Semi-Infinite Algebraic Geometry

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In the most general terms:

Semi-infinite homological algebra = homological theory of mathematical objects of "semi-infinite nature".

Semi-infinite algebraic geometry = semi-infinite homological algebra of "doubly" infinite-dimensional algebraic varieties.

We will come to more specific definitions shortly.

Semi-infinite mathematical objects =

- objects that can be viewed as extending in both a "positive" and a "negative" direction
- with some natural "zero position" in between
- (perhaps defined up to a finite movement).

The roles of the "positive" and the "negative" variables are not symmetric, in that the "positive" coordinates are grouped together in some sense.

The most basic example of a semi-infinite mathematical object is the field of formal Laurent power series $k((z))$ over a ground field k, and many more complicated examples are constructed on the basis of this simplest example.

The field $k((z))$ can be viewed as the field of functions on the punctured formal disc $\operatorname{Spec} k((z)) = \operatorname{Spec} k[[z]] \setminus \{0\}$.

Examples

Semi-infinite algebraic object:

- the Lie algebra of vector fields on the punctured formal disc $k((z))d/dz$
- with its subalgebra $zk[[z]]d/dz \subset k((z))d/dz$ of vector fields that extend to and vanish at the origin.

Semi-infinite geometric object:

- the vector space $k((z))$, viewed as an infinite-dimensional affine space (a “doubly infinite-dimensional” space),
- fibered over its quotient space $k((z))/k[[z]]$, also viewed as an infinite-dimensional affine space,
- with infinite-dimensional affine fibers isomorphic to $k[[z]]$.

More generally, if G is an affine algebraic group over a field k, then the fibration $G(k((z))) \longrightarrow G(k((z)))/G(k[[z]])$ can be viewed as a semi-infinite object in algebraic geometry.

Semi-infinite homological algebra =

- homological algebra in the semiderived categories of modules (comodules, and contramodules),

where the semiderived category =

- derived category of the second kind (the coderived or the contraderived category) along the subalgebra,
- derived category of the first kind (the conventional derived category) in the direction complementary to the subalgebra.

Semi-infinite algebraic geometry =

- homological algebra in the semiderived categories of quasi-coherent sheaves and contraherent cosheaves,

where the semiderived category =

- derived category of the second kind (the coderived or the contraderived category) along the base of the fibration,
- derived category of the first kind (the conventional derived category) along the fibers.

Derived categories of the first and second kind

Classical homological algebra:

two hypercohomology spectral sequences

Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories (assume that $\mathcal{A}$ has enough injectives).
Let $0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$ be a complex in $\mathcal{A}$.
Then there are two spectral sequences converging to the same limit

$$'E_2^{pq} = R^pF(H^qC^\bullet) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet);$$
$$''E_2^{pq} = H^p(R^qF(C^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(C^\bullet).$$

For unbounded complexes $C^\bullet$, the two spectral sequences converge (perhaps in some weak sense) to *two different limits*. The same problem occurs for (even totally finite-dimensional) DG-modules.

Hence differential derived functors of the first and the second kind [Eilenberg–Moore '62 — Husemoller–Moore–Stasheff '74].

Derived categories of the first and second kind

Classical homological algebra

Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then the derived category of complexes over $\mathcal{A}$ bounded above or below can be alternatively described as

- $\mathrm{D}^+(\mathcal{A}) = \mathrm{Hot}^+(\mathcal{A})/\mathrm{Acycl}^+(\mathcal{A}) \simeq \mathrm{Hot}^+(\mathcal{A}_{\mathrm{inj}})$;
- $\mathrm{D}^-(\mathcal{A}) = \mathrm{Hot}^-(\mathcal{A})/\mathrm{Acycl}^-(\mathcal{A}) \simeq \mathrm{Hot}^-(\mathcal{A}_{\mathrm{proj}})$.

Not true for unbounded complexes.

Example: let $\Lambda = k[\varepsilon]/(\varepsilon^2)$ be the exterior algebra in one variable (the ring of dual numbers) over a field k. Then

$$\cdots \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \Lambda \xrightarrow{\ \varepsilon\ } \cdots$$

is an unbounded complex of projective, injective Λ-modules. It is acyclic, but not contractible.

Derived Categories of the First and Second Kind

The complex

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

of modules over $\Lambda = k[\varepsilon]/(\varepsilon^2)$ can be dealt with as

- representing a zero object in the derived category,

  not "projective" or "injective" (not suitable for computing the derived functors) —

  derived category of the first kind (conventional)

- "projective" and/or "injective" (adjusted for computing the derived functors),

  representing a nontrivial object in the derived category —

  derived category of the second kind (exotic)

Derived categories of the first and second kind

Theories of the first kind feature:

- equivalence relation on complexes simply described (being a quasi-isomorphism only depends on the underlying complexes of abelian groups, not on the module structure)
- complicated descriptions of categories of resolutions (homotopy projective, homotopy injective complexes)

[Bernstein, Spaltenstein, Keller, . . . '88 –]

Theories of the second kind feature:

- categories of resolutions simply described (depending only on the underlying graded module structure, irrespective of the differentials on complexes)
- complicated descriptions of equivalence relations on complexes (more delicate than the conventional quasi-isomorphism)

[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, . . . '98 –]

Derived categories of the first and second kind

Philosophical conclusion: in theories of the first kind, a complex is viewed as a deformation of its cohomology.

In theories of the second kind, a complex is viewed as a deformation of itself endowed with the zero differential.

Classical homological algebra is the realm in which there is no difference between the theories of the first and the second kind.

There is a natural way to build derived categories of the first and second kind on top of one another.

Given a ring R with a subring $A \subset R$, the semiderived category of R-modules relative to A is a mixture of

- a derived category of the second kind (the coderived or the contraderived category) along the variables from A and
- the derived category of the first kind (the conventional derived category) along the complementary variables from R.

Conventional derived category

Let $\mathcal{E}$ be an exact category (in the sense of Quillen). Assume for simplicity that $\mathcal{E}$ contains the images of its idempotent endomorphisms. A complex $C^\bullet$ in $\mathcal{E}$ is called acyclic if it is composed of short exact sequences:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{-1} & & \longrightarrow & & C^{0} & & \longrightarrow & & C^{1} & \longrightarrow & \cdots \\ & & & \searrow\!\!\!\!\to & & \nearrow & & \searrow\!\!\!\!\to & & \nearrow & & & \\ & & & & Z^{0} & & & & Z^{1} & & & & \end{array}$$

Denote by $\mathrm{Hot}(\mathcal{E})$ the homotopy category of (unbounded) complexes in $\mathcal{E}$ and by $\mathrm{Acycl}(\mathcal{E})$ its full subcategory consisting of acyclic complexes.

The triangulated quotient category $\mathrm{D}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}(\mathcal{E})$ is called the derived category of an exact category $\mathcal{E}$.

Coderived and contraderived categories

Let $\mathcal{E}$ be an exact category. Suppose $0 \longrightarrow K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow 0$ is a short exact sequence of complexes in $\mathcal{E}$:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{-1} & \longrightarrow & K^0 & \longrightarrow & K^1 & \xrightarrow{d} & K^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \partial & & \downarrow & & \\
\cdots & \longrightarrow & L^{-1} & \longrightarrow & L^0 & \xrightarrow{d} & L^1 & \longrightarrow & L^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \partial & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & M^{-1} & \xrightarrow{d} & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Form the total complex $\mathrm{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.

Coderived and Contraderived Categories

A complex $C^\bullet$ in $\mathcal{E}$ is called absolutely acyclic if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(\mathcal{E})$ containing the complexes $\mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ for all the short exact sequences $0 \longrightarrow K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow 0$ of complexes in $\mathcal{E}$:

$$\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{E}) = \langle \mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet) \rangle \subset \mathrm{Hot}(\mathcal{E}).$$

The triangulated quotient category $\mathrm{D}^{\mathrm{abs}}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}^{\mathrm{abs}}(\mathcal{E})$ is called the absolute derived category of an exact category $\mathcal{E}$.

- The absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathcal{E})$ is defined for any exact category $\mathcal{E}$.
- The coderived category $\mathrm{D}^{\mathrm{co}}(\mathcal{E})$ is defined for any exact category $\mathcal{E}$ with exact functors of infinite direct sum.
- The contraderived category $\mathrm{D}^{\mathrm{ctr}}(\mathcal{E})$ is defined for any exact category $\mathcal{E}$ with exact functors of infinite product.

Coderived and Contraderived Categories

A complex $C^\bullet$ in $\mathcal{E}$ is called coacyclic if it belongs to the minimal triangulated subcategory of the homotopy category $\mathrm{Hot}(\mathcal{E})$ containing the complexes $\mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ and closed under infinite direct sums:

$$\mathrm{Acycl}^{\mathrm{co}}(\mathcal{E}) = \langle \mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet) \rangle_\oplus \subset \mathrm{Hot}(\mathcal{E}).$$

A complex in $\mathcal{E}$ is called contraacyclic if it belongs to the minimal triangulated subcategory of $\mathrm{Hot}(\mathcal{E})$ containing the complexes $\mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet)$ and closed under infinite products:

$$\mathrm{Acycl}^{\mathrm{ctr}}(\mathcal{E}) = \langle \mathsf{Tot}(K^\bullet \to L^\bullet \to M^\bullet) \rangle_\sqcap \subset \mathrm{Hot}(\mathcal{E}).$$

The triangulated quotient category

$$\mathrm{D}^{\mathrm{co}}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}^{\mathrm{co}}(\mathcal{E})$$

is called the coderived category of an exact category $\mathcal{E}$.

The quotient category

$$\mathrm{D}^{\mathrm{ctr}}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}^{\mathrm{ctr}}(\mathcal{E})$$

is called the contraderived category of an exact category $\mathcal{E}$.

Any coacyclic complex is acyclic, and any contraacyclic complex is acyclic, but the converse is not generally true. So the conventional derived category $\mathrm{D}(\mathcal{E})$ is a quotient category of both $\mathrm{D}^{\mathrm{co}}(\mathcal{E})$ and $\mathrm{D}^{\mathrm{ctr}}(\mathcal{E})$ (whenever the latter are defined).

In an exact category $\mathcal{E}$ of finite homological dimension, any acyclic complex is absolutely acyclic (hence also co- and contraacyclic).

Coderived and contraderived categories

Example: the acyclic complex $\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$ of modules over the algebra of dual numbers $\Lambda = k[\varepsilon]/(\varepsilon^2)$ is neither coacyclic, nor contraacyclic.

Let us decompose this complex in two halves. The acyclic complex of Λ-modules

$$\cdots \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \twoheadrightarrow k \to 0$$

is contraacyclic, but not coacyclic.

The acyclic complex of Λ-modules

$$0 \to k \rightarrowtail \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \Lambda \xrightarrow{\varepsilon} \cdots$$

is coacyclic, but not contraacyclic.

Coderived and contraderived categories

Consider the following two conditions on an exact category $\mathcal{E}$:

($*$) There are enough injective objects in $\mathcal{E}$, and countable direct sums of injective objects have finite injective dimension.

($**$) There are enough projective objects in $\mathcal{E}$, and countable products of projective objects have finite projective dimension.

Theorem

(a) For any exact category $\mathcal{E}$ satisfying ($*$), the natural functor from the homotopy category of complexes of injective objects in $\mathcal{E}$ to the coderived category of $\mathcal{E}$ is a triangulated equivalence,

$$\mathrm{Hot}(\mathcal{E}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathcal{E}).$$

(b) For any exact category $\mathcal{E}$ satisfying ($**$), the natural functor from the homotopy category of complexes of projective objects in $\mathcal{E}$ to the contraderived category of $\mathcal{E}$ is a triangulated equivalence,

$$\mathrm{Hot}(\mathcal{E}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathcal{E}).$$

Let $\mathcal{E} \longrightarrow \mathcal{A}$ be an exact functor between exact categories, thought of as a "forgetful" functor. (We will assume the infinite direct sums or infinite products in $\mathcal{E}$ and $\mathcal{A}$ to be exact and preserved by the functor $\mathcal{E} \longrightarrow \mathcal{A}$ as needed.)

The semicoderived category of the exact category $\mathcal{E}$ relative to the exact category $\mathcal{A}$ is defined as the quotient category of the homotopy category $\mathrm{Hot}(\mathcal{E})$ by the triangulated subcategory of complexes that are coacyclic in $\mathcal{A}$

$$\mathrm{D}^{\mathrm{sico}}_{\mathcal{A}}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}^{\mathrm{co}}_{\mathcal{A}}(\mathcal{E}).$$

The semicontraderived category of $\mathcal{E}$ relative to $\mathcal{A}$ is the quotient category of the homotopy category $\mathrm{Hot}(\mathcal{E})$ by the triangulated subcategory of complexes that are contraacyclic in $\mathcal{A}$

$$\mathrm{D}^{\mathrm{sictr}}_{\mathcal{A}}(\mathcal{E}) = \mathrm{Hot}(\mathcal{E})/\mathrm{Acycl}^{\mathrm{ctr}}_{\mathcal{A}}(\mathcal{E}).$$

In particular, let R be an (associative) ring with a subring A. Then there is the exact forgetful functor $R\text{-mod} \longrightarrow A\text{-mod}$ between the abelian categories of modules. Denote the corresponding semiderived categories by $\mathrm{D}^{\mathrm{sico}}_A(R\text{-mod})$ and $\mathrm{D}^{\mathrm{sictr}}_A(R\text{-mod})$.

When $A = R$, one has

$$\mathrm{D}^{\mathrm{sico}}_R(R\text{-mod}) = \mathrm{D}^{\mathrm{co}}(R\text{-mod})$$
$$\mathrm{D}^{\mathrm{sictr}}_R(R\text{-mod}) = \mathrm{D}^{\mathrm{ctr}}(R\text{-mod}).$$

When A is a field or $A = \mathbb{Z}$, one has

$$\mathrm{D}^{\mathrm{sico}}_{\mathbb{Z}}(R\text{-mod}) = \mathrm{D}(R\text{-mod}) = \mathrm{D}^{\mathrm{sictr}}_{\mathbb{Z}}(R\text{-mod}).$$

For a complex of R-modules is acyclic $\Longleftrightarrow$ acyclic as a complex of abelian groups $\Longleftrightarrow$ co/contraacyclic as complex of abelian groups.

So the semiderived category is indeed a mixture of the co/contraderived category along A and the conventional derived category in the direction of R relative to A.

Semi-infinite algebraic varieties

A semi-infinite algebraic variety is a morphism of ind-schemes or ind-stacks $\mathfrak{Y} \longrightarrow \mathfrak{X}$ with, approximately, the following properties:

- $\mathfrak{Y}$ is a large and complicated ind-scheme or ind-stack;
- $\mathfrak{X}$ is built up in a complicated way from small affine pieces: something like an ind-Noetherian or ind-coherent ind-scheme or ind-stack with a dualizing complex;
- the morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is locally well-behaved: at least flat, or perhaps very flat;
- the fibers of the morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ are built up in a simple way from large affine pieces: might be arbitrary affine schemes, or quasi-compact semi-separated schemes, or weakly proregular formal schemes.

Example: semi-infinite algebraic variety

Consider the example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$.

The fiber $k[[z]] = \{a_0 + a_1 z + a_2 z^2 + \cdots\}$ is the set of k-points of the infinite-dimensional affine scheme

$$\operatorname{Spec} k[a_0, a_1, a_2, \ldots, a_n, \ldots].$$

The base $k((z))/k[[z]] = \bigcup_n t^{-n} k[[z]]/k[[z]]]$ is the set of k-points of the ind-Noetherian ind-affine ind-scheme

$$\varinjlim_n \operatorname{Spec} k[a_{-n}, \ldots, a_{-2}, a_{-1}].$$

Can be viewed as the "ind-spectrum" of the pro-Noetherian topological ring $\mathfrak{A} = \varprojlim_n k[a_{-n}, \ldots, a_{-2}, a_{-1}]$.

The total space $k((z)) = \bigcup_n t^{-n} k[[z]]$ is the set of k-points of the ind-affine ind-scheme

$$\varinjlim_n \operatorname{Spec} k[a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots].$$

The homological formalism of semi-infinite algebraic geometry is supposed to feature:

- the geometric derived semico-semicontra correpondence, i.e., a triangulated equivalence between the semiderived categories of quasi-coherent torsion sheaves and contraherent cosheaves of contramodules on $\mathfrak{Y}$ relative to $\mathfrak{X}$:
$$\mathrm{D}^{\mathrm{sico}}_{\mathfrak{X}}(\mathfrak{Y}\text{-qcoh}) \simeq \mathrm{D}^{\mathrm{sictr}}_{\mathfrak{X}}(\mathfrak{Y}\text{-ctrh});$$
- the "semi-infinite quasi-coherent $\mathcal{T}or$ functor", or the double-sided derived functor of semitensor product of quasi-coherent torsion sheaves on $\mathfrak{Y}$, which means a mixture of the cotensor product along $\mathfrak{X}$ and the conventional tensor product along the fibers;
- the double-sided derived functor of semihomomorphisms from quasi-coherent sheaves to contraherent cosheaves on $\mathfrak{Y}$, transformed by the semico-semicontra correspondence into the conventional quasi-coherent internal $\mathbb{R}\mathcal{H}om$.

Known particular cases

The geometric derived semico-semicontra correspondence has been worked out in the following cases:

- theory on the fiber: equivalence of the conventional derived categories of quasi-coherent sheaves and contraherent cosheaves on $\mathfrak{Y}$ over $\mathfrak{X} = \{*\}$, where
  $\mathfrak{Y}$ is a quasi-compact semi-separated scheme;
  $\mathfrak{Y}$ is a Noetherian scheme;
  $\mathfrak{Y}$ is a weakly proregular (e.g., Noetherian) affine formal scheme;
- theory on the base: equivalence between the coderived category of quasi-coherent sheaves and the contraderived category of contraherent cosheaves on $\mathfrak{Y} = \mathfrak{X}$, where
  $\mathfrak{X}$ is a Noetherian scheme with a dualizing complex;
  $\mathfrak{X}$ is a semi-separated Noetherian stack with a dualizing complex;
  $\mathfrak{X}$ is an ind-affine ind-Noetherian ind-scheme with a dualizing complex;

Known particular cases

- relative situation: equivalence between the semicoderived and the semicontraderived category of modules for a flat morphism of affine schemes $\operatorname{Spec} R \longrightarrow \operatorname{Spec} A$, where A is a coherent ring with a dualizing complex.

These results can be found in:

- "Contraherent cosheaves", `arXiv:1209.2995`, Sections 4, 5, Appendices B, C, D (schemes, stacks, ind-affine ind-schemes)
- "Dedualizing complexes and MGM duality", `arXiv:1503.05523` (weakly proregular affine formal schemes)
- "Coherent rings, fp-injective modules, dualizing complexes, and covariant Serre-Grothendieck duality", `arXiv:1504.00700` (overview; relative situation)

Contraherent cosheaves

Let X be a scheme. A quasi-coherent sheaf $\mathcal{M}$ on X = a rule assigning

- an $\mathcal{O}_X(U)$-module $\mathcal{M}(U)$ to every affine open subscheme $U \subset X$
- and an isomorphism of $\mathcal{O}_X(V)$-modules $\mathcal{M}(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)$ to every pair of embedded affine open subschemes $V \subset U \subset X$
- satisfying a compatibility equation for every triple of embedded affine open subschemes $W \subset V \subset U \subset X$.

This definition works well (provides an abelian category of quasi-coherent sheaves $X\text{-}\mathrm{qcoh}$) because $\mathcal{O}_X(V)$ is always a flat $\mathcal{O}_X(U)$-module.

Contraherent cosheaves

A contraherent cosheaf $\mathfrak{P}$ on X = a rule assigning

- an $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ to every affine open subscheme $U \subset X$
- and an isomorphism of $\mathcal{O}_X(V)$-modules $\mathfrak{P}[V] \simeq \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U])$ to every pair of embedded affine open subschemes $V \subset U \subset X$,
- where in addition one has $\mathrm{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$ for all affine $V \subset U \subset X$
- and a compatibility equation for every triple of embedded affine open subschemes $W \subset V \subset U \subset X$ is satisfied.

Here it is important that $\mathcal{O}_X(V)$ is not a projective $\mathcal{O}_X(U)$-module in general, but it always has projective dimension at most 1.

This definition works well enough to provide an exact category of contraherent cosheaves $X\text{-}\mathrm{ctrh}$ on X.

Contraherent cosheaves

The category $X\text{-qcoh}$ of quasi-coherent sheaves on a scheme X is an abelian category with exact functors of infinite direct sum.

Therefore, in addition to the derived category $\mathrm{D}(X\text{-qcoh})$, the coderived category $\mathrm{D}^{\mathrm{co}}(X\text{-qcoh})$ is well defined for it.

The category $X\text{-ctrh}$ of contraherent cosheaves on a scheme X is an exact category with exact functors of infinite product.

Therefore, in addition to the derived category $\mathrm{D}(X\text{-ctrh})$, the contraderived category $\mathrm{D}^{\mathrm{ctr}}(X\text{-ctrh})$ is well defined for it.

The abelian category $X\text{-qcoh}$ always has enough injective objects. When the scheme X is quasi-compact and semi-separated, or Noetherian of finite Krull dimension, the exact category $X\text{-ctrh}$ has enough projective objects.

Example: equivalence on the fiber

Theorem

Let X be a quasi-compact semi-separated scheme, or a Noetherian scheme of finite Krull dimension.

Then there is a natural equivalence of triangulated categories

$$\mathrm{D}(X\text{-qcoh}) \simeq \mathrm{D}(X\text{-ctrh}).$$

Moreover, there are also equivalences of bounded derived categories

$$\mathrm{D}^\star(X\text{-qcoh}) \simeq \mathrm{D}^\star(X\text{-ctrh})$$

for any symbol $\star = +$, $-$, or b.

For a quasi-compact semi-separated scheme X, there is also an equivalence of absolute derived categories

$$\mathrm{D}^{\mathrm{abs}}(X\text{-qcoh}) \simeq \mathrm{D}^{\mathrm{abs}}(X\text{-ctrh}).$$

Example: equivalence on the base

Let X be a Noetherian scheme. Recall that a *dualizing complex* $\mathcal{D}_X^\bullet$ on X is a complex of quasi-coherent sheaves satisfying the following conditions:

- $\mathcal{D}_X^\bullet$ is a finite complex of injective quasi-coherent sheaves;
- the cohomology sheaves $\mathcal{H}^i(\mathcal{D}_X^\bullet)$ are coherent;
- the natural map $\mathcal{O}_X \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet)$ is a quasi-isomorphism, where $\mathcal{H}om_{X\text{-qc}}$ denotes the quasi-coherent internal $\mathcal{H}om$ of quasi-coherent sheaves on X.

In particular, if A is a Noetherian commutative ring, then a dualizing complex of A-modules $D_A^\bullet$ is the same thing as a dualizing complex of quasi-coherent sheaves on $\operatorname{Spec} A$.

Example: equivalence on the base

Proposition

For any Noetherian commutative ring A of finite Krull dimension, the natural functors provide triangulated equivalences

- $\mathrm{Hot}(A\text{-mod}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(A\text{-mod})$;
- $\mathrm{Hot}(A\text{-mod}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(A\text{-mod}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(A\text{-mod})$.

Theorem

The choice of a dualizing complex $D_A^\bullet$ for a Noetherian commutative ring A induces an equivalence of triangulated categories $\mathrm{D}^{\mathrm{co}}(A\text{-mod}) \simeq \mathrm{D}^{\mathrm{ctr}}(A\text{-mod})$.

Here the equivalence is provided by the derived functors $M^\bullet \longmapsto \mathbb{R}\,\mathrm{Hom}_A(D_A^\bullet, M^\bullet)$ and $P^\bullet \longmapsto D_A^\bullet \otimes_A^{\mathbb{L}} P^\bullet$.

[Jørgensen, Krause, Iyengar–Krause '05–'06]

Example: equivalence on the base

Proposition

For any locally Noetherian scheme X, the natural functor provides a triangulated equivalence

- $\mathrm{Hot}(X\text{-qcoh}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(X\text{-qcoh})$.

For any semi-separated Noetherian scheme X of finite Krull dimension, one has

- $\mathrm{D}^{\mathrm{abs}}(X\text{-qcoh}_{\mathrm{flat}}) = \mathrm{D}^{\mathrm{co}}(X\text{-qcoh}_{\mathrm{flat}}) = \mathrm{D}(X\text{-qcoh}_{\mathrm{flat}})$.

Theorem

The choice of a dualizing complex $\mathcal{D}_X^\bullet$ for a semi-separated Noetherian scheme X induces an equivalence of triangulated categories $\mathrm{D}^{\mathrm{co}}(X\text{-qcoh}) \simeq \mathrm{D}^{\mathrm{abs}}(X\text{-qcoh}_{\mathrm{flat}})$.

Here the equivalence is provided by the functors $\mathcal{M}^\bullet \longmapsto \mathbb{R}\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}^\bullet)$ and $\mathcal{F}^\bullet \longmapsto \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$.

[Neeman, Murfet '07–'08]

Example: equivalence on the base

A contraherent cosheaf $\mathfrak{P}$ on a scheme X is called *locally injective* if the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is injective for every affine open subscheme $U \subset X$. The exact category of locally injective contraherent cosheaves on X is denoted by $X\text{-}\mathrm{ctrh}^{\mathrm{lin}}$.

Locally injective contraherent cosheaves are dual-analogous to flat quasi-coherent sheaves.

Proposition

For any Noetherian scheme X of finite Krull dimension, the natural functor provides a triangulated equivalence

- $\mathrm{Hot}(X\text{-}\mathrm{ctrh}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{ctr}}(X\text{-}\mathrm{ctrh})$.

For any quasi-compact semi-separated scheme X, one has

- $\mathrm{D}^{\mathrm{abs}}(X\text{-}\mathrm{ctrh}^{\mathrm{lin}}) = \mathrm{D}^{\mathrm{ctr}}(X\text{-}\mathrm{ctrh}^{\mathrm{lin}}) = \mathrm{D}(X\text{-}\mathrm{ctrh}^{\mathrm{lin}})$.

Example: equivalence on the base

Theorem

The choice of a dualizing complex $\mathcal{D}_X^\bullet$ for a semi-separated Noetherian scheme X provides a commutative diagram of triangulated equivalences

$$\begin{array}{ccc} \mathrm{D^{co}}(X\text{-qcoh}) & = & \mathrm{D^{abs}}(X\text{-qcoh}_{\mathrm{flat}}) \\ \| & & \| \\ \mathrm{D^{abs}}(X\text{-ctrh}^{\mathrm{lin}}) & = & \mathrm{D^{ctr}}(X\text{-ctrh}) \end{array}$$

The vertical equivalences do not depend on the choice of $\mathcal{D}_X^\bullet$; the horizontal and diagonal ones do.

Theorem

The choice of a dualizing complex $\mathcal{D}_X^\bullet$ for a Noetherian scheme X induces a triangulated equivalence $\mathrm{D^{co}}(X\text{-qcoh}) \simeq \mathrm{D^{ctr}}(X\text{-ctrh})$.

Contramodules

Contramodules are module-like objects endowed with infinite summation (or, occasionally, integration) operations, understood algebraically as infinitary (linear) operations subject to natural axioms. Contramodules carry no underlying topologies on them, but feel like being in some sense "complete". For about every class of "discrete" or "torsion" modules, there is an much less familiar, but no less interesting accompanying class of contramodules.

"Discrete" or "torsion" module categories typically have exact functors of filtered inductive limits and enough injective objects, but nonexact functors of infinite product and no projectives.

Contramodule categories have exact functors of infinite product, and typically enough projective objects, but nonexact functors of infinite direct sum and no injectives.

The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

Contramodules over topological rings

Fancy definition of (conventional) modules over a discrete ring R:

- to any set X one assigns the set $R[X]$ of all formal linear combinations of elements of X with coefficients in R;
- the functor $X \longmapsto R[X]$ is a monad on the category of sets
- with the "parentheses opening" map $\phi_X\colon R[R[X]] \longrightarrow R[X]$
- and the "point measure" map $\varepsilon_X\colon X \longrightarrow R[X]$;
- define left R-modules as algebras/modules over this monad on Sets, that is
- a left R-module M is a set
- endowed with a map of sets $m\colon R[M] \longrightarrow M$
- satisfying the associativity equation $m \circ R[m] = m \circ \phi_M$

$$R[R[M]] \rightrightarrows R[M] \longrightarrow M$$

- and the unity equation $m \circ \varepsilon_X = \mathrm{id}_M$

$$M \longrightarrow R[M] \longrightarrow M.$$

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

For any set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x \in X} r_x x$ of elements of X with the coefficients forming a family converging to zero in the topology of $\mathfrak{R}$, i.e., for any neiborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ the set $\{x \mid r_x \notin \mathfrak{U}\}$ must be finite.

It follows from the conditions on the topology of $\mathfrak{R}$ that there is a well-defined "parentheses opening" map

$$\phi_X \colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$$

performing infinite summations in the conventional sense of the topology of $\mathfrak{R}$ to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X \colon X \longrightarrow \mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X \longmapsto \mathfrak{R}[[X]] \colon \mathrm{Sets} \longrightarrow \mathrm{Sets}$.

Contramodules over Topological Rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

A left contramodule over the topological ring $\mathfrak{R}$ is an algebra/module over the monad $X \longmapsto \mathfrak{R}[[X]]$ on Sets, that is

- a set $\mathfrak{P}$
- endowed with a contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$
- satisfying the contraassociativity equation $\pi \circ \mathfrak{R}[[\pi]] = \pi \circ \phi_{\mathfrak{P}}$

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{P}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$$

- and the unity equation $\pi \circ \varepsilon_{\mathfrak{P}} = \mathrm{id}_{\mathfrak{P}}$

$$\mathfrak{P} \longrightarrow \mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{P}.$$

The composition of the contraaction map $\pi\colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[[\mathfrak{P}]]$ defines the underlying left $\mathfrak{R}$-module structure on every left $\mathfrak{R}$-contramodule.

Contramodules over topological rings

Let $\mathfrak{R}$ be a (separated and complete) topological ring where open right ideals form a base of neighborhoods of zero.

Then the category of left $\mathfrak{R}$-contramodules is abelian with exact functors of infinite product and enough projectives (which are the direct summands of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$). The forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products.

A right $\mathfrak{R}$-module $\mathcal{N}$ is called *discrete* if the action map $\mathcal{N} \times \mathfrak{R} \longrightarrow \mathcal{N}$ is continuous in the given topology of $\mathfrak{R}$ and the discrete topology of $\mathcal{N}$, i.e., if the annihilator of any element of $\mathcal{N}$ is open in $\mathfrak{R}$. The category of discrete $\mathfrak{R}$-modules is abelian with exact functors of infinite direct sum and enough injectives.

For any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and any abelian group U, the left $\mathfrak{R}$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{N}, U)$ has a natural left $\mathfrak{R}$-contramodule structure.

Contramodules over topological rings

Example: let $\mathfrak{R} = \mathbb{Z}_\ell$ be the ring of ℓ-adic integers. A discrete $\mathbb{Z}_\ell$-module is just an ℓ^∞-torsion abelian group.

A $\mathbb{Z}_\ell$-contramodule $\mathfrak{P}$ is

- an abelian group endowed with an infinite summation operation assigning to any sequence of elements p_0, p_1, p_2, … $\in \mathfrak{P}$ an element denoted by $\sum_{n=0}^\infty \ell^n p_n \in \mathfrak{P}$
- and satisfying the axioms of linearity:

$$\sum_{n=0}^\infty \ell^n(ap_n + bq_n) = a\sum_{n=0}^\infty \ell^n p_n + b\sum_{n=0}^\infty \ell^n q_n,$$

- unitality + compatibility: $\sum_{n=0}^\infty \ell^n p_n = p_0 + \ell p_1$ when $p_i = 0$ for all $i \geqslant 2$,
- and contraassociativity:

$$\sum_{i=0}^\infty \ell^i \sum_{j=0}^\infty \ell^j p_{ij} = \sum_{n=0}^\infty \ell^n \sum_{i+j=n} p_{ij}.$$

Contramodules over topological rings

Nakayama's lemma: let $\mathfrak{R}$ be a topological ring (complete and separated, with open right ideals forming a base of neighborhoods of zero), and let $\mathfrak{m} \subset \mathfrak{R}$ be an ideal that is topologically nilpotent, i.e., for any neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ there exists an integer $n \geqslant 1$ such that $\mathfrak{m}^n \subset \mathfrak{U}$.

Let $\mathfrak{P}$ be a nonzero left $\mathfrak{R}$-contramodule. Then the contraaction map $\mathfrak{m}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ is not surjective.

Let R be a Noetherian commutative ring with an ideal $I \subset R$ generated by some elements $s_1, \ldots, s_m \in R$, and let $\mathfrak{R} = R_I\widehat{\ }$ be the I-adic completion of R (endowed with the I-adic topology).

Then the forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow R\text{-mod}$ is fully faithful and its image consists of all the modules $P \in R\text{-mod}$ such that $\mathrm{Ext}^*_R(R[s_j^{-1}], P) = 0$ for all $j = 1, \ldots, m$.

In particular, $\mathbb{Z}_\ell$-contramodules = weakly ℓ-complete (Ext-ℓ-complete) abelian groups [Bousfield–Kan '72, Jannsen '88].

Contramodules over Commutative Ring with an Ideal

Let R be a commutative ring and $I \subset R$ be an ideal. An R-module M is said to be I-torsion if for any $s \in I$ and $m \in M$ there exists $n \in \mathbb{N}$ such that $s^n m = 0$.

An abelian group P with an additive operator $s\colon P \longrightarrow P$ is said to be an s-contramodule if for any sequence $p_0, p_1, p_2, \ldots \in P$ the infinite system of nonhomogeneous linear equations

$$q_n = sq_{n+1} + p_n \quad \text{for all } n \geqslant 0$$

has a unique solution $q_0, q_1, q_2, \ldots \in P$.

The infinite summation operation with s-power coefficients in P is defined by the rule

$$\sum_{n=0}^{\infty} s^n p_n = q_0.$$

Contramodules over commutative ring with an ideal

Conversely, given an additive, associative, and unital s-power infinite summation operation

$$(p_n)_{n=0}^{\infty} \longmapsto \sum_{n=0}^{\infty} s^n p_n$$

in P one can uniquely solve the system of equations $q_n = sq_{n+1} + p_n$ by setting

$$q_n = \sum_{i=0}^{\infty} s^i p_{n+i}.$$

A module P over a commutative ring R with an element $s \in R$ is an s-contramodule (i.e., a contramodule with respect to the operator of multiplication with s) if and only if $\mathrm{Ext}_R^i(R[s^{-1}], P) = 0$ for $i = 0$ and 1. (Notice that the R-module $R[s^{-1}]$ has projective dimension at most 1.)

Contramodules over Commutative Ring with an Ideal

Let $I \subset R$ be an ideal and $s_j \in R$ be a set of generators for I. An R-module P is called an *I-contramodule* if it is an s_j-contramodule for every j. This property does not depend on the choice of a set of generators s_j, and only depends on the radical $\sqrt{I} \subset R$ of I.

The full subcategory of I-torsion R-modules $R\text{-}\mathrm{mod}_{I\text{-tors}} \subset R\text{-}\mathrm{mod}$ is closed under the passages to submodules, quotient modules, extensions, and infinite direct sums. So $R\text{-}\mathrm{mod}_{I\text{-tors}}$ is an abelian category with exact functors of infinite direct sum and its embedding $R\text{-}\mathrm{mod}_{I\text{-tors}} \longrightarrow R\text{-}\mathrm{mod}$ is an exact functor preserving infinite direct sums.

The full subcategory of I-contramodule R-modules $R\text{-}\mathrm{mod}_{I\text{-ctra}}$ is closed under the kernels and cokernels of morphisms, extensions, and infinite products in $R\text{-}\mathrm{mod}$. So $R\text{-}\mathrm{mod}_{I\text{-ctra}}$ is an abelian category with exact functors of infinite product and its embedding $R\text{-}\mathrm{mod}_{I\text{-tors}} \longrightarrow R\text{-}\mathrm{mod}$ is an exact functor preserving products.

Contramodules over commutative ring with an ideal

I-torsion R-modules are the same thing as quasi-coherent sheaves on $\operatorname{Spec} R$ vanishing in the restriction to the open subset $U = \operatorname{Spec} R \setminus \operatorname{Spec}(R/I)$.

I-contramodule R-modules are closely related to contraherent cosheaves on $\operatorname{Spec} R$ vanishing in the restriction to the open subset $U = \operatorname{Spec} R \setminus \operatorname{Spec}(R/I)$.

More precisely, a contraherent cosheaf $\mathfrak{P}$ on $\operatorname{Spec} R$ vanishing in the restriction to U is the same thing as an I-contramodule R-module P satisfying the additional condition $\operatorname{Ext}^1_R(R[s^{-1}], P) = 0$ for all $s \in R$ (or for all $s \in R \setminus I$).

Example: equivalence on the fiber

From now on we assume the ideal I to be finitely generated. Let $s_1, \ldots, s_m \in R$ be a set of generators for I. We will denote the sequence $s_1, \ldots, s_m \in R$ by a single letter $\mathbf{s}$.

For any R-module M, consider the following augmented Čech complex $C^\bullet_{\mathbf{s}}(M)$

$$M \longrightarrow \bigoplus_{j=1}^{m} M[s_j^{-1}] \longrightarrow \bigoplus_{j'<j''} M[s_{j'}^{-1}, s_{j''}^{-1}] \\ \longrightarrow \cdots \longrightarrow M[s_1^{-1}, \ldots, s_m^{-1}].$$

One has $C^\bullet_{\mathbf{s}}(M) \simeq C^\bullet_{\mathbf{s}}(R) \otimes_R M$ and

$$C^\bullet_{\mathbf{s}}(R) \simeq C^\bullet_{\{s_1\}}(R) \otimes_R \cdots \otimes_R C^\bullet_{\{s_m\}}(R),$$

where $C^\bullet_{\{s_j\}}(R)$ is the two-term complex $R \longrightarrow R[s_j^{-1}]$.

Example: equivalence on the fiber

The two-term complex $C^\bullet_{\{s_j\}}(R) = (R \to R[s_j^{-1}])$ is the inductive limit of two-term complexes of free R-modules with one generator $K^\bullet(R, s_j^n) = (R \to s_j^{-n}R)$. The dual (Koszul) complexes $K_\bullet(R, s_j^n) = \mathrm{Hom}_R(K^\bullet(R, s_j^n), R)$ form a projective system.

Set

$$K_\bullet(R, \mathbf{s}^n) = K_\bullet(R, s_1^n) \otimes_R \cdots \otimes_R K_\bullet(R, s_m^n),$$

where $\mathbf{s}^n$ denotes the sequence s_1^n, …, s_m^n. The Koszul complexes $K_\bullet(R, \mathbf{s}^n)$ form a projective system as well, and consequently so do their cohomology modules.

A projective system of abelian groups $U_1 \longleftarrow U_2 \longleftarrow U_3 \longleftarrow \cdots$ is said to be *pro-zero* if for every $k \geqslant 1$ there exists $n > k$ such that the composition of maps $U_k \longleftarrow \cdots \longleftarrow U_n$ vanishes.

Example: Equivalence on the Fiber

A finite sequence of elements $s_1, \ldots, s_m$ in a commutative ring R is said to be weakly proregular if either of the following equivalent conditions holds:

- one has $H^i C^\bullet_{\mathbf{s}}(J) = 0$ for any injective R-module J and all $i > 0$, or
- the projective system $H_i K_\bullet(R, \mathbf{s}^n)$ is pro-zero for every $i > 0$.

The weak proregularity property of a sequence of generators $\mathbf{s}$ of a finitely generated ideal $I \subset R$ only depends on the ideal $\sqrt{I} \subset R$ and not on the chosen generators. Hence the notion of a weakly proregular finitely generated ideal $I \subset R$.

Any regular sequence of elements in a commutative ring is weakly proregular. Any finite sequence of elements in a Noetherian commutative ring is weakly proregular.

[Schenzel, Porta–Shaul–Yekutieli, '03–'14]

Example: equivalence on the fiber

Theorem

Let R be a commutative ring and $I \subset R$ be a weakly proregular finitely generated ideal. Then there is a triangulated equivalence between the derived categories of the abelian categories of I-torsion and I-contramodule R-modules

$$\mathrm{D}^\star(R\text{-mod}_{I\text{-tors}}) \simeq \mathrm{D}^\star(R\text{-mod}_{I\text{-ctra}})$$

for every symbol $\star = \mathrm{b}, +, -, \varnothing$, *or* abs.

Without the weak proregularity assumption, for any finitely generated ideal I in a commutative ring R there is an equivalence between the full subcategories in $\mathrm{D}^\star(R\text{-mod})$ consisting of complexes with I-torsion and I-contramodule cohomology modules

$$\mathrm{D}^\star_{I\text{-tors}}(R\text{-mod}) \simeq \mathrm{D}^\star_{I\text{-ctra}}(R\text{-mod}), \qquad \star = \mathrm{b}, +, -, \text{ or } \varnothing.$$

[MGM Duality: Matlis, Greenlees–May, Dwyer–Greenlees, Porta–Shaul–Yekutieli, L.P., '78–'15]

Example: equivalence on the base

Let $A_0 \longleftarrow A_1 \longleftarrow A_2 \longleftarrow \cdots$ be a projective system of Noetherian commutative rings and surjective morphisms between them. Consider the projective limit $\mathfrak{A} = \varprojlim_n A_n$, and endow it with the projective limit topology.

For any $\mathfrak{A}$-contramodule $\mathfrak{P}$, denote by $\overline{\mathfrak{P}}_n$ the maximal quotient $\mathfrak{A}$-contramodule of $\mathfrak{P}$ whose $\mathfrak{A}$-contramodule structure comes from an A_n-module structure. An $\mathfrak{A}$-contramodule $\mathfrak{P}$ is called *flat* if

- the A_n-module $\overline{\mathfrak{P}}_n$ is flat for every $n \geqslant 0$, and
- the natural map $\mathfrak{P} \longrightarrow \varprojlim_n \overline{\mathfrak{P}}_n$ is an isomorphism.

The class $\mathfrak{A}\text{-contra}_{\text{flat}}$ of flat $\mathfrak{A}$-contramodules is closed under extensions, infinite products, and the passage to the kernels of surjective morphisms in $\mathfrak{A}\text{-contra}$, so in particular $\mathfrak{A}\text{-contra}_{\text{flat}}$ inherits an exact category structure from $\mathfrak{A}\text{-contra}$.

Denote by $\mathfrak{A}\text{-discr}$ the abelian category of discrete $\mathfrak{A}$-modules.

Example: equivalence on the base

Let $B \longrightarrow A$ be a surjective morphism of Noetherian commutative rings. Then for any dualizing complex $D_B^\bullet$ for the ring B, the maximal subcomplex of A-modules $\mathrm{Hom}_B(A, D_B^\bullet)$ in $D_B^\bullet$ is a dualizing complex for the ring A.

Let us say that dualizing complexes $D_A^\bullet$ and $D_B^\bullet$ for the rings A and B are *compatible* if a homotopy equivalence of complexes of injective A-modules

$$D_A^\bullet \simeq \mathrm{Hom}_B(A, D_B^\bullet)$$

is fixed.

Example: equivalence on the base

Let $\mathfrak{A} = \varprojlim_n A_n$ be a commutative pro-Noetherian ring (as above).

Proposition

The natural functors provide triangulated equivalences

- $\mathrm{Hot}(\mathfrak{A}\text{-discr}_{\mathrm{inj}}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{A}\text{-discr})$;
- $\mathrm{D}^{\mathrm{ctr}}(\mathfrak{A}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{A}\text{-contra})$.

When the Krull dimensions of the rings A_n are uniformly bounded, one has $\mathrm{Hot}(\mathfrak{A}\text{-contra}_{\mathrm{proj}}) \simeq \mathrm{D}^{\mathrm{abs}}(\mathfrak{A}\text{-contra}_{\mathrm{flat}}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{A}\text{-contra})$. This is not necessary for the following theorem.

Theorem

Any compatible system $\mathcal{D}^\bullet_{\mathfrak{A}}$ of choices of dualizing complexes $D^\bullet_{A_n}$ for the Noetherian rings A_n, $n \geqslant 0$, induces an equivalence of triangulated categories

$$\mathrm{D}^{\mathrm{co}}(\mathfrak{A}\text{-discr}) \simeq \mathrm{D}^{\mathrm{ctr}}(\mathfrak{A}\text{-contra}).$$

To repeat:

- On the fiber, one has an equivalence between the conventional derived categories (sometimes also between the absolute derived categories).
- On the base, one has an equivalence between the coderived category and the contraderived category. One needs a dualizing complex on the base.

The reason for the base and the fiber being this way comes from the definition of the semiderived category, which turns out to be the co/contraderived category along the subring and the conventional derived category in the complementary direction of the ambient ring relative to the subring.

Example: equivalence in the relative situation

Let A be a coherent commutative ring such that fp-injective A-modules have finite injective dimension (e.g., this is so if all ideals in A are at most countably generated).

A bounded complex of fp-injective A-modules $D_A^\bullet$ with finitely presented cohomology A-modules is called a *dualizing complex* for A if the natural map $A \longrightarrow \mathrm{Hom}_{\mathrm{D}(A\text{-mod})}(D_A^\bullet, D_A^\bullet[*])$ is an isomorphism.

Let $A \longrightarrow R$ be a homomorphism of commutative rings such that R is a flat A-module.

Theorem

The choice of a dualizing complex $D_A^\bullet$ for the coherent ring A induces an equivalence between the two semiderived categories of R-modules relative to A,

$$\mathrm{D}_A^{\mathrm{sico}}(R\text{-mod}) \simeq \mathrm{D}_A^{\mathrm{sictr}}(R\text{-mod}).$$

Example: cotensor product along the base

How to define a tensor structure on the category of torsion abelian groups so that $\mathbb{Q}/\mathbb{Z}$ would be the unit object?

For finite abelian groups M and N, set

$$M \mathbin{\square} N = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}),\ \mathbb{Q}/\mathbb{Z}).$$

Then pass to the inductive limit for infinite torsion abelian groups.

Alternatively, use covariant duality instead of contravariant one. For injective torsion abelian groups M and N, set

$$M \mathbin{\square} N = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M) \otimes_{\mathbb{Z}} N \simeq M \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N).$$

Then take the right derived functor.

Example: cotensor product along the base

Let X be a semi-separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$. The derived cotensor product $\square_{\mathcal{D}_X^\bullet}$ of complexes of quasi-coherent sheaves on X is a tensor structure on the coderived category $\mathrm{D^{co}}(X\text{-qcoh})$ with the unit object $\mathcal{D}_X^\bullet \in \mathrm{D^{co}}(X\text{-qcoh})$.

For bounded-below complexes of injective quasi-coherent sheaves with coherent cohomology sheaves $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ on X, one has

$$\mathcal{M}^\bullet \square_{\mathcal{D}_X^\bullet} \mathcal{N}^\bullet = \mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^\bullet, \mathcal{D}_X^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_X^\bullet),\ \mathcal{D}_X^\bullet).$$

For any two complexes of injective quasi-coherent sheaves $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ on X, one has

$$\mathcal{M}^\bullet \square_{\mathcal{D}_X^\bullet} \mathcal{N}^\bullet = \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet \simeq \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{N}^\bullet) \simeq \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{N}^\bullet).$$

Example: cotensor product along the base

Let X be an algebraic variety (separated scheme of finite type) over a field k. Denote the diagonal morphism by $\Delta\colon X \longrightarrow X \times_k X$ and the projection to the point by $p\colon X \longrightarrow \operatorname{Spec} k$.

For any morphism $f: X \longrightarrow Y$ of varieties over k, denote by $f^!$ the Hartshorne–Deligne extraordinary inverse image, which is in fact well-defined as a functor between the coderived categories

$$f^!\colon \mathrm{D^{co}}(Y\text{-qcoh}) \longrightarrow \mathrm{D^{co}}(X\text{-qcoh}).$$

In particular, for any dualizing complex $\mathcal{D}_Y^\bullet$ on Y, the complex $f^!\mathcal{D}_Y^\bullet$ is a dualizing complex on X. So we can set $\mathcal{D}_X^\bullet = p^!\mathcal{O}_{\operatorname{Spec} k}$.

Then for any two complexes of quasi-coherent sheaves $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ on X one has

$$\mathcal{M}^\bullet \mathbin{\square}_{\mathcal{D}_X^\bullet} \mathcal{N}^\bullet = \Delta^!(\mathcal{M}^\bullet \boxtimes_k \mathcal{N}^\bullet),$$

where $\boxtimes_k$ denotes the external tensor product functor, so $\mathcal{M}^\bullet \boxtimes_k \mathcal{N}^\bullet$ is a complex of quasi-coherent sheaves on $X \times_k X$.

This operation of

- derived cotensor product of complexes of quasi-coherent (torsion) sheaves along the base $\mathfrak{X}$

should be somehow mixed with

- the conventional derived tensor product of complexes of quasi-coherent sheaves along the fibers

in order to obtain

- the double-sided derived functor of semitensor product of complexes of quasi-coherent (torsion) sheaves on the total scheme $\mathfrak{Y}$.

The double-sided derived semitensor product operation should provide a tensor structure on the semiderived category of quasi-coherent torsion sheaves $\mathrm{D}^{\mathrm{sico}}_{\mathfrak{X}}(\mathfrak{Y}\text{-qcoh})$ with the unit object $\pi^*\mathcal{D}^\bullet_{\mathfrak{X}}$, where $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ denotes the fibration morphism.

To conclude:

- The fibers are glued in a simple way from large affine pieces.
- The base is glued in a complicated way from small affine pieces (and endowed with a dualizing complex).

The reason for the base and the fibers being like that is because

- The conventional derived category is well-behaved for modules over an arbitrary ring, and by the way of generalization for (co)sheaves on infinite-dimensional quasi-compact semi-separated schemes.
- The co/contraderived category is well-behaved for co/contramodules over a coalgebra, and by the way of generalization for (co)sheaves on finite-dimensional stacks and ind-Noetherian ind-schemes. (A coalgebra is a dualizing complex over itself.)

The example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$ comes as an afterthought.

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