Semi-Infinite Algebraic Geometry

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We will come to more specific definitions shortly.

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More generally, if G is an affine algebraic group over a field k, then the fibration $G(k((z))) \longrightarrow G(k((z)))/G(k[[z]])$ can be viewed as a semi-infinite object in algebraic geometry.



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Classical homological algebra: two hypercohomology spectral sequences

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Then there are two spectral sequences converging to the same limit

$${}^{\prime}E_{2}^{pq} = R^{p}F(H^{q}C^{\bullet}) \Longrightarrow \mathbb{H}^{p+q}(C^{\bullet});$$
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 derived category of the first kind (conventional)
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[Hinich, Lefèvre-Hasegawa, Krause, L.P., H. Becker, ... '98 –]



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There is a natural way to build derived categories of the first and second kind on top of one another.

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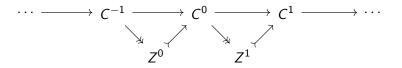
- a derived category of the second kind (the coderived or the contraderived category) along the variables from A and
- the derived category of the first kind (the conventional derived category) along the complementary variables from *R*.

Let ${\mathcal E}$ be an exact category (in the sense of Quillen).

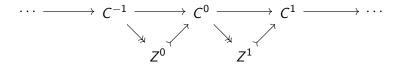
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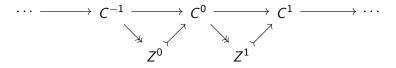


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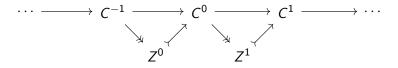
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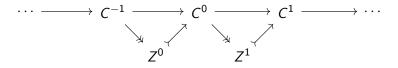


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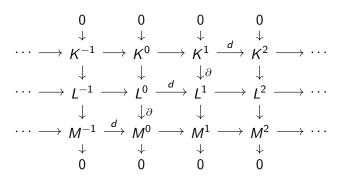
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$$\downarrow \qquad \downarrow \qquad \downarrow \partial \qquad \downarrow$$

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Form the total complex $\operatorname{Tot}(K^{\bullet} \to L^{\bullet} \to M^{\bullet})$ by taking direct sums along the diagonals, with the differential $D = \partial \pm d$.



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So the semiderived category is indeed a mixture of the co/contraderived category along A and the conventional derived category in the direction of R relative to A_{C} .

Semi-infinite algebraic varieties

A semi-infinite algebraic variety is a morphism of ind-schemes or ind-stacks $\mathfrak{Y} \longrightarrow \mathfrak{X}$

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Known particular cases

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This definition works well enough to provide an exact category of contraherent cosheaves X- ctrh on X.



The category X-qcoh of quasi-coherent sheaves on a scheme X

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The abelian category X-qcoh always has enough injective objects. When the scheme X is quasi-compact and semi-separated, or Noetherian of finite Krull dimension, the exact category X-ctrh has enough projective objects.

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Let X be a Noetherian scheme. Recall that a dualizing complex \mathcal{D}_X^{\bullet} on X is a complex of quasi-coherent sheaves satisfying the following conditions:

- ullet \mathcal{D}_{X}^{ullet} is a finite complex of injective quasi-coherent sheaves;
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[Jørgensen, Krause, Iyengar-Krause '05-'06]



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[Neeman, Murfet '07-'08]



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The historical obscurity/neglect of contramodules seems to be the reason why many people believe that projectives are much less common than injectives in "naturally appearing" abelian categories.

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• and the unity equation $m \circ \varepsilon_X = id_M$

$$M \longrightarrow R[M] \longrightarrow M$$
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performing infinite summations in the conventional sense of the topology of \Re to compute the coefficients. There is also the obvious "point measure" map $\varepsilon_X \colon X \longrightarrow \mathfrak{R}[[X]]$. The natural transformations ϕ and ε define the structure of a monad on the functor $X \longmapsto \mathfrak{R}[[X]] : \operatorname{Sets} \longrightarrow \operatorname{Sets}$.

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The composition of the contraaction map $\pi \colon \mathfrak{R}[[\mathfrak{P}]] \longrightarrow \mathfrak{P}$ with the obvious embedding $\mathfrak{R}[\mathfrak{P}] \longrightarrow \mathfrak{R}[\mathfrak{P}]$ defines the underlying left \Re -module structure on every left \Re -contramodule.

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For any discrete right \mathfrak{R} -module \mathcal{N} and any abelian group U, the left \mathfrak{R} -module $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{N},U)$ has a natural left \mathfrak{R} -contramodule structure.

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$$q_n = sq_{n+1} + p_n$$
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An abelian group P with an additive operator $s: P \longrightarrow P$ is said to be an s-contramodule if for any sequence $p_0, p_1, p_2, \ldots \in P$ the infinite system of nonhomogeneous linear equations

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Example: equivalence on the fiber

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where $C^{ullet}_{\{s_j\}}(R)$ is the two-term complex $R \longrightarrow R[s_j^{-1}]$.



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[Schenzel, Porta-Shaul-Yekutieli, '03-'14]



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[MGM Duality: Matlis, Greenlees–May, Dwyer–Greenlees, Porta–Shaul–Yekutieli, L.P., '78–'15]

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Denote by $\mathfrak{A}\text{-}\mathrm{discr}$ the abelian category of discrete $\mathfrak{A}\text{-}\mathrm{modules}.$



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$\mathsf{Theorem}$

Any compatible system $\mathcal{D}_{\mathfrak{A}_n}^{\bullet}$ of choices of dualizing complexes $D_{A_n}^{\bullet}$ for the Noetherian rings A_n , $n \geqslant 0$

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Theorem

Any compatible system $\mathcal{D}_{\mathfrak{A}_n}^{\bullet}$ of choices of dualizing complexes $D_{A_n}^{\bullet}$ for the Noetherian rings A_n , $n \geqslant 0$, induces an equivalence of triangulated categories

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The reason for the base and the fiber being this way comes from the definition of the semiderived category, which turns out to be the co/contraderived category along the subring and the conventional derived category in the complementary direction of the ambient ring relative to the subring.

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The choice of a dualizing complex D_A^{\bullet} for the coherent ring A

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Theorem

The choice of a dualizing complex D_A^{\bullet} for the coherent ring A induces an equivalence between the two semiderived categories of R-modules relative to A,

Let A be a coherent commutative ring such that fp-injective A-modules have finite injective dimension (e.g., this is so if all ideals in A are at most countably generated).

A bounded complex of fp-injective A-modules D_A^{\bullet} with finitely presented cohomology A-modules is called a *dualizing complex* for A if the natural map $A \longrightarrow \operatorname{Hom}_{\mathrm{D}(A\operatorname{-mod})}(D_A^{\bullet}, D_A^{\bullet}[*])$ is an isomorphism.

Let $A \longrightarrow R$ be a homomorphism of commutative rings such that R is a flat A-module.

Theorem

The choice of a dualizing complex D_A^{\bullet} for the coherent ring A induces an equivalence between the two semiderived categories of R-modules relative to A,

$$D_A^{\text{sico}}(R\text{-mod}) \simeq D_A^{\text{sictr}}(R\text{-mod}).$$

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Then take the right derived functor.



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The example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$



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The example of the fibration $k((z)) \longrightarrow k((z))/k[[z]]$ comes as an afterthought.

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