CATEGORICAL BOCKSTEIN SEQUENCES

LEONID POSITSELSKI

Abstract. We construct the reduction of an exact category with a twist functor with respect to an element of its graded center in presence of an exact-conservative forgetful functor annihilating this central element. The construction uses matrix factorizations in a nontraditional way. We obtain the Bockstein long exact sequences for the Ext groups in the exact categories produced by reduction. Our motivation comes from the theory of Artin–Tate motives and motivic sheaves with finite coefficients, and our key techniques generalize those of [12, Section 4].

Contents

Introduction

The goal of this paper is to develop a general categorical framework for the following problem. Let G be a finite group. For any commutative ring k, denote by $\mathcal{F}_k$ the category of representations of G in finitely generated free k-modules. The category $\mathcal{F}_k$ has a natural exact category structure in which a short sequence is exact if and only if it is exact as a sequence of modules over $k[G]$, or equivalently, split exact as a sequence of k-modules. Let $m = l^r$ be a prime power. How does one recover the exact category of modular representations $\mathcal{F}_{\mathbb{Z}/m}$ from the exact category $\mathcal{F}_{\mathbb{Z}_l}$ of representations of G over the l-adic integers?

Notice that the reduction functor $\rho\colon \mathcal{F}_{\mathbb{Z}_l} \longrightarrow \mathcal{F}_{\mathbb{Z}/m}$ taking a free $\mathbb{Z}_l$-module M with an action of G to the free $\mathbb{Z}/m$-module $\rho(M) = M/mM$ with the induced action of G is *not* surjective on the isomorphism classes of objects. E. g., for $m = l^2$ with an odd prime l and a cyclic group $G = \mathbb{Z}/l$, the representation of G in a free $\mathbb{Z}/l^2$-module of rank 1 corresponding to a nontrivial character $\mathbb{Z}/l \longrightarrow (\mathbb{Z}/l^2)^*$ cannot be lifted to a representation of G in a free $\mathbb{Z}_l$-module of rank 1. On the other hand, the regular representation of a finite group G over the residue ring $\mathbb{Z}/m$ can, of course, be lifted to a regular representation of G over the ring $\mathbb{Z}_l$.

Neither is the functor ρ surjective on morphisms. Instead, for any two objects M and $N \in \mathcal{F}_{\mathbb{Z}_l}$ there is a natural *Bockstein long exact sequence*

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) &\longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \\ \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) &\longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \\ \longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) &\longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}_l}}(M,N) \longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m}}(\rho(M),\rho(N)) \longrightarrow \cdots \end{aligned}$$

Moreover, given two prime powers $m' = l^s$ and $m'' = l^t$ with $m = m'm''$, there is a Bockstein long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) &\longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathbb{Z}/m''}}(\rho''(M),\rho''(N)) \\ \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) &\longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathbb{Z}/m''}}(\rho''(M),\rho''(N)) \\ \longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m'}}(\rho'(M),\rho'(N)) &\longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathbb{Z}/m}}(M,N) \longrightarrow \cdots \end{aligned}$$

for the reduction functors $\rho^{(i)}\colon \mathcal{F}_{\mathbb{Z}/m} \longrightarrow \mathcal{F}_{\mathbb{Z}/m^{(i)}}$, $i = 1, 2$, and any two objects M, $N \in \mathcal{F}_{\mathbb{Z}/m}$. We would like to have such long exact sequences coming out from our categorical formalism of reductions.

More generally, let G be a profinite group. Then it is more convenient to consider representations of G in infinitely generated k-modules, particularly when the ring of coefficients k is itself infinite. So, let $\mathcal{F}^+_{\mathbb{Z}/m}$ denote the category of free $\mathbb{Z}/m$-modules endowed with a discrete action of G. For the l-adic coefficients, set $\mathcal{F}^+_{\mathbb{Z}/l}$ to be the category of l-divisible l-primary abelian groups (or, in a different language, injective discrete $\mathbb{Z}_l$-modules) with a discrete action of G.

The reduction functor $\rho\colon \mathcal{F}^+_{\mathbb{Z}_l} \longrightarrow \mathcal{F}^+_{\mathbb{Z}/m}$ acts by assigning to an l-divisible l-primary abelian group M with a discrete action of G the (co)free $\mathbb{Z}/m$-module ${}_mM \subset M$ of all the elements in M annihilated by m, endowed with the restriction of the action of G. Our procedure of reduction of exact categories will produce the exact category $\mathcal{F}^+_{\mathbb{Z}/m}$ with the reduction functor ρ starting from the exact category $\mathcal{F}^+_{\mathbb{Z}_l}$ endowed with the natural tranformation of the identity endofunctor $m\colon \operatorname{Id}_{\mathcal{F}^+_{\mathbb{Z}_l}} \longrightarrow \operatorname{Id}_{\mathcal{F}^+_{\mathbb{Z}_l}}$ (acting by multiplication with m on every object $M \in \mathcal{F}^+_{\mathbb{Z}_l}$) and some additional data.

If one feels a bit uncomfortable about a reduction functor assigning to an injective l^∞-torsion abelian group its subgroup of elements annihilated by $m = l^r$, one can restate the above description of the category $\mathcal{F}^+_{\mathbb{Z}_l}$ in the language of free $\mathbb{Z}_l$-contramodules [14] rather than cofree discrete $\mathbb{Z}_l$-modules. The $\mathbb{Z}_l$-module $\mathbb{Z}_l(G)$ of continuous $\mathbb{Z}_l$-valued functions on a profinite group G is a *$\mathbb{Z}_l$-free coalgebra* in the sense of [14, Subsections 1.6 and 3.1]. The category $\mathcal{F}^+_{\mathbb{Z}_l}$ can be equivalently defined as consisting of free/projective $\mathbb{Z}_l$-contramodules P (i. e., torsion-free abelian groups for which the natural map $P \longrightarrow \varprojlim_n P/l^nP$ is an isomorphism) endowed with a *$\mathbb{Z}_l$-free $\mathbb{Z}_l(G)$-comodule* structure [14, Subsections 3.1 and 3.3] (cf. [11, Example in Subsection 3.1]). The reduction functor $\rho\colon \mathcal{F}^+_{\mathbb{Z}_l} \longrightarrow \mathcal{F}^+_{\mathbb{Z}/m}$ then acts by assigning the $\mathbb{Z}/m$-free discrete G-module P/mP to a $\mathbb{Z}_l$-free $\mathbb{Z}_l(G)$-comodule P. The equivalence between the two definitions of the category $\mathcal{F}^+_{\mathbb{Z}_l}$ is provided by the rules $P = \Psi_{\mathbb{Z}_l}(M) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, M)$ and $M = \Phi_{\mathbb{Z}_l}(P) = \mathbb{Q}_l/\mathbb{Z}_l \otimes_{\mathbb{Z}} P$ (see [5, Proposition 2.1] and [14, Subsection 1.5 and Proposition 3.3.2(b)]).

In fact, the reduction construction presented in this paper is applicable to a wider class of situations than the above discussion might seem to suggest. Invented originally in the author's paper [12], the first version of this reduction procedure was intended to solve the following associated graded category problem, which looks quite different from the above coefficient reduction questions at the first glance.

Let l be a prime number and G be a pro-l-group. We are interested in (say, finite-dimensional) discrete G-modules M over the field $k = \mathbb{Z}/l$ endowed with a finite decreasing filtration by G-submodules $\cdots \supset F^{-1}M \supset F^0M \supset F^1M \supset \cdots$ such that the action of G is trivial on the quotient modules $F^nM/F^{n+1}M$. The question is to define the structure induced on the associated graded vector space $\bigoplus_n F^nM/F^{n+1}M$ by the G-module structure on M. Let us first describe the answer algebraically, and then formulate the problem in categorical terms.

More generally, let C be a coassociative coalgebra over a field k and $0 = F_{-1}C \subset F_0C \subset F_1C \subset F_2C \subset \cdots$ be a comultiplicative increasing filtration on C. Set $F^{-i}C = F_iC$, and consider the category of finite-dimensional left C-comodules M endowed with a decreasing filtration F compatible with the filtration on C. The above example with a pro-l-group G is obtained by setting $C = k(G)$ to be the group coalgebra (coalgebra of locally constant k-valued functions with the convolution comultiplication) of the profinite group G and $F_nC \subset C$ to be the components of the coaugmentation filtration $F_nC = \ker(C \to (C/k)^{\otimes n+1})$ (cf. [12, Section 2]).

In these terms, the induced structure on the graded vector space $\operatorname{gr}_F M = \bigoplus_n F^nM/F^{n+1}M$ is simply that of a graded left comodule over the graded coalgebra $\operatorname{gr}_F C = \bigoplus_n F^nC/F^{n+1}C$. The Ext spaces computed in the categories of filtered C-comodules and graded $\operatorname{gr}_F C$-comodules are related by a Bockstein long exact sequence (cf. the spectral sequence in [11, proof of main theorem]).

Now let $\mathcal{F}$ be the exact category of finite-dimensional left C-comodules M endowed with finite decreasing filtrations F compatible with the filtration F on C. The category $\mathcal{F}$ comes endowed with the *twist functor* $X \longmapsto X(1)$ taking a filtered C-comodule (M, F) to the same C-comodule M with the shifted filtration $F(1)^nM = F^{n-1}M$. One has $F^nM \subset F(1)^nM$ for all M and n; hence for every filtered C-comodule $X = (M, F)$ there is a natural morphism $\mathfrak{s}_X\colon X \longrightarrow X(1)$ in the category $\mathcal{F}$. The natural transformation $\mathfrak{s}\colon \operatorname{Id}_{\mathcal{F}} \longrightarrow (1)$ can be thought of as an element of the *graded center* of the category $\mathcal{F}$ with the twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$. The reduction construction, applied to the exact category $\mathcal{F}$ with the natural transformation $\mathfrak{s}$, should produce the exact (and, in fact, in this case abelian) category $\mathcal{G}$ of finite-dimensional graded modules over the graded coalgebra $\operatorname{gr}_F C$.

A solution to the latter "categorical filtration reduction" problem was worked out in our previous paper [12]. The aim of the present work is to generalize the reduction construction of [12, Section 4] so as to make it also applicable to, e. g., the "categorical coefficient reduction" problem described in the beginning of this introduction. We also generalize the categorical Bockstein sequence construction of [12, Section 4], formulating it in the abstract terms of an exact category with a graded center element and two exact functors to two other exact categories, satisfying appropriate conditions. This allows us to obtain the more complicated "finite-finite-finite" Bockstein long exact sequence for the Ext spaces in our reduced exact categories alongside with the simpler "integral-integral-finite" sequence.

The construction of the reduced category $\mathcal{G} = \mathcal{F}/\mathfrak{s}$ uses matrix factorizations. Indeed, the diagrams $V(-1) \longrightarrow U \longrightarrow V \longrightarrow U(1)$ defining objects of the intermediate category $\widetilde{\mathcal{H}}$ are nothing but matrix factorizations of the natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on the category $\mathcal{F}$ (in the sense of, e. g., [3, Remark 2.7]). However, the category-theoretic procedures that we apply to this category of matrix factorizations are quite different from the ones conventionally employed in the matrix factorization theory. Our aim is also quite different: while in the matrix factorization theory as it is presently known one produces a *triangulated* category out of one's matrix factorizations, in this paper we use matrix factorizations in order to produce an *exact* category (cf. [12, Remark 4.3]).

One of the differences is that our approach requires an exact-conservative functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ annihilating all the morphisms $\mathfrak{s}_X\colon X \longrightarrow X(1)$ (and the ones divisible by these, but no other morphisms) to be given as an additional piece of data. We start by considering the full subcategory $\mathcal{H} \subset \widetilde{\mathcal{H}}$ consisting of all the matrix factorizations that are transformed into exact sequences in $\mathcal{E}$ by the functor π (cf. [10, Lemma 1.5]). This allows to construct the middle cocycles/coboundaries functor $\Delta\colon \mathcal{H} \longrightarrow \mathcal{E}$.

We proceed by passing to the quotient category of the category $\mathcal{H}$ by the ideal $\mathcal{I}$ of all morphisms annihilated by the functor Δ. This serves to make the class $\mathcal{S}$ of all morphisms transformed to isomorphisms by Δ a localizing class in the category $\mathcal{H}/\mathcal{I}$, and we finally set $\mathcal{G} = (\mathcal{H}/\mathcal{I})[\mathcal{S}^{-1}]$. Though this two-step procedure may remind one of (and was indeed inspired by) the two-step construction of the derived category starting from the category of complexes (with passing to the quotient category by the ideal of morphisms homotopic to zero on the first step), our present construction produces, to repeat, an exact category and *not* a triangulated one.

Let us make a few more comments on the difference between the two theories looking from a different angle. The conventional theory of matrix factorizations, from its inception in the works of Eisenbud [4] and Buchweitz [1], was concerned with (local or global) *singularities* of algebraic varieties. When there were, in fact, no singularities, the theory would become largely trivial.

In the form the theory obtained in the work of Orlov [8, 9] (see also [10]), the homotopy/derived category of matrix factorizations was identified with the triangulated category of singularities of the zero locus X_0 of a nonzero-dividing global section $w \in L(X)$ of a line bundle L on a smooth scheme X. In its most advanced present form [9, 3], the theory uses categories of matrix factorizations to describe relative singularies (in one or another sense) of the Cartier divisor X_0 as compared to those of the whole (also singular) scheme X. When the zero locus is, in fact, nonsingular (or if its singularities are no worse than those of the ambient variety), the triangulated categories of matrix factorizations vanish.

To compare, consider an associative ring R with a nonzero-dividing central element $s \in R$, and let $\mathcal{F}$ be the exact category of all left R-modules M without s-torsion for which the natural map $M \longrightarrow \varprojlim_n M/s^n M$ is an isomorphism (or, which is equivalent for R-modules without s-torsion, one has $\mathrm{Ext}^*_R(R[s^{-1}], M) = 0$; cf. [14, Section 1 and Appendix B]). The multiplication with s provides a natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathrm{Id}_{\mathcal{F}}$. Take $\mathcal{E}$ to be the abelian category of abelian groups (or k-vector spaces, if R contains a field k, etc.). One can use the functor assigning the abelian group/vector space M/sM to an s-complete R-module without s-torsion $M \in \mathcal{F}$ in the role of the background exact-conservative functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$.

Applying our reduction procedure to this set of inputs, one obtains the abelian category of left $R/(s)$-modules $\mathcal{G} = \mathcal{F}/\mathfrak{s}$ in the output. Notice that the s-completeness condition is necessary for our construction to work (or otherwise the functor π would not be "exact-conservative"). On the other hand, we obtain the category of $R/(s)$-modules *itself* in the result, rather than any category of singularities of the quotient ring $R/(s)$. Assuming the ring R to be commutative and Noetherian (of finite Krull dimension, etc.), and the ring $R/(s)$ to be regular, etc., does *not* make our reduction construction, or the exact/abelian category produced by it, trivial (in any sense apparent to the author). Of course, this does not mean that the category of $R/(s)$-modules could not be obtained without our formalism.

The importance of the reduction construction worked out in this paper lies, in our view, in its wide applicability, including applicability to complicated exact categories for which its outputs may be hard to produce in any alternative or explicit way. In all the examples discussed above in this introduction, it was known in advance what the reduced exact category $\mathcal{G} = \mathcal{F}/\mathfrak{s}$ is supposed to be. The related Bockstein long exact sequences were not difficult to obtain "by hand", either. This does not seem to be true for the examples that we are really interested in, however.

The latter mostly come from the theory of Artin–Tate motives and motivic sheaves with finite coefficients [12, 13]. The construction of [12, Section 4] was developed and applied for producing the associated graded category $\mathcal{G}$ of the exact category $\mathcal{F}$ of mixed Artin–Tate motives with finite coefficients (and of the similar exact categories of filtered objects in a given exact category with the successive quotients belonging to a given additive subcategory endowed with the split exact structure). We do *not* know of any other way to define this exact category $\mathcal{G}$ (e. g., as the category of graded modules or comodules over anything, etc.).

Our motivation for developing the construction presented below also comes from the theory of Artin–Tate motivic sheaves with finite coefficients. It was shown in the paper [13] that the exact category $\mathcal{F}^m_X$ of Artin–Tate motivic sheaves with the coefficients $\mathbb{Z}/m$ on a smooth algebraic variety X (over a field of characteristic prime to m) has the Ext groups between the Tate objects $\mathrm{Ext}^i_{\mathcal{F}^m_X}(\mathbb{Z}/m, \mathbb{Z}/m(j))$ agreeing with the motivic cohomology groups of the variety X (as described by the Beilinson–Lichtenbaum conjectures) if and only if all the scheme points y of varieties Y étale over X have the same property. Furthermore, for a field K containing a primitive m-root of unity it was proven in [12] that one can express this "$K(\pi,1)$" (Ext agreement) conjecture as the Koszul property of a certain "big graded algebra".

We would like to get rid of the root of unity assumption in the latter theorem. This is straightforward when $m = l$ is a prime, because the cyclotomic modules $\mu_m^{\otimes j}$ over the Galois group G_K are direct summands of permutational modules in this case. So the plan is to reduce the $K(\pi,1)$-conjecture for Artin–Tate motives with $\mathbb{Z}/m$-coefficients to the case of prime coefficients $\mathbb{Z}/l$. This is easily done (using the Koszul algebras language) when the field K contains a primitive m-root of unity [12, Subsections 7.3 and 9.5], but we are interested in the opposite case. In the general situation, we want to use the Bockstein long exact sequences relating the Ext groups in the categories $\mathcal{F}_K^m$ with varying coefficients $\mathbb{Z}/m$.

The existence of such Bockstein exact sequences is itself a nontrivial assertion, and we intend to prove it by constructing exact categories $\mathcal{G}_K^m = \mathcal{F}_K^{l^\infty}/m$ in which the desired long exact sequences are forced to hold first, and comparing the categories $\mathcal{G}_K^m$ with the desired categories $\mathcal{F}_K^m$ later. So in this purported application we know in advance what the reduced exact category $\mathcal{G}$ should be, but the Bockstein exact sequences for the desired exact categories cannot be obtained directly, and proving that the category $\mathcal{G}$ is what it is supposed to be is a nontrivial task. In particular, it appears that the $K(\pi,1)$/Koszulity hypothesis for prime coefficients $\mathbb{Z}/l$ may be *needed* for obtaining the Bockstein long exact sequences.

At the very end of the present version of the paper, the intended main results are formulated as two conjectures.

Acknowledgement. The author's conception of the question of constructing the quotient category of an exact category with a twist functor by a natural transformation goes back to my years as a graduate student at Harvard University in the second half of '90s. My thinking was influenced by conversations with V. Voevodsky and A. Beilinson at the time. The details below were worked out in Moscow in February–March 2010 (as presented in [12, Section 4]) and subsequently in September 2013 (in full generality). The paper was written while I was vacationing in Prague in March–April 2014, visiting Ben Gurion University of the Negev in Be'er Sheva in June–September 2014, and visiting the Technion in Haifa in October 2014–March 2015. The author was supported in part by RFBR grants in Moscow and by a fellowship from the Lady Davis Foundation at the Technion.

0. Preliminaries

0.0. **Notation and terminology.** Throughout this paper, by an *exact category* we mean an exact category in Quillen's sense, i. e., an additive category endowed with a class of short exact sequences satisfying the natural axioms (see, e. g., [6, 7], [2], or [12, Appendix A]). A sequence of objects and morphisms in an exact category is said to be *exact* if it is composed of short exact sequences. A functor between exact categories is called exact if it takes short (or, equivalently, arbitrary) exact sequences in the source category to short (resp., long) exact sequences in the target one.

A *twist functor* on a category $\mathcal{F}$ is an autoequivalence denoted usually by $X \longmapsto X(1)$. The inverse autoequivalence is denoted by $X \longmapsto X(-1)$, and the integral powers of the twist functor are denoted by $X \longmapsto X(n)$, $n \in \mathbb{Z}$. Twist functors on exact categories will be presumed to be exact autoequivalences.

Given two categories $\mathcal{F}$ and $\mathcal{E}$ endowed with twist functors, a functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ is said to commute with the twists if a functorial isomorphism $\pi(X(1)) \simeq \pi(X)(1)$ is fixed for all objects $X \in \mathcal{F}$. Speaking of a commutative diagram of functors $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}$ commuting with the twists, we will always presume that the commutation isomorphisms form commutative diagrams of morphisms.

A morphism of endofunctors $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$, $n \in \mathbb{Z}$ on a category $\mathcal{F}$ with a twist functor $X \longmapsto X(1)$ (i. e., a morphism $\mathfrak{t}_X\colon X \longrightarrow X(n)$ defined for every object $X \in \mathcal{F}$ and functorial with respect to all the morphisms $X \longrightarrow Y$ in $\mathcal{F}$) is said to *commute with the twist* if for every object $X \in \mathcal{F}$ the equation $\mathfrak{t}_{X(1)} = \mathfrak{t}_X(1)$ holds in the set $\mathrm{Hom}_{\mathcal{F}}(X(1), X(n+1))$.

Notice that the endomorphisms of the identity functor on an additive category $\mathcal{F}$ always form a commutative ring, which is called the *center* of the category $\mathcal{F}$. It is the universal object among all the commutative rings k for which $\mathcal{F}$ can be endowed with the structure of a k-linear category. Similarly, given an additive category $\mathcal{F}$ with a twist functor $X \longmapsto X(1)$, morphisms of endofunctors $\mathrm{Id} \longrightarrow (n)$ commuting with the twist form a commutative ring with a $\mathbb{Z}$-grading, which can be called the *graded center* of an (additive) category with a twist functor.

We will say that a morphism $f\colon X \longrightarrow Y$ in $\mathcal{F}$ is *divisible by* a natural transformation $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$ commuting with the twist functor $X \longmapsto X(1)$ on $\mathcal{F}$ if the morphism f factorizes through the morphism $\mathfrak{t}_X\colon X \longrightarrow X(n)$, or equivalently, through the morphism $\mathfrak{t}_{Y(-n)}\colon Y(-n) \longrightarrow Y$. Similarly, a morphism $f\colon X \longrightarrow Y$ is *annihilated by* an element of the graded center $\mathfrak{t}\colon \mathrm{Id} \longrightarrow (n)$ of an additive category $\mathcal{F}$ if the composition $X \longrightarrow Y \longrightarrow Y(n)$ vanishes, or equivalently, the composition $X(-n) \longrightarrow X \longrightarrow Y$ vanishes in $\mathcal{F}$.

Furthermore, suppose that two commuting autoequivalences $X \longmapsto X(1)$ and $X \longmapsto X\{1\}$ are defined on a category $\mathcal{F}$. Then one can consider morphisms of endofunctors $\mathfrak{t}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (n)\{m\}$ commuting with *both* the twist functors (1) and $\{1\}$, i. e., satisfying the equations $\mathfrak{t}_{X(1)} = \mathfrak{t}_X(1)$ and $\mathfrak{t}_{X\{1\}} = \mathfrak{t}_X\{1\}$. For an additive category $\mathcal{F}$, such natural transformations form a bigraded commutative ring, which can be called the *bigraded center* of $\mathcal{F}$.

More generally, let $\mathcal{F}$ and $\mathcal{G}$ be two categories with twist functors $X \longmapsto X(1)$ and $\rho\colon \mathcal{F} \longrightarrow \mathcal{G}$ be a functor commuting with the twists. Then a morphism of functors $\mathfrak{s}\colon \rho \longrightarrow \rho(n)$ acting between the categories $\mathcal{F}$ and $\mathcal{G}$ is said to *commute with the twists* if for every object $X \in \mathcal{F}$ the equation $\mathfrak{s}_{X(1)} = \mathfrak{s}_X(1)$ holds in the set $\mathrm{Hom}_{\mathcal{G}}(\rho(X)(1), \rho(X)(n+1))$. If this is the case and $f\colon X \longrightarrow Y$ is a morphism in the category $\mathcal{F}$, then the composition $\rho(X) \longrightarrow \rho(Y) \longrightarrow \rho(Y)(n)$ of the morphisms $\rho(f)$ and $\mathfrak{s}_Y$, which is equal to the composition $\rho(X) \longrightarrow \rho(X)(n) \longrightarrow \rho(Y)(n)$ of the morphisms $\mathfrak{s}_X$ and $\rho(f(n))$, is called the *product* of the morphism $\rho(f)$ with the natural transformation $\mathfrak{s}$ and denoted by $\mathfrak{s}\rho(f)$.

Given two objects X and Y in the category $\mathcal{F}$, a morphism $g\colon \rho(X) \longrightarrow \rho(Y)$ in the category $\mathcal{G}$ is said to be *divisible by* a natural transformation $\mathfrak{s}\colon \rho \longrightarrow \rho(n)$ commuting with the twists if it has the form $\mathfrak{s}\rho(f)$ for a certain morphism $f\colon X \longrightarrow Y(-n)$ in the category $\mathcal{F}$, that is the morphism g is equal to the composition $\rho(X) \longrightarrow \rho(Y)(-n) \longrightarrow \rho(Y)$, or equivalently, to the composition $\rho(X) \longrightarrow \rho(X)(n) \longrightarrow \rho(Y)$, where the morphisms $\rho(X) \longrightarrow \rho(Y)(-n)$ and $\rho(X)(n) \longrightarrow \rho(Y)$ come from morphisms in the category $\mathcal{F}$ via the functor ρ. Similarly, given a morphism $f\colon X \longrightarrow Y$ in the category $\mathcal{F}$, the morphism $\rho(f)$ is said to be *annihilated by* the natural transformation $\mathfrak{s}$ if the product $\mathfrak{s}\rho(f)$ vanishes, i. e., the composition $\rho(X) \longrightarrow \rho(Y) \longrightarrow \rho(Y)(n)$ is equal to zero, or equivalently, the composition $\rho(X)(-n) \longrightarrow \rho(X) \longrightarrow \rho(Y)$ is equal to zero in the category $\mathcal{G}$.

An exact functor between exact categories $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ is called *exact-conservative* if it reflects admissible monomorphisms, admissible epimorphisms, and exact sequences. In other words, a functor π is said to be exact-conservative if a morphism in $\mathcal{F}$ is an admissible monomorphism or admissible epimorphism, or a sequence in $\mathcal{F}$ is exact, if and only if so is its image with respect to the functor π in the exact category $\mathcal{E}$. Notice that any exact-conservative functor between exact categories is conservative in the conventional sense (i. e., reflects isomorphisms).

0.1. **Exact surjectivity conditions.** Let $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ be an exact functor between two exact categories. The following conditions on a functor η will play a key role in the construction of the Bockstein long exact sequence in Section 1:

- (i′) for any object $X \in \mathcal{F}$ and any admissible epimorphism $T \longrightarrow \eta(X)$ in $\mathcal{G}$ there exists an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ and a morphism $\eta(Z) \longrightarrow T$ in $\mathcal{G}$ making the triangle diagram $\eta(Z) \longrightarrow T \longrightarrow \eta(X)$ commutative;
- (i″) for any object $Y \in \mathcal{F}$ and any admissible monomorphism $\eta(Y) \longrightarrow T$ in $\mathcal{G}$ there exists an admissible monomorphism $Y \longrightarrow Z$ in $\mathcal{F}$ and a morphism $T \longrightarrow \eta(Z)$ in $\mathcal{G}$ making the triangle diagram $\eta(Y) \longrightarrow T \longrightarrow \eta(Z)$ commutative;
- (ii′) for any objects $X, Y \in \mathcal{F}$ and any morphism $\eta(X) \longrightarrow \eta(Y)$ in $\mathcal{G}$ there exists an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Y$ in $\mathcal{F}$ making the triangle diagram $\eta(X') \longrightarrow \eta(X) \longrightarrow \eta(Y)$ commutative in $\mathcal{G}$;
- (ii″) for any objects X, $Y \in \mathcal{F}$ and any morphism $\eta(X) \longrightarrow \eta(Y)$ in $\mathcal{G}$ there exists an admissible monomorphism $Y \longrightarrow Y'$ and a morphism $X \longrightarrow Y'$ in $\mathcal{F}$ making the triangle diagram $\eta(X) \longrightarrow \eta(Y) \longrightarrow \eta(Y')$ commutative in $\mathcal{G}$.

We will say that an exact functor η satisfies the condition (i) if both the dual conditions (i′) and (i″) hold for it. Similarly, we will say that η satisfies the condition (ii) if both the dual conditions (ii′) and (ii″) hold for η. For simple examples of exact functors satisfying the conditions (i-ii), we refer the reader to Subsection 1.2.

For the sake of completeness, consider also the following two easier formulated conditions:

- ($*'$) for any object $T \in \mathcal{G}$ there exists an object $U \in \mathcal{F}$ and an admissible epimorphism $\eta(U) \longrightarrow T$ in $\mathcal{G}$;

($*''$) for any object $T \in \mathcal{G}$ there exists an object $V \in \mathcal{F}$ and an admissible monomorphism $T \longrightarrow \eta(V)$ in $\mathcal{G}$.

We will say that an exact functor η satisfies ($*$) if it satisfies both ($*'$) and ($*''$). The following lemma will be useful in Subsection 0.4, and also relevant in Subsection 2.5.

Lemma 0.1. *If the three conditions ($*'$), (i$'$), (ii$'$) hold for an exact functor η, then the morphism $\eta(Z) \longrightarrow T$ in (i$'$) can be chosen to be an admissible epimorphism, too. Moreover, for a functor η reflecting admissible epimorphisms, the condition (i$'$) entirely follows from ($*'$) and (ii$'$).*

Proof. Indeed, let $T \longrightarrow \eta(X)$ be an admissible epimorphism in $\mathcal{G}$. According to ($*'$), there exists an object $U \in \mathcal{F}$ together with an admissible epimorphism $\eta(U) \longrightarrow T$ in $\mathcal{G}$. The composition $\eta(U) \longrightarrow T \longrightarrow \eta(X)$ is a morphism in $\mathcal{G}$ between two objects coming from $\mathcal{F}$. According to (ii$'$), there exists an admissible epimorphism $U' \longrightarrow U$ and a morphism $U' \longrightarrow X$ in $\mathcal{F}$ making the triangle diagram $\eta(U') \longrightarrow \eta(U) \longrightarrow \eta(X)$ commutative in $\mathcal{G}$. Then the composition of two admissible epimorphisms $\eta(U') \longrightarrow \eta(U) \longrightarrow T$ is an admissible epimorphism in $\mathcal{G}$, and the triangle diagram $\eta(U') \longrightarrow T \longrightarrow \eta(X)$ is commutative.

Now if the functor η reflects admissible epimorphisms, then the morphism $U' \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$, because its image in $\mathcal{G}$ is equal to the composition of admissible epimorphisms $\eta(U') \longrightarrow T \longrightarrow \eta(X)$. In the general case, one applies (i$'$) to find an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ whose image in $\mathcal{G}$ factorizes through the admissible epimorphism $T \longrightarrow \eta(X)$. Then $U' \oplus Z \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$ whose image in $\mathcal{G}$ is the composition of the two admissible epimorphisms $\eta(U') \oplus \eta(Z) \longrightarrow T \longrightarrow \eta(X)$. ☐

Finally, the following pair of conditions on an exact functor $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ will be needed in Subsection 0.4:

($**'$) For any object $X \in \mathcal{F}$ and any morphism $\eta(X) \longrightarrow T$ in $\mathcal{G}$ there exists an admissible epimorphism $X' \longrightarrow X$ in $\mathcal{F}$, a morphism $X' \longrightarrow S$ in $\mathcal{F}$, and an admissible epimorphism $\eta(S) \longrightarrow T$ in $\mathcal{G}$ making the square diagram $\eta(X') \longrightarrow \eta(X) \longrightarrow T$, $\eta(X') \longrightarrow \eta(S) \longrightarrow T$ commutative in $\mathcal{G}$.

($**''$) For any object $Y \in \mathcal{F}$ and any morphism $T \longrightarrow \eta(Y)$ in $\mathcal{G}$ there exists an admissible monomorphism $Y \longrightarrow Y'$ in $\mathcal{F}$, a morphism $S \longrightarrow Y'$ in $\mathcal{F}$, and an admissible monomorphism $T \longrightarrow \eta(S)$ in $\mathcal{G}$ making the square diagram $T \longrightarrow \eta(Y) \longrightarrow \eta(Y')$, $T \longrightarrow \eta(S) \longrightarrow \eta(Y')$ commutative in $\mathcal{G}$.

We will say that an exact functor η satisfies ($**$) if it satisfies both ($**'$) and ($**''$).

0.2. **Ext trivia.** Here we formulate several elementary lemmas about the low-degree Yoneda Ext groups in exact categories which will be useful in Subsections 1.6–1.8. The proofs are left to the reader.

Lemma 0.2. *Let $z \in \operatorname{Ext}^1_{\mathcal{F}}(X,Y)$ and $w \in \operatorname{Ext}^1_{\mathcal{F}}(U,V)$ be two Ext^1 classes in an exact category $\mathcal{F}$ represented by short exact sequences $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow V \longrightarrow W \longrightarrow U \longrightarrow 0$. Let $f\colon Y \longrightarrow V$ and $g\colon X \longrightarrow U$ be two morphisms in $\mathcal{F}$. Then the equation $fz = wg$ holds in the group $\operatorname{Ext}^1_{\mathcal{F}}(X,V)$ if and* *only if there exists a morphism $h\colon Z \longrightarrow W$ in $\mathcal{F}$ extending the given collection of morphisms to a morphism of short exact sequences (i. e., a commutative diagram) $(Y \to Z \to X) \longrightarrow (V \to W \to U)$ in $\mathcal{F}$.* $\square$

Lemma 0.3. *Let $(K \to L \to M) \longrightarrow (X \to Y \to Z) \longrightarrow (U \to V \to W)$ be a 3×3 square commutative diagram in an exact category $\mathcal{F}$, all of whose three rows and three columns are short exact sequences in $\mathcal{F}$. Denote by $l \in \mathrm{Ext}^1_{\mathcal{F}}(M,K)$, $z \in \mathrm{Ext}^1_{\mathcal{F}}(W,M)$, $v \in \mathrm{Ext}^1_{\mathcal{F}}(W,U)$, and $x \in \mathrm{Ext}^1_{\mathcal{F}}(U,K)$ the Ext^1 classes represented by the four short exact sequences along the perimeter. Then the equation $lz + xv = 0$ holds in the group $\mathrm{Ext}^2_{\mathcal{F}}(W,K)$.* $\square$

Lemma 0.4. *Let $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ be two short exact sequences in an exact category $\mathcal{F}$, and let $K \longrightarrow V$ and $L \longrightarrow W$ be two morphisms in $\mathcal{F}$ forming a commutative square diagram $K \longrightarrow L \longrightarrow W$, $K \longrightarrow V \longrightarrow W$. Then*

(a) *denoting by H the cohomology object of the pair of mophisms with zero composition $K \longrightarrow V \oplus L \longrightarrow W$, the short sequence $0 \longrightarrow U \longrightarrow H \longrightarrow M \longrightarrow 0$ of morphisms induced by the morphisms $U \longrightarrow V$ and $L \longrightarrow M$ is exact in $\mathcal{F}$;*

(b) *the short exact sequence constructed in part (a) splits if and only if there exists a morphism $L \longrightarrow V$ in $\mathcal{F}$ making the triangle diagrams $K \longrightarrow L \longrightarrow V$ and $L \longrightarrow V \longrightarrow W$ commutative;*

(c) *given a morphism of short exact sequences $(U \to V \to W) \longrightarrow (U' \to V' \to W')$ in $\mathcal{F}$, and denoting by H' the cohomology object of the pair of morphisms $K \longrightarrow V' \oplus L \longrightarrow W'$, the class in $\mathrm{Ext}^1_{\mathcal{F}}(M,U')$ represented by the short exact sequence $(U' \to H' \to M)$ is equal to the product of the class in $\mathrm{Ext}^1_{\mathcal{F}}(M,U)$ represented by the sequence $(U \to H \to M)$ and the morphism $U \longrightarrow U'$;*

(d) *given two short exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ and $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ as above and two morphisms of morphisms (two commutative squares) $(K \to L) \rightrightarrows (V \to W)$ in $\mathcal{F}$, the class in $\mathrm{Ext}^1_{\mathcal{F}}(M,U)$ assigned to the sum of two morphisms of morphisms by the construction of part (a) is equal to the sum of the classes assigned to the summands.* $\square$

0.3. **Exact surjectivity and the Ext groups.** The proofs of the two parts of the next proposition can be found in [12, Subsection 4.4] (for a discussion of big graded rings, see [12, Subsection A.1]). These results will play a key role in the arguments of Subsections 1.7–1.8. We denote by $\eta^n = \eta^n_{X,Y}\colon \mathrm{Ext}^n_{\mathcal{F}}(X,Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{G}}(\eta(X),\eta(Y))$ the Ext group homomorphisms induced by an exact functor $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$.

Proposition 0.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be two exact categories and $\eta\colon \mathcal{F} \longrightarrow \mathcal{G}$ be an exact functor satisfying the condition (i′). Then*

(a) *for any objects X, $Y \in \mathcal{F}$ and $W \in \mathcal{G}$, and any Ext classes $a \in \mathrm{Ext}^n_{\mathcal{F}}(X,Y)$ and $b \in \mathrm{Ext}^m_{\mathcal{G}}(\eta(Y),W)$ such that $b\eta^n(a) = 0$ and $m \geqslant 1$ there exists an object $Y' \in \mathcal{F}$, a morphism $f\colon Y' \longrightarrow Y$ in $\mathcal{F}$, and a class $a' \in \mathrm{Ext}^n_{\mathcal{F}}(X,Y')$ for which $a = fa'$ and $b\eta(f) = 0$;*

(b) *for any object $W \in \mathcal{G}$ the right graded module $(\mathrm{Ext}^n_{\mathcal{G}}(\eta(X),W))_{X \in \mathcal{F};\, n \geqslant 0}$ over the big graded ring $(\mathrm{Ext}^n_{\mathcal{F}}(X,Y))_{Y,X \in \mathcal{F};\, n \geqslant 0}$ over the set of all objects of $\mathcal{F}$ is induced* *from the right module* $(\mathrm{Hom}_{\mathcal{G}}(\eta(Y),W))_{Y\in\mathcal{F}}$ *over the big subring* $(\mathrm{Hom}_{\mathcal{F}}(X,Y))_{Y,X} \subset (\mathrm{Ext}^n_F(X,Y))_{Y,X;n}$. $\square$

The following two corollaries allow one to prove that certain exact functors between exact categories induce isomorphisms of the Ext groups. They will be useful for us in Subsection 2.7.

Corollary 0.6. *Let* $\mathcal{F}$, $\mathcal{G}'$, $\mathcal{G}''$ *be three exact categories, and let* $\eta'\colon \mathcal{F} \longrightarrow \mathcal{G}'$ *and* $\iota\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ *be two exact functors. Suppose that the composition of exact functors* $\eta'' = \iota\eta'\colon \mathcal{F} \longrightarrow \mathcal{G}''$ *satisfies the condition (i′) and the functor* ι *induces isomorphisms* $\iota\colon \mathrm{Hom}_{\mathcal{G}'}(\eta'(X),W) \simeq \mathrm{Hom}_{\mathcal{G}''}(\eta''(X),\iota(W))$ *for all the objects* $X \in \mathcal{F}$, $W \in \mathcal{G}'$. *Then*

(a) *the functor* ι *induces isomorphisms of Ext groups* $\iota^n\colon \mathrm{Ext}^n_{\mathcal{G}'}(\eta'(X),W) \simeq \mathrm{Ext}^n_{\mathcal{G}''}(\eta''(X),\iota(W))$ *for all the objects* $X \in \mathcal{F}$, $W \in \mathcal{G}'$ *and all integers* $n \geqslant 0$*;*

(b) *if the functor* η' *satisfies the condition (∗′), then the functor* ι *is fully faithful, its image* $\iota(\mathcal{G}')$ *is a full subcategory closed under extensions in* $\mathcal{G}''$*, the exact category structure on* $\mathcal{G}'$ *coincides with the one induced from* $\mathcal{G}''$ *via* ι*, and the functor* ι *induces isomorphisms of the Ext groups* $\iota^n\colon \mathrm{Ext}^n_{\mathcal{G}'}(T,W) \simeq \mathrm{Ext}^n_{\mathcal{G}''}(\iota(T),\iota(W))$ *for all the objects* T, $W \in \mathcal{G}'$ *and all* $n \geqslant 0$.

Proof. It is straighforward to see that, in our assumptions on the functor ι, a functor η' satisfies the condition (i′) whenever the functor η'' does. Applying Proposition 0.5(b) to both the functors η' and η'' and comparing the resulting descriptions of Ext groups in the categories $\mathcal{G}'$ and $\mathcal{G}''$, one obtains the assertion of part (a).

All the assertions in the conclusion of part (b) follow from the latter one. To deduce it from part (a), it suffices to consider a short exact sequence $0 \longrightarrow S \longrightarrow \eta'(X) \longrightarrow T \longrightarrow 0$ in $\mathcal{G}'$, the morphism between the related long exact sequences of Ext groups induced by the functor ι, and argue by induction in n using the 5-lemma, proving the injectivity first and then the surjectivity for every given $n \geqslant 0$. $\square$

Corollary 0.7. *Let* $\mathcal{F}$, $\mathcal{G}'$, $\mathcal{G}''$ *be three exact categories, and let* $\eta'\colon \mathcal{F} \longrightarrow \mathcal{G}'$ *and* $\iota\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ *be two exact functors. Suppose that both functors* η' *and* $\eta'' = \iota\eta'\colon \mathcal{F} \longrightarrow \mathcal{G}''$ *satisfy the condition (i′), and the functor* ι *induces isomorphisms* $\iota\colon \mathrm{Hom}_{\mathcal{G}'}(\eta'(X),\eta'(Y)) \simeq \mathrm{Hom}_{\mathcal{G}''}(\eta''(X),\eta''(Y))$ *for all the objects* X, $Y \in \mathcal{F}$. *Then*

(a) *the functor* ι *induces isomorphisms of Ext groups* $\iota^n\colon \mathrm{Ext}^n_{\mathcal{G}'}(\eta'(X),\eta'(Y)) \simeq \mathrm{Ext}^n_{\mathcal{G}''}(\eta''(X),\eta''(Y))$ *for all the objects* X, $Y \in \mathcal{F}$ *and all integers* $n \geqslant 0$*;*

(b) *if the exact category* $\mathcal{G}'$ *coincides with its minimal full subcategory containing all the objects in the image of the functor* η' *and closed under extensions and direct summands, or alternatively if the functor* η' *satisfies both the conditions (∗′) and (∗″), then the conclusion of Corollary 0.6(b) also holds.*

Proof. Part (a) is provided by Proposition 0.5(b) as above, and part (b) in its second set of assumptions follows from part (a) as above, by resolving both the objects T and W on the appropriate sides. Part (b) in its first set of assumptions also follows by the 5-lemma (no induction in n needed in this situation). $\square$

0.4. **Exact surjectivity in compositions.** Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be three exact categories and $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$, $\eta\colon \mathcal{G} \longrightarrow \mathcal{H}$ be two exact functors. We are interested in proving that the functor η satisfies the above conditions (i) and (ii) provided that both the functors γ and $\eta\gamma$ satisfy certain (stronger) conditions. The next lemma will be important in Subsection 2.9.

Lemma 0.8. (a) *If the functor $\eta\gamma$ reflects admissible epimorphisms, admissible monomorphisms, or exact sequences, then so does the functor γ. If the functor $\eta\gamma$ satisfies the condition ($*'$), then so does the functor η.*

(b) *If the functor γ satisfies the condition ($*'$) and the functor $\eta\gamma$ satisfies the condition (i$'$), then the functor η also satisfies the condition (i$'$).*

(c) *If the functor γ satisfies the condition ($*'$) and the functor $\eta\gamma$ satisfies the condition ($**'$), then the functor η also satisfies the condition ($**'$).*

(d) *If the functor γ satisfies the condition ($*'$), and the functor $\eta\gamma$ satisfies the conditions (i$'$-ii$'$), ($*'$–$**'$) and reflects admissible epimorphisms, then the functor η satisfies the condition (ii$'$).*

Proof. Part (a) is obvious. Part (b): let $T \longrightarrow \eta(X)$ be an admissible epimorphism in $\mathcal{H}$ onto the image of an object $X \in \mathcal{G}$. Pick an admissible epimorphism $\gamma(U) \longrightarrow X$ in $\mathcal{G}$ from the image of an object $U \in \mathcal{F}$, and denote by W the fibered product of T and $\eta\gamma(U)$ over $\eta(X)$ in $\mathcal{H}$. Then $W \longrightarrow \eta\gamma(U)$ is an admissible epimorphism in $\mathcal{H}$ onto the image of an object from $\mathcal{F}$, so there exists an admissible epimorphism $Z \longrightarrow U$ in $\mathcal{F}$ and a morphism $\eta\gamma(Z) \longrightarrow W$ in $\mathcal{H}$ making the triangle $\eta\gamma(Z) \longrightarrow W \longrightarrow \eta\gamma(U)$ commutative in $\mathcal{H}$. Now the composition $\gamma(Z) \longrightarrow \gamma(U) \longrightarrow X$ is an admissible epimorphism in $\mathcal{G}$ whose image in $\mathcal{H}$ decomposes as $\eta\gamma(Z) \longrightarrow W \longrightarrow T \longrightarrow \eta(X)$ and therefore factorizes through the original admissible epimorphism $T \longrightarrow \eta(X)$. Notice also that the morphism $\eta\gamma(Z) \longrightarrow T$ is an admissible epimorphism in $\mathcal{H}$ whenever the morphism $\eta\gamma(Z) \longrightarrow W$ is.

Part (c): let $\eta(X) \longrightarrow T$ be a morphism in $\mathcal{H}$ from the image of an object $X \in \mathcal{G}$. Pick an admissible epimorphism $\gamma(U) \longrightarrow X$ in $\mathcal{G}$ and consider the composition $\eta\gamma(U) \longrightarrow \eta(X) \longrightarrow T$ in $\mathcal{H}$. Then there exists an admissible epimorphism $U' \longrightarrow U$ in $\mathcal{F}$, a morphism $U' \longrightarrow S$ in $\mathcal{F}$, and an admissible epimorphism $\eta\gamma(S) \longrightarrow T$ in $\mathcal{H}$ making the square diagram $\eta\gamma(U') \longrightarrow \eta\gamma(U) \longrightarrow T$, $\eta\gamma(U') \longrightarrow \eta\gamma(S) \longrightarrow T$ commutative in $\mathcal{H}$. Now the composition $\gamma(U') \longrightarrow \gamma(U) \longrightarrow X$ is an admissible epimorphism in $\mathcal{G}$ whose image under η makes a commutative square diagram with the image of the morphism $\gamma(U') \longrightarrow \gamma(S)$, the original morphism $\eta(X) \longrightarrow T$, and the admissible epimorphism $\eta\gamma(S) \longrightarrow T$ in $\mathcal{H}$.

Part (d): let $\eta(X) \longrightarrow \eta(Y)$ be a morphism in $\mathcal{H}$ between two objects coming from $\mathcal{G}$. Pick an admissible epimorphism $\gamma(U) \longrightarrow X$ in $\mathcal{G}$ and consider the composition $\eta\gamma(U) \longrightarrow \eta(X) \longrightarrow \eta(Y)$ in $\mathcal{H}$. According to the condition ($**'$), there exists an admissible epimorphism $U' \longrightarrow U$ in $\mathcal{F}$, a morphism $U' \longrightarrow S$ in $\mathcal{F}$, and an admissible epimorphism $\eta\gamma(S) \longrightarrow \eta(Y)$ in $\mathcal{H}$ such that the composition $\eta\gamma(U') \longrightarrow \eta\gamma(U) \longrightarrow \eta(X) \longrightarrow \eta(Y)$ is equal to the composition $\eta\gamma(U') \longrightarrow \eta\gamma(S) \longrightarrow \eta(Y)$ in $\mathcal{H}$. According to part (b), there exists an admissible epimorphism $Z \longrightarrow Y$ in $\mathcal{G}$ and a morphism $\eta(Z) \longrightarrow \eta\gamma(S)$ in $\mathcal{H}$ making the triangle $\eta(Z) \longrightarrow \eta\gamma(S) \longrightarrow \eta(Y)$

commutative in $\mathcal{H}$. Moreover, in view of the remark after the above proof of part (b) and the first assertion of Lemma 0.1 applied to the functor $\eta\gamma$, one can choose the morphism $\eta(Z) \longrightarrow \eta\gamma(S)$ to be an admissible epimorphism, too.

Pick an admissible epimorphism $\gamma(V) \longrightarrow Z$ in $\mathcal{G}$ and consider the composition $\eta\gamma(V) \longrightarrow \eta(Z) \longrightarrow \eta\gamma(S)$ in $\mathcal{H}$. According to the condition (ii′) for the functor $\eta\gamma$, there exists an admissible epimorphism $V' \longrightarrow V$ and a morphism $V' \longrightarrow S$ in $\mathcal{F}$ such that the composition $\eta\gamma(V') \longrightarrow \eta\gamma(V) \longrightarrow \eta\gamma(S)$ in $\mathcal{H}$ comes from the morphism $V' \longrightarrow S$ via the functor $\eta\gamma$. The morphism $\eta\gamma(V') \longrightarrow \eta\gamma(S)$ being a composition of three admissible epimorphisms in $\mathcal{H}$, we can conclude that the morphism $V' \longrightarrow S$ is an admissible epimorphism in $\mathcal{F}$. Now let W be the fibered product of the objects V' and U' over the object S in the category $\mathcal{F}$. Then the composition $\gamma(W) \longrightarrow \gamma(U') \longrightarrow \gamma(U) \longrightarrow X$ is an admissible epimorphism in $\mathcal{G}$ whose image under η composed with the original morphism $\eta(X) \longrightarrow \eta(Y)$ in $\mathcal{H}$ is equal to the image of the composition of morphisms $\gamma(W) \longrightarrow \gamma(V') \longrightarrow \gamma(V) \longrightarrow Z \longrightarrow Y$ in $\mathcal{G}$. ☐

1. The Bockstein Sequence

1.0. **Toy version: two-category sequence.** Let $\mathcal{F}$ and $\mathcal{F}_\mathfrak{s}$ be two exact categories endowed with twist functors (exact autoequivalences) $X \longmapsto X(1)$. Suppose that we are given an exact functor $\eta_\mathfrak{s}\colon \mathcal{F} \longrightarrow \mathcal{F}_\mathfrak{s}$ commuting with the twists. Assume that the functor $\eta_\mathfrak{s}$ satisfies the conditions (i-ii) of Subsection 0.1.

Suppose also that we are given a natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on the category $\mathcal{F}$ commuting with the twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ as explained in Subsection 0.0. Assume the following further conditions to be satisfied:

(ııı) for any object $X \in \mathcal{F}$, the morphism $\mathfrak{s}_X\colon X \longrightarrow X(1)$ is injective and surjective; in other words, no nonzero morphism in the category $\mathcal{F}$ is annihilated by the natural transformation $\mathfrak{s}$;

(ıııı) a morphism in the category $\mathcal{F}$ is annihilated by the functor $\eta_\mathfrak{s}$ if and only if it is divisible by the natural transformation $\mathfrak{s}$.

In this section we will construct, in the assumption of conditions (i-ii) and (ııı-ıııı), the following Bockstein long exact sequence of Ext groups

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_\mathcal{F}(X, Y(-1)) \longrightarrow \mathrm{Hom}_\mathcal{F}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X), \eta_\mathfrak{s}(Y)) \\ &\longrightarrow \mathrm{Ext}^1_\mathcal{F}(X, Y(-1)) \longrightarrow \mathrm{Ext}^1_\mathcal{F}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X), \eta_\mathfrak{s}(Y)) \\ &\longrightarrow \mathrm{Ext}^2_\mathcal{F}(X, Y(-1)) \longrightarrow \mathrm{Ext}^2_\mathcal{F}(X, Y) \longrightarrow \mathrm{Ext}^2_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X), \eta_\mathfrak{s}(Y)) \longrightarrow \cdots \end{aligned}$$

for any two objects X, $Y \in \mathcal{F}$. The differentials in this long exact sequence have the following properties:

(a) the maps $\eta_\mathfrak{s} = \eta_\mathfrak{s}^n\colon \mathrm{Ext}^n_\mathcal{F}(X, Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X), \eta_\mathfrak{s}(Y))$ are induced by the exact functor $\eta_\mathfrak{s}\colon \mathcal{F} \longrightarrow \mathcal{F}_\mathfrak{s}$;

(b) the maps $\mathfrak{s} = \mathfrak{s}_n\colon \mathrm{Ext}^n_\mathcal{F}(X, Y(-1)) \longrightarrow \mathrm{Ext}^n_\mathcal{F}(X, Y)$ are provided by the composition with the natural transformation $\mathfrak{s}\colon \mathrm{Id}_\mathcal{F} \longrightarrow (1)$;

(c) the maps $\partial = \partial^n\colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}}(X,Y)(-1))$ satisfy the equation
$$\partial^{i+n+j}(\eta^i_{\mathfrak{s}}(a) z \eta^j_{\mathfrak{s}}(b)) = (-1)^i a(-1)\partial^n(z)b$$
for any objects U, X, Y, $V \in \mathcal{F}$ and any Ext classes $b \in \operatorname{Ext}^j_{\mathcal{F}}(U,X)$, $z \in \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$, and $a \in \operatorname{Ext}^i_{\mathcal{F}}(Y,V)$.

This long exact sequence is a more immediate generalization of the long exact sequence of [12, Section 4] than the more elaborated constructions of Subsections 1.1 and 1.3 below. It is also essentially the particular case of the long exact sequence of the next Subsection 1.1 corresponding to the situation with $\mathcal{F}_{\mathfrak{t}} = \mathcal{F}_{\mathfrak{st}} = \mathcal{F}$ and the identity functor $\eta_{\mathfrak{t}} = \operatorname{Id}_{\mathcal{F}}$ (or, if one wishes, the particular case of the even more general long exact sequence of Subsection 1.3 corresponding to the situation with $\mathcal{F}_{\mathfrak{t}} = \mathcal{F} = \mathcal{F}_{\mathfrak{st}}$ and the identity functors $\eta_{\mathfrak{t}} = \operatorname{Id}_{\mathcal{F}} = \eta_{\mathfrak{st}}$).

Notice that in presence of the above condition (ııı) the condition (iv) of Subsection 1.1 becomes equivalent to its apparently stronger form (ıııı). One can see this, e. .g., by comparing the initial three-term fragments of the Bockstein sequences in this subsection and in Subsection 1.1 (cf. Lemmas 1.3(d) and 1.4(c) below).

1.1. **Posing the problem: three-category sequence.** Let $\mathcal{F}_{\mathfrak{t}}$, $\mathcal{F}_{\mathfrak{s}}$, and $\mathcal{F}_{\mathfrak{st}}$ be three exact categories endowed with twist functors (exact autoequivalences) $X \longmapsto X(1)$. Suppose that we are given exact functors $\eta_{\mathfrak{t}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{t}}$ and $\eta_{\mathfrak{s}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{s}}$, both commuting with the twists. Assume that both the functors $\eta_{\mathfrak{t}}$ and $\eta_{\mathfrak{s}}$ satisfy the conditions (i-ii) of Subsection 0.1.

Furthermore, suppose that we are given a natural transformation $\mathfrak{s}\colon \operatorname{Id} \longrightarrow (1)$ on the category $\mathcal{F}_{\mathfrak{st}}$ commuting with the twist functor $(1)\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ as explained in Subsection 0.0. We assume the following further conditions to be satisfied:

(iii) a morphism $X \longrightarrow Y$ in the category $\mathcal{F}_{\mathfrak{st}}$ is annihilated by the functor $\eta_{\mathfrak{t}}$ if and only if it is annihilated by the natural transformation $\mathfrak{s}$ in $\mathcal{F}_{\mathfrak{st}}$;

(iv) a morphism $X \longrightarrow Y$ in the category $\mathcal{F}_{\mathfrak{st}}$ is annihilated by the functor $\eta_{\mathfrak{s}}$ if and only if there exists an admissible epimorphism $X' \longrightarrow X$ such that the composition $X' \longrightarrow X \longrightarrow Y$ is divisible by $\mathfrak{s}$ in $\mathcal{F}_{\mathfrak{st}}$, or equivalently, if and only if there exists an admissible monomorphism $Y \longrightarrow Y'$ such that the composition $X \longrightarrow Y \longrightarrow Y'$ is divisible by $\mathfrak{s}$ in $\mathcal{F}_{\mathfrak{st}}$.

We will see below in Subsection 1.4 that the two dual formulations of the condition (iv) are equivalent modulo our previous assumptions (specifically, the argument is based on the condition (ii) for the functor $\eta_{\mathfrak{t}}$ and the condition (iii)).

Our goal in this section is to construct, in the assumption of the conditions (i-iv), the following Bockstein long exact sequence for the Ext groups
$$\begin{aligned}
0 &\longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \\
&\longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(X,Y) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \\
&\longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \operatorname{Ext}^2_{\mathcal{F}_{\mathfrak{st}}}(X,Y) \longrightarrow \cdots
\end{aligned}$$
for any two objects X, $Y \in \mathcal{F}_{\mathfrak{st}}$. The differentials in this long exact sequence have the following properties:

(a) the maps $\eta_{\mathfrak{s}} = \eta_{\mathfrak{s}}^n \colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X),\eta_{\mathfrak{s}}(Y))$ are induced by the exact functor $\eta_{\mathfrak{s}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{s}}$;

(b) the maps $\mathfrak{s} = \mathfrak{s}_n \colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X),\eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(X,Y)$ satisfy the equations $\mathfrak{s}_0(\mathrm{id}_{\eta_{\mathfrak{t}}(E)}) = \mathfrak{s}_E \in \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(E,E(1))$ and

$$\mathfrak{s}_{i+n+j}(\eta_{\mathfrak{t}}^i(a(-1))z\eta_{\mathfrak{t}}^j(b)) = a\mathfrak{s}_n(z)b$$

for any objects E, U, X, Y, $V \in \mathcal{F}_{\mathfrak{st}}$ and any Ext classes $b \in \operatorname{Ext}^j_{\mathcal{F}_{\mathfrak{st}}}(U,X)$, $z \in \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X),\eta_{\mathfrak{t}}(Y)(-1))$, and $a \in \operatorname{Ext}^i_{\mathcal{F}_{\mathfrak{st}}}(Y,V)$;

(c) the maps $\partial = \partial^n \colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X),\eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X),\eta_{\mathfrak{t}}(Y)(-1))$ satisfy the equation

$$\partial^{i+n+j}(\eta_{\mathfrak{s}}^i(a)z\eta_{\mathfrak{s}}^j(b)) = (-1)^i\eta_{\mathfrak{t}}^i(a(-1))\partial^n(z)\eta_{\mathfrak{t}}^j(b)$$

for any objects U, X, Y, $V \in \mathcal{F}_{\mathfrak{st}}$ and any Ext classes $b \in \operatorname{Ext}^j_{\mathcal{F}_{\mathfrak{st}}}(U,X)$, $z \in \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X),\eta_{\mathfrak{s}}(Y))$, and $a \in \operatorname{Ext}^i_{\mathcal{F}_{\mathfrak{st}}}(Y,V)$.

1.2. **Examples.** The "real" examples of exact functors $\eta_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{s}}$ and natural transformations $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ satisfying together the conditions (i-ɪɪɪ), as well as those of pairs of exact functors $\eta_{\mathfrak{s}}$, $\eta_{\mathfrak{t}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{s}}$, $\mathcal{F}_{\mathfrak{t}}$ and natural transformations $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}_{\mathfrak{st}}} \longrightarrow (1)$ satisfying the conditions (i-iv), will appear in connection with the reduction construction of Section 2, where the results of the present section will be applied (see Subsections 2.6 and 2.9; cf. Subsection 2.8, where the even more general setting of Subsection 1.3 below will appear). Nevertheless, let us present two very simple explicit examples here.

Example 1.1. Let G be a finite group, l be a prime number, and s, $t \geqslant 2$ be any two powers of l. For any prime power $m \geqslant 2$, let $\mathcal{F}_{\mathbb{Z}/m} = \mathcal{F}^G_{\mathbb{Z}/m}$ be the exact category of finitely generated free $\mathbb{Z}/m$-modules with an action of G.

Set $\mathcal{F}_{\mathfrak{s}} = \mathcal{F}_{\mathbb{Z}/s}$, $\mathcal{F}_{\mathfrak{t}} = \mathcal{F}_{\mathbb{Z}/t}$, and $\mathcal{F}_{\mathfrak{st}} = \mathcal{F}_{\mathbb{Z}/st}$. Let the functors $\eta_{\mathfrak{s}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{t}}$ and $\eta_{\mathfrak{t}}\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{t}}$ take a $\mathbb{Z}/st$-free G-module M to the $\mathbb{Z}/s$-free and $\mathbb{Z}/t$-free G-modules $M/sM = tM$ and $M/tM = sM$, respectively. Set all the twists (1) to be the identity functors, and the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}_{\mathfrak{st}}} \longrightarrow \mathrm{Id}_{\mathcal{F}_{\mathfrak{st}}}$ to act on all the $\mathbb{Z}/st$-free G-modules by the operator of multiplication with s.

Then we claim that all the conditions (i-iv) of Subsections 0.1 and 1.1 are satisfied for the exact functors $\eta_{\mathfrak{s}}$, $\eta_{\mathfrak{t}}$ and the center element $\mathfrak{s}$. The assertion is also true for the exact categories of arbitrary (infinitely generated) free $\mathbb{Z}/m$-modules with an action of G. Indeed, the functor $\eta_{\mathfrak{s}}$ is exact-conservative, since so is the forgetful functor $\mathcal{F}^G_{\mathbb{Z}/st} \longrightarrow \mathcal{F}^{\{e\}}_{\mathbb{Z}/st}$ and the reduction functor $\mathcal{F}^{\{e\}}_{\mathbb{Z}/st} \longrightarrow \mathcal{F}^{\{e\}}_{\mathbb{Z}/s}$ acting between the categories of free modules over $\mathbb{Z}/st$ and $\mathbb{Z}/s$ without any group action.

To check the conditions ($*'$) and ($*''$) for the functor $\eta_{\mathfrak{s}}$, one simply notices that any $\mathbb{Z}/m$-free G-module can be presented as the image of a surjective homomorphism from, and embedded into, a (co)free G-module over $\mathbb{Z}/m$ (i. e., a direct sum of copies of $\mathbb{Z}/m[G] = \mathbb{Z}/m(G)$). Furthermore, (co)free G-modules over $\mathbb{Z}/s$ can be obtained as the reductions of similar G-modules over $\mathbb{Z}/st$. Using the projectivity/injectivity properties of (co)free G-modules, one can also check the conditions (ii′) and (ii″), as well as the other conditions of Subsection 0.1 for the functor $\eta_{\mathfrak{s}}$.

The condition (iii) is obvious: a morphism $f\colon M \longrightarrow N$ in the category $\mathcal{F}_{\mathbb{Z}/st}$ is annihilated by the functor $\eta_{\mathfrak{t}}$ if and only if its image is contained in tN (or its kernel contains sM), which equivalently means that $sf = 0$. Finally, to check the condition (iv) one notices that any morphism $f\colon M \longrightarrow N$ in $\mathcal{F}_{\mathbb{Z}/st}$ with the image contained in sN is divisible by s in the group $\mathrm{Hom}_{\mathcal{F}_{\mathbb{Z}/st}}(M, N)$ whenever at least one of the G-modules M and N is (co)free over $\mathbb{Z}/st$.

Example 1.2. Let G be a finite group and $m = l^r$ be a prime power. Keeping the notation $\mathcal{F}_{\mathbb{Z}/m} = \mathcal{F}^G_{\mathbb{Z}/m}$ for the exact category of finitely generated free $\mathbb{Z}/m$-modules with an action of G, set also $\mathcal{F}_{\mathbb{Z}_l} = \mathcal{F}^G_{\mathbb{Z}_l}$ to be the category of finitely generated free $\mathbb{Z}_l$-modules with a G-action. Set $\mathcal{F} = \mathcal{F}_{\mathbb{Z}_l}$ and $\mathcal{F}_{\mathfrak{s}} = \mathcal{F}_{\mathbb{Z}/m}$.

Let the functor $\eta_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{s}}$ take a $\mathbb{Z}_l$-free G-module M to the $\mathbb{Z}/m$-free G-module M/mM. Set all the twists (1) to be the identity functors, and the map $\mathfrak{s}\colon M \longrightarrow M$ to be the multiplication with m for every module $M \in \mathcal{F} = \mathcal{F}_{\mathbb{Z}_l}$. Then it is claimed that all the conditions (i-ɪɪɪɪ) of Subsections 0.1 and 1.0 are satisfied for the exact functor $\eta_{\mathfrak{s}}$ and the natural transformation $\mathfrak{s}$.

The functor $\eta_{\mathfrak{s}}$ is exact-conservative, since so are the forgetful functor $\mathcal{F}^G_{\mathbb{Z}_l} \longrightarrow \mathcal{F}^{\{e\}}_{\mathbb{Z}_l}$ and the reduction functor $\mathcal{F}^{\{e\}}_{\mathbb{Z}_l} \longrightarrow \mathcal{F}^{\{e\}}_{\mathbb{Z}/m}$. The "exact surjectivity" conditions of Subsection 0.1 can be proven in the same way as in Example 1.1. To compensate for an apparent loss of symmetry between the injectivity and surjectivity properties of the (co)free G-modules $\mathbb{Z}_l[G] = \mathbb{Z}_l(G)$ with l-adic coefficients, one can use the alternative interpretation of $\mathcal{F}_{\mathbb{Z}_l}$ as the category of l-divisible l^∞-torsion abelian groups of finite rank endowed with an action of G.

The condition (ɪɪɪ) is obvious: no nonzero morphism in the category $\mathcal{F}^G_{\mathbb{Z}_l}$ is annihilated by the multiplication with m (since the same is true in the category $\mathcal{F}^{\{e\}}_{\mathbb{Z}_l}$). The condition (ɪɪɪɪ) holds, since any morphism $f\colon M \longrightarrow N$ in the category $\mathcal{F}^G_{\mathbb{Z}_l}$ that is annihilated by the functor $\eta_{\mathfrak{s}}$ (i. e., has the image contained in mN) is divisible by m in the group $\mathrm{Hom}_{\mathcal{F}^G_{\mathbb{Z}_l}}(M, N)$.

Applying the categorical Bockstein long exact sequence construction of this section to these two examples, one obtains the "finite-finite-finite" and "integral-integral-finite" Bockstein sequences for the Ext groups in the categories of finite group representations written down in the beginning of the introduction.

Examples of explicit sets of data that can be straightforwardly shown to satisfy "a half of" the conditions (i-ɪɪɪɪ) or (i-iv) are more numerous, and one can easily work out some of them on the basis of arguments similar to the above.

1.3. **Further generalization: four-category sequence.** Let $\mathcal{F}$, $\mathcal{F}_{\mathfrak{t}}$, $\mathcal{F}_{\mathfrak{s}}$, and $\mathcal{F}_{\mathfrak{st}}$ be four exact categories endowed with twist functors $X \longrightarrow X(1)$. Suppose we are given exact functors $\eta_{\mathfrak{t}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{t}}$, $\eta_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{s}}$, and $\eta_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{st}}$, all of them commuting with the twists. Assume that all the three functors $\eta_{\mathfrak{t}}$, $\eta_{\mathfrak{s}}$, $\eta_{\mathfrak{st}}$ satisfy the conditions (i-ii) of Subsection 0.1.

Suppose further that we are given a natural transformation $\mathfrak{s}\colon \eta_{\mathfrak{st}} \longrightarrow \eta_{\mathfrak{st}}(1)$ of functors $\mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ which commutes with the twist functors $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ and $(1)\colon \mathcal{F}_{\mathfrak{st}} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ in the sense of Subsection 0.0. Assume that the following further conditions are satisfied by this set of data:

(III) a morphism $f\colon X \longrightarrow Y$ in the category $\mathcal{F}$ is annihilated by the functor $\eta_{\mathfrak{t}}$ if and only if the morphism $\eta_{\mathfrak{st}}(f)$ is annihilated by the natural transformation $\mathfrak{s}$;

(IV) a morphism $f\colon X \longrightarrow Y$ in the category $\mathcal{F}$ is annihilated by the functor $\eta_{\mathfrak{s}}$ if and only if there exists an admissible epimorphism $X' \longrightarrow X$ in $\mathcal{F}$ such that the composition $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$ is divisible by the natural transformation $\mathfrak{s}$, or equivalently, if and only if there exists an admissible monomorphism $Y \longrightarrow Y'$ in $\mathcal{F}$ such that the composition $\eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y) \longrightarrow \eta_{\mathfrak{st}}(Y')$ is divisible by $\mathfrak{s}$.

We will see below in Subsection 1.4 that the two dual formulations of the condition (IV) are equivalent modulo the previous assumptions (the argument is based on the condition (ii) for the functor $\eta_{\mathfrak{t}}$ and the condition (III)).

Assuming the conditions (i-ii) and (III–IV), we will construct a Bockstein long exact sequence of the form

$$\begin{aligned}
0 &\longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^2_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \cdots
\end{aligned}$$

for any two objects X, $Y \in \mathcal{F}$. The differentials in this long exact sequence have the following properties:

(a) the maps $r = r^n_{\mathfrak{t}}\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ satisfty the equations

$$r^n_{\mathfrak{t}}(\eta^n_{\mathfrak{st}}(a)) = \eta^n_{\mathfrak{s}}(a) \quad \text{and} \quad r^{i+j}_{\mathfrak{t}}(xy) = r^i_{\mathfrak{t}}(x) r^j_{\mathfrak{t}}(y)$$

for any objects X, Y, U, V, $W \in \mathcal{F}$ and any Ext classes $a \in \mathrm{Ext}^n_{\mathcal{F}}(X, Y)$, $y \in \mathrm{Ext}^j_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(U), \eta_{\mathfrak{st}}(V))$, and $x \in \mathrm{Ext}^i_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(V), \eta_{\mathfrak{st}}(W))$;

(b) the maps $\mathfrak{s} = \mathfrak{s}_n\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$ satisfy the equations $\mathfrak{s}_0(\mathrm{id}_{\eta_{\mathfrak{t}}(E)}) = \mathfrak{s}_E \in \mathrm{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(E), \eta_{\mathfrak{st}}(E)(1))$ and

$$\mathfrak{s}_{i+n+j}(\eta^i_{\mathfrak{t}}(a(-1)) z \eta^j_{\mathfrak{t}}(b)) = \eta^i_{\mathfrak{st}}(a) \mathfrak{s}_n(z) \eta^j_{\mathfrak{st}}(b)$$

for any objects E, U, X, Y, $V \in \mathcal{F}$ and any Ext classes $b \in \mathrm{Ext}^j_{\mathcal{F}}(U, X)$, $z \in \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$, and $a \in \mathrm{Ext}^i_{\mathcal{F}}(Y, V)$;

(c) the maps $\partial = \partial^n\colon \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ satisfy the equation

$$\partial^{i+n+j}(\eta^i_{\mathfrak{s}}(a) z \eta^j_{\mathfrak{s}}(b)) = (-1)^i \eta^i_{\mathfrak{t}}(a(-1)) \partial^n(z) \eta^j_{\mathfrak{t}}(b)$$

for any objects U, X, Y, $V \in \mathcal{F}$ and any Ext classes $b \in \mathrm{Ext}^j_{\mathcal{F}}(U, X)$, $z \in \mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$, and $a \in \mathrm{Ext}^i_{\mathcal{F}}(Y, V)$.

The Bockstein long exact sequence described in Subsection 1.1 is a particular case of the long exact sequence of the present subsection corresponding to the situation when the functor $\eta_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ is an equivalence of exact categories.

1.4. **The first term.** Now we proceed to construct the Bockstein long exact sequence promised in Subsection 1.3. We start with constructing the map $\mathfrak{s}_0$, proving that it is injective, and describing its image.

Let X and Y be two objects of the category $\mathcal{F}$, and $p : \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ be a morphism in the category $\mathcal{F}_{\mathfrak{t}}$. According to the condition (ii) for the functor $\eta_{\mathfrak{t}}$, there exist an admissible epimorphism $X' \longrightarrow X$, an admissible monomorphism $Y(-1) \longrightarrow Y'(-1)$, and morphisms $X' \longrightarrow Y(-1)$ and $X \longrightarrow Y'(-1)$ in the category $\mathcal{F}$ whose images under the functor $\eta_{\mathfrak{t}}$ together with the morphism p form a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{t}}$.

The square diagram of morphisms $X' \longrightarrow X \longrightarrow Y'(-1)$, $X' \longrightarrow Y(-1) \longrightarrow Y'(-1)$ in the category $\mathcal{F}$ becomes commutative after applying the functor $\eta_{\mathfrak{t}}$, hence it follows from the condition (III) that it is commutative modulo the ideal of morphisms whose images under the functor $\eta_{\mathfrak{st}}$ are annihilated by the natural transformation $\mathfrak{s}$. Multiplying by $\mathfrak{s}$ the images of both morphisms $X \longrightarrow Y'(-1)$ and $X' \longrightarrow Y(-1)$ under the functor $\eta_{\mathfrak{st}}$, we therefore obtain a commutative square $\eta_{\mathfrak{st}}(X)' \longrightarrow \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y')$, $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(Y) \longrightarrow \eta_{\mathfrak{st}}(Y')$ in the category $\mathcal{F}_{\mathfrak{st}}$.

Since the morphism $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(X)$ is an admissible epimorphism and the morphism $\eta_{\mathfrak{st}}(Y) \longrightarrow \eta_{\mathfrak{st}}(Y')$ is an admissible monomorphism, it follows that there exists a unique morphism $f\colon \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$ complementing the latter square to a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{st}}$. By the definition, we set $\mathfrak{s}_0(p) = f$. As the morphisms $X' \longrightarrow X$ and $Y \longrightarrow Y'$ can be chosen independently and the choice of either one of them is sufficient to determine the morphism f, it does not depend on these choices.

Lemma 1.3. *Assuming the condition (ii) for the functor $\eta_{\mathfrak{t}}$ and the condition (III), the map $\mathfrak{s}_0\colon \mathrm{Hom}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$ has the following properties:*

(a) *the equation $\mathfrak{s}_0(\mathrm{id}_{\eta_{\mathfrak{t}}(E)}) = \mathfrak{s}_E \in \mathrm{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(E), \eta_{\mathfrak{st}}(E)(1))$ describing the image of the identity endomorphism in the category $\mathcal{F}_{\mathfrak{t}}$ under the map $\mathfrak{s}_0$ in terms of the natural transformation $\mathfrak{s}\colon \eta_{\mathfrak{st}} \longrightarrow \eta_{\mathfrak{st}}(1)$ of functors $\mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ holds in the category $\mathcal{F}_{\mathfrak{st}}$ for every object E of the category $\mathcal{F}$;*

(b) *the equation $\mathfrak{s}_0(\eta_{\mathfrak{t}}(g(-1))p\eta_{\mathfrak{t}}(h)) = \eta_{\mathfrak{st}}(g)\mathfrak{s}_0(p)\eta_{\mathfrak{st}}(h)$ holds in the category $\mathcal{F}_{\mathfrak{st}}$ for any two morphisms $h\colon U \longrightarrow X$, $g\colon Y \longrightarrow V$ in the category $\mathcal{F}$ and any morphism $p\colon \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$;*

(c) *the map $\mathfrak{s}_0$ is injective for any objects X, $Y \in \mathcal{F}$;*

(d) *a morphism $\eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$ belongs to the image of the map $\mathfrak{s}_0$ if and only if there exists an admissible epimorphism $X' \longrightarrow X$ in the category $\mathcal{F}$ such that the composition $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$ is divisible by the natural transformation $\mathfrak{s}$, and if and only if there exists an admissible monomorphism $Y \longrightarrow Y'$ such that the composition $\eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y) \longrightarrow \eta_{\mathfrak{st}}(Y')$ is divisible by $\mathfrak{s}$.*

Proof. Part (a) is immediate from the construction. In part (b), one checks the equations $\mathfrak{s}_0(\eta_\mathfrak{t}(g(-1))p) = \eta_{\mathfrak{st}}(g)\mathfrak{s}_0(p)$ and $\mathfrak{s}_0(p\eta_\mathfrak{t}(h)) = \mathfrak{s}_0(p)\eta_{\mathfrak{st}}(h)$ separately, using the construction of the morphism $\mathfrak{s}_0(p)$ in terms of an admissible epimorphism $X' \longrightarrow X$ in the former case and in terms of an admissible monomorphism $Y \longrightarrow Y'$ in the latter one. Part (c) holds, since the morphisms $X' \longrightarrow Y$ and $X \longrightarrow Y'$ are annihilated by the functor $\eta_\mathfrak{t}$ whenever their images under the functor $\eta_{\mathfrak{st}}$ are annihilated by the multiplication with the natural transformation $\mathfrak{s}$, according to the condition (III).

The assertions "only if" in part (d) are obvious from the construction of the map $\mathfrak{s}_0$. To prove the "if", suppose that we are given a morphism $X' \longrightarrow Y(-1)$ in the category $\mathcal{F}$ whose image under the functor $\eta_{\mathfrak{st}}$, multiplied with $\mathfrak{s}$, is equal to the composition $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$. Denote by K the kernel of the morphism $X' \longrightarrow X$ and consider the composition $K \longrightarrow X' \longrightarrow Y(-1)$. The image of this composition under the functor $\eta_{\mathfrak{st}}$ is annihilated by the multiplication with $\mathfrak{s}$, and consequently, according to (III), the morphism $K \longrightarrow Y(-1)$ is annihilated by the functor $\eta_\mathfrak{t}$. The short sequence $0 \longrightarrow \eta_\mathfrak{t}(K) \longrightarrow \eta_\mathfrak{t}(X') \longrightarrow \eta_\mathfrak{t}(X) \longrightarrow 0$ being exact in $\mathcal{F}_\mathfrak{t}$, one obtains the desired morphism $\eta_\mathfrak{t}(X) \longrightarrow \eta_\mathfrak{t}(Y)(-1)$. ☐

In particular, it follows from part (d) of the Lemma that the two formulations of the condition (IV) in Subsection 1.3 are equivalent to each other, as are the two formulations of the condition (iv) in Subsection 1.1.

1.5. **The second term.** Let us construct the map $r_\mathfrak{t}^0$ and verify exactness of our sequence at its second nontrivial term.

Let X and Y be two objects of the category $\mathcal{F}$, and $f\colon \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(Y)$ be a morphism in the category $\mathcal{F}_{\mathfrak{st}}$. According to the condition (ii) for the functor $\eta_{\mathfrak{st}}$, there exist an admissible epimorphism $X' \longrightarrow X$, an admissible monomorphism $Y \longrightarrow Y'$, and morphisms $X' \longrightarrow Y$ and $X \longrightarrow Y'$ in the category $\mathcal{F}$ whose images under the functor $\eta_{\mathfrak{st}}$ together with the morphism f form a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{st}}$.

Let $K \longrightarrow X'$ be the kernel of the admissible epimorphism $X' \longrightarrow X$ and $Y' \longrightarrow C$ be the cokernel of the admissible monomorphism $Y \longrightarrow Y'$. Then the compositions of morphisms $K \longrightarrow X' \longrightarrow Y$ and $X \longrightarrow Y' \longrightarrow C$ in the category $\mathcal{F}$ are annihilated by the functor $\eta_{\mathfrak{st}}$, so it follows from (either formulation of) the condition (IV) that they are also annihilated by the functor $\eta_\mathfrak{s}$. The same applies to the difference of the two compositions $X' \longrightarrow X \longrightarrow Y'$ and $X' \longrightarrow Y \longrightarrow Y'$ in the category $\mathcal{F}$. Hence the image of our square of morphisms in the category $\mathcal{F}$ with respect to the functor $\eta_\mathfrak{s}$ is commutative in the category $\mathcal{F}_\mathfrak{s}$, and there is a unique morphism $q\colon \eta_\mathfrak{s}(X) \longrightarrow \eta_\mathfrak{s}(Y)$ complementing this square to a commutative diagram of two triangles with a common edge in $\mathcal{F}_\mathfrak{s}$. By the definition, we set $r_\mathfrak{t}^0(f) = q$.

Lemma 1.4. *Assuming the condition (ii) for the functors $\eta_\mathfrak{t}$, $\eta_{\mathfrak{st}}$ and the conditions (III–IV), the map $r_\mathfrak{t}^0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Hom}_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X), \eta_\mathfrak{s}(Y))$ has the following properties:*

(a) *the equation $r_\mathfrak{t}^0(\eta_{\mathfrak{st}}(e)) = \eta_\mathfrak{s}(e)$ and holds in the category $\mathcal{F}_\mathfrak{s}$ for any morphism $e\colon X \longrightarrow Y$ in the category $\mathcal{F}$;*

(b) *the equation* $r^0_{\mathfrak t}(gh)=r^0_{\mathfrak t}(g)r^0_{\mathfrak t}(h)$ *holds in the category* $\mathcal F_{\mathfrak s}$ *for any two morphisms* $g\colon\eta_{\mathfrak{st}}(V)\longrightarrow\eta_{\mathfrak{st}}(W)$ *and* $h\colon\eta_{\mathfrak{st}}(U)\longrightarrow\eta_{\mathfrak{st}}(V)$ *in the category* $\mathcal F_{\mathfrak{st}}$*;*

(c) *for any two objects* X, Y *in the category* $\mathcal F$, *the image of the injection* $\mathfrak s_0\colon\operatorname{Hom}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$ *coincides with the kernel of the map* $r^0_{\mathfrak t}\colon\operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))\longrightarrow\operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$.

Proof. Part (a) is obvious from the construction. In part (b), one first considers the situation when one of the morphisms g or h comes from a morphism in the category $\mathcal F$ via the functor $\eta_{\mathfrak{st}}$, proving the equation $r^0_{\mathfrak t}(\eta_{\mathfrak{st}}(j)h)=\eta_{\mathfrak s}(j)r^0_{\mathfrak t}(h)$ using the construction of the morphism $r^0_{\mathfrak t}(h)$ in terms of an admissible epimorphism $U'\longrightarrow U$ and the equation and $r^0_{\mathfrak t}(g\eta_{\mathfrak{st}}(k))=r^0_{\mathfrak t}(g)\eta_{\mathfrak s}(k)$ using the construction of the morphism $r^0_{\mathfrak t}(g)$ in terms of an admissible monomorphism $W\longrightarrow W'$. Then one deduces and applies a common generalization of our two constructions of the map $r^0_{\mathfrak t}$, in which, to obtain the morphism $r^0_{\mathfrak t}(f)\in\operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$ for a given morphism $f\in\operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$, one picks an admissible epimorphism $X'\longrightarrow X$ and an admissible monomorphism $Y\longrightarrow Y'$ in the category $\mathcal F$ for which the composition $\eta_{\mathfrak{st}}(X')\longrightarrow\eta_{\mathfrak{st}}(X)\longrightarrow\eta_{\mathfrak{st}}(Y)\longrightarrow\eta_{\mathfrak{st}}(Y')$ comes from a morphism $X'\longrightarrow Y'$ in the category $\mathcal F$. The morphism $r^0_{\mathfrak t}(f)$ is the unique morphism $\eta_{\mathfrak s}(X)\longrightarrow\eta_{\mathfrak s}(Y)$ complementing to a commutative diagram the images of the morphisms $X'\longrightarrow X$, $Y\longrightarrow Y'$, and $X'\longrightarrow Y'$ under the functor $\eta_{\mathfrak s}$.

To prove part (c), consider a morphism $f\colon\eta_{\mathfrak{st}}(X)\longrightarrow\eta_{\mathfrak{st}}(Y)$ in the category $\mathcal F_{\mathfrak{st}}$. Let $X'\longrightarrow X$ and $X'\longrightarrow Y$ be an admissible epimorphism and a morphism in the category $\mathcal F$ whose images under the functor $\eta_{\mathfrak{st}}$ form a commutative diagram with the morphism f. The equation $r^0_{\mathfrak t}(f)=0$ means that the morphism $X'\longrightarrow Y$ is annihilated by the functor $\eta_{\mathfrak s}$. According to the condition (IV), this is equivalent to the existence of an admissible epimorphism $X''\longrightarrow X'$ in the category $\mathcal F$ for which the composition $\eta_{\mathfrak{st}}(X'')\longrightarrow\eta_{\mathfrak{st}}(X')\longrightarrow\eta_{\mathfrak{st}}(Y)$ is divisible by $\mathfrak s$. If this is the case, then it follows by the way of Lemma 1.3(d) that the morphism f belongs to the image of the map $\mathfrak s_0$ (as the composition $X''\longrightarrow X'\longrightarrow X$ is also an admissible epimorphism in $\mathcal F$). Conversely, by Lemma 1.3(b) the composition $\eta_{\mathfrak{st}}(X')\longrightarrow\eta_{\mathfrak{st}}(X)\longrightarrow\eta_{\mathfrak{st}}(Y)$ belongs to the image of the map $\mathfrak s_0$ whenever the morphism f does. Then it remains to apply Lemma 1.3(d) in order to deduce the existence of an admissible epimorphism $X''\longrightarrow X'$ with the desired property. ☐

The result of Lemma 1.4(b) says that for any commutative diagram in the category $\mathcal F_{\mathfrak{st}}$ with the objects in the vertices coming from the category $\mathcal F$ via the functor $\eta_{\mathfrak{st}}$, one can apply the maps $r^0_{\mathfrak t}$ to every arrow in the diagram, obtaining a commutative diagram in the category $\mathcal F_{\mathfrak s}$. By part (a) of the Lemma, when some of the arrows in the original diagram come from arrows in the category $\mathcal F_{\mathfrak{st}}$, the procedure is compatible with the action of the functors $\eta_{\mathfrak{st}}$ and $\eta_{\mathfrak s}$ on such arrows.

1.6. **The third term.** Now we construct the map ∂^0 and check exactness of the sequence at its third term. The construction and arguments largely follow those in [12, Subsections 4.5–4.6].

Let X and Y be two objects of the category $\mathcal{F}$, and $q\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ be a morphism in the category $\mathcal{F}_{\mathfrak{s}}$. According to the condition (ii) for the functor $\eta_{\mathfrak{s}}$, there exist an admissible epimorphism $X' \longrightarrow X$, an admissible monomorphism $Y \longrightarrow Y'$, and morphisms $X' \longrightarrow Y$ and $X \longrightarrow Y'$ in the category $\mathcal{F}$ whose images under the functor $\eta_{\mathfrak{s}}$ together with the morphism q form a commutative diagram of two triangles with a common edge in the category $\mathcal{F}_{\mathfrak{s}}$.

The square diagram of morphisms $X' \longrightarrow X \longrightarrow Y'$, $X' \longrightarrow Y \longrightarrow Y'$ becomes commutative after applying the functor $\eta_{\mathfrak{s}}$, and consequently, according to Lemma 1.4(a,c), its image under the functor $\eta_{\mathfrak{st}}$ is commutative in the category $\mathcal{F}_{\mathfrak{st}}$ up to a morphism coming from a morphism in $\mathcal{F}_{\mathfrak{t}}$ via the map $\mathfrak{s}_0$.

Let $K \longrightarrow X'$ be the kernel of the morphism $X' \longrightarrow X$ and $Y' \longrightarrow C$ be the cokernel of the morphism $Y \longrightarrow Y'$ in $\mathcal{F}$. Then the compositions $K \longrightarrow X' \longrightarrow Y$ and $X \longrightarrow Y' \longrightarrow C$ are annihilated by the functor $\eta_{\mathfrak{s}}$, and therefore their images under the functor $\eta_{\mathfrak{st}}$ come from morphisms $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ and $\eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(C)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$. The difference of the two compositions in the square diagram of morphisms in $\mathcal{F}_{\mathfrak{st}}$ comes from a morphism $\eta_{\mathfrak{t}}(X') \longrightarrow \eta_{\mathfrak{t}}(Y')(-1)$ in $\mathcal{F}_{\mathfrak{t}}$.

Together with the images of the short exact sequences $0 \longrightarrow K \longrightarrow X' \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y(-1) \longrightarrow Y'(-1) \longrightarrow C(-1) \longrightarrow 0$ with respect to the functor $\eta_{\mathfrak{t}}$, these three morphisms form a diagram of two squares, one of which is commutative and the other one anticommutative (as one can check using Lemma 1.3(b-c)). Such a diagram defines an element of the group $\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ in any one of the two dual ways differing by the minus sign (cf. Lemma 0.2).

Namely, the desired element can be obtained either as the composition of the Ext^1 class of the sequence $0 \longrightarrow \eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(X') \longrightarrow \eta_{\mathfrak{t}}(X) \longrightarrow 0$ with the morphism $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$, or as the composition of the morphism $\eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(C)(-1)$ with the Ext^1 class of the sequence $0 \longrightarrow \eta_{\mathfrak{t}}(Y)(-1) \longrightarrow \eta_{\mathfrak{t}}(Y')(-1) \longrightarrow \eta_{\mathfrak{t}}(C)(-1) \longrightarrow 0$ in the exact category $\mathcal{F}_{\mathfrak{t}}$. By the definition, we set this element to be the value $\partial^0(q)$ of the map $\partial^0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ at the morphism $q\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$.

Lemma 1.5. *Assuming the condition (ii) for the functors $\eta_{\mathfrak{t}}$, $\eta_{\mathfrak{st}}$, $\eta_{\mathfrak{s}}$ and the conditions (III–IV), the map $\partial^0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ has the following properties:*

(a) *the equation $\partial^0(\eta_{\mathfrak{s}}(g)q\eta_{\mathfrak{s}}(h)) = \eta_{\mathfrak{t}}(g(-1))\partial^0(q)\eta_{\mathfrak{t}}(h)$ holds in the category $\mathcal{F}_{\mathfrak{t}}$ for any two morphisms $h\colon U \longrightarrow X$, $g\colon Y \longrightarrow V$ in the category $\mathcal{F}$ and any morphism $q\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ in the category $\mathcal{F}_{\mathfrak{s}}$;*

(b) *for any two objects X, Y in the category $\mathcal{F}$, the kernel of the map $\partial^0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ coincides with the image of the map $r^0_{\mathfrak{t}}\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$.*

Proof. To prove part (a), one checks the equations $\partial^0(\eta_{\mathfrak{s}}(g)q) = \eta_{\mathfrak{t}}(g(-1))\partial^0(q)$ and $\partial^0(q\eta_{\mathfrak{s}}(h)) = \partial^0(q)\eta_{\mathfrak{t}}(h)$ separately, using the construction of the element $\partial^0(q)$ (as the product of a morphism and and an Ext^1 class in $\mathcal{F}_{\mathfrak{t}}$) in terms of an admissible epimorphism $X' \longrightarrow X$ in the former case and in terms of an admissible monomorphism $Y \longrightarrow Y'$ in the latter one, together with the result of Lemma 1.3(b).

To prove part (b), consider a morphism $q\colon \eta_{\mathfrak s}(X)\longrightarrow \eta_{\mathfrak s}(Y)$ in the category $\mathcal F_{\mathfrak s}$, and let $X'\longrightarrow X$ and $X'\longrightarrow Y$ be an admissible epimorphism and a morphism in the category $\mathcal F$ whose images under the functor $\eta_{\mathfrak s}$ form a commutative diagram together with the morphism q. Let $K\longrightarrow X'$ be the kernel of the morphism $X'\longrightarrow X$. According to the above, the image of the composition of morphisms $K\longrightarrow X'\longrightarrow Y$ in the category $\mathcal F$ under the functor $\eta_{\mathfrak{st}}$ comes from a morphism $\eta_{\mathfrak t}(K)\longrightarrow \eta_{\mathfrak t}(Y)(-1)$ in the category $\mathcal F_{\mathfrak t}$ via the map $\mathfrak s_0$.

The class $\partial^0(q)\in \operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$ is induced from the Ext^1 class of the short exact sequence $0\longrightarrow\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(X')\longrightarrow\eta_{\mathfrak t}(X)\longrightarrow 0$ using the morphism $\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(Y)(-1)$. Hence one has $\partial^0(q)=0$ if and ony if the latter morphism factorizes through the admissible monomorphism $\eta_{\mathfrak t}(K)\longrightarrow\eta_{\mathfrak t}(X')$.

Subtracting the image of the related morphism $\eta_{\mathfrak t}(X')\longrightarrow\eta_{\mathfrak t}(Y)(-1)$ under the map $\mathfrak s_0$ from the image of the morphism $X'\longrightarrow Y$ under the functor $\eta_{\mathfrak{st}}$, we obtain a morphism $f'\colon\eta_{\mathfrak{st}}(X')\longrightarrow\eta_{\mathfrak{st}}(Y)$ in the category $\mathcal F_{\mathfrak{st}}$ whose composition with the image of the admissible monomorphism $K\longrightarrow X'$ under the functor $\eta_{\mathfrak{st}}$ vanishes (as one can compute using Lemma 1.3(b)). Hence the desired morphism $f\colon\eta_{\mathfrak{st}}(X)\longrightarrow\eta_{\mathfrak{st}}(Y)$ in the category $\mathcal F_{\mathfrak{st}}$. Since $r^0_{\mathfrak t}\circ\mathfrak s_0=0$, one has $r^0_{\mathfrak t}(f')=\eta_{\mathfrak s}(X'\to Y)$ in the category $\mathcal F_{\mathfrak s}$, and it follows that $r^0_{\mathfrak t}(f)=q$. ☐

1.7. **Construction of higher differentials.** The constructions of the maps $\mathfrak s_n$, $r^n_{\mathfrak t}$, and ∂^n for $n\geqslant 1$ are based on the result of Proposition 0.5(b). We continue to follow [12, Subsection 4.5].

Lemma 1.6. *Assuming the conditions (i–IV), there exists a unique way to extend the maps* $\mathfrak s_0\colon \operatorname{Hom}_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$ *of Subsection 1.4 to maps* $\mathfrak s_n\colon \operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$ *defined for all objects* X, $Y\in\mathcal F$ *and all integers* $n\geqslant 0$ *and satisfying the equations (b) of Subsection 1.3.*

Proof. Consider the two equations $\mathfrak s_{i+n}(\eta^i_{\mathfrak t}(a(-1))z)=\eta^i_{\mathfrak{st}}(a)\mathfrak s_n(z)$ and $\mathfrak s_{n+j}(z\eta^j_{\mathfrak t}(b))=\mathfrak s_n(z)\eta^j_{\mathfrak{st}}(b)$ separately. In view of Proposition 0.5(b) and the dual result, based on the conditions (i′) and (i″) for the functor $\eta_{\mathfrak t}$, it follows from Lemma 1.3(b) that there exists a unique collection of maps $\mathfrak s''_n\colon \operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$ extending the maps $\mathfrak s_0$ and satisfying the former system of equations, and also a unique collection of maps $\mathfrak s'_n\colon \operatorname{Ext}^n_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))\longrightarrow\operatorname{Ext}^n_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))$ extending the maps $\mathfrak s_0$ and satisfying the latter system of equations.

It remains to show that $\mathfrak s'_n=\mathfrak s''_n$; here it suffices to check that $\mathfrak s'_1=\mathfrak s''_1$. Suppose that we are given two short exact sequences $0\longrightarrow V\longrightarrow P\longrightarrow X\longrightarrow 0$ and $0\longrightarrow Y\longrightarrow Q\longrightarrow U\longrightarrow 0$ in the category $\mathcal F$ representing the Ext^1 classes $b\in\operatorname{Ext}^1_{\mathcal F}(X,V)$ and $a\in\operatorname{Ext}^1_{\mathcal F}(U,Y)$. Suppose further that we are given two morphisms $w\colon\eta_{\mathfrak t}(V)\longrightarrow\eta_{\mathfrak t}(Y)(-1)$ and $z\colon\eta_{\mathfrak t}(X)\longrightarrow\eta_{\mathfrak t}(U)(-1)$ in the category $\mathcal F_{\mathfrak t}$ such that the equation $\eta^1_{\mathfrak t}(a(-1))z=w\eta^1_{\mathfrak t}(b)$ holds in the group $\operatorname{Ext}^1_{\mathcal F_{\mathfrak t}}(\eta_{\mathfrak t}(X),\eta_{\mathfrak t}(Y)(-1))$. Then the morphisms w and z can be extended to a morphism of short exact sequences (that is a diagram of two commutative squares) $(\eta_{\mathfrak t}(V)\to\eta_{\mathfrak t}(P)\to\eta_{\mathfrak t}(X))\longrightarrow(\eta_{\mathfrak t}(Y)(-1)\to\eta_{\mathfrak t}(Q)(-1)\to\eta_{\mathfrak t}(U)(-1))$ in the category $\mathcal F_{\mathfrak t}$ (see Lemma 0.2).

Applying the maps $\mathfrak{s}_0$ to the morphisms $w\colon \eta_{\mathfrak{t}}(V) \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$, $\eta_{\mathfrak{t}}(P) \longrightarrow \eta_{\mathfrak{t}}(Q)(-1)$, and $z\colon \eta_{\mathfrak{t}}(X) \longrightarrow \eta_{\mathfrak{t}}(U)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$, we obtain, in view of Lemma 1.3(b), a morphism of short exact sequences $(\eta_{\mathfrak{st}}(V) \to \eta_{\mathfrak{st}}(P) \to \eta_{\mathfrak{st}}(X)) \longrightarrow (\eta_{\mathfrak{st}}(Y) \to \eta_{\mathfrak{st}}(Q) \to \eta_{\mathfrak{st}}(U))$ in the category $\mathcal{F}_{\mathfrak{st}}$. The commutativity of this diagram of two squares proves the desired equation $\eta^1_{\mathfrak{st}}(a)\mathfrak{s}_0(z) = \mathfrak{s}_0(w)\eta^1_{\mathfrak{st}}(b)$ in the group $\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$. ☐

Lemma 1.7. *Assuming the conditions (i–IV), there exists a unique way to extend the maps* $r^0_{\mathfrak{t}}\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ *of Subsection 1.5 to maps* $r^n_{\mathfrak{t}}\colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ *defined for all objects X, $Y \in \mathcal{F}$ and all integers $n \geqslant 0$ and satisfying the equations (a) of Subsection 1.3.*

Proof. Consider first two weaker systems of equations $r^{i+n}_{\mathfrak{t}}(\eta^i_{\mathfrak{st}}(a)z) = \eta^i_{\mathfrak{s}}(a)r^n_{\mathfrak{t}}(z)$ and $r^{n+j}_{\mathfrak{t}}(z\eta^j_{\mathfrak{st}}(b)) = r^n_{\mathfrak{t}}(z)\eta^j_{\mathfrak{s}}(b)$ for any objects U, X, Y, $V \in \mathcal{F}$ and any Ext classes $a \in \operatorname{Ext}^i_{\mathcal{F}}(Y, V)$, $b \in \operatorname{Ext}^j_{\mathcal{F}}(U, X)$, and $z \in \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$. By the way of Proposition 0.5(b) and the dual result, based on the conditions (i′) and (i″) for the functor $\eta_{\mathfrak{st}}$, it follows from Lemma 1.4(a-b) that there exists a unique collection of maps ${}''r^n_{\mathfrak{t}}\colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ extending the maps $r^0_{\mathfrak{t}}$ and satisfying the former system of equations, and also a unique collection of maps ${}'r^n_{\mathfrak{t}}\colon \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y)) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ extending the maps $r^0_{\mathfrak{t}}$ and satisfying the latter system of equations.

In order to show that ${}'r^n_{\mathfrak{t}} = {}''r^n_{\mathfrak{t}}$ for all $n \geqslant 1$, it suffices to check that ${}'r^1_{\mathfrak{t}} = {}''r^1_{\mathfrak{t}}$. As in the previous proof, we have two short exact sequences $0 \longrightarrow V \longrightarrow P \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Q \longrightarrow U \longrightarrow 0$ in the category $\mathcal{F}$ representing the Ext^1 classes $b \in \operatorname{Ext}^1_{\mathcal{F}}(X, V)$ and $a \in \operatorname{Ext}^1_{\mathcal{F}}(U, Y)$. We also have two morphisms $w\colon \eta_{\mathfrak{st}}(V) \longrightarrow \eta_{\mathfrak{st}}(Y)$ and $z\colon \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(U)$ in the category $\mathcal{F}_{\mathfrak{st}}$ for which the equation $\eta^1_{\mathfrak{st}}(a)z = w\eta^1_{\mathfrak{st}}(b)$ holds in $\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$. Then there is a morphism of short exact sequences $(\eta_{\mathfrak{st}}(V) \to \eta_{\mathfrak{st}}(P) \to \eta_{\mathfrak{st}}(X)) \longrightarrow (\eta_{\mathfrak{st}}(Y) \to \eta_{\mathfrak{st}}(Q) \to \eta_{\mathfrak{st}}(U))$ in the category $\mathcal{F}_{\mathfrak{st}}$.

Applying the maps $r^0_{\mathfrak{t}}$ to this commutative diagram in the category $\mathcal{F}_{\mathfrak{st}}$ with the objects in the vertices coming from the category $\mathcal{F}$ via the functor $\eta_{\mathfrak{st}}$, we obtain, in view of Lemma 1.4(a-b), a morphism of short exact sequences $(\eta_{\mathfrak{s}}(V) \to \eta_{\mathfrak{s}}(P) \to \eta_{\mathfrak{s}}(X)) \longrightarrow (\eta_{\mathfrak{s}}(Y) \to \eta_{\mathfrak{s}}(Q) \to \eta_{\mathfrak{s}}(U))$ in the category $\mathcal{F}_{\mathfrak{s}}$. Commutativity of the latter diagram in the category $\mathcal{F}_{\mathfrak{s}}$ proves the desired equation $\eta^1_{\mathfrak{s}}(a)r^0_{\mathfrak{t}}(z) = r^0_{\mathfrak{t}}(w)\eta^1_{\mathfrak{s}}(b)$ in the group $\operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$.

Now, again by Proposition 0.5(b) and its dual assertion for the functor $\eta_{\mathfrak{st}}$, for any three objects U, V, W in the category $\mathcal{F}$ and any two Ext classes $y \in \operatorname{Ext}^j_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(U), \eta_{\mathfrak{st}}(V))$ and $x \in \operatorname{Ext}^i_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(V), \eta_{\mathfrak{st}}(W))$ in the category $\mathcal{F}_{\mathfrak{st}}$, one can find two morphisms $g\colon \eta_{\mathfrak{st}}(U') \longrightarrow \eta_{\mathfrak{st}}(V)$ and $f\colon \eta_{\mathfrak{st}}(V) \longrightarrow \eta_{\mathfrak{st}}(W')$ in $\mathcal{F}_{\mathfrak{st}}$ and two Ext classes $b \in \operatorname{Ext}^j_{\mathcal{F}}(U, U')$ and $a \in \operatorname{Ext}^i_{\mathcal{F}}(W', W)$ in $\mathcal{F}$ such that $y = g\eta^j_{\mathfrak{st}}(b)$ and $x = \eta^i_{\mathfrak{st}}(a)f$. Finally, we have $r^{i+j}_{\mathfrak{t}}(xy) = r^n_{\mathfrak{t}}\big(\eta^i_{\mathfrak{st}}(a)fg\eta^j_{\mathfrak{st}}(b)\big) = \eta^i_{\mathfrak{s}}(a)r^0_{\mathfrak{t}}(fg)\eta^j_{\mathfrak{s}}(b) = \eta^i_{\mathfrak{s}}(a)r^0_{\mathfrak{t}}(f)r^0_{\mathfrak{t}}(g)\eta^j_{\mathfrak{s}}(b) = r^i_{\mathfrak{t}}(x)r^j_{\mathfrak{t}}(y)$ in $\operatorname{Ext}^{i+j}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(U), \eta_{\mathfrak{s}}(W))$. ☐

Lemma 1.8. *Assuming the conditions (i–IV), there exists a unique way to extend the maps* $\partial^0\colon \operatorname{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y)) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ *of Subsection 1.6 to* *maps* $\partial^n\colon \operatorname{Ext}^n_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X),\eta_\mathfrak{s}(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}_\mathfrak{t}}(\eta_\mathfrak{t}(X),\eta_\mathfrak{t}(Y)(-1))$ *defined for all objects* X, $Y\in\mathcal{F}$ *and all integers* $n\geqslant 0$ *and satisfying the equations (c) of Subsection 1.3.*

Proof. As in the proof of Lemma 1.6, we consider the two equations $\partial^{i+n}(\eta^i_\mathfrak{s}(a)z) = (-1)^i\eta^i_\mathfrak{t}(a(-1))\partial^n(z)$ and $\partial^{n+j}(z\eta^j_\mathfrak{s}(b)) = \partial^n(z)\eta^j_\mathfrak{t}(b)$ separately. In view of Proposition 0.5(b) and the dual result, based on the conditions (i′) and (i″) for the functor $\eta_\mathfrak{s}$, it follows from Lemma 1.5(a) that there exists a unique collection of maps $''\partial^n\colon \operatorname{Ext}^n_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X),\eta_\mathfrak{s}(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}_\mathfrak{t}}(\eta_\mathfrak{t}(X),\eta_\mathfrak{t}(Y)(-1))$ extending the maps ∂^0 and satisfying the former system of equations, and also a unique collection of maps $'\partial^n\colon \operatorname{Ext}^n_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X),\eta_\mathfrak{s}(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}_\mathfrak{t}}(\eta_\mathfrak{t}(X),\eta_\mathfrak{t}(Y)(-1))$ extending the maps ∂^0 and satisfying the latter system of equations.

In order to show that $'\partial^n = {}''\partial^n$ for all $n\geqslant 1$, it suffices to check that $'\partial^1={}''\partial^1$. As in the proofs of the preceding lemmas in this subsection, we have two short exact sequences $0\longrightarrow V\longrightarrow P\longrightarrow X\longrightarrow 0$ and $0\longrightarrow Y\longrightarrow Q\longrightarrow U\longrightarrow 0$ in the category $\mathcal{F}$ representing the Ext^1 classes $b\in\operatorname{Ext}^1_\mathcal{F}(X,V)$ and $a\in\operatorname{Ext}^1_\mathcal{F}(U,Y)$. We have two morphisms $w\colon \eta_\mathfrak{s}(V)\longrightarrow\eta_\mathfrak{s}(Y)$ and $z\colon\eta_\mathfrak{s}(X)\longrightarrow\eta_\mathfrak{s}(U)$ in the category $\mathcal{F}_\mathfrak{s}$ for which the equation $\eta^1_\mathfrak{s}(a)z = w\eta^1_\mathfrak{s}(b)$ holds in $\operatorname{Ext}^1_{\mathcal{F}_\mathfrak{s}}(\eta_\mathfrak{s}(X),\eta_\mathfrak{s}(Y))$. Then there is a morphism of short exact sequences $(\eta_\mathfrak{s}(V)\to\eta_\mathfrak{s}(P)\to\eta_\mathfrak{s}(X))\longrightarrow(\eta_\mathfrak{s}(Y)\to\eta_\mathfrak{s}(Q)\to\eta_\mathfrak{s}(U))$ in the category $\mathcal{F}_\mathfrak{s}$.

According to the condition (ii) for the functor $\eta_\mathfrak{s}$, there exists an admissible epimorphism $X'\longrightarrow X$ and a morphism $X'\longrightarrow U$ in the category $\mathcal{F}$ whose images under the functor $\eta_\mathfrak{s}$ form a commutative diagram with the morphism $\eta_\mathfrak{s}(X)\longrightarrow\eta_\mathfrak{s}(U)$ in the category $\mathcal{F}_\mathfrak{s}$. Denote by P''' the fibered product of the objects P and X' over X in the category $\mathcal{F}$. Choose an admissible epimorphism $P''\longrightarrow P'''$ and a morphism $P''\longrightarrow Q$ in the category $\mathcal{F}$ whose images under $\eta_\mathfrak{s}$ form a commutative diagram with the composition of morphisms $\eta_\mathfrak{s}(P''')\longrightarrow\eta_\mathfrak{s}(P)\longrightarrow\eta_\mathfrak{s}(Q)$ in $\mathcal{F}_\mathfrak{s}$.

Consider the difference of the compositions of morphisms $P''\longrightarrow P'''\longrightarrow X\longrightarrow U$ and $P''\longrightarrow Q\longrightarrow U$ in the category $\mathcal{F}$. It is annihilated by the functor $\eta_\mathfrak{s}$, and consequently, its image under the functor $\eta_{\mathfrak{st}}$ comes from a morphism $\eta_\mathfrak{t}(P'')\longrightarrow\eta_\mathfrak{t}(U)(-1)$ in the category $\mathcal{F}_\mathfrak{t}$ via the map $\mathfrak{s}_0$. Denote by T the fibered product of the objects $\eta_\mathfrak{t}(P'')$ and $\eta_\mathfrak{t}(Q)(-1)$ over $\eta_\mathfrak{t}(U)(-1)$ in the category $\mathcal{F}_\mathfrak{t}$. The morphism $T\longrightarrow\eta_\mathfrak{t}(P'')$ is an admissible epimorphism in $\mathcal{F}_\mathfrak{t}$; hence, according to the condition (i) for the functor $\eta_\mathfrak{t}$, there exists an admissible epimorphism $P'\longrightarrow P''$ in the category $\mathcal{F}$ and a morphism $\eta_\mathfrak{t}(P')\longrightarrow T$ in the category $\mathcal{F}_\mathfrak{t}$ making the triangle diagram $\eta_\mathfrak{t}(P')\longrightarrow T\longrightarrow\eta_\mathfrak{t}(P'')$ commutative in $\mathcal{F}_\mathfrak{t}$.

Applying the map $\mathfrak{s}_0$ to the composition of morphisms $\eta_\mathfrak{t}(P')\longrightarrow T\longrightarrow\eta_\mathfrak{t}(Q)(-1)$ in the category $\mathcal{F}_\mathfrak{t}$, we obtain a morphism $f\colon\eta_{\mathfrak{st}}(P')\longrightarrow\eta_{\mathfrak{st}}(Q)$ entering into a commutative square of morphisms $\eta_{\mathfrak{st}}(P')\longrightarrow\eta_{\mathfrak{st}}(Q)\longrightarrow\eta_{\mathfrak{st}}(U)$ and $\eta_{\mathfrak{st}}(P')\longrightarrow\eta_{\mathfrak{st}}(P'')\longrightarrow\eta_{\mathfrak{st}}(U)$ in the category $\mathcal{F}_{\mathfrak{st}}$. Here the morphisms $\eta_{\mathfrak{st}}(Q)\longrightarrow\eta_{\mathfrak{st}}(U)$ and $\eta_{\mathfrak{st}}(P')\longrightarrow\eta_{\mathfrak{st}}(P'')\longrightarrow\eta_{\mathfrak{st}}(U)$ come from morphisms in the category $\mathcal{F}$ via the functor $\eta_{\mathfrak{st}}$, while at the same time the morphisms $\eta_{\mathfrak{st}}(P'')\longrightarrow\eta_{\mathfrak{st}}(U)$ and $f\colon\eta_{\mathfrak{st}}(P')\longrightarrow\eta_{\mathfrak{st}}(Q)$ come from morphisms in the category $\mathcal{F}_\mathfrak{t}$ via the maps $\mathfrak{s}_0$, and are, consequently, annihilated by the maps $r^0_\mathfrak{t}$.

Define the morphism $P' \longrightarrow P$ as the composition $P' \longrightarrow P'' \longrightarrow P''' \longrightarrow P$ and the morphism $P' \longrightarrow X'$ as the composition $P' \longrightarrow P'' \longrightarrow P''' \longrightarrow X'$ of morphisms in the category $\mathcal{F}$. Let $\eta_{\mathfrak{st}}(P') \longrightarrow \eta_{\mathfrak{st}}(X')$, $\eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(U)$, and $\eta_{\mathfrak{st}}(Q) \longrightarrow \eta_{\mathfrak{st}}(U)$ be the images of the morphisms $P' \longrightarrow X'$, $X' \longrightarrow U$, and $Q \longrightarrow U$ under the functor $\eta_{\mathfrak{st}}$. Furthermore, set the new morphism $\eta_{\mathfrak{st}}(P') \longrightarrow \eta_{\mathfrak{st}}(Q)$ to be the sum of the image of the composition $P' \longrightarrow P'' \longrightarrow Q$ under the functor $\eta_{\mathfrak{st}}$ and the morphism f. Then the square diagram formed by the morphisms $\eta_{\mathfrak{st}}(P') \longrightarrow \eta_{\mathfrak{st}}(X') \longrightarrow \eta_{\mathfrak{st}}(U)$ and $\eta_{\mathfrak{st}}(P') \longrightarrow \eta_{\mathfrak{st}}(Q) \longrightarrow \eta_{\mathfrak{st}}(U)$ is commutative in the category $\mathcal{F}_{\mathfrak{st}}$, while the triangle diagram $\eta_{\mathfrak{s}}(P') \longrightarrow \eta_{\mathfrak{s}}(P) \longrightarrow \eta_{\mathfrak{s}}(Q)$ formed by the morphism $\eta_{\mathfrak{s}}(P) \longrightarrow \eta_{\mathfrak{s}}(Q)$, the image of the morphism $P' \longrightarrow P$ under the functor $\eta_{\mathfrak{s}}$, and the image of the morphism $\eta_{\mathfrak{st}}(P') \longrightarrow \eta_{\mathfrak{st}}(Q)$ under the map $r^0_{\mathfrak{t}}$ is commutative in the category $\mathcal{F}_{\mathfrak{s}}$.

Let $V' \longrightarrow P'$ be the kernel of the admissible epimorphism $P' \longrightarrow X'$ in the category $\mathcal{F}$. Then there is an admissible epimorphism of short exact sequences $(V' \to P' \to X') \longrightarrow (V \to P \to X)$ in the category $\mathcal{F}$ and a morphism of short exact sequences $(\eta_{\mathfrak{st}}(V') \to \eta_{\mathfrak{st}}(P') \to \eta_{\mathfrak{st}}(X')) \longrightarrow (\eta_{\mathfrak{st}}(Y) \to \eta_{\mathfrak{st}}(Q) \to \eta_{\mathfrak{st}}(U))$ in the category $\mathcal{F}_{\mathfrak{st}}$ whose images under the functor $\eta_{\mathfrak{s}}$ and the maps $r^0_{\mathfrak{t}}$ form a commutative triangle with the morphism of short exact sequences $(\eta_{\mathfrak{s}}(V) \to \eta_{\mathfrak{s}}(P) \to \eta_{\mathfrak{s}}(X)) \longrightarrow (\eta_{\mathfrak{s}}(Y) \to \eta_{\mathfrak{s}}(Q) \to \eta_{\mathfrak{s}}(U))$ in the category $\mathcal{F}_{\mathfrak{s}}$. Let $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ be the kernel of the admissible epimorphism $(V' \to P' \to X') \longrightarrow (V \to P \to X)$ (in the exact category) of short exact sequences in $\mathcal{F}$. Then the composition of morphisms of short exact sequences $(\eta_{\mathfrak{st}}(K) \to \eta_{\mathfrak{st}}(L) \to \eta_{\mathfrak{st}}(M)) \longrightarrow (\eta_{\mathfrak{st}}(V') \to \eta_{\mathfrak{st}}(P') \to \eta_{\mathfrak{st}}(X')) \longrightarrow (\eta_{\mathfrak{st}}(Y) \to \eta_{\mathfrak{st}}(Q) \to \eta_{\mathfrak{st}}(U))$ is annihilated by the maps $r^0_{\mathfrak{t}}$, so, by Lemmas 1.4(a-c) and Lemma 1.3(b-c), it comes from a (uniquely defined) morphism of short exact sequences $(\eta_{\mathfrak{t}}(K) \to \eta_{\mathfrak{t}}(L) \to \eta_{\mathfrak{t}}(M)) \longrightarrow (\eta_{\mathfrak{t}}(Y)(-1) \to \eta_{\mathfrak{t}}(Q)(-1) \to \eta_{\mathfrak{t}}(U)(-1))$ in the category $\mathcal{F}_{\mathfrak{t}}$ via the maps $\mathfrak{s}_0$.

Consider the extension of short exact sequences $0 \longrightarrow \eta_{\mathfrak{t}}(V) \longrightarrow \eta_{\mathfrak{t}}(P) \longrightarrow \eta_{\mathfrak{t}}(X) \longrightarrow 0$ and $0 \longrightarrow \eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(L) \longrightarrow \eta_{\mathfrak{t}}(M) \longrightarrow 0$ with the middle term $0 \longrightarrow \eta_{\mathfrak{t}}(V') \longrightarrow \eta_{\mathfrak{t}}(P') \longrightarrow \eta_{\mathfrak{t}}(X') \longrightarrow 0$ in (the exact category of short exact sequences in) the category $\mathcal{F}_{\mathfrak{t}}$, and induce from it an extension of the exact sequences $0 \longrightarrow \eta_{\mathfrak{t}}(V) \longrightarrow \eta_{\mathfrak{t}}(P) \longrightarrow \eta_{\mathfrak{t}}(X) \longrightarrow 0$ and $0 \longrightarrow \eta_{\mathfrak{t}}(Y)(-1) \longrightarrow \eta_{\mathfrak{t}}(Q)(-1) \longrightarrow \eta_{\mathfrak{t}}(U)(-1) \longrightarrow 0$ using the above-constructed morphism of short exact sequences in $\mathcal{F}_{\mathfrak{t}}$. We have obtained a commutative 3×3 square formed by short exact sequences in the exact category $\mathcal{F}_{\mathfrak{t}}$. For any such square, the two Ext^2 classes between the objects at the opposite vertices obtained by composing the Ext^1 classes along the perimeter differ by the minus sign (see Lemma 0.3). This proves the desired equation $-\eta^1_{\mathfrak{t}}(a(-1))\partial^0(z) = \partial^0(w)\eta^1_{\mathfrak{t}}(b)$ in the group $\mathrm{Ext}^2_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$ (cf. [12, Subsection 4.5]). $\square$

1.8. **Exactness of the long sequence.** Here we partly follow [12, Subsection 4.6]. The argument is based on Proposition 0.5(a). We start with the following lemma, which is in some way similar to Lemma 1.3(b).

Lemma 1.9. *Let* $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ *and* $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ *be two short exact sequences in the category* $\mathcal{F}$*, and let* $f\colon \eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(V)(-1)$ *and*

$g\colon \eta_{\mathsf t}(L)\longrightarrow \eta_{\mathsf t}(W)(-1)$ be two morphisms in the category $\mathcal F_{\mathsf t}$ forming a commutative square diagram with the images of the morphisms $K\longrightarrow L$ and $V(-1)\longrightarrow W(-1)$ under the functor $\eta_{\mathsf t}$. Let $w\in \operatorname{Ext}^1_{\mathcal F_{\mathsf t}}(\eta_{\mathsf t}(M),\eta_{\mathsf t}(U)(-1))$ denote the related extension class provided by the construction of Lemma 0.4(a), and let $z\in \operatorname{Ext}^1_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(M),\eta_{\mathfrak{st}}(U))$ be the similar extension class produced by the same construction applied to the images of the short exact sequences $0\longrightarrow K\longrightarrow L\longrightarrow M\longrightarrow 0$ and $0\longrightarrow U\longrightarrow V\longrightarrow W\longrightarrow 0$ under the functor $\eta_{\mathfrak{st}}$ and the morphisms $\mathfrak s_0(f)\colon \eta_{\mathfrak{st}}(K)\longrightarrow \eta_{\mathfrak{st}}(V)$ and $\mathfrak s_0(g)\colon \eta_{\mathfrak{st}}(L)\longrightarrow \eta_{\mathfrak{st}}(W)$ in the category $\mathcal F_{\mathfrak{st}}$. Then one has $z=\mathfrak s_1(w)$.

Proof. Choose an admissible epimorphism $L'\longrightarrow L$ and a morphism $L'\longrightarrow W(-1)$ in the category $\mathcal F$ making the triangle diagram $\eta_{\mathsf t}(L')\longrightarrow \eta_{\mathsf t}(L)\longrightarrow \eta_{\mathsf t}(W)(-1)$ commutative in the category $\mathcal F_{\mathsf t}$. Denote by L'' the fibered product of the objects L' and $V(-1)$ over $W(-1)$ in $\mathcal F$. Let K'' denote the kernel of the composition of admissible epimorphisms $L''\longrightarrow L'\longrightarrow L\longrightarrow M$; then there is a natural morphism of short exact sequences $(K''\to L''\to M)\longrightarrow (K\to L\to M)$ in the category $\mathcal F$. By Lemma 0.4(c), both the classes $w\in \operatorname{Ext}^1_{\mathcal F_{\mathsf t}}(\eta_{\mathsf t}(M),\eta_{\mathsf t}(U)(-1))$ and $z\in \operatorname{Ext}^1_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(M),\eta_{\mathfrak{st}}(U))$ can be obtained by applying the construction of Lemma 0.4(a) to the images of the short exact sequences $0\longrightarrow K''\longrightarrow L''\longrightarrow M\longrightarrow 0$ and $0\longrightarrow U\longrightarrow V\longrightarrow W\longrightarrow 0$ under the functors $\eta_{\mathsf t}$ and $\eta_{\mathfrak{st}}$ together with the composition of morphisms of morphisms $(\eta_{\mathsf t}(K'')\to\eta_{\mathsf t}(L''))\longrightarrow(\eta_{\mathsf t}(K)\to\eta_{\mathsf t}(L))\longrightarrow (\eta_{\mathsf t}(V)(-1)\to\eta_{\mathsf t}(W)(-1)$ and its image under the maps $\mathfrak s_0$.

On the other hand, the images of the composition $K''\longrightarrow L''\longrightarrow V(-1)$ and the morphism $L''\longrightarrow W(-1)$ under the functor $\eta_{\mathsf t}$ provide another commutative square $(\eta_{\mathsf t}(K'')\to\eta_{\mathsf t}(L''))\longrightarrow(\eta_{\mathsf t}(V)(-1)\to\eta_{\mathsf t}(W)(-1)$ in the category $\mathcal F_{\mathsf t}$. By Lemma 0.4(b), the Ext^1 classes assigned to this morphism of morphisms and its image under the maps $\mathfrak s_0$ by the construction of Lemma 0.4(a) vanish. Subtracting one morphism of morphisms from the other one and applying Lemma 0.4(d), we reduce the original problem to the case of a morphism of morphisms (f,g) with a vanishing second component $g=0$. In this case, the construction of Lemma 0.4(a) reduces to that of the product of a morphism with an Ext^1 class, so the desired assertion holds by Lemma 1.3(b) (or rather, even by the definition of the map $\mathfrak s_1$). ☐

Lemma 1.10. *The initial segment*

$$0\longrightarrow \operatorname{Hom}_{\mathcal F_{\mathsf t}}(\eta_{\mathsf t}(X),\eta_{\mathsf t}(Y)(-1))\longrightarrow \operatorname{Hom}_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))\longrightarrow \operatorname{Hom}_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$$
$$\longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathsf t}}(\eta_{\mathsf t}(X),\eta_{\mathsf t}(Y)(-1))\longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X),\eta_{\mathfrak{st}}(Y))\longrightarrow \operatorname{Ext}^1_{\mathcal F_{\mathfrak s}}(\eta_{\mathfrak s}(X),\eta_{\mathfrak s}(Y))$$

of the long sequence that we have constructed is exact for any two objects X and Y in the category $\mathcal F$.

Proof. Exactness at the first three nontrivial terms has been proven already in Subsections 1.4–1.6. Let us prove exactness at the term $\operatorname{Ext}^1_{\mathcal F_{\mathsf t}}(\eta_{\mathsf t}(X),\eta_{\mathsf t}(Y)(-1))$.

According to Proposition 0.5(b) and the condition (i) for the functor $\eta_{\mathsf t}$, any element z in our Ext^1 group in the category $\mathcal F_{\mathsf t}$ is equal to the product $p\eta^1_{\mathsf t}(b)$ of the image $\eta^1_{\mathsf t}(b)$ of an Ext^1 class b represented by a short exact sequence $0\longrightarrow$

$Y' \longrightarrow Z' \longrightarrow X \longrightarrow 0$ in the category $\mathcal{F}$ under the functor $\eta_{\mathfrak{t}}$ and a morphism $p\colon \eta_{\mathfrak{t}}(Y') \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ in the category $\mathcal{F}_{\mathfrak{t}}$.

By the definition, the element $\mathfrak{s}_1(z) \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$ is constructed as the composition $\mathfrak{s}_0(p)\eta^1_{\mathfrak{st}}(b)$ of the Ext^1 class $\eta^1_{\mathfrak{st}}(b) \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y'))$ and the morphism $\mathfrak{s}_0(p)\colon \eta_{\mathfrak{st}}(Y') \longrightarrow \eta_{\mathfrak{st}}(Y)$ in the category $\mathcal{F}_{\mathfrak{st}}$. The equation $\mathfrak{s}_1(z) = \mathfrak{s}_0(p)\eta^1_{\mathfrak{st}}(b) = 0$ means that the morphism $\mathfrak{s}_0(p)$ factorizes through the morphism $\eta_{\mathfrak{st}}(Y') \longrightarrow \eta_{\mathfrak{st}}(Z')$ in the category $\mathcal{F}_{\mathfrak{st}}$, i. e., there exists a morphism $\eta_{\mathfrak{st}}(Z') \longrightarrow \eta_{\mathfrak{st}}(Y)$ in $\mathcal{F}_{\mathfrak{st}}$ making the triangle $\eta_{\mathfrak{st}}(Y') \longrightarrow \eta_{\mathfrak{st}}(Z') \longrightarrow \eta_{\mathfrak{st}}(Y)$ commutative.

Applying the maps $r^0_{\mathfrak{t}}$ to the whole diagram in the category $\mathcal{F}_{\mathfrak{st}}$ with the objects in the vertices coming from the category $\mathcal{F}$ via the functor $\eta_{\mathfrak{st}}$, we see that the morphism $r^0_{\mathfrak{t}}\mathfrak{s}_0(p)\colon \eta_{\mathfrak{s}}(Y') \longrightarrow \eta_{\mathfrak{s}}(Y)$ vanishes, so the morphism $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(Y)$ factorizes through the admissible epimorphism $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(X)$ and there exists a morphism $q\colon \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ in the category $\mathcal{F}_{\mathfrak{s}}$ making the triangle $\eta_{\mathfrak{s}}(Z') \longrightarrow \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ commutative. Let us check that $\partial^0(q) = z$.

Pick an admissible epimorphism $Z'' \longrightarrow Z'$ and a morphism $Z'' \longrightarrow Y$ in the category $\mathcal{F}$ making the triangle diagram $\eta_{\mathfrak{st}}(Z'') \longrightarrow \eta_{\mathfrak{st}}(Z') \longrightarrow \eta_{\mathfrak{st}}(Y)$ commutative in the category $\mathcal{F}_{\mathfrak{st}}$. Let Y'' denote the kernel of the composition of admissible epimorphisms $Z'' \longrightarrow Z' \longrightarrow X$; then there is a natural morphism of short exact sequences $(Y'' \to Z'' \to X) \longrightarrow (Y' \to Z' \to X)$ in the category $\mathcal{F}$. Set $b' \in \mathrm{Ext}^1_{\mathcal{F}}(X, Y'')$ to be the Ext^1 class represented by the short exact sequence $0 \longrightarrow Y'' \longrightarrow Z'' \longrightarrow X \longrightarrow 0$ in $\mathcal{F}$, and denote by $p'\colon \eta_{\mathfrak{t}}(Y'') \longrightarrow \eta_{\mathfrak{t}}(Y)(-1)$ the composition of the image of the morphism $Y'' \longrightarrow Y'$ under the functor $\eta_{\mathfrak{t}}$ with the morphism p. Then the triangle diagram $\eta_{\mathfrak{s}}(Z'') \longrightarrow \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Y)$ is commutative in $\mathcal{F}_{\mathfrak{s}}$ and the morphism $\mathfrak{s}_0(p')\colon \eta_{\mathfrak{st}}(Y'') \longrightarrow \eta_{\mathfrak{st}}(Y)$ is equal to the image of the composition $Y'' \longrightarrow Z'' \longrightarrow Y$ under the functor $\eta_{\mathfrak{st}}$. By the definition, one has $\partial^0(q) = p'\eta^1_{\mathfrak{t}}(b') = p\eta^1_{\mathfrak{t}}(b) = z$.

It remains to prove exactness of our sequence at the term $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$. According to Proposition 0.5(b) and the condition (i) for the functor $\eta_{\mathfrak{st}}$, any element $z \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$ can be presented as a product of the form $z = \eta^1_{\mathfrak{st}}(a)f$, where a is an Ext^1 class represented by a short exact sequence $0 \longrightarrow Y \longrightarrow Z' \longrightarrow X' \longrightarrow 0$ in the category $\mathcal{F}$ and f is a morphism $\eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(X')$ in the category $\mathcal{F}_{\mathfrak{st}}$. The equation $r^1_{\mathfrak{t}}(z) = \eta^1_{\mathfrak{s}}(a)r^0_{\mathfrak{t}}(f) = 0$ in $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ means that there exists a morphism $\eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Z')$ in the category $\mathcal{F}_{\mathfrak{s}}$ forming a commutative triangle with the image of the morphism $Z' \longrightarrow X'$ under the functor $\eta_{\mathfrak{s}}$ and the image of the morphism $f\colon \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(X')$ under the map $r^0_{\mathfrak{t}}$.

Applying the condition (ii) for the functors $\eta_{\mathfrak{s}}$ and $\eta_{\mathfrak{st}}$, one can find an admissible epimorphism $X'' \longrightarrow X$ and morphisms $X'' \longrightarrow X'$, $X'' \longrightarrow Z'$ in the category $\mathcal{F}$ making the triangle diagram $\eta_{\mathfrak{st}}(X'') \longrightarrow \eta_{\mathfrak{st}}(X) \longrightarrow \eta_{\mathfrak{st}}(X')$ commutative in the category $\mathcal{F}_{\mathfrak{st}}$ and the triangle diagram $\eta_{\mathfrak{s}}(X'') \longrightarrow \eta_{\mathfrak{s}}(X) \longrightarrow \eta_{\mathfrak{s}}(Z')$ commutative in the category $\mathcal{F}_{\mathfrak{s}}$. Let $K \longrightarrow X''$ denote the kernel of the morphism $X'' \longrightarrow X$ in the category $\mathcal{F}$; then the composition $K \longrightarrow X'' \longrightarrow X'$ is annihilated by the functor $\eta_{\mathfrak{st}}$ and the composition $K \longrightarrow X'' \longrightarrow Z'$ is annihilated by the functor $\eta_{\mathfrak{s}}$. Furthermore, it follows that the difference of the compositions $X'' \longrightarrow Z' \longrightarrow X'$ and $X'' \longrightarrow X \longrightarrow X'$ in the category $\mathcal{F}$ is also annihilated by the functor $\eta_{\mathfrak{s}}$, and the image of this difference under the functor $\eta_{\mathfrak{st}}$ forms a commutative square with the image of the composition $K \longrightarrow X'' \longrightarrow Z'$ and the images of the morphisms $K \longrightarrow X''$ and $Z' \longrightarrow X'$ under the same functor.

By Lemma 1.4(a,c), the images of our morphisms $K \longrightarrow Z'$ and $X'' \longrightarrow X'$ under the functor $\eta_{\mathfrak{st}}$ are consequently equal to the images of (uniquely defined) morphisms $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Z')(-1)$ and $\eta_{\mathfrak{t}}(X'') \longrightarrow \eta_{\mathfrak{t}}(X')(-1)$ under the map $\mathfrak{s}_0$. Using Lemma 1.3(b-c), one checks that the square diagram $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(Z')(-1) \longrightarrow \eta_{\mathfrak{t}}(X')(-1)$, $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(X'') \longrightarrow \eta_{\mathfrak{t}}(X')(-1)$, where the morphisms $\eta_{\mathfrak{t}}(Z') \longrightarrow \eta_{\mathfrak{t}}(X')$ and $\eta_{\mathfrak{t}}(K) \longrightarrow \eta_{\mathfrak{t}}(X'')$ come from the given morphisms in the category $\mathcal{F}$ via the functor $\eta_{\mathfrak{t}}$, is commutative in the category $\mathcal{F}_{\mathfrak{t}}$.

Applying the construction of Lemma 0.4(a) to the images of the short exact sequences $0 \longrightarrow K \longrightarrow X'' \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow Y \longrightarrow Z' \longrightarrow X' \longrightarrow 0$ under the functor $\eta_{\mathfrak{t}}$ and the above commutative square of morphisms in the category $\mathcal{F}_{\mathfrak{t}}$ produces the desired class $w \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$. Indeed, the result of applying the same construction to the images of the same short exact sequences under the functor $\eta_{\mathfrak{st}}$ and the commutative square $K \longrightarrow Z' \longrightarrow X'$, $K \longrightarrow X'' \longrightarrow X'$ in the category $\mathcal{F}_{\mathfrak{st}}$ is easily compared to the original class $z \in \mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{st}}}(\eta_{\mathfrak{st}}(X), \eta_{\mathfrak{st}}(Y))$, so it remains to make use of Lemma 1.9. ☐

Corollary 1.11. *The whole long sequence of* Ext *groups from Subsection 1.3, as constructed in Subsection 1.7, is exact for any two objects X, Y of the category $\mathcal{F}$.*

Proof. The assertion follows formally from the construction and Lemma 1.10 in view of Proposition 0.5(a) applied to the functors $\eta_{\mathfrak{s}}$, $\eta_{\mathfrak{t}}$, and $\eta_{\mathfrak{st}}$.

E. g., let us prove exactness at the terms $\mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$ for all $n \geqslant 1$. Let z be an element of our Ext group in the category $\mathcal{F}_{\mathfrak{s}}$. By Proposition 0.5(b) applied to the functor $\eta_{\mathfrak{s}}$, there exists an Ext class $b \in \mathrm{Ext}^n_{\mathcal{F}}(X, X')$ in the category $\mathcal{F}$ and a morphism $q\colon \eta_{\mathfrak{s}}(X') \longrightarrow \eta_{\mathfrak{s}}(Y)$ in the category $\mathcal{F}_{\mathfrak{s}}$ such that the element z is equal to the product $q\eta^n_{\mathfrak{s}}(b)$ in the group $\mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$. By the definition, one has $\partial^n(z) = \partial^0(q)\eta^n_{\mathfrak{t}}(b)$ in $\mathrm{Ext}^{n+1}_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X), \eta_{\mathfrak{t}}(Y)(-1))$.

Now assume that $\partial^0(q)\eta^n_{\mathfrak{t}}(b) = 0$. By Proposition 0.5(a) applied to the functor $\eta_{\mathfrak{t}}$, there exists a morphism $f\colon X'' \longrightarrow X'$ and an Ext class $b' \in \mathrm{Ext}^n_{\mathcal{F}}(X, X'')$ in the category $\mathcal{F}$ such that $b = fb'$ in $\mathrm{Ext}^n_{\mathcal{F}}(X, X')$ and $\partial^0(q)\eta_{\mathfrak{t}}(f) = 0$ in $\mathrm{Ext}^1_{\mathcal{F}_{\mathfrak{t}}}(\eta_{\mathfrak{t}}(X''), \eta_{\mathfrak{t}}(Y)(-1))$. By Lemma 1.8 (or, actually, even Lemma 1.5(a)) one has $\partial^0(q\eta_{\mathfrak{s}}(f)) = \partial^0(q)\eta_{\mathfrak{t}}(f) = 0$, and by Lemma 1.10 (or, actually, Lemma 1.5(b)), there exists a morphism $g\colon \eta_{\mathfrak{st}}(X'') \longrightarrow \eta_{\mathfrak{st}}(Y)$ in the category $\mathcal{F}_{\mathfrak{st}}$ such that $q\eta_{\mathfrak{s}}(f) = r^0_{\mathfrak{t}}(g)$ in the group $\mathrm{Hom}_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X''), \eta_{\mathfrak{s}}(Y))$. Finally, we have $z = q\eta^n_{\mathfrak{s}}(b) = q\eta_{\mathfrak{s}}(f)\eta^n_{\mathfrak{s}}(b') = r^0_{\mathfrak{t}}(g)\eta^n_{\mathfrak{s}}(b') = r^n_{\mathfrak{t}}(g\eta^n_{\mathfrak{st}}(b'))$ in $\mathrm{Ext}^n_{\mathcal{F}_{\mathfrak{s}}}(\eta_{\mathfrak{s}}(X), \eta_{\mathfrak{s}}(Y))$. ☐

2. The Matrix Factorization Construction

2.1. **Posing the problem.** Let $\mathcal{F}$ be an exact category endowed with a twist functor (exact autoequivalence) $X \longmapsto X(1)$. Let $\mathfrak{s}$ be a morphism of endofunctors $\mathrm{Id} \longrightarrow (1)$

on the category $\mathcal{F}$ commuting with the twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ (see Subsection 0.0 for the precise definitions and discussion).

Let $\mathcal{E}$ be another exact category endowed with a twist functor $(1)\colon \mathcal{E} \longrightarrow \mathcal{E}$. Suppose that we are given an exact functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ commuting with the twists on $\mathcal{F}$ and $\mathcal{E}$, and that the following conditions are satisfied:

(v) the functor π is exact-conservative;

(vi) the functor π takes all the morphisms $\mathfrak{s}_X\colon X \longrightarrow X(1)$ in the category $\mathcal{F}$ to zero morphisms in the category $\mathcal{E}$;

(vii) any morphism in the category $\mathcal{F}$ annihilated by the functor π is divisible by the natural transformation $\mathfrak{s}$;

(viii) for any object $X \in \mathcal{F}$, the morphism $\mathfrak{s}_X\colon X \longrightarrow X(1)$ is injective and surjective; in other words, no nonzero morphism in the category $\mathcal{F}$ is annihilated by the natural transformation $\mathfrak{s}$.

The conditions (vi) and (vii) taken together can be restated by saying that a morphism in the category $\mathcal{F}$ is annihilated by the functor π if and only if it is divisible by the natural transformation $\mathfrak{s}$. The conditions (vii) and (viii) taken together can be reformulated by saying that any morphism in the category $\mathcal{F}$ annihilated by the functor π is uniquely divisible by the natural transformation $\mathfrak{s}$ Taken together, these are simply a restament of the conditions (iii-iiii) of Subsection 1.0 (notice, however, that we also presume the exact-conservativity of our functor π here, and most importantly, do *not* assume the conditions (i-ii)).

Our goal in this section is to construct an exact category $\mathcal{G}$ which we will call the *reduction of exact category $\mathcal{F}$ by the natural transformation $\mathfrak{s}$ taken on the background of the functor π*. The category $\mathcal{G}$ comes endowed with exact-conservative functors $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ and $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ whose composition $\epsilon\gamma$ is identified with π. The functor γ annihilates all the morphisms $\mathfrak{s}_X$, while the functor ϵ reflects zero morphisms (i. e., it is faithful). The category $\mathcal{G}$ is also endowed with a twist functor $(1)\colon \mathcal{G} \longrightarrow \mathcal{G}$, and the functors γ and ϵ commute with the twists.

The Ext groups computed in the categories $\mathcal{F}$ and $\mathcal{G}$ are related by the following Bockstein long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_{\mathcal{F}}(X, Y(-1)) &\longrightarrow \operatorname{Hom}_{\mathcal{F}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{G}}(\gamma(X), \gamma(Y)) \\ \longrightarrow \operatorname{Ext}^1_{\mathcal{F}}(X, Y(-1)) &\longrightarrow \operatorname{Ext}^1_{\mathcal{F}}(X, Y) \longrightarrow \operatorname{Ext}^1_{\mathcal{G}}(\gamma(X), \gamma(Y)) \\ \longrightarrow \operatorname{Ext}^2_{\mathcal{F}}(X, Y(-1)) &\longrightarrow \operatorname{Ext}^2_{\mathcal{F}}(X, Y) \longrightarrow \operatorname{Ext}^2_{\mathcal{G}}(\gamma(X), \gamma(Y)) \longrightarrow \cdots \end{aligned}$$

for any two objects X, $Y \in \mathcal{F}$ (cf. [12, Subsection 4.1]).

Here the map $\mathfrak{s}_n\colon \operatorname{Ext}^n_{\mathcal{F}}(X, Y(-1)) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}}(X, Y)$ is provided by the composition with the morphism $\mathfrak{s}_{Y(-1)}\colon Y(-1) \longrightarrow Y$ (or, equivalently, the twist by (1) and the composition with the morphism $\mathfrak{s}_X\colon X \longrightarrow X(1)$) in the category $\mathcal{F}$. The map $\gamma^n\colon \operatorname{Ext}^n_{\mathcal{F}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{G}}(\gamma(X), \gamma(Y))$ is induced by the exact functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$. Finally, the boundary map $\partial^n\colon \operatorname{Ext}^n_{\mathcal{G}}(\gamma(X), \gamma(Y)) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}}(X, Y(-1))$ is defined by the construction of Subsections 1.6–1.7 and satisfies the equation

$$\partial^{i+n+j}(\gamma^i(a) z \gamma^j(b)) = (-1)^i a(-1) \partial^n(z) b$$

for any objects U, X, Y, $V \in \mathcal{F}$ and any Ext classes $b \in \operatorname{Ext}^j_{\mathcal{F}}(U,X)$, $z \in \operatorname{Ext}^n_{\mathcal{G}}(\gamma(X),\gamma(Y))$, and $a \in \operatorname{Ext}^i_{\mathcal{F}}(Y,V)$.

2.2. **Examples.** The following examples illustrate the ways of choosing a background exact-conservative functor π satisfying the conditions (vi-vii) for a given natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on an exact category $\mathcal{F}$.

Example 2.1. (a) Let G be a finite group and $m = l^r$ be a prime power. As in Example 1.2, let $\mathcal{F} = \mathcal{F}^G_{\mathbb{Z}_l}$ be the exact category of finitely generated free $\mathbb{Z}_l$-modules with an action of G. Set the twist $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ to be the identity functor and the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \mathrm{Id}_{\mathcal{F}}$ to act by the multiplication with m.

Let $\mathcal{E}$ be the category of finitely generated free $\mathbb{Z}/m$-modules with the split exact category structure, and $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ be the functor taking a finitely generated free $\mathbb{Z}_l$-module M with an action of G to the $\mathbb{Z}/m$-module M/mM. Then π is an exact-conservative functor satisfying the conditions (vi-vii) for the center element $\mathfrak{s}$ in $\mathcal{F}$.

(b) More generally, let G be a profinite group. As in the introduction, let $\mathcal{F}^+ = \mathcal{F}^{G+}_{\mathbb{Z}_l}$ be the exact category of l-divisible l^∞-torsion (l-primary) abelian groups with a discrete action of G. Set the twist (1) on $\mathcal{F}^+$ to be the identity functor and the natural transformation $\mathfrak{s}$ to act by the multiplication with m.

Let $\mathcal{E}^+$ be the category of free $\mathbb{Z}/m$-modules with the split exact category structure, and $\pi\colon \mathcal{F}^+ \longrightarrow \mathcal{E}^+$ be the functor taking an l-divisible l^∞-torsion abelian group M with a discrete action of G to the $\mathbb{Z}/m$-module ${}_mM$ of all the elements annihilated by m in M. Then π is an exact-conservative functor satisfying the conditions (vi-vii) for the center element $\mathfrak{s}$ in $\mathcal{F}^+$ (which satisfies the condition (viii)). (See Subsection 3.2 below for further discussion of this example.)

Example 2.2. (a) Let C be a coassociative coalgebra over a field k endowed with a comultiplicative increasing filtration $0 = F_{-1}C \subset F_0C \subset F_1C \subset F_2C \subset \cdots$. Set $F^{-n}C = F_nC$, and let $\mathcal{F}$ be the exact category of finite-dimensional left C-comodules M endowed with finite decreasing filtrations $\cdots \supset F^{n-1}M \supset F^nM \supset F^{n+1}M \supset \cdots$, $F^{-N}M = M$ and $F^NM = 0$ for $N \gg 0$, compatible with the filtration F on C.

Let the twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ take a filtered C-comodule (M,F) to the same C-comodule M with the shifted filtration $F(1)^nM = F^{n-1}M$, and let $\mathfrak{s}_{(M,F)}\colon (M,F) \longrightarrow (M,F(1))$ be the identity map $M \longrightarrow M$ viewed as a morphism of filtered comodules.

Let $\mathcal{E}$ be the category of finite-dimensional graded k-vector spaces (with the split exact category structure), and let $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ be the functor taking a filtered C-comodule (M,F) to its associated graded vector space $\bigoplus_n F^nM/F^{n+1}M$. Then π is an exact-conservative functor satisfying the conditions (vi-vii) for the natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on $\mathcal{F}$.

In particular, when the coalgebra C is endowed with a coaugmentation (coalgebra morphism) $k \longrightarrow C$ and F is the filtration by the kernels of iterated comultiplication maps $F_nC = \ker(C \to (C/k)^{\otimes n+1})$ [12, Section 2], this example is naturally generalized to the next example (b). Specifically, consider the abelian category $\mathcal{A}$ of left C-comodules and the exact functor $\phi\colon \mathcal{E}_0 \longrightarrow \mathcal{A}$ of fully faithful embedding of the split exact category $\mathcal{E}_0$ of finite-dimensional k-vector spaces endowed with trivial C-comodule structures. Then the following filtered exact category construction yields the above exact category $\mathcal{F}$ of filtered C-comodules.

(b) Let $\mathcal{A}$ and $\mathcal{E}_0$ be exact categories and $\phi\colon \mathcal{E}_0 \longrightarrow \mathcal{A}$ be an exact functor. Set $\mathcal{F}$ to be the category whose objects are the triples (M, Q, q), where $M = (M, F)$ is a finitely filtered object of the exact category $\mathcal{A}$, $Q = (Q_i)$ is a finitely supported object of the Cartesian product $\prod_{i\in\mathbb{Z}} \mathcal{E}_0$, and q is an isomorphism $\mathrm{gr}_F M \simeq \phi(Q)$ of graded objects in $\mathcal{A}$ [12, Section 3 and Examples A.5(4-5)].

The category $\mathcal{F}$ is endowed with an exact category structure in which a short sequence with zero composition is exact if the related short sequence of the objects Q_i is exact in $\mathcal{E}_0$ for each $i \in \mathbb{Z}$. The twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ shifts the indices of both the filtration F on M and the sequence of objects Q_i, and the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ acts by identity on the underlying objects $M \in \mathcal{A}$ of filtered objects (M, F) and by zero on the graded objects $Q \in \prod_{i\in\mathbb{Z}} \mathcal{E}_0$.

Let $\mathcal{E}$ be the full exact subcategory of finitely supported objects in $\prod_{i\in\mathbb{Z}} \mathcal{E}_0$ and $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ be the functor taking a triple (M, Q, q) to the graded object Q. Then π is an exact-conservative functor satisfying the condition (vi) for the graded center element $\mathfrak{s}$ in $\mathcal{F}$ (cf. [12, Section 4]). When the functor $\phi\colon \mathcal{E}_0 \longrightarrow \mathcal{A}$ is fully faithful, the conditions (vii-viii) are also satisfied [12, Example 4.1].

The categorical reduction construction of this section applied to the exact categories, natural transformations, and background exact-conservative functors in Example 2.1(a-b), produces the exact categories $\mathcal{G} = \mathcal{F}^G_{\mathbb{Z}/m}$ and $\mathcal{G}^+ = \mathcal{F}^{G+}_{\mathbb{Z}/m}$ of finitely and infinitely generated free $\mathbb{Z}/m$-modules with a (discrete) action of G. The same construction in Example 2.2(a) produces the abelian category $\mathcal{G}$ of finite-dimensional graded comodules over the graded coalgebra $\mathrm{gr}_F C = \bigoplus_n F_n C/F_{n-1} C$.

We are not aware of any alternative way to define or produce the exact category $\mathcal{G}$ that one obtains as the output of the categorical reduction construction applied in the case of Example 2.2(b) in general (even when, e. g., $\mathcal{E}_0$ is a split exact category and $\mathcal{A}$ is an abelian category).

Example 2.3. The following example appears in connection with coefficient reduction and Bockstein sequences in exact categories of Artin–Tate motives with finite coefficients. Let G be a profinite group, k be a complete Noetherian local ring, and $c\colon G \longrightarrow k^*$ be a continuous group homomorphism; set $k(n)$ to denote the free k-module k in which G acts by the character c^n.

For any integer n, denote by $\mathcal{E}_n^+ = \mathcal{E}_{k,n}^{G+}$ the category of all injective discrete k-modules E with a discrete action of G for which the G-module $E(-n) = k(-n)\otimes_k E$ is permutational (i. e., isomorphic to a direct sum of G-modules induced from trivial representations of open subgroups $H \subset G$ in injective discrete k-modules). Endow the additive category $\mathcal{E}_n^+$ with the split exact category structure.

Let $\mathcal{F}_k^+ = \mathcal{F}_k^{G+}$ denote the exact category of finitely filtered discrete G-modules (M, F) whose associated graded modules $\mathrm{gr}_F^n M = F^n M/F^{n+1} M$ belong to $\mathcal{E}_n^+$ (cf. the construction of exact category $\mathcal{F}$ in Example 2.2(b)). Set the twist $(1)\colon \mathcal{F}_k^+ \longrightarrow \mathcal{F}_k^+$ to be the identity functor and the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}_k^+} \longrightarrow \mathrm{Id}_{\mathcal{F}_k^+}$ to act by the multiplication with a fixed nonzero-dividing, noninvertible element $s \in k$.

The quotient ring $k/s = k/(s)$ is also a complete Noetherian local ring, and the submodule ${}_sM$ of all elements annihilated by s in any injective discrete k-module M is an injective discrete module over k/s. Let ${}'\mathcal{E}_{k/s}^+$ denote the full subcategory of finitely supported graded objects in $\prod_{n\in\mathbb{Z}} \mathcal{E}_{k/s,\,n}^{G\,+}$, and let ${}''\mathcal{E}_{k/s}^+$ denote the category of discrete injective modules over $k/(s)$. Notice that the objects of ${}'\mathcal{E}_{k/s}^+$ are collections of c^n-twisted permutational G-modules over k/s, while the objects of ${}''\mathcal{E}_{k/s}^+$ are simply k/s-modules without any action of G.

Endow both the additive categories ${}'\mathcal{E}_{k/s}^+$ and ${}''\mathcal{E}_{k/s}^+$ with split exact category structures, and denote by $\mathcal{E}_{k/s}^+$ their Cartesian product ${}'\mathcal{E}_{k/s}^+ \times {}''\mathcal{E}_{k/s}^+$. Let the functor $\pi\colon \mathcal{F}_k^+ \longrightarrow \mathcal{E}_{k/s}^+$ take a finitely filtered discrete G-module (M,F) over k with twisted-permutational k-injective k-discrete associated quotient modules $\mathrm{gr}_F^n M$ to the pair formed by the collection of permutational representations ${}_s\mathrm{gr}_F^n M = \mathrm{gr}_F^n\, {}_sM \in \mathcal{E}_{k/s,\,n}^{G\,+}$ of the group G over the ring k/s and the k/s-module ${}_sM \in {}''\mathcal{E}_{k/s}^+$.

It is claimed that π is an exact-conservative functor satisfying the conditions (vi-vii) for the center element $\mathfrak{s}$ of the exact category $\mathcal{F}$. The condition (viii) is also satisfied.

In this example one would *like* to obtain the exact category $\mathcal{F}_{k/s}^+ = \mathcal{F}_{k/s}^{G\,+}$ in the output of the reduction procedure (implying, in particular, the Bockstein long exact sequences relating the Ext groups in the exact categories $\mathcal{F}_k^+$ and $\mathcal{F}_{k/s}^+$), but this is not always true (see Subsection 3.4 below for further details).

2.3. **Matrix factorizations.** In order to construct the desired exact category $\mathcal{G}$, consider first the following category $\widetilde{\mathcal{H}}$. The objects of $\widetilde{\mathcal{H}}$ are the diagrams (U,V) of the form

$$V(-1) \longrightarrow U \longrightarrow V \longrightarrow U(1)$$

in the category $\mathcal{F}$, where the morphism $V \longrightarrow U(-1)$ is obtained from the morphism $V(-1) \longrightarrow U$ by applying the twist functor (1), while the two compositions $V(-1) \longrightarrow U \longrightarrow V$ and $U \longrightarrow V \longrightarrow U(1)$ are equal to the maps $\mathfrak{s}_{V(-1)}$ and $\mathfrak{s}_U$, respectively. Morphisms $(U',V') \longrightarrow (U'',V'')$ in the category $\mathcal{H}$ are the pairs of morphisms $U' \longrightarrow U''$ and $V' \longrightarrow V''$ in $\mathcal{F}$ making the whole diagram $(V'(-1) \to U' \to V' \to U'(1)) \longrightarrow (V''(-1) \to U'' \to V'' \to U''(1))$ commutative.

Furthermore, consider the following full subcategory $\mathcal{H} \subset \widetilde{\mathcal{H}}$. By the definition, a diagram $(U,V) \in \widetilde{\mathcal{H}}$ belongs to the category $\mathcal{H}$ if the functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ (which, as we recall, takes the morphisms $\mathfrak{s}_{V(-1)}$ and $\mathfrak{s}_U$ in $\mathcal{F}$ to zero morphisms in $\mathcal{E}$) transforms it into an exact sequence $\pi(V(-1)) \longrightarrow \pi(U) \longrightarrow \pi(V) \longrightarrow \pi(U(1))$ in the exact category $\mathcal{E}$. The functor $\Delta\colon \mathcal{H} \longrightarrow \mathcal{E}$ assigns to a diagram $(U,V) \in \mathcal{H}$ the image of the morphism $\pi(U) \longrightarrow \pi(V)$ in $\mathcal{E}$ (which is well-defined due to the exactness condition imposed on the objects of $\mathcal{H}$).

The category $\widetilde{\mathcal{H}}$ has a natural exact category structure in which a short sequence of diagrams is exact if it is exact in $\mathcal{F}$ at every term of the diagrams. The full subcategory $\mathcal{H} \subset \widetilde{\mathcal{H}}$ is closed under the operations of passage to the cokernels of admissible monomorphisms, the kernels of admissible epimorphisms, and extensions; so in particular it inherits the induced exact category structure. The functor $\Delta\colon \mathcal{H} \longrightarrow \mathcal{E}$ is an exact functor between exact categories.

Let $\mathcal{I} \subset \mathcal{H}$ denote the ideal of morphisms in $\mathcal{H}$ annihilated by Δ. Consider the quotient category $\mathcal{H}/\mathcal{I}$ of the category $\mathcal{H}$ by this ideal of morphisms, and let $\mathcal{S} \subset \mathcal{H}/\mathcal{I}$ denote the multiplicative class of morphisms which the functor $\Delta\colon \mathcal{H}/I \longrightarrow \mathcal{E}$ transforms to isomorphisms in $\mathcal{E}$.

Lemma 2.4. *Assuming the conditions (v-vi) of Subsection 2.1, the class of morphisms $\mathcal{S}$ is localizing in the category $\mathcal{H}/\mathcal{I}$ (i. e., it satisfies the left and right Ore conditions).*

Proof. The argument follows that in [12, Subsection 4.2]. It is clear from the definitions of the classes $\mathcal{I}$ and $\mathcal{S}$ that if any two morphisms $X \rightrightarrows Y$ in $\mathcal{H}/\mathcal{I}$ have equal compositions with a morphism $X' \longrightarrow X$ or $Y \longrightarrow Y'$ belonging to $\mathcal{S}$, then these two morphisms $X \rightrightarrows Y$ are equal to each other in $\mathcal{H}/\mathcal{I}$.

Let $(S,T) \longrightarrow (K,L) \longleftarrow (U,V)$ be a pair of morphisms in $\mathcal{H}$ such that $\Delta((U,V) \longrightarrow (K,L))$ is an admissible epimorphism in $\mathcal{E}$. Then the morphism $U \oplus L(-1) \longrightarrow K$ is an admissible epimorphism in $\mathcal{F}$ (since so is its image under π). Consider the fibered product $P = S \sqcap_K (U \oplus L(-1))$ in $\mathcal{F}$, and set $Q = T \oplus V \in \mathcal{F}$.

Let the map $S \sqcap_K (U \oplus L(-1)) \longrightarrow T \oplus V$ be defined as the composition $S \sqcap_K (U \oplus L(-1)) \longrightarrow S \oplus U \longrightarrow T \oplus V$ and the map $T(-1) \oplus V(-1) \longrightarrow S \sqcap_K (U \oplus L(-1))$ be induced by the maps $T(-1) \longrightarrow S$, $T(-1) \longrightarrow L(-1)$, $V(-1) \longrightarrow U$, and minus the map $V(-1) \longrightarrow L(-1)$. Then the diagram

$$T(-1) \oplus V(-1) \longrightarrow S \sqcap_K (U \oplus L(-1)) \longrightarrow T \oplus V \longrightarrow S(1) \sqcap_{K(1)} (U(1) \oplus L)$$

is an object (P,Q) of the category $\mathcal{H}$. Indeed, one easily checks that the diagram (P,Q) belongs to $\widetilde{\mathcal{H}}$; and to prove the exactness condition, it suffices to notice that (P,Q) is the kernel of an admissible epimorphism $(S,T) \oplus (U,V) \oplus (L(-1),L) \longrightarrow (K,L)$ between two objects of $\mathcal{H}$ in $\widetilde{\mathcal{H}}$.

There are natural morphisms $(S,T) \longleftarrow (P,Q) \longrightarrow (U,V)$ in the category $\mathcal{H}$; the square diagram formed by these two morphisms and the morphisms $(S,T) \longrightarrow (K,L) \longleftarrow (U,V)$ is commutative modulo $\mathcal{I}$. The object $\Delta(P,Q)$ is the fibered product of $\Delta(S,T)$ and $\Delta(U,V)$ over $\Delta(K,L)$. In particular, if the morphism $(U,V) \longrightarrow (K,L)$ belongs to $\mathcal{S}$, then so does the morphism $(P,Q) \longrightarrow (S,T)$. This proves a half of the Ore conditions, and the dual half can be proven in the dual way. ☐

2.4. **Exact category structure.** We define the category $\mathcal{G}$ as the localization $(\mathcal{H}/\mathcal{I})[\mathcal{S}^{-1}]$. By Lemma 2.4, $\mathcal{G}$ is an additive category and the localization $\mathcal{H} \longrightarrow \mathcal{G}$ is an additive functor. The twist functor $(1)\colon \mathcal{G} \longrightarrow \mathcal{G}$ is induced by the obvious twist functor $(U,V) \longmapsto (U(1),V(1))$ on the category $\mathcal{H}$. The functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ assigns to an object $X \in \mathcal{F}$ the diagram (X, X) with the identity morphism $X \longrightarrow X$ in the middle. The functor $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ is induced by the functor Δ.

Lemma 2.5. *In the assumption of the conditions (v-viii) from Subsection 2.1, the rule according to which a short sequence in the category $\mathcal{G}$ is said to be exact if its image under the functor ϵ is exact in $\mathcal{E}$ defines an exact category structure on $\mathcal{G}$. Moreover, a morphism is an admissible monomorphism (resp., admissible epimorphism) in $\mathcal{G}$ if and only if its image under ϵ is an admissible monomorphism (resp., admissible epimorphism) in $\mathcal{E}$.*

Proof. We follow the argument in [12, Subsection 4.3]. Consider a morphism f in $\mathcal{G}$ such that $\epsilon(f)$ is an admissible epimorphism in $\mathcal{E}$. Then, clearly, f is a surjective morphism in $\mathcal{G}$. Represent f by a morphism $(U, V) \longrightarrow (K, L)$ in $\mathcal{H}$ and apply the construction from the proof of Lemma 2.4 to the pair of morphisms $(0, 0) \longrightarrow (K, L) \longleftarrow (U, V)$. We obtain a morphism $(P, Q) = (\ker(U \oplus L(-1) \to K),\, V) \longrightarrow (U, V)$ in $\mathcal{H}$ whose image g in $\mathcal{G}$ completes the morphism f to a short sequence $0 \longrightarrow (P, Q) \longrightarrow (U, V) \longrightarrow (K, L) \longrightarrow 0$ that is exact in $\mathcal{G}$ (in the sense of our definition; i. e., its image under ϵ is exact in $\mathcal{E}$).

Let us check that the morphism g is the kernel of f in $\mathcal{G}$. Any morphism with the target (U, V) in $\mathcal{G}$ can be represented by a morphism $(X, Y) \longrightarrow (U, V)$ in $\mathcal{H}$. Assume that the composition $(X, Y) \longrightarrow (U, V) \longrightarrow (K, L)$ is annihilated by Δ. Then the composition $X \longrightarrow K \longrightarrow L$ is annihilated by π, so the morphism $X \longrightarrow K$ factorizes through the morphism $L(-1) \longrightarrow K$ by the conditions (vii-viii). This allows to lift the morphism $(X, Y) \longrightarrow (U, V)$ to a morphism $(X, Y) \longrightarrow (P, Q)$ in $\mathcal{H}$.

The morphism g being injective in $\mathcal{G}$, the above lifting is unique as a morphism in $\mathcal{G}$. Furthermore, the short sequence $0 \longrightarrow \Delta(P, Q) \longrightarrow \Delta(U, V) \longrightarrow \Delta(K, L) \longrightarrow 0$ is exact in $\mathcal{E}$ according to the proof of Lemma 2.4. Recalling that the morphisms taken to isomorphisms in $\mathcal{E}$ by the functor Δ have been inverted in $\mathcal{G}$, one concludes that any morphism $(S, T) \longrightarrow (U, V)$ with the latter property is a cokernel of f in $\mathcal{G}$.

This suffices to check the axioms Ex0–Ex1 and Ext3 from [12, Subsection A.3] for the category $\mathcal{G}$; it remains to prove Ex2. Suppose that we are given a short exact sequence in $\mathcal{G}$; it can be represented by a sequence of morphisms $(S, T) \longrightarrow (U, V) \longrightarrow (K, L)$ in $\mathcal{H}$. Any morphism with the target (K, L) in $\mathcal{G}$ can be represented by a morphism $(X, Y) \longrightarrow (K, L)$ in $\mathcal{H}$. Applying the construction from the proof of Lemma 2.4 again, we obtain an object $(M, N) = (X \sqcap_K (U \oplus L(-1)),\, Y \oplus V)$ in $\mathcal{H}$ together with a pair of morphisms $(U, V) \longleftarrow (M, N) \longrightarrow (X, Y)$.

Setting, as above, $(P, Q) = (\ker(U \oplus L(-1) \to K),\, V)$, we have the above-constructed morphism $(S, T) \longrightarrow (P, Q)$ in $\mathcal{H}$, whose composition with the natural admissible monomorphism $(P, Q) \longrightarrow (M, N)$ provides a morphism $(S, T) \longrightarrow (M, N)$ in $\mathcal{H}$. The triangle $(S, T) \longrightarrow (M, N) \longrightarrow (U, V)$ is commutative already in $\mathcal{H}$, and the short sequence $0 \longrightarrow (S, T) \longrightarrow (M, N) \longrightarrow (X, Y) \longrightarrow 0$ is exact in $\mathcal{G}$, as so is its image under ϵ in $\mathcal{E}$. The exact category axioms are verified. $\square$

It follows immediately from the above description of the exact category structure on $\mathcal{G}$ that the functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ is exact and exact-conservative (since both the functors $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ and $\pi = \epsilon\gamma\colon \mathcal{F} \longrightarrow \mathcal{E}$ are). The functor $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ is faithful, because the functor $\Delta\colon \mathcal{H}/\mathcal{I} \longrightarrow \mathcal{E}$ is faithful by the definition of $\mathcal{I}$. It is also clear that the functors ϵ and γ commute with the twists. The more advanced properties of our reduction construction are discussed in the next subsection.

2.5. **Properties of the reduction functor.** The following lemma shows that the reduction functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ satisfies the conditions (i-ii) and ($*$–$**$) from Subsection 0.1 (and even a stronger form of the last one of these).

Lemma 2.6. *The exact functor $\gamma\colon \mathcal{F} \longrightarrow \mathcal{G}$ constructed above has the following "exact surjectivity" properties:*

- (a′) *for any object $X \in \mathcal{F}$ and any admissible epimorphism $T \longrightarrow \gamma(X)$ in $\mathcal{G}$ there exists an admissible epimorphism $Z \longrightarrow X$ in $\mathcal{F}$ and a morphism $\gamma(Z) \longrightarrow T$ in $\mathcal{G}$ making the triangle diagram $\gamma(Z) \longrightarrow T \longrightarrow \gamma(X)$ commutative;*
- (b′) *for any object $T \in \mathcal{G}$ there exists an object $U \in \mathcal{F}$ and an admissible epimorphism $\gamma(U) \longrightarrow T$ in $\mathcal{G}$;*
- (c′) *for any objects X, $Y \in \mathcal{F}$ and any morphism $\gamma(X) \longrightarrow \gamma(Y)$ in $\mathcal{G}$ there exists an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Y$ in $\mathcal{F}$ making the triangle diagram $\gamma(X') \longrightarrow \gamma(X) \longrightarrow \gamma(Y)$ commutative in $\mathcal{G}$;*
- (d′) *for any object $X \in \mathcal{F}$ and any morphism $\gamma(X) \longrightarrow T$ in $\mathcal{G}$ there exists a morphism $X \longrightarrow S$ in $\mathcal{F}$ and an admissible epimorphism $\gamma(S) \longrightarrow T$ in $\mathcal{G}$ such that the triangle diagram $\gamma(X) \longrightarrow \gamma(S) \longrightarrow T$ is commutative;*

as well as the properties (a″-d″) dual to (a′-d′).

Proof. Part (b′): let the object $T \in \mathcal{G}$ be represented by a diagram $(U,V) = (V(-1) \to U \to V \to U(1)$ in $\mathcal{H}$; then there is a natural admissible epimorphism $\gamma(U) \longrightarrow T$ in $\mathcal{G}$. (Indeed, $\pi(U) \longrightarrow \Delta(U,V)$ is an admissible epimorphism in $\mathcal{E}$.) Part (c′): let the morphism $\gamma(X) \longrightarrow \gamma(Y)$ be represented by a fraction $(X,X) \longleftarrow (U,V) \longrightarrow (Y,Y)$ of two morphisms in $\mathcal{H}$, where the morphism $(U,V) \longrightarrow (X,X)$ belongs to $\mathcal{S}$ (modulo $\mathcal{I}$). Then there is a natural admissible epimorphism $U \longrightarrow X$ and a natural morphism $U \longrightarrow Y$ in $\mathcal{F}$ making the diagram $\gamma(U) \longrightarrow \gamma(X) \longrightarrow \gamma(Y)$ commutative in $\mathcal{G}$ by the definition. (Indeed, the morphism $\pi(U) \longrightarrow \Delta(U,V) \simeq \Delta(X,X) = \pi(X)$ is an admissible epimorphism in $\mathcal{E}$.)

Part (d′): one can represent the morphism $\gamma(X) \longrightarrow T$ in $\mathcal{G}$ by a morphism of diagrams $(X,X) \longrightarrow (U,V)$ in $\mathcal{H}$. Then there is a morphism $X \longrightarrow U$ in $\mathcal{F}$ and an admissible epimorphism $\gamma(U) \longrightarrow (U,V)$ in $\mathcal{G}$, while the triangle $(X,X) \longrightarrow (U,U) \longrightarrow (U,V)$ is commutative already in $\mathcal{H}$. Part (a′) follows from (b′) and (c′) according to Lemma 0.1; to prove it directly, represent the morphism $T \longrightarrow \gamma(X)$ in $\mathcal{G}$ by a morphism of diagrams $(U,V) \longrightarrow (X,X)$ in $\mathcal{H}$. Then $U \longrightarrow X$ is an admissible epimorphism in $\mathcal{F}$ (since $\pi(U) \longrightarrow \Delta(U,V) \longrightarrow \pi(X)$ is a composition of admissible epimorphisms in $\mathcal{E}$) and $(U,U) \longrightarrow (U,V)$ is an admissible epimorphism in $\mathcal{G}$, while the triangle $(U,U) \longrightarrow (U,V) \longrightarrow (X,X)$ is commutative in $\mathcal{H}$. ☐

2.6. **The first Bockstein sequence.** Now we are in the position to construct the Bockstein long exact sequence for the Ext groups in the categories $\mathcal{F}$ and $\mathcal{G}$ promised in Subsection 2.1. Namely, we obtain the desired construction as a particular case of the general construction of a Bockstein long exact sequence from Section 1, or more specifically, of its version from Subsection 1.0.

To compare our present notation with that in Section 1, set $\mathcal{F}_{\mathfrak{t}} = \mathcal{F}_{\mathfrak{st}} = \mathcal{F}$ and $\mathcal{F}_{\mathfrak{s}} = \mathcal{G}$. The exact category structures on $\mathcal{F}_{\mathfrak{t}}$, $\mathcal{F}_{\mathfrak{st}}$, and $\mathcal{F}_{\mathfrak{s}}$ are the same as the exact category structures on $\mathcal{F}$ and $\mathcal{G}$. Let $\eta_{\mathfrak{t}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{t}}$ and $\eta_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{st}}$ be the identity functor $\mathrm{Id}_{\mathcal{F}}$, and $\eta_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{F}_{\mathfrak{s}}$ be our reduction functor γ. Finally, let the natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on the category $\mathcal{F}$ from Subsection 1.0 be our natural transformation $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ on the category $\mathcal{F}$ from Subsection 2.1.

Let us check that the conditions (i-ıııı) of Subsections 0.1–1.0 hold in the situation at hand. The functor γ satisfies the conditions (i′-ii′) according to Lemma 2.6(a′,c′), and the dual conditions (i″-ii″) according to the dual assertions of the same Lemma. The condition (ııı) coincides with the condition (viii); and the condition (ıııı) of Subsection 1.0 for the functor γ follows from the conditions (vi-vii) of Subsection 2.1 for the functor π (since the functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$ decomposes as $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}$ and the functor $\epsilon\colon \mathcal{G} \longrightarrow \mathcal{E}$ does not annihilate any morphisms).

The Bockstein long exact sequence promised in Subsection 2.1 is now obtained. The equation for the maps ∂^n from Subsection 2.1 is the equation (c) of Subsection 1.0.

2.7. **Independence of the background functor.** The following arguments show that the construction of the reduction $\mathcal{G}$ of an exact category $\mathcal{F}$ with a twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ by a natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ "almost" does not depend on the background exact-conservative functor $\pi\colon \mathcal{F} \longrightarrow \mathcal{E}$.

Let $\mathcal{F}$ be an exact category with a twist functor $(1)\colon \mathcal{F} \longrightarrow \mathcal{F}$ and a natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ commuting with the twist. Suppose that $\mathcal{G}'$ and $\mathcal{G}''$ are two exact categories with the twist functors, and $\gamma'\colon \mathcal{F} \longrightarrow \mathcal{G}'$ and $\gamma''\colon \mathcal{F} \longrightarrow \mathcal{G}''$ are two exact functors, commuting with the twists, for which the boundary maps ∂^n are defined between the Ext groups in the categories $\mathcal{G}'$, $\mathcal{G}''$ and $\mathcal{F}$ making the long sequence of Subsection 2.1 exact in both cases.

Suppose further that there is an exact functor $\lambda\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ commuting with the twists, making a commutative triangle diagram with the functors γ' and γ'', and inducing a morphism between the long exact sequences of Subsection 2.1 corresponding to the functors γ' and γ''. Let $\overline{\mathcal{G}}'$ and $\overline{\mathcal{G}}''$ denote the minimal full subcategories in $\mathcal{G}'$ and $\mathcal{G}''$, closed under extensions and containing the all the objects in the image of the functors γ' and γ'', respectively. Endow the full subcategories $\overline{\mathcal{G}}' \subset \mathcal{G}'$ and $\overline{\mathcal{G}}'' \subset \mathcal{G}''$ with the induced exact category structures.

Lemma 2.7. (a) *The exact functor $\lambda\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ restricts to an equivalence $\overline{\mathcal{G}}' \simeq \overline{\mathcal{G}}''$ between the full exact subcategories $\overline{\mathcal{G}}'$ and $\overline{\mathcal{G}}''$ in $\mathcal{G}'$ and $\mathcal{G}''$.*

(b) *Assume that the functor γ' or γ'' satisfies one of the conditions (i′) or (i″) of Subsection 0.1. Then the Ext groups between the objects of the corresponding subcategory $\overline{\mathcal{G}}'$ or $\overline{\mathcal{G}}''$ computed in the exact subcategory $\overline{\mathcal{G}}'$ or $\overline{\mathcal{G}}''$ concide with those computed in the whole exact category $\mathcal{G}'$ or $\mathcal{G}''$, respectively.*

(c) *Assume that both functors γ' and γ'' simultaneously satisfy one of the conditions (i′) or (i″), and the functor γ' satisfies both the conditions (∗′) and (∗″) of Subsection 0.1. Then the functor λ is fully faithful, its image $\lambda(\mathcal{G}')$ is a full subcategory closed under extensions in $\mathcal{G}''$, the exact category structure on $\mathcal{G}'$ coincides with the one induced from $\mathcal{G}''$ via λ, and the functor λ induces isomorphisms $\lambda^n\colon \operatorname{Ext}^n_{\mathcal{G}'}(Z,W)\simeq \operatorname{Ext}^n_{\mathcal{G}''}(\lambda(Z),\lambda(W))$ for all objects Z, $W\in\mathcal{G}'$ and all $n\geqslant 0$.*

Proof. Notice first of all that the maps $\lambda^n\colon \operatorname{Ext}^n_{\mathcal{G}'}(\gamma'(X),\gamma'(Y))\longrightarrow \operatorname{Ext}^n_{\mathcal{G}''}(\gamma''(X),\gamma''(Y))$ are isomorphisms for all X, $Y\in\mathcal{F}$ and $n\geqslant 0$ by 5-lemma.

In the situation of part (a), the maps $\operatorname{Ext}^n_{\overline{\mathcal{G}}'}(Z,W)\longrightarrow \operatorname{Ext}^n_{\mathcal{G}'}(Z,W)$ are isomorphisms for all Z, $W\in\overline{\mathcal{G}}'$ and $n=0$, 1, and monomorphisms for $n=2$, because $\overline{\mathcal{G}}'$ is a full subcategory closed under extensions in the exact category $\mathcal{G}'$, with the induced exact category structure [12, Subsection A.8]. The same applies to the maps of Ext groups induced by the embedding $\overline{\mathcal{G}}''\longrightarrow\mathcal{G}''$; and consequently the maps $\lambda^n\colon \operatorname{Ext}^n_{\overline{\mathcal{G}}'}(\gamma'(X),\gamma'(Y))\longrightarrow \operatorname{Ext}^n_{\overline{\mathcal{G}}''}(\gamma''(X),\gamma''(Y))$ are also isomorphisms for all X, $Y\in\mathcal{F}$ and $n=0$, 1, and monomorphisms for $n=2$. The desired assertion now follows by the general criterion of [12, Lemma 3.2].

To obtain part (b), one applies Corollary 0.7(b) in its first set of assuptions, and to prove part (c), in the second one. □

Given an exact category $\mathcal{F}$ with a twist functor (1) and a natural transformation $\mathfrak{s}$ commuting with the twist functor and satisfying the condition (viii), let $\mathcal{E}'$ and $\mathcal{E}''$ be two exact categories endowed with twist functors, and let $\pi'\colon\mathcal{F}\longrightarrow\mathcal{E}'$ and $\pi''\colon\mathcal{F}\longrightarrow\mathcal{E}''$ be two exact functors, both commuting with the twists and satisfying the conditions (v-vii) of Subsection 2.1. Let $\mathcal{G}'$ and $\mathcal{G}''$ denote the corresponding two exact categories obtained by the reduction procedure of Subsections 2.3–2.4, and $\gamma'\colon\mathcal{F}\longrightarrow\mathcal{G}'$, $\gamma''\colon\mathcal{F}\longrightarrow\mathcal{G}''$ be the two related exact functors.

Suppose first that there exists an exact functor $\mathcal{E}'\longrightarrow\mathcal{E}''$ commuting with the twists and making a commutative triangle diagram with the functors π' and π''. Then one easily constructs the induced exact functor $\lambda\colon\mathcal{G}'\longrightarrow\mathcal{G}''$, commuting with the twists, making a commutative triangle diagram with the functors γ' and γ'', and satisfying the conditions of Lemma 2.7(c). All the conclusions of Lemma 2.7(a-c) accordingly apply.

More generally, denote by $\mathcal{E}=\mathcal{E}'\times\mathcal{E}''$ the Cartesian product of the two exact categories $\mathcal{E}'$ and $\mathcal{E}''$. The objects of $\mathcal{E}$ are pais (E',E''), where E' is an object of $\mathcal{E}'$ and E'' is an object of $\mathcal{E}''$; morphisms $(E_1',E_1'')\longrightarrow(E_2',E_2'')$ are pairs of morphisms $E_1'\longrightarrow E_2'$ and $E_1''\longrightarrow E_2''$ in $\mathcal{E}'$ and $\mathcal{E}''$; and short exact sequences in $\mathcal{E}$ are pairs of short exact sequences in the exact categories $\mathcal{E}'$ and $\mathcal{E}''$ (cf. [12, Example A.5(4)]).

Let $\pi=(\pi',\pi'')\colon\mathcal{F}\longrightarrow\mathcal{E}$ denote the functor taking an object $X\in\mathcal{F}$ to the pair $(\pi'(X),\pi''(X))$; then π is also an exact functor satisfying the conditions (v-vii) for the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}}\longrightarrow(1)$. Let $\mathcal{G}$ denote the reduction of the exact category $\mathcal{F}$ by the natural transformation $\mathfrak{s}$ taken on the background of the functor π, and let $\gamma\colon\mathcal{F}\longrightarrow\mathcal{G}$ be the corresponding exact functor.

Then the natural projections $\mathcal{E} \longrightarrow \mathcal{E}'$, $\mathcal{E}''$ forming commutative triangle diagrams with the exact-conservative functors π', π'', and π induce exact functors $\lambda'\colon \mathcal{G} \longrightarrow \mathcal{G}'$ and $\lambda''\colon \mathcal{G} \longrightarrow \mathcal{G}''$ between the reduced exact categories. The functors λ' and λ'' form commutative triangle diagrams with the reduction functors γ', γ'', and γ. All the conclusions of Lemma 2.7(a-c) apply to both the functors λ' and λ'', leading in particular to the following corollary.

Corollary 2.8. *The exact functors $\lambda'\colon \mathcal{G} \longrightarrow \mathcal{G}'$ and $\lambda''\colon \mathcal{G} \longrightarrow \mathcal{G}''$ are fully faithful, and their images are full subcategories closed under extensions in $\mathcal{G}'$ and $\mathcal{G}''$. The exact category structure on $\mathcal{G}$ coincides with the exact category structures induced on the full subcategories $\lambda'(\mathcal{G}) \subset \mathcal{G}'$ and $\lambda''(\mathcal{G}) \subset \mathcal{G}''$ by the exact category structures on $\mathcal{G}'$ and $\mathcal{G}''$. The induced maps of the Ext groups $\lambda'^n\colon \operatorname{Ext}^n_{\mathcal{G}}(Z,W) \longrightarrow \operatorname{Ext}^n_{\mathcal{G}'}(\lambda'(Z),\lambda'(W))$ and $\lambda''^n\colon \operatorname{Ext}^n_{\mathcal{G}}(Z,W) \longrightarrow \operatorname{Ext}^n_{\mathcal{G}''}(\lambda''(Z),\lambda''(W))$ are isomorphisms for all the objects Z, $W \in \mathcal{G}$ and all integers $n \geqslant 0$.*

The restrictions of the functors λ' and λ'' to the minimal full subcategories $\overline{\mathcal{G}} \subset \mathcal{G}$, $\overline{\mathcal{G}}' \subset \mathcal{G}'$, $\overline{\mathcal{G}}'' \subset \mathcal{G}''$ containing all the objects in the images of the functors γ, γ', γ'' and closed under extensions are equivalences of exact categories $\overline{\mathcal{G}}' \simeq \overline{\mathcal{G}} \simeq \overline{\mathcal{G}}''$ (with the exact category structures induced from $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$). The Ext groups between the objects of the subcategories $\overline{\mathcal{G}}$, $\overline{\mathcal{G}}'$, $\overline{\mathcal{G}}''$ computed in these exact subcategories coincide with the Ext groups computed in the whole exact categories $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$. $\square$

Based on the above "almost uniqueness"/"almost independence" results, we will sometimes denote the exact category obtained by the reduction procedure developed in this section by $\mathcal{G} = \mathcal{F}/\mathfrak{s}$. When a specific choice of the background exact-concervative functor needs to be mentioned, we will write $\mathcal{G} = \mathcal{F}/_{\mathcal{E}}\,\mathfrak{s} = \mathcal{F}/_{\pi}\,\mathfrak{s}$.

2.8. **The second Bockstein sequence.** Let $\mathcal{F}$ be an exact category with two commuting exact autoequivalences $X \longmapsto X(1)$ and $X \longmapsto X\{1\}$. Let $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ and $\mathfrak{t}\colon \mathrm{Id} \longrightarrow \{1\}$ be two natural tranformations of endofunctors on $\mathcal{F}$, both commuting with both the twist functors (1) and $\{1\}$ and acting by injective and surjective morphisms $\mathfrak{s}_X\colon X \longrightarrow X(1)$ and $\mathfrak{t}_X\colon X \longrightarrow X\{1\}$ on all the objects $X \in \mathcal{F}$. Denote by $\mathfrak{st}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)\{1\}$ the product of these two bigraded center elements of $\mathcal{F}$.

Let $\mathcal{E}_{\mathfrak{s}}$, $\mathcal{E}_{\mathfrak{t}}$, and $\mathcal{E}_{\mathfrak{st}}$ be three exact categories endowed with commuting exact autoequivalences (1) and $\{1\}$, and let $\pi_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{E}_{\mathfrak{s}}$, $\pi_{\mathfrak{t}}\colon \mathcal{F} \longrightarrow \mathcal{E}_{\mathfrak{t}}$, and $\pi_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{E}_{\mathfrak{st}}$ be three exact-conservative functors, commuting with the twist functors (1) and $\{1\}$ and satisfying the conditions (vi-vii) of Subsection 2.1 for the natural transformations $\mathfrak{s}$, $\mathfrak{t}$, and $\mathfrak{st}$, respectively. Consider the three reduced exact categories $\mathcal{G}_{\mathfrak{s}} = \mathcal{F}/_{\pi_{\mathfrak{s}}}\mathfrak{s}$, $\mathcal{G}_{\mathfrak{t}} = \mathcal{F}/_{\pi_{\mathfrak{t}}}\mathfrak{t}$, and $\mathcal{G}_{\mathfrak{st}} = \mathcal{F}/_{\pi_{\mathfrak{st}}}\mathfrak{st}$ with the corresponding reduction functors $\gamma_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{s}}$, $\gamma_{\mathfrak{t}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{t}}$, and $\gamma_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{st}}$ and exact-conservative faithful functors $\epsilon_{\mathfrak{s}}\colon \mathcal{G}_{\mathfrak{s}} \longrightarrow \mathcal{E}_{\mathfrak{s}}$, $\epsilon_{\mathfrak{t}}\colon \mathcal{G}_{\mathfrak{t}} \longrightarrow \mathcal{E}_{\mathfrak{t}}$, and $\epsilon_{\mathfrak{st}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{E}_{\mathfrak{st}}$. Clearly, there are induced exact autoequivalences (1) and $\{1\}$ on all the three categories $\mathcal{G}_{\mathfrak{s}}$, $\mathcal{G}_{\mathfrak{t}}$, and $\mathcal{G}_{\mathfrak{st}}$, and all the functors γ and ϵ commute with both of the twists.

Denote by $\widetilde{\mathcal{H}}_{\mathfrak{st}}$ and $\mathcal{H}_{\mathfrak{st}}$ the intermediate categories employed in the construction of the category $\mathcal{G}_{\mathfrak{st}} = \mathcal{F}/_{\pi_{\mathfrak{st}}}\mathfrak{st}$ in Subsections 2.3–2.4. Then the twist functors (1) and $\{1\}$ act naturally on the categories $\widetilde{\mathcal{H}}_{\mathfrak{st}}$ and $\mathcal{H}_{\mathfrak{st}}$; and the natural transformation

$\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ induces bigraded center elements $\mathfrak{s}\colon \mathrm{Id} \longrightarrow (1)$ in the categories $\widetilde{\mathcal{H}}_{\mathfrak{st}}$, $\mathcal{H}_{\mathfrak{st}}$, and $\mathcal{G}_{\mathfrak{st}}$. Composing the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{G}_{\mathfrak{st}}} \longrightarrow (1)$ with the functor $\gamma_{\mathfrak{st}}$, or equivalently, the functor $\gamma_{\mathfrak{st}}$ with the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$ produces a morphism $\mathfrak{s}\colon \gamma_{\mathfrak{st}} \longrightarrow \gamma_{\mathfrak{st}}(1)$ of functors $\mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{st}}$ commuting with both the twists (1) and $\{1\}$ in the sense of Subsection 0.0.

Lemma 2.9. *The conditions (i–IV) of Subsection 1.3 are satisfied for the exact categories $\mathcal{F}$, $\mathcal{F}_{\mathfrak{t}} = \mathcal{G}_{\mathfrak{t}}$, $\mathcal{F}_{\mathfrak{s}} = \mathcal{G}_{\mathfrak{s}}$, and $\mathcal{F}_{\mathfrak{st}} = \mathcal{G}_{\mathfrak{st}}$ and the exact functors $\gamma_{\mathfrak{t}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{t}}$, $\gamma_{\mathfrak{s}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{s}}$, and $\gamma_{\mathfrak{st}}\colon \mathcal{F} \longrightarrow \mathcal{G}_{\mathfrak{st}}$ together with the natural transformation $\mathfrak{s}\colon \gamma_{\mathfrak{st}} \longrightarrow \gamma_{\mathfrak{st}}(1)$ commuting with the twist functors (1) on the categories $\mathcal{F}$, $\mathcal{G}_{\mathfrak{t}}$, $\mathcal{G}_{\mathfrak{s}}$, and $\mathcal{G}_{\mathfrak{st}}$.*

Proof. The conditions (i-ii) hold for the functors $\gamma_{\mathfrak{t}}$, $\gamma_{\mathfrak{s}}$, and $\gamma_{\mathfrak{st}}$ by Lemma 2.6(a,c). To prove the condition (III), recall that a morphism $f\colon X \longrightarrow Y$ in the category $\mathcal{F}$ is annihilated by the functor $\gamma_{\mathfrak{t}}$ if and only if it is divisible by the natural transformation $\mathfrak{t}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow \{1\}$. Similarly, the morphism $\gamma_{\mathfrak{st}}(f)$ is annihilated by the natural transformation $\mathfrak{s}\colon \gamma_{\mathfrak{st}} \longrightarrow \gamma_{\mathfrak{st}}(1)$ if and only if the morphism $\mathfrak{s}f = \mathfrak{s}_Y f\colon X \longrightarrow Y(1)$ in the category $\mathcal{F}$ is annihilated by the functor $\gamma_{\mathfrak{st}}$, that is, if and only if the morphism $\mathfrak{s}f$ is divisible by the natural transformation $\mathfrak{st}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)\{1\}$. In the latter case, let $g\colon X \longrightarrow Y\{-1\}$ be a morphism in $\mathcal{F}$ such that $\mathfrak{s}_Y f = \mathfrak{s}_Y \mathfrak{t}_{Y\{-1\}} g$. The morphism $\mathfrak{s}_Y$ being injective in the category $\mathcal{F}$, one has $f = \mathfrak{t}_{Y\{-1\}} g$.

A morphism $f\colon X \longrightarrow Y$ in the category $\mathcal{F}$ is annihilated by the functor $\gamma_{\mathfrak{s}}$ if and only if it is divisible by the natural transformation $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{F}} \longrightarrow (1)$, so the “only if” assertion in the condition (IV) follows immediately (and even in a stronger form). To prove the “if”, suppose there is an admissible epimorphism $X' \longrightarrow X$ and a morphism $X' \longrightarrow Y(-1)$ in the category $\mathcal{F}$ making the square diagram $\gamma_{\mathfrak{st}}(X') \longrightarrow \gamma_{\mathfrak{st}}(X) \longrightarrow \gamma_{\mathfrak{st}}(Y)$, $\gamma_{\mathfrak{st}}(X') \longrightarrow \gamma_{\mathfrak{st}}(Y)(-1) \longrightarrow \gamma_{\mathfrak{st}}(Y)$ commutative in the category $\mathcal{G}_{\mathfrak{st}}$. This diagram being the image of the diagram $X' \longrightarrow X \longrightarrow Y$, $X' \longrightarrow Y(-1) \longrightarrow Y$ in the category $\mathcal{F}$ under the functor $\gamma_{\mathfrak{st}}$, we conclude that the diagram in the category $\mathcal{F}$ commutes up to a morphism divisible by $\mathfrak{st}$. Hence one can make the square diagram commutative in the category $\mathcal{F}$ by adding a summand divisible by $\mathfrak{t}$ to the morphism $X' \longrightarrow Y(-1)$. So the composition $X' \longrightarrow X \longrightarrow Y$ is divisible by $\mathfrak{s}$ in the category $\mathcal{F}$, and consequently annihilated by the functor $\gamma_{\mathfrak{s}}$. The morphism $X' \longrightarrow X$ being an admissible epimorphism, it follows that $\gamma_{\mathfrak{s}}(f) = 0$. $\square$

Therefore, the construction of Section 1 applies and we obtain a natural Bockstein long exact sequence

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Hom}_{\mathcal{G}_{\mathfrak{t}}}(\gamma_{\mathfrak{t}}(X), \gamma_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Hom}_{\mathcal{G}_{\mathfrak{st}}}(\gamma_{\mathfrak{st}}(X), \gamma_{\mathfrak{st}}(Y)) \longrightarrow \mathrm{Hom}_{\mathcal{G}_{\mathfrak{s}}}(\gamma_{\mathfrak{s}}(X), \gamma_{\mathfrak{s}}(Y)) \\
&\longrightarrow \mathrm{Ext}^1_{\mathcal{G}_{\mathfrak{t}}}(\gamma_{\mathfrak{t}}(X), \gamma_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}_{\mathfrak{st}}}(\gamma_{\mathfrak{st}}(X), \gamma_{\mathfrak{st}}(Y)) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}_{\mathfrak{s}}}(\gamma_{\mathfrak{s}}(X), \gamma_{\mathfrak{s}}(Y)) \\
&\longrightarrow \mathrm{Ext}^2_{\mathcal{G}_{\mathfrak{t}}}(\gamma_{\mathfrak{t}}(X), \gamma_{\mathfrak{t}}(Y)(-1)) \longrightarrow \mathrm{Ext}^2_{\mathcal{G}_{\mathfrak{st}}}(\gamma_{\mathfrak{st}}(X), \gamma_{\mathfrak{st}}(Y)) \longrightarrow \cdots
\end{aligned}
$$

for any two objects X and Y in the category $\mathcal{F}$. The differentials in this long exact sequence have the properties (a-c) of Subsection 1.3.

2.9. **The third Bockstein sequence.** The above construction of a “finite-finite-finite” Bockstein long exact sequence for three reductions of a given exact category

$\mathcal{F}$ still leaves somewhat more to be desired. One would like to have such a sequence defined and exact for any two objects of the exact category $\mathcal{G}_{\mathfrak{st}}$, and functorial with respect to all the morphisms in the category $\mathcal{G}_{\mathfrak{st}}$.

Keeping the setting and assumptions of Subsection 2.8 in place, suppose additionally that we are given two exact functors $\upsilon_{\mathfrak{s}}\colon \mathcal{E}_{\mathfrak{st}} \longrightarrow \mathcal{E}_{\mathfrak{s}}$ and $\upsilon_{\mathfrak{t}}\colon \mathcal{E}_{\mathfrak{st}} \longrightarrow \mathcal{E}_{\mathfrak{t}}$, commuting with the twists (1) and $\{1\}$ and forming commutative triangle diagrams with the background functors $\pi_{\mathfrak{s}}$, $\pi_{\mathfrak{t}}$, and $\pi_{\mathfrak{st}}$. In this situation, one would expect existence of exact functors $\eta_{\mathfrak{s}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{s}}$ and $\eta_{\mathfrak{t}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{t}}$ forming a commutative diagram will all the functors above. However, there does *not* seem to be a natural way to construct a matrix factorization of the natural transformation $\mathfrak{s}$ or $\mathfrak{t}$ on $\mathcal{F}$ starting from a matrix factorization of the central element $\mathfrak{st}$.

Therefore, let us simply *assume* that we are given exact functors $\eta_{\mathfrak{s}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{s}}$ and $\eta_{\mathfrak{t}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{t}}$, commuting with the twist functors (1) and $\{1\}$ and making a commutative diagram of seven categories and ten functors with the above functors γ, ϵ, and υ. Assume further that there is a bigraded center element $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{E}_{\mathfrak{st}}} \longrightarrow (1)$ in the category $\mathcal{E}_{\mathfrak{st}}$ agreeing with the natural transformation $\mathfrak{s}$ on $\mathcal{F}$, and that a morphism in $\mathcal{E}_{\mathfrak{st}}$ is annihilated by the functor $\upsilon_{\mathfrak{t}}$ if and only if it is annihilated by $\mathfrak{s}$.

Lemma 2.10. *The conditions (i-iv) of Subsection 1.1 are satisfied for the exact functors $\eta_{\mathfrak{s}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{s}}$ and $\eta_{\mathfrak{t}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{t}}$ together with the bigraded center element $\mathfrak{s}\colon \mathrm{Id}_{\mathcal{G}_{\mathfrak{st}}} \longrightarrow (1)$ in the category $\mathcal{G}_{\mathfrak{st}}$.*

Proof. The functor $\gamma_{\mathfrak{st}}$ satisfies the condition $(*)$ by Lemma 2.6(b), and the functors $\gamma_{\mathfrak{s}}$ and $\gamma_{\mathfrak{t}}$ satisfy the conditions (i-ii) and $(*$–$**)$ of Subsection 0.1 by Lemma 2.6(a-d). The latter two functors also reflect admissible epimorphisms and admissible monomorphisms, as explained in the end of Subsection 2.4. According to Lemma 0.4(b,d), it follows that the functors $\eta_{\mathfrak{s}}$ and $\eta_{\mathfrak{t}}$ satisfy the conditions (i-ii).

To check the condition (iii) of Subsection 1.1, consider a morphism $g\colon X \longrightarrow Y$ in the category $\mathcal{G}_{\mathfrak{st}}$. If it is annihilated by the functor $\eta_{\mathfrak{t}}$, then $0 = \epsilon_{\mathfrak{t}}\eta_{\mathfrak{t}}(g) = \upsilon_{\mathfrak{t}}\epsilon_{\mathfrak{st}}(g)$ implies, according to our condition on the functor $\upsilon_{\mathfrak{t}}$, the equations $0 = \mathfrak{s}_{\epsilon_{\mathfrak{st}}(Y)}\epsilon_{\mathfrak{st}}(g) = \epsilon_{\mathfrak{st}}(\mathfrak{s}_Y g)$ in $\mathcal{E}_{\mathfrak{st}}$ and $\mathfrak{s}_Y g = 0$ in $\mathcal{G}_{\mathfrak{st}}$, since the functor $\epsilon_{\mathfrak{st}}$ is faithful. Conversely, the equation $\mathfrak{s}_Y g = 0$ in $\mathcal{G}_{\mathfrak{st}}$ implies $\mathfrak{s}_{\epsilon_{\mathfrak{st}}(Y)}\epsilon_{\mathfrak{st}}(g) = 0$ in $\mathcal{E}_{\mathfrak{st}}$, hence $0 = \upsilon_{\mathfrak{t}}\epsilon_{\mathfrak{st}}(g) = \epsilon_{\mathfrak{t}}\eta_{\mathfrak{t}}(g)$ and $\eta_{\mathfrak{t}}(g) = 0$, since the functor $\epsilon_{\mathfrak{t}}$ is faithful, too.

To prove the remaining condition (iv), represent a morphism g in the category $\mathcal{G}_{\mathfrak{st}}$ by a morphism of diagrams $(K, L) \longrightarrow (P, Q)$ in the category $\mathcal{H} = \mathcal{H}_{\mathfrak{st}}$ (as defined in Subsection 2.3). Then the composition $(K, K) \longrightarrow (K, L) \longrightarrow (P, Q) \longrightarrow (Q, Q)$ comes from a morphism $f\colon K \longrightarrow L$ in the category $\mathcal{F}$ via the functor $\gamma_{\mathfrak{st}}$. Hence the equation $\eta_{\mathfrak{s}}(g) = 0$ impies the equation $\gamma_{\mathfrak{s}}(f) = 0$ in the category $\mathcal{G}_{\mathfrak{s}}$. According to the condition (vii) of Subsection 2.1, it follows that the morphism f is divisible by the natural transformation $\mathfrak{s}$ in the category $\mathcal{F}$.

We have shown that for any morphism $g\colon X \longrightarrow Y$ in the category $\mathcal{G}_{\mathfrak{st}}$ annihilated by the functor $\eta_{\mathfrak{s}}\colon \mathcal{G}_{\mathfrak{st}} \longrightarrow \mathcal{G}_{\mathfrak{s}}$ there exist an admissible epimorphism $X' \longrightarrow X$ and an admissible monomophism $Y \longrightarrow Y'$ in the category $\mathcal{G}_{\mathfrak{st}}$ such that the composition $X' \longrightarrow X \longrightarrow Y \longrightarrow Y'$ is divisible by the natural transformation $\mathfrak{s}$ in $\mathcal{G}_{\mathfrak{st}}$. Since the conditions (ii-iii) of Subsection 1.1 are proven already, it follows by the way of Lemma 1.3(d) applied to the composition of morphisms $X' \longrightarrow X \longrightarrow Y$ (and the functors $\eta_{\mathfrak t}\colon \mathcal G_{\mathfrak{st}} \longrightarrow \mathcal G_{\mathfrak t}$, $\eta_{\mathfrak s}\colon \mathcal G_{\mathfrak{st}} \longrightarrow \mathcal G_{\mathfrak s}$, and $\eta_{\mathfrak{st}} = \mathrm{Id}_{\mathcal G_{\mathfrak{st}}}$) that there exists an admissible epimorphism $X'' \longrightarrow X$ in the category $\mathcal G_{\mathfrak{st}}$ for which the composition $X'' \longrightarrow X' \longrightarrow X \longrightarrow Y$ is divisible by $\mathfrak s$. This proves the condition (iv). $\square$

Consequently, the construction of Section 1 is applicable in our assumptions and we obtain a natural long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{\mathcal G_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1)) \longrightarrow \mathrm{Hom}_{\mathcal G_{\mathfrak{st}}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal G_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y)) \\ &\longrightarrow \mathrm{Ext}^1_{\mathcal G_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1)) \longrightarrow \mathrm{Ext}^1_{\mathcal G_{\mathfrak{st}}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal G_{\mathfrak s}}(\eta_{\mathfrak s}(X), \eta_{\mathfrak s}(Y)) \\ &\longrightarrow \mathrm{Ext}^2_{\mathcal G_{\mathfrak t}}(\eta_{\mathfrak t}(X), \eta_{\mathfrak t}(Y)(-1)) \longrightarrow \mathrm{Ext}^2_{\mathcal G_{\mathfrak{st}}}(X, Y) \longrightarrow \cdots \end{aligned}$$

for any two objects X and Y in the category $\mathcal G_{\mathfrak{st}}$. The differentials in this long exact sequence have the properties (a-c) of Subsection 1.1.

3. Reduction of Coefficients in Artin–Tate Motives

3.1. **The main hypothesis.** Let $\mathcal A$ be an exact category, $\mathcal E_i$, $i \in \mathbb Z$, be a sequence of additive categories, and $\phi_i\colon \mathcal E_i \longrightarrow \mathcal A$ be additive functors. We will view the categories $\mathcal E_i$ as exact categories with split exact category structures. Consider the category $\mathcal F$ whose objects are the triples (M, Q, q), where $M = (M, F)$ is a finitely filtered object of the exact category $\mathcal A$, $Q = (Q_i)$ is a finitely supported object of the Cartesian product of additive categories $\prod_{i\in\mathbb Z} \mathcal E_i$, and q is a collection of isomorphisms $q_i\colon \mathrm{gr}^i_F M \simeq \phi_i(Q_i)$ of objects in the category $\mathcal A$. The category $\mathcal F$ is endowed with the exact category structure in which a short sequence with zero composition is exact if the related short sequence of the objects Q_i is split exact in $\mathcal E_i$ for each $i \in \mathbb Z$ (see [12, Section 3 and Examples A.5(4-5)]; cf. Example 2.2(b) above).

There are natural embedding functors $\mathcal E_i \longrightarrow \mathcal F$ identifying the category $\mathcal E_i$ with the full exact subcategory of $\mathcal F$ consisting of all the objects (M, Q, q) such that $Q_j = 0$ for all $i \neq j$. For the sake of simplicity of the terminology and notation, assume further that there are equivalences of categories $(1)\colon \mathcal E_i \longrightarrow \mathcal E_{i+1}$ and a twist functor (exact autoequivalence) $(1)\colon \mathcal A \longrightarrow \mathcal A$ forming commutative diagrams with the functors ϕ_i. Then there is a naturally induced twist functor $(1)\colon \mathcal F \longrightarrow \mathcal F$ taking an object (M, Q, q) to the object $(M(1), Q(1), q(1))$, where $F^{i+1}(M(1)) = (F^iM)(1)$, $Q(1)_{i+1} = Q_i(1)$, and $q(1)_{i+1} = q_i(1)$.

Denote by $\psi\colon \mathcal F \longrightarrow \mathcal A$ the forgetful exact functor taking an object $(M, Q, q) = (M, F, Q, q) \in \mathcal F$ to the object $M \in \mathcal A$. Let $\mathcal J$ be a full subcategory of $\mathcal E_0$ such that any object of $\mathcal E_0$ is a finite direct sum of objects from $\mathcal J$.

Proposition 3.1. (a) *One has* $\mathrm{Ext}^n_{\mathcal F}(X, Y) = 0$ *for any two objects* $X \in \mathcal E_i$ *and* $Y \in \mathcal E_j \subset \mathcal F$ *and any* $n > j - i$*;*

(b) *the maps* $\mathrm{Ext}^n_{\mathcal F}(X, Y) \longrightarrow \mathrm{Ext}^n_{\mathcal A}(\phi_i(X), \phi_j(X))$ *are isomorphisms for all* $i < j$*,* $n = 0$ *or* 1 *and monomorphisms for all* $i, j \in \mathbb Z$*,* $n = 2$*;*

(c) *the big graded ring of diagonal cohomology* $\mathrm{Ext}^n_{\mathcal F}(X, Y(n))_{Y,X\in\mathcal J;\,n\geqslant 0}$ *is quadratic (see* [12, Subsections A.1 and 6.1] *for the definitions).*

Proof. The cases $n = 0$ and 1 in part (a) are obvious. Applying the result of [12, Theorem 6.1] to the derived category $\mathcal{D} = \mathcal{D}^b(\mathcal{F})$ and its full subcategories $\mathcal{E}_i \subset \mathcal{F} \subset \mathcal{D}$, one obtains the rest of part (a) together with part (c). Applying the result of [12, Theorem 3.1(2)] to the sequence of exact functors $\phi_i\colon \mathcal{E}_i \longrightarrow \mathcal{A}$ provides part (b). $\square$

We will say that the exact category $\mathcal{A}$ together with the additive categories $\mathcal{E}_i$ and additive functors $\phi_i\colon \mathcal{E}_i \longrightarrow \mathcal{A}$ *satisfy the main hypothesis* if the maps

$$\psi^n\colon \operatorname{Ext}^n_{\mathcal{F}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(\phi_i(X),\phi_j(Y))$$

are isomorphisms for all the objects $X \in \mathcal{E}_i$ and $Y \in \mathcal{E}_j \subset \mathcal{F}$ and all integers $n \leqslant j-i$. In particular, this condition for $n = 0$, $i = j$ means that the functors ϕ_i are fully faithful. With this fact in mind, we will often consider the categories $\mathcal{E}_i$ as full additive subcategories in the exact category $\mathcal{A}$ and the functors $\mathcal{E}_i \longrightarrow \mathcal{A}$ as identity embeddings, dropping the notation ϕ_i. Abusing terminology, we will simply speak of the exact category $\mathcal{F}$ or, sometimes, the exact functor $\psi\colon \mathcal{F} \longrightarrow \mathcal{A}$ as satisfying (or not satisfying) the main hypothesis.

It follows from Proposition 3.1(c) that the main hypothesis implies quadraticity of the big graded ring $(\operatorname{Ext}_{\mathcal{A}}(X,Y(n))_{Y,X\in\mathcal{J};\,n\geqslant 0}$. When the twist $(1)\colon \mathcal{A} \longrightarrow \mathcal{A}$ is the identity functor, the main result of the paper [12] allows one to say more.

Theorem 3.2. *Assume that all the functors $\phi_i\colon \mathcal{E}_i \longrightarrow \mathcal{A}$ are fully faithful embeddings of one and the same full additive subcategory $\mathcal{E}_0 \subset \mathcal{A}$. Then the main hypothesis holds if and only if the big graded ring* $\operatorname{Ext}^n_{\mathcal{A}}(X,Y(n))_{Y,X\in\mathcal{J};\,n\geqslant 0}$ *is Koszul (see* [12, Section 7] *for the definition and discussion).*

Proof. This is essentially the assertion of [12, Theorem 9.1]. $\square$

In the rest of this section we discuss a specific class of examples of exact categories $\mathcal{A}$ with additive functors $\phi_i\colon \mathcal{E}_i \longrightarrow \mathcal{A}$ for which we would like to be able to show that an appropriate Koszulity assumption guarantees validity of the main hypothesis, even though the twist functor $(1)\colon \mathcal{A} \longrightarrow \mathcal{A}$ is not isomorphic to the identity, but only becomes so after the passage to the reduction of the exact category $\mathcal{A}$ by a certain element of its center.

3.2. **Reduction of representation categories.** Let k be a complete Noetherian local ring with the maximal ideal $l \subset k$. Pick an injective hull I of the k-module k/l in the abelian category of k-modules. A k-module M is said to be *discrete* if for every element $m \in M$ there exists an integer $N \geqslant 1$ such that $l^N m = 0 \subset M$. In particular, the k-module I is discrete. A discrete k-module M is said to be *of finite type* if its submodule of elements annihilated by l is a finite-dimensional k-vector space.

The category of discrete k-modules is a locally Noetherian Grothendieck abelian category with a single isomorphism class of indecomposable injective modules formed by the k-module I; every injective object in the category of discrete k-modules is a direct sum of copies of the module I. We will denote by $\mathcal{A}_k^{\{e\}+}$ the additive category of injective discrete k-modules and by $\mathcal{A}_k^{\{e\}} \subset \mathcal{A}_k^{\{e\}+}$ its full subcategory of injective discrete k-modules of finite type. The subcategory $\mathcal{A}_k^{\{e\}}$ consists of all the objects in $\mathcal{A}_k^{\{e\}+}$ isomorphic to finite direct sums of the object I.

Furthermore, let G be a profinite group. Let us denote by $\mathcal{A}_k^{G+}$ the category of injective discrete k-modules endowed with a discrete action of the group G, and by $\mathcal{A}_k^G$ the full subcategory of $\mathcal{A}_k^{G+}$ formed by injective discrete k-modules of finite type endowed with a discrete G-action. The categories $\mathcal{A}_k^G$ and $\mathcal{A}_k^{G+}$ are endowed with exact category structures in which a short sequence is exact if it is (split) exact as a short sequence of discrete k-modules.

Let $s \in k$ be a noninvertible, nonzero-dividing element and $k/s = k/(s)$ be the quotient ring by the principal ideal generated by s. Denote by $\pi\colon \mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}+}$ the exact functor assigning to a discrete G-module $M \in \mathcal{A}_k^{G+}$ over k the k/s-module $\pi(M) = {}_sM \subset M$ of s-torsion elements in M. So the functor π is the diagonal composition in the commutative square diagram of the reduction and forgetful functors $\mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_{k/s}^{G+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}+}$ and $\mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_k^{\{e\}+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}+}$.

An alternative description of the category $\mathcal{A}_k^{G+}$ and the functor π is provided by the theory of k-contramodule coalgebras and comodules developed in the paper [14, Sections 1 and 3]. The ring k can be considered as (a very particular case of) a pro-Artinian topological local ring [14, Section 1 and Appendix B]. Continuous functions $G \longrightarrow k$ in the l-adic topology of k form a k-free k-contramodule coalgebra $k(G)$, and the category $\mathcal{A}_k^{G+}$ is that of k-cofree k-comodule $k(G)$-comodules, or, equivalently, k-free k-contramodule $k(G)$-comodules.

The choice of an injective k-module I as above fixes an equivalence between these two representations of objects of the category $\mathcal{A}_k^{G+}$ given by the usual functors $P = \Psi_I(M) = \mathrm{Hom}_k(I, M)$ and $M = \Phi_I(P) = I \otimes_k P$. While on the level of k- and G-discrete G-modules over k the functor π assigns to a module M its maximal submodule annihilated by s, on the level of k-free k-contramodule $k(G)$-comodules it assigns to a module P its quotient module P/sP.

Proposition 3.3. *Let $\mathcal{A}_k^{G+}/s$ denote the reduction of the exact category $\mathcal{A}_k^{G+}$ by the natural transformation $s\colon \mathrm{Id} \longrightarrow \mathrm{Id}$ taken in presence of the background functor $\pi\colon \mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}+}$. Then the natural exact functor $\mathcal{A}_k^{G+}/s \longrightarrow \mathcal{A}_{k/s}^{G+}$ is an equivalence of exact categories.*

Proof. The natural exact functor assigns to a diagram $V \longrightarrow U \longrightarrow V \longrightarrow U$ the image of the map ${}_sU \longrightarrow {}_sV$ in the category $\mathcal{A}_{k/s}^{G+}$. To construct the inverse functor, consider an object $M \in \mathcal{A}_{k/s}^{G+}$. Viewed as an object of the abelian category of discrete k-modules with a discrete G-action, it can be embedded into a k- and $k(G)$-injective k-discrete $k(G)$-comodule V. The element s being a nonzero-divizor, the quotient module $U = V/M$ is also k-injective. The multiplication map $s\colon V \longrightarrow V$ annihilates M, so it factorizes naturally through the surjection $V \longrightarrow U$. The composition $U \longrightarrow V \longrightarrow U$ is also equal to the multiplication with s, as one can see by composing it with the same surjection $V \longrightarrow U$. We have constructed the desired matrix factorization in $\mathcal{A}_k^{G+}$; since it consists of surjective maps, the passage to the submodules of elements annihilated by s transforms it into an exact sequence of k/s-injective objects in the abelian category of discrete k/s-modules with a discrete action of G. The k/s-modules of cocycles in this exact sequence are injective, since such is the one of them that is isomorphic to M by construction. $\square$

Let $H \subset G$ be a fixed closed normal subgroup in the profinite group G. Denote by $\mathcal{E}_{k,0}^{G/H}$ the category of k-injective permutational G/H-modules of finite type, i. e., the full subcategory of $\mathcal{A}_k^G$ consisting of modules induced from trivial representations of open subgroups in G containing H in injective k-modules of finite type. Similarly, let $\mathcal{E}_{k,0}^{G/H+}$ denote the category of arbitrary k-injective permutational discrete G/H-modules, i. e., the full subcategory in $\mathcal{A}_k^{G+}$ whose objects are the direct sums of objects from $\mathcal{E}_{k,0}^{G/H}$. The categories $\mathcal{E}_{k,0}^{G/H}$ and $\mathcal{E}_{k,0}^{G/H+}$ are endowed with split exact category structures.

Consider the reduction functor $\pi\colon \mathcal{E}_{k,0}^{G/H} \longrightarrow \mathcal{E}_{k/s,0}^{G/H}$ taking a k-injective permutational G/H-module M to the k/s-injective permutational G/H-module ${}_sM$, and denote by $f \longmapsto \bar f$ its action on morphisms in the two categories; and similarly for the reduction functor $\pi\colon \mathcal{E}_{k,0}^{G/H+} \longrightarrow \mathcal{E}_{k/s,0}^{G/H+}$ and its action on morphisms.

Lemma 3.4. (a) *Let M and N be k-injective permutational G/H-modules of finite type over k, i. e., objects of the category $\mathcal{E}_{k,0}^{G/H}$. Then every morphism $\bar f\colon {}_sM \longrightarrow {}_sN$ in the category the category $\mathcal{E}_{k/s,0}^{G/H}$ can be lifted to a morphism $f\colon M \longrightarrow N$. Moreover, a morphism f is an admissible monomorphism or admissible epimorphism if and only if the morphism $\bar f$ is, and a sequence of permutational G/H-modules with zero composition is exact in $\mathcal{E}_{k,0}^{G/H}$ if and only if its reduction modulo s is.*

(b) *Let M and N be arbitrary k-injective discrete permutational G/H-modules over k, i. e., objects of the category $\mathcal{E}_{k,0}^{G/H+}$. Then every morphism $\bar f\colon {}_sM \longrightarrow {}_sN$ in the category $\mathcal{E}_{k/s,0}^{G/H+}$ can be lifted to a morphism $f\colon M \longrightarrow N$. Moreover, a morphism f is an admissible monomorphism or admissible epimorphism if and only if the morphism $\bar f$ is, and a sequence of permutational G/H-modules with zero composition is exact in $\mathcal{E}_{k,0}^{G/H+}$ if and only if its reduction modulo s is.*

Proof. Part (a): the G-module $\mathrm{Hom}_k(M,N)$ is a permutational k-free G/H-module of finite rank, while the G-module $\mathrm{Hom}_{k/s}({}_sM, {}_sN)$ is its reduction modulo s. Clearly, the reduction map $\mathrm{Hom}_k(M,N) \longrightarrow \mathrm{Hom}_{k/s}({}_sM, {}_sN)$ identifies the submodule of G-invariants in $\mathrm{Hom}_{k/s}({}_sM, {}_sN)$ with the reduction of the submodule of G-invariants in $\mathrm{Hom}_k(M,N)$, proving the first assertion. Furthermore, by a version of Nakayama's lemma a morphism in the category $\mathcal{E}_{k,0}^{G/H}$ is an isomorphism if and only if so is its image in the category $\mathcal{E}_{k/s,0}^{G/H}$. The remaining assertions now follow from the fact any k/s-injective permutational G/H-module of finite type can be lifted to a similar k-injective permutational G/H-module.

Part (b): in this case, the G-module $\mathrm{Hom}_k(M,N)$ is an infinite product of infinite direct sums (in the k-contramodule category) of permutational k-free G/H-modules of finite rank, while the G-module $\mathrm{Hom}_{k/s}({}_sM, {}_sN)$ is the reduction of this infinite product of infinite direct sums modulo s [14, Subsections 1.2–1.3 and 1.5]. Otherwise, the argument works in the same way. One only has to check that the passage to G-invariants commutes with contramodule infinite direct sums of permutational representations and the reductions of such contramodule representations by the ideals in k. Indeed, the direct sum of a family of free k-contramodules P_i is the set of all families of elements $p_i \in P_i$ such that for any $n > 0$ all but a finite number of p_i belong to $l^n P_i$. The key observation is that for a permutational k-free G-module P (of finite rank over k), the k-submodules $(l^n P)^G$ and $l^n(P^G)$ coincide in P. $\square$

Corollary 3.5. (a) *The reduction functor* $\pi\colon \mathcal{E}_{k,0}^{G/H} \longrightarrow \mathcal{E}_{k/s,0}^{G/H}$ *satisfies the conditions (v-viii) of Subsection 2.1. One obtains the category* $\mathcal{E}_{k/s,0}^{G/H}$ *as the outcome of the reduction procedure.*

(b) *The reduction functor* $\pi\colon \mathcal{E}_{k,0}^{G/H+} \longrightarrow \mathcal{E}_{k/s,0}^{G/H+}$ *satisfies the conditions (v-viii) of Subsection 2.1. One obtains the category* $\mathcal{E}_{k/s,0}^{G/H+}$ *as the outcome of the reduction procedure.*

Proof. By Lemma 3.4, the reduction functors π are exact-conservative. The remaining verifications are easy. $\square$

3.3. **Large and small filtered representation categories.** Let $\bar k$ be a discrete Artinian ring and G be a profinite group. As above, we denote by $\mathcal{A}_{\bar k}^{G+}$ the exact category of $\bar k$-injective discrete G-modules over $\bar k$, and by $\mathcal{A}_{\bar k}^{G}$ its full exact subcategory formed by injective $\bar k$-modules of finite type endowed with a discrete action of G.

Lemma 3.6. (a) *The embedding functor* $\mathcal{A}_{\bar k}^{G} \longrightarrow \mathcal{A}_{\bar k}^{G+}$ *satisfies the condition (i′) of Subsection 0.1; in particular, it induces isomorphisms on all the Ext groups.*

(b) *The functor Ext in the exact category* $\mathcal{A}_{\bar k}^{G+}$ *transforms infinite direct sums in its first argument into infinite products. When its first argument belongs to* $\mathcal{A}_{\bar k}^{G}$*, the functor Ext in the exact category* $\mathcal{A}_{\bar k}^{G+}$ *transforms infinite direct sums in the second argument into infinite direct sums.*

Proof. Choosing an injective $\bar k$-module I as in Subsection 3.2 allows to identify $\mathcal{A}_{\bar k}^{G+}$ with the category of $\bar k$-projective discrete G-modules over $\bar k$ and $\mathcal{A}_{\bar k}^{G}$ with its exact subcategory of projective $\bar k$-modules of finite rank endowed with a discrete G-action. In this setting, the assertions of Lemma become true for any discrete commutative ring $\bar k$. To prove part (a), notice that any element in a discrete G-module over $\bar k$ generates a submodule isomorphic to a quotient module of a (permutational) $\bar k$-projective discrete G-module of finite rank over $\bar k$.

The first assertion in part (b) holds for any exact category with enough injective objects, and the second one follows from the similar property of the groups Hom and the first assertion of part (a) (alternatively, it can be deduced from the fact that infinite direct sums are exact and preserve injective objects in $\mathcal{A}_{\bar k}^{G+}$). Another alternative way of arguing is to compare the groups Ext in the exact categories $\mathcal{A}_{\bar k}^{G+}$

and $\mathcal{A}_{\bar{k}}^G$ with the similar groups in the abelian categories of arbitrary and $\bar{k}$-finitely generated discrete G-modules over $\bar{k}$ (assuming $\bar{k}$ Noetherian). $\square$

Let $H \subset G$ be a closed normal subgroup and $\bar{c}\colon G \longrightarrow \bar{k}^*$ be a discrete multiplicative character of the profinite group G.

As in Subsection 3.2, we denote by $\mathcal{E}_{\bar{k},0}^{G/H} \subset \mathcal{A}_{\bar{k}}^G$ the full subcategory of $\bar{k}$-injective permutational G/H-modules of finite type and by $\mathcal{E}_{\bar{k},0}^{G/H\,+} \subset \mathcal{A}_{\bar{k}}^{G+}$ the full subcategory of $\bar{k}$-injective permutational discrete G/H-modules of possibly infinite type. For any $i \in \mathbb{Z}$, denote by $\mathcal{E}_{\bar{k},i}^{G/H} \subset \mathcal{A}_{\bar{k}}^G$ and $\mathcal{E}_{\bar{k},i}^{G/H\,+} \subset \mathcal{A}_{\bar{k}}^{G+}$ the full subcategories of objects obtained by twisting objects of the full subcategories $\mathcal{E}_{\bar{k},0}^{G/H}$ and $\mathcal{E}_{\bar{k},0}^{G/H\,+}$, respectively, with the character $\bar{c}^i$ (the i-th power of the character $\bar{c}$).

Let $\mathcal{E}_{\bar{k}}^G$ denote the category of finitely supported graded objects $Q = (Q_i)$ in the category $\mathcal{A}_{\bar{k}}^G$ in which the object Q_i belongs to the subcategory $\mathcal{E}_{\bar{k},i}^{G/H}$. Similarly, $\mathcal{E}_{\bar{k}}^{G+}$ denotes the category of finitely supported graded objects $Q = (Q_i)$ in the category $\mathcal{A}_{\bar{k}}^{G+}$ in which the object Q_i belongs to the subcategory $\mathcal{E}_{\bar{k},i}^{G/H\,+}$. The categories $\mathcal{E}_{\bar{k}}^G$ and $\mathcal{E}_{\bar{k}}^{G+}$ are endowed with split exact category structures.

As in Subsection 3.1, we consider the exact category $\mathcal{F}_{\bar{k}}^G$ of finitely filtered objects of the category $\mathcal{A}_{\bar{k}}^G$ with the associated quotient modules belonging to $\mathcal{E}_{\bar{k}}^G$. The twist functor (1) is the twist with the character $\bar{c}$. Similarly, $\mathcal{F}_{\bar{k}}^{G+}$ is the exact category of finitely filtered objects of the category $\mathcal{A}_{\bar{k}}^{G+}$ with the associated quotient modules belonging to $\mathcal{E}_{\bar{k}}^{G+}$. Finally, let $\mathcal{F}_{\bar{k},[j',j'']}^{G+} \subset \mathcal{F}_{\bar{k}}^{G+}$ denote the full exact subcategory formed by all the triples (M, Q, q) where the graded object $Q = (Q_i)$ is supported in the segment of degrees $j' \leqslant i \leqslant j''$.

Proposition 3.7. (a) *Any object of the category $\mathcal{F}_{\bar{k}}^{G+}$ is the inductive limit of a directed diagram of objects from $\mathcal{F}_{\bar{k}}^G$ and admissible monomorphisms between them. In particular, the embedding functor $\mathcal{F}_{\bar{k}}^G \longrightarrow \mathcal{F}_{\bar{k}}^{G+}$ satisfies the condition (i′) of Subsection 0.1, so it induces isomorphisms on all the Ext groups.*

(b) *The functor Ext in the exact category $\mathcal{F}_{\bar{k},[j',j'']}^{G+}$ transforms infinite direct sums in its first argument into infinite products. When its first argument belongs to $\mathcal{F}_{\bar{k}}^G$, the functor Ext in the exact category $\mathcal{F}_{\bar{k},[j',j'']}^{G+}$ transforms infinite direct sums in the second argument into infinite direct sums.*

Proof. Part (a): let (M, Q, q) be an object of the category $\mathcal{F}_{\bar{k},[j',j'']}^{G+} \subset \mathcal{F}_{\bar{k}}^{G+}$. Let us fix decompositions of the objects Q_i into direct sums of twists of $\bar{k}$-injective permutational representations of G/H of finite type over $\bar{k}$, and show that for any finitely generated G-submodule $K \subset M$ there exists a finitely generated G-submodule $K \subset N \subset M$ such that the associated graded modules $\mathrm{gr}_F^i N \subset Q_i$ are direct sums of finite subcollections of direct summands in our fixed direct sum decompositions. Proceeding by induction in $j'' - j'$, consider the object $M_1 = M/F^{j''}M \in \mathcal{F}_{\bar{k},[j',j''-1]}^{G+}$ and the submodule $K_1 = K/F^{j''}M \cap K \subset M_1$; let $K_1 \subset N_1 \subset M_1$ be the related intermediate submodule satisfying the above condition. We have obtained an extension class in $\mathrm{Ext}^1_{\mathcal{A}_{\bar k}^{G+}}(N_1, Q_{j''})$, and now it suffices to show that it is induced from an extension class with a direct sum of a finite subcollection of the direct summands in the second argument. This is a particular case of the second assertion of Lemma 3.6(b).

To prove the first assertion of part (b), it suffices to show that there are enough injective objects in the exact category $\mathcal{F}_{\bar k,\,[j',j'']}^{G+}$. To construct such injective objects, we proceed again by induction in the length of the filtration $j''-j'$. Suppose J_1 is an injective object in $\mathcal{F}_{\bar k,\,[j'+1,j'']}^{G+}$. For every $\bar c^{j'}$-twisted $\bar k$-injective permutational representation R of G/H of finite type over $\bar k$ assign a nonempty multiplicity set to every element of the extension group $\mathrm{Ext}^1_{\mathcal{A}_{\bar k}^{G+}}(R, \psi(J_1))$, where $\psi\colon \mathcal{F}_{\bar k}^{G+} \longrightarrow \mathcal{A}_{\bar k}^{G+}$ is the forgetful functor, and denote by Q_R the direct sum of copies of R over the disjoint union of these sets. By the first assertion of Lemma 3.6(b), there is a naturally induced element in $\mathrm{Ext}^1_{\mathcal{A}_{\bar k}^{G+}}(\bigoplus_R Q_R, \psi(J_1))$, where the direct sum is taken over all twisted permutational representations R of finite type (or just twisted representations induced from the standard trivial modules I over open subgroups in G containing G/H). This extension class corresponds to an injective object $J \in \mathcal{F}_{\bar k,\,[j',j'']}^{G+}$, and choosing the multiplicity sets big enough one can construct an admissible monomorphism from any object of $\mathcal{F}_{\bar k,\,[j',j'']}^{G+}$ into an injective object constructed in this way. The second assertion of part (b) follows from the second assertion of part (a). $\square$

Remark 3.8. Applied to $\bar k$-projective instead of $\bar k$-injective G-modules over $\bar k$ and permutational representations of the related kind, the above arguments hold for any discrete Noetherian ring $\bar k$. However, they fail over a complete Noetherian local ring k (as in Subsection 3.2), because injective k-modules of finite type are not finitely generated, while infinite direct sums in the category of projective k-contramodules are not preserved by the forgetful functor to k-modules (or just abelian groups).

Corollary 3.9. *The forgetful functor $\mathcal{F}_{\bar k}^{G+} \longrightarrow \mathcal{A}_{\bar k}^{G+}$ satisfies the main hypothesis of Subsection 3.1 if and only if the functor $\mathcal{F}_{\bar k}^{G} \longrightarrow \mathcal{A}_{\bar k}^{G+}$ does, and if and only if the functor $\mathcal{F}_{\bar k}^{G} \longrightarrow \mathcal{A}_{\bar k}^{G}$ does.*

Proof. Follows from Lemma 3.6(a-b) and Proposition 3.7(a-b). $\square$

Now let us suppose that $\bar k$ is a field and the character $\bar c\colon G \longrightarrow \bar k^*$ annihilates the closed subgroup $H \subset G$. Let $\mathcal{J} \subset \mathcal{E}_{\bar k,\,0}^{G}$ denote the full subcategory of representations induced from trivial representations $\bar k$ of open subgroups in G containing H.

Lemma 3.10. *The exact category $\mathcal{F}_{\bar k}^{G}$ satisfies the main hypothesis if and only if the big graded ring $\mathrm{Ext}^n_{\mathcal{A}_{\bar k}^{G}}(X, Y(n))_{Y,X\in\mathcal{J},\,n\geqslant0}$ is Koszul. In this case, the big graded ring $\mathrm{Ext}^n_{\mathcal{F}_{\bar k}^{G}}(X,Y)_{Y\in\mathcal{E}_{\bar k,\,j}^{G},\,X\in\mathcal{E}_{\bar k,\,i}^{G};\,i,j\in\mathbb{Z};\,n\geqslant0}$ is generated by its Ext^0 and the diagonal ($n=j-i$) extension classes. The latter property also hods for the extension classes between objects of the full subcategories $\mathcal{E}_{\bar k,\,i}^{G+}$ and $\mathcal{E}_{\bar k,\,j}^{G+}$ in the exact category $\mathcal{F}_{\bar k}^{G+}$.*

Proof. The image of the character $\bar{c}$, being a compact subgroup of a discrete group, is therefore a finite subgroup of the multiplicative group of the field $\bar{k}$, whose order is consequently prime to the characteristic. Therefore, the representation $\bar{k}(1)$ of the group G corresponding to the character $\bar{c}$ is a direct summand of a permutational one. Since we have assumed that $\bar{c}$ annihilates H, this is (can be chosen to be) also a permutational representation of G/H. Twisting with the character $\bar{c}$ transforms permutational representations of G/H over $\bar{k}$ into direct summands of permutational representations of G/H.

So we see that the category $\mathcal{F}_{\bar{k}}^{G}$ only differs from the similar category constructed using the trivial character in place of $\bar{c}$ by adjoining some direct summands and removing some others (cf. [12, Section 5]). Now the first assertion follows from Theorem 3.2, and the second and third ones are also clear. □

3.4. **Reduction of filtered representation category.** Let G be a profinite group, $H \subset G$ be a closed normal subgroup, k be a complete Noetherian local ring with the maximal ideal l, and $c\colon G \longrightarrow k^*$ be a continuous multiplicative character of the profinite group G in the l-adic topology of the ring k.

As in Subsections 3.2–3.3 (see also Example 2.3 in Subsection 2.2), we denote by $\mathcal{A}_k^{G+}$ the category of injective discrete k-modules endowed with a discrete action of the group G, and by $\mathcal{E}_{k,\,i}^{G/H\,+} \subset \mathcal{A}_k^{G+}$ its full subcategory of c^i-twisted k- and G-discrete k-injective permutational G/H-modules (of possibly infinite type over k). Let $\mathcal{E}_k^{G+}$ denote the category of finitely supported graded objects $Q = (Q_i)$ in the category $\mathcal{A}_k^{G+}$ in which the object Q_i belongs to the the subcategory $\mathcal{E}_{k,\,i}^{G/H\,+}$.

As in Subsection 3.1, we consider the exact category $\mathcal{F}_k^{G+}$ of finitely filtered objects (M, F) of the exact category $\mathcal{A}_k^{G+}$ with the associated graded modules $Q = (Q_i)$, $q_i\colon \mathrm{gr}_F^i M \simeq Q_i$ belonging to $\mathcal{E}_k^{G+}$. The twist functor $(1)\colon \mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_k^{G+}$ is the twist with the character c. We denote by $\mathrm{gr}_F\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{E}_k^{G+}$ the exact functor assigning the graded object Q to a filtered object (M, F) and by $\psi\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{A}_k^{G+}$ the forgetful functor taking (M, F) to M.

Let $s \in k$ be a nonzero-dividing, noninvertible element. The quotient ring k/s is endowed with a continuous multiplicative character $c/s\colon G \longrightarrow (k/s)^*$, so the above exact categories have their natural versions with coefficients in k/s.

Denote by $\pi'\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{E}_{k/s}^{G\,+}$ the composition of the functor gr_F with the reduction functor $\mathcal{E}_k^{G+} \longrightarrow \mathcal{E}_{k/s}^{G\,+}$ taking (Q_i) to $({}_sQ_i)$. Furthermore, let $\mathcal{A}_k^{\{e\}\,+}$ denote the additive category of injective discrete k-modules. Denote by $\pi''\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}\,+}$ the composition of the forgetful functor ψ with the forgetful functor $\mathcal{A}_k^{G+} \longrightarrow \mathcal{A}_k^{\{e\}\,+}$ and the reduction functor $\mathcal{A}_k^{\{e\}\,+} \longrightarrow \mathcal{A}_{k/s}^{\{e\}\,+}$ taking M to ${}_sM$.

Both $\mathcal{E}_{k/s}^{G\,+}$ and $\mathcal{A}_{k/s}^{\{e\}\,+}$ are naturally endowed with split exact category structures. Consider the Cartesian product $\mathcal{E}_{k/s}^{G\,+} \times \mathcal{A}_{k/s}^{\{e\}\,+}$, and let $\pi = (\pi', \pi'')\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{E}_{k/s}^{G\,+} \times \mathcal{A}_{k/s}^{\{e\}\,+}$ denote the related diagonal exact functor.

Lemma 3.11. *The exact functor $\pi = (\pi', \pi'')\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{E}_{k/s}^{G+} \times \mathcal{A}_{k/s}^{\{e\}+}$ satisfies the conditions (v-vii) from Subsection 2.1 on a background exact functor for the reduction construction of the exact category $\mathcal{F} = \mathcal{F}_k^{G+}$ with respect to the natural transformation $s\colon \mathrm{Id}_\mathcal{F} \longrightarrow \mathrm{Id}_\mathcal{F}$. The condition (viii) is also satisfied.*

Proof. The condition (viii) on the natural transformation s holds because $s \in k$ is a nonzero-dividing element. The condition (vi) is obvious from the construction. The condition (v) holds due to the presence of the component π' in the functor π and by Lemma 3.4(b), while the condition (vii) is true due to the presence of the component π'' in π. $\square$

Applying the construction of Section 2, we obtain the reduced exact category $\mathcal{F}_k^{G+}/s$. There is a natural exact comparison functor $\varkappa\colon \mathcal{F}_k^{G+}/s \longrightarrow \mathcal{F}_{k/s}^{G+}$ assigning to a matrix factorization diagram $V \longrightarrow U \longrightarrow V \longrightarrow U$ for the center element s in the exact category $\mathcal{F}_k^{G+}$ the image of the morphism ${}_sU \longrightarrow {}_sV$ in the sequence ${}_sV \longrightarrow {}_sU \longrightarrow {}_sV \longrightarrow {}_sU$ in the exact category $\mathcal{F}_{k/s}^{G+}$ (the sequence being exact, because so is its associated graded sequence by the filtration F).

Furthermore, every object of the exact category $\mathcal{F}_k^{G+}/s$ comes endowed with a natural finite filtration F given by the natural filtration on the matrix factorization diagrams. The filtrations F on the objects of $\mathcal{F}_k^{G+}/s$ are preserved by all morphisms in this category. The objects of $\mathcal{F}_k^{G+}/s$ concentrated in a single filtration degree i form a full exact subcategory $\mathcal{E}_{k,i}^{G+}/s \simeq \mathcal{E}_{k/s,i}^{G+}$ (see Corollary 3.5(b)), so the categories $\mathcal{E}_{k/s,i}^{G+}$ are embedded into $\mathcal{F}_k^{G+}/s$. The triangle diagrams of exact functors $\mathcal{E}_{k/s,i}^{G+} \longrightarrow \mathcal{F}_k^{G+}/s \longrightarrow \mathcal{F}_{k/s}^{G+}$ are commutative.

Proposition 3.12. *The comparison functor $\varkappa\colon \mathcal{F}_k^{G+}/s \longrightarrow \mathcal{F}_{k/s}^{G+}$ is fully faithful.*

Proof. Essentially, the claim is that the induced morphisms of the Ext groups

$$\varkappa^n\colon \operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{k/s}^{G+}}(X,Y)$$

are isomorphisms for $n=1$ and monomorphisms for $n=2$. In view of the 5-lemma, it suffices to show this for objects $X \in \mathcal{E}_{k/s,i}^{G+}$ and $Y \in \mathcal{E}_{k/s,j}^{G+}$ (cf. [12, Lemma 3.2]).

Taking into account Proposition 3.3, the forgetful functor $\psi\colon \mathcal{F}_k^{G+} \longrightarrow \mathcal{A}_k^{G+}$ induces a morphism between the long exact sequences

$$\operatorname{Ext}^n_{\mathcal{F}_k^{G+}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_k^{G+}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX,{}_sY) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{F}_k^{G+}}(X,Y)$$

and

$$\operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX,{}_sY) \longrightarrow \operatorname{Ext}^{n+1}_{\mathcal{A}_k^{G+}}(X,Y)$$

from Subsection 2.1 for any two objects $X \in \mathcal{E}_{k,i}^{G+}$ and $Y \in \mathcal{E}_{k,j}^{G+}$. According to Proposition 3.1(a), one has $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}}(X,Y)=0$ and consequently $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}(X,Y)=0$ for $n > j-i$. By Proposition 3.1(b), the maps $\psi^n\colon \operatorname{Ext}^n_{\mathcal{F}_k^{G+}}(X,Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X,Y)$ are isomorphisms for $i<j$ and $n=0$ or 1 and monomorphisms for $n=2$. Applying

the 5-lemma, we conclude that the map $\mathrm{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX,{}_sY)\longrightarrow\mathrm{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX,{}_sY)$ is an isomorphism for $n=0$ and a monomorphism for $n=1$ whenever $i<j$. It remains to compare these computations with the description of $\mathrm{Ext}^n_{\mathcal{F}_{k/s}^{G+}}({}_sX,{}_sY)$ provided by the same Proposition, keeping in mind the commutative triangle of exact functors $\mathcal{F}_k^{G+}/s\longrightarrow\mathcal{F}_{k/s}^{G+}\longrightarrow\mathcal{A}_{k/s}^{G+}$. ☐

Example 3.13. The following example shows that the comparison functors $\varkappa$ are not equivalences of categories in general. Let l be a prime number and $k=\mathbb{Z}_l$ be the ring of l-adic integers. Let s and t be two powers of l. Then for any $X\in\mathcal{E}_{k,i}^{G+}$ and $Y\in\mathcal{E}_{k,j}^{G+}$ there is the long exact sequence of Subsection 2.8

$$\begin{aligned}0&\longrightarrow\mathrm{Hom}_{\mathcal{F}_k^{G+}/t}({}_tX,{}_tY)\longrightarrow\mathrm{Hom}_{\mathcal{F}_k^{G+}/st}({}_{st}X,{}_{st}Y)\longrightarrow\mathrm{Hom}_{\mathcal{F}_k^{G+}/s}({}_sX,{}_sY)\\&\longrightarrow\mathrm{Ext}^1_{\mathcal{F}_k^{G+}/t}({}_tX,{}_tY)\longrightarrow\mathrm{Ext}^1_{\mathcal{F}_k^{G+}/st}({}_{st}X,{}_{st}Y)\longrightarrow\mathrm{Ext}^1_{\mathcal{F}_k^{G+}/s}({}_sX,{}_sY)\\&\longrightarrow\mathrm{Ext}^2_{\mathcal{F}_k^{G+}/t}({}_tX,{}_tY)\longrightarrow\mathrm{Ext}^2_{\mathcal{F}_k^{G+}/st}({}_{st}X,{}_{st}Y)\longrightarrow\cdots\end{aligned}$$

Set $X=\mathbb{Q}_l/\mathbb{Z}_l\in\mathcal{E}_{k,0}^{G+}$ and $Y=(\mathbb{Q}_l/\mathbb{Z}_l)(1)\in\mathcal{E}_{k,1}^{G+}$; according to the above, one then has $\mathrm{Ext}^2_{\mathcal{F}_k^{G+}/t}({}_tX,{}_tY)=0$, so the map $\mathrm{Ext}^1_{\mathcal{F}_k^{G+}/st}({}_{st}X,{}_{st}Y)\longrightarrow\mathrm{Ext}^1_{\mathcal{F}_k^{G+}/s}({}_sX,{}_sY)$ is surjective. Assuming the equivalences $\mathcal{F}_k^{G+}/s\simeq\mathcal{F}_{k/s}^{G+}$ and $\mathcal{F}_k^{G+}/st\simeq\mathcal{F}_{k/st}^{G+}$, this would simply mean that the natural cohomology map $H^1(G,\mathbb{Z}/st(1))\longrightarrow H^1(G,\mathbb{Z}/s(1))$ is surjective for the profinite group G with the character $c\colon G\longrightarrow\mathbb{Z}_l^*$ (irrespectively of the closed subgroup H). However, this is not always true. When G is the absolute Galois group of a field and c is its cyclotomic character, the surjectivity of such maps follows from Hilbert's Theorem 90.

As it was mentioned in Subsection 2.9, there is apparently no straightforward way to construct reduction functors $\eta\colon\mathcal{F}_k^{G+}/st\longrightarrow\mathcal{F}_k^{G+}/s$ on the matrix factorization diagrams level. However, there *are* natural functors $\rho\colon\mathcal{F}_{k/st}^{G+}\longrightarrow\mathcal{F}_{k/s}^{G+}$ assigning to an injective discrete k/st-module N with a filtration F and a discrete action of the group G the injective discrete k/s-module ${}_sN$ of elements annihilated by s in N with the induced filtration and the group action.

Proposition 3.14. *The functor $\rho\colon\mathcal{F}_{k/st}^{G+}\longrightarrow\mathcal{F}_{k/s}^{G+}$ takes the full subcategory $\mathcal{F}_k^{G+}/st\subset\mathcal{F}_{k/st}^{G+}$ into the full subcategory $\mathcal{F}_k^{G+}/s\subset\mathcal{F}_{k/s}^{G+}$.*

Proof. An object $N=(N,F)\in\mathcal{F}_{k/st}^{G+}$ belongs to $\mathcal{F}_k^{G+}/st$ if and only if there exists an object $M=(M,F)\in\mathcal{F}_k^{G+}$ such that the injective k/st-module N can be embedded into the injective k-module M in a way strictly compatible with the filtrations, compatible with the actions of k and G, and such that the induced embeddings $\mathrm{gr}_F^iN\longrightarrow{}_{st}\mathrm{gr}_F^iM$ are admissible monomorphisms in $\mathcal{E}_{k/st,i}^{G+}$. (Cf. the proof of Proposition 3.3.) Indeed, the "only if" part is obvious from the construction, and to prove the "if", one recalls that by Lemma 3.4(b) any admissible monomorphism in $\mathcal{E}_{k/st,i}^{G+}$ can be lifted to an admissible monomorphism in $\mathcal{E}_{k,i}^{G+}$.

Whenever such an embedding $N \longrightarrow M$ exists, its composition ${}_sN \longrightarrow N \longrightarrow M$ with the identity embedding ${}_sN \longrightarrow N$ provides an embedding ${}_sN \longrightarrow M$ showing that the object $({}_sN, F)$ belongs to $\mathcal{F}_k^{G+}/s$. $\square$

Theorem 3.15. *Suppose that the maps*

$$\operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$$

are surjective for all $X \in \mathcal{E}_{k,i}^{G+}$ and $Y \in \mathcal{E}_{k,j}^{G+}$ with $j - i = n$ and $n \geqslant 1$. Assume that the exact category $\mathcal{F}_k^{G+}$ satisfies the main hypothesis of Subsection 3.1. Then the comparison functor $\varkappa\colon \mathcal{F}_k^{G+}/s \longrightarrow \mathcal{F}_{k/s}^{G+}$ is an equivalence of exact categories, and the main hypothesis for the exact category $\mathcal{F}_{k/s}^{G+}$ is also satisfied.

Proof. From the morphism of long exact sequences considered in the proof of Proposition 3.12 and the main hypothesis for the exact category $\mathcal{F}_k^{G+}$ one can see by the way of the 5-lemma that the map $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ is an isomorphism for $n < j - i$ and a monomorphism for $n = j - i$. Furthermore, since the map $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X, Y)$ is surjective by the main hypothesis, surjectivity of the map $\operatorname{Ext}^n_{\mathcal{A}_k^{G+}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ implies surjectivity of the map $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ for $n = j - i$.

By Proposition 3.1(b), the map $\operatorname{Ext}^n_{\mathcal{F}_{k/s}^{G+}}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ is an isomorphism for $n \leqslant 1$, $n \leqslant j - i$ and a monomorphism for $n = 2$. From commutativity of the triangle diagram $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{k/s}^{G+}}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ we conclude that the map $\operatorname{Ext}^n_{\mathcal{F}_k^{G+}/s}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{F}_{k/s}^{G+}}({}_sX, {}_sY)$ is an isomorphism for $n \leqslant 1$ and a monomorphism for $n = 2$ (and all i, $j \in \mathbb{Z}$). According to [12, Lemma 3.2], it follows that the comparison functor $\varkappa\colon \mathcal{F}_k^{G+}/s \longrightarrow \mathcal{F}_{k/s}^{G+}$ is an equivalence of exact categories. Now we have proven that the map $\operatorname{Ext}^n_{\mathcal{F}_{k/s}^{G+}}({}_sX, {}_sY) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/s}^{G+}}({}_sX, {}_sY)$ is an isomorphism for $n \leqslant j - i$, that is the main hypothesis holds for the exact category $\mathcal{F}_{k/s}^{G+}$. $\square$

In order to formulate our main conjectures, let us pass to the following particular case of our general setting. Suppose that k is a complete discrete valuation ring with a uniformizing element $l \in k$. Let $c\colon G \longrightarrow k^*$ be a continuous multiplicative character of a profinite group G and $H \subset G$ be a closed normal subgroup annihilated by the reduced character $c/l\colon G \longrightarrow (k/l)^*$.

Conjecture 3.16. *Suppose that the natural maps*

$$\operatorname{Ext}^n_{\mathcal{A}_{k/l^{r+1}}^{G+}}({}_{l^{r+1}}X, {}_{l^{r+1}}Y(n)) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}_{k/l^r}^{G+}}({}_{l^r}X, {}_{l^r}Y(n))$$

are surjective for all $r > 0$ and $X, Y \in \mathcal{E}_{k,0}^{G/H+}$. Assume that the exact category $\mathcal{F}_{k/l}^{G+}$ satisfies the main hypothesis. Then the comparison functors

$$\varkappa\colon \mathcal{F}_k^{G+}/l^r \longrightarrow \mathcal{F}_{k/l^r}^{G+}$$

are equivalences of exact categories for all $r > 0$.

Conjecture 3.17. *Suppose that the natural maps*

$$\mathrm{Ext}^n_{\mathcal{A}^G_{k/l^{r+1}}}({}_{l^{r+1}}X,\, {}_{l^{r+1}}Y(n)) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}^G_{k/l^r}}({}_{l^r}X,\, {}_{l^r}Y(n))$$

are surjective for all $r > 0$ and X, $Y \in \mathcal{E}^{G/H}_{k,0}$. Then the exact categories $\mathcal{F}^G_{k/l^r}$ satisfy the main hypothesis whenever so does the exact category $\mathcal{F}^G_{k/l}$.

It is not difficult to deduce Conjecture 3.17 from Conjecture 3.16. Their surjectivity assumptions are equivalent by Lemma 3.6(a-b), while the Ext groups in the categories $\mathcal{F}^G_{k/l^r}$ and $\mathcal{F}^{G+}_{k/l^r}$ agree by Proposition 3.7(a). Assuming the comparison functors $\varkappa$ to be equivalences, the forgetful functor $\psi\colon \mathcal{F}^{G+}_k \longrightarrow \mathcal{A}^{G+}_k$ induces a morphism between the long exact sequences

$$\mathrm{Ext}^n_{\mathcal{F}^G_{k/l^{r'}}}({}_{l^{r'}}X,\, {}_{l^{r'}}Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}^G_{k/l^r}}({}_{l^r}X,\, {}_{l^r}Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{F}^G_{k/l^{r''}}}({}_{l^{r''}}X,\, {}_{l^{r''}}Y) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{F}^G_{k/l^{r'}}}({}_{l^{r'}}X,\, {}_{l^{r'}}Y)$$

and

$$\mathrm{Ext}^n_{\mathcal{A}^G_{k/l^{r'}}}({}_{l^{r'}}X,\, {}_{l^{r'}}Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}^G_{k/l^r}}({}_{l^r}X,\, {}_{l^r}Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}^G_{k/l^{r''}}}({}_{l^{r''}}X,\, {}_{l^{r''}}Y) \longrightarrow \mathrm{Ext}^{n+1}_{\mathcal{A}^G_{k/l^{r'}}}({}_{l^{r'}}X,\, {}_{l^{r'}}Y)$$

of Subsection 2.8 or 2.9 for any positive integers $r' + r'' = r$ and any objects $X \in \mathcal{E}^G_{k,i}$ and $Y \in \mathcal{E}^G_{k,j}$. Applying the 5-lemma, the main hypothesis for $\mathcal{F}^G_{k/l^r}$ follows from that for $\mathcal{F}^G_{k/l}$ by trivial induction in r.

In the situation relevant for Artin–Tate motives [12, Subsections 9.2–9.5], the coefficient ring $k = \mathbb{Z}_l$ is the ring of l-adic integers, the profinite group $G = G_K$ is the absolute Galois group of a field K of characteristic different from l, the character c is the cyclotomic one, and the closed normal subgroup $H \subset G$ corresponds to a Galois extension M/K with the field M containing a primitive l-root of unity. The surjectivity assumption of the Conjectures is then a reformulation of the Milnor–Bloch–Kato conjecture (proven in [15]), while the main hypothesis of Subsection 3.1 is the silly filtration conjecture [12, Conjecture 9.2].

The main hypothesis for the exact category $\mathcal{F}^G_{k/l}$ can be interpreted as a Koszulity condition by Lemma 3.10. The described setting includes the case of $H = \{e\}$ that appears in connection with Artin–Tate motivic sheaves [13].

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Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127994; and

Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312, Russia; and

Mathematics Department, Technion — Israel Institute of Technology, Haifa 32000, Israel

E-mail address: `posic@mccme.ru`