CATEGORICAL BOCKSTEIN SEQUENCES

LEONID POSITSELSKI

Abstract. We construct the reduction of an exact category with a twist functor with respect to an element of its graded center in presence of an exact-conservative forgetful functor annihilating this central element. The construction uses matrix factorizations in a nontraditional way. We obtain the Bockstein long exact sequences for the Ext groups in the exact categories produced by reduction. Our motivation comes from the theory of Artin–Tate motives and motivic sheaves with finite coefficients, and our key techniques generalize those of [4, Section 4].

Introduction

The goal of this paper is to develop a general categorical framework for the following problem. Let G be a finite group. For any commutative ring k, denote by $\mathcal{F}_k$ the category of representations of G in finitely generated free k-modules. The category $\mathcal{F}_k$ has a natural exact category structure in which a short sequence is exact if and only if it is exact as a sequence of modules over $k[G]$, or equivalently, split exact as a sequence of k-modules. Let $m = l^r$ be a prime power. How does one recover the exact category of modular representations $\mathcal{F}_{\mathbb{Z}/m}$ from the exact category $\mathcal{F}_{\mathbb{Z}_l}$ of representations of G over the l-adic integers?

Notice that the reduction functor $\rho\colon \mathcal{F}_{\mathbb{Z}_l} \longrightarrow \mathcal{F}_{\mathbb{Z}/m}$ taking a free $\mathbb{Z}_l$-module M with an action of G to the free $\mathbb{Z}/m$-module $\rho(M) = M/mM$ with the induced action of G is *not* surjective on the isomorphism classes of objects. E. g., for $m = l^2$ with an odd prime l and a cyclic group $G = \mathbb{Z}/l$, the representation of G in a free $\mathbb{Z}/l^2$-module of rank 1 corresponding to a nontrivial character $\mathbb{Z}/l \longrightarrow (\mathbb{Z}/l^2)^*$ cannot be lifted to a representation of G in a free $\mathbb{Z}_l$-module of rank 1. On the other hand, the regular representation of a finite group G over the residue ring $\mathbb{Z}/m$ can, of course, be lifted to a regular representation of G over the ring $\mathbb{Z}_l$.

Neither is the functor ρ surjective on morphisms. Instead, for any two objects M and $N \in \mathcal{F}_{Z_l}$ there is a natural *Bockstein long exact sequence*

$$\begin{aligned}
0 &\longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Ext}^2_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \mathrm{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \longrightarrow \cdots
\end{aligned}$$

Moreover, given two prime powers $m' = l^s$ and $m'' = l^t$ with $m = m'm''$, there is a Bockstein long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}/m''}}(\rho''(M),\rho''(N))$$

$$\longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m''}}(\rho''(M),\rho''(N))$$
$$\longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) \longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \cdots$$

for the reduction functors $\rho^{(i)}\colon \mathcal{F}_{\mathbb{Z}/m} \longrightarrow \mathcal{F}_{\mathbb{Z}/m^{(i)}}$, $i = 1, 2$, and any two objects M, $N \in \mathcal{F}_{\mathbb{Z}/m}$. We would like to have such long exact sequences coming out from our categorical formalism of reductions.

Acknowledgement. The author was partially supported by RFBR grants while working on this project.

1. The Bockstein Sequence

1.0. **Notation and terminology.** Throughout this paper, by an *exact category* we mean an exact category in Quillen's sense, i. e., an additive category endowed with a class of short exact sequences satisfying the natural axioms (see, e. g., [2, 3], [1], or [4, Appendix A]). A sequence of objects and morphisms in an exact category is said to be exact if it is composed of short exact sequences. A functor between exact categories is said to be exact if it takes short (or, equivalently, arbitrary) exact sequences in the source category to short (resp., long) exact sequences in the target one.

A *twist functor* on a category $\mathcal{F}$ is an autoequivalence denoted usually by $X \longmapsto X(1)$. The inverse autoequivalence is denoted by $X \longmapsto X(-1)$, and the integral powers of the twist functor are denoted by $X \longmapsto X(n)$, $n \in \mathbb{Z}$. Twist functors on exact categories will be presumed to be exact autoequivalences.

Given two categories $\mathcal{F}$ and $\mathcal{E}$ endowed with twist functors, a functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ is said to commute with the twists if a functorial isomorphism $\pi(X(1)) \simeq \pi(X)(1)$ is fixed for all objects $X \in \mathcal{F}$. Speaking of a commutative diagram of functors $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}$ commuting with the twists, we will always presume that the commutation isomorphisms form commutative diagrams of morphisms.

A morphism of endofunctors $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$, $n \in \mathbb{Z}$ on a category $\mathcal{F}$ with a twist functor $X \longmapsto X(1)$ (i. e., a morphism $\mathfrak{t}_X\colon X \longrightarrow X(n)$ defined for every object $X \in \mathcal{F}$ and functorial with respect to all the morphisms $X \longrightarrow Y$ in $\mathcal{F}$) is said to *commute with the twist* if for any object $X \in \mathcal{F}$ the equation $\mathfrak{t}_{X(1)} = \mathfrak{t}_X(1)$ holds in the set $\operatorname{Hom}_{\mathcal{F}}(X(1), X(n+1))$.

Notice that the endomorphisms of the identity functor on an additive category $\mathcal{F}$ always form a commutative ring, which is called the *center* of the category $\mathcal{F}$. It is the universal object among all the commutative rings k for which $\mathcal{F}$ can be endowed with the structure of a k-linear category. Similarly, given an additive category $\mathcal{F}$ with a twist functor $X \longmapsto X(1)$, morphisms of endofunctors $\mathrm{Id} \longrightarrow (n)$ commuting with the twist form a commutative ring with a $\mathbb{Z}$-grading, which can be called the *graded center* of an (additive) category with a twist functor.

We will say that a morphism $f\colon X \longrightarrow Y$ in $\mathcal{F}$ is *divisible by* a natural transformation $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$ commuting with the twist functor $X \longmapsto X(1)$ on $\mathcal{F}$ if the morphism f factorizes through the morphism $\mathfrak{t}_X\colon X \longrightarrow X(n)$, or equivalently, through the morphism $\mathfrak{t}_{Y(-n)}\colon Y(-n) \longrightarrow Y$. Similarly, a morphism $f\colon X \longrightarrow Y$ is *annihilated by* an element of the graded center $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$ of an additive category $\mathcal{F}$ if the composition $X(-n) \longrightarrow X \longrightarrow Y$ vanishes, or equivalently, the composition $X \longrightarrow Y \longrightarrow Y(n)$ vanishes in $\mathcal{F}$.

An exact functor between exact categories $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ is called *exact-conservative* if it reflects admissible monomorphisms, admissible epimorphisms, and exact sequences. In other words, a functor π is said to be exact-conservative if a morphism in $\mathcal{F}$ is an admissible monomorphism or admissible epimorphism, or a sequence in $\mathcal{F}$ is exact, if and only if so is its image with respect to the functor π in the exact category $\mathcal{E}$. Notice that any exact-conservative functor between exact categories is conservative in the conventional sense (i. e., reflects isomorphisms).

1.1. **Exact surjectivity conditions.** Let $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ be an exact functor between two exact categories. The following conditions on a functor η will play a key role in the constructions below in this section:

- (i$'$) for any object $X \in \mathcal{F}$ and any admissible epimorphism $T \longrightarrow \eta(X)$ in $\mathcal{G}$ there exists an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ and a morphism $\eta(Z) \longrightarrow T$ in $\mathcal{G}$ making the triangle diagram $\eta(Z) \longrightarrow T \longrightarrow \eta(X)$ commutative;
- (i$''$) for any object $X \in \mathcal{F}$ and any admissible monomorphism $\eta(X) \longrightarrow T$ in $\mathcal{G}$ there exists an admissible monomorphism $X \longrightarrow Z$ in $\mathcal{F}$ and a morphism $T \longrightarrow \eta(Z)$ in $\mathcal{G}$ making the triangle diagram $\eta(X) \longrightarrow T \longrightarrow \eta(Z)$ commutative;
- (ii$'$) for any objects $X, Y \in \mathcal{F}$ and any morphism $\eta(X) \longrightarrow \eta(Y)$ in $\mathcal{G}$ there exists an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Y$ in $\mathcal{F}$ making the triangle diagram $\eta(X') \longrightarrow \eta(X) \longrightarrow \eta(Y)$ commutative in $\mathcal{G}$;
- (ii$''$) for any objects X, $Y \in \mathcal{F}$ and any morphism $\eta(X) \longrightarrow \eta(Y)$ in $\mathcal{G}$ there exists an admissible monomorphism $Y \longrightarrow Y'$ and a morphism $X \longrightarrow Y'$ in $\mathcal{F}$ making the triangle diagram $\eta(X) \longrightarrow \eta(Y) \longrightarrow \eta(Y')$ commutative in $\mathcal{G}$.

We will say that an exact functor η satisfies the condition (i) if both the dual conditions (i$'$) and (i$''$) hold for it. Similarly, we will say that η satisfies the condition (ii) if both the dual conditions (ii$'$) and (ii$''$) hold for η.

The proofs of the two parts of the next proposition can be found in [4, Subsection 4.4] (for a discussion of big graded rings, see [4, Subsection A.1]). We denote by $\eta^n = \eta^n_{X,Y}\colon \mathrm{Ext}^n_{\mathcal{F}}(X,Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{G}}(\eta(X),\eta(Y))$ the Ext group homomorphisms induced by an exact functor $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$.

Proposition 1.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two exact categories and $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ be an exact functor satisfying the condition (i$'$). Then*

(a) *for any objects X, $Y \in \mathcal{F}$ and $W \in \mathcal{G}$, and any Ext classes $a \in \mathrm{Ext}^n_{\mathcal{F}}(X,Y)$ and $b \in \mathrm{Ext}^m_{\mathcal{G}}(\eta(Y),W)$ such that $b\eta^n(a) = 0$ and $m \geqslant 1$ there exists an object $Y' \in \mathcal{F}$, a morphism $f\colon Y' \longrightarrow Y$ in $\mathcal{F}$, and a class $a' \in \mathrm{Ext}^n_{\mathcal{F}}(X,Y')$ for which $a = fa'$ and $b\eta(f) = 0$;*

(b) *for any object $W \in \mathcal{G}$ the right graded module $(\mathrm{Ext}^n_{\mathcal{G}}(\eta(X),W))_{X\in\mathcal{F};\,n\geqslant 0}$ over the big graded ring $(\mathrm{Ext}^n_{\mathcal{F}}(X,Y))_{Y,X\in\mathcal{F};\,n\geqslant 0}$ over the set of all objects of $\mathcal{F}$ is induced* *from the right module* $(\mathrm{Hom}_{\mathcal G}(\eta(Y),W))_{Y\in\mathcal F}$ *over the big subring* $(\mathrm{Hom}_{\mathcal F}(X,Y))_{Y,X}\subset(\mathrm{Ext}^n_F(X,Y))_{Y,X;\,n}$. ☐

1.2. **Posing the problem.** Let $\mathcal F_{\mathfrak t}$, $\mathcal F_{\mathfrak s}$, and $\mathcal F_{\mathfrak{st}}$ be three exact categories endowed with twist functors (exact autoequivalences) $X\longmapsto X(1)$. Suppose that we are given an exact functor $\eta_{\mathfrak t}\colon\mathcal F_{\mathfrak{st}}\longrightarrow\mathcal F_{\mathfrak t}$ and an exact-conservative functor $\eta_{\mathfrak s}\colon\mathcal F_{\mathfrak{st}}\longrightarrow\mathcal F_{\mathfrak s}$ (see Subsection 1.0 for the definitions), both commuting with the twists. Assume that both the functors $\eta_{\mathfrak t}$ and $\eta_{\mathfrak s}$ satisfy the conditions (i-ii) of Subsection 1.1.

Furthermore, suppose that we are given a natural transformation $\mathfrak s\colon\mathrm{Id}\longrightarrow(1)$ on the category $\mathcal F_{\mathfrak{st}}$ commuting with the twist functor $(1)\colon\mathcal F_{\mathfrak{st}}\longrightarrow\mathcal F_{\mathfrak{st}}$ as explained in Subsection 1.0. We assume the following further conditions to be satisfied:

- (iii) a morphism $X\longrightarrow Y$ in the category $\mathcal F_{\mathfrak{st}}$ is annihilated by the functor $\eta_{\mathfrak t}$ if and only if it is annihilated by the natural transformation $\mathfrak s$ in $\mathcal F_{\mathfrak{st}}$;
- (iv) a morphism $X\longrightarrow Y$ in the category $\mathcal F_{\mathfrak{st}}$ is annihilated by the functor $\eta_{\mathfrak s}$ if and only if there exists an admissible epimorphism $X'\longrightarrow X$ such that the composition $X'\longrightarrow X\longrightarrow Y$ is divisible by $\mathfrak s$ in $\mathcal F_{\mathfrak{st}}$, or equivalently, if and only if there exists an admissible monomorphism $Y\longrightarrow Y'$ such that the composition $X\longrightarrow Y\longrightarrow Y'$ is divisible by $\mathfrak s$ in $\mathcal F_{\mathfrak{st}}$.

We will see below in Subsection 1.3 that the two dual formulations of the condition (iv) are equivalent modulo our previous assumptions (specifically, the argument is based on the condition (ii) for the functor $\eta_{\mathfrak t}$ and the condition (iii)).

Our goal in this section is to construct, in the assumption of the conditions (i-iv), the following Bockstein long exact sequence for the Ext groups

$$\begin{aligned}0&\longrightarrow\mathrm{Hom}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\mathrm{Hom}_{\mathcal F_{\mathfrak{st}}}(X,Y)\longrightarrow\mathrm{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))\\&\longrightarrow\mathrm{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\mathrm{Ext}^1_{\mathcal F_{\mathfrak{st}}}(X,Y)\longrightarrow\mathrm{Ext}^1_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))\\&\longrightarrow\mathrm{Ext}^2_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\mathrm{Ext}^2_{\mathcal F_{\mathfrak{st}}}(X,Y)\longrightarrow\cdots\end{aligned}$$

for any two objects X, $Y\in\mathcal F_{\mathfrak{st}}$. The differentials in this long exact sequence have the following properties:

- (a) the maps $\eta_{\mathfrak s}=\eta_{\mathfrak s}^n\colon\mathrm{Ext}^n_{\mathcal F_{\mathfrak{st}}}(X,Y)\longrightarrow\mathrm{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$ are induced by the exact functor $\eta_{\mathfrak s}\colon\mathcal F_{\mathfrak{st}}\longrightarrow\mathcal F_{\mathfrak s}$;
- (b) the maps $\mathfrak s=\mathfrak s_n\colon\mathrm{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\mathrm{Ext}^n_{\mathcal F_{\mathfrak{st}}}(X,Y)$ satisfy the equation
$$\mathfrak s_{i+n+j}(\eta_{\mathfrak t}^i(a(-1))z\eta_{\mathfrak t}^j(b))=a\mathfrak s_n(z)b$$
for any objects U, X, Y, $V\in\mathcal F_{\mathfrak{st}}$ and any Ext classes $b\in\mathrm{Ext}^j_{\mathcal F_{\mathfrak{st}}}(U,X)$, $z\in\mathrm{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$, and $a\in\mathrm{Ext}^i_{\mathcal F_{\mathfrak{st}}}(Y,V)$;
- (c) the maps $\partial=\partial^n\colon\mathrm{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))\longrightarrow\mathrm{Ext}^{n+1}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$ satisfy the equation
$$\partial^{i+n+j}(\eta_{\mathfrak s}^i(a)z\eta_{\mathfrak s}^j(b))=(-1)^i\eta_{\mathfrak t}^i(a(-1))\partial^n(z)\eta_{\mathfrak t}^j(b)$$
for any objects U, X, Y, $V\in\mathcal F_{\mathfrak{st}}$ and any Ext classes $b\in\mathrm{Ext}^j_{\mathcal F_{\mathfrak{st}}}(U,X)$, $z\in\mathrm{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$, and $a\in\mathrm{Ext}^i_{\mathcal F_{\mathfrak{st}}}(Y,V)$.

1.3. **The first two terms.** We start with constructing the map $\mathfrak{s}_0$ and verifying exactness of our sequence at its first two nontrivial terms.

Let X and Y be two objects of the category $\mathcal{F}_{\mathfrak{st}}$, and $p : \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ be a morphism in the category $\mathcal{F}_{\mathfrak{t}}$. According to the condition (ii) for the functor $\eta_{\mathfrak{t}}$, there exist an admissible epimorphism $X' \longrightarrow X$, an admissible monomorphism $Y(-1) \longrightarrow Y'(-1)$, and morphisms $X' \longrightarrow Y(-1)$ and $X \longrightarrow Y'(-1)$ in the category $\mathcal{F}_{\mathfrak{st}}$ whose images under the functor $\eta_{\mathfrak{t}}$ together with the morphism p form a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{t}}$.

The square diagram of morphisms $X' \longrightarrow X \longrightarrow Y'(-1)$, $X' \longrightarrow Y(-1) \longrightarrow Y'(-1)$ becomes commutative after applying the functor $\eta_{\mathfrak{t}}$, hence it follows from the condition (iii) that it is commutative modulo the ideal of morphisms annihilated by the natural transformation $\mathfrak{s}$ in the category $\mathcal{F}_{\mathfrak{st}}$. Multiplying both morphisms $X \longrightarrow Y'(-1)$ and $X' \longrightarrow Y(-1)$ by $\mathfrak{s}$, we therefore obtain a commutative square $X' \longrightarrow X \longrightarrow Y'$, $X' \longrightarrow Y \longrightarrow Y'$ in the category $\mathcal{F}_{\mathfrak{st}}$.

Since the morphism $X' \longrightarrow X$ is an admissible epimorphism and the morphism $Y \longrightarrow Y'$ is an admissible monomorphism, it follows that there exists a unique morphism $f\colon X \longrightarrow Y$ complementing the latter square to a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{st}}$. By the definition, we set $f = \mathfrak{s}_0(p)$. As the morphisms $X' \longrightarrow X$ and $Y \longrightarrow Y'$ can be chosen independently and the choice of either one of them is sufficient to determine the morphism f, it does not depend on these choices.

Lemma 1.2. *Assuming the condition (ii) for the functor $\eta_{\mathfrak{t}}$ and the condition (iii), the map $\mathfrak{s}_0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(X, Y)$ has the following properties:*

(a) *the equation $\mathfrak{s}_0(\eta_{\mathfrak{t}}(g(-1))p\eta_{\mathfrak{t}}(h)) = g\mathfrak{s}_0(p)h$ holds for any morphisms $h\colon U \longrightarrow X$, $g\colon Y \longrightarrow V$ in the category $\mathcal{F}_{\mathfrak{st}}$ and any morphism $p\colon \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$;*

(b) *the map $\mathfrak{s}_0$ is injective for any objects X, $Y \in \mathcal{F}_{\mathfrak{st}}$;*

(c) *a morphism $X \longrightarrow Y$ in the category $\mathcal{F}_{\mathfrak{st}}$ belongs to the image of the map $\mathfrak{s}_0$ if and only if there exists an admissible epimorphism $X' \longrightarrow X$ such that the composition $X' \longrightarrow X \longrightarrow Y$ is divisible by the natural transformation $\mathfrak{s}$ in the category $\mathcal{F}_{\mathfrak{st}}$, and if and only if there exists an admissible monomorphism $Y \longrightarrow Y'$ such that the composition $X \longrightarrow Y \longrightarrow Y'$ is divisible by $\mathfrak{s}$ in $\mathcal{F}_{\mathfrak{st}}$.*

Proof. In part (a), one checks the equations $\mathfrak{s}_0(\eta_{\mathfrak{t}}(g(-1))p) = g\mathfrak{s}_0(p)$ and $\mathfrak{s}_0(p\eta_{\mathfrak{t}}(h)) = \mathfrak{s}_0(p)h$ separately, using the construction of the morphism $\mathfrak{s}_0(p)$ in terms of an admissible epimorphism $X' \longrightarrow X$ in the former case and in terms of an admissible monomorphism $Y \longrightarrow Y'$ in the latter one. Part (b) holds, since the morphisms $X' \longrightarrow Y$ and $X \longrightarrow Y'$ are annihilated by the functor $\eta_{\mathfrak{t}}$ whenever they are annihilated by the multiplication with the natural transformation $\mathfrak{s}$ in the category $\mathcal{F}_{\mathfrak{st}}$, according to the condition (iii). The assertions "only if" in part (c) are obvious from the construction of the map $\mathfrak{s}_0$; and the prove the "if", denote by K the kernel of the morphism $X' \longrightarrow X$. Then the composition $K \longrightarrow X' \longrightarrow Y(-1)$ is annihilated by the multiplication with $\mathfrak{s}$, and consequently, according to (iii), it is also annihilated by the functor $\eta_{\mathfrak t}$. The short sequence $0 \longrightarrow \eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(X') \longrightarrow \eta_{\mathfrak t}(X) \longrightarrow 0$ being exact in $\mathcal F_{\mathfrak t}$, one obtains the desired morphism $\eta_{\mathfrak t}(X) \longrightarrow \eta_{\mathfrak t}(Y)(-1)$. $\square$

It follows that the two formulations of the condition (iv) are equivalent in Subsection 1.2. Assuming this condition, we conclude that the sequence $0 \longrightarrow \operatorname{Hom}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1)) \longrightarrow \operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y))$ is exact.

1.4. **The third term.** Now we construct the map ∂^0 and check exactness of the sequence at its third term. The construction and arguments largely follow those in [4, Subsections 4.5–4.6].

Let X and Y be two objects of the category $\mathcal F_{\mathfrak{st}}$, and $q\colon \eta_{\mathfrak s}(X) \longrightarrow \eta_{\mathfrak s}(Y)$ be a morphism in the category $\mathcal F_{\mathfrak s}$. According to the condition (ii) for the functor $\eta_{\mathfrak s}$, there exist an admissible epimorphism $X' \longrightarrow X$, an admissible monomorphism $Y \longrightarrow Y'$, and morphisms $X' \longrightarrow Y$ and $X \longrightarrow Y'$ in the category $\mathcal F_{\mathfrak{st}}$ whose images under the functor $\eta_{\mathfrak s}$ together with the morphism q form a commutative diagram of two triangles with a common edge in the category $\mathcal F_{\mathfrak s}$.

The square diagram of morphisms $X' \longrightarrow X \longrightarrow Y'$, $X' \longrightarrow Y \longrightarrow Y'$ becomes commutative after applying the functor $\eta_{\mathfrak s}$, and consequently, according to the condition (iv) and the above discussion, is commutative in the category $\mathcal F_{\mathfrak{st}}$ modulo the ideal of morphisms coming from morphisms in $\mathcal F_{\mathfrak t}$ via the maps $\mathfrak s_0$.

Let $K \longrightarrow X'$ be the kernel of the morphism $X' \longrightarrow X$ and $Y' \longrightarrow C$ be the cokernel of the morphism $Y \longrightarrow Y'$ in $\mathcal F_{\mathfrak{st}}$. Then the compositions $K \longrightarrow X' \longrightarrow Y$ and $X \longrightarrow Y' \longrightarrow C$ are annihilated by the functor $\eta_{\mathfrak s}$, and therefore come from morphisms $\eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(Y)(-1)$ and $\eta_{\mathfrak t}(X) \longrightarrow \eta_{\mathfrak t}(C)(-1)$ in the category $\mathcal F_{\mathfrak t}$. The difference of the two compositions in the square diagram of morphisms in the category $\mathcal F_{\mathfrak{st}}$ also comes from a certain morphism $\eta_{\mathfrak t}(X') \longrightarrow \eta_{\mathfrak t}(Y')(-1)$ in the category $\mathcal F_{\mathfrak t}$.

Together with the images of the short exact sequences $0 \longrightarrow K \longrightarrow X' \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y(-1) \longrightarrow Y'(-1) \longrightarrow C(-1) \longrightarrow 0$ with respect to the functor $\eta_{\mathfrak t}$, these three morphisms form a diagram of two squares, one of which is commutative and the other one anticommutative (as one can check using Lemma 1.2(a-b)). Such a diagram defines an element of the group $\operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1))$ in any one of the two dual ways differing by the minus sign.

Namely, the desired element can be obtained either as the composition of the Ext^1 class of the sequence $0 \longrightarrow \eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(X') \longrightarrow \eta_{\mathfrak t}(X) \longrightarrow 0$ with the morphism $\eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(Y)(-1)$, or as the composition of the morphism $\eta_{\mathfrak t}(X) \longrightarrow \eta_{\mathfrak t}(C)(-1)$ with the Ext^1 class of the sequence $0 \longrightarrow \eta_{\mathfrak t}(Y)(-1) \longrightarrow \eta_{\mathfrak t}(Y')(-1) \longrightarrow \eta_{\mathfrak t}(C)(-1) \longrightarrow 0$ in the exact category $\mathcal F_{\mathfrak t}$. By the definition, we set this element to be the value $\partial^0(q)$ of the map $\partial^0\colon \operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1))$ at the morphism $q\colon \eta_{\mathfrak s}(X) \longrightarrow \eta_{\mathfrak s}(Y)$.

Lemma 1.3. *Assuming the condition (ii) for the functors $\eta_{\mathfrak s}$, $\eta_{\mathfrak t}$ and the conditions (iii-iv), the map $\partial^0\colon \operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1))$ has the following properties:*

(a) *the equation* $\partial^0(\eta_{\mathfrak s}(g)q\eta_{\mathfrak s}(h)) = \eta_{\mathfrak t}(g(-1))\partial^0(q)\eta_{\mathfrak t}(h)$ *holds for any morphisms* $h\colon U\longrightarrow X$, $g\colon Y\longrightarrow V$ *in the category* $\mathcal F_{\mathfrak{st}}$ *and any morphism* $q\colon \eta_{\mathfrak s}(X)\longrightarrow\eta_{\mathfrak s}(Y)$ *in the category* $\mathcal F_{\mathfrak s}$*;*

(b) *for any two objects* X, Y *in the category* $\mathcal F_{\mathfrak{st}}$*, the kernel of the map* $\partial^0\colon \operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))\longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$ *coincides with the image of the map* $\eta_{\mathfrak s}\colon \operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(X,Y)\longrightarrow\operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$*.*

Proof. To prove part (a), one checks the equations $\partial^0(\eta_{\mathfrak s}(g)q) = \eta_{\mathfrak t}(g(-1))\partial^0(q)$ and $\partial^0(q\eta_{\mathfrak s}(h)) = \partial^0(q)\eta_{\mathfrak t}(h)$ separately, using the construction of the element $\partial^0(q)$ (as the product of a morphism and and an Ext^1 class in $\mathcal F_{\mathfrak t}$) in terms of an admissible epimorphism $X'\longrightarrow X$ in the former case and in terms of an admissible monomorphism $Y\longrightarrow Y'$ in the latter one, together with the result of Lemma 1.2(a).

To prove part (b), consider a morphism $q\colon\eta_{\mathfrak s}(X)\longrightarrow\eta_{\mathfrak s}(Y)$ in the category $\mathcal F_{\mathfrak s}$, and let $X'\longrightarrow X$ and $X'\longrightarrow Y$ be an admissible epimorphism and a morphism in the category $\mathcal F_{\mathfrak{st}}$ whose images under the functor $\eta_{\mathfrak s}$ form a commutative diagram together with the morphism q. Let $K\longrightarrow X'$ be the kernel of the morphism $X'\longrightarrow X$; according to the above, the composition $K\longrightarrow X'\longrightarrow Y$ comes from a morphism $\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(Y)(-1)$ in the category $\mathcal F_{\mathfrak t}$ via the map $\mathfrak s_0$.

The class $\partial^0(q)\in\operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$ is induced from the Ext^1 class of the short exact sequence $0\longrightarrow\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(X')\longrightarrow\eta_{\mathfrak t}(X)\longrightarrow 0$ using the morphism $\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(Y)(-1)$. Hence one has $\partial^0(q)=0$ if and ony if the latter morphism factorizes through the admissible monomorphism $\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(X')$.

Subtracting the image of the related morphism $\eta_{\mathfrak t}(X')\longrightarrow\eta_{\mathfrak t}(Y)(-1)$ under the map $\mathfrak s_0$ from the original morphism $X'\longrightarrow Y$ in the category $\mathcal F_{\mathfrak{st}}$, we obtain a new morphism $X'\longrightarrow Y$ with the same image under the functor $\eta_{\mathfrak s}$ and the additional property that the composition $K\longrightarrow X'\longrightarrow Y$ vanishes. This allows to lift the morphism $q\colon\eta_{\mathfrak s}(X)\longrightarrow\eta_{\mathfrak s}(Y)$ to a morphism $X\longrightarrow Y$ in the category $\mathcal F_{\mathfrak{st}}$. ☐

1.5. **Construction of higher differentials.** The constructions of the maps $\mathfrak s_n$ and ∂^n for $n\geqslant 1$ are based on the result of Proposition 1.1(b). We continue to follow [4, Subsection 4.5].

Lemma 1.4. *Assuming the conditions (i-iv), there exists a unique way to extend the above-defined maps* $\mathfrak s_0\colon\operatorname{Hom}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(X,Y)$ *to maps* $\mathfrak s_n\colon\operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(X,Y)$ *defined for all objects* X, $Y\in\mathcal F_{\mathfrak{st}}$ *and all integers* $n\geqslant 0$ *and satisfying the equations (b) of Subsection 1.2.*

Proof. Consider the two equations $\mathfrak s_{i+n}(\eta_{\mathfrak t}^i(a(-1))z) = a\mathfrak s_n(z)$ and $\mathfrak s_{n+j}(z\eta_{\mathfrak t}^j(b)) = \mathfrak s_n(z)b$ separately. In view of Proposition 1.1(b) and the dual result, based on the conditions (i′) and (i″) for the functor $\eta_{\mathfrak t}$, it follows from Lemma 1.2(a) that there exists a unique collection of maps $\mathfrak s_n''\colon\operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(X,Y)$ extending the maps $\mathfrak s_0$ and satisfying the former system of equations, and also a unique collection of maps $\mathfrak s_n'\colon\operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(X,Y)$ extending the maps $\mathfrak s_0$ and satisfying the latter system of equations.

It remains to show that $\mathfrak s_n'=\mathfrak s_n''$; here it suffices to check that $\mathfrak s_1'=\mathfrak s_1''$. Suppose that we are given two short exact sequences $0\longrightarrow V\longrightarrow P\longrightarrow X\longrightarrow 0$ and

$0 \longrightarrow Y \longrightarrow Q \longrightarrow U \longrightarrow 0$ in the category $\mathcal{F}_{\mathfrak{st}}$ representing the Ext^1 classes $b \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X,V)$ and $a \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(U,Y)$. Suppose further that we are given two morphisms $w\colon \eta_{\mathfrak{t}}(V) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ and $z\colon \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(U)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$ such that the equation $\eta^1_{\mathfrak{t}}(a(-1))z = w\eta^1_{\mathfrak{t}}(b)$ holds in the group $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$. Then the morphisms w and z can be extended to a morphism of short exact sequences (that is a diagram of two commutative squares) $(\eta_{\mathfrak{t}}(V) \to \eta_{\mathfrak{t}}(P) \to \eta_{\mathfrak{t}}(X)) \longrightarrow (\eta_{\mathfrak{t}}(Y)(-1) \to \eta_{\mathfrak{t}}(Q)(-1) \to \eta_{\mathfrak{t}}(U)(-1))$ in the category $\mathcal{F}_{\mathfrak{t}}$.

Applying the maps $\mathfrak{s}_0$ to the morphisms $w\colon \eta_{\mathfrak{t}}(V) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$, $\eta_{\mathfrak{t}}(P) \longrightarrow \eta_{\mathfrak{t}}(Q)(-1)$, and $z\colon \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(U)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$, we obtain, in view of Lemma 1.2(a), a morphism of short exact sequences $(V \to P \to X) \longrightarrow (Y \to Q \to U)$ in the category $\mathcal{F}_{\mathfrak{st}}$. The commutativity of this diagram of two squares proves the equation $a\mathfrak{s}_0(z) = \mathfrak{s}_0(w)b$ in the group $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X,Y)$. ☐

Lemma 1.5. *Assuming the conditions (i-iv), there exists a unique way to extend the above-defined maps* $\partial^0\colon \mathrm{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ *to maps* $\partial^n\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ *defined for all objects* X, $Y \in \mathcal{F}_{\mathfrak{st}}$ *and all integers* $n \geqslant 0$ *and satisfying the equations (c) of Subsection 1.2.*

Proof. As in the proof of Lemma 1.4, we consider the two equations $\partial^{i+n}(\eta^i_{\mathfrak{s}}(a)z) = (-1)^i\eta^i_{\mathfrak{t}}(a(-1))\partial^n(z)$ and $\partial^{n+j}(z\eta^j_{\mathfrak{s}}(b)) = \partial^n(z)\eta^j_{\mathfrak{t}}(b)$ separately. In view of Proposition 1.1(b) and the dual result, based on the conditions (i′) and (i″) for the functor $\eta_{\mathfrak{s}}$, it follows from Lemma 1.3(a) that there exists a unique collection of maps $''\partial^n\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ extending the maps ∂^0 and satisfying the former system of equations, and also a unique collection of maps $'\partial^n\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ extending the maps ∂^0 and satisfying the latter system of equations.

In order to show that $'\partial^n = {}''\partial^n$ for all $n \geqslant 1$, it suffices to check that $'\partial^1 = {}''\partial^1$. As in the previous proof, we have two short exact sequences $0 \longrightarrow V \longrightarrow P \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Q \longrightarrow U \longrightarrow 0$ in the category $\mathcal{F}_{\mathfrak{st}}$ representing the Ext^1 classes $b \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X,V)$ and $a \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(U,Y)$. We also have two morphisms $w\colon \eta_{\mathfrak{s}}(V) \longrightarrow \eta_{\mathfrak{s}}(Y)$ and $z\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(U)$ in the category $\mathcal{F}_{\mathfrak{s}}$ for which the equation $\eta^1_{\mathfrak{s}}(a)z = w\eta^1_{\mathfrak{s}}(b)$ holds in $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$. Then there is a morphism of short exact sequences $(\eta_{\mathfrak{s}}(V) \to \eta_{\mathfrak{s}}(P) \to \eta_{\mathfrak{s}}(X)) \longrightarrow (\eta_{\mathfrak{s}}(Y) \to \eta_{\mathfrak{s}}(Q) \to \eta_{\mathfrak{s}}(U))$ in the category $\mathcal{F}_{\mathfrak{s}}$.

According to the condition (ii) for the functor $\eta_{\mathfrak{s}}$, there exists an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow U$ in the category $\mathcal{F}_{\mathfrak{st}}$ whose images under the functor $\eta_{\mathfrak{s}}$ form a commutative diagram with the morphism $\eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(U)$ in the category $\mathcal{F}_{\mathfrak{s}}$. Denote by P''' the fibered product of the objects P and X' over X in the category $\mathcal{F}_{\mathfrak{st}}$. Choose an admissible epimorphism $P'' \longrightarrow P'''$ and a morphism $P'' \longrightarrow Q$ in the category $\mathcal{F}_{\mathfrak{st}}$ whose images under $\eta_{\mathfrak{s}}$ form a commutative diagram with the composition of morphisms $\eta_{\mathfrak{s}}(P''') \longrightarrow \eta_{\mathfrak{s}}(P) \longrightarrow \eta_{\mathfrak{s}}(Q)$ in $\mathcal{F}_{\mathfrak{s}}$.

Consider the difference of the compositions of morphisms $P'' \longrightarrow P''' \longrightarrow X \longrightarrow U$ and $P'' \longrightarrow Q \longrightarrow U$ in the category $\mathcal{F}_{\mathfrak{st}}$. It is annihilated by the functor $\eta_{\mathfrak{s}}$, and consequently, comes from a morphism $\eta_{\mathfrak{t}}(P'') \longrightarrow \eta_{\mathfrak{t}}(U)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$ via the map $\mathfrak{s}_0$. Denote by T the fibered product of the objects $\eta_{\mathfrak{t}}(P'')$ and $\eta_{\mathfrak{t}}(Q)(-1)$

over $\eta_\mathfrak{t}(U)(-1)$ in the category $\mathcal{F}_\mathfrak{t}$. The morphism $T \longrightarrow \eta_\mathfrak{t}(P'')$ is an admissible epimorphism in $\mathcal{F}_\mathfrak{t}$; hence, according to the condition (i) for the functor $\eta_\mathfrak{t}$, there exists an admissible epimorphism $P' \longrightarrow P''$ in the category $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$ and a morphism $\eta_\mathfrak{t}(P') \longrightarrow T$ in the category $\mathcal{F}_\mathfrak{t}$ making the triangle diagram $\eta_\mathfrak{t}(P') \longrightarrow T \longrightarrow \eta_\mathfrak{t}(P'')$ commutative in $\mathcal{F}_\mathfrak{t}$.

Applying the map $\mathfrak{s}_0$ to the composition of morphisms $\eta_\mathfrak{t}(P') \longrightarrow T \longrightarrow \eta_\mathfrak{t}(Q)(-1)$ in the category $\mathcal{F}_\mathfrak{t}$, we obtain a morphism $f\colon P' \longrightarrow Q$ entering into a commutative square of morphisms $P' \longrightarrow Q \longrightarrow U$ and $P' \longrightarrow P'' \longrightarrow U$ in the category $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$ with both morphisms $P'' \longrightarrow U$ and $f\colon P' \longrightarrow Q$ being annihilated by the functor $\eta_\mathfrak{s}$. Define the morphism $P' \longrightarrow P$ as the composition $P' \longrightarrow P'' \longrightarrow P''' \longrightarrow P$, the morphism $P' \longrightarrow X'$ as the composition $P' \longrightarrow P'' \longrightarrow P''' \longrightarrow X'$, and the new morphism $P' \longrightarrow Q$ as the sum of the composition $P' \longrightarrow P'' \longrightarrow Q$ and the morphism f. Then the square diagram formed by the morphisms $P' \longrightarrow X' \longrightarrow U$ and $P' \longrightarrow Q \longrightarrow U$ is commutative in the category $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$, while the triangle $\eta_\mathfrak{s}(P') \longrightarrow \eta_\mathfrak{s}(P) \longrightarrow \eta_\mathfrak{s}(Q)$ is commutative in the category $\mathcal{F}_\mathfrak{s}$.

Let $V' \longrightarrow P'$ be the kernel of the admissible epimorphism $P' \longrightarrow X'$ in the category $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$. Then there is an admissible epimorphism of short exact sequences $(V' \to P' \to X') \longrightarrow (V \to P \to X)$ and a morphism of short exact sequences $(V' \to P' \to X') \to (Y \to Q \to U)$ in $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$ whose images under the functor $\eta_\mathfrak{s}$ form a commutative triangle with the morphism of short exact sequences $(\eta_\mathfrak{s}(V) \to \eta_\mathfrak{s}(P) \to \eta_\mathfrak{s}(X)) \longrightarrow (\eta_\mathfrak{s}(Y) \to \eta_\mathfrak{s}(Q) \to \eta_\mathfrak{s}(U))$ in the category $\mathcal{F}_\mathfrak{s}$. Let $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ be the kernel of the admissible epimorphism $(V' \to P' \to X') \longrightarrow (V \to P \to X)$ (in the exact category) of short exact sequences in $\mathcal{F}_{\mathfrak{s}\mathfrak{t}}$. Then the composition of morphisms of short exact sequences $(K \to L \to M) \longrightarrow (V' \to P' \to X') \longrightarrow (Y \to Q \to U)$ is annihilated by the functor $\eta_\mathfrak{s}$, so, by Lemma 1.2(a-c) and the condition (iv), it comes from a (uniquely defined) morphism of short exact sequences $(\eta_\mathfrak{t}(K) \to \eta_\mathfrak{t}(L) \to \eta_\mathfrak{t}(M)) \longrightarrow (\eta_\mathfrak{t}(Y)(-1) \to \eta_\mathfrak{t}(Q)(-1) \to \eta_\mathfrak{t}(U)(-1))$ in the category $\mathcal{F}_\mathfrak{t}$ via the maps $\mathfrak{s}_0$.

Consider the extension of short exact sequences $0 \longrightarrow \eta_\mathfrak{t}(V) \longrightarrow \eta_\mathfrak{t}(P) \longrightarrow \eta_\mathfrak{t}(X) \longrightarrow 0$ and $0 \longrightarrow \eta_\mathfrak{t}(K) \longrightarrow \eta_\mathfrak{t}(L) \longrightarrow \eta_\mathfrak{t}(M) \longrightarrow 0$ with the middle term $0 \longrightarrow \eta_\mathfrak{t}(V') \longrightarrow \eta_\mathfrak{t}(P') \longrightarrow \eta_\mathfrak{t}(X') \longrightarrow 0$ in (the exact category of short exact sequences in) the category $\mathcal{F}_\mathfrak{t}$, and induce from it an extension of the exact sequences $0 \longrightarrow \eta_\mathfrak{t}(V) \longrightarrow \eta_\mathfrak{t}(P) \longrightarrow \eta_\mathfrak{t}(X) \longrightarrow 0$ and $0 \longrightarrow \eta_\mathfrak{t}(Y)(-1) \longrightarrow \eta_\mathfrak{t}(Q)(-1) \longrightarrow \eta_\mathfrak{t}(U)(-1) \longrightarrow 0$ using the above-constructed morphism of short exact sequences in $\mathcal{F}_\mathfrak{t}$. We have obtained a commutative 3×3 square formed by short exact sequences in the exact category $\mathcal{F}_\mathfrak{t}$. For any such square, the two Ext^2 classes between the objects at the opposite vertices obtained by composing the Ext^1 classes along the perimeter differ by the minus sign. This proves the desired equation $-a\partial^0(z) = \partial^0(w)b$ in the group $\operatorname{Ext}^2_{\mathcal{F}_\mathfrak{t}}(X, Y(-1))$ (cf. [4, Subsection 4.5]). ☐

1.6. **Exactness of the long sequence.** Here we largely follow [4, Subsection 4.6]. The argument is based on Proposition 1.1(a). We start with the following lemma, in whose proof the condition that the functor $\eta_\mathfrak{s}$ is exact-conservative (rather than only exact) will be used for the first time in this section.

Lemma 1.6. *The initial segment*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{st}}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$$
$$\longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$$

of the long sequence that we have constructed is exact for any two objects X and Y in the category $\mathcal{F}_{\mathfrak{st}}$.

Proof. Exactness at the first three nontrivial terms has been proven already in Subsections 1.3–1.4. Let us check exactness at the term $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$.

According to Proposition 1.1(b) and the condition (i) for the functor $\eta_{\mathfrak{t}}$, any element z in our Ext^1 group in the category $\mathcal{F}_{\mathfrak{t}}$ is equal to the product $p\eta^1_{\mathfrak{t}}(b)$ of the image $\eta^1_{\mathfrak{s}}(b)$ of an Ext^1 class b represented by a short exact sequence $0 \longrightarrow Y' \longrightarrow Z' \longrightarrow X \longrightarrow 0$ in the category $\mathcal{F}_{\mathfrak{st}}$ under the functor $\eta_{\mathfrak{t}}$ and a morphism $p\colon \eta_{\mathfrak{t}}(Y') \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$.

By the definition, the element $\mathfrak{s}_1(z) \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X, Y)$ is constructed as the composition $\mathfrak{s}_0(p)b$ of the Ext^1 class b and the morphism $\mathfrak{s}_0(p)\colon Y' \longrightarrow Y$ in the category $\mathcal{F}_{\mathfrak{st}}$. The equation $\mathfrak{s}_1(z) = \mathfrak{s}_0(p)b = 0$ means that the morphism $\mathfrak{s}_0(p)$ factorizes through the morphism $Y' \longrightarrow Z$ in the category $\mathcal{F}_{\mathfrak{st}}$, i. e., there exists a morphism $Z' \longrightarrow Y$ in $\mathcal{F}_{\mathfrak{st}}$ making the triangle $Y' \longrightarrow Z' \longrightarrow Y$ commutative.

Applying the functor $\eta_{\mathfrak{s}}$ to the whole diagram in the category $\mathcal{F}_{\mathfrak{st}}$, we see that the morphism $\eta_{\mathfrak{s}}(Y') \longrightarrow \eta_{\mathfrak{s}}(Y)$ vanishes, so the morphism $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(Y)$ factorizes through the admissible epimorphism $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(X)$ and there exists a morphism $q\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ in the category $\mathcal{F}_{\mathfrak{s}}$ making the triangle $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ commutative. By the definition, one has $\partial^0(q) = z$.

It remains to check exactness at the term $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X, Y)$. Suppose that the image of a short exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in the category $\mathcal{F}_{\mathfrak{st}}$ under the functor $\eta_{\mathfrak{s}}$ splits in the category $\mathcal{F}_{\mathfrak{s}}$. So there exists a splitting morphism $\eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Z)$. Then, according to the condition (ii) for the functor $\eta_{\mathfrak{s}}$, one can find an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Z$ in the category $\mathcal{F}_{\mathfrak{st}}$ making the triangle $\eta_{\mathfrak{s}}(X') \longrightarrow \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Z)$ commutative in $\mathcal{F}_{\mathfrak{s}}$.

Now, applying the functor $\eta_{\mathfrak{s}}$ to the composition $X' \longrightarrow Z \longrightarrow X$ and the admissible epimorphism $X' \longrightarrow X$, we get the same morphism in the category $\mathcal{F}_{\mathfrak{s}}$. Since the functor $\eta_{\mathfrak{s}}$ preserves and reflects admissible epimorphisms, it follows that the composition $X' \longrightarrow Z \longrightarrow X$ is an admissible epimorphism, too. So we can replace our original admissible epimorphism $X' \longrightarrow X$ with this composition and make the diagram commutative in the category $\mathcal{F}_{\mathfrak{st}}$.

Let $K \longrightarrow X'$ denote the kernel of the (new) admissible epimorphism $X' \longrightarrow X$. Then the composition $K \longrightarrow X' \longrightarrow Z$ is annihilated by the functor $\eta_{\mathfrak{s}}$ (since the morphism $\eta_{\mathfrak{s}}(X') \longrightarrow \eta_{\mathfrak{s}}(Z)$ factorizes through the admissible epimorphism $\eta_{\mathfrak{s}}(X') \longrightarrow \eta_{\mathfrak{s}}(X)$ by construction), and therefore, comes from a certain morphism $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Z)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$ via the map $\mathfrak{s}_0$. The morphism $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Z)(-1)$ is annihilated by the composition with the image of the admissible epimorphism $Z(-1) \longrightarrow X(-1)$ under the functor $\eta_{\mathfrak{t}}$, because the composition $K \longrightarrow X' \longrightarrow Z \longrightarrow X$ vanishes in the category $\mathcal{F}_{\mathfrak{st}}$ (see Lemma 1.2(a-b)), and therefore, factorizes through the image of the admissible monomorphism $Y(-1) \longrightarrow Z(-1)$ under the same functor $\eta_{\mathfrak t}$.

Consider the element in $\operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1))$ induced from the class of the short exact sequence $0 \longrightarrow \eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(X') \longrightarrow \eta_{\mathfrak t}(X) \longrightarrow 0$ using the morphism $\eta_{\mathfrak t}(K) \longrightarrow \eta_{\mathfrak t}(Y)(-1)$ that we just constucted. Commutativity of the diagram with five vertices Y, Z, X, X', K (a morphism of short exact sequences with a common third object X) in the category $\mathcal F_{\mathfrak s\mathfrak t}$ proves that the image of this Ext^1 class in the category $\mathcal F_{\mathfrak t}$ under the map $\mathfrak s_1$ is equal to the class of our original short exact sequence $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ in $\mathcal F_{\mathfrak s\mathfrak t}$. □

Corollary 1.7. *The whole long sequence of Ext groups from Subsection 1.2, as constructed in Subsection 1.5, is exact for any two objects X, Y of the category $\mathcal F_{\mathfrak s\mathfrak t}$.*

Proof. The assertion follows formally from the construction and Lemma 1.6 in view of Proposition 1.1(a) (applied to the functors $\eta_{\mathfrak s}$, $\eta_{\mathfrak t}$, and the identity functor $\mathrm{Id}_{\mathcal F_{\mathfrak s\mathfrak t}}$).

E. g., let us prove exactness at the terms $\operatorname{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y))$ for all $n \geqslant 1$. Let z be an element of our Ext group in the category $\mathcal F_{\mathfrak s}$. By Proposition 1.1(b) applied to the functor $\eta_{\mathfrak s}$, there exists an Ext class $b \in \operatorname{Ext}^n_{\mathcal F_{\mathfrak s\mathfrak t}}(X, X')$ in the category $\mathcal F_{\mathfrak s\mathfrak t}$ and a morphism $q\colon \eta_{\mathfrak s}(X') \longrightarrow \eta_{\mathfrak s}(Y)$ in the category $\mathcal F_{\mathfrak s}$ such that the element z is equal to the product $q\eta^n_{\mathfrak s}(b)$ in the group $\operatorname{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y))$. By the definition, one has $\partial^n(z) = \partial^0(q)\eta^n_{\mathfrak t}(b)$ in $\operatorname{Ext}^{n+1}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1))$.

Now assume that $\partial^0(q)\eta^n_{\mathfrak t}(b) = 0$. By Proposition 1.1(a) applied to the functor $\eta_{\mathfrak t}$, there exists a morphism $f\colon X'' \longrightarrow X'$ and an Ext class $b' \in \operatorname{Ext}^n_{\mathcal F_{\mathfrak s\mathfrak t}}(X, X'')$ in the category $\mathcal F_{\mathfrak s\mathfrak t}$ such that $b = fb'$ in $\operatorname{Ext}^n_{\mathcal F_{\mathfrak s\mathfrak t}}(X, X')$ and $\partial^0(q)\eta_{\mathfrak t}(f) = 0$ in $\operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X''), \eta_{\mathfrak t}(Y)(-1))$. By Lemma 1.5 (or, actually, even Lemma 1.3(a)) one has $\partial^0(q\eta_{\mathfrak s}(f)) = \partial^0(q)\eta_{\mathfrak t}(f) = 0$, and by Lemma 1.6 (or, actually, Lemma 1.3(b)), there exists a morphism $g\colon X'' \longrightarrow Y$ in the category $\mathcal F_{\mathfrak s\mathfrak t}$ such that $q\eta_{\mathfrak s}(f) = \eta_{\mathfrak s}(g)$ in the group $\operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X''), \eta_{\mathfrak s}(Y))$. Finally, we have $z = q\eta^n_{\mathfrak s}(b) = q\eta_{\mathfrak s}(f)\eta^n_{\mathfrak s}(b') = \eta_{\mathfrak s}(g)\eta^n_{\mathfrak s}(b') = \eta^n_{\mathfrak s}(gb')$ in $\operatorname{Ext}^n_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y))$. □

2. The Matrix Factorization Construction

2.1. **Posing the problem.** Let $\mathcal F$ be an exact category endowed with a twist functor (exact autoequivalence) $X \longmapsto X(1)$. Let $\mathfrak s$ be a morphism of endofunctors $\mathrm{Id} \longrightarrow (1)$ on the category $\mathcal F$ commuting with the twist functor $(1)\colon \mathcal F \longrightarrow \mathcal F$ (see Subsection 1.0 for the precise definitions and discussion).

Let $\mathcal E$ be another exact category endowed with a twist functor $(1)\colon \mathcal E \longrightarrow \mathcal E$. Suppose that we are given an exact functor $\pi\colon \mathcal F \longrightarrow \mathcal E$ commuting with the twists on $\mathcal F$ and $\mathcal E$, and that the following conditions are satisfied:

(i) the functor π is exact-conservative;
(ii) the functor π takes all the morphisms $\mathfrak s_X\colon X \longrightarrow X(1)$ in the category $\mathcal F$ to zero morphisms in the category $\mathcal E$;

(iii) any morphism in the category $\mathcal{F}$ annihilated by the functor π is divisible by the natural transformation $\mathfrak{s}$;

(iv) for any object $X \in \mathcal{F}$, the morphism $\mathfrak{s}_X : X \longrightarrow X(1)$ is injective and surjective; in other words, no nonzero morphism in the category $\mathcal{F}$ is annihilated by the natural transformation $\mathfrak{s}$.

The conditions (ii) and (iii) taken together can be restated by saying that a morphism in the category $\mathcal{F}$ is annihilated by the functor π if and only if it is divisible by the natural transformation $\mathfrak{s}$. The conditions (iii) and (iv) taken together can be reformulated by saying that any morphism in the category $\mathcal{F}$ annihilated by the functor π is uniquely divisible by the natural transformation $\mathfrak{s}$.

Our goal in this section is to construct an exact category $\mathcal{G}$ which we will call the *reduction of exact category $\mathcal{F}$ by the natural transformation $\mathfrak{s}$ taken on the background of the functor π*. The category $\mathcal{G}$ comes endowed with exact-conservative functors $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ and $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ whose composition $\epsilon\gamma$ is identified with π. The functor γ annihilates all the morphisms $\mathfrak{s}_X$, while the functor ϵ reflects zero morphisms (i. e., it is faithful). The category $\mathcal{G}$ is also endowed with a twist functor $(1)\colon \mathcal{G} \longrightarrow \mathcal{G}$, and the functors γ and ϵ commute with the twists.

The Ext groups computed in the categories $\mathcal{F}$ and $\mathcal{G}$ are related by the following Bockstein long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{\mathcal{F}}(X, Y(-1)) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{G}}(\gamma(X), \gamma(Y)) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal{F}}(X, Y(-1)) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}}(\gamma(X), \gamma(Y)) \\ &\longrightarrow \mathrm{Ext}^2_{\mathcal{F}}(X, Y(-1)) \longrightarrow \mathrm{Ext}^2_{\mathcal{F}}(X, Y) \longrightarrow \mathrm{Ext}^2_{\mathcal{G}}(\gamma(X), \gamma(Y)) \longrightarrow \cdots \end{aligned}$$

for any two objects X, $Y \in \mathcal{F}$ (cf. [4, Subsection 4.1]).

Here the map $\mathfrak{s}_n\colon \mathrm{Ext}^n_{\mathcal{F}}(X, Y(-1)) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}}(X, Y)$ is provided by the composition with the morphism $\mathfrak{s}_{Y(-1)}\colon Y(-1) \longrightarrow Y$ (or, equivalently, the twist by (1) and the composition with the morphism $\mathfrak{s}_X\colon X \longrightarrow X(1)$) in the category $\mathcal{F}$. The map $\gamma^n\colon \mathrm{Ext}^n_{\mathcal{F}}(X, Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{G}}(\gamma(X), \gamma(Y))$ is induced by the exact functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$. Finally, the boundary map $\partial^n\colon \mathrm{Ext}^n_{\mathcal{G}}(\gamma(X), \gamma(Y)) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}}(X, Y(-1))$ is defined by the construction of Subsections 1.4–1.5 and satisfies the equation

$$\partial^{i+n+j}(\gamma^i(a) z \gamma^j(b)) = (-1)^i a(-1) \partial^n(z) b$$

for any objects U, X, Y, $V \in \mathcal{F}$ and any Ext classes $b \in \mathrm{Ext}^j_{\mathcal{F}}(U, X)$, $z \in \mathrm{Ext}^n_{\mathcal{G}}(\gamma(X), \gamma(Y))$, and $a \in \mathrm{Ext}^i_{\mathcal{F}}(Y, V)$.

2.2. **Matrix factorizations.** In order to construct the desired exact category $\mathcal{G}$, consider first the following category $\widetilde{\mathcal{H}}$. The objects of $\widetilde{\mathcal{H}}$ are the diagrams (U, V) of the form

$$V(-1) \longrightarrow U \longrightarrow V \longrightarrow U(1)$$

in the category $\mathcal{F}$, where the morphism $V \longrightarrow U(-1)$ is obtained from the morphism $V(-1) \longrightarrow U$ by applying the twist functor (1), while the two compositions $V(-1) \longrightarrow U \longrightarrow V$ and $U \longrightarrow V \longrightarrow U(1)$ are equal to the maps $\mathfrak{s}_{V(-1)}$ and $\mathfrak{s}_U$, respectively. Morphisms $(U', V') \longrightarrow (U'', V'')$ in the category $\mathcal{H}$ are the pairs of morphisms $U' \longrightarrow U''$ and $V' \longrightarrow V''$ in $\mathcal{F}$ making the whole diagram $(V'(-1) \to U' \to V' \to U'(1)) \longrightarrow (V''(-1) \to U'' \to V'' \to U''(1))$ commutative.

Furthermore, consider the following full subcategory $\mathcal{H} \subset \widetilde{\mathcal{H}}$. By the definition, a diagram $(U, V) \in \widetilde{\mathcal{H}}$ belongs to the category $\mathcal{H}$ if the functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ (which, as we recall, takes the morphisms $\mathfrak{s}_{V(-1)}$ and $\mathfrak{s}_U$ in $\mathcal{F}$ to zero morphisms in $\mathcal{E}$) transforms it into an exact sequence $\pi(V(-1)) \longrightarrow \pi(U) \longrightarrow \pi(V) \longrightarrow \pi(U(1))$ in the exact category $\mathcal{E}$. The functor $\Delta\colon \mathcal{H} \longrightarrow \mathcal{E}$ assigns to a diagram $(U, V) \in \mathcal{H}$ the image of the morphism $\pi(U) \longrightarrow \pi(V)$ in $\mathcal{E}$ (which is well-defined due to the exactness condition imposed on the objects of $\mathcal{H}$).

The category $\widetilde{\mathcal{H}}$ has a natural exact category structure in which a short sequence of diagrams is exact if it is exact in $\mathcal{F}$ at every term of the diagrams. The full subcategory $\mathcal{H} \subset \widetilde{\mathcal{H}}$ is closed under the operations of passage to the cokernels of admissible monomorphisms, the kernels of admissible epimorphisms, and extensions; so in particular it inherits the induced exact category structure. The functor $\Delta\colon \mathcal{H} \longrightarrow \mathcal{E}$ is an exact functor between exact categories.

Let $\mathcal{I} \subset \mathcal{H}$ denote the ideal of morphisms in $\mathcal{H}$ annihilated by Δ. Consider the quotient category $\mathcal{H}/\mathcal{I}$ of the category $\mathcal{H}$ by this ideal of morphisms, and let $\mathcal{S} \subset \mathcal{H}/\mathcal{I}$ denote the multiplicative class of morphisms which the functor $\Delta\colon \mathcal{H}/I \longrightarrow \mathcal{E}$ transforms to isomorphisms in $\mathcal{E}$.

Lemma 2.1. *Assuming the conditions (i-ii) of Subsection 2.1, the class of morphisms $\mathcal{S}$ is localizing in the category $\mathcal{H}/\mathcal{I}$ (i. e., it satisfies the left and right Ore conditions).*

Proof. The argument follows that in [4, Subsection 4.2]. It is clear from the definitions of the classes $\mathcal{I}$ and $\mathcal{S}$ that if any two morphisms $X \rightrightarrows Y$ in $\mathcal{H}/\mathcal{I}$ have equal compositions with a morphism $X' \longrightarrow X$ or $Y \longrightarrow Y'$ belonging to $\mathcal{S}$, then these two morphisms $X \rightrightarrows Y$ are equal to each other in $\mathcal{H}/\mathcal{I}$.

Let $(S, T) \longrightarrow (K, L) \longleftarrow (U, V)$ be a pair of morphisms in $\mathcal{H}$ such that $\Delta((U, V) \longrightarrow (K, L))$ is an admissible epimorphism in $\mathcal{E}$. Then the morphism $U \oplus L(-1) \longrightarrow K$ is an admissible epimorphism in $\mathcal{F}$ (since so is its image under π). Consider the fibered product $P = S \sqcap_K (U \oplus L(-1))$ in $\mathcal{F}$, and set $Q = T \oplus V \in \mathcal{F}$.

Let the map $S \sqcap_K (U \oplus L(-1)) \longrightarrow T \oplus V$ be defined as the composition $S \sqcap_K (U \oplus L(-1)) \longrightarrow S \oplus U \longrightarrow T \oplus V$ and the map $T(-1) \oplus V(-1) \longrightarrow S \sqcap_K (U \oplus L(-1))$ be induced by the maps $T(-1) \longrightarrow S$, $T(-1) \longrightarrow L(-1)$, $V(-1) \longrightarrow U$, and minus the map $V(-1) \longrightarrow L(-1)$. Then the diagram

$$T(-1) \oplus V(-1) \longrightarrow S \sqcap_K (U \oplus L(-1)) \longrightarrow T \oplus V \longrightarrow S(1) \sqcap_{K(1)} (U(1) \oplus L)$$

is an object (P, Q) of the category $\mathcal{H}$. Indeed, one easily checks that the diagram (P, Q) belongs to $\widetilde{\mathcal{H}}$; and to prove the exactness condition, it suffices to notice that (P, Q) is the kernel of an admissible epimorphism $(S, T) \oplus (U, V) \oplus (L(-1), L) \longrightarrow (K, L)$ between two objects of $\mathcal{H}$ in $\widetilde{\mathcal{H}}$.

There are natural morphisms $(S, T) \longleftarrow (P, Q) \longrightarrow (U, V)$ in the category $\mathcal{H}$; the square diagram formed by these two morphisms and the morphisms $(S, T) \longrightarrow (K, L) \longleftarrow (U, V)$ is commutative modulo $\mathcal{I}$. The object $\Delta(P, Q)$ is the fibered product of $\Delta(S,T)$ and $\Delta(U,V)$ over $\Delta(K,L)$. In particular, if the morphism $(U,V) \longrightarrow (K,L)$ belongs to $\mathcal{S}$, then so does the morphism $(P,Q) \longrightarrow (S,T)$. This proves a half of the Ore conditions, and the dual half can be proven in the dual way. $\square$

2.3. **Exact category structure.** We define the category $\mathcal{G}$ as the localization $(\mathcal{H}/\mathcal{I})[\mathcal{S}^{-1}]$. By Lemma 2.1, $\mathcal{G}$ is an additive category and the localization $\mathcal{H} \longrightarrow \mathcal{G}$ is an additive functor. The twist functor $(1)\colon \mathcal{G} \longrightarrow \mathcal{G}$ is induced by the obvious twist functor $(U,V) \longmapsto (U(1),V(1))$ on the category $\mathcal{H}$. The functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ assigns to an object $X \in \mathcal{F}$ the diagram (X,X) with the identity morphism $X \longrightarrow X$ in the middle. The functor $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ is induced by the functor Δ.

Lemma 2.2. *In the assumption of the conditions (i-iv) from Subsection 2.1, the rule according to which a short sequence in the category $\mathcal{G}$ is said to be exact if its image under the functor ϵ is exact in $\mathcal{E}$ defines an exact category structure on $\mathcal{G}$. Moreover, a morphism is an admissible monomorphism (resp., admissible epimorphism) in $\mathcal{G}$ if and only if its image under ϵ is an admissible monomorphism (resp., admissible epimorphism) in $\mathcal{E}$.*

Proof. We follow the argument in [4, Subsection 4.3]. Consider a morphism f in $\mathcal{G}$ such that $\epsilon(f)$ is an admissible epimorphism in $\mathcal{E}$. Then, clearly, f is a surjective morphism in $\mathcal{G}$. Represent f by a morphism $(U,V) \longrightarrow (K,L)$ in $\mathcal{H}$ and apply the construction from the proof of Lemma 2.1 to the pair of morphisms $(0,0) \longrightarrow (K,L) \longleftarrow (U,V)$. We obtain a morphism $(P,Q) = (\ker(U \oplus L(-1) \to K),\ V) \longrightarrow (U,V)$ in $\mathcal{H}$ whose image g in $\mathcal{G}$ completes the morphism f to a short sequence $0 \longrightarrow (P,Q) \longrightarrow (U,V) \longrightarrow (K,L) \longrightarrow 0$ that is exact in $\mathcal{G}$ (in the sense of our definition; i. e., its image under ϵ is exact in $\mathcal{E}$).

Let us check that the morphism g is the kernel of f in $\mathcal{G}$. Any morphism with the target (U,V) in $\mathcal{G}$ can be represented by a morphism $(X,Y) \longrightarrow (U,V)$ in $\mathcal{H}$. Assume that the composition $(X,Y) \longrightarrow (U,V) \longrightarrow (K,L)$ is annihilated by Δ. Then the composition $X \longrightarrow K \longrightarrow L$ is annihilated by π, so the morphism $X \longrightarrow K$ factorizes through the morphism $L(-1) \longrightarrow K$ by the conditions (iii-iv). This allows to lift the morphism $(X,Y) \longrightarrow (U,V)$ to a morphism $(X,Y) \longrightarrow (P,Q)$ in $\mathcal{H}$.

The morphism g being injective in $\mathcal{G}$, the above lifting is unique as a morphism in $\mathcal{G}$. Furthermore, the short sequence $0 \longrightarrow \Delta(P,Q) \longrightarrow \Delta(U,V) \longrightarrow \Delta(K,L) \longrightarrow 0$ is exact in $\mathcal{E}$ according to the proof of Lemma 2.1. Recalling that the morphisms taken to isomorphisms in $\mathcal{E}$ by the functor Δ have been inverted in $\mathcal{G}$, one concludes that any morphism $(S,T) \longrightarrow (U,V)$ with the latter property is a cokernel of f in $\mathcal{G}$.

This suffices to check the axioms Ex0–Ex1 and Ext3 from [4, Subsection A.3] for the category $\mathcal{G}$; it remains to prove Ex2. Suppose that we are given a short exact sequence in $\mathcal{G}$; it can be represented by a sequence of morphisms $(S,T) \longrightarrow (U,V) \longrightarrow (K,L)$ in $\mathcal{H}$. Any morphism with the target (K,L) in $\mathcal{G}$ can be represented by a morphism $(X,Y) \longrightarrow (K,L)$ in $\mathcal{H}$. Applying the construction from the proof of Lemma 2.1 again, we obtain an object $(M,N) = (X \sqcap_K (U \oplus L(-1)),\ Y \oplus V)$ in $\mathcal{H}$ together with a pair of morphisms $(U,V) \longleftarrow (M,N) \longrightarrow (X,Y)$.

Setting, as above, $(P, Q) = (\ker(U \oplus L(-1) \to K),\ V)$, we have the above-constructed morphism $(S, T) \longrightarrow (P, Q)$ in $\mathcal{H}$, whose composition with the natural admissible monomorphism $(P, Q) \longrightarrow (M, N)$ provides a morphism $(S, T) \longrightarrow (M, N)$ in $\mathcal{H}$. The triangle $(S, T) \longrightarrow (M, N) \longrightarrow (U, V)$ is commutative already in $\mathcal{H}$, and the short sequence $0 \longrightarrow (S, T) \longrightarrow (M, N) \longrightarrow (X, Y) \longrightarrow 0$ is exact in $\mathcal{G}$, as so is its image under ϵ in $\mathcal{E}$. The exact category axioms are verified. $\square$

It follows immediately from the above description of the exact category structure on $\mathcal{G}$ that the functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ is exact and exact-conservative (since both the functors $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ and $\pi = \epsilon\gamma\colon \mathcal{F} \longrightarrow \mathcal{E}$ are). The functor $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ is faithful, because the functor $\Delta\colon \mathcal{H}/\mathcal{I} \longrightarrow \mathcal{E}$ is faithful by the definition of $\mathcal{I}$. It is also clear that the functors ϵ and γ commute with the twists. The more advanced properties of our reduction construction are discussed below in this section.

2.4. **Properties of the reduction functor.** The following lemma shows that the reduction functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ satisfies the conditions (i-ii) from Subsection 1.1, and in fact, certain even stronger conditions.

Lemma 2.3. *The exact functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ constructed above has the following "exact surjectivity" properties:*

- (a′) *for any object $X \in \mathcal{F}$ and any admissible epimorphism $T \longrightarrow \gamma(X)$ in $\mathcal{G}$ there exists an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ and a morphism $\gamma(Z) \longrightarrow T$ in $\mathcal{G}$ making the triangle diagram $\gamma(Z) \longrightarrow T \longrightarrow \gamma(X)$ commutative;*
- (b′) *for any object $T \in \mathcal{G}$ there exists an object $U \in \mathcal{F}$ and an admissible epimorphism $\gamma(U) \longrightarrow T$ in $\mathcal{G}$;*
- (c′) *for any objects X, $Y \in \mathcal{F}$ and any morphism $\gamma(X) \longrightarrow \gamma(Y)$ in $\mathcal{G}$ there exists an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Y$ in $\mathcal{F}$ making the triangle diagram $\gamma(X') \longrightarrow \gamma(X) \longrightarrow \gamma(Y)$ commutative in $\mathcal{G}$;*
- (d′) *for any object $X \in \mathcal{F}$ and any morphism $\gamma(X) \longrightarrow T$ in $\mathcal{G}$ there exists a morphism $X \longrightarrow Z$ in $\mathcal{F}$ and an admissible epimorphism $\gamma(Z) \longrightarrow T$ in $\mathcal{G}$ such that the triangle diagram $\gamma(X) \longrightarrow \gamma(Z) \longrightarrow T$ is commutative;*

as well as the properties (a″-d″) dual to (a′-d′).

Notice that it follows from the properties (b′-c′) that the morphism $\gamma(Z) \longrightarrow T$ in (a′) can be chosen to be an admissible epimorphism, too. Moreover, for an exact-conservative functor γ, the property (a′) can be entirely deduced from (b′-c′).

Indeed, let $T \longrightarrow \gamma(X)$ be an admissible epimorphism in $\mathcal{G}$. According to (b′), there exists an object $U \in \mathcal{F}$ together with an admissible epimorphism $\gamma(U) \longrightarrow T$ in $\mathcal{G}$. The composition $\gamma(U) \longrightarrow T \longrightarrow \gamma(X)$ is a morphism in $\mathcal{G}$ between two objects coming from $\mathcal{F}$. According to (c′), there exists an admissible epimorphism $U' \longrightarrow U$ and a morphism $U' \longrightarrow X$ in $\mathcal{F}$ making the triangle diagram $\gamma(U') \longrightarrow \gamma(U) \longrightarrow \gamma(X)$ commutative in $\mathcal{G}$.

Now if the functor γ is exact-conservative, then the morphism $U' \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$, because its image in $\mathcal{G}$ is equal to the composition of admissible epimorphisms $\gamma(U') \longrightarrow \gamma(U) \longrightarrow T \longrightarrow X$. In the general case, pick an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ whose image in $\mathcal{G}$ factorizes through the admissible epimorphism $T \longrightarrow \gamma(X)$. Then $U' \oplus Z \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$ whose image in $\mathcal{G}$ is the composition of two admissible epimorphisms $\gamma(U') \oplus \gamma(Z) \longrightarrow T \longrightarrow \gamma(X)$.

Proof of Lemma 2.3. Part (b$'$): let the object $T \in \mathcal{G}$ be represented by a diagram $(U, V) = (V(-1) \to U \to V \to U(1)$ in $\mathcal{H}$; then there is a natural admissible epimorphism $\gamma(U) \longrightarrow T$ in $\mathcal{G}$. (Indeed, $\pi(U) \longrightarrow \Delta(U, V)$ is an admissible epimorphism in $\mathcal{E}$.) Part (c$'$): let the morphism $\gamma(X) \longrightarrow \gamma(Y)$ be represented by a fraction $(X, X) \longleftarrow (U, V) \longrightarrow (Y, Y)$ of two morphisms in $\mathcal{H}$, where the morphism $(U, V) \longrightarrow (X, X)$ belongs to $\mathcal{S}$ (modulo $\mathcal{I}$). Then there is a natural admissible epimorphism $U \longrightarrow X$ and a natural morphism $U \longrightarrow Y$ in $\mathcal{F}$ making the diagram $\gamma(U) \longrightarrow \gamma(X) \longrightarrow \gamma(Y)$ commutative in $\mathcal{G}$ by the definition. (Indeed, the morphism $\pi(U) \longrightarrow \Delta(U, V) \simeq \Delta(X, X) = \pi(X)$ is an admissible epimorphism in $\mathcal{E}$.)

Part (d$'$): one can represent the morphism $\gamma(X) \longrightarrow T$ in $\mathcal{G}$ by a morphism of diagrams $(X, X) \longrightarrow (U, V)$ in $\mathcal{H}$. Then there is a morphism $X \longrightarrow U$ in $\mathcal{F}$ and an admissible epimorphism $\gamma(U) \longrightarrow (U, V)$ in $\mathcal{G}$, while the triangle $(X, X) \longrightarrow (U, U) \longrightarrow (U, V)$ is commutative already in $\mathcal{H}$. Part (a$'$) follows from (b$'$) and (c$'$) according to the above comments; to prove it directly, represent the morphism $T \longrightarrow \gamma(X)$ in $\mathcal{G}$ by a morphism of diagrams $(U, V) \longrightarrow (X, X)$ in $\mathcal{H}$. Then $U \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$ (since $\pi(U) \longrightarrow \Delta(U, V) \longrightarrow \pi(X)$ is a composition of admissible epimorphisms in $\mathcal{E}$) and $(U, U) \longrightarrow (U, V)$ is an admissible epimorphism in $\mathcal{G}$, while the triangle $(U, U) \longrightarrow (U, V) \longrightarrow (X, X)$ is commutative in $\mathcal{H}$. $\square$

2.5. **The first Bockstein sequence.**

References

[1] T. Bühler. Exact categories. *Expositiones Math.* **28**, #1, p. 1–69, 2010. `arXiv:0811.1480 [math.HO]`

[2] B. Keller. Exact categories. Appendix A to the paper: Chain complexes and stable categories. *Manuscripta Math.* **67**, #4, p. 379–417, 1990.

[3] B. Keller. On Gabriel–Roiter's axioms for exact categories. Appendix to the paper: P. Dräxler, I. Reiten, S. O. Smalø, Ø. Solberg. Exact categories and vector space categories. *Trans. Amer. Math. Soc.* **351**, #2, p. 647–692, 1999.

[4] L. Positselski. Mixed Artin–Tate motives with finite coefficients. *Moscow Math. Journal* **11**, #2, p. 317–402, 2011. `arXiv:1006.4343 [math.KT]`

Faculty of Mathematics and Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312; and

Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127994, Russia

E-mail address: `posic@mccme.ru`