KOSZULITY OF COHOMOLOGY $= K(\pi,1)$-NESS + QUASI-FORMALITY

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Abstract. This paper is a greatly expanded version of [36, Section 9.11]. A series of definitions and results illustrating the thesis in the title (where quasi-formality means vanishing of a certain kind of Massey multiplications in the cohomology) is presented. In particular, we include a categorical interpretation of the "Koszulity implies $K(\pi,1)$" claim, discuss the differences between two versions of Massey operations, and apply the derived nonhomogeneous Koszul duality theory in order to deduce the main theorem. In the end we demonstrate a counterexample providing a negative answer to a question of Hopkins and Wickelgren about formality of the cochain DG-algebras of absolute Galois groups, thus showing that quasi-formality cannot be strengthened to formality in the title assertion.

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Introduction

A *quadratic algebra* is an associative algebra defined by homogeneous quadratic relations. In other words, a positively graded algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ over a field k is called quadratic if it is generated by its first-degree component A_1 with relations in degree 2. A positively graded associative algebra A is called *Koszul* [39, 4, 32] if one has $\operatorname{Tor}^A_{ij}(k,k) = 0$ for all $i \neq j$, where the first grading i on the Tor spaces is the usual homological grading and the second grading j, called the *internal* grading, is induced by the grading of A. In particular, this condition for $i = 1$ means that the algebra A is generated by A_1, and the conditions for $i = 1$ and 2 taken together mean that A is quadratic.

Conversely, for any positively graded algebra A with finite-dimensional components A_n the diagonal part $\bigoplus_n \operatorname{Ext}^{n,n}_A(k,k)$ of the algebra $\operatorname{Ext}^*_A(k,k) \simeq \operatorname{Tor}^A_*(k,k)^*$

is a quadratic algebra. When the algebra A is quadratic, the two quadratic algebras A and $\bigoplus_n \operatorname{Ext}_A^{n,n}(k,k)$ are called *quadratic dual* to each other. Without the finite-dimensionality assumption on the grading components, the quadratic duality connects quadratic graded algebras with quadratic graded coalgebras [32, 33].

When one attempts to deform quadratic algebras by considering algebras with *nonhomogeneous* quadratic relations, one discovers that there are two essentially different ways of doing so. One can either consider relations with terms of the degrees not greater than 2, that is

$$q_2(x) + q_1(x) + q_0 = 0, \tag{1}$$

where $x = (x_\alpha)$, $\deg x_\alpha = 1$ denotes the set of generators and $\deg q_n = n$; or relations with terms of the degrees not less than 2, that is

$$q_2(x) + q_3(x) + q_4(x) + q_5(x) + \cdots = 0. \tag{2}$$

In the latter case, it is natural to allow the relations to be infinite power series, that is consider the algebra they define as a quotient algebra of the algebra of formal Taylor power series in noncommuting variables x_α, rather than the algebra of noncommutative polynomials. Almost equivalently, this means considering a set of relations of the type (2) as defining a conilpotent *coalgebra*, while a set of relations of the type (1) defines a filtered *algebra*. More precisely, of course, one should say that the complete topological algebra defined by the relations (2) is the dual vector space to a discrete conilpotent coalgebra. This is one of the simplest ways to explain the importance of coalgebras in Koszul duality.

The alternative between considering nonhomogeneous quadratic relations of the types (1) and (2) roughly leads to a division of the Koszul duality theory into two streams, the former of them going back to the classical paper [39] and the present author's work [31], and the latter one originating in the paper [32]. The former theory, invented originally for the purposes of computing the cohomology of associative algebras generally and the Steenrod algebra in particular, eventually found its applications in semi-infinite homological algebra. The latter point of view was being applied to Galois cohomology, the conjectures about absolute Galois groups, and the theory of motives with finite coefficients. As a general rule, the author's subsequent papers vaguely associated with relations of the type (1) were published on the `arXiv` in the subject area `[math.CT]`, while the papers having to do with relations of the type (2) were put into the area `[math.KT]`.

This paper is concerned with relations of the type (2). Its subject can be roughly described as cohomological characterization of the coalgebras C defined by the relations (2) with the quadratic principal parts

$$q_2(x) = 0 \tag{3}$$

of the relations definining a Koszul graded coalgebra. In fact, according to the main theorem of [32] (see also [23]) a conilpotent coalgebra C is defined by a self-consistent system of relations (2) with Koszul quadratic principal part (3) if and only if its cohomology algebra

$$H^*(C) = \mathrm{Ext}^*_C(k,k)$$

is Koszul. Moreover, a certain weaker set of conditions on the algebra $H^* = H^*(C)$ is sufficient, and implies Koszulity of algebras of the form $H^*(C)$. When the algebra $H^*(C)$ is Koszul, it is simply the dual quadratic algebra to the quadratic coalgebra defined by the relations (3).

Given an arbitrary (not necessarily conilpotent) coaugmented coalgebra D with the maximal conilpotent subcoalgebra $C = \mathrm{Nilp}\, D \subset D$, the algebra $H^*(D) = \mathrm{Ext}^*_D(k,k)$ is Koszul if and only if the following two conditions hold [36, Section 9.11]:

(i) the homomorphism of cohomology algebras $H^*(C) \longrightarrow H^*(D)$ induced by the embedding of coalgebras $C \longrightarrow D$ is an isomorphism;

(ii) a certain family of higher Massey products in the cohomology algebra $H^*(D)$ vanishes.

In this paper we provide a detailed proof of this result, and discuss at length its constituting components.

In particular, the condition (i) and the implication "Koszulity of $H^*(D)$ implies (i)" allow numerous analogues and generalizations, including such assertions as

- for any discrete group Γ whose cohomology algebra $H^*(\Gamma, k)$ with coefficients in a field k is Koszul, the cohomology algebra $H^*(\Gamma^{\frown}_k, k)$ of the k-completion of the group Γ is isomorphic to the algebra $H^*(\Gamma, k)$ [33, Section 5]; or
- any rational homotopy type X with a Koszul cohomology algebra $H^*(X, \mathbb{Q})$ is a rational $K(\pi, 1)$ space [30].

That is why we call the condition (i) "the $K(\pi, 1)$ condition".

More generally, in place of the cochain DG-algebra of a coaugmented coalgebra D consider an arbitrary nonnegatively cohomologically graded augmented DG-algebra $0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow \cdots$ over a field k with $H^0(A^\bullet) \simeq k$. Then the cohomology algebra $H^*(A^\bullet)$ of the DG-algebra $A^\bullet$ is Koszul if and only if the following two conditions hold:

(i) the cohomology coalgebra of the bar-construction of the augmented DG-algebra $A^\bullet$ is concentrated in cohomological degree 0;

(ii) a certain family of higher Massey products in the cohomology algebra $H^*(A^\bullet)$ vanishes.

Once again, we call the condition (i) "the $K(\pi, 1)$ condition".

Furthermore, relation sets of the type (2) are naturally viewed up to variable changes

$$x_\alpha \longrightarrow x_\alpha + p_{2,\alpha}(x) + p_{3,\alpha}(x) + p_{4,\alpha}(x) + \cdots, \tag{4}$$

where $\deg p_{n,\alpha} = n$. A natural question is whether or when a system of relations (2) can be homogenized, i. e., transformed into the system (3) by a variable change (4). We show that a variable change (4) homogenizing a given system of relations (2) defining a conilpotent coalgebra C with Koszul cohomology algebra $H^*(C)$ exists if and only if the cochain DG-algebra computing $H^*(C)$ is *formal*, i. e., can be connected with its cohomology algebra by a chain of multiplicative quasi-isomorphisms.

Obviously, formality implies the Massey product vanishing condition (ii), which we accordingly call the *quasi-formality* condition. Not distinguishing formality from quasi-formality seems to be a common misconception. The above explanations suggest that the cochain DG-algebras of most conilpotent coalgebras C with Koszul cohomology algebras $H^*(C)$ should not be formal but only quasi-formal, as the possibility of homogenizing a system of relations (2) looks unlikely, generally speaking.

Indeed, we provide a simple counterexample of a pro-l-group H whose cohomology algebra $H^*(H, \mathbb{Z}/l)$ is Koszul, while the cochain DG-algebra computing it is not formal, as the relations in the group coalgebra $\mathbb{Z}/l(H)$ cannot be homogenized by variable changes. It was conjectured in the papers [32, 36] that the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$ is Koszul for the absolute Galois group G_F of any field F containing a primitive l-root of unity; and the question was asked in the paper [16] whether the cochain DG-algebra of the group G_F with coefficients in $\mathbb{Z}/l$ is formal. As the group H in our counterexample is the maximal quotient pro-l-group of the absolute Galois group G_F of an appropriate p-adic field F containing a primitive l-root of unity, our results provide a negative answer to this question of Hopkins and Wickelgren.

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1. Koszulity Implies $K(\pi, 1)$-ness

Postponing the discussion of DG-algebras, DG-coalgebras, and derived nonhomogeneous Koszul duality to Sections 2–4, we devote this section to the formulation of a categorical version of the "Koszulity implies $K(\pi, 1)$" claim. We begin our discussion with recalling some basic definitions and results from [32] and [33, Section 5].

A coassociative counital coalgebra D over a field k is said to be *coaugmented* if it is endowed with a coalgebra morphism $k \longrightarrow D$ (called the *coaugmentation*). The quotient coalgebra (without counit) of a coaugmented coalgebra D by the image of the coaugmentation morphism is denoted by $D_+ = D/k$.

A coaugmented coalgebra C is called *conilpotent* if for any element $c \in C$ there exists an integer $m \geqslant 1$ such that c is annihilated by the iterated comultiplication map $C \longrightarrow C_+^{\otimes m+1}$. (Several references and terminological comments related to this definition can be found in [34, Remark D.6.1].) The maximal conilpotent subcoalgebra $\bigcup_m \ker(D \to D_+^{\otimes m+1})$ of a coaugmented coalgebra D is denoted by $\operatorname{Nilp} D \subset D$.

The cohomology algebra of a coaugmented coalgebra D is defined as the Ext algebra $H^*(D) = \operatorname{Ext}_D^*(k,k)$, where the field k is endowed with a left D-comodule structure via the coaugmentation map. The cohomology algebra $H^*(D)$ is computed by the reduced cochain DG-algebra of the coalgebra D

$$k \longrightarrow D_+ \longrightarrow D_+ \otimes_k D_+ \longrightarrow D_+ \otimes_k D_+ \otimes_k D_+ \longrightarrow \cdots,$$

which is otherwise known as the *reduced cobar-complex* or the *cobar construction* of the coaugmented coalgebra D and denoted by $\operatorname{Cob}^\bullet(D)$.

The following result can be found in [33, Corollary 5.3].

Theorem 1.1. *Let D be a coaugmented coalgebra over a field k and $\operatorname{Nilp} D \subset D$ be its maximal conilpotent subcoalgebra. Assume that the cohomology algebra $H^*(D) = \operatorname{Ext}_D^*(k,k)$ is Koszul. Then the embedding $\operatorname{Nilp} D \longrightarrow D$ induces a cohomology isomorphism $H^*(\operatorname{Nilp} D) \simeq H^*(D)$.*

Theorem 1.1 has a version with an augmented algebra R replacing the coaugmented coalgebra D [33, Remark 5.6]. Let $R_+ = \ker(R \to k)$ denote the augmentation ideal, and let I run over all the ideals $I \subset R_+$ in R for which the quotient algebra R/I is finite-dimensional and its augmentation ideal R_+/I is nilpotent. The coalgebra of pronilpotent completion $R\widehat{\ }$ of the augmented algebra R is defined as the filtered inductive limit $R\widehat{\ } = \varinjlim_I (R/I)^*$ of the coalgebras $(R/I)^*$ dual to the finite-dimensional algebras R/I. Clearly, the coalgebra $R\widehat{\ }$ is conilpotent.

The cohomology algebra $H^*(R) = \operatorname{Ext}_R^*(k,k)$ of an augmented algebra R is computed by its reduced cobar-complex $\operatorname{Cob}^\bullet(R)$

$$k \longrightarrow R_+^* \longrightarrow (R_+ \otimes_k R_+)^* \longrightarrow (R_+ \otimes_k R_+ \otimes_k R_+)^* \longrightarrow \cdots$$

The natural injective morphism of cobar-complexes $\operatorname{Cob}^\bullet(R\widehat{\ }) \longrightarrow \operatorname{Cob}^\bullet(R)$ induces a natural morphism of cohomology algebras $H^*(R\widehat{\ }) \longrightarrow H^*(R)$.

Theorem 1.2. *Let R be an augmented algebra over a field k and $R\widehat{\ }$ be the coalgebra of its pronilpotent completion. Assume that the cohomology algebra $H^*(R) = \operatorname{Ext}_R^*(k,k)$ is Koszul. Then the natural morphism of the cohomology algebras $H^*(R\widehat{\ }) \longrightarrow H^*(R)$ is an isomorphism.*

The proofs of Theorems 1.1 and 1.2 are based on the following result about the cohomology of conilpotent coalgebras [32, Main Theorem 3.2]. For any positively graded algebra H^* over a field k, we denote by $\operatorname{q} H^*$ the quadratic part of the algebra H^*, i. e., the universal final object in the category of quadratic algebras over k endowed with a morphism into H^*. The quadratic algebra $\operatorname{q} H^*$ is uniquely defined by the condition that the morphism of graded algebras $\operatorname{q} H^* \longrightarrow H^*$ is an isomorphism in degree 1 and a monomorphism in degree 2.

Theorem 1.3. *Let C be a conilpotent coaugmented coalgebra, i. e.,* $\operatorname{Nilp} C = C$. *Assume that*

- *the quadratic part* $\mathrm{q}\, H^*(C)$ *of the graded algebra* $H^*(C)$ *is Koszul; and*
- *the morphism of graded algebras* $\mathrm{q}\, H^*(C) \longrightarrow H^*(C)$ *is an isomorphism in degree* 2 *and a monomorphism in degree* 3.

Then the graded algebra $H^*(C)$ *is quadratic (and consequently, Koszul).* □

The proof of Theorem 1.1 can be found in [33, Theorem 5.2 and Corollary 5.3]. The proof of Theorem 1.2 is very similar; let us briefly explain how it works.

Proof of Theorem 1.2. One notices that for any augmented algebra R the morphism of cohomology algebras $H^*(R^\frown) \longrightarrow H^*(R)$ is an isomorphism in degree 1 and a monomorphism in degree 2. Indeed, the category of left comodules over $R^\frown$ is isomorphic to the full subcategory in the category of left R-modules consisting of all the ind-nilpotent R-modules (direct limits of iterated extensions of the trivial R-module k, the latter being defined in terms of the augmentation of R). This is a full subcategory closed under subobjects, quotient objects, and extensions in the abelian category of left R-modules; so the argument of [33, Lemma 5.1] applies.

Now if the algebra $H^*(R)$ is Koszul, then it follows that the maps $H^1(R^\frown) \longrightarrow H^1(R)$ and $H^2(R^\frown) \longrightarrow H^2(R)$ are isomorphisms, the composition $\mathrm{q}\, H^*(R^\frown) \longrightarrow H^*(R^\frown) \longrightarrow H^*(R)$ is an isomorphism of graded algebras, and the algebra $H^*(R^\frown)$ satisfies the conditions of Theorem 1.3. Hence we conclude that the algebra $H^*(R^\frown)$ is quadratic and the morphism $H^*(R^\frown) \longrightarrow H^*(R)$ is an isomorphism. □

A generalization of the results of Theorems 1.1 and 1.2 to t-structures in triangulated categories [1, n° 1.3] was announced in [33, Remark 5.6]. The idea of this generalization can be described as follows.

Recall that for any t-structure $(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ on a triangulated category D with the core $\mathsf{C} = \mathsf{D}^{\leqslant 0} \cap \mathsf{D}^{\geqslant 0}$ and for any two objects X, $Y \in \mathsf{C}$ there are natural maps

$$\theta^n_{\mathsf{C},\mathsf{D}}(X,Y)\colon \operatorname{Ext}^n_{\mathsf{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathsf{D}}(X, Y[n]), \qquad n \geqslant 0,$$

from the Ext groups in the abelian category C to the Hom groups in the triangulated category D. The maps $\theta^n_{\mathsf{C},\mathsf{D}} = \theta^n_{\mathsf{C},\mathsf{D}}(X,Y)$ transform the compositions of Yoneda Ext classes in C into the compositions of morphisms in D. Furthermore, the maps $\theta^n_{\mathsf{C},\mathsf{D}}$ are always isomorphisms for $n \leqslant 1$ and monomorphisms for $n = 2$ (see [1, Remarque 3.1.17], [3, Section 4.0], or [36, Corollary A.17]).

Starting with a coaugmented coalgebra D, consider the conilpotent coalgebra $C = \operatorname{Nilp} D$ and the abelian category C of finite-dimensional left C-comodules. Consider the bounded derived category of left D-comodules $\mathsf{D}^{\mathsf{b}}(D\text{–}\mathsf{comod})$, and set D to be the full subcategory of $\mathsf{D}^{\mathsf{b}}(D\text{–}\mathsf{comod})$ generated by the abelian subcategory $\mathsf{C} \subset D\text{–}\mathsf{comod}$. Then C is the core of a bounded t-structure on D.

Analogously, starting with an augmented algebra R, consider the conilpotent coalgebra $C = R^\frown$ and the abelian category C of finite-dimensional left C-comodules (or, which is the same, finite-dimensional nilpotent R-modules). Consider the bounded derived category of left R-modules $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$, and set D to be the full triangulated subcategory of $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ generated by the abelian subcategory $\mathsf{C} \subset R\text{–}\mathsf{mod}$. Once again, C is the core of a bounded t-structure on D.

In both cases, the assertions of Theorems 1.1 and 1.2 claim that the map

$$\theta^n_{\mathsf{C},\mathsf{D}}(k,k)\colon\ \mathrm{Ext}^n_{\mathsf{C}}(k,k) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(k,k[n])$$

is an isomorphism for all n, provided that the graded algebra $\mathrm{Hom}_{\mathsf{D}}(k,k[*])$ is Koszul. Here the trivial D-comodule or R-module k is the only irreducible object in C. All objects of the abelian category C being of finite length, it follows that all the morphisms $\theta^n_{\mathsf{C},\mathsf{D}}$ are isomorphisms for the t-structures under consideration.

A t-structure for which all the maps $\theta^n_{\mathsf{C},\mathsf{D}}$ are isomorphisms is called a "t-structure of derived type" [3, Section 4.0]. This condition is also known as the "$K(\pi,1)$ condition of Bloch and Kriz" [7] and, in somewhat larger generality, as the "silly filtration condition" [36, Sections 0.2–0.5]. Proving that a given t-structure is of derived type is sometimes an important and difficult problem (see, e. g., [2]). A standard approach working in some particular cases can be found in [22, Section 12] (see also [37, Section A.2]); the results below in this section provide an alternative way.

Let $S=\{\alpha\}$ be a set of indices. A *big ring* (or a "ring with many objects") A is a collection of abelian groups $A^n_{\alpha\beta}$ endowed with the multiplication maps $A_{\alpha\beta}\times A_{\beta\gamma}\longrightarrow A_{\alpha\gamma}$ and the unit elements $1_\alpha\in A_{\alpha\alpha}$ satisfying the conventional associativity and unit axioms. A big ring with a set of indices S is the same thing as a preadditive category with the objects indexed by S (see [29] or [36, Section A.1]).

The categories of left and right modules over a big ring are defined in the obvious way. A left (resp., right) A-module is the same thing as a covariant (resp., contravariant) additive functor from the preadditive category corresponding to A to the category of abelian groups. The constructions of the functor of tensor product of right and left modules over A and its left derived functor Tor^A are also straightforward.

A *big graded ring* (or a "graded ring with many objects") A is a big ring in which every group $A_{\alpha\beta}$ is graded and the multiplication maps are homogeneous. In other words, it is a collection of abelian groups $A^n_{\alpha\beta}$ endowed with the multiplication maps $A^p_{\alpha\beta}\times A^q_{\beta\gamma}\longrightarrow A^{p+q}_{\alpha\gamma}$ and the unit elements $1_\alpha\in A^0_{\alpha\alpha}$ satisfying the associativity and unit axioms. We will assume that $A^n_{\alpha\beta}=0$ for $n<0$ or $\alpha\neq\beta$ and $n=0$, and that the rings $A^0_{\alpha\alpha}$ are (classically) semisimple.

The definition of the Koszul property of a nonnegatively graded ring $A=A_0\oplus A_1\oplus A_2\oplus\cdots$ with a semisimple degree-zero component A_0 is pretty well known [4], and the definition of a Koszul big graded ring in the above generality is its straightforward extension. Specifically, a big graded ring A is called *Koszul* if one has $\mathrm{Tor}^A_{ij}(A^0_{\alpha\alpha},A^0_{\beta\beta})=0$ for all α, $\beta\in S$ and all $i\neq j$. One can also define quadratic big graded rings and the quadratic part of a big graded ring, etc. A discussion of the Koszul property of big graded rings in a greater generality with the semisimplicity condition replaced by a flatness condition can be found in [36, Section 7.4], and even without the flatness condition, in the rest of [36, Section 7].

The following theorem is the main result of this section.

Theorem 1.4. *Let* C *be the core of a t-structure on a small triangulated category* D. *Suppose that every object of* C *has finite length and let* I_α *be the irredicible objects of* C. *Assume that the big graded ring* A *with the components* $A^n_{\alpha,\beta} = \mathrm{Hom}_{\mathsf{D}}(I_\alpha, I_\beta[n])$ *is Koszul. Then for any two objects* X, $Y \in \mathsf{C}$ *and any* $n \geqslant 0$ *the natural map* $\theta^n_{\mathsf{C},\mathsf{D}}(X,Y)\colon \mathrm{Ext}^n_{\mathsf{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(X,Y[n])$ *is an isomorphism.*

The above theorem is deduced from the following result about the Ext rings between irreducible objects in abelian categories, which is a categorical generalization of Theorem 1.3.

Theorem 1.5. *Let* C *be a small abelian category such that every object of* C *has finite length and let* I_α *be the irreducible objects of* C. *Consider the big graded ring* B *with the components* $B^n_{\alpha,\beta} = \mathrm{Ext}^n_{\mathsf{C}}(I_\alpha, I_\beta)$. *Assume that*

(I) *the quadratic part* $\mathrm{q}\,B$ *of the big graded ring* B *is Koszul; and*

(II) *the morphism of big graded rings* $\mathrm{q}\,B \longrightarrow B$ *is an isomorphism in the degree* $n = 2$ *and a monomorphism in the degree* $n = 3$.

Then the big graded ring B *is quadratic (and consequently, Koszul).*

Actually, the following slightly stronger form of Theorem 1.4, generalizing both Theorems 1.4 and 1.5, can be obtained from Theorem 1.5. It is a categorical generalization of [33, Theorem 5.2].

Theorem 1.6. *Let* C *be the core of a t-structure on a small triangulated category* D. *Suppose that every object of* C *has finite length and let* I_α *be the irredicible objects of* C. *Consider the big graded rings* A *and* B *with the components* $A^n_{\alpha,\beta} = \mathrm{Ext}^n_{\mathsf{C}}(I_\alpha, I_\beta)$ *and* $B^n_{\alpha,\beta} = \mathrm{Hom}_{\mathsf{D}}(I_\alpha, I_\beta[n])$. *Then whenever the big graded ring* B *satisfies the assumptions (I) and (II) of Theorem 1.5, the natural morphism of big graded rings* $A \longrightarrow B$ *induces an isomorphism* $A \simeq \mathrm{q}\,B$.

Let us clarify the following point, which otherwise might become a source of confusion. It is well-known [1, 3, 36] that for a t-structure with the core C on a triangulated category D the natural maps $\theta^n_{\mathsf{C},\mathsf{D}}(X,Y)\colon \mathrm{Ext}^n_{\mathsf{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(X,Y[n])$ are isomorphisms for all X, $Y \in \mathsf{C}$, $n \geqslant 0$ if and only if any element in $\mathrm{Hom}_{\mathsf{D}}(X,Y[n])$ can be decomposed into a product of n elements from the groups $\mathrm{Hom}_{\mathsf{D}}(U,V[1])$ with U, $V \in \mathsf{C}$. However, this is only true because one considers the degree-one generation condition for X and Y running over *all* the objects of C and not just the irreducible objects (cf. [36, Proposition B.1]). The Koszulity condition in Theorem 1.4 is much stronger than a degree-one generation condition, but it is applied to a much smaller algebra of homomorphisms between the irreducible objects of C. On the other hand, the big graded ring of Yoneda extensions between all the objects of a given abelian category is always Koszul in an appropriate sense [36, Example 8.3].

Example 1.7. Given a discrete group Γ and a field k, set C to be the coalgebra of (functions on) the proalgebraic completion of Γ over k. Then the category C of finite-dimensional representations of a group Γ over a field k is isomorphic to the category of finite-dimensional left C-comodules. Let D denote the full triangulated subcategory of the bounded derived category $\mathsf{D^b}(k[\Gamma]\text{–}\mathsf{mod})$ of arbitrary Γ-modules over k generated by the subcategory of finite-dimensional modules $\mathsf{C} \subset k[\Gamma]\text{–}\mathsf{mod}$. Then C is the core of a (bounded) t-structure on D, so Theorems 1.4–1.6 are applicable whenever the Koszulity assumption or the assumptions (I–II) are satisfied. This would allow to obtain a comparison between the cohomology of (the proalgebraic group corresponding to) the coalgebra/commutative Hopf algebra C and the discrete group Γ with finite-dimensional coefficients.

Let us emphasize that we do *not* assume the abelian category C or the triangulated category D to be linear over a field in the above theorems. E. g., the category of finite modules over a profinite group is fine in the role of C, as is the category of modules of finite length over any complete commutative local ring, etc.

Proof of Theorem 1.6. This is a particular case of [36, Corollary 8.5]. By a general property of t-structures (see [1, Remarque 3.1.17] or [36, Corollary A.17]), the morphism of big graded rings $A \longrightarrow B$ is an isomorphism in degree 1 and a monomorphism in degree 2 (cf. the proof of Theorem 1.2). It follows that if the ring B satisfies the conditions (I) and (II), then so does the ring A. By Theorem 1.5, one can then conclude that the big graded ring A is quadratic, and hence $A \simeq \mathrm{q}\,B$. ☐

Proof of Theorem 1.4. According to Theorem 1.6, the map $\theta^n_{\mathsf{C},\mathsf{D}}(X,Y)$ is an isomorphism whenever the objects X and $Y \in \mathsf{C}$ are irreducible. The general case follows by induction on the lengths. ☐

Brief sketch of proof of Theorem 1.5. This is a particular case of [36, Theorem 8.4].

Let us start with a comment on the proof of Theorem 1.3. It is based on two ingredients: the basic theory of quadratic and Koszul graded algebras and coalgebras over a field, and the spectral sequence connecting the cohomology of a conilpotent coalgebra C with the cohomology of its associated graded coalgebra with respect to the coaugmentation filtration. In the case of Theorem 1.5, first of all one needs to develop the basic theory of quadratic and Koszul big graded rings, which is done (in a greater generality) in [36, Sections 6–7]. Then one has to work out the passage from the ungraded category C to its graded version.

One possible approach to the latter task would be to associate a coalgebra-like algebraic structure with an abelian category consisting of objects of finite length, then filter that structure and pass to the associated graded one. The required class of algebraic structures was introduced in [10, § IV.3-4]. An abelian category consisting of objects of finite length is equivalent to the category of finitely generated discrete modules over a *pseudo-compact* topological ring. One would have to embed the category of finitely generated discrete modules into the category of pseudo-compact modules in order to do cohomology computations with projective resolutions.

There is a more delicate approach developed in [36, Sections 3–4], which is purely categorical. One associates with an abelian category C consisting of objects of finite length the exact category F whose objects are the objects of C endowed with a finite filtration for which all the successive quotient objects are semisimple. Then one needs to pass from the filtered exact category F to the associated graded abelian category G. The main property connecting the categories C and F is the natural isomorphism

$$\mathrm{Ext}^n_{\mathsf{C}}(u(X), u(Y)) \simeq \varinjlim\nolimits_{m\to+\infty} \mathrm{Ext}^n_{\mathsf{F}}(X, Y(m)), \tag{5}$$

where $u\colon \mathsf{F} \longrightarrow \mathsf{C}$ is the functor of forgetting the filtration and $Z \longmapsto Z(m)$ denotes the filtration shift. The main property connecting the categories F and G is the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}^n_{\mathsf{F}}(X, Y(-1)) \longrightarrow \mathrm{Ext}^n_{\mathsf{F}}(X, Y) \\ \longrightarrow \mathrm{Ext}^n_{\mathsf{G}}(\mathrm{gr}\, X,\ \mathrm{gr}\, Y) \longrightarrow \mathrm{Ext}^{n+1}_{\mathsf{F}}(X, Y(-1)) \longrightarrow \cdots \end{aligned} \tag{6}$$

for any two objects X, $Y \in \mathsf{F}$, where $\mathrm{gr}\colon \mathsf{F} \longrightarrow \mathsf{G}$ is the functor assigning to a filtered object its associated graded object. The construction of the category G is a particular case of a general construction of the reduction of an exact category by a graded center element developed in the paper [38].

The isomorphism (5) and the long exact sequence (6) taken together are used in lieu of the spectral sequence connecting the cohomology of a coalgebra C and its associated graded coalgebra $\mathrm{gr}_N C$ by the coaugmentation filtration N in order to extend the argument from [32, Main Theorem 3.2] from conilpotent coalgebras to abelian categories consisting of objects of finite length. ☐

2. Koszulity Implies Quasi-Formality

Generally speaking, *Massey products* are natural partially defined multivalued polylinear operations in the cohomology algebra of a DG-algebra which are preserved by quasi-isomorphisms of DG-algebras. There are several different ways to construct such operations. We start with introducing the construction most relevant for our purposes, and later explain how it is related to a more elementary construction. The most relevant reference for us is [25]; see also the earlier paper [41], the heavier [26], and the later work [12, Section 5].

Let $A^\bullet = (A^*,\, d\colon A^i \to A^{i+1})$ be a nonzero DG-algebra over a field k; suppose that it is endowed with an augmentation (DG-algebra morphism) $A^\bullet \longrightarrow k$ and denote the augmentation kernel ideal by $A^\bullet_+ = \ker(A^\bullet \to k)$. By the definition, the *bar construction* $\mathrm{Bar}^\bullet(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ is the tensor coalgebra

$$\mathrm{Bar}(A) = \textstyle\bigoplus_{n=0}^\infty A_+[1]^{\otimes n}$$

of the graded vector space $A_+[1]$ obtained by shifting by 1 the cohomological grading of the augmentation ideal A_+. The grading on $\mathrm{Bar}(A)$ is induced by the grading of $A_+[1]$. Alternatively, one can define the bar construction $\mathrm{Bar}(A)$ as the direct sum of the tensor powers $A_+^{\otimes n}$ of the vector space A_+ and endow it with the total grading equal to the difference $i - n$ of the grading i induced by the grading of A_+ and the grading n by the number of tensor factors.

The differential on $\mathrm{Bar}^\bullet(A^\bullet)$ is the sum of two summands $d+\partial$, the former of them induced by the differential on $A^\bullet_+$ and the latter one by the multiplication in $A^\bullet_+$.

One has to work out the plus/minus signs in order to make the total differential on $\mathrm{Bar}^\bullet(A^\bullet)$ square to zero; this is a standard exercise.

One defines a natural increasing filtration on the complex $\mathrm{Bar}^\bullet(A^\bullet)$ by the rule

$$F_p \,\mathrm{Bar}^\bullet(A^\bullet) = \bigoplus\nolimits_{n=0}^p A_+[1]^{\otimes n}.$$

The associated graded complex $\mathrm{gr}^F \mathrm{Bar}^\bullet(A^\bullet)$ of the complex $\mathrm{Bar}^\bullet(A^\bullet)$ by the filtration F is naturally identified with the graded vector space $\mathrm{Bar}(A)$ endowed with the differential d induced by the differential on $A_+^\bullet$. The spectral sequence E_r^{pq} of the filtered complex $\mathrm{Bar}^\bullet(A^\bullet)$ has the initial page

$$E_1^{pq} = (H^*(A_+^\bullet)^{\otimes p})^q,$$

where the grading q on the tensor power $H^*(A_+^\bullet)^{\otimes p}$ of the augmentation ideal $H^*(A_+^\bullet)$ of the cohomology algebra $H^*(A^\bullet)$ of the DG-algebra A is induced by the cohomological grading on $H^*(A_+^\bullet)$. The differentials are

$$d_r^{pq}\colon E_r^{pq} \longrightarrow E_r^{p-r,q-r+1},$$

and the limit page is given by the rule

$$E_\infty^{pq} = \mathrm{gr}_p^F H^{q-p}\,\mathrm{Bar}^\bullet(A^\bullet).$$

The cohomology of the complex $\mathrm{Bar}^\bullet(A^\bullet)$ are known as the *differential Tor spaces* [9, 12] $H^*\,\mathrm{Bar}^\bullet(A^\bullet) = \mathrm{Tor}_*^{A^\bullet}(k,k)$ ("of the first kind" [17]) over the DG-algebra $A^\bullet$. This is the derived functor of tensor product of DG-modules over $A^\bullet$ defined on the "conventional" derived categories of DG-modules (obtained by inverting the DG-module morphisms inducing isomorphisms of the cohomology modules); see [21], [14], or [35, Section 1]. The spectral sequence E_r^{pq} is called the *algebraic Eilenberg–Moore spectral sequence* associated with a DG-algebra $A^\bullet$ [9, 12, 13].

The differential d_1 is induced by the multiplication in the cohomology algebra $H^*(A^\bullet)$, and whole the page E_1 is simply the bar-complex of the cohomology algebra $H^*(A^\bullet)$. Hence the page E_2 can be computed as

$$E_2^{pq} = \mathrm{Tor}_{pq}^{H^*(A^\bullet)}(k,k),$$

where the first grading p on the Tor spaces in the right-hand side is the conventional homological grading of the Tor and the second grading q is the "internal" grading induced by the cohomological grading on the algebra $H^*(A^\bullet)$.

The differentials d_r^{pq}, $r \geqslant 2$, in the algebraic Eilenberg–Moore spectral sequence $E_r^{pq} = E_r^{pq}(A^\bullet)$ associated with an augmented DG-algebra $A^\bullet$ are, by the definition, the *Massey products* in the cohomology algebra $H^*(A^\bullet)$ that we are interested in. An augmented DG-algebra $A^\bullet$ is called *quasi-formal* if the spectral sequence $E_r^{pq}(A^\bullet)$ degenerates at the page E_2, that is all the Massey products d_r^{pq}, $r \geqslant 2$, vanish.

An augmented DG-algebra $A^\bullet$ is called *formal* (in the class of augmented DG-algebras) if it can be connected with its cohomology algebra $H^*(A^\bullet)$, viewed as an augmented DG-algebra with zero differential and the augmentation induced by that of $A^\bullet$, by a chain of quasi-isomorphisms of augmented DG-algebras.

Proposition 2.1. *Any formal augmented DG-algebra is quasi-formal.*

Proof. Notice that any morphism of augmented DG-algebras $A^\bullet \longrightarrow B^\bullet$ induces a morphism of spectral sequences $E_r^{pq}(A^\bullet) \longrightarrow E_r^{pq}(B^\bullet)$. When the morphism $A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism, the induced morphism of spectral sequences in an isomorphism on the pages E_1, and consequenly also on all the higher pages. So the Massey products are preserved by quasi-isomorphisms of augmented DG-algebras.

Therefore, whenever an augmented DG-algebra $A^\bullet$ is connected with an augmented DG-algebra $B^\bullet$ by a chain of quasi-isomorphisms of augmented DG-algebras, an augmented DG-algebra $A^\bullet$ is quasi-formal if and only if an augmented DG-algebra $B^\bullet$ is. Since the Massey products in a DG-algebra with zero differential clearly vanish (as do the higher differentials in the spectral sequence of any bicomplex with one of the two differentials vanishing), the desired assertion follows. □

An extension of (a part of) the canonical, partially defined, multivalued Massey operations on the cohomology algebra of a DG-algebra $A^\bullet$ to total, single-valued polylinear maps, which taken together are defined up to certain transformations, is called the A_∞-*algebra* structure on the cohomology algebra $H^*(A^\bullet)$ of a DG-algebra $A^\bullet$ [41, 19, 20, 11, 24]. The A_∞-algebra structure on $H^*(A^\bullet)$ determines a DG-algebra $A^\bullet$ up to quasi-isomorphism. Thus, while vanishing of the Massey operations in the cohomology only makes a DG-algebra $A^\bullet$ *quasi-formal*, vanishing of the higher operations in (a certain representative of the A_∞-isomorphism class of) the A_∞-algebra structure on $H^*(A^\bullet)$ would actually mean that the DG-algebra $A^\bullet$ is *formal*.

Let $m \neq 0$ be an integer. Suppose that the cohomology algebra $H^*(A^\bullet)$ is concentrated in the cohomological gradings $q = mn,\ n = 0, 1, 2, \dots$, one has $H^0(A^\bullet) = k$, and the algebra $H^*(A^\bullet)$ is Koszul in the grading rescaled by m, i. e., one has

$$\operatorname{Tor}_{pq}^{H^*(A^\bullet)}(k,k) = 0 \qquad \text{for } mp \neq q.$$

Then all the differentials d_r^{pq}, $r \geqslant 2$, vanish for "dimension" (bigrading) reasons, and the DG-algebra $A^\bullet$ is quasi-formal (cf. [5]). One can say that this is an instance of *intrinsic quasi-formality*, i. e., a situation when any augmented DG-algebra with a given cohomology algebra is quasi-formal. In this paper, we are interested in the case $m = 1$, i. e., the situation when the augmentation ideal $H^*(A^\bullet_+)$ of the cohomology algebra $H^*(A^\bullet)$ is concentrated in the cohomological degrees 1, 2, 3 …

Corollary 2.2. *Suppose that the cohomology algebra $H^*(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ is positively cohomologically graded and Koszul in its cohomological grading. Then the augmented DG-algebra $A^\bullet$ is quasi-formal.*

Proof. This is a corollary of the definitions, as explained above. □

For lack of a better term, let us call the Massey products discussed above the *tensor Massey products*. Our next aim is to compare these with a more elementary construction that we call the *tuple Massey products*.

One reason for our interest in tuple Massey products and this comparison comes from the application to the absolute Galois groups and Galois cohomology. A conjecture of ours claims that the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$ of the absolute Galois group G_F of a field F containing a primitive l-root of unity is Koszul [32, 36]. On the other hand, there is a series of recent papers [16, 27, 8, 28] discussing and partially proving the conjecture that tuple Massey products of degree-one elements vanish in the cohomology algebra $H^*(G_F, \mathbb{Z}/l)$. The results above in this section and the discussion below show that the Koszulity conjecture implies vanishing of the tensor Massey products in $H^*(G_F, \mathbb{Z}/l)$, but may have no direct implications concerning the problem of vanishing of the tuple Massey products.

Let $A^\bullet = (A^*, d\colon A^i \to A^{i+1})$ be a DG-algebra over a field k; assume for simplicity that $A^i = 0$ for $i < 0$ and $A^0 = k$ (so in particular $d^0\colon A^0 \longrightarrow A^1$ is a zero map and the DG-algebra $A^\bullet$ has a natural augmentation $A^\bullet \longrightarrow k$). Let $B^i \subset Z^i \subset A^i$ denote the subspaces of coboundaries and cocycles in $A^\bullet$, so that $H^i = H^i(A^\bullet) = Z^i/B^i$. The simplest possible construction of a 3-tuple Massey product of degree-one elements in the cohomology algebra $H^*(A^\bullet)$ proceeds as follows.

Let x, y, $z \in H^1(A^\bullet)$ be three elements for which $xy = 0 = yz$ in $H^2(A^\bullet)$. Pick some preimages $\tilde{x}$, $\tilde{y}$, $\tilde{z} \in Z^1$ of the elements x, y, $z \in H^1$. Then the products $\tilde{x}\tilde{y}$ and $\tilde{y}\tilde{z}$ are coboundaries in A^2; so there exist elements ζ and $\xi \in A^1$ such that $d\zeta = \tilde{x}\tilde{y}$ and $d\xi = \tilde{y}\tilde{z}$ in A^2. Hence one has

$$d(\tilde{x}\xi + \zeta\tilde{z}) = -\tilde{x}d(\xi) + d(\zeta)\tilde{z} = -\tilde{x}\tilde{y}\tilde{z} + \tilde{x}\tilde{y}\tilde{z} = 0,$$

so the element $\tilde{x}\xi + \zeta\tilde{z}$ is a cocycle in A^2. By the definition, one sets the 3-tuple Massey product $\langle x, y, z\rangle \in H^2(A^\bullet)$ to be equal to the cohomology class of the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$.

We have made some arbitrary choices along the way, so it is important to find out how does the output depend on these choices. Replacing the cochain ζ by a different cochain ζ' with the same differential $d\zeta' = \tilde{x}\tilde{y} \in A^2$ adds the product of two cocycles $(\zeta' - \zeta)\tilde{z} \in Z^1 \cdot \tilde{z} \subset Z^2$ to the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$. This means adding an element of the subspace $H^1 \cdot z \subset H^2$ to our 3-tuple Massey product $\langle x, y, z\rangle \in H^2(A^\bullet)$.

Similarly, replacing the cochain ξ by a different cochain ξ' with the same differential $d\xi' = \tilde{y}\tilde{z}$ adds the product of two cocycles $\tilde{x}(\xi' - \xi) \in \tilde{x} \cdot Z^1 \subset Z^2$ to the cocycle $\tilde{x}\xi + \zeta\tilde{z} \in Z^2$, which means adding an element of the subspace $x \cdot H^1 \subset H^2$ to the 3-tuple Massey product $\langle x, y, z\rangle$.

Furthermore, since we have assumed that $A^0 = k$, the preimages $\tilde{x}$, $\tilde{y}$, $\tilde{z} \in Z^1$ of given elements x, y, $z \in H^1$ are uniquely defined. However, even without the $A^0 = k$ assumption, one easily checks that the choice of the preimages $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ does not introduce any new indeterminacies into the output of our 3-tuple Massey product construction as compared to the ones we already described.

To conclude, the tuple Massey product of three elements x, y, $z \in H^1(A^\bullet)$ with $xy = 0 = yz$ in $H^2(A^\bullet)$ is well-defined as an element of the quotient space

$$\langle x, y, z\rangle \in H^2(A^\bullet)/(x \cdot H^1 + H^1 \cdot z).$$

Now let us describe the connection with the tensor Massey products. Suppose that we want to extend the above construction to elements of the tensor product space $H^1(A^\bullet)^{\otimes 3} = H^1(A^\bullet) \otimes_k H^1(A^\bullet) \otimes_k H^1(A^\bullet)$. With any three vectors x, y, $z \in H^1(A^\bullet)$

one can associate the decomposable tensor $x\otimes y\otimes z\in H^1(A^\bullet)^{\otimes 3}$; however, not every tensor is decomposable.

Let $m\colon A^*\otimes_k A^*\longrightarrow A^*$ denote the multiplication map in the DG-algebra $A^\bullet$. We denote the induced (conventional) multiplication on the cohomology algebra by $m_2\colon H^*(A^\bullet)\otimes_k H^*(A^\bullet)\longrightarrow H^*(A^\bullet)$. Let $K^2\subset H^1(A^\bullet)\otimes_k H^1(A^\bullet)$ denote the kernel of the multiplication map $m_2\colon H^1(A^\bullet)\otimes_k H^1(A^\bullet)\longrightarrow H^2(A^\bullet)$. We would like to have our triple Massey product defined on the subspace

$$K^2\otimes_k H^1(A^\bullet)\cap H^1(A^\bullet)\otimes_k K^2\,\subset\, H^1(A^\bullet)^{\otimes 3}.$$

The construction proceeds as follows. Given a tensor $\theta\in K^2\otimes H^1\cap H^1\otimes K^2\subset H^1\otimes H^1\otimes H^1$, we lift it to a tensor $\widetilde\theta$ in $Z^1\otimes Z^1\otimes Z^1$ and apply the maps of multiplication of the first two and the last two tensor factors $m^{(12)}=m\otimes\mathrm{id}$ and $m^{(23)}=\mathrm{id}\otimes m$ to obtain a pair of elements $m^{(12)}(\widetilde\theta)\in B^2\otimes Z^1$ and $m^{(23)}(\widetilde\theta)\in Z^1\otimes B^2$. Then we lift these two elements arbitrarily to elements in $A^1\otimes Z^1$ and $Z^1\otimes A^1$, respectively, and finally apply the product map m to each of them and add the results in order to obtain an element in A^2. By virtue of a computation similar to the above, this turns out to be an element of Z^2. Its image in $H^2(A^\bullet)$, denoted by $m_3(\theta)$, is the triple tensor Massey product of our tensor θ.

What is the subspace in $H^2(A^\bullet)$ up to which the element $m_3(\theta)$ is defined? Let $W_l\subset H^1$ be the minimal vector subspace for which $\theta\in W_l\otimes H^1\otimes H^1$, and let W_r be the similar minimal subspace for which $\theta\in H^1\otimes H^1\otimes W_r$ (hence in fact $\theta\in W_l\otimes H^1\otimes W_r$). If one is careful, one can make the Massey product $m_3(\theta)$ well-defined up to elements of $W_l\cdot H^1+H^1\cdot W_r\subset H^2(A^\bullet)$. However, generally speaking, for "most" tensors $\theta\in K^2\otimes H^1\cap H^1\otimes K^2$ (and certainly for "most" tensors in $H^1\otimes H^1\otimes H^1$) one would expect $W_l=H^1=W_r$. So the triple Massey product that we have constructed is most simply viewed as a linear map

$$m_3\colon K^2\otimes_k H^1(A^\bullet)\cap H^1(A^\bullet)\otimes_k K^2\longrightarrow H^2(A^\bullet)/m_2(H^1(A^\bullet)\otimes_k H^1(A^\bullet)),$$
$$K^2=\ker(m_2\colon H^1(A^\bullet)\otimes H^1(A^\bullet)\longrightarrow H^2(A^\bullet)).$$

Notice that one has

$$K^2\otimes_k H^1(A^\bullet)\cap H^1(A^\bullet)\otimes_k K^2\,\simeq\,\mathrm{Tor}^{H^*(A^\bullet)}_{3,3}(k,k)\,=\,E_2^{3,3}$$

and

$$H^2(A^\bullet)/m_2(H^1(A^\bullet)\otimes_k H^1(A^\bullet))\,\simeq\,\mathrm{Tor}^{H^*(A^\bullet)}_{1,2}(k,k)\,=\,E_2^{1,2}$$

in the Eilenberg–Moore spectral sequence. We have obtained an explicit construction of the differential

$$d_2^{3,3}\colon E_2^{3,3}\longrightarrow E_2^{1,2},$$

which is the simplest example of a tensor Massey product in the sense of our definition.

How is this triple tensor Massey product construction related to the 3-tuple Massey product defined above? On the one hand, a subspace $K^2\subset H^1\otimes H^1$ may well contain no nonzero decomposable tensors at all, while containing many nontrivial indecomposable tensors. Then there may be also many nontrivial indecomposable tensors in $K^1\otimes H^1\cap H^1\otimes K^2$. So the domain of definition of the tensor Massey product may be essentially much wider than that of the tuple Massey product. On the other hand, the latter, more elementary construction may produce its outputs with better precision, i. e., modulo a smaller subspace in $H^2(A^\bullet)$. Thus the map m_3 carries both more and less information about the DG-algebra $A^\bullet$ than the operation $\langle x, y, z\rangle$ in the cohomology algebra $H^*(A^\bullet)$.

Similarly, let x_1, x_2, x_3, $x_4 \in H^1(A^\bullet)$ be four elements for which $x_1x_2 = x_2x_3 = x_3x_4 = 0$ in $H^2(A^\bullet)$. Since we have assumed that $A^0 = k$, these elements have uniquely defined defined preimages in $Z^1 \simeq H^1$, which we will denote by $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, $\tilde{x}_4$. The products $\tilde{x}_1\tilde{x}_2$, $\tilde{x}_2\tilde{x}_3$, and $\tilde{x}_3\tilde{x}_4$ are coboundaries in A^2, so there exist three elements η_{12}, η_{23}, $\eta_{34} \in A^1$ such that $d\eta_{rs} = \tilde{x}_r\tilde{x}_s$ for all $1 \leqslant r < s \leqslant 4$, $s - r = 1$. The elements

$$\tilde{x}_1\eta_{23} + \eta_{12}\tilde{x}_3 \quad \text{and} \quad \tilde{x}_2\eta_{34} + \eta_{12}\tilde{x}_4 \in Z^2$$

represent the 3-tuple Massey products $\langle x_1, x_2, x_3\rangle$ and $\langle x_2, x_3, x_4\rangle$. Suppose that these two cocycles are coboundaries, i. e., there exist two elements ζ_{123} and $\zeta_{234} \in A^1$ such that $d\zeta_{rst} = \tilde{x}_r\eta_{st} + \eta_{rs}\tilde{x}_t$ for $1 \leqslant r < s < t \leqslant 4$, $t - s = s - r = 1$. One has

$$\begin{aligned} &d(\tilde{x}_1\zeta_{234} + \eta_{12}\eta_{34} + \zeta_{123}\tilde{x}_4) \\ &\qquad = -\tilde{x}_1\tilde{x}_2\eta_{34} - \tilde{x}_1\eta_{23}\tilde{x}_4 + \tilde{x}_1\tilde{x}_2\eta_{34} - \eta_{12}\tilde{x}_3\tilde{x}_4 + \tilde{x}_1\eta_{23}\tilde{x}_4 + \eta_{12}\tilde{x}_3\tilde{x}_4 = 0, \end{aligned}$$

so the element $\tilde{x}_1\zeta_{234} + \eta_{12}\eta_{34} + \zeta_{123}\tilde{x}_4$ is a cocycle in A^2. By the definition, one sets the 4-tuple Massey product $\langle x_1, x_2, x_3, x_4\rangle \in H^2(A^\bullet)$ to be equal to the cohomology class of this cocycle.

Let us briefly describe the tensor version of the quadruple Massey product. Let $K^3 \subset K^2 \otimes H^1 \cap H^1 \otimes K^2 \subset H^1(A^\bullet)^{\otimes 3}$ denote the kernel of the above map m_3. Consider the intersection of two vector subspaces $K^3 \otimes H^1 \cap H^1 \otimes K^3$ inside $H^1(A^\bullet)^{\otimes 4}$. Then the desired map is

$$m_4\colon K^3 \otimes H^1(A^\bullet) \cap H^1(A^\bullet) \otimes K^3 \longrightarrow (H^2(A^\bullet)/\operatorname{im} m_2)/\operatorname{im} m_3.$$

Its explicit construction is based on the same formulas as the above construction of the 4-tuple Massey product. This is the differential

$$d_3^{4,4}\colon E_3^{4,4} \longrightarrow E_3^{1,2}$$

in the Eilenberg–Moore spectral sequence.

The n-ary tensor Massey product of degree-one elements is a partially defined multivalued linear map

$$m_n\colon H^1(A^\bullet) \otimes H^1(A^\bullet) \otimes \cdots \otimes H^1(A^\bullet) \dashrightarrow H^2(A^\bullet)$$

that can de identified (up to a possible plus/minus sign) with the differential

$$d_{n-1}^{n,n}\colon E_{n-1}^{n,n} \longrightarrow E_{n-1}^{1,2}.$$

As in the case of triple Massey products, the constructions of the n-tuple and n-ary tensor Massey products agree where the former is defined up to elements of the subspace up to which the latter is defined. However, the domain of definition of the tensor Massey product may be wider than that of the tuple Massey product, while the tuple Massey product may produce its outputs with better precision.

Now let us consider the case when the cohomology algebra $H^*(A^\bullet)$ is generated by H^1 (as an associative algebra with the conventional multiplication m_2). Then, the map $m_2\colon H^1(A^\bullet)\otimes H^1(A^\bullet)\longrightarrow H^2(A^\bullet)$ being surjective, the above tensor Massey product maps m_3, m_4, … vanish automatically (as their target spaces are zero). So do the similar Massey products

$$d_{p-1}^{p,\,j_1+\dots+j_p}\colon H^{j_1}(A^\bullet)\otimes\dots\otimes H^{j_p}(A^\bullet) \dashrightarrow H^{j_1+\dots+j_p-p+2}(A^\bullet),$$

j_1, …, $j_p\geqslant 1$, $p\geqslant 3$, in the higher cohomology.

Does it mean that all the differentials d_r^{pq} in the Eilenberg–Moore spectral sequence vanish for $r\geqslant 2$? Not necessarily. The first possibly nontrivial example would be

$$d_2^{4,4}\colon H^1(A^\bullet)^{\otimes 4}\dashrightarrow H^1(A^\bullet)\otimes H^2(A^\bullet)\oplus H^2(A^\bullet)\otimes H^1(A^\bullet).$$

This is the map whose source space is actually the kernel $\mathrm{Tor}_{4,4}^{H^*(A^\bullet)}(k,k)$ the differential $d_1^{4,4}\colon H^1(A^\bullet)^{\otimes 4}\longrightarrow H^1(A^\bullet)^{\otimes 3}$, that is, the subspace

$$K^2\otimes H^1\otimes H^1\cap H^1\otimes K^2\otimes H^1\cap H^1\otimes H^1\otimes K^2\subset H^1(A^\bullet)^{\otimes 4}$$

and whose target space is the middle homology space $\mathrm{Tor}_{2,3}^{H^*(A^\bullet)}(k,k)$ of the sequence

$$H^1\otimes H^1\otimes H^1\longrightarrow H^2\otimes H^1\oplus H^1\otimes H^2\longrightarrow H^3$$

formed by the differentials $d_1^{3,3}$ and $d_1^{2,3}$. The latter vector space is otherwise known as the space of relations of degree 3 in the graded algebra $H^*(A^\bullet)$.

What does the map $d_2^{4,4}$ do? Its source space can be otherwise described as the intersection

$$(K^2\otimes H^1\cap H^1\otimes K^2)\otimes H^1\,\cap\, H^1\otimes(K^2\otimes H^1\cap H^1\otimes K^2).$$

The map $(m_3\otimes\mathrm{id},\ \mathrm{id}\otimes m_3)$ acts from this subspace to the quotient space of the vector space $H^2\otimes H^1\oplus H^1\otimes H^2$ by the image of the map $(m_2\otimes\mathrm{id},\ \mathrm{id}\otimes m_3)$ coming from the direct sum of two copies of $H^1\otimes H^1\otimes H^1$. It is claimed that the map $(m_3\otimes\mathrm{id},\ \mathrm{id}\otimes m_3)$ can be naturally lifted to the quotient space of $H^2\otimes H^1\oplus H^1\otimes H^2$ by the image of only one (diagonal) copy of $H^1\otimes H^1\otimes H^1$, as one can see from the explicit constructon of m_3.

Indeed, let us restrict ourselves to decomposable tensors now (for simplicity of notation). Let x_1, x_2, x_3, x_4 be four elements in $H^1(A^\bullet)$ for which $x_1x_2=x_2x_3=x_3x_4=0$ in $H^2(A^\bullet)$, and let $\tilde x_1$, $\tilde x_2$, $\tilde x_3$, $\tilde x_4$ be the liftings of these elements to $Z^1\simeq H^1$. Let η_{12}, η_{23}, η_{34} be three elements in A^1 such that $d\eta_{rs}=\tilde x_r\tilde x_s$ in $B^2\subset A^2$ for all $1\leqslant r<s\leqslant 4$, $s-r=1$. Then the triple Massey products are $\langle x_1,x_2,x_3\rangle=(\tilde x_1\eta_{23}+\eta_{12}\tilde x_3 \bmod B^2)$ and $\langle x_2,x_3,x_4\rangle=(\tilde x_2\eta_{34}+\eta_{23}\tilde x_4 \bmod B^2)\in H^2$. Replacing η_{rs} with $\eta'_{rs}=\eta_{rs}+\tilde y_{rs}$ with $\tilde y_{rs}\in Z^1$ for all $1\leqslant r<s\leqslant 4$, $s-r=1$, one obtains $\langle x_1,x_2,x_3\rangle'=\tilde x_1\eta'_{23}+\eta'_{12}\tilde x_3 \bmod B^2=\langle x_1,x_2,x_3\rangle+x_1y_{23}+y_{12}x_3$ and $\langle x_2,x_3,x_4\rangle'=$

$\tilde{x}_2\eta'_{34}+\eta'_{23}\tilde{x}_4 \bmod B^2 = \langle x_2, x_3, x_4\rangle + x_2y_{34} + y_{23}x_4$, where $y_{rs} \in H^1$ are the cohomology classes of the elements $\tilde{y}_{rs} \in Z^1$. Finally, one has

$$(\langle x_1, x_2, x_3\rangle' \otimes x_4,\ x_1 \otimes \langle x_2, x_3, x_4\rangle') = (\langle x_1, x_2, x_3\rangle \otimes x_4,\ x_1 \otimes \langle x_2, x_3, x_4\rangle) \\ + ((x_1y_{23} + y_{12}x_3) \otimes x_4,\ x_1 \otimes (x_2y_{34} + y_{23}x_4))$$

and

$$((x_1y_{23} + y_{12}x_3) \otimes x_4,\ x_1 \otimes (x_2y_{34} + y_{23}x_4)) \\ = d_1^{3,3}(x_1 \otimes y_{23} \otimes x_4 + y_{12} \otimes x_3 \otimes x_4 + x_1 \otimes x_2 \otimes y_{34})$$

in $H^2 \otimes H^1 \oplus H^1 \otimes H^2$, because $x_3x_4 = x_1x_2 = 0$ in $H^2(A^\bullet)$ by assumption.

What if the cohomology algebra $H^*(A^\bullet)$ is not only generated by H^1, but also defined by quadratic relations? There still can be nontrivial tensor Massey operations (i. e., the differentials d_r^{pq} with $r \geqslant 2$), starting from

$$d_2^{5,5}\colon H^1(A^\bullet)^{\otimes 5} \dashrightarrow H^2 \otimes H^1 \otimes H^1 \oplus H^1 \otimes H^2 \otimes H^1 \oplus H^1 \otimes H^1 \otimes H^2.$$

This is actually well-defined as a linear map from the source space

$$\operatorname{Tor}_{5,5}^{H^*(A^\bullet)}(k,k) \simeq \textstyle\bigcap_{i=1}^4 H^1(A^\bullet)^{\otimes i-1} \otimes K^2 \otimes H^1(A^\bullet)^{\otimes 4-i} \subset H^1(A^\bullet)^{\otimes 5}$$

to a target space isomorphic to $\operatorname{Tor}_{3,4}^{H^*(A^\bullet)}(k,k)$. The latter Tor space is the first obstruction to Koszulity of a quadratic graded algebra $H^*(A^\bullet)$.

3. Noncommutative (Rational) Homotopy Theory

From an algebraist's point of view, *rational homotopy theory* is an equivalence between categories of commutative and Lie DG-(co)algebras satisfying appropriate boundedness conditions and viewed up to quasi-isomorphism. The classical formulation [40] claims an equivalence between the categories of negatively cohomologically graded Lie DG-algebras and augmented cocommutative DG-coalgebras with the augmentation ideals concentrated in the cohomological degrees $\leqslant -2$. The localizations of such two categories of DG-(co)algebras by the classes of (co)multiplicative quasi-isomorphisms are equivalent over any field of characteristic 0; over the field of rational numbers, these are also identified with the localization of the category of connected, simply connected topological spaces by the class of rational equivalences.

Attempting to include a nontrivial fundamental group into the picture, people usually consider nilpotent topological spaces, nilpotent groups, and Malcev completions. Here a discrete group is called nilpotent if its lower central series converges to zero in a finite number of steps. Yet the most natural setup for the nilpotency condition is that of coalgebras rather than algebras, as it allows for infinitary, or "ind"-conilpotency [32, 15]. Thus it appears that the maximal natural generality for "an algebraist's version of rational homotopy theory" is that of an equivalence between the categories of nonnegatively cohomologically graded conilpotent Lie DG-coalgebras and positively cohomologically graded commutative DG-algebras, considered up to quasi-isomorphism over a field of characteristic 0.

Here the commutative DG-algebra computes the cohomology algebra of the would-be topological space, while the conilpotent Lie DG-coalgebra is, roughly speaking, dual to the derived rational completion of its homotopy groups with their Whitehead bracket (notice the passage to the dual coalgebra in the completion construction of Theorem 1.2 and [33, Remark 5.6]). A Lie DG-coalgebra is called conilpotent if its underlying graded Lie (super)coalgebra is conilpotent; a nonnegatively graded Lie supercoalgebra is conilpotent if it is a union of dual coalgebras to finite-dimensional nilpotent nonpositively graded Lie superalgebras; and a finite-dimensional nonpositively graded Lie superalgebra is nilpotent if its degree-zero component is nilpotent Lie algebra and its action in the components of other degrees is nilpotent. (See [33, Section 8] and [34, Section D.6.1] for a discussion of conilpotent Lie coalgebras and their conilpotent coenveloping coalgebras.)

In this section, we make yet another algebraic generalization/simplification and replace the pair of dual operads $\mathcal{C}om$-$\mathcal{L}ie$ with that of $\mathcal{A}ss$-$\mathcal{A}ss$. In other words, we consider a noncommutative version of the above-described theory with commutative DG-algebras replaces by associative ones and conilpotent Lie DG-coalgebras replaced by conilpotent coassociative ones. In this setting, the characteristic 0 restriction becomes unnecessary and one can work over an arbitrary ground field k. Thus our aim is to construct an equivalence between the categories of nonnegatively cohomologically graded conilpotent (coassociative) DG-coalgebras and positively cohomologically graded (associative) DG-algebras.

Let $C^\bullet$ be a nonzero DG-coalgebra over a field k; suppose that it is endowed with a coaugmentation (DG-coalgebra morphism) $k \longrightarrow C^\bullet$ and denote the quotient DG-coalgebra (without counit) by $C^\bullet_+ = C^\bullet/k$. By the definition, the *cobar construction* $\mathrm{Cob}^\bullet(C^\bullet)$ of a coaugmented DG-coalgebra $C^\bullet$ is the free associative algebra

$$\mathrm{Cob}(C) = \bigoplus\nolimits_{n=0}^{\infty} C_+[-1]^{\otimes n}$$

generated by the graded vector space $C_+[-1]$ obtained by shifting by -1 the cohomological grading of the coaugmentation cokernel C_+. The grading on $\mathrm{Cob}(C)$ is induced by the grading of $C_+[-1]$. Alternatively, one can define the cobar construction $\mathrm{Cob}(C)$ as the direct sum of the tensor powers $C_+^{\otimes n}$ of the vector space C_+ and endow it with the total grading equal to the sum $i+n$ of the grading i induced by the grading of C_+ and the grading n by the number of tensor factors.

The differential on $\mathrm{Cob}^\bullet(C^\bullet)$ is the sum of two summands $d+\partial$, the former of them induced by the differential on $C^\bullet_+$ and the latter one by the comultiplication in $C^\bullet_+$. One has to work out the plus/minus signs in order to make the total differential on $\mathrm{Cob}^\bullet(C^\bullet)$ square to zero (cf. the definition of the bar construction in Section 2).

A quasi-isomorphism of augmented DG-algebras $A^\bullet \longrightarrow B^\bullet$ induces a quasi-isomorphism of their bar constructions $\mathrm{Bar}^\bullet(A^\bullet) \longrightarrow \mathrm{Bar}^\bullet(B^\bullet)$ (as one can see from the filtration and the Eilenberg–Moore spectral sequence discussed in Section 2). However, the morphism of cobar constructions $\mathrm{Cob}^\bullet(C^\bullet) \longrightarrow \mathrm{Cob}^\bullet(D^\bullet)$ induced by a quasi-isomorphism of coaugmented (even conilpotent) DG-coalgebras $C^\bullet \longrightarrow D^\bullet$ may not be a quasi-isomorphism (see Remark 3.6 at the end of this section). The reason is that the filtration of the bar construction by the number of tensor factors is an increasing one, while the similar filtration of the cobar construction is a decreasing one (cf. the discussion of two kinds of differential Cotor functors in [9], [17], and [34, Section 0.2.10]).

As we are interested in the cobar construction as defined above (i. e., the direct sum of the tensor powers of $C^\bullet_+[-1]$) rather than its completion by this filtration (which would mean the direct product of such tensor powers), the related spectral sequence can be viewed as converging to the cohomology of the cobar construction only when it is in some sense locally finite. This includes two separate cases considered below, which roughly correspond to the "conilpotent" and "simply connected" versions of noncommutative homotopy theory as discussed above.

The conilpotent version of the theory, which is of primary interest to us, is based on the following assertion (which does not yet presume conilpotency, but it will be needed further on).

Proposition 3.1. *Let $C^\bullet = (C^0 \to C^1 \to C^2 \to \cdots)$ and $D^\bullet = (D^0 \to D^1 \to D^2 \to \cdots)$ be two nonnegatively cohomologically graded coaugmented DG-coalgebras. Then any comultiplicative quasi-isomorphism $f\colon C^\bullet \longrightarrow D^\bullet$, i. e., a morphism of DG-coalgebras inducing an isomorphism $H^*(C^\bullet) \simeq H^*(D^\bullet)$ of their cohomology coalgebras, induces a quasi-isomorphism of the cobar constructions* $\mathrm{Cob}^\bullet(f)\colon \mathrm{Cob}^\bullet(C^\bullet) \longrightarrow \mathrm{Cob}^\bullet(D^\bullet)$.

Proof. For any coaugmented DG-coalgebra $E^\bullet$, set $G^p \,\mathrm{Cob}^\bullet(E^\bullet) = \bigoplus_{n=p}^\infty E_+[-1]^{\otimes n}$. This is a decreasing filtration of the DG-algebra $\mathrm{Cob}^\bullet(E^\bullet)$ compatible with the multiplication and the differential. Clearly, a quasi-isomorphism of coaugmented DG-coalgebras $C^\bullet \longrightarrow D^\bullet$ induces a quasi-isomorphism of the associated graded algebras $\mathrm{gr}_G \mathrm{Cob}^\bullet(C^\bullet) \longrightarrow \mathrm{gr}_G \mathrm{Cob}^\bullet(D^\bullet)$. It remains to observe that when the DG-coalgebras $C^\bullet$ and $D^\bullet$ are nonnegatively cohomologically graded, the filtrations $G^\bullet$ on $\mathrm{Cob}^\bullet(C^\bullet)$ and $\mathrm{Cob}^\bullet(D^\bullet)$ are finite at every (total) cohomological degree. Indeed, one has $G^{n+1}\,\mathrm{Cob}^n(C^\bullet) = 0 = G^{n+1}\,\mathrm{Cob}^n(D^\bullet)$ for every integer n. □

A coaugmented DG-coalgebra $C^\bullet$ is called *conilpotent* if its underlying coaugmented (graded) coalgebra is conilpotent (see Section 1 for the definition). The cobar construction $\mathrm{Cob}^\bullet(C^\bullet)$ of a coaugmented DG-coalgebra $C^\bullet$ is naturally an augmented DG-algebra; and the bar construction $\mathrm{Bar}^\bullet(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ is naturally a conilpotent DG-coalgebra.

The two constructions $C^\bullet \longmapsto \mathrm{Cob}^\bullet(C^\bullet)$ and $A^\bullet \longmapsto \mathrm{Bar}^\bullet(A^\bullet)$, viewed as functors between the categories of augmented DG-algebras and conilpotent DG-coalgebras, are adjoint functors: for any conilpotent DG-coalgebra $C^\bullet$ and augmented DG-algebra $A^\bullet$, there is a bijective correspondence between morphisms of augmented DG-algebras $\mathrm{Cob}^\bullet(C^\bullet) \longrightarrow A^\bullet$ and morphisms of coaugmented DG-coalgebras $C^\bullet \longrightarrow \mathrm{Bar}^\bullet(A^\bullet)$. The proofs of these results, as well as of the following ones, can be found, e. g., in [35, Section 6.10] (see also [17, Sections II.3-4] and [15, Section 3]).

Proposition 3.2. (a) *For any augmented DG-algebra $A^\bullet$, the adjunction morphism $\mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet)) \longrightarrow A^\bullet$ is a quasi-isomorphism of augmented DG-algebras.*

(b) *For any conilpotent DG-coalgebra $C^\bullet$, the adjunction morphism $C^\bullet \longrightarrow \mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(C^\bullet))$ is a quasi-isomorphism of (conilpotent) DG-coalgebras.*

Warning: the assertion of part (b) does *not* hold without the conilpotency assumption on $C^\bullet$, and in fact, the adjunction morphism does not even *exist* without this assumption (cf. Remark 4.5 below).

Proof. Part (a): define an increasing filtration on a DG-algebra $A^\bullet$ by the rules $F_0A = k$ and $F_nA = A$ for $n \geqslant 1$. This filtration is compatible with the multiplication on $A^\bullet$, and so it induces a filtration F compatible with the comultiplication on $\mathrm{Bar}^\bullet(A^\bullet)$, and further, a filtration F compatible with the multiplication on $\mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet))$ as well as with the adjunction morphism $\mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet)) \longrightarrow A^\bullet$. The passage to the associated graded DG-(co)algebras with respect to the filtration F gets rid of all the information about the multiplication in $A^\bullet$; so the DG-algebra $\mathrm{gr}^F A^\bullet$ is obtained from the DG-algebra $A^\bullet$ by setting the multiplication on $A_+^\bullet$ to be zero, and the DG-algebra $\mathrm{gr}^F \mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet))$ is the cobar-construction of the bar-construction of the DG-algebra $\mathrm{gr}^F A^\bullet$. From this point one can proceed further and rid oneself also of the differential on $A^\bullet$ in addition to the multiplication; but this is unnecessary. It suffices to notice that the component of degree n of the complex $\mathrm{gr}^F \mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet))$ with respect to the grading by the indices of the filtration F is the total complex of a bicomplex composed of 2^{n-1} copies of the complex $A_+^{\otimes n}$. Proving that such complexes are acyclic for $n \geqslant 2$ is elementary combinatorics.

Part (b): the argument dual to the one in part (b) is not immediately applicable, as the related filtration G on the DG-coalgebra $C^\bullet$ and the induced filtrations on its cobar and bar constructions would be decreasing ones. Instead, consider the canonical increasing filtration $N_mC^\bullet = \ker(C^\bullet \to C_+^{\otimes m+1})$ on the conilpotent DG-coalgebra $C^\bullet$. The associated graded DG-coalgebra $\mathrm{gr}^N \mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(C^\bullet))$ of the DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(C^\bullet))$ by the increasing filtration induced by the filtration N on $C^\bullet$ is identified with the DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(\mathrm{gr}^n C^\bullet))$. This reduces the question to the case of the DG-coalgebra $\mathrm{gr}^N C^\bullet$, which endowed with an additional positive grading by the indices of the filtration N. Now one endows the DG-coalgebra $\mathrm{gr}^N C^\bullet$ with the decreasing filtration G with $G^0 \mathrm{gr}^N C^\bullet = \mathrm{gr}^N C^\bullet$, the component $G^1 \mathrm{gr}^N C^\bullet$ being the kernel of the counit map $\mathrm{gr}^N C^\bullet \longrightarrow k$, and $G^n \mathrm{gr}^N C^\bullet = 0$ for $n \geqslant 2$. The induced decreasing filtration G on the DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(\mathrm{gr}^N C^\bullet))$ is locally finite in the grading by the indices m of the filtration N. This reduces the question to proving that the morphism of DG-coalgebras $\mathrm{gr}_G \mathrm{gr}^N C^\bullet \longrightarrow \mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(\mathrm{gr}_G \mathrm{gr}^N C^\bullet))$ is a quasi-isomorphism, which can be done by a combinatorial argument similar to the one in part (a). □

We recall that an augmented DG-algebra $A^\bullet$ is called *positively cohomologically graded* if its augmentation ideal is concentrated in the positive cohomological degrees, that is $A_+^i = 0$ for all $i \leqslant 0$. Equivalently, a DG-algebra $A^\bullet$ is positively cohomologically graded if $A^i = 0$ for all $i < 0$ and A^0 is a one-dimensional vector space generated by the unit element of A; any such DG-algebra $A^\bullet$ has a unique augmentation (DG-algebra morphism) $A^\bullet \longrightarrow k$.

Theorem 3.3. *The functors $A^\bullet \longmapsto \mathrm{Bar}^\bullet(A^\bullet)$ and $C^\bullet \longmapsto \mathrm{Cob}^\bullet(C^\bullet)$ induce mutually inverse equivalences between the category of positively cohomologically graded DG-algebras $A^\bullet$ with quasi-isomorphisms inverted and the category of nonnegatively cohomologically graded conilpotent DG-coalgebras $C^\bullet$ with quasi-isomorphisms inverted.*

Proof. Having in mind the results of Propositions 3.1 and 3.2, it suffices to notice that the bar construction takes positively cohomologically graded augmented DG-algebras to nonnegatively cohomologically graded conilpotent DG-coalgebras, while the cobar construction takes nonnegatively cohomologically graded coaugmented DG-coalgebras to positively cohomologically graded augmented DG-algebras. □

For comparison, let us now present the simply connected version of the theory, that is the direct noncommutative analogue of "the algebraic part" of [40, Theorem I]. A coaugmented DG-coalgebra $C^\bullet$ is called *negatively cohomologically graded* if its coaugmentation cokernel $C_+^\bullet$ is concentrated in the negative cohomological degrees, that is $C_+^i = 0$ for all $i \geqslant 0$. Clearly, any negatively cohomologically graded DG-coalgebra is conilpotent. Let us call a negatively cohomologically graded DG-coalgebra *simply connected* if $C^{-1} = 0$, i. e., $C_+^i = 0$ for all $i \geqslant -1$.

Proposition 3.4. *Any quasi-isomorphism $f\colon C^\bullet \longrightarrow D^\bullet$ between two simply connected negatively cohomologically graded DG-coalgebras $C^\bullet = (\cdots \to C^{-3} \to C^{-2} \to 0 \to k)$ and $D^\bullet = (\cdots \to D^{-3} \to D^{-2} \to 0 \to k)$ induces a quasi-isomorphism of the cobar constructions $\mathrm{Cob}^\bullet(f)\colon \mathrm{Cob}^\bullet(C^\bullet) \longrightarrow \mathrm{Cob}^\bullet(D^\bullet)$.*

Proof. The argument is similar to the proof of Proposition 3.1. Once again, one observes that the decreasing filtrations G on the cobar constructions $\mathrm{Cob}^\bullet(C^\bullet)$ and $\mathrm{Cob}^\bullet(D^\bullet)$ are finite at every cohomological degree in our assumptions. Indeed, one has $G^{n+1}\,\mathrm{Cob}^{-n}(C^\bullet) = 0 = G^{n+1}\,\mathrm{Cob}^{-n}(D^\bullet)$ for every integer n. □

An augmented DG-algebra $A^\bullet$ is called *negatively cohomologically graded* if one has $A_+^i = 0$ for all $i \geqslant 0$.

Theorem 3.5. *The functors $A^\bullet \longmapsto \mathrm{Bar}^\bullet(A^\bullet)$ and $C^\bullet \longmapsto \mathrm{Cob}^\bullet(C^\bullet)$ induce mutually inverse equivalences between the category of negatively cohomologically graded augmented DG-algebras $A^\bullet$ with quasi-isomorphisms inverted and the category of simply connected negatively cohomologically graded DG-coalgebras $C^\bullet$ with quasi-isomorphisms inverted.*

Proof. Here one has to notice that the bar construction takes negatively cohomologically graded augmented DG-algebras to simply connected negatively cohomologically graded DG-coalgebras, while the cobar construction takes simply connected negatively cohomologically graded DG-coalgebras to negatively cohomologically graded augmented DG-algebras. Otherwise the argument is similar to the proof of Theorem 3.3 and based on the result of Proposition 3.4. □

Remark 3.6. The assertions of the above theorems can be modified so as to hold for arbitrary augmented DG-algebras and conilpotent DG-coalgebras. One just has to replace the class of quasi-isomorphisms of DG-coalgebras with a finer class of *filtered quasi-isomorphisms* of conilpotent DG-coalgebras, which are to be inverted in order to obtain a category equivalent to the category of augmented DG-algebras with quasi-isomorphisms inverted (see [15, Section 4] or [35, Section 6.10]). In the form stated above, on the other hand, the assertions of the theorems do *not* hold already for the negatively cohomologically graded DG-coalgebras that are not simply connected—in fact, this is the class of DG-coalgebras for which the difference between quasi-isomorphisms and filtered quasi-isomorphisms becomes essential. It suffices to consider a morphism between two different augmented k-algebras (viewed as DG-algebras concentrated in cohomological degree zero) $A \longrightarrow B$ inducing an isomorphism of the Tor spaces $\mathrm{Tor}^A_*(k,k) \simeq \mathrm{Tor}^B_*(k,k)$. Then the induced morphism of the bar constructions $\mathrm{Bar}^\bullet(A) \longrightarrow \mathrm{Bar}^\bullet(B)$ is a quasi-isomorphism of negatively cohomologically graded conilpotent DG-coalgebras that is transformed by the cobar construction into a morphism of DG-algebras $\mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A)) \longrightarrow \mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(B))$ with two different cohomology algebras A and B [35, Remark 6.10].

4. $K(\pi,1)$-ness + Quasi-Formality Imply Koszulity

We refer for the definitions of the bar construction $\mathrm{Bar}^\bullet(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ to Section 2 and of the cobar construction $\mathrm{Cob}^\bullet(C^\bullet)$ of a coaugmented DG-coalgebra $C^\bullet$ to Section 3. The definition of the cobar construction $\mathrm{Cob}^\bullet(D)$ of a coalgebra D was given previously in Section 1; and the definition of the conilpotency property of a coalgebra C can be found in the same section.

The construction of the (tensor) Massey operations on the cohomology algebra of an augmented DG-algebra $A^\bullet$, understood as the higher differentials in the algebraic Eilenberg–Moore spectral sequence (associated with a natural increasing filtration on the bar-complex $\mathrm{Bar}^\bullet(A^\bullet)$), was introduced and discussed in Section 2. An augmented DG-algebra $A^\bullet$ is called *quasi-formal* if all these Massey operations vanish.

Finally, we recall that a graded algebra H^* over a field k is called *Koszul* if it is concentrated in the positive degrees, that is $H^i = 0$ for $i < 0$ and $H^0 = k$, and its bigraded Tor coalgebra (computed by the internally graded DG-coalgebra $\mathrm{Bar}^\bullet(H^*)$) is concentrated in the diagonal grading, i. e., $\mathrm{Tor}^{H^*}_{ij}(k,k) = 0$ for $i \neq j$.

Let $A^\bullet$ be an augmented DG-algebra over a field k. We will say that a DG-algebra $A^\bullet$ is *of the* $K(\pi,1)$ *type* (or just simply "a $K(\pi,1)$") if there exists a conilpotent coalgebra C over k such that the DG-algebra $A^\bullet$ can be connected with the DG-algebra $\mathrm{Cob}^\bullet(C)$ by a chain of quasi-isomorphisms of augmented DG-algebras. Here the (arbitrary coassociative and counital) conilpotent coalgebra C plays the role of the conilpotent coenveloping coalgebra of the conilpotent Lie coalgebra of a k-complete fundamental group π; so one could as well write "$A^\bullet = K(C,1)$".

The connection between Koszulity and the Massey operation vanishing was first pointed out by Priddy in [39, Section 8] (cf. [24]). In our language, the result of [39, Proposition 8.1] can be reformulated as follows. We refer to [32, Section 2] for the background material about positively graded coalgebras.

Theorem 4.1. *Let $C = k \oplus C_1 \oplus C_2 \oplus C_3 \oplus \cdots$ be a positively graded coalgebra cogenerated by its first-degree component C_1 over a field k, and let $A^\bullet = \operatorname{Cob}^\bullet(C)$ be the cobar DG-algebra of C. Then the graded coalgebra C is Koszul if and only if the differentials $d_{p-1}^{p,q}\colon E_{p-1}^{p,q} \longrightarrow E_{p-1}^{1,q-p+2}$*

$$(H^*(A^\bullet_+)^{\otimes p})^q \dashrightarrow H^{q-p+2}(A^\bullet_+)$$

in the Eilenberg–Moore spectral sequence of the DG-algebra $A^\bullet$ vanish for $p \geqslant 3$.

Proof. By [32, Propositions 1 and 2], a positively graded coalgebra C cogenerated by its first-degree component is Koszul if and only if its cohomology algebra $\operatorname{Ext}_C^*(k,k) = H^*(A^\bullet)$ is generated by $\operatorname{Ext}_C^1(k,k) = H^1(A^\bullet)$. Hence it remains to apply part (a) of the following proposition. (Notice also that, by [32, Proposition 3], a coalgebra C is Koszul if and only if the algebra $H^*(A^\bullet)$ is Koszul.) □

Proposition 4.2. *Let $A^\bullet = \operatorname{Cob}^\bullet(C)$ be the cobar DG-algebra of a conilpotent coalgebra C. Then*

(a) *the cohomology algebra $H^*(A^\bullet)$ is multiplicatively generated by $H^1(A^\bullet)$ if and only if the differentials $d_{p-1}^{p,q}\colon E_{p-1}^{p,q} \longrightarrow E_{p-1}^{1,q-p+2}$ vanish for $p \geqslant 3$;*

(b) *the cohomology algebra $H^*(A^\bullet)$ is quadratic if and only if the differentials $d_{p-1}^{p,q}$ as well as the differentials $d_{p-1}^{p+1,q}\colon E_{p-1}^{p+1,q} \longrightarrow E_{p-1}^{2,q-p+2}$*

$$(H^*(A^\bullet_+)^{\otimes p+1})^q \dashrightarrow (H^*(A^\bullet_+) \otimes H^*(A^\bullet_+))^{q-p+2}$$

vanish for $p \geqslant 3$.

Proof. Part (a): if the algebra $H^*(A^\bullet)$ is multiplicatively generated by H^1, then $E_2^{1,n} = \operatorname{Tor}_{1,n}^{H^*(A^\bullet)}(k,k) = 0$ for all $n \geqslant 2$; since one also has $E_1^{p,q} = (H^*(A^\bullet)^{\otimes p})^q = 0$ for all $p > q$, it follows that the differentials $d_{p-1}^{p,q}$ vanish for $p \geqslant 3$.

Conversely, by Proposition 3.2(b) the DG-coalgebra $\operatorname{Bar}^\bullet(A^\bullet)$ is quasi-isomorphic to C, so $E_\infty^{p,q} = \operatorname{gr}_p^F H^{p-q} \operatorname{Bar}^\bullet(A^\bullet) = 0$ for $p \neq q$, and in particular $E_\infty^{1,n} = 0$ for $n \geqslant 2$. This is the situation which people colloquially describe as "the cohomology $H^*(A^\bullet)$ is generated by $H^1(A^\bullet)$ using Massey products". If all the differentials $d_r^{p,q}$ landing in $E_r^{1,n}$ vanish for $r \geqslant 2$, it follows that $E_2^{1,n} = 0$ for $n \geqslant 2$, so $H^*(A^\bullet)$ is generated by H^1 using the conventional multiplication.

Part (b): we can assume that the algebra $H^*(A^\bullet)$ is generated by H^1. If this algebra is also quadratic, then $E_2^{2,n} = \operatorname{Tor}_{2,n}^{H^*(A^\bullet)}(k,k) = 0$ for all $n \geqslant 3$, so it follows that the differentials $d_{p-1}^{p+1,q}$ vanish for $p \geqslant 3$. Conversely, as we explained above, $E_\infty^{2,n} = 0$ for $n \geqslant 3$, so if all the differentials landing in $E_r^{2,n}$ vanish for $r \geqslant 2$, then we can conclude that $E_2^{2,n} = 0$ for $n \geqslant 3$. □

In the nonhomogeneous conilpotent setting we are working in, the implication "$K(\pi,1)$-ness + quasi-formality imply Koszulity" becomes a bit more complicated than in Theorem 4.1, as the cohomology algebra $H^*(A^\bullet) = H^* \operatorname{Cob}^\bullet(C)$ being generated by H^1 no longer implies it being Koszul (cf. the final paragraphs of Section 2). The following theorem is the main result of this paper.

Theorem 4.3. *The cohomology algebra $H^* = H^*(A^\bullet)$ of an augmented DG-algebra $A^\bullet$ is Koszul if and only if the augmented DG-algebra $A^\bullet$ is simultaneously quasi-formal and of the $K(\pi, 1)$ type.*

Proof. It was explained in Section 2 that Koszulity of the cohomology algebra $H^*(A^\bullet)$ implies vanishing of the Massey products. The assertion that $A^\bullet$ is a $K(\pi, 1)$ whenever $H^*(A^\bullet)$ is Koszul, announced in the title of Section 1, was not actually proven there (in our present setting) but rather postponed; so we have to prove it now. We start with the following lemma.

Lemma 4.4. *Let $A^\bullet$ be an augmented DG-algebra whose cohomology algebra $H^*(A^\bullet)$ is concentrated in the positive cohomological degrees. Then there exists a positively cohomologically graded DG-algebra $P^\bullet$ together with a quasi-isomorphism of augmented DG-algebras $P^\bullet \longrightarrow A^\bullet$.*

Proof. The construction of a cofibrant resolution of the DG-algebra $A^\bullet$ in the conventional model structure on the category of augmented DG-algebras (see [14], [18], or [35, Section 9.1]) provides the desired DG-algebra $P^\bullet$. One starts from a free graded algebra with generators corresponding to representative cocycles of a chosen basis in $H^*(A^\bullet_+)$, and then iteratively adds to it new free generators whose differentials kill the cohomology classes annihilated by the morphism into $A^\bullet$. □

Thus we can assume our DG-algebra $A^\bullet$ to be positively cohomologically graded; then its bar construction $\mathrm{Bar}^\bullet(A^\bullet)$ is nonnegatively cohomologically graded. Now if the cohomology algebra $H^*(A^\bullet)$ is Koszul, then it follows from the Eilenberg–Moore spectral sequence that the cohomology coalgebra $H^* \mathrm{Bar}^\bullet(A^\bullet)$ of the DG-coalgebra $\mathrm{Bar}^\bullet(A^\bullet)$ is concentrated in cohomological degree 0.

Hence the embedding $C \longrightarrow \mathrm{Bar}^\bullet(A^\bullet)$ of the subcoalgebra $C = \ker(d^0\colon \mathrm{Bar}^0(A^\bullet) \to \mathrm{Bar}^1(A^\bullet))$ of the DG-coalgebra $\mathrm{Bar}^\bullet(A^\bullet)$ is a quasi-isomorphism. By Proposition 3.1, the induced morphism of the cobar constructions $\mathrm{Cob}^\bullet(C) \longrightarrow \mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet))$ is a quasi-isomorphism, too. By Proposition 3.2(a), so is the adjunction morphism $\mathrm{Cob}^\bullet(\mathrm{Bar}^\bullet(A^\bullet)) \longrightarrow A^\bullet$. Finally, the coalgebra C is conilpotent, since its ambient DG-coalgebra $\mathrm{Bar}^\bullet(A^\bullet)$ is. We have shown that the DG-algebra $A^\bullet$ is a $K(\pi, 1)$.

Now suppose, as the title of this section suggests, that the augmented DG-algebra $A^\bullet$ is a $K(\pi, 1)$ and the Massey products in its cohomology algebra $H^*(A^\bullet)$ vanish. Then the augmented DG-algebra $A^\bullet$ is connected by a chain of quasi-isomorphisms with the DG-algebra $\mathrm{Bar}^\bullet(C)$ for a certain conilpotent coalgebra C; we can simply assume that $A^\bullet = \mathrm{Cob}^\bullet(C)$. In particular, the cohomology algebra $H^*(A^\bullet)$ is concentrated in the positive cohomological degrees.

Applying Proposition 3.2(b), we can conclude that the DG-coalgebra $\mathrm{Bar}^\bullet(A^\bullet)$ is quasi-isomorphic to C, so its cohomology coalgebra $H^* \mathrm{Bar}^\bullet(A^\bullet)$ is concentrated in cohomological degree 0. On the other hand, the Massey product vanishing means that one has $E_2^{pq} = E_\infty^{pq}$ in the Eilenberg–Moore spectral sequence. As $E_\infty^{pq} = \mathrm{gr}_p^F H^{q-p} \mathrm{Bar}^\bullet(A^\bullet) = 0$ for $p \neq q$, it follows that $E_2^{pq} = \mathrm{Tor}_{pq}^{H^*(A^\bullet)}(k, k) = 0$. We have proven that the cohomology algebra $H^*(A^\bullet)$ is Koszul. □

Remark 4.5. Applying the cobar and bar constructions to a nonconilpotent coaugmented coalgebra D produces a conilpotent DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(D))$ with the zero-degree cohomology algebra $H^0\,\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(D))$ isomorphic to the maximal conilpotent subcoalgebra $C = \mathrm{Nilp}\, D$ of the coaugmented coalgebra D (see Section 1 for the definitions and notation here and below). The DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(D))$ can be called the DG-coalgebra of *derived conilpotent completion* of a coaugmented coalgebra D. The DG-algebra $\mathrm{Cob}^\bullet(D)$ is a $K(\pi,1)$ (i. e., the cohomology coalgebra of the DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(D))$ is concentrated in cohomological degree 0) if and only if the embedding $C \longrightarrow D$ induces a cohomology isomorphism $\mathrm{Ext}^*_C(k,k) \simeq \mathrm{Ext}^*_D(k,k)$.

Similarly, applying the cobar and bar constructions to an augmented algebra R produces a conilpotent DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(R))$ with the zero cohomology algebra $H^0\,\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(R))$ isomorphic to the coalgebra of pronilpotent completion $C = R\,\hat{}$ of the augmented DG-algebra R. The DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(R))$ can be called the DG-coalgebra of *derived pronilpotent completion* of an augmented algebra R. The DG-algebra $\mathrm{Cob}^\bullet(R)$ is a $K(\pi,1)$ (i. e., the cohomology coalgebra of the DG-coalgebra $\mathrm{Bar}^\bullet(\mathrm{Cob}^\bullet(R))$ is concentrated in cohomological degree 0) if and only if the natural map of the cohomology algebras $\mathrm{Ext}^*_C(k,k) \longrightarrow \mathrm{Ext}^*_R(k,k)$ is an isomorphism. These are noncommutative analogues of the procedure of rational completion of the space $K(\Gamma,1)$ with a discrete group Γ in rational homotopy theory.

These observations show, in particular, how to deduce the assertions of Theorems 1.1 and 1.2 from the "Koszulity implies $K(\pi,1)$-ness" claim in Theorem 4.3.

5. Koszulity Does Not Imply Formality

Examples of quasi-formal DG-algebras that are not formal are well known in the conventional (commutative) rational homotopy theory [13, Examples 8.13]. In this section we, working in the noncommutative homotopy theory of Section 3, over a field of prime characteristic, present a series of counterexamples of quasi-formal, nonformal DG-algebras *with Koszul cohomology algebras*. We also present a family of commutative DG-algebras with the similar properties defined over an arbitrary field (of zero or prime characteristic).

Recall that a DG-algebra $A^\bullet$ is called *formal* if it can be connected by a chain of quasi-isomorphisms of DG-algebras with its cohomology algebra $H^*(A^\bullet)$, viewed as a DG-algebra with zero differential (cf. Section 2). The following lemma shows that there is no ambiguity in this definition as applied to DG-algebras with the cohomology algebras concentrated in the positive cohomological degrees. We refer to Section 3 for a short discussion of positively cohomologically graded DG-algebras.

Lemma 5.1. *Let $A^\bullet$ and $B^\bullet$ be two augmented DG-algebras with the cohomology algebras concentrated in the positive cohomological degrees, connected by a chain of quasi-isomorphisms of DG-algebras over a field k. Then there exists a positively* *cohomologically graded DG-algebra $P^\bullet$ together with two quasi-isomorphisms of augmented DG-algebras $P^\bullet \longrightarrow A^\bullet$ and $P^\bullet \longrightarrow B^\bullet$. In particular, the chain of quasi-isomorphisms between the DG-algebras $A^\bullet$ and $B^\bullet$ can be made to consist of augmented quasi-isomorphisms of augmented DG-algebras.*

Proof. It suffices to choose a positively cohomologically graded cofibrant model of either DG-algebra $A^\bullet$ or $B^\bullet$ in the role of $P^\bullet$ (see Lemma 4.4). $\square$

Recall that any DG-algebra $A^\bullet$ with a Koszul cohomology algebra $H^*(A^\bullet)$ is "a $K(\pi,1)$", i. e., admits a quasi-isomorphism $\mathrm{Cob}^\bullet(C) \longrightarrow A^\bullet$ from the cobar construction of a conilpotent coalgebra C (see Theorem 4.3 and its proof). The conilpotent coalgebra C can be recovered as the degree-zero cohomology coalgebra of the bar construction of the DG-algebra $A^\bullet$, i. e., $C = H^0 \,\mathrm{Bar}^\bullet(A^\bullet)$.

Let $N_mC = \ker(C \to (C/k)^{\otimes m+1})$ denote the canonical increasing filtration on a conilpotent coalgebra C (see the definition in Section 1; cf. the proof of Proposition 3.2(b)). The filtration N is compatible with the comultiplication on C, so the associated graded vector space $\mathrm{gr}^N C = \bigoplus_m N_mC/N_{m-1}C$ is endowed with a natural coalgebra structure. The following theorem characterizes those DG-algebras with Koszul cohomology algebras that are not only quasi-formal but actually formal.

Theorem 5.2. *Let $A^\bullet$ be an augmented DG-algebra with a Koszul cohomology algebra $H^*(A^\bullet)$. Then the DG-algebra $A^\bullet$ is formal if and only if the conilpotent coalgebra $C = H^0\,\mathrm{Bar}(A^\bullet)$ is isomorphic to its associated graded coalgebra $\mathrm{gr}^N C$ with respect to the canonical increasing filtration N.*

Proof. By (the proof of) Theorem 4.3, the DG-coalgebra $\mathrm{Bar}(A^\bullet)$ is quasi-isomorphic to its degree-zero cohomology coalgebra C. The coalgebra C is conilpotent, and its cohomology algebra $\mathrm{Ext}^*_C(k,k) = H^*\,\mathrm{Cob}^\bullet(C)$, being isomorphic to the algebra $H^*(A^\bullet)$, is Koszul. By [32, Main Theorem 3.2], it follows that the graded coalgebra $\mathrm{gr}^N C$ is Koszul and quadratic dual to $H^*(A^\bullet)$. By the definition of a Koszul graded coalgebra, there is a natural quasi-isomorphism $\mathrm{Cob}^\bullet(\mathrm{gr}^N C) \longrightarrow H^*(A^\bullet)$.

Hence, whenever the coalgebras C and $\mathrm{gr}^N C$ are isomorphic, the DG-algebras $A^\bullet$ and $H^*(A^\bullet)$ are connected by a pair of quasi-isomorphisms $\mathrm{Cob}^\bullet(C) \longrightarrow A^\bullet$ and $\mathrm{Cob}^\bullet(C) \longrightarrow H^*(A^\bullet)$. Conversely, suppose that there is a chain of quasi-isomorphisms of DG-algebras connecting $A^\bullet$ with $H^*(A^\bullet)$. By Lemma 5.1, this can be assumed to be a chain of quasi-isomorphisms of augmented DG-algebras. Applying the bar construction, we obtain a chain of comultiplicative quasi-isomorphisms connecting the DG-coalgebras $\mathrm{Bar}^\bullet(A^\bullet)$ and $\mathrm{Bar}^\bullet(H^*(A^\bullet))$. It follows that the degree-zero cohomology coalgebras $H^0\,\mathrm{Bar}^\bullet(A^\bullet) = C$ and $H^0\,\mathrm{Bar}^\bullet(H^*(A^\bullet)) = \mathrm{gr}^N C$ of these two DG-coalgebras are isomorphic. $\square$

The following series of examples [36, Section 9.11] provides a negative answer to a question of Hopkins and Wickelgren [16, Question 1.4].

Example 5.3. Let l be a prime number and G be a profinite group; denote by $G^{(l)}$ the maximal quotient pro-l-group of G. Let k be a field of characteristic l; then the k-vector space $D = k(G)$ of locally constant k-valued functions on G is endowed with a natural structure of coalgebra over k with respect to the convolution comultiplication. We will call this coalgebra the *group coalgebra* of a profinite group G over a field k. The maximal conilpotent subcoalgebra $C = \operatorname{Nilp} D \subset D$ is naturally identified with the group coalgebra $k(G^{(l)})$ of the pro-l-group $G^{(l)}$. The cohomology map $H^*(G^{(l)}, k) \longrightarrow H^*(G, k)$ is known to be an isomorphism, at least, whenever either the cohomology algebra $H^*(G, k)$ is Koszul [33, Corollary 5.5], or $G = G_F$ is the absolute Galois group of a field F containing a primitive l-root of unity [42].

Let $l \neq p$ be two prime numbers and F be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ or the field of formal Laurent power series $\mathbb{F}_p((z))$ with coefficients in the prime field $\mathbb{F}_p$. Assume that the field F contains a primitive l-root of unity if l is odd, or a square root of -1 if $l = 2$. In other words, the cardinality q of the residue field $f = \mathcal{O}_F/\mathfrak{m}_F$ of the field F should be such that $q - 1$ is divisible by l if l is odd and by 4 if $l = 2$. Then the maximal quotient pro-l-group $G_F^{(l)}$ of the absolute Galois group G_F is isomorphic to the semidirect product of two copies of the group of l-adic integers $\mathbb{Z}_l$ with one of them acting in the other one by the multiplication with q.

So, in the exponential notation, the group $H = G_F^{(l)}$ is generated by two symbols s and t with the relation $sts^{-1} = t^q$, or, redenoting $s = 1 + x$ and $t = 1 + y$ and recalling that we are working over a field of characterictic l,

$$(7) \qquad \begin{aligned} &(1+x)(1+y)(1+x)^{-1}(1+y)^{-1} = (1+y^l)^{\frac{q-1}{l}} && \text{for } l \text{ odd, or} \\ &(1+x)(1+y)(1+x)^{-1}(1+y)^{-1} = (1+y^4)^{\frac{q-1}{4}} && \text{for } l = 2. \end{aligned}$$

This is a single nonhomogeneous quadratic relation of the type (2) defining the conilpotent group coalgebra $C = k(H)$. The quadratic principal part (3) of this relation is simply $xy - yx = 0$; this defines the associated graded coalgebra $\operatorname{gr}^N C$, which turns out to be the symmetric coalgebra in two variables.

Alternatively, one can easily compute the cohomology algebra $H^*(H, k) \simeq H^*(G_F, k)$ to be the exterior algebra in two generators of degree 1; then the graded coalgebra $\operatorname{gr}^N C$ is recovered as the quadratic dual. Either way, the coalgebra $\operatorname{gr}^N C$ is cocommutative and the coalgebra C is not (as the group H is not commutative), so C cannot be isomorphic to $\operatorname{gr}^N C$. Applying Theorem 5.2, we conclude that the cochain DG-algebra $\operatorname{Cob}^\bullet(C)$ of the pro-l-group H is not formal. The cochain DG-algebra $\operatorname{Cob}^\bullet(D) = \operatorname{Cob}^\bullet(k(G_F))$ of the absolute Galois group G_F, being quasi-isomorphic to the DG-algebra $\operatorname{Cob}^\bullet(C)$ via the natural quasi-isomorphism $\operatorname{Cob}^\bullet(C) \longrightarrow \operatorname{Cob}^\bullet(D)$ induced by the embedding of coalgebras $C \longrightarrow D$, is consequently not formal, either.

We are not aware of any example of a field F *containing all the l-power roots of unity* (that is, all the roots of unity of the powers l^n, $n \geqslant 1$) whose cochain DG-algebra $\operatorname{Cob}^\bullet(\mathbb{Z}/l(G_F))$ over the coefficient field $\mathbb{Z}/l$ is not formal. In particular, if would be interesting to know if there is a field F containing an algebraically closed subfield such that the DG-algebra $\operatorname{Cob}^\bullet(\mathbb{Z}/l(G_F))$ is not formal for some prime l. Our expectation is that such fields do exist, but we cannot pinpoint any.

The following family of examples of nonformal commutative DG-algebras over an arbitrary field is obtained by a slight modification of Example 5.3.

Example 5.4. Consider a single nonhomogeneous quadratic Lie relation of the type (2) for two variables x and y

$$[x,y] + q_3(x,y) + q_4(x,y) + q_5(x,y) + \cdots = 0, \tag{8}$$

where q_n are homogeneous Lie expressions of degree n in the variables x and y over a field k. The relation (8) can be viewed as defining a pronilpotent Lie algebra L, or its dual conilpotent Lie coalgebra, or its conilpotent coenveloping coalgebra C, or its dual topological associative algebra, which is simply the quotient algebra of the algebra of noncommutative formal Taylor power series in the two variables x, y by the closed ideal generated by the single power series (8).

The homogeneous part (3) of the relation (8) has the form $[x,y] = 0$, and the relation (8) is self-consistent, i. e., the associated graded coalgebra $\mathrm{gr}^N C$ is indeed the symmetric coalgebra in two variables defined by $xy - yx = 0$ (and not a smaller coalgebra). One can check this, e. g., by a trivial application of the Diamond Lemma [6] for noncommutative power series (the single relation (8) starts with xy, so there are no ambiguities to resolve). The graded coalgebra $\mathrm{gr}^N C$ is Koszul, so $H^*(C) \simeq H^*(\mathrm{gr}^N C)$ is the exterior algebra in two generators of degree 1.

Now setting $A^\bullet = (\bigwedge(L^*), d)$ to be the Chevalley–Eilenberg complex of the profinite-dimensional Lie algebra L (i. e., the inductive limit of the Chevalley-Eilenberg cohomological complexes of the finite-dimensional quotient Lie algebras of L by its open ideals), one obtains a commutative DG-algebra endowed with a natural quasi-isomorphism $\mathrm{Cob}^\bullet(C) \longrightarrow A^\bullet$ from the cobar construction of the coalgebra C. The cohomology algebra $H^*(A^\bullet)$ is the exterior algebra in two generators of degree 1, while the bar construction $\mathrm{Bar}^\bullet(A^\bullet)$ is quasi-isomorphic to C. So the commutative DG-algebra $A^\bullet$ cannot be connected with its cohomology algebra $H^*(A^\bullet)$ by a chain of quasi-isomorphisms, even in the class of noncommutative DG-algebras, unless the coalgebra C is isomorphic to $\mathrm{gr}^N C$. The latter would mean that C is cocommutative, which only happens when all the Lie forms $q_n(x,y)$ vanish for $n \geqslant 3$.

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