}^-(\mathsf{P}) \longrightarrow \mathsf{D}^-(\mathsf{A})$. Taking into account the triangulated equivalence $\mathsf{D}^-(\mathsf{P}) \simeq \mathsf{D}^-(\mathsf{F})$ provided by [13, Proposition A.3.1(a)], we obtain the derived functor $\mathbb{L}\Phi\colon \mathsf{D}^-(\mathsf{F}) \longrightarrow \mathsf{D}^-(\mathsf{A})$. Our assumptions on Φ imply that this functor $\mathbb{L}\Phi$ takes the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \subset \mathsf{D}^-(\mathsf{F})$ into the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \subset \mathsf{D}^{\mathsf{b}}(\mathsf{A}) \subset \mathsf{D}^-(\mathsf{A})$; hence the triangulated functor

$$\mathbb{L}\Phi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E}). \tag{26}$$

Passing from the DG-functor $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$ to the induced DG-functor between the DG-categories of unbounded complexes in the given DG-categories, as explained in Section A.2, and restricting to the full DG-subcategories of uniformly bounded bicomplexes, one obtains the DG-functor

$$\Phi_{\mathsf{C}}\colon \mathsf{C}^-(\mathsf{C}(\mathsf{P})) \longrightarrow \mathsf{C}^-(\mathsf{C}(\mathsf{A})).$$

The functor Φ_{C} takes acyclic complexes in the exact category $\mathsf{C}(\mathsf{P})$ to acyclic complexes in the exact category $\mathsf{C}(\mathsf{A})$. It also takes the full subcategory ${}_{\mathsf{C}(\mathsf{F})}\mathsf{C}^{\leqslant 0}(\mathsf{C}(\mathsf{P})) \subset \mathsf{C}^-(\mathsf{C}(\mathsf{P}))$ into the full subcategory $\mathsf{C}^{\leqslant l_2}_{\mathsf{C}(\mathsf{E})}(\mathsf{C}(\mathsf{A}))^{\geqslant -l_1} \subset \mathsf{C}^-(\mathsf{C}(\mathsf{A}))$. So we can apply the construction of the derived functor (26) to the DG-functor Φ_{C} in place of Φ, and produce a triangulated functor

$$\mathbb{L}\Phi_{\mathsf{C}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{F})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{E})). \tag{27}$$

Using Proposition A.1 and Corollary A.2, one shows that the triangulated functor (27) descends to a triangulated functor (25) between the absolute or conventional derived categories, $\star = \mathsf{abs}$ or $\varnothing$. The cases of bounded above or below absolute or conventional derived categories, $\star = +, -, \mathsf{abs}+$, or $\mathsf{abs}-$ can be treated as explained in Section A.2. Under the respective assumptions, one can also descend from the absolute derived categories to the coderived or contraderived categories, producing the derived functor (25) for $\star = \mathsf{co}$ or ctr.

A.4. **Deriving adjoint functors.** Let A and B be exact categories containing the images of idempotent endomorphisms of its objects, let $\mathsf{J} \subset \mathsf{E} \subset \mathsf{A}$ be coresolving subcategories in A, and let $\mathsf{P} \subset \mathsf{F} \subset \mathsf{B}$ be resolving subcategories in B.

Let $\Psi\colon \mathsf{C}^+(\mathsf{J}) \longrightarrow \mathsf{C}^+(\mathsf{B})$ be a DG-functor satisfying the conditions of Section A.1, and let $\Phi\colon \mathsf{C}^-(\mathsf{P}) \longrightarrow \mathsf{C}^-(\mathsf{A})$ be a DG-functor satisfying the conditions of Section A.3. Suppose that the DG-functors Φ and Ψ are partially adjoint, in the sense that for any two complexes $J^\bullet \in \mathsf{C}^+(\mathsf{J})$ and $P^\bullet \in \mathsf{C}^-(\mathsf{P})$ there is a natural isomorphism of complexes of abelian groups

$$\mathrm{Hom}_{\mathsf{A}}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_{\mathsf{B}}(P^\bullet, \Psi(J^\bullet)), \tag{28}$$

where $\mathrm{Hom}_{\mathsf{A}}$ and $\mathrm{Hom}_{\mathsf{B}}$ denote the complexes of morphisms in the DG-categories of unbounded complexes $\mathsf{C}(\mathsf{A})$ and $\mathsf{C}(\mathsf{B})$.

Our aim is to show that the triangulated functor $\mathbb{L}\Phi$ (25) is left adjoint to the triangulated functor $\mathbb{R}\Phi$ (19), for any symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-$, or abs. When the functors $\mathbb{L}\Phi$ and $\mathbb{R}\Psi$ acting between the categories $\mathsf{D^{co}}$ or $\mathsf{D^{ctr}}$ are defined (i. e., the related conditions in Sections A.1 and A.3 are satisfied), the former of them is also left adjoint to the latter one.

Our first step is the following lemma.

Lemma A.3. *In the assumptions above, the induced triangulated functors $\Phi\colon \mathsf{D}^-(\mathsf{P}) \longrightarrow \mathsf{D}^-(\mathsf{A})$ and $\Psi\colon \mathsf{D}^+(\mathsf{J}) \longrightarrow \mathsf{D}^+(\mathsf{B})$ are partially adjoint, in the sense that for any complexes $J^\bullet \in \mathsf{C}^+(\mathsf{J})$ and $P^\bullet \in \mathsf{C}^-(\mathsf{P})$ there is a natural isomorphism of abelian groups of morphisms in the unbounded derived categories*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{A})}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}(\mathsf{B})}(P^\bullet, \Psi(J^\bullet)).$$

Proof. Passing to the cohomology groups in the DG-adjunction isomorphism (28), one obtains an isomorphism of the groups of morphisms in the homotopy categories

$$\mathrm{Hom}_{\mathsf{Hot}(\mathsf{A})}(\Phi(P^\bullet), J^\bullet) \simeq \mathrm{Hom}_{\mathsf{Hot}(\mathsf{B})}(P^\bullet, \Psi(J^\bullet)).$$

In order to pass from this to the desired isomorphism of the groups of morphisms in the unbounded derived categories, one can notice that for any (unbounded) complex $A^\bullet \in \mathsf{C}(\mathsf{A})$ endowed with a quasi-isomorphism $J^\bullet \longrightarrow A^\bullet$ of complexes in A there exists a bounded below complex $I^\bullet \in \mathsf{C}^+(\mathsf{J})$ together with a quasi-isomorphism $A^\bullet \longrightarrow I^\bullet$ of complexes in A. The composition $J^\bullet \longrightarrow A^\bullet \longrightarrow I^\bullet$ is then a quasi-isomorphism of bounded below complexes in J. Similarly, for any (unbounded) complex $B^\bullet \in \mathsf{C}(\mathsf{B})$ endowed with a quasi-isomorphism $B^\bullet \longrightarrow P^\bullet$ of complexes in B there exists a bounded above complex $Q^\bullet \in \mathsf{C}^-(\mathsf{P})$ together with a quasi-isomorphism $Q^\bullet \longrightarrow B^\bullet$ of complexes in B. The composition $Q^\bullet \longrightarrow B^\bullet \longrightarrow P^\bullet$ is then a quasi-isomorphism of bounded above complexes in P. ☐

Restricting to the full subcategories $\mathsf{D^b}(\mathsf{E}) \subset \mathsf{D}^-(\mathsf{J}) \subset \mathsf{D}(\mathsf{A})$ and $\mathsf{D^b}(\mathsf{F}) \subset \mathsf{D}^-(\mathsf{P}) \subset \mathsf{D}(\mathsf{B})$, we conclude that the derived functor $\mathbb{L}\Phi\colon \mathsf{D^b}(\mathsf{F}) \longrightarrow \mathsf{D^b}(\mathsf{E})$ (26) is left adjoint to the derived functor $\mathbb{R}\Psi\colon \mathsf{D^b}(\mathsf{E}) \longrightarrow \mathsf{D^b}(\mathsf{F})$ (20). Replacing all the exact categories with the categories of unbounded complexes in them, we see that the derived functor $\mathbb{L}\Phi_{\mathsf{C}}\colon \mathsf{D^b}(\mathsf{C}(\mathsf{F})) \longrightarrow \mathsf{D^b}(\mathsf{C}(\mathsf{E}))$ (27) is left adjoint to the derived functor $\mathbb{R}\Psi_{\mathsf{C}}\colon \mathsf{D^b}(\mathsf{C}(\mathsf{E})) \longrightarrow \mathsf{D^b}(\mathsf{C}(\mathsf{F}))$ (21).

In order to pass to the desired adjunction between the derived functors $\mathbb{R}\Psi\colon \mathsf{D}^\star(\mathsf{E}) \longrightarrow \mathsf{D}^\star(\mathsf{F})$ (19) and $\mathbb{L}\Phi\colon \mathsf{D}^\star(\mathsf{F}) \longrightarrow \mathsf{D}^\star(\mathsf{E})$ (25), it remains to apply the next (well-known) lemma.

Lemma A.4. *Suppose that we are given two commutative diagrams of triangulated functors*

$$\begin{tikzcd} \mathsf{D}_1 \arrow[r, "G"] \arrow[d, two heads] & \mathsf{D}_2 \arrow[d, two heads] \\ \overline{\mathsf{D}}_1 \arrow[r, "\overline{G}"] & \overline{\mathsf{D}}_2 \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D}_1 \arrow[d, two heads] & \mathsf{D}_2 \arrow[l, "F"'] \arrow[d, two heads] \\ \overline{\mathsf{D}}_1 & \overline{\mathsf{D}}_2 \arrow[l, "\overline{F}"'] \end{tikzcd}$$

where the vertical arrows are Verdier quotient functors. Suppose further that the functor $F\colon \mathsf{D}_2 \longrightarrow \mathsf{D}_1$ *is left adjoint to the functor* $G\colon \mathsf{D}_1 \longrightarrow \mathsf{D}_2$*. Then the functor* $\overline{F}\colon \overline{\mathsf{D}}_2 \longrightarrow \overline{\mathsf{D}}_1$ *is also naturally left adjoint to the functor* $\overline{G}\colon \overline{\mathsf{D}}_1 \longrightarrow \overline{\mathsf{D}}_2$*.*

Proof. The adjunction morphisms $F\circ G \longrightarrow \mathrm{Id}_{\mathsf{D}_1}$ and $\mathrm{Id}_{\mathsf{D}_2} \longrightarrow G\circ F$ induce adjunction morphisms $\overline{F}\circ\overline{G} \longrightarrow \mathrm{Id}_{\overline{\mathsf{D}}_1}$ and $\mathrm{Id}_{\overline{\mathsf{D}}_2} \longrightarrow \overline{G}\circ\overline{F}$. $\square$

A.5. **Triangulated equivalences.** The following theorem describes the situation in which the adjoint triangulated functors $\mathbb{R}\Psi$ and $\mathbb{L}\Phi$ turn out to be triangulated equivalences (cf. the proofs of [15, Theorems 4.9 and 5.10], [19, Theorems 2.6 and 3.3], and [18, Theorem 7.6], where this technique was used).

Theorem A.5. *In the context of Section A.4, suppose that the adjoint derived functors* $\mathbb{R}\Psi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{F})$ (20) *and* $\mathbb{L}\Phi\colon \mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E})$ (26) *are mutually inverse triangulated equivalences. Then so are the adjoint derived functors* $\mathbb{R}\Psi\colon \mathsf{D}^{\star}(\mathsf{E}) \longrightarrow \mathsf{D}^{\star}(\mathsf{F})$ (19) *and* $\mathbb{L}\Phi\colon \mathsf{D}^{\star}(\mathsf{F}) \longrightarrow \mathsf{D}^{\star}(\mathsf{E})$ (25) *for all the symbols* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *and also for any one of the symbols* $\star = \mathsf{co}$ *or* ctr *for which these two functors are defined by the constructions of Sections A.2–A.3.*

Moreover, assume that the adjunction morphisms $\mathbb{L}\Phi(\Psi(J)) \longrightarrow J$ *and* $P \longrightarrow \mathbb{R}\Psi(\Phi(P))$ *are isomorphisms in* $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathsf{F})$ *for all objects* $J \in \mathsf{J}$ *and* $P \in \mathsf{P}$*. Then the adjoint derived functors* (19) *and* (25) *are mutually inverse triangulated equivalences for all the symbols* $\star$ *for which they are defined.*

Proof. A complex of complexes in an exact category G is acyclic if and only if it is termwise acyclic. In other words, one can consider the family of functors $\Theta^n_{\mathsf{G}}\colon \mathsf{C}(\mathsf{G})) \longrightarrow \mathsf{G}$, indexed by the integers n, assigning to a complex $G^\bullet$ its n-th term G^n. Then the family of induced triangulated functors $\Theta^n_{\mathsf{G}}\colon \mathsf{D}(\mathsf{C}(\mathsf{G})) \longrightarrow \mathsf{D}(\mathsf{G})$ is conservative in total. This means that for any nonzero object $G^{\bullet,\bullet} \in \mathsf{D}(\mathsf{C}(\mathsf{G}))$ there exists $n \in \mathbb{Z}$ such that $\Theta^n_{\mathsf{G}}(G^{\bullet,\bullet}) \neq 0$ in $\mathsf{D}(\mathsf{G})$.

Now the two such functors $\Theta^n_{\mathsf{E}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{E})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{E})$ and $\Theta^n_{\mathsf{F}}\colon \mathsf{D}^{\mathsf{b}}(\mathsf{C}(\mathsf{F})) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{F})$ form commutative diagrams with the adjoint derived functors (20–21) and (26–27). Therefore, the adjoint functors (21) and (27) are mutually inverse equivalences whenever so are the adjoint functors (20) and (26). It remains to point out that, in the context of Lemma A.4, the two adjoint functors $\overline{F}$ and $\overline{G}$ are mutually inverse equivalences whenever so are the two adjoint functors F and G.

This proves the first assertion of the theorem, and in fact somewhat more than that. We have shown that the adjunction morphism $\mathbb{L}\Phi(\mathbb{R}\Psi(E^\bullet)) \longrightarrow E^\bullet$ is an isomorphism in $\mathsf{D}^{\star}(\mathsf{E})$ whenever for every $n \in \mathbb{Z}$ the adunction morphism $\mathbb{L}\Phi(\mathbb{R}\Psi(E^n)) \longrightarrow E^n$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$. Now, replacing an object $E \in \mathsf{E}$ by its coresolution $J^\bullet$ by objects from J, viewed as an object in $\mathsf{D}^{\star}(\mathsf{E})$ with $\star = +$, we see that it suffices to check that the adjunction morphism is an isomorphism for an object $J \in \mathsf{J}$. Similarly, the adjunction morphism $F^\bullet \longrightarrow \mathbb{R}\Psi(\mathbb{L}\Phi(F^\bullet))$ is an isomorphism in $\mathsf{D}^{\star}(\mathsf{E})$ whenever for every $n \in \mathbb{Z}$ the adjunction morphism $F^n \longrightarrow \mathbb{R}\Psi(\mathbb{L}\Phi(F^n))$ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(\mathsf{F})$. Replacing an object $F \in \mathsf{F}$ by its resolution $P^\bullet$ by objects from P, viewed as an object in $\mathsf{D}^{\star}(\mathsf{F})$ with $\star = -$, we see that it suffices to check that the adjunction morphism is an isomorphism for an object $P \in \mathsf{P}$. $\square$

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