QUASI-COHERENT TORSION SHEAVES, THE SEMIDERIVED CATEGORY, AND THE SEMITENSOR PRODUCT

SEMI-INFINITE ALGEBRAIC GEOMETRY OF QUASI-COHERENT SHEAVES ON IND-SCHEMES

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Abstract. We construct the semi-infinite tensor structure on the semiderived category of quasi-coherent torsion sheaves on an ind-scheme endowed with a flat affine morphism into an ind-Noetherian ind-scheme with a dualizing complex. The semitensor product is "a mixture of" the cotensor product along the base and the derived tensor product along the fibers. The inverse image of the dualizing complex is the unit object. This construction is a partial realization of the Semi-infinite Algebraic Geometry program, as outlined in the introduction to [47].

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Introduction

0.0. The aim of this paper is to extend the Semi-infinite Homological Algebra, as developed in the author's monograph [40], to the realm of Algebraic Geometry. According to the philosophy elaborated in the preface to [40], semi-infinite homological theories are assigned to "semi-infinite algebrac and geometric objects". A detailed explanation of what should be understood by a "semi-infinite algebraic variety" was suggested in the introduction to the author's paper [47] (see also the presentation [48]). In the present work, we develop a part of the Semi-infinite Algebraic Geometry program along the lines of [47, 48].

To be more precise, following the approach of [40], one has to distinguish between the semi-infinite *homology* and *cohomology* theories. The semi-infinite homology groups (SemiTor) are assigned to a pair of *semimodules*, which means roughly "comodules along a half of the variables and modules along the other half". The semi-infinite cohomology groups (SemiExt) are assigned to one semimodule and one *semicontramodule*; the latter means "a contramodule along a half of the variables and a module along the other half".

In the context of algebraic geometry, quasi-coherent sheaves on nonaffine schemes and quasi-coherent torsion sheaves on ind-schemes play the role of the "comodules along some of the variables", while for contramodules one has to consider the *contraherent cosheaves* [44]. In the present paper, we restrict ourselves to the semi-infinite homology, the SemiTor; so contraherent cosheaves do not appear.

According to the philosophy of [40], the key technical concept for semi-infinite homological algebra is the *semiderived category.* This means "the coderived category along a half of the variables mixed with the derived category along the other half". One is supposed to take the coderived category along the coalgebra variables and the derived category along the ring/algebra variables.

How does one interpret this prescription in the context of algebraic geometry, or more specifically, e. g., quasi-coherent sheaves on algebraic varieties? Is one supposed to take the coderived or the derived category of quasi-coherent sheaves, or how does it depend on the nature of the variety at hand? This paper grew out of the author's attempts to find an answer to this question, which spanned more than a decade since about 2009. The paper [47] and the presentation [48] were the first formulations of the conclusions I had arrived at.

0.1. So, what is semi-infinite geometry? Before attempting to answer this question, let us discuss *semi-infinite set theory* first.

Let S be a set. A *semi-infinite structure* on S is the datum of a set of *semi-infinite subsets* S^+ in S such that, for any two semi-infinite subsets $S^{+\prime}$ and $S^{+\prime\prime} \subset S$, the set-theoretic difference $S^{+\prime} \setminus S^{+\prime\prime}$ is a *finite set.* Furthermore, if a subset $S^+ \subset S$ is semi-infinite and $S^{+\prime} \subset S$ is a subset for which both the sets $S^{+\prime} \setminus S^+$ and $S^+ \setminus S^{+\prime}$ are finite, then $S^{+\prime}$ should be also a semi-infinite subset.

So, in order to specify a semi-infinite structure on S, it suffices to point out one particular semi-infinite subset in S; then it becomes clear what the other semi-infinite subsets are. In the *standard semi-infinite structure* on the set of integers $\mathbb{Z}$, the subset of positive integers is a semi-infinite subset.

Given a semi-infinite structure on S, the *dual* semi-infinite structure is formed by the set of all subsets $S^- = S \setminus S^+$, where $S^+ \subset S$ are the semi-infinite subsets. The subsets $S^- \subset S$ are called *co-semi-infinite*. Every infinite set has the *trivial* semi-infinite structure, in which the semi-infinite subsets are precisely the finite subsets, and the *cotrivial* semi-infinite structure, in which the co-semi-infinite subsets are precisely the finite ones. On a finite set, the trivial and cotrivial semi-infinite structures coincide, and there are no other semi-infinite structures; but any infinite set admits infinitely many semi-infinite structures.

The datum of a semi-infinite structure on S is equivalent to the datum of a compact, Hausdorff topology on the set $S_\infty = S \sqcup \{-\infty, +\infty\}$ for which the induced topology on the subset $S \subset S_\infty$ is discrete. A subset $S_+ \subset S$ contains a semi-infinite subset if and only if $S_+ \sqcup \{+\infty\}$ is a neighborhood of the point $+\infty$ in S_∞, while a subset $S_- \subset S$ contains a co-semi-infinite subset if and only if $S_- \sqcup \{-\infty\}$ is a neighborhood of the point $-\infty \in S_\infty$. In other words, a semi-infinite structure on a set is the same thing as the datum of a two-point compactification.

Notice that there is a notion of "relative cardinality", a well-defined integer, for a pair of semi-infinite subsets in S. Given two semi-infinite subsets $S^{+\prime}$ and $S^{+\prime\prime} \subset S$, put "$|S^{+\prime}|-|S^{+\prime\prime}|$" $= |\widetilde{S}^+ \setminus S^{+\prime\prime}| - |\widetilde{S}^+ \setminus S^{+\prime}| = |S^{+\prime} \setminus \overline{S}^+| - |S^{+\prime\prime} \setminus \overline{S}^+| \in \mathbb{Z}$, where $\widetilde{S}^+$ and $\overline{S}^+$ are any semi-infinite subsets in S such that $\overline{S} \subset S^{+\prime} \cap S^{+\prime\prime}$ and $S^{+\prime} \cup S^{+\prime\prime} \subset \widetilde{S}^+$. So it makes sense to say that "there are more elements in $S^{+\prime}$ than in $S^{+\prime\prime}$" (and how many more, precisely), even though none of the two sets may be contained in the other one, and their (infinite) cardinalities are the same.

0.2. The notion of a *locally linearly compact* topological vector space (otherwise known as a *Tate vector space*) is the central concept of *semi-infinite linear algebra*.

Here is the formal definition: a complete, separated topological vector space W over a field $\Bbbk$ is said to be *linearly compact* if open vector subspaces of finite codimension form a base of neighborhoods of zero in W. Equivalently, this means that W is isomorphic to the projective limit of a (directed, if one wishes) projective system of discrete finite-dimensional vector spaces, endowed with the projective limit topology; or the product of discrete one-dimensional vector spaces, endowed with the product topology. A topological vector space V is said to be *locally linearly compact* if it has a linearly compact open subspace.

Informally, one can say a locally linearly compact vector space V is a topological vector space whose topological basis is indexed by a set S with a natural semi-infinite structure. A topological basis of a linearly compact open subspace $W \subset V$ is a semi-infinite subset in S, while a basis of the discrete quotient space V/W is the complementary co-semi-infinite subset in S.

To a set S with a semi-infinite structure, one can assign the locally linearly compact topological vector space $V = \bigoplus_{t \in S \setminus S^+} \Bbbk t \oplus \prod_{s \in S^+} \Bbbk s$, where $S^+ \subset S$ is a semi-infinite subset. Here the topology is discrete on the first direct summand and linearly compact on the second one. Obviously, the (topological) vector space V does not depend on the choice of a particular semi-infinite subset S^+ within the given semi-infinite structure on the set S.

Similarly to the relative cardinality of a pair of semi-infinite subsets, one can speak about the "relative dimension" (a well-defined integer) for a pair of linearly compact open subspaces W' and $W'' \subset V$ in a given locally linearly compact topological vector space V. There is also the notion of a "relative determinant" (a functorially defined one-dimensional $\Bbbk$-vector space) for W' and W''.

The topological $\Bbbk$-vector space of formal Laurent power series $\Bbbk((t))$ in one variable t with the coefficients in $\Bbbk$ is the thematic example of a locally linearly compact topological vector space. The subspace of formal Taylor power series $\Bbbk[[t]] \subset \Bbbk((t))$ is a linearly compact open subspace.

0.3. Now we can offer the following very rough and imprecise definition of a *semi-infinite geometric object*, or a "semi-infinitely structured space". A semi-infinitely structured space is an (infinite-dimensional) space Y with local or global coordinates $(x_s)_{s\in S}$ indexed by a set S with a natural semi-infinite structure.

It seems to makes sense to require that, for every point $p \in Y$, the subset $S(p) \subset S$ of all indices $s \in S$ for which $x_s(p) \neq 0$ be contained in a semi-infinite subset, $S(p) \subset S^+$, in the set S. So all the coordinates may vanish simulaneously, but at most a semi-infinite subset of the coordinates may be simultaneously nonzero at any given point on Y.

Then one can say that a closed subvariety $Y^+ \subset Y$ is a *semi-infinite subvariety* if in local coordinates it can be defined by a system of equations prescribing a fixed value to every coordinate x_s with $s \in S^-$, for some co-semi-infinite subset $S^- \subset S$. One can think of a "semi-infinite homology theory", in which semi-infinite subvarieties would be cycles of homological dimension $\infty/2 + n$, $n \in \mathbb{Z}$ (where one postulates $|S^+| = \infty/2$ for one fixed semi-infinite subset $S^+ \subset S$, and then has $|S^{+\prime}| \in \infty/2+\mathbb{Z}$ for every other semi-infinite subset $S^{+\prime} \subset S$).

One can think of a "semi-infinite intersection theory", in which subvarieties of finite codimension in Y would form a graded ring, with respect to the (properly understood) intersection, and semi-infinite subvarieties would form a $\mathbb{Z}$-graded (or a "$(\infty/2 + \mathbb{Z})$-graded") module over this graded ring. The intersection of a semi-infinite subvariety with a subvariety of finite codimension would produce another semi-infinite subvariety, interpreted as their product.

This is the kind of geometric speculation which inspired the present work. Moving closer to the setting in this paper, one can consider a particular case when the coordinates x_s with $s \in S^-$ can be somehow globally separated and grouped together, producing a fibration $\pi\colon Y \longrightarrow X$. So the local coordinates on X are indexed by a co-semi-infinite subset of the varibles, while the local coordinates on the fibers are indexed by a semi-infinite subset (thus the fibers $Y_q = \pi^{-1}(q)$, $q \in X$, are "semi-infinite subvarieties" in Y in the above sense).

0.4. The cornerstone technical, homological principle of our approach to semi-infinite algebra and geometry is that one is supposed to work with the *semiderived category.* This means a mixture of the derived category along a semi-infinite subset of the variables and the coderived (or contraderived) category along a co-semi-infinite subset of the variables.

One reason why this is important is because the delicate choice of an exotic derived category to work with dictates the derived functors which are naturally produced or well-defined. One notices this when one starts working systematically with doubly unbounded complexes. The left derived tensor product is well-behaved on the derived category, while the right derived cotensor product behaves well as a functor on the coderived category. In the context of algebraic geometry, this means that the left derived functor of $*$-restriction onto a closed subscheme is well-defined on the derived category, while the right derived functor of !-restriction (or, in our notation, $+$-restriction) onto a locally closed subscheme is well-behaved as a functor between the coderived categories.

How does one know along which coordinates the derived category needs to be taken, and along which ones the coderived category? One answer to this question is that, given a fibration, there is a natural way to define a semiderived category that is a mixture of the coderived category along the base and the derived category along the fibers. Another heuristic is that, given an infinite set of coordinates which are allowed to be nonzero all simultaneously, one should take the derived category along these; but if the condition is imposed that only a finite subset of the coordinates may differ from zero at any given point, one should take the coderived category.

0.5. So, what is the coderived category? There are several alternative answers to this question offered in the contemporary literature. The simplest definition says the the coderived category of an abelian or exact category E is the *homotopy category of unbounded complexes of injective objects* in E. It is presumed that there are enough injective objects in E. This approach was initiated by Krause [23] and developed by Becker [6] (see also [37], [62], and [56]).

In the present author's work, the coderived categories first appeared in connection with derived nonhomogeneous Koszul duality [41] and were subsequently used for the purposes of semi-infinite homological algebra in [40] (see also [47, 51, 55]). Our approach emphasizes a construction of the coderived category as the *quotient category* of the homotopy category by what we call the full subcategory of *coacyclic* complexes. The coacyclic complexes are defined as the ones that can be obtained from the *totalizations of short exact sequences of complexes* using the operations of cone and infinite coproduct. Any coacyclic complex (in an abelian/exact category with exact coproducts) is acyclic, but the converse is not generally true.

The example of the abelian category $X\text{–}\mathsf{qcoh}$ of quasi-coherent sheaves on a semi-separated Noetherian scheme X is instructive. For this class of schemes, both the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ and the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ are perfectly well-behaved. In fact, both of them are compactly generated. The compact objects in $\mathsf{D}(X\text{–}\mathsf{qcoh})$ are the *perfect complexes*, while the compact objects in $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ are all the bounded complexes of coherent sheaves. In this connection, the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ has been called the category of "ind-coherent sheaves" in [16]. Both the derived and the coderived category can be properly considered for much wider classes of algebro-geometric objects, but the natural directions for generalization differ: while the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$ makes perfect sense for an arbitrary quasi-compact semi-separated scheme X, the natural generality for the coderived category is that of an ind-Noetherian ind-scheme (or ind-stack) $\mathfrak{X}$.

0.6. As we have mentioned above, one important way to think of the distinction between the derived and the coderived category in the algebraic geometry context is in connection with the *left derived inverse image functor* vs. the *extraordinary inverse image functor*. This observation seems to be due to Gaitsgory [16].

Given a morphism of (good enough) schemes $f\colon Y \longrightarrow X$, the direct image functor $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ has finite homological dimension, so its right derived functor is equally well-defined on the unbounded derived and the coderived categories, $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathbb{R}f_*\colon \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$. The inverse image functor $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$, however, has infinite homological dimension in general. As infinite left resolutions are problematic in the coderived categories, the left derived inverse image $\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ is well-defined on the convenional unbouded derived categories, but *not* on the coderived categories (unless the morphism f has finite Tor-dimension). Whenever it exists, the left derived inverse image $\mathbb{L}f^*$ is left adjoint to $\mathbb{R}f_*$.

The functor $\mathbb{R}f_*$ preserves coproducts both in the unbounded derived and the coderived categories; so, by the Brown representability theorem, it has a right adjoint functor in both the contexts. This right adjoint functor, which we, following the terminology of [44], call the *extraordinary inverse image functor in the sense of Neeman* (with the reference to [33]) and denote by $f^!$, may be not as important for algebraic geometry as the *extraordinary inverse image functor in the sense of Deligne*, which we denote by f^+ (it is denoted by $f^!$ in [19] and Deligne's appendix to [19]).

The functor f^+ is defined by the conditions that $(fg)^+ \simeq g^+f^+$ for any pair of composable morphisms f and g; one has $f^+ = f^!$ for a proper morphism f; and one has $j^+ = j^*$ for an open immersion j. The functor $f^+\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–}\mathsf{qcoh})$ is well-defined on the coderived categories [44, Section 5.16], but *not* on the unbounded derived categories. In particular, even for locally closed immersions f, it is *impossible* to define a functor $f^+\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ in such a way that one would have $j^+ = j^*$ for open immersions, while $i^+ = \mathbb{R}i^!$ would be the functor of right derived restriction with supports for closed immersions i, and $(fg)^+ \simeq g^+f^+$ for all composable pairs of morphisms [33, Example 6.5]. The punchline: *the unbounded derived category works well with the left derived inverse image $\mathbb{L}f^*$; the coderived category words well with the extraordinary inverse image f^+.*

0.7. What is the semiderived category? To answer this question properly, it is better to step back and ask the most basic question: *What is the derived category?* The derived category is usually defined as the result of localizing the homotopy category of complexes by the class of quasi-isomorphisms. What are the quasi-isomorphisms? Let R be an associative algebra over a field $\Bbbk$, and let $f\colon C^\bullet \longrightarrow D^\bullet$ be a morphism of complexes of A-modules. What does it mean that f is a quasi-isomorphism? Here is the answer which we suggest: let us apply the forgetful functor and view f as a morphism of complexes of $\Bbbk$-vector spaces. The map f is a quasi-isomorphism of complexes of R-modules *if and only if* it is a homotopy equivalence of complexes of $\mathbb{k}$-vector spaces. In other words, a complex of R-modules is acyclic if and only if its underlying complex of $\mathbb{k}$-vector spaces is contractible.

The previous paragraph implies that one should think of the derived category in the context of a forgetful functor. But one does not have to go all the way and forget the whole action of the algebra R, staying only with vector spaces. One can forget the action of a half of the variables in R, while keeping the other half. Let $A \longrightarrow R$ be a ring homomorphism. One defines the *R/A-semiderived category* of R-modules $\mathsf{D}^{\mathsf{si}}_A(R\text{–}\mathsf{mod})$ as the quotient category of the homogopy category of (unbounded complexes of) R-modules by the thick subcategory of those complexes which are *coacyclic as complexes of A-modules*. This definition of the semiderived category can be found in [47, Section 5] (see [40, Section 2.3] for the original definition of the *semiderived category of semimodules*). Thus the semiderived category is a mixture of the "coderived category along A" and the "derived category in the direction of R relative to A". This is the way to mix the coderived category with the derived category alluded to several paragraphs above.

In the context of algebraic geometry, the forgetful functor $R\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ is interpreted geometrically as the direct image functor. To have the direct image functor exact and faithful (as needed for the definition of a semiderived category), it is simplest to assume the geometric morphism in question to be *affine*. Thus one can speak of the *semiderived category of quasi-coherent sheaves* $\mathsf{D}^{\mathsf{si}}_X(\mathbf{Y}\text{–}\mathsf{qcoh})$ for an affine morphism of schemes $\pi\colon \mathbf{Y} \longrightarrow X$. More specifically, following the discussion above, one may want to restrict generality to Noetherian schemes X, and then expand it to ind-Noetherian ind-schemes $\mathfrak{X}$. Then one assumes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ to be a morphism of ind-schemes with (possibly infinite-dimensional) affine scheme fibers, and considers the semiderived category of what we call *quasi-coherent torsion sheaves on $\mathfrak{Y}$*.

0.8. What can one do with the semiderived category of quasi-coherent torsion sheaves? Our suggestion is to construct a tensor category structure on it. We start with defining the *cotensor product* operation on the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{X}$, and proceed further to construct the *semitensor product* on the $\mathfrak{Y}/\mathfrak{X}$-semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$. For this purpose, we need to choose a *dualizing complex* on $\mathfrak{X}$.

The notion of a dualizing complex of quasi-coherent sheaves on an ind-scheme is itself a perfect illustration of the importance of the coderived categories. Notice that there is some affinity between the dualizing complexes and the extraordinary inverse images in the sense of Deligne: for a morphism of schemes $f\colon Y \longrightarrow X$, given a dualizing complex $\mathcal{D}^\bullet_X$ on X, the complex $f^+(\mathcal{D}^\bullet_X)$ is a dualizing complex on Y. A dualizing complex on a scheme, however, is bounded; so one does not feel the distinction between the derived and the coderived category in connection with it (in fact, there is no difference between the derived and the coderived category for bounded below complexes). A dualizing complex on an ind-scheme, on the other hand, is usually *not* bounded below; so it is important that a dualizing complex of quasi-coherent torsion sheaves $\mathscr{D}^\bullet_{\mathfrak{X}}$ on an ind-scheme $\mathfrak{X}$ has to be viewed as an object of the coderived category $\mathsf{D^{co}}(\mathfrak{X}\text{–tors})$. We explain in Section 11.1(7) that, for about the simplest example of an infinite-dimensional ind-scheme $\mathfrak{X}$ (the "discrete affine space over a field"), the dualizing complex $\mathscr{D}_{\mathfrak{X}}^\bullet$ on $\mathfrak{X}$ is an acyclic complex!

0.9. Let us have a more detailed discussion of the cotensor and semitensor products, whose definitions are the main objectives of this paper. The functor $\mathrm{Tor}_1^{\mathbb{Z}}$, known classically as the *torsion product* of abelian groups [25, 39, 22], defines a tensor structure on the category of torsion abelian groups with $\mathbb{Q}/\mathbb{Z}$ as the unit object. The torsion product of p-primary abelian groups, for a fixed prime number p, is very similar to the cotensor product of comodules over the coalgebra $\mathscr{C}$ dual to the topological algebra $\Bbbk[[z]]$ of formal Taylor power series in one variable over a field.

The first aim of this paper is to extend these constructions to complexes of quasi-coherent torsion sheaves on an ind-Noetherian ind-scheme $\mathfrak{X}$ with a dualizing complex. The dualizing complex $\mathscr{D}_{\mathfrak{X}}^\bullet$ is the unit object of this tensor category structure, which is defined on the coderived category [41] $\mathsf{D^{co}}(\mathfrak{X}\text{–tors})$ of quasi-coherent torsion sheaves on $\mathfrak{X}$. The case of a Noetherian scheme X with a dualizing complex $\mathcal{D}_X^\bullet$ was considered by Murfet in [31, Proposition B.6] and in our paper [12, Section B.2.5], and the case of an ind-affine ind-Noetherian (or ind-coherent) $\aleph_0$-ind-scheme $\mathfrak{X}$ with a dualizing complex, in [44, Section D.3].

Furthermore, pursuing the Semi-infinite Algebraic Geometry program as outlined in the introduction to the paper [47] and in the presentation [48], we consider a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. The fibers of the morphism π are, generally speaking, infinite-dimensional affine schemes. Then the semiderived category [40, 47] $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$ of the abelian category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ relative to the direct image functor $\pi_*\colon \mathfrak{Y}\text{–tors} \longrightarrow \mathfrak{X}\text{–tors}$ carries the operation of *semitensor product*, making it a tensor category whose unit object is the inverse image $\pi^*\mathscr{D}_{\mathfrak{X}}^\bullet$ of the dualizing complex on $\mathfrak{X}$ to the ind-scheme $\mathfrak{Y}$. The case of an affine Noetherian (or coherent) scheme $\operatorname{Spec} A$ in the role of $\mathfrak{X}$ and an affine scheme $\operatorname{Spec} R$ in the role of $\mathfrak{Y}$ was considered in [47, Section 6].

The cotensor product $\square_{\mathscr{D}_{\mathfrak{X}}^\bullet}$ of complexes of quasi-coherent torsion sheaves on $\mathfrak{X}$ is similar to a right derived functor, in that (under mild assumptions on $\mathscr{D}_{\mathfrak{X}}^\bullet$) the cotensor product of two bounded complexes is a complex bounded from below. The semitensor product $\lozenge_{\pi^*\mathscr{D}_{\mathfrak{X}}^\bullet}$ of complexes of quasi-coherent torsion sheaves of $\mathfrak{Y}$ resembles a double-sided derived functor, as the semitensor product of two bounded complexes is, generally speaking, a doubly unbounded complex. Thus our semitensor product construction can be viewed as a version of *semi-infinite homology theory* for quasi-coherent sheaves; and indeed, it reduces to a particular case of the semi-infinite homology functor SemiTor from the book [40] in the case of an ind-zero-dimensional (ind-Artinian) ind-scheme $\mathfrak{X}$ of ind-finite type over a field.

0.10. From the perspective of algebraic geometry, the most natural way to think about the conventional left derived functor of tensor product of quasi-coherent sheaves on a scheme may be the following one. Let Y be a scheme over a field $\Bbbk$, and let $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ be two complexes of quasi-coherent sheaves over it. Consider the

Cartesian product $Y \times_{\Bbbk} Y$, and consider the external tensor product $\mathcal{M}^\bullet \boxtimes_{\Bbbk} \mathcal{N}^\bullet$ of the complexes $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$; this is a complex of quasi-coherent sheaves on $Y \times_{\Bbbk} Y$. Let $\Delta_Y : Y \longrightarrow Y \times_{\Bbbk} Y$ be the diagonal morphism. Then the derived tensor product of $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ on Y can be defined as the left derived restriction of the external tensor product to the diagonal, $\mathcal{M}^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_Y} \mathcal{N}^\bullet = \mathbb{L}\Delta_Y^*(\mathcal{M}^\bullet \boxtimes_{\Bbbk} \mathcal{N}^\bullet)$.

In this spirit, if one is interested in alternative tensor product operations on quasi-coherent sheaves, the most natural approach may be to keep the external tensor product unchanged, but tweak the restriction to the diagonal. Following the discussion above, one is supposed to take $\mathbb{R}\Delta^!$ (or Δ^+) for the cotensor product and some "combination of $\mathbb{L}\Delta^*$ with $\mathbb{R}\Delta^!$" for the semitensor product.

It was shown already in [12, end of Section B.2.5] that Murfet's tensor structure [31] (which we call the *cotensor product*) on the coderived category of quasi-coherent sheaves on a separated scheme X of finite type over a field $\Bbbk$ can be computed as $\mathcal{M}^\bullet \,\square_{\mathcal{D}_X^\bullet}\, \mathcal{N}^\bullet \simeq \mathbb{R}\Delta_X^!(\mathcal{M}^\bullet \boxtimes_{\Bbbk} \mathcal{N}^\bullet)$ for any two complexes of quasi-coherent sheaves $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ on X. This presumes the choice of the dualizing complex $\mathcal{D}_X^\bullet = p^+(\Bbbk)$ on the scheme X, where $p\colon X \longrightarrow \operatorname{Spec} \Bbbk$ is the morphism of finite type (this is called the *rigid dualizing complex* in [66, 65, 59]). In this paper, we extend this computation to ind-separated ind-schemes $\mathfrak{X}$ of ind-finite type, and further to the relative/semi-infinite context of an affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$.

0.11. The first three sections of this paper develop the basic language of ind-schemes, quasi-coherent torsion sheaves, and pro-quasi-coherent pro-sheaves. The pro-quasi-coherent pro-sheaves are used in the subsequent sections as a key technical tool for our constructions involving quasi-coherent torsion sheaves. In the next three sections, we consider an ind-Noetherian ind-scheme $\mathfrak{X}$, the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, and the cotensor product operation $\square_{\mathscr{D}_{\mathfrak{X}}^\bullet}$ on it. In the last five sections, we study the relative (or properly semi-infinite) situation: a flat affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$, and the semitensor product functor $\between_{\pi^*\mathscr{D}_{\mathfrak{X}}^\bullet}$.

We discuss the basics of the theory of (strict ind-quasi-compact ind-quasi-separated) ind-schemes in Section 1. The additive/exact/abelian categories of module objects over ind-schemes (namely, the quasi-coherent torsion sheaves and the pro-quasi-coherent pro-sheaves) are defined and discussed in Sections 2–3. In particular, we show that quasi-coherent torsion sheaves on a reasonable ind-scheme (in the sense of [7]) form a Grothendieck category.

The equivalence between the coderived category of quasi-coherent torsion sheaves and the derived category of flat pro-quasi-coherent pro-sheaves on an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex is constructed in Section 4. The cotensor product functor over such an ind-scheme is defined and many (particularly ind-Artinian) examples of it are considered in Section 5. In particular, we establish the comparisons with the derived cotensor product of comodules over a cocommutative coalgebra over a field and the torsion product of torsion modules over a Dedekind domain.

The particular case of an ind-separated ind-scheme $\mathfrak{X}$ of ind-finite type over a field $\Bbbk$ is considered in Section 6. In this setting, we compute the cotensor product of complexes of quasi-coherent torsion sheaves as the right derived !-restriction of the external tensor product to the diagonal closed immersion $\Delta_{\mathfrak{X}}\colon \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\Bbbk} \mathfrak{X}$.

In the relative context of a flat affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ into an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex, the equivalence between the semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ relative to $\mathfrak{X}$ and the derived category of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is constructed in Section 7. In particular, the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ itself is introduced in Section 7.1. The functor of semitensor product of complexes of quasi-coherent torsion sheaves over $\mathfrak{Y}$ relative to $\mathfrak{X}$ is defined in Section 8. The special case of a flat affine morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ with an ind-zero-dimensional ind-scheme of ind-finite type $\mathfrak{X}$ over a field $\Bbbk$ is considered in Section 8.5, and the comparison with the functor SemiTor from the book [40] is discussed.

In Section 9 we compute the semiderived product as a combination of two kinds of derived restrictions to the diagonal. Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over a field $\Bbbk$, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism. The diagonal map $\Delta_{\mathfrak{Y}}\colon \mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$ factorizes naturally into two "partial diagonals" $\mathfrak{Y} \xrightarrow{\delta} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y} \xrightarrow{\eta} \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$; both δ and η are closed immersions. Let $\mathscr{D}^{\bullet}_{\mathfrak{X}}$ be a rigid dualizing complex on $\mathfrak{X}$. For any two complexes $\mathcal{M}^{\bullet}$ and $\mathcal{N}^{\bullet}$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$, we construct a natural isomorphism $\mathcal{M}^{\bullet} \lozenge_{\pi^*\mathscr{D}^{\bullet}_{\mathfrak{X}}} \mathcal{N}^{\bullet} \simeq \mathbb{L}\delta^* \, \mathbb{R}\eta^!(\mathcal{M}^{\bullet} \boxtimes_{\Bbbk} \mathcal{N}^{\bullet})$ in $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$. In Section 10 we show that both the semiderived category and the semitensor product operation on it are preserved when a flat affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is replaced with a flat affine morphism $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ such that $\pi = \tau\pi'$, where $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ is a smooth (or "weakly smooth") affine morphism of finite type between ind-semi-separated ind-Noetherian ind-schemes.

Several simple, but geometrically nontrivial examples of flat affine morphisms of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ with an ind-separated ind-scheme $\mathfrak{X}$ of ind-finite type over a field are considered in Section 11. In particular, in Section 11.1, we discuss the example of the flat affine morphism of ind-affine ind-schemes $\mathfrak{Y} \longrightarrow \mathfrak{X}$ corresponding to a surjective open linear map of locally linearly compact topological vector spaces $V \longrightarrow V/W$ with a discrete target V/W and a linearly compact kernel W, such as $\Bbbk((t)) \longrightarrow \Bbbk((t))/\Bbbk[[t]]$. Here $\Bbbk((t))$ is the topological vector space of formal Laurent power series in a variable t over a field $\Bbbk$, and $\Bbbk[[t]] \subset \Bbbk((t))$ is the open vector subspace of Taylor power series. It follows from the results of Section 10 that the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ in this example does *not* depend on the choice of a linearly compact open subspace $W \subset V$, but is entirely determined by the locally linearly compact vector space V. The semitensor product operation on $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ also does not depend on W up to a dimensional cohomological shift and a determinantal twist.

This paper does not strive for maximal natural generality. Instead, our aim is to demonstrate certain phenomena in a suitable context where they manifest themselves in a fully nontrivial way. If one is interested in generalizations, one of the first ideas would be to replace ind-schemes with ind-stacks. Another and perhaps even more important direction for possible generalization is to replace an affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ with a quasi-compact, semi-separated one. We attempt to make the first step in this direction in the appendix, where a definition of the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ for a nonaffine morphism π is worked out.

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1. Ind-Schemes and their Morphisms

1.1. **Ind-objects.** Let K be a small category. Consider the category $\mathsf{Sets}^{\mathsf{K}^{\mathsf{op}}}$ of contravariant functors from K to the category of sets. The category $\mathsf{Ind}(\mathsf{K})$ of *ind-objects in* K can be defined as the full subcategory in $\mathsf{Sets}^{\mathsf{K}^{\mathsf{op}}}$ consisting of the (filtered) direct limits of representable functors $\mathrm{Mor}_{\mathsf{K}}(-,K)$, $K\in\mathsf{K}$.

Explicitly, this means that the objects of $\mathsf{Ind}(\mathsf{K})$ are represented by inductive systems $(K_\gamma\in\mathsf{K})_{\gamma\in\Gamma}$ indexed by directed posets Γ. For any γ', $\gamma''\in\Gamma$ there exists $\gamma\in\Gamma$ such that $\gamma'\le\gamma$, $\gamma''\le\gamma$; and for every $\beta<\gamma\in\Gamma$ the transition morphism $K_\beta\longrightarrow K_\gamma$ in K is given in such a way that the triangle diagrams $K_\alpha\longrightarrow K_\beta\longrightarrow K_\gamma$ are commutative for all $\alpha<\beta<\gamma\in\Gamma$. The object of $\mathsf{Ind}(\mathsf{K})$ represented by an inductive system $(K_\gamma)_{\gamma\in\Gamma}$ is denoted by "$\varinjlim$"$_{\gamma\in\Gamma}K_\gamma\in\mathsf{Ind}(\mathsf{K})$. The set of morphisms in $\mathsf{Ind}(\mathsf{K})$ between the two objects represented by two inductive systems $(K_\gamma)_{\gamma\in\Gamma}$ and $(L_\delta)_{\delta\in\Delta}$ is computed by the formula

$$(1)\qquad \mathrm{Mor}_{\mathsf{Ind}(\mathsf{K})}(\text{“}\varinjlim\text{”}_{\gamma\in\Gamma}K_\gamma,\ \text{“}\varinjlim\text{”}_{\delta\in\Delta}L_\delta)=\varprojlim\nolimits^{\mathsf{Sets}}_{\gamma\in\Gamma}\varinjlim\nolimits^{\mathsf{Sets}}_{\delta\in\Delta}\mathrm{Mor}_{\mathsf{K}}(K_\gamma,L_\delta),$$

where the inductive and projective limits in the right-hand side are taken in the category of sets.

One can (and we will) consider K as a full subcategory in $\mathsf{Ind}(\mathsf{K})$, embedded by the functor assigning to an object $M\in\mathsf{K}$ the related inductive system $(M_\epsilon)_{\epsilon\in\mathrm{E}}$ indexed by the singleton poset $\mathrm{E}=\{*\}$ with $M_*=M$. Then the formula (1) essentially means that, firstly, "$\varinjlim$"$_{\gamma\in\Gamma}K_\gamma=\varinjlim^{\mathsf{Ind}(\mathsf{K})}_{\gamma\in\Gamma}K_\gamma$, and secondly, $\mathrm{Mor}_{\mathsf{Ind}(\mathsf{K})}(K,\text{“}\varinjlim\text{”}_{\delta\in\Delta}L_\delta)=\varinjlim^{\mathsf{Sets}}_{\delta\in\Delta}\mathrm{Mor}_{\mathsf{K}}(K,L_\delta)$ for all $K\in\mathsf{K}$ and "$\varinjlim$"$_{\gamma\in\Gamma}K_\gamma$, "$\varinjlim$"$_{\delta\in\Delta}L_\delta\in\mathsf{Ind}(\mathsf{K})$.

One can drop the assumption that the category K be small and, for any category K, define the category of ind-objects $\mathsf{Ind}(\mathsf{K})$ by saying that the objects of $\mathsf{Ind}(\mathsf{K})$ correspond to directed inductive systems in K and the morphisms are given by the formula (1).

Let K be a category with fibered products (i. e., every pair of morphisms with the same target $K\longrightarrow M$ and $L\longrightarrow M$ has a pullback in K). Then the category $\mathsf{Ind}(\mathsf{K})$ also has fibered products, which can be constructed as follows.

Let $f\colon$ "$\varinjlim$"$_{\gamma\in\Gamma} K_\gamma \longrightarrow$ "$\varinjlim$"$_{\epsilon\in\mathrm{E}} M_\epsilon$ and $g\colon$ "$\varinjlim$"$_{\delta\in\Delta} L_\delta \longrightarrow$ "$\varinjlim$"$_{\epsilon\in\mathrm{E}} M_\epsilon$ be a pair of morphisms with the same target in $\mathsf{Ind}(\mathsf{K})$. Denote by Ξ the set of all quintuples $(\gamma',\delta',\epsilon',f_{\gamma'\epsilon'},g_{\delta'\epsilon'})$ such that $\gamma'\in\Gamma$, $\delta'\in\Delta$, $\epsilon'\in\mathrm{E}$, while $f_{\gamma'\epsilon'}\colon K_{\gamma'}\longrightarrow M_{\epsilon'}$ and $g_{\delta'\epsilon'}\colon L_{\delta'}\longrightarrow M_{\epsilon'}$ are morphisms in K forming two commutative square diagrams with the morphisms f, g and the natural morphisms $K_{\gamma'}\longrightarrow$ "$\varinjlim$"$_{\gamma\in\Gamma} K_\gamma$, $L_{\delta'}\longrightarrow$ "$\varinjlim$"$_{\delta\in\Delta} L_\delta$, $M_{\epsilon'}\longrightarrow$ "$\varinjlim$"$_{\epsilon\in\mathrm{E}} M_\epsilon$ in $\mathsf{Ind}(\mathsf{K})$.

The set Ξ is directed in the natural partial order, and the projection maps $\Xi\longrightarrow\Gamma$, $\Xi\longrightarrow\Delta$, $\Xi\longrightarrow\mathrm{E}$ are cofinal maps of directed posets. Put $K_\xi=K_{\gamma'}$, $L_\xi=L_{\delta'}$, $M_\xi=M_{\epsilon'}$ for $\xi=(\gamma',\delta',\epsilon',f_{\gamma'\epsilon'},g_{\delta'\epsilon'})\in\Xi$. Then our pair of morphisms with the same target (f,g) in $\mathsf{Ind}(\mathsf{K})$ can be obtained by applying the functor "$\varinjlim$"$_{\xi\in\Xi}$ to the Ξ-indexed inductive system of pairs of morphisms with the same target $f'_\xi\colon K_\xi\longrightarrow M_\xi$ and $g'_\xi\colon L_\xi\longrightarrow M_\xi$ in K (where $f'_\xi=f_{\gamma'\epsilon'}$ and $g'_\xi=g_{\delta'\epsilon'}$).

Finally, let $N_\xi\in\mathsf{K}$ denote the fibered product of the pair of morphisms f'_ξ and g'_ξ. Then "$\varinjlim$"$_{\xi\in\Xi} N_\xi\in\mathsf{Ind}(\mathsf{K})$ is the fibered product of the pair of morphisms f and g. In other words, for any directed poset Ξ, the functor "$\varinjlim$"$_{\xi\in\Xi}\colon \mathsf{K}^\Xi\longrightarrow\mathsf{Ind}(\mathsf{K})$ (acting from the category K^Ξ of Ξ-indexed inductive systems in K to the category of ind-objects) preserves fibered products, as one can easily see.

In fact, stronger assertions hold. First of all, for any category K, the category $\mathsf{Ind}(\mathsf{K})$ can be equivalently defined using inductive systems indexed by filtered categories rather than filtered posets [1, Theorem 1.5]. Furthermore, the full subcategory $\mathsf{Ind}(\mathsf{K})$ is closed under (filtered) direct limits in $\mathsf{Sets}^{\mathsf{K}^{\mathsf{op}}}$ [21, Theorem 6.1.8] (so direct limits exist in $\mathsf{Ind}(\mathsf{K})$). Returning to the particular case of a category K with fibered products, one observes that $\mathsf{Ind}(\mathsf{K})$ is closed under fibered products in $\mathsf{Sets}^{\mathsf{K}^{\mathsf{op}}}$ in this case. Since finite limits commute with (filtered) direct limits in $\mathsf{Sets}^{\mathsf{K}^{\mathsf{op}}}$, it follows that fibered products commute with direct limits in $\mathsf{Ind}(\mathsf{K})$.

1.2. **Ind-schemes.** Ind-schemes are the main object of study in this paper. Contrary to what the name seems to suggest, in the advanced contemporary point of view ind-schemes are *not* defined as ind-objects in the category of schemes (or in any other category). Rather, one considers the category of affine schemes AffSch (so $\mathsf{AffSch}\simeq\mathsf{CRings}^{\mathsf{op}}$, where CRings is the category of commutative rings). Then the ind-schemes are those contravariant functors $\mathsf{AffSch}^{\mathsf{op}}\longrightarrow\mathsf{Sets}$ ("presheaves of sets") on the category of affine schemes which can be obtained as direct limits of functors representable by schemes [7, Section 7.11], [58, Section 1]. So the category of ind-schemes is a certain full subcategory in the category of presheaves of sets on AffSch.

Let Sch denote the category of schemes, and let $\mathsf{CSch}\subset\mathsf{Sch}$ be the full subcategory of concentrated (that is, quasi-compact quasi-separated) schemes. Then, for any $X\in\mathsf{Sch}$, the functor $\mathrm{Mor}_{\mathsf{Sch}}(-,X)$ is actually a *sheaf* of sets in the Zariski topology on Sch. In particular, the restrictions of $\mathrm{Mor}_{\mathsf{Sch}}(-,X)$ to CSch and AffSch are sheaves in the Zariski topology on the categories of concentrated schemes and affine schemes, respectively. Furthermore, the filtered direct limit functors in the categories of presheaves and Zariski sheaves of sets on CSch agree, due to the finite nature of the Zariski topology on a concentrated scheme. The same applies to the prescheaves and Zariski sheaves on AffSch. For these reasons, one can consider ind-schemes as Zariski sheaves of sets on AffSch, which are the same thing as Zariski sheaves of sets on CSch. These arguments explain that ind-schemes can be equivalently viewed as functors $\mathsf{CSch}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ instead of $\mathsf{AffSch}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$.

In view of the previous paragraph, one can develop the following simplified, naïve approach to the definition of *ind-concentrated ind-schemes*, which we will use. An (*ind-concentrated*) *ind-scheme* $\mathfrak{X}$ is defined as an ind-object in the category of concentrated schemes CSch. In the rest of this paper, all "schemes" will be presumed concentrated, and all "ind-schemes" will be ind-concentrated. An ind-scheme is said to be *strict* if it can be represented by an inductive system of closed immersions, i. e., $\mathfrak{X} = \text{"}\varinjlim\text{"}_{\gamma\in\Gamma} X_\gamma$, where the morphism of schemes $X_\gamma \longrightarrow X_\delta$ in the inductive system is a closed immersion for every $\gamma < \delta \in \Gamma$.

An $\aleph_0$-*ind-scheme* $\mathfrak{X}$ is an ind-scheme which can be represented by a countable inductive system of schemes, i. e., $\mathfrak{X} = \text{"}\varinjlim\text{"}_{\gamma\in\Gamma} X_\gamma$, where the poset Γ is has (at most) countable cartinality. It is not difficult to see that any strict $\aleph_0$-ind-scheme $\mathfrak{X}$ can be represented by a sequence of closed immersions indexed by the natural numbers, $\mathfrak{X} = \text{"}\varinjlim\text{"}(X_0 \to X_1 \to X_2 \to \cdots)$, where the transition morphisms $X_n \longrightarrow X_{n+1}$ are closed immersions of schemes. The fact that closed immersions are monomorphisms in Sch is helpful to keep in mind here (and elsewhere below).

Following the discussion in Section 1.1, fibered products exist in the category of (ind-concentrated) ind-schemes (because they exist in CSch, as the full subcategory CSch is closed under fibered products in Sch). Furthermore, the full subcategory of strict ind-schemes is closed under fibered products in the category of ind-schemes. Moreover, looking into the above construction of the fibered product in the case of three ind-objects $\text{"}\varinjlim\text{"}_{\gamma\in\Gamma} K_\gamma$, $\text{"}\varinjlim\text{"}_{\delta\in\Delta} L_\delta$, and $\text{"}\varinjlim\text{"}_{\epsilon\in\mathrm{E}} M_\epsilon$ represented by inductive systems of monomorphisms, one can see that the morphisms $f_{\gamma'\epsilon'}$ and $g_{\delta'\epsilon'}$ are determined by the indices γ', δ', and ϵ'; so the set Ξ is at most countable whenever the sets Γ, Δ, and E are. Hence the full subcategory of strict $\aleph_0$-ind-schemes is closed under fibered products in the category of strict ind-schemes.

It is also useful to observe that, considering CSch as a full subcategory in strict ind-schemes, the fibered product of any two schemes over a strict ind-scheme is a scheme. Indeed, given a strict ind-scheme $\mathfrak{X} = \text{"}\varinjlim\text{"}_{\gamma\in\Gamma} X_\gamma$ with closed immersions $X_\gamma \longrightarrow X_\delta$ for $\gamma < \delta \in \Gamma$, for any two schemes Y and Z and morphisms $Y \longrightarrow \mathfrak{X} \longleftarrow Z$, the fibered product $Y \times_{\mathfrak{X}} Z$ is isomorphic to $Y \times_{X_\gamma} Z$, where $\gamma \in \Gamma$ is any index for which both the morphisms $Y \longrightarrow \mathfrak{X}$ and $Z \longrightarrow \mathfrak{X}$ factorize through $X_\gamma \longrightarrow \mathfrak{X}$.

Remark 1.1. The definition of a strict ind-scheme raises an obvious question. Let $\mathfrak{X} \in \mathsf{Ind}(\mathsf{CSch})$ be an ind-scheme which can be represented by an inductive system of closed immersions in CSch. In what sense is such a representation unique if it exists? Suppose that $\mathfrak{X} \simeq \text{"}\varinjlim\text{"}_{\gamma\in\Gamma} Y_\gamma \simeq \text{"}\varinjlim\text{"}_{\delta\in\Delta} Z_\delta$, where $(Y_\gamma)_{\gamma\in\Gamma}$ and $(Z_\delta)_{\delta\in\Delta}$ are inductive systems of closed immersions of schemes. In what sense are these two "strict" representations of $\mathfrak{X}$ equivalent?

A more advanced approach is to consider the poset $\mathrm{E} = \Gamma \times \Delta$ with the product order, and write $\mathfrak{X} \simeq$ "$\varinjlim$"$_{(\gamma,\delta)\in\mathrm{E}} W_{\gamma,\delta}$, where $W_{\gamma,\delta} = Y_\gamma \times_{\mathfrak{X}} Z_\delta$. Still one may be interested in having a straightforward answer to the following straightforward question. Suppose that, for some $\gamma \in \Gamma$ and $\delta \in \Delta$, the morphism $Y_\gamma \longrightarrow \mathfrak{X}$ factorizes as $Y_\gamma \longrightarrow Z_\delta \longrightarrow \mathfrak{X}$. How do we know that $Y_\gamma \longrightarrow Z_\delta$ is a closed immersion?

The next two lemmas (in which the schemes do not need to be concentrated) provide an explanation in a natural generality.

Lemma 1.2. (a) *Let $X \longrightarrow Y \longrightarrow Z$ be morphisms of schemes such that the composition $X \longrightarrow Z$ is a locally closed immersion. Then the morphism $X \longrightarrow Y$ is a locally closed immersion.*

(b) *Let $X \longrightarrow Y \longrightarrow Z$ be morphisms of schemes such that the composition $X \longrightarrow Y \longrightarrow Z$ is a closed immersion and the morphism $Y \longrightarrow Z$ is separated. Then $X \longrightarrow Y$ is a closed immersion.*

(c) *In particular, if the composition $X \longrightarrow Y \longrightarrow Z$ is a closed immersion and Y is a separated scheme, then the morphism $X \longrightarrow Y$ is a closed immersion.*

Proof. Parts (a) and (b) are [20, Tag 07RK]. The morphism $X \longrightarrow Y$ is the composition $X = X \times_Y Y \longrightarrow X \times_Z Y \longrightarrow Y$, where the morphism $X \times_Z Y \longrightarrow Y$ is a base change of the morphism $X \longrightarrow Z$ (by the morphism $Y \longrightarrow Z$). If $X \longrightarrow Z$ is a locally closed immersion, then $X \times_Z Y \longrightarrow Y$ is a locally closed immersion; and if $X \longrightarrow Z$ is a closed immersion, then $X \times_Z Y \longrightarrow Y$ is a closed immersion [20, Tag 01JU]. The morphism $X \times_Y Y \longrightarrow X \times_Z Y$ is a locally closed immersion; and moreover if the morphism $Y \longrightarrow Z$ is separated, then $X \times_Y Y \longrightarrow X \times_Z Y$ is a closed immersion, by [20, Tag 01KR]. Finally, a composition of locally closed immersions is a locally closed immersion, and a composion of closed immersions is a closed immersion [20, Tag 02V0]. Part (c) follows from (b), since any scheme morphism from a separated scheme is separated [20, Tag 01KV]. ☐

Lemma 1.3. *Let $X \longrightarrow Y \longrightarrow Z \longrightarrow W$ be morphisms of schemes such that the compositions $X \longrightarrow Z$ and $Y \longrightarrow W$ are closed immersions. Then the morphism $X \longrightarrow Y$ is a closed immersion.*

Proof. By Lemma 1.2(a) applied to the pair of morphisms $Y \longrightarrow Z \longrightarrow W$, the morphism $Y \longrightarrow Z$ is a locally closed immersion. Any locally closed immersion is a separated morphism; so Lemma 1.2(b) is applicable to the pair of morphisms $X \longrightarrow Y \longrightarrow Z$, proving that $X \longrightarrow Y$ is a closed immersion. ☐

Returning to the question posed in Remark 1.1, one can choose an index $\epsilon \in \Gamma$ such that the morphism $Z_\delta \longrightarrow \mathfrak{X}$ factorizes as $Z_\delta \longrightarrow Y_\epsilon \longrightarrow \mathfrak{X}$, and then one can choose an index $\eta \in \Delta$ such that the morphism $Y_\epsilon \longrightarrow \mathfrak{X}$ factorizes as $Y_\epsilon \longrightarrow Z_\eta \longrightarrow \mathfrak{X}$. The triangle diagrams $Y_\gamma \longrightarrow Z_\delta \longrightarrow Y_\epsilon$ and $Z_\delta \longrightarrow Y_\epsilon \longrightarrow Z_\eta$ are commutative, because the transition morphisms in both the inductive systems are monomorphisms. By Lemma 1.3, it follows that $Y_\gamma \longrightarrow Z_\delta$ is a closed immersion.

1.3. **Morphisms of ind-schemes.** *In the sequel, we will presume all our ind-schemes to be (ind-concentrated and) strict.* Let us start from the observation that any morphism of ind-schemes can be obtained by applying the functor "$\varinjlim$" to some morphism of inductive systems of (closed immersions of) schemes.

Indeed, let $f\colon$ "$\varinjlim$"$_{\delta\in\Delta} Y_\delta = \mathfrak{Y} \longrightarrow \mathfrak{X} =$ "$\varinjlim$"$_{\gamma\in\Gamma} X_\gamma$ be a morphism of ind-schemes represented by inductive systems of closed immersions $(Y_\delta)_{\delta\in\Delta}$ and $(X_\gamma)_{\gamma\in\Gamma}$. Arguing as in Section 1.1 and keeping in mind that closed immersions are monomorphisms in CSch, one can consider the directed poset Ξ of all pairs $(\delta\in\Delta,\ \gamma\in\Gamma)$ such that the composition $Y_\delta \longrightarrow \mathfrak{Y} \longrightarrow \mathfrak{X}$ factorizes through the morphism $X_\gamma \longrightarrow \mathfrak{X}$. Then one has $f =$ "$\varinjlim$"$_{\xi\in\Xi} f_\xi$, where $f_\xi\colon Y_\xi \longrightarrow X_\xi$ is the related morphism between $Y_\xi = Y_\delta$ and $X_\xi = X_\gamma$.

Alternatively, consider the product of two posets $\mathrm{E} = \Delta\times\Gamma$ with the product order, and for every $\epsilon = (\delta,\gamma)\in\mathrm{E}$ put $Y_\epsilon = Y_\delta \times_{\mathfrak{X}} X_\gamma$ and $X_\epsilon = X_\gamma$. Then $f =$ "$\varinjlim$"$_{\epsilon\in\mathrm{E}} f_\epsilon$, where $f_\epsilon\colon Y_\epsilon \longrightarrow X_\epsilon$.

The following representation of morphisms of schemes may be even more useful. Following [58, Lemma 1.7 and Definition 1.8], one says that a morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is "representable by schemes" if, for every scheme T and a morphism of ind-schemes $T \longrightarrow \mathfrak{X}$, the fibered product $\mathfrak{Y}\times_{\mathfrak{X}} T$ is a scheme. In this case, let $\mathfrak{X} =$ "$\varinjlim$"$_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes. Put $Y_\gamma = \mathfrak{Y}\times_{\mathfrak{X}} X_\gamma$. Then $\mathfrak{Y} =$ "$\varinjlim$"$_{\gamma\in\Gamma} Y_\gamma$ is a representation of $\mathfrak{Y}$ by an inductive system of closed immersions of schemes, and the morphism f is represented as $f =$ "$\varinjlim$"$_{\gamma\in\Gamma} f_\gamma$, where $f_\gamma\colon Y_\gamma \longrightarrow X_\gamma$.

A morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is said to be *affine* if, for any scheme T and a morphism of ind-schemes $T \longrightarrow \mathfrak{X}$, the ind-scheme $\mathfrak{Y}\times_{\mathfrak{X}} T$ is a scheme *and* the morphism of schemes $\mathfrak{Y}\times_{\mathfrak{X}} T \longrightarrow T$ is affine. Equivalently (in view of [58, Lemma 1.7]), this means that $\mathfrak{Y}\times_{\mathfrak{X}} T$ is an affine scheme whenever T is an affine scheme. Let $\mathfrak{X} =$ "$\varinjlim$"$_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes. Then a morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is affine if and only if, for every $\gamma\in\Gamma$, the fibered product $\mathfrak{Y}\times_{\mathfrak{X}} X_\gamma$ is a scheme and the morphism of schemes $\mathfrak{Y}\times_{\mathfrak{X}} X_\gamma \longrightarrow X_\gamma$ is affine.

A morphism of ind-schemes $\mathfrak{Z} \longrightarrow \mathfrak{X}$ is said to be a *closed immersion* if, for every scheme T and a morphism of ind-schemes $T \longrightarrow \mathfrak{X}$, the ind-scheme $\mathfrak{Z}\times_{\mathfrak{X}} T$ is a scheme *and* the morphism $\mathfrak{Z}\times_{\mathfrak{X}} T \longrightarrow T$ is a closed immersion of schemes. Obviously, any closed immersion of ind-schemes is an affine morphism.

In particular, given a scheme Z and an ind-scheme $\mathfrak{X}$, a morphism of ind-schemes $Z \longrightarrow \mathfrak{X}$ is said to be a closed immersion if, for every scheme T and a morphism of ind-schemes $T \longrightarrow \mathfrak{X}$, the morphism of schemes $Z\times_{\mathfrak{X}} T \longrightarrow T$ is a closed immersion. In this case, one says that the morphism $Z \longrightarrow \mathfrak{X}$ makes Z a *closed subscheme* in $\mathfrak{X}$.

Let $\mathfrak{X} =$ "$\varinjlim$"$_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes $X_\gamma \longrightarrow X_\delta$, $\gamma < \delta \in \Gamma$. Let $i\colon Z \longrightarrow \mathfrak{X}$ be a morphism into $\mathfrak{X}$ from a scheme Z, and let $\gamma\in\Gamma$ be an index such that i factorizes as $Z \longrightarrow X_\gamma \longrightarrow \mathfrak{X}$. Then the morphism i is a closed immersion if and only if the morphism $Z \longrightarrow X_\gamma$ is a closed immersion of schemes. This observation provides another way to answer the question from Remark 1.1.

A morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is said to be *flat* if, for any scheme T and a morphism of ind-schemes $T \longrightarrow \mathfrak{X}$, the fibered product $\mathfrak{Y} \times_{\mathfrak{X}} T$ is a scheme *and* the morphism of schemes $\mathfrak{Y} \times_{\mathfrak{X}} T \longrightarrow T$ is flat. Let $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes. Then a morphism $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is flat if and only if, for every $\gamma \in \Gamma$, the fibered product $\mathfrak{Y} \times_{\mathfrak{X}} X_\gamma$ is a scheme and the morphism of schemes $\mathfrak{Y} \times_{\mathfrak{X}} X_\gamma \longrightarrow X_\gamma$ is flat.

1.4. **Ind-affine examples.** An ind-scheme is said to be *ind-affine* if it can be represented by an inductive system of affine schemes. It follows from Lemma 1.2(c) (with an affine scheme Y) that any closed subscheme of a strict ind-affine scheme is affine. Thus any (strict) ind-affine ind-scheme can be represented by an inductive system of closed immersions of affine schemes.

Examples 1.4. (1) Consider the directed poset of all positive integers $\mathbb{Z}_{>0}$ in the divisibility order. To every $n \in \mathbb{Z}_{>0}$, assign the affine scheme $X_n = \operatorname{Spec} \mathbb{Z}/n\mathbb{Z}$. Whenever m divides n, there is a unique (surjective) ring homomorphism $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$, and accordingly a unique morphism (closed immersion) of affine schemes $X_m \longrightarrow X_n$. The inductive system of schemes $(X_n)_{n\in\mathbb{Z}_{>0}}$ represents a strict ind-affine $\aleph_0$-ind-scheme $\mathfrak{X}$, which we will denote by $\mathfrak{X} = \operatorname{Spi} \widehat{\mathbb{Z}}$. Here $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the profinite completion of the ring of integers, or equivalently, the product of the rings of p-adic integers $\mathbb{Z}_p$ taken over all the prime numbers p.

(2) Choose a prime number p, and consider the directed poset of all nonnegative integers $\mathbb{Z}_{\ge0}$ in the usual linear order. To every $r \in \mathbb{Z}_{\ge0}$, assign the affine scheme $X_r = \operatorname{Spec} \mathbb{Z}/p^r\mathbb{Z}$. Whenever $r \le s$, there is a unique (surjective) ring homomorphism $\mathbb{Z}/p^s\mathbb{Z} \longrightarrow \mathbb{Z}/p^r\mathbb{Z}$, and accordingly a unique morphism (closed immersion) of affine schemes $X_r \longrightarrow X_s$. The inductive system of schemes $(X_r)_{r\in\mathbb{Z}_{\ge0}}$ represents a strict ind-affine $\aleph_0$-ind-scheme $\mathfrak{X}$, which we will denote by $\mathfrak{X} = \operatorname{Spi} \mathbb{Z}_p$.

The ind-scheme $\operatorname{Spi} \widehat{\mathbb{Z}}$ is the coproduct (“disjoint union”) of the ind-schemes $\operatorname{Spi} \mathbb{Z}_p$ in the category of ind-schemes, taken over all the prime numbers p.

Examples 1.5. (1) Pick a field $\Bbbk$, and consider the directed poset of all nonnegative integers $\mathbb{Z}_{\ge0}$ in the usual linear order. To every $r \in \mathbb{Z}_{\ge0}$, assign the affine scheme $X_r = \operatorname{Spec} \Bbbk[x]/x^r\Bbbk[x]$. Whenever $r \le s$, there is a unique (surjective) homomorphism of $\Bbbk$-algebras $\Bbbk[x]/x^s\Bbbk[x] \longrightarrow \Bbbk[x]/x^r\Bbbk[x]$ taking the coset $x + x^s\Bbbk[x]$ to the coset $x + x^r\Bbbk[x]$. Let $X_r \longrightarrow X_s$ be the related closed immersion of affine schemes. The inductive system of schemes $(X_r)_{r\in\mathbb{Z}_{\ge0}}$ represents a strict ind-affine $\aleph_0$-ind-scheme $\mathfrak{X}$, which we will denote by $\mathfrak{X} = \operatorname{Spi} \Bbbk[[x]]$. This ind-scheme comes endowed with a morphism of ind-schemes $\operatorname{Spi} \Bbbk[[x]] \longrightarrow \operatorname{Spec} \Bbbk$.

Example 1.5(1) is a close analogue of Example 1.4(2).

(2) Let $\mathscr{C}$ be a coassociative, cocommutative, counital coalgebra over a field $\Bbbk$. Any coassociative coalgebra over a field is the union of its finite-dimensional subcoalgebras (which form a directed poset by inclusion). All the subcoalgebras of $\mathscr{C}$ are also coassociative, cocommutative, and counital. Let Γ denote the poset of all finite-dimensional subcoalgebras of $\mathscr{C}$ in the inclusion order. For a finite-dimensional

subcoalgebra $\mathcal{E} \subset \mathscr{C}$ (so $\mathcal{E} \in \Gamma$), the dual vector space $\mathcal{E}^*$ is an associative, commutative, and unital finite-dimensional $\Bbbk$-algebra.

For every pair of finite-dimensional subcoalgebras $\mathcal{E}'$, $\mathcal{E}'' \subset \mathscr{C}$ such that $\mathcal{E}' \subset \mathcal{E}''$, the dual map $\mathcal{E}''^* \longrightarrow \mathcal{E}'^*$ to the inclusion $\mathcal{E}' \longrightarrow \mathcal{E}''$ is a surjective homomorphism of commutative rings. Consider the related closed immersion of affine schemes $\operatorname{Spec} \mathcal{E}'^* \longrightarrow \operatorname{Spec} \mathcal{E}''^*$. The inductive system of schemes $(\operatorname{Spec} \mathcal{E}^*)_{\mathcal{E}\in\Gamma}$ represents a strict ind-affine ind-scheme $\mathfrak{X}$, which we will denote by $\mathfrak{X} = \operatorname{Spi} \mathscr{C}^*$. This ind-scheme comes endowed wih a morphism of ind-schemes $\operatorname{Spi} \mathscr{C}^* \longrightarrow \operatorname{Spec} \Bbbk$.

Here the notation $\mathscr{C}^*$ stands for the associative, commutative, and unital topological $\Bbbk$-algebra $\mathscr{C}^* = \varprojlim_{\mathcal{E}\in\Gamma} \mathcal{E}^*$, with the topology of projective limit of discrete finite-dimensional vector spaces/algebras $\mathcal{E}^*$. Example 1.5(1) is the particular case of Example 1.5(2) corresponding to the choice of the coalgebra $\mathscr{C}$ such that $\mathscr{C}^* = \Bbbk[[x]]$ (with the x-adic topology on $\Bbbk[[x]]$). The coalgebra $\mathscr{C}$ is a $\Bbbk$-vector space with the basis $\{1^*, x^*, x^{2*}, \dots, x^{n*}, \dots\}$, $n \in \mathbb{Z}_{\ge 0}$, with the counit map $\mathscr{C} \ni 1^* \longmapsto 1 \in \Bbbk$, $x^{n*} \longmapsto 0$ for $n > 0$, and the comultiplication map $\mathscr{C} \ni x^{n*} \longmapsto \sum_{p+q=n} x^{p*} \otimes x^{q*} \in \mathscr{C} \otimes_\Bbbk \mathscr{C}$.

Examples 1.6. This example is taken from [7, Example 7.11.2(i)].

(1) Let $\mathfrak{R}$ be an associative, commutative, unital ring endowed with a complete, separated topology with a base of neighborhoods of zero consisting of open ideals. For every open ideal $\mathfrak{I} \subset \mathfrak{R}$, consider the quotient ring $\mathfrak{R}/\mathfrak{I}$ and the affine scheme $\operatorname{Spec} \mathfrak{R}/\mathfrak{I}$. Whenever $\mathfrak{I}$, $\mathfrak{J} \subset \mathfrak{R}$ are open ideals such that $\mathfrak{J} \subset \mathfrak{I}$, there exists a unique (surjective) morphism of commutative rings $\mathfrak{R}/\mathfrak{J} \longrightarrow \mathfrak{R}/\mathfrak{I}$ making the triangle diagram $\mathfrak{R} \longrightarrow \mathfrak{R}/\mathfrak{J} \longrightarrow \mathfrak{R}/\mathfrak{I}$ commutative. Let $\operatorname{Spec} \mathfrak{R}/\mathfrak{I} \longrightarrow \operatorname{Spec} \mathfrak{R}/\mathfrak{J}$ be the related closed immersion of affine schemes. The inductive system of schemes $(\operatorname{Spec} \mathfrak{R}/\mathfrak{I})_{\mathfrak{I}\subset\mathfrak{R}}$ indexed by the directed poset of open ideals in $\mathfrak{R}$ with the reverse inclusion order represents a strict ind-affine ind-scheme $\mathfrak{X}$, which we will denote by $\operatorname{Spi} \mathfrak{R}$. For any base of neighborhoods of zero B consisting of open ideals in $\mathfrak{R}$, one has $\operatorname{Spi} \mathfrak{R} = \text{“}\varinjlim\text{”}_{\mathfrak{I}\in B} \operatorname{Spec} \mathfrak{R}/\mathfrak{I}$ (where B is viewed as a directed poset in the reverse inclusion order).

(2) In particular, assume in the context of (1) that $\mathfrak{R}$ has a *countable* base of neighborhoods of zero. Then one can choose a countable base of neighborhoods of zero B consisting of open ideals in $\mathfrak{R}$, hence $\operatorname{Spi} \mathfrak{R} = \text{“}\varinjlim\text{”}_{\mathfrak{I}\in B} \operatorname{Spi} \mathfrak{R}/\mathfrak{I}$ is an ind-affine $\aleph_0$-ind-scheme. The functor $\mathfrak{R} \longmapsto \operatorname{Spi} \mathfrak{R}$ establishes an anti-equivalence between the category of topological commutative rings with a countable neighborhood of zero consisting of open ideals and the category of ind-affine $\aleph_0$-ind-schemes.

In the more general context of (1), the assignment $\mathfrak{R} \longmapsto \operatorname{Spi} \mathfrak{R}$ is a *fully faithful* contravariant functor from the category of topological commutative rings where open ideals form a base of neighborhoods of zero to the category of ind-affine ind-schemes. Its essential image consists of all the ind-affine ind-schemes $\text{“}\varinjlim\text{”}_{\gamma\in\Gamma} \operatorname{Spec} R_\gamma$ for which the projection map $\varprojlim_{\gamma\in\Gamma} R_\gamma \longrightarrow R_\delta$ is surjective for all $\delta \in \Gamma$.

Examples 1.4 and 1.5(1) are particular cases of Example 1.6(2). In particular, the ind-affine ind-scheme $\operatorname{Spi} \widehat{\mathbb{Z}}$ in Example 1.4(1) corresponds to the topological ring $\mathfrak{R} = \widehat{\mathbb{Z}}$ with its profinite topology, while the ind-affine ind-scheme $\operatorname{Spi} \mathbb{Z}_p$ in

Example 1.4(2) corresponds to the topological ring $\mathfrak{R} = \widehat{\mathbb{Z}}_p$ with its p-adic topology. The ind-affine ind-scheme $\operatorname{Spi} \Bbbk[[x]]$ in Example 1.5(1) corresponds to the topological ring or $\Bbbk$-algebra $\mathfrak{R} = \Bbbk[[x]]$ with its x-adic topology.

Example 1.5(2) is a particular case of Example 1.6(1) corresponding to the topological ring $\mathfrak{R} = \mathscr{C}^*$ with its profinite-dimensional (linearly compact) topology. The ind-affine ind-scheme $\operatorname{Spec} \mathscr{C}^*$ in Example 1.5(2) corresponds to a topological ring with a countable base of neighborhoods of zero (i. e., $\mathscr{C}^*$ has a countable base of neighborhoods of zero) if and only if $\operatorname{Spi} \mathscr{C}^*$ is an $\aleph_0$-ind-scheme, and if and only if the underlying vector space of the coalgebra $\mathscr{C}$ has at most countable dimension over $\Bbbk$.

For a further discussion of specific examples of ind-affine ind-schemes, see Section 11.1 below.

2. Quasi-Coherent Torsion Sheaves

In this section and below, as mentioned in Sections 1.2–1.3, all the *schemes* are concentrated, and all the *ind-schemes* are ind-concentrated and strict.

2.1. **Reasonable ind-schemes.** Here we largely follow [7, Section 7.11.1].

Let X be a concentrated scheme with the structure sheaf $\mathcal{O}_X$. Notice that any closed subscheme of a concentrated scheme is concentrated. Let $Z \subset X$ be a closed subscheme, and let $\mathcal{I}_{Z,X} \subset \mathcal{O}_X$ denote the quasi-coherent sheaf of ideals in $\mathcal{O}_X$ corresponding to Z. So $\mathcal{I}_{Z,X}$ is the kernel of the natural surjective morphism of quasi-coherent sheaves $\mathcal{O}_X \longrightarrow k_*\mathcal{O}_Z$, where $k\colon Z \longrightarrow X$ is the closed immersion. The closed subscheme $Z \subset X$ is said to be *reasonable* if $\mathcal{I}_{Z,X}$ is generated (as a quasi-coherent sheaf on X) by a finite set of local sections.

Let $Y \subset X$ be a closed subscheme such that $Z \subset Y \subset X$. Then Z is also a closed subscheme in Y. Denote the closed immersion morphism by $i\colon Y \longrightarrow X$. Then the natural surjective morphism $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ of quasi-coherent sheaves on X restricts to a surjective morphism $\mathcal{I}_{Z,X} \longrightarrow i_*\mathcal{I}_{Z,Y}$. It follows that Z is a reasonable closed subscheme in Y whenever Z is a reasonable closed subcheme in X.

Part (a) of the following lemma is more general.

Lemma 2.1. (a) *Let $Y \longrightarrow X$ be a morphism of schemes and $Z \subset X$ be a reasonable closed subscheme. Then $Z \times_X Y$ is a reasonable closed subscheme in Y.*

(b) *Let Y and Z be closed subschemes in a scheme X such that $Z \subset Y \subset X$. Assume that Z is a reasonable closed subscheme in Y, and Y is a reasonable closed subscheme in X. Then Z is a reasonable closed subscheme in X.*

Proof. Both the assertions are essentially local (as the schemes are presumed to be concentrated) and reduce to the affine case. Part (a): let $R \longrightarrow S$ be a homomorphism of commutative rings and $I \subset R$ be a finitely generated ideal. Then the claim is that the kernel of the surjective map $S \longrightarrow R/I \otimes_R S$ is a finitely generated ideal in S. Part (b): let $R \longrightarrow S \longrightarrow T$ be surjective homomorphisms of commutative rings such that the kernel of $R \longrightarrow S$ is a finitely generated ideal in R and the kernel of $S \longrightarrow T$ is a finitely generated ideal in T. Then the claim is that the kernel of $R \longrightarrow T$ is a finitely generated ideal in R. $\square$

Let $\mathfrak{X} = $ "$\varinjlim$"$_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes. A closed subscheme $Z \subset \mathfrak{X}$ is said to be *reasonable* if, for every closed subscheme $Y \subset \mathfrak{X}$ such that $Z \subset Y$, the closed subcheme Z in Y is reasonable. A closed subscheme $Z \subset \mathfrak{X}$ is reasonable if and only if, for every index $\gamma \in \Gamma$ such that $Z \subset X_\gamma$, the closed subcheme Z in X_γ is reasonable.

An ind-scheme $\mathfrak{X}$ is said to be *reasonable* if it is a filtered direct limit of its reasonable closed subschemes. Equivalently, $\mathfrak{X}$ is reasonable if and only if there exists a (filtered) inductive system of closed immersions of schemes $(X_\gamma)_{\gamma\in\Gamma}$ such that, for every $\gamma < \delta \in \Gamma$, the closed subscheme X_γ in X_δ is reasonable. Clearly, in the latter case, X_γ is a reasonable closed subscheme in $\mathfrak{X}$ for every $\gamma \in \Gamma$.

Let $\mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes which is "representable by schemes" in the sense of Section 1.3. Then it follows from Lemma 2.1(a) that, for any reasonable closed subscheme $Z \subset \mathfrak{X}$, the fibered product $Z \times_{\mathfrak{X}} \mathfrak{Y}$ is a reasonable closed subscheme in $\mathfrak{Y}$. Therefore, the ind-scheme $\mathfrak{Y}$ is reasonable if the ind-scheme $\mathfrak{X}$ is.

2.2. **Quasi-coherent sheaves and functors.** Given a scheme X, we denote by X–qcoh the abelian (Grothendieck) category of quasi-coherent sheaves on X. For every morphism of (concentrated) schemes $f\colon Y \longrightarrow X$, we have the direct and inverse image functors $f_*\colon Y$–$\mathsf{qcoh} \longrightarrow X$–$\mathsf{qcoh}$ and $f^*\colon X$–$\mathsf{qcoh} \longrightarrow Y$–$\mathsf{qcoh}$; the functor f^* is left adjoint to f_*. The functors f_* and f^* are the restrictions (to the full subcategories of quasi-coherent sheaves) of the similar functors acting between the ambient abelian categories of sheaves of $\mathcal{O}_X$-modules and sheaves of $\mathcal{O}_Y$-modules.

The category X–qcoh is a tensor subcategory of the (associative, commutative, and unital) tensor category of sheaves of $\mathcal{O}_X$-modules, with respect to the tensor product functor $-\otimes_{\mathcal{O}_X}-$. The structure sheaf $\mathcal{O}_X$ is the unit object. The inverse image $f^*\colon X$–$\mathsf{qcoh} \longrightarrow Y$–$\mathsf{qcoh}$ is a tensor functor. The next lemma is very well-known; it is called the "projection formula".

Lemma 2.2. *Let $f\colon Y \longrightarrow X$ be a morphism of (concentrated) schemes, $\mathcal{M}$ be a quasi-coherent sheaf on X, and $\mathcal{N}$ be a quasi-coherent sheaf on Y. Then there is a natural morphism of quasi-coherent sheaves on X*

$$\mathcal{M} \otimes_{\mathcal{O}_X} f_*\mathcal{N} \longrightarrow f_*(f^*\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}),$$

which is an isomorphism if the morphism f is affine.

Proof. The morphism in question is adjoint to the morphism $f^*(\mathcal{M} \otimes_{\mathcal{O}_X} f_*\mathcal{N}) \simeq f^*\mathcal{M} \otimes_{\mathcal{O}_Y} f^*f_*\mathcal{N} \longrightarrow f^*\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ induced by the adjunction morphism $f^*f_*\mathcal{N} \longrightarrow \mathcal{N}$ in Y–qcoh. The second assertion is local in X, so it reduces to the case of affine schemes, for which it means the following. Let $R \longrightarrow S$ be a commutative ring homomorphism, M be an R-module, and N be an S-module. Then there is a natural isomorphism of R-modules $M \otimes_R N \simeq (S \otimes_R M) \otimes_S N$. $\square$

For any two sheaves of $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, we denote by $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ the sheaf of $\mathcal{O}_X$-modules with the modules of sections $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{M}|_U,\mathcal{N}|_U)$ for all open subschemes $U \subset X$. The sheaf of $\mathcal{O}_X$-modules $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ is quasi-coherent whenever the sheaf $\mathcal{N}$ is quasi-coherent and the sheaf $\mathcal{M}$ is locally (i. e., in restriction to a small enough Zariski neighborhood of every point of X) the cokernel of a morphism between finite direct sums of copies of the structure sheaf $\mathcal{O}$.

Let $Z \subset X$ be a closed subscheme with the closed immersion morphism $i\colon Z \longrightarrow X$. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$, denote by $i^!\mathcal{M}$ the sheaf of $\mathcal{O}_Z$-modules defined by the property that $i_*i^!\mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z,\mathcal{M})$ is the subsheaf of $\mathcal{M}$ consisting of all the local sections annihilated by $\mathcal{I}_{Z,X}$. The sheaf of $\mathcal{O}_Z$-modules $i^!\mathcal{M}$ is quasi-coherent whenever the sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ is *and* the closed subscheme $Z \subset X$ is reasonable. In this case, the functor $i^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ is right adjoint to the direct image functor $i_*\colon Z\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$.

Lemma 2.3. *Let $f\colon Y \longrightarrow X$ be a morphism of (concentrated) schemes and $Z \subset X$ be a reasonable closed subscheme with the closed immersion $i\colon Z \longrightarrow X$. Consider the pullback diagram*

$$\begin{array}{ccc} Z\times_X Y & \xrightarrow{k} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

Then there are natural isomorphisms
(a) $i^!f_* \simeq g_*k^!$ *of functors* $Y\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$;
(b) $f^*i_* \simeq k_*g^*$ *of functors* $Z\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$.

Proof. Parts (a) and (b) are adjoint to each other, so it suffices to check any one of them. Both the assertions (particularly clearly (b)) are essentially local and reduce to the case of affine schemes, for which they mean the following. Let $R \longrightarrow S$ be a commutative ring homomorphism and $I \subset R$ be a finitely generated ideal. Then (a) for any S-module N, there is a natural isomorphism of R/I-modules $\mathrm{Hom}_R(R/I,N) \simeq \mathrm{Hom}_S(S/IS,N)$; and (b) for any R/I-module M, there is a natural isomorphism of S-modules $S\otimes_R M \simeq S/IS\otimes_{R/I} M$.

The assumption of the closed subscheme $Z \subset X$ being reasonable is only needed in part (a). For a further generalization of part (b), see Lemma 3.3(a) below. □

More generally, for an arbitrary closed immersion of schemes $i\colon Z \longrightarrow X$, the direct image functor $i_*\colon Z\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ has a right adjoint (whose existence follows already from the facts that $Z\text{–}\mathsf{qcoh}$ is a Grothendieck abelian category and i_* is an exact functor preserving coproducts). This functor, which could be properly called the "quasi-coherent $i^!$", can be constructed by applying the *coherator* functor [63, Sections B.12–B.14] to the "$\mathcal{O}$-module $i^!$" mentioned above. In fact, a more specific description is available [20, Tag 01R0].

In the sequel, the notation $i^!$ for a closed immersion i will always stand for the "quasi-coherent $i^!$". For the closed immersion of a reasonable closed subscheme $i\colon Z \longrightarrow X$, the "$\mathcal{O}$-module $i^!$" and the "quasi-coherent $i^!$" agree.

Notice that, for any closed immersion i, the direct image functor i_* is fully faithful; so the adjunction morphism $\mathcal{N} \longrightarrow i^! i_* \mathcal{N}$ is an isomorphism for all $\mathcal{N} \in Z\text{–}\mathsf{qcoh}$. The adjunction morphism $i_* i^! \mathcal{M} \longrightarrow \mathcal{M}$ is a monomorphism for all $\mathcal{M} \in X\text{–}\mathsf{qcoh}$ (cf. Lemma 2.14 below).

2.3. **Quasi-coherent torsion sheaves.** We follow [7, Sections 7.11.3–4]. Let $\mathfrak{X}$ be a reasonable ind-scheme. A *quasi-coherent torsion sheaf* $\mathscr{M}$ on $\mathfrak{X}$ (called an "$\mathcal{O}^!$-module" in [7]) is the following set of data:

(i) to every reasonable closed subscheme $Y \subset \mathfrak{X}$, a quasi-coherent sheaf $\mathscr{M}_{(Y)}$ on Y is assigned;

(ii) to every pair of reasonable closed subschemes Y, $Z \subset \mathfrak{X}$, $Z \subset Y$ with the closed immersion morphism $i_{ZY}\colon Z \longrightarrow Y$, a morphism $i_{ZY*}\mathscr{M}_{(Z)} \longrightarrow \mathscr{M}_{(Y)}$ of quasi-coherent sheaves on Y is assigned;

(iii) such that the corresponding morphism $\mathscr{M}_{(Z)} \longrightarrow i_{ZY}^! \mathscr{M}_{(Y)}$ of quasi-coherent sheaves on Z is an isomorphism;

(iv) and, for every triple of reasonable closed subschemes Y, Z, $W \subset \mathfrak{X}$, $W \subset Z \subset Y$, the triangle diagram $i_{WY*}\mathscr{M}_{(W)} \longrightarrow i_{ZY*}\mathscr{M}_{(Z)} \longrightarrow \mathscr{M}_{(Y)}$ is commutative in $Y\text{–}\mathsf{qcoh}$.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of reasonable closed subschemes. Then, in order to construct a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$, it suffices to specify the quasi-coherent sheaves $\mathscr{M}_{(X_\gamma)} \in X_\gamma\text{–}\mathsf{qcoh}$ for every $\gamma \in \Gamma$ and the morphisms $i_{X_\gamma X_\delta *}\mathscr{M}_{(X_\gamma)} \longrightarrow \mathscr{M}_{(X_\delta)}$ for every $\gamma < \delta \in \Gamma$ satisfying conditions (iii–iv) for $W = X_\beta$, $Z = X_\gamma$, $Y = X_\delta$, $\beta < \gamma < \delta \in \Gamma$. The quasi-coherent sheaves $\mathscr{M}_{(Y)}$ for all the other reasonable closed subschemes $Y \subset \mathfrak{X}$ and the related morphisms (ii) can then be uniquely recovered so that conditions (iii–iv) are satisfied for all reasonable closed subschemes in $\mathfrak{X}$.

Morphisms of quasi-coherent torsion sheaves $f\colon \mathscr{M} \longrightarrow \mathscr{N}$ on $\mathfrak{X}$ are defined in the obvious way. We denote the category of quasi-coherent torsion sheaves on $\mathfrak{X}$ by $\mathfrak{X}\text{–}\mathsf{tors}$. In the rest of Section 2, our main aim is to prove the following theorem.

Theorem 2.4. *For any reasonable strict ind-concentrated ind-scheme $\mathfrak{X}$, the category of quasi-coherent torsion sheaves $\mathfrak{X}\text{–}\mathsf{tors}$ is a Grothendieck abelian category.*

The proof of Theorem 2.4 will be given at the end of Section 2.7.

2.4. **Ind-affine examples.** (1) Let $\mathfrak{X} = \operatorname{Spi}\widehat{\mathbb{Z}}$ be the ind-affine ind-scheme from Example 1.4(1). Then $\mathfrak{X}$ is a reasonable ind-scheme, and the category $\mathfrak{X}\text{–}\mathsf{tors}$ is equivalent to the category of torsion abelian groups.

Indeed, the closed subschemes of $\operatorname{Spi}\widehat{\mathbb{Z}}$ are precisely the schemes $X_n = \operatorname{Spec}\mathbb{Z}/n\mathbb{Z}$. All of them are reasonable. Let M be a torsion abelian group; then the corresponding quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\widehat{\mathbb{Z}}$ is defined by the rule that $\mathscr{M}_{(X_n)} \in X_n\text{–}\mathsf{qcoh}$ is the quasi-coherent sheaf corresponding to the $\mathbb{Z}/n\mathbb{Z}$-module $M_{(n)} \subset M$ of all the elements annihilated by n in M. The morphisms of quasi-coherent sheaves $i_{ZY*}\mathscr{M}_{(Z)} \longrightarrow \mathscr{M}_{(Y)}$ correspond to the inclusion maps of abelian groups $M_{(m)} \longrightarrow M_{(n)}$ for m dividing n. Conversely, given a quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\widehat{\mathbb{Z}}$, the related torsion abelian group M is the direct limit $M = \varinjlim_{n\in\mathbb{Z}_{>0}} \mathscr{M}_{(X_n)}(X_n)$.

(2) Let $\mathfrak{X} = \operatorname{Spi}\mathbb{Z}_p$ be the ind-affine ind-scheme from Example 1.4(2). Then $\mathfrak{X}$ is a reasonable ind-scheme, and the category $\mathfrak{X}$–tors is equivalent to the category of p-primary (torsion) abelian groups.

Indeed, the closed subschemes of $\operatorname{Spi}\widehat{\mathbb{Z}}$ are precisely the schemes $X_r = \operatorname{Spec}\mathbb{Z}/p^r\mathbb{Z}$. All of them are reasonable. Let M be a p-primary abelian group; then the corresponding quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathbb{Z}_p$ is defined by the rule that $\mathscr{M}_{(X_r)} \in X_r$–$\mathsf{qcoh}$ is the quasi-coherent sheaf corresponding to the $\mathbb{Z}/p^r\mathbb{Z}$-module $M_{(p^r)} \subset M$ of all the elements annihilated by p^r in M. Conversely, given a quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathbb{Z}_p$, the related p-primary abelian group M is the direct limit $M = \varinjlim_{r\in\mathbb{Z}_{\geq 0}} \mathscr{M}_{(X_r)}(X_r)$.

(3) Let $\mathfrak{X} = \operatorname{Spi}\Bbbk[[x]]$ be the ind-affine ind-scheme from Example 1.5(1). Then, similarly to (2), one can describe the closed subschemes in $\mathfrak{X}$ and see that all of them are reasonable. The category $\mathfrak{X}$–tors is equivalent to the category of x-primary torsion $\Bbbk[x]$-modules, i. e., the full subcategory in the abelian category of all $\Bbbk[x]$-modules consisting of all the $\Bbbk[x]$-modules M such that for every element $b \in M$ there exists an integer $r \geq 1$ for which $x^r b = 0$ in M.

(4) Let $\mathfrak{X} = \operatorname{Spi}\mathscr{C}^*$ be the ind-affine ind-scheme from Example 1.5(2). Then $\mathfrak{X}$ is a reasonable ind-scheme, and the category $\mathfrak{X}$–tors is equivalent to the category of comodules over the coalgebra $\mathscr{C}$.

Indeed, the closed subchemes of $\operatorname{Spi}\mathscr{C}^*$ are precisely the schemes $X_{\mathcal{E}} = \operatorname{Spec}\mathcal{E}^*$, where $\mathcal{E} \subset \mathscr{C}$ are the finite-dimensional subcoalgebras. All the closed subschemes in $\mathfrak{X}$ are reasonable. For any $\mathscr{C}$-comodule M, denote by $M_{(\mathcal{E})} \subset M$ the maximal $\mathscr{C}$-subcomodule of M whose $\mathscr{C}$-comodule structure comes from an $\mathcal{E}$-comodule structure. Simply put, $M_{(\mathcal{E})}$ is the kernel of the composition $M \longrightarrow \mathscr{C}\otimes_\Bbbk M \longrightarrow \mathscr{C}/\mathcal{E}\otimes_\Bbbk M$ of the coaction map $M \longrightarrow \mathscr{C}\otimes_\Bbbk M$ with the map $\mathscr{C}\otimes_\Bbbk M \longrightarrow \mathscr{C}/\mathcal{E}\otimes_\Bbbk M$ induced by the natural surjection $\mathscr{C} \longrightarrow \mathscr{C}/\mathcal{E}$.

Notice that the category of $\mathcal{E}$-comodules is naturally equivalent to the category of $\mathcal{E}^*$-modules. The quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathscr{C}^*$ corresponding to a $\mathscr{C}$-comodule M is defined by the rule that $\mathscr{M}_{(X_{\mathcal{E}})} \in X_{\mathcal{E}}$–$\mathsf{qcoh}$ is the quasi-coherent sheaf corresponding to the $\mathcal{E}^*$-module $M_{(\mathcal{E})}$. The morphisms of quasi-coherent sheaves $i_{X_{\mathcal{E}'}X_{E''}*}\mathscr{M}_{(X_{\mathcal{E}'})} \longrightarrow \mathscr{M}_{(X_{\mathcal{E}''})}$ for $\mathcal{E}' \subset \mathcal{E}''$ correspond to the inclusion maps $M_{(\mathcal{E}')} \longrightarrow M_{(\mathcal{E}'')}$. Conversely, given a quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathscr{C}^*$, the related $\mathscr{C}$-comodule M is the direct limit $M = \varinjlim_{\mathcal{E}\subset\mathscr{C}} \mathscr{M}_{(X_{\mathcal{E}})}(X_{\mathcal{E}})$.

(5) Let $\mathfrak{R}$ be a complete, separated topological commutative ring where open ideals form a base of neighborhoods of zero, as in Example 1.6(1). We will say that an open ideal $\mathfrak{I} \subset \mathfrak{R}$ is *reasonable* (with respect to the given topology on $\mathfrak{R}$) if, for any open ideal $\mathfrak{J} \subset \mathfrak{R}$ such that $\mathfrak{J} \subset \mathfrak{I}$, the kernel of the natural surjective ring homomorphism $\mathfrak{R}/\mathfrak{J} \longrightarrow \mathfrak{R}/\mathfrak{I}$ is a finitely generated ideal in the discrete ring $\mathfrak{R}/\mathfrak{J}$.

A topological ring $\mathfrak{R}$ is said to be *reasonable* if reasonable open ideals form a base of neighborhoods of zero in $\mathfrak{R}$. Equivalently, $\mathfrak{R}$ is reasonable if and only if there exists a base of neighborhoods of zero B consisting of open ideals in $\mathfrak{R}$ such that for all $\mathfrak{J} \subset \mathfrak{I} \in B$ the kernel of the natural surjective ring homomorphism $\mathfrak{R}/\mathfrak{J} \longrightarrow \mathfrak{R}/\mathfrak{I}$ is a finitely generated ideal in $\mathfrak{R}/\mathfrak{J}$.

Let $\mathfrak{X} = \operatorname{Spi}\mathfrak{R}$ be the ind-affine ind-scheme from Example 1.6(1). Then $\mathfrak{X}$ is a reasonable ind-scheme if and only if $\mathfrak{R}$ is a reasonable topological ring.

(6) Let $\mathfrak{R}$ be a reasonable topological commutative ring; so $\operatorname{Spi}\mathfrak{R}$ is a reasonable ind-scheme. Then the category $\mathfrak{X}\text{–}\mathsf{tors}$ is equivalent to the category $\mathfrak{R}\text{–}\mathsf{discr}$ of *discrete* $\mathfrak{R}$-*modules*. This means the full subcategory $\mathfrak{R}\text{–}\mathsf{discr} \subset \mathfrak{R}\text{–}\mathsf{mod}$ in the category of abelian category of $\mathfrak{R}$-modules $\mathfrak{R}\text{–}\mathsf{mod}$ consisting of all the $\mathfrak{R}$-modules M such that for every element $b \in M$ the annihilator of b is an open ideal in $\mathfrak{R}$.

Indeed, the closed subschemes of $\operatorname{Spi}\mathfrak{R}$ are precisely the schemes $X_\mathfrak{I} = \operatorname{Spec}\mathfrak{R}/\mathfrak{I}$, where $\mathfrak{I} \subset \mathfrak{R}$ ranges over the open ideals. Let us say that $\mathfrak{I} \subset \mathfrak{R}$ is a *reasonable open ideal* if the closed subscheme $X_\mathfrak{I} \subset \mathfrak{X}$ is reasonable. This means that, for every open ideal $\mathfrak{J} \subset \mathfrak{I} \subset \mathfrak{R}$, the kernel of the map $\mathfrak{R}/\mathfrak{J} \longrightarrow \mathfrak{R}/\mathfrak{I}$ is a finitely generated ideal.

For any discrete $\mathfrak{R}$-module M, denote by $M_{(\mathfrak{I})} \subset M$ the submodule of all elements annihilated by $\mathfrak{I}$. Then the quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathfrak{R}$ corresponding to M is defined by the rule that, for any reasonable open ideal $\mathfrak{I} \subset \mathfrak{R}$, the quasi-coherent torsion sheaf $\mathscr{M}_{(X_\mathfrak{I})} \in X_\mathfrak{I}\text{–}\mathsf{qcoh}$ corresponds to the $\mathfrak{R}/\mathfrak{I}$-module $M_{(\mathfrak{I})}$. The morphisms of quasi-coherent sheaves $i_{X_\mathfrak{J}X_\mathfrak{I}*}\mathscr{M}_{(X_\mathfrak{J})} \longrightarrow \mathscr{M}_{(X_\mathfrak{I})}$ for reasonable open ideals $\mathfrak{J} \subset \mathfrak{I} \subset \mathfrak{R}$ correspond to the inclusion maps $M_{(\mathfrak{J})} \longrightarrow M_{(\mathfrak{I})}$. Conversely, given a quasi-coherent torsion sheaf $\mathscr{M}$ on $\operatorname{Spi}\mathfrak{R}$, the related discrete $\mathfrak{R}$-module M is the direct limit $M = \varinjlim_{\mathfrak{I}\subset\mathfrak{R}} \mathscr{M}_{(X_\mathfrak{I})}(X_\mathfrak{I})$ taken over the directed poset of reasonable open ideals $\mathfrak{I} \subset \mathfrak{R}$ (in the reverse inclusion order).

The above examples explain the terminology "quasi-coherent torsion sheaves". One can also observe that, while in every one of the examples (1–5) the category $\mathfrak{X}\text{–}\mathsf{tors}$ is indeed abelian as Theorem 2.4 claims, the forgetful functors $\mathfrak{X}\text{–}\mathsf{tors} \longrightarrow Z\text{–}\mathsf{qcoh}$ assigning to a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$ the quasi-coherent sheaf $\mathscr{M}_{(Z)} \in Z\text{–}\mathsf{qcoh}$ for a reasonable closed subscheme $Z \subset \mathfrak{X}$ are *not* exact. In fact, the functors $\mathscr{M} \longmapsto \mathscr{M}_{(Z)} \colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow Z\text{–}\mathsf{qcoh}$ are left, but not necessarily right exact (see Section 2.8 below for a further discussion). This explains why Theorem 2.4 is nontrivial and its proof is not straightforward.

2.5. **Direct limits.** Recall that the functor of global sections of quasi-coherent sheaves over a concentrated scheme preserves (filtered) direct limits. It follows that so does the direct image functor f_* for a morphism of concentrated schemes $f \colon Y \longrightarrow X$. The inverse image functor f^*, being a left adjoint, obviously preserves direct limits. Furthermore, for any reasonable closed subscheme $Z \subset X$ with the closed immersion morphism $i \colon Z \longrightarrow X$, the functor $i^! \colon X\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ preserves direct limits.

Let $\mathfrak{X}$ be a reasonable ind-scheme and $(\mathscr{M}_\theta)_{\theta\in\Theta}$ be an inductive system of quasi-coherent torsion sheaves on $\mathfrak{X}$, indexed by a directed poset Θ. For every reasonable closed subscheme $Z \subset \mathfrak{X}$, put $\mathscr{M}_{(Z)} = \varinjlim_{\theta\in\Theta}(\mathscr{M}_\theta)_{(Z)}$ (where the direct limit is taken in the category of quasi-coherent sheaves on Z). Then the collection of quasi-coherent sheaves $\mathscr{M}_{(Z)}$ with the obvious maps $i_{ZY*}\mathscr{M}_{(Z)} \longrightarrow \mathscr{M}_{(Y)}$ for $Z \subset Y \subset \mathfrak{X}$ is a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$ (as one can see from the previous paragraph). One has $\mathscr{M} = \varinjlim_{\theta\in\Theta}\mathscr{M}_\theta$ in the category $\mathfrak{X}$–tors.

2.6. **Direct images.** Let $\mathfrak{X}$ be a reasonable ind-scheme and $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes which is "representable by schemes". According to Section 2.1, the ind-scheme $\mathfrak{Y}$ is also reasonable.

Let $\mathscr{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}$. For every reasonable closed subscheme $Z \subset \mathfrak{X}$, put $\mathscr{M}_{(Z)} = f_{Z*}(\mathscr{N}_{(W)}) \in Z$–$\mathsf{qcoh}$, where f_Z is the morphism $W = Z \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow Z$ and $f_{Z*}\colon W$–$\mathsf{qcoh} \longrightarrow Z$–$\mathsf{qcoh}$ is the direct image functor of quasi-coherent sheaves. Then it is clear from Lemma 2.3(a) that the collection of quasi-coherent sheaves $\mathscr{M}_{(Z)}$ with the natural maps $i_{Z'Z''*}\mathscr{M}_{(Z')} \longrightarrow \mathscr{M}_{(Z'')}$ for $Z' \subset Z'' \subset \mathfrak{X}$ is a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$.

Put $f_*\mathscr{N} = \mathscr{M}$. This construction defines the functor of *direct image of quasi-coherent torsion sheaves* $f_*\colon \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$.

2.7. **Γ-systems.** Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme represented by an inductive system of closed immersions of reasonable closed subschemes. The definition of what we will call a Γ*-system* on $\mathfrak{X}$ (which is a shorthand for "$(X_\gamma)_{\gamma\in\Gamma}$-system of quasi-coherent sheaves") is obtained from the definition of a quasi-coherent torsion sheaf in Section 2.3 by restricting the reasonable subschemes under consideration to those belonging to the inductive system $(X_\gamma)_{\gamma\in\Gamma}$ *and* dropping the condition (iii).

In other words, a Γ-system $\mathbb{M}$ on $\mathfrak{X}$ is the following set of data:

(i) to every index $\gamma \in \Gamma$, a quasi-coherent sheaf $\mathbb{M}_{(\gamma)}$ on X_γ is assigned;

(ii) to every pair of indices $\gamma < \delta \in \Gamma$ with the related transition morphism $i_{\gamma\delta}\colon X_\gamma \longrightarrow X_\delta$, a morphism $i_{\gamma\delta*}\mathbb{M}_{(\gamma)} \longrightarrow \mathbb{M}_{(\delta)}$ of quasi-coherent sheaves on X_δ, or equivalently, a morphism $\mathbb{M}_{(\gamma)} \longrightarrow i^!_{\gamma\delta}\mathbb{M}_{(\delta)}$ of quasi-coherent sheaves on X_γ, is assigned;

(iv) such that for every triple of indices $\beta < \gamma < \delta$, the triangle diagram $i_{\beta\delta*}\mathbb{M}_{(\beta)} \longrightarrow i_{\gamma\delta*}\mathbb{M}_{(\gamma)} \longrightarrow \mathbb{M}_\delta$ is commutative in X_δ–qcoh.

Morphisms of Γ*-systems* $f\colon \mathbb{M} \longrightarrow \mathbb{N}$ on $\mathfrak{X}$ are defined in the obvious way. We denote the category of Γ-systems on $\mathfrak{X}$ by $(\mathfrak{X},\Gamma)$–syst.

For every $\gamma \in \Gamma$, denote by $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ the natural closed immersion. As a particular case of the construction of Section 2.6, we have the direct image functor $i_{\gamma*}\colon X_\gamma$–$\mathsf{qcoh} = X_\gamma$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$. For every Γ-system $\mathbb{M}$ on $\mathfrak{X}$, we put

$$\mathbb{M}^+ = \varinjlim_{\gamma\in\Gamma} i_{\gamma*}\mathbb{M}_{(\gamma)} \in \mathfrak{X}\text{–}\mathsf{tors},$$

where $\mathbb{M}_{(\gamma)} \in X_\gamma$–$\mathsf{qcoh}$, $i_{\gamma*}\mathbb{M}_{(\gamma)} \in \mathfrak{X}$–$\mathsf{tors}$, and the direct limit is taken in the category $\mathfrak{X}$–tors. Notice that, according to Section 2.5, all (filtered) direct limits exist in $\mathfrak{X}$–tors.

The quasi-coherent torsion sheaf $\mathbb{M}^+$ on $\mathfrak{X}$ can be described more explicitly as follows. For any reasonable closed subscheme $Z \subset \mathfrak{X}$, one has

$$(\mathbb{M}^+)_{(Z)} = \varinjlim\nolimits_{\gamma\in\Gamma: Z\subset X_\gamma} i^!_{Z,\gamma}\mathbb{M}_{(\gamma)},$$

where the direct limit in Z–qcoh is taken over the cofinal subset of all $\gamma \in \Gamma$ such that $Z \subset X_\gamma$, and $i_{Z,\gamma}\colon Z \longrightarrow X_\gamma$ is the closed immersion.

Conversely, given a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$, the rule $\mathbb{M}_{(\gamma)} = \mathscr{M}_{(X_\gamma)}$ defines a Γ-system $\mathbb{M}$ on $\mathfrak{X}$, which we will denote by $\mathscr{M}|_\Gamma = \mathbb{M}$.

Lemma 2.5. *The functor* $\mathbb{M} \longmapsto \mathbb{M}^+\colon (\mathfrak{X},\Gamma)$–$\mathsf{syst} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$ *is left adjoint to the functor* $\mathscr{M} \longmapsto \mathscr{M}|_\Gamma\colon \mathfrak{X}$–$\mathsf{tors} \longrightarrow (\mathfrak{X},\Gamma)$–$\mathsf{syst}$.

Proof. Let $\mathbb{M}$ be a Γ-system and $\mathscr{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$. Then the abelian group of morphisms $\mathbb{M}^+ \longrightarrow \mathscr{N}$ in $\mathfrak{X}$–tors is isomorphic to the group of all compatible collections of morphisms $i^!_{Z,\gamma}\mathbb{M}_{(\gamma)} \longrightarrow \mathscr{N}_{(Z)}$ in Z–qcoh, defined for all reasonable closed subschemes $Z \subset \mathfrak{X}$ and indices $\gamma \in \Gamma$ such that $Z \subset X_\gamma$. On the other hand, the abelian groups of morphisms $\mathbb{M} \longrightarrow \mathscr{N}|_\Gamma$ in $(\mathfrak{X},\Gamma)$–syst is isomorphic to the group of all compatible collections of morphisms $\mathbb{M}_{(\gamma)} \longrightarrow \mathscr{N}_{(X_\gamma)}$ in X_γ–qcoh, defined for all $\gamma \in \Gamma$. These are equivalent sets of data, as the morphism $i^!_{Z,\gamma}\mathbb{M}_{(\gamma)} \longrightarrow \mathscr{N}_{(Z)}$ is uniquely recoverable by applying the functor $i^!_{Z,\gamma}\colon X_\gamma$–$\mathsf{qcoh} \longrightarrow Z$–$\mathsf{qcoh}$ to the morphism $\mathbb{M}_{(\gamma)} \longrightarrow \mathscr{N}_{(X_\gamma)}$. □

Proposition 2.6. *The category* $(\mathfrak{X},\Gamma)$–syst *of* Γ*-systems on* $\mathfrak{X}$ *is a Grothendieck abelian category.*

Proof. The assertion that $(\mathfrak{X},\Gamma)$–syst is an abelian category with exact direct limit functors is straightforward. Moreover, the forgetful functor $(\mathfrak{X},\Gamma)$–$\mathsf{syst} \longrightarrow X_\gamma$–$\mathsf{qcoh}$ taking a Γ-system $\mathbb{M}$ to the quasi-coherent sheaf $\mathbb{M}_{(\gamma)}$ preserves the kernels, cokernels, and direct limits. To show that the category $(\mathfrak{X},\Gamma)$–syst has a set of generators, choose for every $\gamma \in \Gamma$ a set of generators $\mathsf{S}_\gamma \subset X_\gamma$–$\mathsf{qcoh}$ in the Grothendieck category of quasi-coherent sheaves on X_γ. For every quasi-coherent sheaf $\mathcal{K}$ on X_γ, define the Γ-system $\mathbb{M}(\gamma,\mathcal{K})$ by the rules $\mathbb{M}(\gamma,\mathcal{K})_{(\delta)} = i_{\gamma\delta*}\mathcal{K}$ for $\gamma \le \delta \in \Gamma$ and $\mathbb{M}(\gamma,\mathcal{K})_{(\delta)} = 0$ for $\gamma \not\le \delta \in \Gamma$. Then all the objects of the form $\mathbb{M}(\gamma,S)$ with $\gamma \in \Gamma$ and $S \in \mathsf{S}_\gamma$ form a set of generators of the category $(\mathfrak{X},\Gamma)$–syst. □

Lemma 2.7. *The functor* $\mathscr{M} \longmapsto \mathscr{M}|_\Gamma\colon \mathfrak{X}$–$\mathsf{tors} \longrightarrow (\mathfrak{X},\Gamma)$–$\mathsf{syst}$ *is fully faithful. The endofunctor* $\mathbb{M} \longmapsto \mathbb{M}^+|_\Gamma\colon (\mathfrak{X},\Gamma)$–$\mathsf{syst} \longrightarrow (\mathfrak{X},\Gamma)$–$\mathsf{syst}$ *is left exact.*

Proof. The first assertion holds essentially because one can equivalently define a quasi-coherent torsion sheaf on $\mathfrak{X}$ as a collection of quasi-coherent sheaves on the schemes X_γ, $\gamma \in \Gamma$, endowed with the usual maps and satisfying the usual conditions, as per the discussion in Section 2.3. To check the second assertion, one computes that

$$(\mathbb{M}^+|_\Gamma)_{(\gamma)} = \varinjlim\nolimits_{\delta\in\Gamma:\gamma\le\delta} i^!_{\gamma\delta}\mathbb{M}_{(\delta)}$$

and recalls that the functor $i^!_{\gamma\delta}$ is left exact (since it is a right adjoint). It is important here that the forgetful functor assigning to a Γ-system $\mathbb{N}$ the collection of quasi-coherent sheaves $\mathbb{N}_{(\gamma)}$ (viewed as an object of the Cartesian product of the categories X_γ–qcoh) is exact and faithful. □

Proposition 2.8. *Let* B *be an abelian category and* $\mathsf{A}\subset\mathsf{B}$ *be a full subcategory whose inclusion functor* $G\colon\mathsf{A}\longrightarrow\mathsf{B}$ *has a left adjoint functor* $F\colon\mathsf{B}\longrightarrow\mathsf{A}$*. Assume that the composition* $GF\colon\mathsf{B}\longrightarrow\mathsf{B}$ *is a left exact functor. Then the category* A *is abelian and the functor* F *is exact. If* B *is a Grothendieck category, then so is* A*.*

Proof. This well-known result describes a familiar setting which occurs, e. g., when A is a sheaf category and B is the related presheaf category (so G is the inclusion of the sheaves into the presheaves and F is the sheafification), or when an arbitrary Grothendieck category A is represented as a localization of a module category B via the Gabriel–Popescu theorem. The claim is that, in the context of the proposition, the functor F represents A as a quotient category of B by its Serre subcategory of all objects annihilated by F. Moreover, if A has coproducts, then the subcategory of objects annihilated by F is closed under coproducts; and it follows that the direct limits are exact in A whenever they are exact in B, and a set of generators exists in A whenever such a set exists in B. The full subcategory $\mathsf{A}\subset\mathsf{B}$ is called a *Giraud subcategory* [60, Section X.1]; notice that the inclusion functor $G\colon\mathsf{A}\longrightarrow\mathsf{B}$ is left exact, but *not* exact in general. □

Proof of Theorem 2.4. Put $\mathsf{A}=\mathfrak{X}$–tors and $\mathsf{B}=(\mathfrak{X},\Gamma)$–$\mathsf{syst}$. Furthermore, put $G(\mathscr{M})=\mathscr{M}|_\Gamma$ and $F(\mathbb{M})=\mathbb{M}^+$. Then the category B is Grothendieck by Proposition 2.6, the functor F is left adjoint to G by Lemma 2.5, the functor G is fully faithful by Lemma 2.7, and the functor GF is left exact by the same Lemma 2.7. Thus Proposition 2.8 is applicable, implying that A is a Grothendieck category. □

Question 2.9. Is the reasonableness assumption needed for the validity of Theorem 2.4 ? Is the category of quasi-coherent torsion sheaves on an arbitrary (strict) ind-scheme abelian? Notice that the category of discrete $\mathfrak{R}$-modules, as defined in Section 2.4(6), is a Grothendieck abelian category for any topological ring $\mathfrak{R}$.

2.8. **Inverse images.** Let $\mathfrak{X}=$ “$\varinjlim$”$_{\gamma\in\Gamma}X_\gamma$ be a reasonable ind-scheme represented by an inductive system of closed immersions of reasonable closed subschemes. Let $f\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be a morphism of ind-schemes which is “representable by schemes”. Put $Y_\gamma=\mathfrak{Y}\times_\mathfrak{X}X_\gamma$; then Y_γ are reasonable closed subschemes in $\mathfrak{Y}$ and $\mathfrak{Y}=$ “$\varinjlim$”$_{\gamma\in\Gamma}Y_\gamma$.

Let $\mathbb{M}=(\mathbb{M}_{(\gamma)}\in X_\gamma$–$\mathsf{qcoh})_{\gamma\in\Gamma}$ be a Γ-system on $\mathfrak{X}$. For every $\gamma\in\Gamma$, put $\mathbb{N}_{(\gamma)}=f_\gamma^*\mathbb{M}_{(\gamma)}\in Y_\gamma$–$\mathsf{qcoh}$, where $f_\gamma\colon Y_\gamma\longrightarrow X_\gamma$. Then it is clear from Lemma 2.3(b) that the collection of quasi-coherent cosheaves $\mathbb{N}_{(\gamma)}\in Y_\gamma$–$\mathsf{qcoh}$ has a natural structure of a Γ-system $\mathbb{N}$ on $\mathfrak{Y}$. We put $f^*\mathbb{M}=\mathbb{N}$.

The functor of inverse image of quasi-coherent torsion sheaves $f^*\colon\mathfrak{X}$–$\mathsf{tors}\longrightarrow\mathfrak{Y}$–$\mathsf{tors}$ is defined by the rule

$$f^*(\mathscr{M})=(f^*(\mathscr{M}|_\Gamma))^+.$$

One check directly or deduce from Lemma 2.10(b) that this construction of the functor $f^*\colon \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{Y}\text{–tors}$ does not depend on the choice of a representation of a reasonable ind-scheme $\mathfrak{X}$ by an inductive system of closed immersions of reasonable closed subschemes $(X_\gamma)_{\gamma\in\Gamma}$. (See Remark 7.4 below for a simpler construction of the functor f^* in the case of a flat morphism f.)

Let us also define the functor of direct image of Γ-systems $f_*\colon (\mathfrak{Y},\Gamma)\text{–syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–syst}$. Put $f_*\mathbb{N} = \mathbb{M}$, where $\mathbb{N}_{(\gamma)} = f_*\mathbb{M}_{(\gamma)}$ for all $\gamma\in\Gamma$. It is clear that the direct images of torsion sheaves and Γ-systems agree in the sense that $(f_*\mathscr{N})|_\Gamma \simeq f_*(\mathscr{N}|_\Gamma)$ for all $\mathscr{N}\in\mathfrak{Y}\text{–tors}$ (cf. Section 2.6).

Lemma 2.10. (a) *The functor* $f^*\colon (\mathfrak{X},\Gamma)\text{–syst} \longrightarrow (\mathfrak{Y},\Gamma)\text{–syst}$ *is left adjoint to the functor* $f_*\colon (\mathfrak{Y},\Gamma)\text{–syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–syst}$.

(b) *The functor* $f^*\colon \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{Y}\text{–tors}$ *is left adjoint to the functor* $f_*\colon \mathfrak{Y}\text{–tors} \longrightarrow \mathfrak{X}\text{–tors}$.

Proof. Part (a): let $\mathbb{M}$ be a Γ-system on $\mathfrak{X}$ and $\mathbb{N}$ be a Γ-system on $\mathfrak{Y}$. Then the group of morphisms $\mathbb{M} \longrightarrow f_*\mathbb{N}$ in $(\mathfrak{X},\Gamma)\text{–syst}$ is isomorphic to the group of all compatible collections of morphisms $\mathbb{M}_{(\gamma)} \longrightarrow f_{\gamma*}\mathbb{N}_{(\gamma)}$ in $X_\gamma\text{–qcoh}$, defined for all $\gamma\in\Gamma$, while the group of morphism $f^*\mathbb{M} \longrightarrow \mathbb{N}$ in $(\mathfrak{Y},\Gamma)\text{–syst}$ is isomorphic to the group of all compatible collections of morphisms $f_\gamma^*\mathbb{M}_{(\gamma)} \longrightarrow \mathbb{N}_{(\gamma)}$ in $Y_\gamma\text{–qcoh}$, defined for all $\gamma\in\Gamma$. In view of the adjunction of the functors $f_{\gamma*}\colon Y_\gamma\text{–qcoh} \longrightarrow X_\gamma\text{–qcoh}$ and $f_\gamma^*\colon X_\gamma\text{–qcoh} \longrightarrow Y_\gamma\text{–qcoh}$, these are two equivalent sets of data.

Part (b): let $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$ and $\mathscr{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}$. Then we have

$$\begin{aligned}\operatorname{Hom}_{\mathfrak{X}\text{–tors}}(\mathscr{M}, f_*\mathscr{N}) &\simeq \operatorname{Hom}_{(\mathfrak{X},\Gamma)\text{–syst}}(\mathscr{M}|_\Gamma, (f_*\mathscr{N})|_\Gamma)\\ &\simeq \operatorname{Hom}_{(\mathfrak{X},\Gamma)\text{–syst}}(\mathscr{M}|_\Gamma, f_*(\mathscr{N}|_\Gamma)) \simeq \operatorname{Hom}_{(\mathfrak{Y},\Gamma)\text{–syst}}(f^*(\mathscr{M}|_\Gamma), \mathscr{N}|_\Gamma)\\ &\simeq \operatorname{Hom}_{\mathfrak{Y}\text{–tors}}(f^*(\mathscr{M}|_\Gamma)^+, \mathscr{N}) = \operatorname{Hom}_{\mathfrak{Y}\text{–tors}}(f^*\mathscr{M}, \mathscr{N}),\end{aligned}$$

where the first isomorphism holds by Lemma 2.7, the middle one by part (a), and the next one by Lemma 2.5. □

Lemma 2.11. *The functors* $(-)^+$ *commute with inverse images; in other words, for any* Γ*-system* $\mathbb{M}$ *on* $\mathfrak{X}$ *there is a natural isomorphism of quasi-coherent torsion sheaves* $f^*(\mathbb{M}^+) \simeq (f^*\mathbb{M})^+$ *on* $\mathfrak{Y}$.

Proof. In view of the adjunctions of Lemma 2.10, the desired natural isomorphism is adjoint to the natural isomorphism of $(f_*\mathscr{N})|_\Gamma \simeq (f_*\mathscr{N}|_\Gamma)$ of Γ-systems on $\mathfrak{X}$ for a quasi-coherent torsion sheaf $\mathscr{N}$ on $\mathfrak{Y}$. □

Let $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of ind-schemes. Assume that $\mathfrak{X}$ is a reasonable ind-scheme; then so is $\mathfrak{Z}$. The functor $i^!\colon \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{Z}\text{–tors}$ is defined by the rule $(i^!\mathscr{M})_{(W)} = k^!(\mathscr{M}_{(Y)})$ for all $\mathscr{M}\in\mathfrak{X}\text{–tors}$, where $W\subset\mathfrak{Z}$ is an arbitrary reasonable closed subscheme and $Y\subset\mathfrak{X}$ is a reasonable closed subscheme such that the composition $W \longrightarrow \mathfrak{Z} \xrightarrow{i} \mathfrak{X}$ factorizes as $W \xrightarrow{k} Y \longrightarrow \mathfrak{X}$.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of reasonable closed subschemes. Put $Z_\gamma = \mathfrak{Z}\times_\mathfrak{X} X_\gamma$; then Z_γ are reasonable closed subschemes in $\mathfrak{Z}$ and $\mathfrak{Z} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} Z_\gamma$. The functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ can be described in these terms by the rule $(i^!\mathscr{M})_{(Z_\gamma)} = i_\gamma^!\mathscr{M}_{(X_\gamma)}$, where i_γ denotes the closed immersion of schemes $i_\gamma\colon Z_\gamma \longrightarrow X_\gamma$.

Lemma 2.12. *The functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ is right adjoint to the direct image functor $i_*\colon \mathfrak{Z}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$.*

Proof. Similar to the proof of Lemma 2.10(a). □

In particular, let $Z \subset \mathfrak{X}$ be a reasonable closed subscheme with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$. Then, by the definition, one has $i^!\mathscr{M} = \mathscr{M}_{(Z)} \in Z\text{–}\mathsf{qcoh}$ for every $\mathscr{M} \in \mathfrak{X}\text{–}\mathsf{tors}$.

Notice that the functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow Z\text{–}\mathsf{qcoh}$ preserves direct limits when Z is a reasonable closed subscheme in $\mathfrak{X}$ (as it is clear from the discussion in Section 2.5). For nonreasonable closed subschemes $Z \subset \mathfrak{X}$, this is not true in general.

2.9. **Injective quasi-coherent torsion sheaves.** Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme represented by an inductive system of closed immersions of reasonable closed subschemes. Let $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ denote the natural closed immersions.

According to Theorem 2.4, the category of quasi-coherent torsion sheaves $\mathfrak{X}\text{–}\mathsf{tors}$ is a Grothendieck abelian category; so it has enough injective objects. Let us describe these injectives.

Lemma 2.13. (a) *A morphism $f\colon \mathscr{M} \longrightarrow \mathscr{N}$ in the abelian category $\mathfrak{X}\text{–}\mathsf{tors}$ is a monomorphism if and only if, for every $\gamma \in \Gamma$, the morphism $i_\gamma^!f\colon i_\gamma^!\mathscr{M} \longrightarrow i_\gamma^!\mathscr{N}$ is a monomorphism in the abelian category $X_\gamma\text{–}\mathsf{qcoh}$.*

(b) *A morphism $f\colon \mathscr{M} \longrightarrow \mathscr{N}$ in the abelian category $\mathfrak{X}\text{–}\mathsf{tors}$ is an epimorphism whenever, for every $\gamma \in \Gamma$, the morphism $i_\gamma^!f\colon i_\gamma^!\mathscr{M} \longrightarrow i_\gamma^!\mathscr{N}$ is an epimorphism in the abelian category $X_\gamma\text{–}\mathsf{qcoh}$.*

Proof. Part (a): for any closed immersion of ind-schemes $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$, the functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ is left exact, since it has a left adjoint functor $i_*\colon \mathfrak{Z}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$. In particular, for a closed subscheme Z in $\mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$, the functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow Z\text{–}\mathsf{qcoh}$ is left exact. This proves the “only if” assertion.

To prove the “if”, assume that the morphism $i_\gamma^!f$ is a monomorphism in $X_\gamma\text{–}\mathsf{qcoh}$ for every $\gamma \in \Gamma$. Let $\mathscr{K}$ be the kernel of f in $\mathfrak{X}\text{–}\mathsf{tors}$. Then $i_\gamma^!\mathscr{K}$ is the kernel of the morphism $i_\gamma^!f$ in $X_\gamma\text{–}\mathsf{qcoh}$, since the functor $i_\gamma^!$ is left exact. So we have $\mathscr{K}_{(X_\gamma)} = i_\gamma^!\mathscr{K} = 0$ for all $\gamma \in \Gamma$, and it follows immediately that $\mathscr{K} = 0$.

Part (b): assume that the morphism $i_\gamma^!f$ is an epimorphism in $X_\gamma\text{–}\mathsf{qcoh}$ for every $\gamma \in \Gamma$. This means that $f|_\Gamma\colon \mathscr{M}|_\Gamma \longrightarrow \mathscr{N}|_\Gamma$ is an epimorphism of Γ-systems on $\mathfrak{X}$. Since the functor $(-)^+\colon (\mathfrak{X},\Gamma)\text{–}\mathsf{syst} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is (right) exact, it follows that $(f_\Gamma)^+\colon (\mathscr{M}|_\Gamma)^+ \longrightarrow (\mathscr{N}|_\Gamma)^+$ is an epimorphism in $\mathfrak{X}\text{–}\mathsf{tors}$. It remains to recall that the adjunction $(f_\Gamma)^+ \longrightarrow f$ is an isomorphism. □

Lemma 2.14. *Let A, B be an abelian categories and $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a fully faithful exact functor which has a right adjoint functor $H\colon \mathsf{B} \longrightarrow \mathsf{A}$. Then the essential image*

$F(\mathsf{A}) \subset \mathsf{B}$ *is a full subcategory closed under subobjects and quotients in* B *if and only if the adjunction morphism* $FH(B) \longrightarrow B$ *is a monomorphism in* B *for every* $B \in \mathsf{B}$.

Proof. Notice that the essential image of a fully faithful exact functor between abelian categories is always a full subcategory closed under kernels and cokernels. Hence $f(\mathsf{A})$ is closed under subobjects in B *if and only if* it is closed under quotients. Now we can proceed with a proof of the lemma.

"If": let $A \in F(\mathsf{A})$ be an object and $A \longrightarrow B$ be an epimorphism in B. Then the adjunction morphism $FH(A) \longrightarrow A$ is an isomorphism, and it follows from commutativity of the obvious diagram that the adjunction morphism $FH(B) \longrightarrow B$ is an epimorphism. Since the morphism $FH(B) \longrightarrow B$ is a monomorphism by assumption, it is an isomorphism. Thus $B \in F(\mathsf{A})$.

"Only if": let $B \in \mathsf{B}$ be an object and $c\colon FH(B) \longrightarrow B$ be the adjunction morphism. Let $C \in \mathsf{B}$ be the image of c. Then C is a quotient object of an object from $F(\mathsf{A})$; by assumption, it follows that $C \in F(\mathsf{A})$. Given an object $B \in \mathsf{B}$, the object $FH(B) \in F(\mathsf{A})$ together with the morphism c is characterized by the universal property that any morphism into B from an object of $F(\mathsf{A})$ factorizes uniquely through c. Now the monomorphism $C \longrightarrow B$ has the same universal property; hence the epimorphism $FH(B) \longrightarrow C$ is an isomorphism and c is a monomorphism. $\square$

Lemma 2.15. *Let* A *be a Grothendieck abelian category with a set of generators* $\mathsf{S} \subset \mathsf{A}$. *Then an object* $J \in \mathsf{A}$ *is injective if and only if any morphism into* J *from a subobject of any object* $S \in \mathsf{S}$ *can be extended to a morphism* $S \longrightarrow J$ *in* A.

Proof. This is a categorical version of the Baer criterion of injectivity of modules, provable in the same way using the Zorn lemma. $\square$

Lemma 2.16. (a) *For any closed subscheme* $Z \subset \mathfrak{X}$ *with the closed immersion morphism* $i\colon Z \longrightarrow \mathfrak{X}$, *the direct image functor* $i_*\colon Z\text{–}\mathsf{qcoh} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ *is exact and fully faithful. Its essential image is closed under subobjects and quotients in the abelian category* $\mathfrak{X}\text{–}\mathsf{tors}$.

(b) *For every* $\gamma \in \Gamma$, *choose a set of generators* $\mathsf{S}_\gamma \subset X_\gamma\text{–}\mathsf{qcoh}$ *of the abelian category of quasi-coherent sheaves on* X_γ. *Then the set* $\mathsf{S} \subset \mathfrak{X}\text{–}\mathsf{tors}$ *of all quasi-coherent torsion sheaves of the form* $i_{\gamma*}\mathcal{S}$, *where* $\gamma \in \Gamma$ *and* $\mathcal{S} \in \mathsf{S}_\gamma$, *is a set of generators of the abelian category* $\mathfrak{X}\text{–}\mathsf{tors}$.

Proof. Part (a): more generally, let $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of (reasonable) ind-schemes. Then, following the discussion in Section 2.8, the direct image functor $i_*\colon \mathfrak{Z}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ has adjoints on both sides, i^* and $i^!$; so i_* is an exact functor.

Furthermore, let $\mathcal{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{Z}$. Then, following the construction in Section 2.6, the quasi-coherent torsion sheaf $i_*\mathcal{N}$ on $\mathfrak{X}$ is defined by the rule $(i_*\mathcal{N})_{(Y)} = i_{Y*}(\mathcal{N}_{(W)})$ for all reasonable closed subschemes $Y \subset \mathfrak{X}$, where $i_Y\colon W = Y \times_{\mathfrak{X}} \mathfrak{Z} \longrightarrow Y$. The quasi-coherent torsion sheaf $i^!i_*\mathcal{N}$ on $\mathfrak{Z}$ is described by the rule $(i^!i_*\mathcal{N})_{(W')} = k^!i_{Y*}(\mathcal{N}_{(W)})$ for all reasonable closed subschemes $W' \subset W$, where $k\colon W' \longrightarrow Y$ is the composition $W' \longrightarrow W \longrightarrow Y$ of the closed immersion $W' \longrightarrow W$ and the morphism $i_Y\colon W \longrightarrow Y$ (which is also a closed immersion, by the definition of a closed immersion of ind-schemes). Clearly, the adjunction morphism $\mathscr{N} \longrightarrow i^! i_* \mathscr{N}$ is an isomorphism in $\mathfrak{Z}$–tors (because the adjunction morphism $\mathscr{N}_{(W)} \longrightarrow i_Y^! i_{Y*} \mathscr{N}_{(W)}$ is an isomorphism in W–qcoh for every Y; cf. Section 2.2). It follows that the direct image functor i_* is fully faithful.

Finally, let $\mathscr{M}$ be a quasi-coherent torsion sheaf on X. Then the quasi-coherent torsion sheaf $i^! \mathscr{M}$ on $\mathfrak{Z}$ is defined as spelled out in Section 2.8. Hence the quasi-coherent torsion sheaf $i_* i^! \mathscr{M}$ on $\mathfrak{X}$ is described by the rule $(i_* i^! \mathscr{M})_{(Y)} = i_{Y*} i_Y^! \mathscr{M}_{(Y)}$ for all reasonable closed subschemes $Y \subset \mathfrak{X}$, where $i_Y \colon W = Y \times_{\mathfrak{X}} \mathfrak{Z} \longrightarrow Y$. The adjunction morphisms $i_{Y*} i_Y^! \mathscr{M}_{(Y)} \longrightarrow \mathscr{M}_{(Y)}$ are monomorphisms in Y–qcoh for all Y (since i_Y is a closed immersion of schemes). By Lemma 2.13(a), it follows that the adjunction morphism $i_* i^! \mathscr{M} \longrightarrow \mathscr{M}$ is a monomorphism in $\mathfrak{X}$–tors. It remains to apply Lemma 2.14 in order to conclude that the essential image of the functor $i_* \colon \mathfrak{Z}$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$ is a full subcategory closed under subobjects and quotients.

Part (b): a set of generators of the abelian category $(\mathfrak{X}, \Gamma)$–syst was constructed in the proof of Proposition 2.6. The subset $\mathsf{S} \subset \mathfrak{X}$–$\mathsf{tors}$ is the image of this set of generators under the functor $\mathbb{M} \longmapsto \mathbb{M}^+ \colon (X, \Gamma)$–$\mathsf{syst} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$. This functor is essentially surjective on objects, exact, and preserves coproducts (being a left adjoint); hence the image of any set of generators under this functor is a set of generators. (See the arguments in Section 2.7 for the details.)

Alternatively, one can notice that, for every quasi-coherent torsion sheaf $\mathscr{M} \in \mathfrak{X}$–$\mathsf{tors}$, the natural morphism

$$\coprod\nolimits_{\gamma \in \Gamma} i_{\gamma*} i_\gamma^! \mathscr{M} \longrightarrow \mathscr{M}$$

is an epimorphism in $\mathfrak{X}$–tors (by Lemma 2.13(b)). Then the assertion easily follows. □

Proposition 2.17. (a) *For any closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i \colon Z \longrightarrow \mathfrak{X}$, the functor $i^! \colon \mathfrak{X}$–$\mathsf{tors} \longrightarrow Z$–$\mathsf{qcoh}$ takes injective objects to injective objects.*

(b) *A quasi-coherent torsion sheaf $\mathscr{J} \in \mathfrak{X}$–$\mathsf{tors}$ is an injective object in $\mathfrak{X}$–tors if and only if, for every $\gamma \in \Gamma$, the quasi-coherent sheaf $i_\gamma^! \mathscr{J} \in X_\gamma$–$\mathsf{qcoh}$ is an injective object in X_γ–qcoh.*

Proof. Part (a): more generally, for any closed immersion of (reasonable) ind-schemes $i \colon \mathfrak{Z} \longrightarrow \mathfrak{X}$, the functor $i_* \colon \mathfrak{Z}$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$ is exact, as explained in the proof of Lemma 2.16(a). The functor $i^! \colon \mathfrak{X}$–$\mathsf{tors} \longrightarrow \mathfrak{Z}$–$\mathsf{tors}$ is right adjoint to i_*; so it takes injectives to injectives. Part (b): the "only if" assertion is provided by part (a). To prove the "if", one can apply Lemma 2.15 to the set of generators of the Grothendieck category $\mathfrak{X}$–tors provided by Lemma 2.16(b). This shows that a quasi-coherent torsion sheaf $\mathscr{J}$ on $\mathfrak{X}$ is injective whenever, for any $\mathcal{M} \in X_\gamma$–qcoh and a subobject $\mathscr{K} \subset i_{\gamma*}\mathcal{M}$, $\mathscr{K} \in \mathfrak{X}$–$\mathsf{tors}$, any morphism $\mathscr{K} \longrightarrow \mathscr{J}$ can be extended to a morphism $i_{\gamma*}\mathcal{M} \longrightarrow \mathscr{J}$ in $\mathfrak{X}$–tors. By Lemma 2.16(a), there is a quasi-coherent subsheaf $\mathcal{N} \subset \mathcal{M}$ on X_γ such that $\mathscr{K} = i_{\gamma*}\mathcal{N}$. Now it suffices to extend a given morphism $\mathcal{N} \longrightarrow i_\gamma^! \mathscr{J}$ to a morphism $\mathcal{M} \longrightarrow i_\gamma^! \mathscr{J}$ in X_γ–qcoh. □

3. Flat Pro-Quasi-Coherent Pro-Sheaves

In this section we continue to follow [7, Sections 7.11.3–4].

3.1. **Pro-quasi-coherent pro-sheaves.** Let $\mathfrak{X}$ be an ind-scheme. A *pro-quasi-coherent pro-sheaf* $\mathfrak{P}$ on $\mathfrak{X}$ (called an "$\mathcal{O}^p$-module" in [7]) is the following set of data:

(i) to every closed subscheme $Y \subset \mathfrak{X}$, a quasi-coherent sheaf $\mathfrak{P}^{(Y)}$ on Y is assigned;

(ii) to every pair of closed subschemes Y, $Z \subset \mathfrak{X}$, $Z \subset Y$ with the closed immersion morphism $i_{ZY}\colon Z \longrightarrow Y$, a morphism $\mathfrak{P}^{(Y)} \longrightarrow i_{ZY*}\mathfrak{P}^{(Z)}$ of quasi-coherent sheaves on Y is assigned;

(iii) such that the corresponding morphism $i_{ZY}^*\mathfrak{P}^{(Y)} \longrightarrow \mathfrak{P}^{(Z)}$ of quasi-coherent sheaves on Z is an isomorphism;

(iv) and, for every triple of closed subschemes Y, Z, $W \subset \mathfrak{X}$, $W \subset Z \subset Y$, the triangle diagram $\mathfrak{P}^{(Y)} \longrightarrow i_{ZY*}\mathfrak{P}^{(Z)} \longrightarrow i_{WY*}\mathfrak{P}^{(W)}$ is commutative in Y–qcoh.

Let $\mathfrak{X} = \text{"}\varinjlim\text{"}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by a inductive system of closed immersions of schemes. Then, in order to construct a pro-quasi-coherent pro-sheaf $\mathfrak{P}$ on $\mathfrak{X}$, it suffices to specify the quasi-coherent sheaves $\mathfrak{P}^{(X_\gamma)} \in X_\gamma$–$\mathsf{qcoh}$ for every $\gamma \in \Gamma$ and the morphisms $\mathfrak{P}^{(X_\delta)} \longrightarrow i_{X_\gamma X_\delta *}\mathfrak{P}^{(X_\gamma)}$ for every $\gamma < \delta \in \Gamma$ satisfying conditions (iii–iv) for $W = X_\beta$, $Z = X_\gamma$, $Y = X_\delta$, $\beta < \gamma < \delta \in \Gamma$. The quasi-coherent sheaves $\mathfrak{P}^{(Y)}$ for all the other closed subsechemes $Y \subset \mathfrak{X}$ and the related morphisms (ii) can then be uniquely recovered so that conditions (iii–iv) are satisfied for all closed subschemes in $\mathfrak{X}$.

Morphisms of pro-quasi-coherent pro-sheaves $f\colon \mathfrak{P} \longrightarrow \mathfrak{Q}$ on $\mathfrak{X}$ are defined in the obvious way. We denote the additive category of quasi-coherent torsion sheaves on $\mathfrak{X}$ by $\mathfrak{X}$–pro. The following example shows that the category $\mathfrak{X}$–pro is usually *not* abelian, and generally not homologically well-behaved. In this paper, we will be interested in certain (better behaved) full subcategories in $\mathfrak{X}$–pro.

Example 3.1. Let $\mathfrak{X} = \operatorname{Spi}\mathbb{Z}_p$ be the ind-affine ind-scheme from Example 1.4(2). Then the category $\mathfrak{X}$–pro is equivalent to the category of p-adically separated and complete abelian groups (cf. Section 2.4(2)). Here an abelian group P is said to be *p-adically separated and complete* if its natural map to its p-adic completion $P \longrightarrow \varprojlim_{r\ge0} P/p^rP$ is an isomorphism. The equivalence of categories assigns to every p-adically separated and complete abelian group P the pro-quasi-coherent pro-sheaf $\mathfrak{P}$ with the quasi-coherent sheaf $\mathfrak{P}^{(X_r)}$ corresponding to the $\mathbb{Z}/p^r\mathbb{Z}$-module P/p^rP (for the closed subscheme $X_r = \operatorname{Spec}\mathbb{Z}/p^r\mathbb{Z} \subset \operatorname{Spi}\mathbb{Z}_p = \mathfrak{X}$). Conversely, to every pro-quasi-coherent pro-sheaf $\mathfrak{P}$ on $\operatorname{Spi}\mathbb{Z}_p$, the p-adically separated and complete abelian group $P = \varprojlim_{r\ge0}\mathfrak{P}^{(X_r)}(X_r)$ is assigned.

The category of p-adically separated and complete abelian groups (known also as *separated p-contramodules*) is *not* abelian [49, Example 2.7(1)]. So the category

$(\operatorname{Spi}\mathbb{Z}_p)$–$\mathsf{pro}$ is not abelian. Similarly, the category $(\operatorname{Spi}\Bbbk[[x]])$–$\mathsf{pro}$ (for the ind-affine ind-scheme $\mathfrak{X} = \operatorname{Spi}\Bbbk[[x]]$ from Example 1.5(1)) is *not* abelian, either.

On the other hand, for any ind-scheme $\mathfrak{X}$, the category $\mathfrak{X}$–pro has a natural (associative, commutative, and unital) tensor category structure. The tensor product $\mathfrak{P} \otimes^{\mathfrak{X}} \mathfrak{Q} \in \mathfrak{X}$–$\mathsf{pro}$ of two pro-quasi-coherent pro-sheaves $\mathfrak{P}$ and $\mathfrak{Q} \in \mathfrak{X}$–$\mathsf{pro}$ is defined by the rule $(\mathfrak{P} \otimes^{\mathfrak{X}} \mathfrak{Q})^{(Z)} = \mathfrak{P}^{(Z)} \otimes_{\mathcal{O}_Z} \mathfrak{Q}^{(Z)} \in Z$–$\mathsf{qcoh}$ for all closed subschemes $Z \subset \mathfrak{X}$. As the inverse images of quasi-coherent sheaves preserve their tensor products, the construction of the structure (iso)morphism (ii–iii) for $\mathfrak{P} \otimes^{\mathfrak{X}} \mathfrak{Q}$ is obvious. The unit object of this tensor structure is the "pro-structure pro-sheaf" $\mathfrak{O}_{\mathfrak{X}} \in \mathfrak{X}$–$\mathsf{pro}$, defined by the rule $(\mathfrak{O}_{\mathfrak{X}})^{(Z)} = \mathcal{O}_Z$ for all closed subschemes $Z \subset \mathfrak{X}$.

The aim of the next Section 3.2 is to construct a structure of module category over $\mathfrak{X}$–pro on the category of quasi-coherent torsion sheaves $\mathfrak{X}$–tors.

3.2. **Action of pro-sheaves in torsion sheaves.** Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme represented by an inductive system of closed immersions of reasonable closed subschemes. We start with considering the category $(\mathfrak{X},\Gamma)$–syst of Γ-systems on $\mathfrak{X}$ (as defined in Section 2.7) and constructing a structure of module category over $\mathfrak{X}$–pro on $(\mathfrak{X},\Gamma)$–syst.

Let $\mathfrak{P} \in \mathfrak{X}$–$\mathsf{pro}$ be a pro-quasi-coherent pro-sheaf and $\mathbb{M} \in (\mathfrak{X},\Gamma)$–$\mathsf{syst}$ be a Γ-system on $\mathfrak{X}$. The Γ-system $\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M}$ on $\mathfrak{X}$ is defined by the rule

$$(\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M})_{(\gamma)} = \mathfrak{P}^{(X_\gamma)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)}.$$

The structure morphism $i_{\gamma\delta*}((\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M})_{(\gamma)}) \longrightarrow (\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M})_{(\delta)}$ from Section 2.7, item (ii), is constructed as the composition

$$\begin{aligned} i_{\gamma\delta*}(\mathfrak{P}^{(X_\gamma)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)}) \;\simeq\; & i_{\gamma\delta*}(i_{\gamma\delta}^*\mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)}) \\ & \simeq \mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\delta}} i_{\gamma\delta*}\mathbb{M}_{(\gamma)} \longrightarrow \mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\delta}} \mathbb{M}_{(\delta)} \end{aligned}$$

of the isomorphism $i_{\gamma\delta*}(\mathfrak{P}^{(X_\gamma)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)}) \simeq i_{\gamma\delta*}(i_{\gamma\delta}^*\mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)})$ induced by the structure isomorphism $\mathfrak{P}^{(X_\gamma)} \simeq i_{\gamma\delta}^*\mathfrak{P}^{(X_\delta)}$, the "projection formula" isomorphism $i_{\gamma\delta*}(i_{\gamma\delta}^*\mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\gamma}} \mathbb{M}_{(\gamma)}) \simeq \mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\delta}} i_{\gamma\delta*}\mathbb{M}_{(\gamma)}$, and the morphism $\mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\delta}} i_{\gamma\delta*}\mathbb{M}_{(\gamma)} \longrightarrow \mathfrak{P}^{(X_\delta)} \otimes_{\mathcal{O}_{X_\delta}} \mathbb{M}_{(\delta)}$ induced by the structure morphism $i_{\gamma\delta*}\mathbb{M}_{(\gamma)} \longrightarrow \mathbb{M}_{(\delta)}$.

The tensor product functor

$$\otimes_{\mathfrak{X}} \colon \mathfrak{X}\text{–}\mathsf{pro} \times (\mathfrak{X},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–}\mathsf{syst}$$

endows the category of Γ-systems on $\mathfrak{X}$ with the structure of an associative, unital *module category* over the tensor category of pro-quasi-coherent pro-sheaves $\mathfrak{X}$–pro.

The following lemma plays a key role.

Lemma 3.2. *Let $\mathfrak{P} \in \mathfrak{X}$–$\mathsf{pro}$ be a pro-quasi-coherent prosheaf and $\mathbb{M}$ be a Γ-system on $\mathfrak{X}$ whose associated quasi-coherent torsion sheaf vanishes, $\mathbb{M}^+ = 0$. Then the quasi-coherent torsion sheaf associated with the Γ-system $\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M}$ also vanishes, $(\mathfrak{P} \otimes_{\mathfrak{X}} \mathbb{M})^+ = 0$.*

Proof. The proof is straightforward. $\square$

Recall from the proof of Proposition 2.8 that the functor $\mathbb{M} \longmapsto \mathbb{M}^+\colon (\mathfrak{X}, \Gamma)\text{–}\mathsf{syst} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ represents the category of quasi-coherent torsion sheaves $\mathfrak{X}\text{–}\mathsf{tors}$ as the abelian quotient category of the category of Γ-systems $(\mathfrak{X}, \Gamma)\text{–}\mathsf{syst}$ by the Serre subcategory of all Γ-systems annihilated by this functor. In view of Lemma 3.2, it follows that, for any $\mathfrak{P} \in \mathfrak{X}\text{–}\mathsf{pro}$, the tensor product functor $\mathfrak{P} \otimes_{\mathfrak{X}} -\,\colon (\mathfrak{X}, \Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{X}, \Gamma)\text{–}\mathsf{syst}$ descends uniquely along the functor $\mathbb{M} \longrightarrow \mathbb{M}^+$, leading to a tensor product functor $\mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$, which we denote by the same symbol $\mathfrak{P} \otimes_{\mathfrak{X}} -$.

Explicitly, for any $\mathfrak{P} \in \mathfrak{X}\text{–}\mathsf{pro}$ and $\mathscr{M} \in \mathfrak{X}\text{–}\mathsf{tors}$ we put

$$\mathfrak{P} \otimes_{\mathfrak{X}} \mathscr{M} = (\mathfrak{P} \otimes_{\mathfrak{X}} \mathscr{M}|_{\Gamma})^+ = \varinjlim\nolimits_{\gamma\in\Gamma} i_{\gamma*}(\mathfrak{P}^{(X_\gamma)} \otimes_{\mathcal{O}_{X_\gamma}} \mathscr{M}_{(X_\gamma)}) \in \mathfrak{X}\text{–}\mathsf{tors},$$

where $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ is the closed immersion morphism and $i_{\gamma*}\colon X_\gamma\text{–}\mathsf{qcoh} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is the direct image functor. Clearly, this construction of the tensor product of a pro-quasi-coherent pro-sheaf and a quasi-coherent torsion sheaf does not depend on the choice of a representation of a reasonable ind-scheme $\mathfrak{X}$ by an inductive system $(X_\gamma)_{\gamma\in\Gamma}$ of closed immersions of reasonable closed subschemes.

The resulting tensor product functor

$$\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{pro} \times \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$$

endows the category of quasi-coherent torsion sheaves $\mathfrak{X}\text{–}\mathsf{tors}$ with the structure of an associative, unital module category over the tensor category of pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{–}\mathsf{pro}$.

In the sequel, we will sometimes switch the two arguments of the functor $\otimes_{\mathfrak{X}}$, writing $\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{tors} \times \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$.

3.3. **Inverse and direct images.** Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes. The functor of inverse image of pro-quasi-coherent pro-sheaves $f^*\colon \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{Y}\text{–}\mathsf{pro}$ is defined by the rule $(f^*\mathfrak{P})^{(W)} = g^*(\mathfrak{P}^{(Z)})$ for all $\mathfrak{P} \in \mathfrak{X}\text{–}\mathsf{pro}$, where $W \subset \mathfrak{Y}$ in an arbitrary closed subscheme and $Z \subset \mathfrak{X}$ is a closed subscheme such that the composition $W \longrightarrow \mathfrak{Y} \xrightarrow{f} \mathfrak{X}$ factorizes as $W \xrightarrow{g} Z \longrightarrow \mathfrak{X}$.

The inverse image functor

$$f^*\colon \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{Y}\text{–}\mathsf{pro}$$

is a tensor functor between the tensor categories $\mathfrak{X}\text{–}\mathsf{pro}$ and $\mathfrak{Y}\text{–}\mathsf{pro}$, taking the unit object $\mathfrak{O}_{\mathfrak{X}} \in \mathfrak{X}\text{–}\mathsf{pro}$ to the unit object $\mathfrak{O}_{\mathfrak{Y}} \in \mathfrak{Y}\text{–}\mathsf{pro}$.

In particular, if $Y \subset \mathfrak{X}$ is a closed subscheme with the closed immersion morphism $i\colon Y \longrightarrow \mathfrak{X}$, then one has $i^*\mathfrak{P} = \mathfrak{P}^{(Y)}$ in $Y\text{–}\mathsf{qcoh}$ for every $\mathfrak{P} \in \mathfrak{X}\text{–}\mathsf{pro}$.

Part (a) of the following lemma is a generalization of Lemma 2.3(b) (with the roles of the schemes Y and Z switched).

Lemma 3.3. *Let $f\colon Y \longrightarrow X$ and $h\colon Z \longrightarrow X$ be morphisms of (concentrated) schemes. Consider the pullback diagram*

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{k} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{h} & X \end{array}$$

Assume that either
(a) *the morphism f is affine, or*
(b) *the morphism h is flat.*
Then there is a natural isomorphism $h^*f_* \simeq g_*k^*$ of functors Y–$\mathsf{qcoh} \longrightarrow Z$–$\mathsf{qcoh}$.

Proof. Part (a) is [20, Tag 02KG]. The assertion is local in X and Z, so it reduces to the case of affine schemes, for which it means the following. Let $R \longrightarrow S$ and $R \longrightarrow T$ be homomorphisms of commutative rings. Then, for any S-module N, there is a natural isomorphism of T-modules $T \otimes_R N \simeq (T \otimes_R S) \otimes_S N$. Part (b) is a particular case of [20, Tag 02KH]. □

Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes (as defined in Section 1.3). Let $\mathfrak{Q}$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$. For every closed subscheme $Z \subset \mathfrak{X}$, put $\mathfrak{P}^{(Z)} = f_{Z*}(\mathfrak{Q}^{(W)}) \in Z$–$\mathsf{qcoh}$, where f_Z is the affine morphism of schemes $W = Z \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow Z$. Then it is clear from Lemma 3.3(a) that the collection of quasi-coherent sheaves $\mathfrak{P}^{(Z)}$ with the natural maps $\mathfrak{P}^{(Z'')} \longrightarrow i_{Z'Z''*}\mathfrak{P}^{(Z')}$ for $Z' \subset Z'' \subset \mathfrak{X}$ is a pro-quasi-coherent pro-sheaf $\mathfrak{P}$ on $\mathfrak{X}$.

Put $f_*\mathfrak{Q} = \mathfrak{P}$. This construction defines the functor of direct image of pro-quasi-coherent pro-sheaves $f_*\colon \mathfrak{Y}$–$\mathsf{pro} \longrightarrow \mathfrak{X}$–$\mathsf{pro}$ with respect to an affine morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. The functor f_* is right adjoint to the inverse image functor $f^*\colon \mathfrak{X}$–$\mathsf{pro} \longrightarrow \mathfrak{Y}$–$\mathsf{pro}$ (as one can show similarly to the proof of Lemma 2.10(a)).

Furthermore, for any affine morphism of ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the following projection formula isomorphism holds naturally

$$f_*(f^*\mathfrak{P} \otimes^{\mathfrak{Y}} \mathfrak{Q}) \simeq \mathfrak{P} \otimes^{\mathfrak{X}} f_*\mathfrak{Q} \tag{2}$$

for all $\mathfrak{P} \in \mathfrak{X}$–$\mathsf{pro}$ and $\mathfrak{Q} \in \mathfrak{Y}$–$\mathsf{pro}$.

Lemma 3.4. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of reasonable ind-schemes which is "representable by schemes". Let $\mathfrak{P}$ a pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ and $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$. Then there is a natural isomorphism*

$$f^*(\mathfrak{P} \otimes_{\mathfrak{X}} \mathscr{M}) \simeq f^*\mathfrak{P} \otimes_{\mathfrak{Y}} f^*\mathscr{M}$$

of quasi-coherent torsion sheaves on $\mathfrak{Y}$.

Proof. The isomorphism $f^*(\mathfrak{P}\otimes_{\mathfrak{X}}\mathbb{M})\simeq f^*\mathfrak{P}\otimes_{\mathfrak{Y}} f^*\mathbb{M}$ of Γ-systems on $\mathfrak{Y}$ for any Γ-system $\mathbb{M}$ on $\mathfrak{X}$ is obvious from the definitions. Now we have

$$\begin{aligned} f^*(\mathfrak{P}\otimes_{\mathfrak{X}}\mathscr{M}) = f^*((\mathfrak{P}\otimes_{\mathfrak{X}}\mathscr{M}|_\Gamma)^+) \simeq (f^*(\mathfrak{P}\otimes_{\mathfrak{X}}\mathscr{M}|_\Gamma))^+ \\ \simeq (f^*\mathfrak{P}\otimes_{\mathfrak{Y}} f^*(\mathscr{M}|_\Gamma))^+ \simeq f^*\mathfrak{P}\otimes_{\mathfrak{Y}} (f^*(\mathscr{M}|_\Gamma))^+ \\ \simeq f^*\mathfrak{P}\otimes_{\mathfrak{Y}} f^*((\mathscr{M}|_\Gamma)^+) \simeq f^*\mathfrak{P}\otimes_{\mathfrak{Y}} f^*\mathscr{M} \end{aligned}$$

by the definition of the functors $\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{pro}\times\mathfrak{X}\text{–}\mathsf{tors}\longrightarrow\mathfrak{X}\text{–}\mathsf{tors}$ and $\otimes_{\mathfrak{Y}}\colon \mathfrak{Y}\text{–}\mathsf{pro}\times\mathfrak{Y}\text{–}\mathsf{tors}\longrightarrow\mathfrak{Y}\text{–}\mathsf{tors}$, and by Lemma 2.11. The point is that both the inverse image and the tensor product functors in question commute with the functors $(-)^+$. ☐

For two versions of projection formula related to Lemma 3.4, see Lemmas 7.5 and 8.2 below.

3.4. **Flat pro-quasi-coherent pro-sheaves.** For any scheme X, let us denote by $X\text{–}\mathsf{flat}$ the full subcategory of flat quasi-coherent sheaves in $X\text{–}\mathsf{qcoh}$. Then, for any morphism of schemes $k\colon Z\longrightarrow X$, the inverse images functor $k^*\colon X\text{–}\mathsf{qcoh}\longrightarrow Z\text{–}\mathsf{qcoh}$ takes $X\text{–}\mathsf{flat}$ into $Z\text{–}\mathsf{flat}$. Furthermore, for any short exact sequence $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}\longrightarrow\mathcal{H}\longrightarrow 0$ in $X\text{–}\mathsf{qcoh}$ with $\mathcal{F},\ \mathcal{G},\ \mathcal{H}\in X\text{–}\mathsf{flat}$, the short sequence $0\longrightarrow k^*\mathcal{F}\longrightarrow k^*\mathcal{G}\longrightarrow k^*\mathcal{H}\longrightarrow 0$ is exact in $Z\text{–}\mathsf{qcoh}$.

Let $\mathfrak{X}=\text{“}\varinjlim\text{''}_{\gamma\in\Gamma}X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes. A pro-quasi-coherent pro-sheaf $\mathfrak{F}$ on $\mathfrak{X}$ is said to be *flat* if the quasi-coherent sheaf $\mathfrak{F}^{(Z)}$ on Z is flat for every closed subscheme $Z\subset\mathfrak{X}$. It is clear from the previous paragraph that $\mathfrak{F}$ is flat whenever the quasi-coherent sheaf $\mathfrak{F}^{(X_\gamma)}$ on X_γ is flat for every $\gamma\in\Gamma$. We denote the full subcategory of flat pro-quasi-coherent pro-sheaves by $\mathfrak{X}\text{–}\mathsf{flat}\subset\mathfrak{X}\text{–}\mathsf{pro}$.

Let $0\longrightarrow\mathfrak{F}\longrightarrow\mathfrak{G}\longrightarrow\mathfrak{H}\longrightarrow 0$ be a short sequence of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. We say that this is an (*admissible*) short *exact* sequence in $\mathfrak{X}\text{–}\mathsf{flat}$ if, for every closed subscheme $Z\subset\mathfrak{X}$, the sequence of quasi-coherent sheaves $0\longrightarrow\mathfrak{F}^{(Z)}\longrightarrow\mathfrak{G}^{(Z)}\longrightarrow\mathfrak{H}^{(Z)}\longrightarrow 0$ is exact in the abelian category $Z\text{–}\mathsf{qcoh}$. It suffices to check this condition for the closed subschemes $Z=X_\gamma,\ \gamma\in\Gamma$, belonging to any chosen representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes.

An *exact category* (in the sense of Quillen) is an additive category endowed with a class of *admissible short exact sequences* (also called *conflations*) satisfying natural axioms. For a reference, see [8] or [42, Appendix A].

Proposition 3.5. *The category* $\mathfrak{X}\text{–}\mathsf{flat}$ *of flat pro-quasi-coherent pro-sheaves on* $\mathfrak{X}$*, endowed with the class of admissible short exact sequences defined above, is an exact category in the sense of Quillen.*

Proof. First of all, in the particular case when $\mathfrak{X}=X$ is a scheme, the full subcategory $X\text{–}\mathsf{flat}\subset X\text{–}\mathsf{qcoh}$, endowed with the class of all short sequences that are exact in the abealian category $X\text{–}\mathsf{qcoh}$, is an exact category, since $X\text{–}\mathsf{flat}$ is closed under extensions in $X\text{–}\mathsf{qcoh}$ (see, e. g., [8, Lemma 10.20] or [42, Example A.5(3)(a)]).

Moreover, the restriction functor $k^*\colon X\text{–}\mathsf{flat} \longrightarrow Z\text{–}\mathsf{flat}$ is exact (i. e., takes admissible short exact sequences to admissible short exact sequences) for any morphism of schemes $k\colon Z \longrightarrow X$.

To prove the assertion for an ind-scheme $\mathfrak{X}$, consider the category of "generalized pro-quasi-coherent pro-sheaves" on $\mathfrak{X}$, defined by items (i–iii) of Section 3.1 (with the transitivity condition (iv) dropped). The definitions of a flat generalized pro-quasi-coherent pro-sheaf and a short exact sequence of flat generalized pro-quasi-coherent pro-sheaves are similar to the above. Then the category of flat generalized pro-quasi-coherent pro-sheaves is exact, since it can be obtained by the construction of [42, Example A.5(4)] applied to the following pair of exact functors

$$\prod\nolimits_{Z\subset\mathfrak{X}} Z\text{–}\mathsf{flat} \longrightarrow \prod\nolimits_{Z\subset Y\subset\mathfrak{X}} Z\text{–}\mathsf{flat} \times \prod\nolimits_{Z\subset Y\subset\mathfrak{X}} Z\text{–}\mathsf{flat} \longleftarrow \prod\nolimits_{Z\subset Y\subset\mathfrak{X}} Z\text{–}\mathsf{flat}.$$

Here Z and Y range over the closed subschemes in $\mathfrak{X}$. All the categories in the diagram are exact, with termwise exact structures on the Cartesian products. The first component $\prod_{Z\subset\mathfrak{X}} Z\text{–}\mathsf{flat} \longrightarrow \prod_{Z\subset Y\subset\mathfrak{X}} Z\text{–}\mathsf{flat}$ of the leftmost functor assigns to a collection of flat quasi-coherent sheaves $(\mathcal{F}_Z)_{Z\subset\mathfrak{X}}$ the collection of flat quasi-coherent sheaves $(\mathcal{F}_Z)_{Z\subset Y\subset\mathfrak{X}}$. The second component $\prod_{Z\subset\mathfrak{X}} Z\text{–}\mathsf{flat} \longrightarrow \prod_{Z\subset Y\subset\mathfrak{X}} Z\text{–}\mathsf{flat}$ of the leftmost functor assigns to a collection $(\mathcal{F}_Z)_{Z\subset\mathfrak{X}}$ the collection of flat quasi-coherent sheaves $(i_{ZY}^*\mathcal{F}_Y)_{Z\subset Y\in\mathfrak{X}}$. The rightmost functor is the diagonal one.

Finally, the full subcategory of flat pro-quasi-coherent pro-sheaves is closed under admissible subobjects and admissible quotients in the ambient exact category of flat generalized pro-quasi-coherent pro-sheaves. So this full subcategory inherits an exact category structure by [42, Example A.5(3)(b)]). ☐

Let $f\colon\mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes. Then the inverse image functor $f^*\colon \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{Y}\text{–}\mathsf{pro}$ takes the full subcategory $\mathfrak{X}\text{–}\mathsf{flat} \subset \mathfrak{X}\text{–}\mathsf{pro}$ into the full subcategory $\mathfrak{Y}\text{–}\mathsf{flat} \subset \mathfrak{Y}\text{–}\mathsf{pro}$. The functor

$$f^*\colon \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{Y}\text{–}\mathsf{flat}$$

is exact with respect to the exact category structures of $\mathfrak{X}\text{–}\mathsf{flat}$ and $\mathfrak{Y}\text{–}\mathsf{flat}$ (defined above).

Let $f\colon\mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat, affine morphism of ind-schemes (as defined in Section 1.3). Then the direct image functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$ takes the full subcategory $\mathfrak{Y}\text{–}\mathsf{flat} \subset \mathfrak{Y}\text{–}\mathsf{pro}$ into the full subcategory $\mathfrak{X}\text{–}\mathsf{flat} \subset \mathfrak{X}\text{–}\mathsf{pro}$. The functor

$$f_*\colon \mathfrak{Y}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat}$$

is exact with respect to the exact category structures on $\mathfrak{Y}\text{–}\mathsf{flat}$ and $\mathfrak{X}\text{–}\mathsf{flat}$.

The tensor product of two flat pro-quasi-coherent pro-sheaves is flat. So the tensor product functor $\otimes^{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{pro} \times \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$ restricts to a functor

$$\otimes^{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{flat} \times \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat},$$

defining a tensor category structure on $\mathfrak{X}\text{–}\mathsf{flat}$. Notice that the "pro-structure pro-sheaf" $\mathfrak{O}_{\mathfrak{X}}$ (which is the unit of the tensor structure on $\mathfrak{X}\text{–}\mathsf{pro}$) belongs to $\mathfrak{X}\text{–}\mathsf{flat}$. The tensor product $\otimes^{\mathfrak{X}}$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ is an exact functor with respect to the exact category structure of $\mathfrak{X}\text{–}\mathsf{flat}$.

Assume that the ind-scheme $\mathfrak{X}$ is reasonable. Then, restricting the tensor product functor $\mathfrak{X}\text{–pro}\times\mathfrak{X}\text{–tors}\longrightarrow\mathfrak{X}\text{–tors}$ to flat pro-quasi-coherent pro-sheaves, one obtains a tensor product functor

$$\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–flat}\times\mathfrak{X}\text{–tors}\longrightarrow\mathfrak{X}\text{–tors},$$

which is exact with respect to the exact category structure of $\mathfrak{X}$–flat and the abelian exact structure on $\mathfrak{X}$–tors.

Lemma 3.6. *Let $i\colon Z\longrightarrow X$ be a closed immersion of schemes and $\mathcal{F}$, $\mathcal{M}$ be quasi-coherent sheaves on X. Then there is a natural morphism of quasi-coherent sheaves on Z*

$$i^*\mathcal{F}\otimes_{\mathcal{O}_Z} i^!\mathcal{M}\longrightarrow i^!(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M}),$$

which is an isomorphism whenever $i(Z)$ is a reasonable closed subscheme in X and $\mathcal{F}$ is a flat quasi-coherent sheaf on X.

Proof. When $i(Z)$ is a reasonable closed subscheme in X, both the assertions are local in X. So they reduce to the case of affine schemes, for which they mean the following. Let $R\longrightarrow S$ be a surjective homomorphism of commutative rings and F, M be R-modules. Then there is a natural homomorphism of S-modules

$$(S\otimes_R F)\otimes_S \operatorname{Hom}_R(S,M)=F\otimes_R\operatorname{Hom}_R(S,M)\longrightarrow\operatorname{Hom}_R(S,\,F\otimes_R M)$$

which is an isomorphism whenever S is a finitely presented R-module and F is a flat R-module.

To construct the desired morphism in the general case, let $\mathcal{L}$ be an arbitrary quasi-coherent sheaf on Z, and let $\mathcal{L}\longrightarrow i^*\mathcal{F}\otimes_{\mathcal{O}_Z}i^!\mathcal{M}$ be a morphism in Z–qcoh. Applying i_*, we produce a morphism $i_*\mathcal{L}\longrightarrow i_*(i^*\mathcal{F}\otimes_{\mathcal{O}_Z}i^!\mathcal{M})\simeq\mathcal{F}\otimes_{\mathcal{O}_X}i_*i^!\mathcal{M}$ in X–qcoh, where the isomorphism holds by Lemma 2.2. Composing with the morphism $\mathcal{F}\otimes_{\mathcal{O}_X}i_*i^!\mathcal{M}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M}$ induced by the adjunction morphism $i_*i^!\mathcal{M}\longrightarrow\mathcal{M}$, we obtain a morphism $i_*\mathcal{L}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M}$ in X–qcoh, which corresponds by adjunction to a morphism $\mathcal{L}\longrightarrow i^!(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{M})$ in Z–qcoh. ☐

Proposition 3.7. *Let $\mathfrak{X}$ be a reasonable ind-scheme, and let $Z\subset\mathfrak{X}$ be a reasonable closed subscheme with the closed immersion morphism $i\colon Z\longrightarrow X$. Let $\mathfrak{F}$ be flat pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ and $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$. Then there is a natural isomorphism $i^!(\mathfrak{F}\otimes_{\mathfrak{X}}\mathscr{M})\simeq i^*\mathfrak{F}\otimes_{\mathcal{O}_Z}i^!\mathscr{M}=\mathfrak{F}^{(Z)}\otimes_{\mathcal{O}_Z}\mathscr{M}_{(Z)}$ in Z–qcoh.*

Proof. It follows from Lemma 3.6 that the rule $\mathscr{N}_{(Z)}=\mathfrak{F}^{(Z)}\otimes\mathscr{M}_{(Z)}$ defines a quasi-coherent torsion sheaf $\mathscr{N}$ on $\mathfrak{X}$. Now it is clear that $\mathfrak{F}\otimes_{\mathfrak{X}}\mathscr{M}\simeq\mathscr{N}$. ☐

Examples 3.8. (1) This is a generalization of Example 3.1. Let $\mathfrak{R}$ be complete, separated topological commutative ring with a countable base of neighborhoods of zero consisting of open ideals, and let $\mathfrak{X}=\operatorname{Spi}\mathfrak{R}$ be the related ind-affine $\aleph_0$-ind-scheme, as in Example 1.6(2). Then the category $\mathfrak{X}$–pro is equivalent to the category of *separated $\mathfrak{R}$-contramodules*, as defined in [43, Section 1.2], [53, Sections 1.2 and 5], or [54, Section 6.2], and discussed in [44, Section D.1]. The category of separated $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{separ}$ is a full subcategory in the abelian category of $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$; so we have $\operatorname{Spi}\mathfrak{R}\text{–}\mathsf{pro} \simeq \mathfrak{R}\text{–}\mathsf{separ} \subset \mathfrak{R}\text{–}\mathsf{contra}$.

The equivalence assigns to a (separated) $\mathfrak{R}$-contramodule $\mathfrak{C}$ the pro-quasi-coherent pro-sheaf $\mathfrak{P}$ with the components $\mathfrak{P}^{(X_\mathfrak{I})} = \mathfrak{C}/(\mathfrak{I} \leftthreetimes \mathfrak{C})$, and conversely, to a pro-quasi-coherent pro-sheaf $\mathfrak{P}$ the separated $\mathfrak{R}$-contramodule $\mathfrak{C} = \varprojlim_{\mathfrak{I}\subset\mathfrak{R}} \mathfrak{P}^{(X_\mathfrak{I})}$ is assigned (where $\mathfrak{I}$ ranges over the open ideals in $\mathfrak{R}$, and the projective limit is taken in the category of $\mathfrak{R}$-contramodules, which agrees with the projective limit in the category of abelian groups). It is clear from [44, Lemma D.1.3] (see also [53, Lemma 6.3(a,c)]) that this is an equivalence of categories.

The generality level in [44, Section D.1] is that of complete, separated topological associative rings with a countable base of neighborhoods of zero consisting of open two-sided ideals. In [53, Section 6], only a (countable) base of open *right* ideals is assumed, which makes the exposition more complicated.

(2) The equivalence of categories described in (1) restricts to an equivalence between their full subcategories of flat pro-quasi-coherent pro-sheaves and *flat* $\mathfrak{R}$-*contramodules*. Notice that any flat $\mathfrak{R}$-contramodule is separated [44, Corollary D.1.7] (see also [53, Corollary 6.15]). The full subcategory of flat contramodules $\mathfrak{R}\text{–}\mathsf{flat}$ (*unlike* the larger full subcategory of separated ones) is closed under extensions in the abelian category of $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$ [44, Lemma D.1.5] (see also [53, Corollary 6.13 or Corollary 7.1(b)]), so it inherits an exact category structure from the abelian exact category structure on $\mathfrak{R}\text{–}\mathsf{contra}$. The equivalence $\operatorname{Spi}\mathfrak{R}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{flat}$ is an equivalence of exact categories [44, Lemma D.1.4] (see also [53, Lemma 6.7 or 6.10]).

(3) In the context of (1), assume that $\mathfrak{R}$ is a reasonable topological ring in the sense of Section 2.4(5). Then, according to Section 2.4(6), the category of quasi-coherent torsion sheaves $\operatorname{Spi}\mathfrak{R}\text{–}\mathsf{tors}$ is equivalent to the category of discrete $\mathfrak{R}$-modules $\mathfrak{R}\text{–}\mathsf{discr}$. This equivalence of categories, together with the equivalence $\operatorname{Spi}\mathfrak{R}\text{–}\mathsf{pro} \simeq \mathfrak{R}\text{–}\mathsf{separ}$ from (1), transforms the tensor product functor $\otimes_\mathfrak{X}\colon \operatorname{Spi}\mathfrak{R}\text{–}\mathsf{pro} \times \operatorname{Spi}\mathfrak{R}\text{–}\mathsf{tors} \longrightarrow \operatorname{Spi}\mathfrak{R}\text{–}\mathsf{tors}$ into the functor of *contratensor product* $\odot_\mathfrak{R}\colon \mathfrak{R}\text{–}\mathsf{contra} \times \mathfrak{R}\text{–}\mathsf{discr} \longrightarrow \mathfrak{R}\text{–}\mathsf{discr}$ (restricted to separated $\mathfrak{R}$-contramodules).

We refer to [44, Section D.2], [53, Definition 5.4], or [54, Section 7.2] for the definition of the contratensor product of a discrete module and a contramodule over a topological ring. In our context, the contratensor product takes values in discrete $\mathfrak{R}$-modules (rather than just abelian groups) because the ring $\mathfrak{R}$ is commutative.

(4) The *contramodule tensor product* functor $\otimes^\mathfrak{R}\colon \mathfrak{R}\text{–}\mathsf{contra} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ on the category of contramodules over a commutative topological ring $\mathfrak{R}$ was defined in [43, Section 1.6]. According to [44, Lemma D.3.1] (which presumes a countable base of neighborhoods of zero in $\mathfrak{R}$), this functor restricts to an exact functor $\otimes^\mathfrak{R}\colon \mathfrak{R}\text{–}\mathsf{flat} \times \mathfrak{R}\text{–}\mathsf{flat} \longrightarrow \mathfrak{R}\text{–}\mathsf{flat}$, which agrees with the functor of tensor product of flat pro-quasi-coherent pro-sheaves $\otimes^\mathfrak{X}\colon \mathfrak{X}\text{–}\mathsf{flat} \times \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat}$ under the equivalence of (exact) categories $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{flat}$.

3.5. **Coproducts and colimits.** Let $\mathfrak{X}$ be an ind-scheme. The coproduct of any family of objects exists in the category $\mathfrak{X}\text{–}\mathsf{pro}$. Specifically, let $(\mathfrak{P}_\theta)_{\theta\in\Theta}$ be a family of pro-quasi-coherent pro-sheaves on X, indexed by a set Θ. For every closed subscheme $Z\subset\mathfrak{X}$, put $\mathfrak{Q}^{(Z)}=\bigoplus_{\theta\in\Theta}\mathfrak{P}_\theta^{(Z)}$ (where the direct sum is taken in the category of quasi-coherent sheaves on Z). Then the collection of quasi-coherent sheaves $\mathfrak{Q}^{(Z)}$ with the obvious isomorphisms $i_{ZY}^*\mathfrak{Q}^{(Y)}\simeq\mathfrak{Q}^{(Z)}$ for $Z\subset Y\subset\mathfrak{X}$ is a pro-quasi-coherent pro-sheaf $\mathfrak{Q}$ on $\mathfrak{X}$. One has $\mathfrak{Q}=\coprod_{\theta\in\Theta}\mathfrak{P}_\theta$ in the category $\mathfrak{X}\text{–}\mathsf{pro}$.

More generally, the colimit of any diagram of objects, indexed by a small category, exists in $\mathfrak{X}\text{–}\mathsf{pro}$. Let $(\mathfrak{P}_\theta)_{\theta\in\Theta}$ be an inductive system of pro-quasi-coherent pro-sheaves on X, indexed by a small category Θ. For every closed subscheme $Z\subset\mathfrak{X}$, put $\mathfrak{Q}^{(Z)}=\varinjlim_{\theta\in\Theta}\mathfrak{P}_\theta^{(Z)}$ (where the colimit is taken in the category of quasi-coherent sheaves $Z\text{–}\mathsf{qcoh}$). Notice that the inverse image functor with respect to any morphism of schemes preserves all colimits of quasi-coherent sheaves. So the collection of quasi-coherent sheaves $\mathfrak{Q}^{(Z)}$ with the induced isomorphisms $i_{ZY}^*\mathfrak{Q}^{(Y)}\simeq\mathfrak{Q}^{(Z)}$ for $Z\subset Y\subset\mathfrak{X}$ is a pro-quasi-coherent pro-sheaf $\mathfrak{Q}$ on $\mathfrak{X}$. One has $\mathfrak{Q}=\varinjlim_{\theta\in\Theta}\mathfrak{P}_\theta$ in $\mathfrak{X}\text{–}\mathsf{pro}$.

Both the tensor product functor $\otimes^{\mathfrak{X}}\colon\mathfrak{X}\text{–}\mathsf{pro}\times\mathfrak{X}\text{–}\mathsf{pro}\longrightarrow\mathfrak{X}\text{–}\mathsf{pro}$ (on any ind-scheme $\mathfrak{X}$) and the action functor $\otimes_{\mathfrak{X}}\colon\mathfrak{X}\text{–}\mathsf{pro}\times\mathfrak{X}\text{–}\mathsf{tors}\longrightarrow\mathfrak{X}\text{–}\mathsf{tors}$ (on a reasonable ind-scheme $\mathfrak{X}$) preserve coproducts in both the categories. In fact, they even preserve all colimits (as the action functor for Γ-systems $\otimes_{\mathfrak{X}}\colon\mathfrak{X}\text{–}\mathsf{pro}\times(\mathfrak{X},\Gamma)\text{–}\mathsf{syst}\longrightarrow(\mathfrak{X},\Gamma)\text{–}\mathsf{syst}$ can be easily seen to preserve colimits).

The full subcategory $\mathfrak{X}\text{–}\mathsf{flat}$ is closed under coproducts in $\mathfrak{X}\text{–}\mathsf{pro}$. In fact, it is closed under all direct limits (of inductive systems indexed by directed posets), because flatness of quasi-coherent sheaves is preserved by such direct limits. So all coproducts and direct limits exist in $\mathfrak{X}\text{–}\mathsf{flat}$. The direct limit (in particular, coproduct) functors are exact in the exact category $\mathfrak{X}\text{–}\mathsf{flat}$.

The inverse image functor $f^*\colon\mathfrak{X}\text{–}\mathsf{pro}\longrightarrow\mathfrak{Y}\text{–}\mathsf{pro}$ with respect to any morphism of ind-schemes $f\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ preserves all colimits. So does the direct image functor $f_*\colon\mathfrak{Y}\text{–}\mathsf{pro}\longrightarrow\mathfrak{X}\text{–}\mathsf{pro}$ with respect to an affine morphism of ind-schemes f. In particular, both the functors preserve coproducts.

3.6. **Pro-quasi-coherent commutative algebras.** Let X be a scheme. A *quasi-coherent sheaf of algebras* (or a *quasi-coherent algebra* for brevity) $\mathcal{A}$ over X is an (associative and unital) algebra object in the tensor category $X\text{–}\mathsf{qcoh}$. In other words, $\mathcal{A}$ is a quasi-coherent sheaf on X endowed with morphisms of quasi-coherent sheaves $\mathcal{O}_X\longrightarrow\mathcal{A}$ and $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{A}\longrightarrow\mathcal{A}$ satisfying the usual associativity and unitality equations. The underlying sheaf of abelian groups of a quasi-coherent algebra has a natural structure of a sheaf of rings.

Let $f\colon Y\longrightarrow X$ be a morphism of (concentrated) schemes. Then the inverse image $f^*\colon X\text{–}\mathsf{qcoh}\longrightarrow Y\text{–}\mathsf{qcoh}$ is a tensor functor, so the inverse image $f^*\mathcal{A}$ of any quasi-coherent sheaf of algebras $\mathcal{A}$ on X is a quasi-coherent sheaf of algebras on Y.

Furthermore, for any quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{N}$ on Y, there is a natural morphism of quasi-coherent sheaves $f_*\mathcal{M}\otimes_{\mathcal{O}_X}f_*\mathcal{N}\longrightarrow f_*(\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N})$ on X, corresponding by adjunction to the morphism of quasi-coherent sheaves $f^*(f_*\mathcal{M}\otimes_{\mathcal{O}_X}f_*\mathcal{N})\simeq$

$f^*f_*\mathcal{M} \otimes_{\mathcal{O}_Y} f^*f_*\mathcal{N} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on Y induced by the adjunction morphisms $f^*f_*\mathcal{M} \longrightarrow \mathcal{M}$ and $f^*f_*\mathcal{N} \longrightarrow \mathcal{N}$. In particular, for any quasi-coherent sheaf of algebras $\mathcal{B}$ on Y, the composition $f_*\mathcal{B} \otimes_{\mathcal{O}_X} f_*\mathcal{B} \longrightarrow f_*(\mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{B}) \longrightarrow f_*\mathcal{B}$ endows the direct image $f_*\mathcal{B}$ of the quasi-coherent sheaf $\mathcal{B}$ with the structure of a quasi-coherent sheaf of algebras on X.

A *quasi-coherent commutative algebra* (or a *quasi-coherent sheaf of commutative algebras*) on X is a commutative (associative, and unital) algebra object in X–qcoh. Clearly, the inverse and direct images preserve commutativity of quasi-coherent algebras. The next lemma is quite standard and well-known.

Lemma 3.9. *For any scheme X, there is a natural anti-equivalence between the category of schemes Y endowed with an affine morphism $Y \longrightarrow X$ and the category of quasi-coherent commutative algebras on X. To an affine morphism $f\colon Y \longrightarrow X$, the quasi-coherent commutative algebra $f_*\mathcal{O}_Y$ on X is assigned.*

Proof. The inverse functor assigns to a quasi-coherent sheaf of algebras $\mathcal{A}$ on X the scheme Y covered by the following affine schemes. For any affine open subscheme $U \subset X$, the affine scheme $\operatorname{Spec}\mathcal{A}(U)$ is an open subscheme in Y; in fact, one has $\operatorname{Spec}\mathcal{A}(U) = U \times_X Y$, so $\operatorname{Spec}\mathcal{A}(U)$ is the preimage of U in Y under f. The map of the rings of sections $\mathcal{O}_X(U) \longrightarrow \mathcal{A}(U)$ induced by the unit morphism $\mathcal{O}_X \longrightarrow \mathcal{A}$ of the quasi-coherent sheaf of algebras $\mathcal{A}$ induces the morphism of schemes $\operatorname{Spec}\mathcal{A}(U) \longrightarrow U$. For any pair of affine open subschemes U and V in X such that $V \subset U$, the restriction map of the rings of sections $\mathcal{A}(U) \longrightarrow \mathcal{A}(V)$ of the sheaf of rings $\mathcal{A}$ on X induces an open immersion $\operatorname{Spec}\mathcal{A}(V) \longrightarrow \operatorname{Spec}\mathcal{A}(U)$; this is the map $V \times_X Y \longrightarrow U \times_X Y$ induced by the open immersion $V \longrightarrow U$. The key observation is that $V \times_U \operatorname{Spec}\mathcal{A}(U) \simeq \operatorname{Spec}\mathcal{A}(V)$ (as it should be), since $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{A}(U) \simeq \mathcal{A}(V)$. We leave further details to the reader. □

We will denote the affine scheme over X corresponding to a quasi-coherent algebra $\mathcal{A}$ under the construction of Lemma 3.9 by $Y = \operatorname{Spec}_X \mathcal{A}$. Notice that an affine morphism of schemes $f\colon Y \longrightarrow X$ is flat if and only if the quasi-coherent sheaf $\mathcal{A} = f_*\mathcal{O}_Y$ on X is flat.

Lemma 3.10. *Let $h\colon Z \longrightarrow X$ be a morphism of schemes and $\mathcal{A}$ be a quasi-coherent commutative algebra over X. Then there is a natural isomorphism of schemes $\operatorname{Spec}_Z h^*\mathcal{A} \simeq Z \times_X \operatorname{Spec}_X \mathcal{A}$. In other words, there is a natural pullback diagram of schemes*

$$\begin{array}{ccc} \operatorname{Spec}_Z h^*\mathcal{A} & \xrightarrow{\quad k \quad} & \operatorname{Spec}_X \mathcal{A} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{\quad h \quad} & X \end{array}$$

where f and g are the structure morphisms of the schemes $\operatorname{Spec}_X \mathcal{A}$ and $\operatorname{Spec}_Z h^*\mathcal{A}$ affine over the schemes X and Z, respectively.

Proof. The question is essentially local in X and Z, so it reduces to the case of affine schemes, for which it is obvious from the constructions. □

Let $\mathcal{A}$ be a quasi-coherent algebra on a scheme X. Then a *quasi-coherent module* (or a *quasi-coherent sheaf of modules*) over $\mathcal{A}$ is a module object over the algebra object $\mathcal{A}$ in the tensor category $X\text{–}\mathsf{qcoh}$. In other words, a quasi-coherent module $\mathcal{M}$ over $\mathcal{A}$ is a quasi-coherent sheaf $\mathcal{M} \in X\text{–}\mathsf{qcoh}$ endowed with a morphism of quasi-coherent sheaves $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{M}$ satisfying the usual associativity and unitality equations. The underlying sheaf of abelian groups of a quasi-coherent module $\mathcal{M}$ over $\mathcal{A}$ is a sheaf of modules over the underlying sheaf of rings of $\mathcal{A}$.

Let $f\colon Y \longrightarrow X$ be a morphism of schemes. Then, for any quasi-coherent module $\mathcal{M}$ over a quasi-coherent algebra $\mathcal{A}$ on X, the inverse image $f^*\mathcal{M}$ is a quasi-coherent module over the quasi-coherent algebra $f^*\mathcal{A}$ on Y. For any quasi-coherent module $\mathcal{N}$ over a quasi-coherent algebra $\mathcal{B}$ on Y, the direct image $f_*\mathcal{N}$ is a quasi-coherent module over the quasi-coherent algebra $f_*\mathcal{B}$ on X. The constructions are similar to the above constructions for algebras.

Furthermore, let $i\colon Z \longrightarrow X$ be a closed immersion of schemes, and let $\mathcal{M}$ be a quasi-coherent module over a quasi-coherent algebra $\mathcal{A}$ on X. Then the composition $i^*\mathcal{A} \otimes_{\mathcal{O}_Z} i^!\mathcal{M} \longrightarrow i^!(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M}) \longrightarrow i^!\mathcal{M}$ of the morphism $i^*\mathcal{A} \otimes_{\mathcal{O}_Z} i^!\mathcal{M} \longrightarrow i^!(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M})$ from Lemma 3.6 and the morphism $i^!(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M}) \longrightarrow i^!\mathcal{M}$ induced by the structure morphism $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \mathcal{M}$ endows the quasi-coherent sheaf $i^!\mathcal{M}$ with the structure of a quasi-coherent module over the quasi-coherent algebra $i^*\mathcal{A}$ on Z.

Lemma 3.11. *Let $f\colon Y \longrightarrow X$ be an affine morphism of schemes. Then the category of quasi-coherent sheaves on Y is equivalent to the category of quasi-coherent modules over the quasi-coherent algebra $f_*\mathcal{O}_Y$ on X. To a quasi-coherent sheaf $\mathcal{N}$ on Y, the quasi-coherent $f_*\mathcal{O}_Y$-module $f_*\mathcal{N} \in X\text{–}\mathsf{qcoh}$ is assigned.*

Proof. Put $\mathcal{A} = f_*\mathcal{O}_Y$. The inverse functor assigns to a quasi-coherent $\mathcal{A}$-module $\mathcal{M}$ on X the following quasi-coherent sheaf $\mathcal{N}$ on Y. For any affine open subscheme $U \subset X$, the restriction of $\mathcal{N}$ to the affine open subscheme $U \times_X Y = \operatorname{Spec} \mathcal{A}(U)$ in Y is the quasi-coherent sheaf on $\operatorname{Spec} \mathcal{A}(U)$ corresponding to the $\mathcal{A}(U)$-module $\mathcal{M}(U)$. We omit further details. □

Let $\mathfrak{X}$ be an ind-scheme. A *pro-quasi-coherent pro-sheaf of algebras* (or a *pro-quasi-coherent algebra*) $\mathfrak{A}$ over $\mathfrak{X}$ is algebra object in the tensor category $\mathfrak{X}\text{–}\mathsf{pro}$. In other words, $\mathfrak{A}$ is a pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ endowed with morphisms of pro-quasi-coherent pro-sheaves $\mathfrak{O}_{\mathfrak{X}} \longrightarrow \mathfrak{A}$ and $\mathfrak{A} \otimes_{\mathfrak{X}} \mathfrak{A} \longrightarrow \mathfrak{A}$ satisfying the usual (associativity and unitality, and if mentioned, commutativity) equations.

It is clear from the construction of the tensor product of pro-quasi-coherent pro-sheaves in Section 3.1 that the datum of a pro-quasi-coherent algebra $\mathfrak{A}$ on $\mathfrak{X}$ is equivalent to the datum of a quasi-coherent algebra $\mathfrak{A}^{(Z)}$ on every closed subscheme $Z \subset \mathfrak{X}$ together with a compatible system of isomorphisms of quasi-coherent algebras $\mathfrak{A}^{(Z')} \simeq i_{Z'Z''}^*\mathfrak{A}^{(Z'')}$ for every pair of closed subschemes $Z' \subset Z'' \subset \mathfrak{X}$ with the closed immersion $i_{Z'Z''}\colon Z'' \longrightarrow Z''$.

Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes and $\mathfrak{A}$ be a pro-quasi-coherent algebra on $\mathfrak{X}$. Then the inverse image $f^*\colon \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{Y}\text{–}\mathsf{pro}$ is a tensor functor, hence $f^*\mathfrak{A}$ is a pro-quasi-coherent algebra on $\mathfrak{Y}$.

Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes, and let $\mathfrak{B}$ be a pro-quasi-coherent algebra on $\mathfrak{Y}$. Then an adjunction argument similar to the above one for schemes shows how to construct an algebra structure on the pro-quasi-coherent pro-sheaf $f_*\mathfrak{B}$ on $\mathfrak{X}$.

Proposition 3.12. *For any ind-scheme $\mathfrak{X}$, there is a natural anti-equivalence between the category of ind-schemes $\mathfrak{Y}$ endowed with an affine morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ and the category of pro-quasi-coherent commutative algebras on $\mathfrak{X}$. To an affine morphism $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$, the pro-quasi-coherent commutative algebra $f_*\mathfrak{O}_{\mathfrak{Y}}$ on $\mathfrak{X}$ is assigned.*

Proof. Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be an inductive system of closed immersions of schemes representing $\mathfrak{X}$. The inverse functor to the one mentioned in the proposition assigns to a pro-quasi-coherent algebra $\mathfrak{A}$ on $\mathfrak{X}$ the ind-scheme $\mathfrak{Y}$ represented by the inductive system of closed immersions of schemes $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} Y_\gamma$, where $Y_\gamma = \operatorname{Spec}_{X_\gamma} \mathfrak{A}^{(X_\gamma)}$ (see Lemma 3.9). The transition morphisms $Y_\gamma \longrightarrow Y_\delta$ for $\gamma < \delta \in \Gamma$ are provided by Lemma 3.10, and it is also clear from Lemma 3.10 that the morphism of ind-schemes $\mathfrak{Y} \longrightarrow \mathfrak{X}$ constructed in this way is affine. □

We will denote the affine ind-scheme over $\mathfrak{X}$ corresponding to a pro-quasi-coherent algebra $\mathfrak{A}$ under the construction of Proposition 3.12 by $\mathfrak{Y} = \operatorname{Spi}_{\mathfrak{X}} \mathfrak{A}$. Notice that an affine morphism of ind-schemes $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is flat if and only the pro-quasi-coherent pro-sheaf $\mathfrak{A} = f_*\mathfrak{O}_Y$ on $\mathfrak{X}$ is flat.

Let $\mathfrak{A}$ be a pro-quasi-coherent algebra over an ind-scheme $\mathfrak{X}$. Then a *pro-quasi-coherent module* (or a *pro-quasi-coherent pro-sheaf of modules*) over $\mathfrak{A}$ is a module over the algebra object $\mathfrak{A}$ in the tensor category $\mathfrak{X}$–pro. In other words, a pro-quasi-coherent module $\mathfrak{M}$ over $\mathfrak{A}$ is a pro-quasi-coherent pro-sheaf $\mathfrak{M} \in \mathfrak{X}$–pro endowed with a morphism of pro-quasi-coherent pro-sheaves $\mathfrak{A} \otimes^{\mathfrak{X}} \mathfrak{M} \longrightarrow \mathfrak{M}$ satisfying the usual associativity and unitality equations.

The datum of a pro-quasi-coherent module $\mathfrak{M}$ over a given pro-quasi-coherent algebra $\mathfrak{A}$ on $\mathfrak{X}$ is equivalent to the datum of a quasi-coherent module $\mathfrak{M}^{(Z)}$ over the quasi-coherent algebra $\mathfrak{A}^{(Z)}$ on every closed subscheme $Z \subset \mathfrak{X}$ together with a compatible system of isomorphisms of quasi-coherent modules $\mathfrak{M}^{(Z')} \simeq i^*_{Z'Z''}\mathfrak{M}^{(Z'')}$ for every pair of closed subschemes $Z' \subset Z''$.

Let $\mathfrak{X}$ be a reasonable ind-scheme and $\mathfrak{A}$ be a pro-quasi-coherent algebra over $\mathfrak{X}$. Then a *quasi-coherent torsion module* (or a *quasi-coherent torsion sheaf of modules*) over $\mathfrak{A}$ is a module object in the module category $\mathfrak{X}$–tors over the algebra object $\mathfrak{A}$ in the tensor category $\mathfrak{X}$–pro. In other words, a quasi-coherent torsion module $\mathscr{M}$ over $\mathfrak{A}$ is a quasi-coherent torsion sheaf $\mathscr{M} \in \mathfrak{X}$–tors endowed with a morphism of quasi-coherent torsion sheaves $\mathfrak{A} \otimes_{\mathfrak{X}} \mathscr{M} \longrightarrow \mathscr{M}$ satisfying the usual associativity and unitality equations.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of reasonable closed subschemes. Then, for any quasi-coherent sheaf $\mathscr{M}$

on $\mathfrak{X}$, one has

$$\operatorname{Hom}_{\mathfrak{X}\text{–tors}}(\mathfrak{A}\otimes_{\mathfrak{X}}\mathscr{M},\ \mathscr{M})\simeq\operatorname{Hom}_{\mathfrak{X}\text{–tors}}((\mathfrak{A}\otimes_{\mathfrak{X}}\mathscr{M}|_\Gamma)^+,\ \mathscr{M})$$
$$\simeq\operatorname{Hom}_{(\mathfrak{X},\Gamma)\text{–syst}}(\mathfrak{A}\otimes_{\mathfrak{X}}\mathscr{M}|_\Gamma,\ \mathscr{M}|_\Gamma).$$

Hence the datum of a quasi-coherent torsion module $\mathscr{M}$ over a given quasi-coherent algebra $\mathfrak{A}$ on $\mathfrak{X}$ is equivalent to the datum of a quasi-coherent module $\mathscr{M}_{(X_\gamma)}$ over the quasi-coherent algebra $\mathfrak{A}^{(X_\gamma)}$ on the scheme X_γ for every $\gamma\in\Gamma$, together with a compatible system of quasi-isomorphisms of quasi-coherent modules $\mathscr{M}_{(X_\gamma)}\simeq i_{\gamma\delta}^!\mathscr{M}_{(X_\delta)}$ for every $\gamma<\delta\in\Gamma$ (where $i_{\gamma\delta}\colon X_\gamma\longrightarrow X_\delta$ is the closed immersion).

Let $f\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be a morphism of ind-schemes. Let $\mathfrak{A}$ be a pro-quasi-coherent algebra on $\mathfrak{X}$ and $\mathfrak{M}$ be a pro-quasi-coherent module over $\mathfrak{A}$. Then the inverse image $f^*\mathfrak{M}$ is a pro-quasi-coherent module over the pro-quasi-coherent algebra $f^*\mathfrak{M}$ on $\mathfrak{Y}$. Furthermore, assume that f is "representable by schemes" and $\mathfrak{X}$ is a reasonable ind-scheme, and let $\mathscr{M}$ be a quasi-coherent torsion module over $\mathfrak{A}$. Then it is clear from Lemma 3.4 that the inverse image $f^*\mathscr{M}$ is a quasi-coherent torsion module over the pro-quasi-coherent algebra $f^*\mathfrak{A}$.

Let $f\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be an affine morphism of ind-schemes. Let $\mathfrak{B}$ be a pro-quasi-coherent algebra on $\mathfrak{Y}$ and $\mathfrak{N}$ be a pro-quasi-coherent module over $\mathfrak{B}$. Then the direct image $f_*\mathfrak{N}$ is a pro-quasi-coherent module over the pro-quasi-coherent algebra $f_*\mathfrak{B}$ on $\mathfrak{X}$. Furthermore, assume that $\mathfrak{X}$ is a reasonable ind-scheme, and let $\mathscr{N}$ be a quasi-coherent torsion modules over $\mathfrak{B}$. Then the direct image $f_*\mathscr{N}$ is a quasi-coherent torsion module over the pro-quasi-coherent algebra $f_*\mathfrak{B}$.

Proposition 3.13. *Let $f\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be an affine morphism of ind-schemes. Then*

(a) *the category of pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is equivalent to the category of pro-quasi-coherent modules over the pro-quasi-coherent algebra $f_*\mathfrak{O}_{\mathfrak{Y}}$ on $\mathfrak{X}$;*

(b) *if the ind-scheme $\mathfrak{X}$ is reasonable, then the category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is equivalent to the category of quasi-coherent torsion modules over the pro-quasi-coherent algebra $f_*\mathfrak{O}_{\mathfrak{Y}}$ on $\mathfrak{X}$.*

Proof. In part (a), to a pro-quasi-coherent pro-sheaf $\mathfrak{N}$ on $\mathfrak{Y}$, the pro-quasi-coherent $f_*\mathfrak{O}_{\mathfrak{Y}}$-module $f_*\mathfrak{N}$ is assigned. To construct the inverse functor, let $\mathfrak{M}$ be a pro-quasi-coherent module over $f_*\mathfrak{O}_{\mathfrak{Y}}$ on $\mathfrak{X}$, and let $Z\subset\mathfrak{X}$ be a closed subscheme; put $W=Z\times_{\mathfrak{X}}\mathfrak{Y}$, and denote by $f_Z\colon W\longrightarrow Z$ the natural morphism. Then, for the corresponding pro-quasi-coherent pro-sheaf $\mathfrak{N}$ on $\mathfrak{Y}$, the quasi-coherent sheaf $\mathfrak{N}^{(W)}\in W\text{–}\mathsf{qcoh}$ corresponds to the quasi-coherent module $\mathfrak{M}^{(Z)}$ over the quasi-coherent algebra $f_{Z*}\mathcal{O}_W\simeq(f_*\mathfrak{O}_{\mathfrak{Y}})^{(Z)}$ on Z under the equivalence of categories from Lemma 3.11. One can use Lemma 3.3(a) to construct the compatibility isomorphisms related to pairs of closed subschemes $Z'\subset Z''\subset\mathfrak{X}$.

In part (b), to a quasi-coherent torsion sheaf $\mathscr{N}$ on Y, the quasi-coherent torsion module $f_*\mathscr{N}$ over the pro-quasi-coherent algebra $f_*\mathfrak{O}_{\mathfrak{Y}}$ is assigned. To construct the inverse functor, let $\mathscr{M}$ be a quasi-coherent torsion module over the quasi-coherent algebra $f_*\mathfrak{O}_Y$ on $\mathfrak{X}$, and let $Z\subset\mathfrak{X}$ be a reasonable closed subscheme. We keep the same notation $f_Z\colon W\longrightarrow Z$ as above. Then, for the corresponding quasi-coherent torsion sheaf $\mathscr{N}$ on $\mathfrak{Y}$, the quasi-coherent sheaf $\mathscr{N}_{(W)} \in W\mathrm{-qcoh}$ corresponds to the quasi-coherent module $\mathscr{M}_{(Z)}$ over the quasi-coherent algebra $f_{Z*}\mathcal{O}_W \simeq (f_*\mathfrak{O}_{\mathfrak{Y}})^{(Z)}$ on Z. One can use Lemma 2.3(a) to construct the compatibility isomorphisms related to pairs of reasonable closed subschemes $Z' \subset Z'' \subset \mathfrak{X}$. $\square$

4. Dualizing Complexes on Ind-Noetherian Ind-Schemes

For any additive category A, we denote by $\mathsf{C}(\mathsf{A})$ the category of complexes in A and by $\mathsf{K}(\mathsf{A})$ the homotopy category of (complexes in) A. The notation $\mathsf{C}^+(\mathsf{A})$, $\mathsf{C}^-(\mathsf{A})$, and $\mathsf{C}^{\mathsf{b}}(\mathsf{A}) \subset \mathsf{C}(\mathsf{A})$ stands for the categories of bounded below, bounded above, and bounded (on both sides) complexes in A, as usual; and similarly for $\mathsf{K}^+(\mathsf{A})$, $\mathsf{K}^-(\mathsf{A})$, and $\mathsf{K}^{\mathsf{b}}(\mathsf{A}) \subset \mathsf{K}(\mathsf{A})$. For an abelian (or exact) category A, the full subcategory of injective objects in A is denoted by $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{A}$.

4.1. **Ind-Noetherian ind-schemes.** An ind-scheme is said to be *ind-Noetherian* if it can be represented by an inductive system of Noetherian schemes. It follows from Lemma 1.2(a) that any closed subscheme of a (strict) ind-Noetherian ind-scheme is Noetherian (since any locally closed subscheme of a Noetherian scheme is Noetherian). Thus any ind-Noetherian ind-scheme can be represented by an inductive system of closed immersions of Noetherian schemes.

Let S be a Noetherian scheme and $\mathfrak{X} \longrightarrow S$ be a morphism of ind-schemes. One says that $\mathfrak{X}$ is an ind-scheme of *ind-finite type* over S if $\mathfrak{X}$ can be represented by an inductive system of schemes of finite type over S. Similarly to the previous paragraph, any closed subscheme of an ind-scheme of ind-finite type is a scheme of finite type (over the Noetherian base scheme S). Thus any ind-scheme of ind-finite type can be represented by an inductive system of closed immersions of schemes of finite type.

Let $\Bbbk$ be a Noetherian commutative ring. Speaking about schemes of finite type and ind-schemes of ind-finite type over $\operatorname{Spec}\Bbbk$, we will say simply “over $\Bbbk$” instead of “over $\operatorname{Spec}\Bbbk$”, for brevity.

Examples 4.1. (1) A scheme X is said to be *Artinian* if it has a finite open covering by spectra of Artinian rings; equivalently, this means that X is the spectum of an Artinian ring. So any Artinian scheme is affine and Noetherian.

An ind-scheme is said to be *ind-Artinian* if it can be represented by an inductive system of Artinian schemes. Any closed subscheme of an ind-Artinian ind-scheme is Artinian; so an ind-Artinian ind-scheme can be represented by an inductive system of closed immersions of Artinian schemes. Any ind-Artinian ind-scheme is ind-affine and ind-Noetherian.

(2) Let $\Bbbk$ be a field. Notice that any Artinian scheme of finite type over $\Bbbk$ is finite over $\Bbbk$ (in other words, any finitely generated commutative Artinian $\Bbbk$-algebra is finite-dimensional over $\Bbbk$).

The construction of Example 1.5(2) establishes an equivalence between the category of ind-Artinian ind-schemes of ind-finite type over $\Bbbk$ and the category of cocommutative coalgebras over $\Bbbk$. To a (coassociative, counital) cocommutative coalgebra $\mathscr{C}$ over $\Bbbk$, the ind-Artinian ind-scheme of ind-finite type $\operatorname{Spi}\mathscr{C}^*$ is assigned.

(3) The ind-schemes $\operatorname{Spi}\widehat{\mathbb{Z}}$ and $\operatorname{Spi}\mathbb{Z}_p$ from Examples 1.4 are ind-Artinian.

Quite generally, the category of ind-Artinian ind-schemes is anti-equivalent to the category of *pro-Artinian topological commutative rings* in the sense of [43, Section 1.1]. The functor $\mathfrak{R} \longmapsto \operatorname{Spi}\mathfrak{R}$ from Example 1.6(1) restricts to the desired anti-equivalence. It is important here that, for any directed projective system of $(R_\gamma)_{\gamma\in\Gamma}$ of Artinian commutative rings R_γ and surjective maps between them, the projection map $\mathfrak{R} = \varprojlim_{\gamma\in\Gamma} R_\gamma \longrightarrow R_\delta$ is surjective for all $\delta \in \Gamma$ [43, Corollary A.2.1].

4.2. **Definition of a dualizing complex.** Notice that any closed subscheme of a Noetherian scheme is reasonable (in the sense of Section 2.1). Hence any ind-Noetherian ind-scheme is reasonable, and any closed subscheme of an ind-Noetherian ind-scheme is reasonable.

Let X be a scheme. Recall the notation $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ for the internal Hom of sheaves of $\mathcal{O}_X$-modules (see Section 2.2). The *quasi-coherent internal Hom* of quasi-coherent sheaves on X is defined as follows. For any two quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{N} \in X\text{–}\mathsf{qcoh}$, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{N}) \in X\text{–}\mathsf{qcoh}$ is the object for which a natural isomorphism of the abelian groups of morphisms

$$\operatorname{Hom}_{X\text{–}\mathsf{qcoh}}(\mathcal{L}, \mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{N})) \simeq \operatorname{Hom}_{X\text{–}\mathsf{qcoh}}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M},\ \mathcal{N})$$

holds for all $\mathcal{L} \in X\text{–}\mathsf{qcoh}$. The quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{N})$ can be obtained by applying the *coherator* functor [63, Sections B.12–B.14] to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$. In particular, one has $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{N}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ whenever the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$ is quasi-coherent (e. g., this holds when the scheme X is Noetherian and the sheaf $\mathcal{M}$ is coherent).

For any two complexes $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ of quasi-coherent sheaves on X, the complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^\bullet,\mathcal{N}^\bullet)$ of quasi-coherent sheaves on X is constructed by totalizing the bicomplex of quasi-coherent sheaves with the components $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p,\mathcal{N}^q)$, p, $q \in \mathbb{Z}$, by taking infinite products in the Grothendieck category of quasi-coherent sheaves $X\text{–}\mathsf{qcoh}$ along the diagonals of the bicomplex.

Lemma 4.2. *Let $f\colon Y \longrightarrow X$ be a morphism of schemes, $\mathcal{M}$ be a quasi-coherent sheaf on X, and $\mathcal{N}$ be a quasi-coherent sheaf on Y. Then there is a natural isomorphism of quasi-coherent sheaves on X*

$$f_*\,\mathcal{H}om_{Y\text{-qc}}(f^*\mathcal{M},\mathcal{N}) \simeq \mathcal{H}om_{X\text{-qc}}(\mathcal{M}, f_*\mathcal{N}).$$

Proof. Let $\mathcal{L}$ be an arbitrary quasi-coherent sheaf on X. Then we have

$$\begin{aligned}\operatorname{Hom}_X(\mathcal{L}, f_*\,\mathcal{H}om_{Y\text{-qc}}(f^*\mathcal{M},\mathcal{N})) &\simeq \operatorname{Hom}_Y(f^*\mathcal{L}, \mathcal{H}om_{Y\text{-qc}}(f^*\mathcal{M},\mathcal{N})) \\ &\simeq \operatorname{Hom}_Y(f^*\mathcal{L} \otimes_{\mathcal{O}_Y} f^*\mathcal{M},\ \mathcal{N}) \simeq \operatorname{Hom}_Y(f^*(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}),\ \mathcal{N}) \\ &\simeq \operatorname{Hom}_X(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M},\ f_*\mathcal{N}) \simeq \operatorname{Hom}_X(\mathcal{L}, \mathcal{H}om_{X\text{-qc}}(\mathcal{M}, f_*\mathcal{N})),\end{aligned}$$

where $\mathrm{Hom}_X(-,-)$ is a shorthand notation for the abelian group $\mathrm{Hom}_{X\text{–}\mathsf{qcoh}}(-,-)$, and similarly for Y. ☐

Lemma 4.3. *Let $f\colon Y \longrightarrow X$ be a morphism of schemes, $\mathcal{M}^\bullet$ be a complex of quasi-coherent sheaves on X, and $\mathcal{N}^\bullet$ be a complex of quasi-coherent sheaves on Y. Then there is a natural isomorphism of complexes of quasi-coherent sheaves on X*

$$f_*\,\mathcal{H}om_{Y\text{-qc}}(f^*\mathcal{M}^\bullet,\mathcal{N}^\bullet) \simeq \mathcal{H}om_{X\text{-qc}}(\mathcal{M}^\bullet, f_*\mathcal{N}^\bullet).$$

Proof. Follows from Lemma 4.2 and the fact that the direct image functor f_*, being a right adjoint, preserves infinite products of quasi-coherent sheaves. ☐

Lemma 4.4. (a) *For any injective quasi-coherent sheaf $\mathcal{J}$ over a quasi-compact semi-separated scheme X, the functor $\mathcal{H}om_{X\text{-qc}}(-,\mathcal{J})\colon X\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ is exact.*

(b) *For any flat quasi-coherent sheaf $\mathcal{F}$ and injective quasi-coherent sheaf $\mathcal{J}$ over a Noetherian scheme X, the quasi-coherent sheaf $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{J}$ is injective.*

(c) *For any flat quasi-coherent sheaf $\mathcal{F}$ and injective quasi-coherent sheaf $\mathcal{J}$ over a scheme X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{J})$ is injective.*

(d) *For any injective quasi-coherent sheaves $\mathcal{J}'$ and $\mathcal{J}$ over a semi-separated Noetherian scheme X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{J}',\mathcal{J})$ is flat.*

Proof. This is [31, Lemma 8.7] or [12, Lemma 2.5]. ☐

Let X be a semi-separated Noetherian scheme. For us, a *dualizing complex* $\mathcal{D}^\bullet$ on X is a complex of injective quasi-coherent sheaves, $\mathcal{D}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$, satisfying the following conditions:

(i) the complex $\mathcal{D}^\bullet$ is homotopy equivalent to a bounded complex of injective quasi-coherent sheaves on X;

(ii) the cohomology sheaves of the complex $\mathcal{D}^\bullet$ are coherent sheaves on X;

(iii) the natural morphism of complexes of quasi-coherent sheaves $\mathcal{O}_X \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet,\mathcal{D}^\bullet)$ is a quasi-isomorphism (of complexes in the abelian category $X\text{–}\mathsf{qcoh}$).

This definition is equivalent to the one in [19, Section V.2], with the only difference that a dualizing complex is viewed as a derived category object in [19], while we presume a complex of injectives representing this derived category object to be chosen. The complex of injectives $\mathcal{D}^\bullet$ does not have to be bounded, but it must be homotopy equivalent to a bounded complex of injectives. In view of the semi-separatedness assumption in Lemma 4.4(a), we are imposing the semi-separatedness assumption on the scheme X in the definition above in order to be able to use the quasi-coherent internal Hom in the formulation of condition (iii). In particular, being a dualizing complex is a local property of a complex of quasi-coherent sheaves.

Lemma 4.5. *Let $i\colon Z \longrightarrow X$ be a closed immersion of semi-separated Noetherian schemes and $\mathcal{D}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ be a dualizing complex on X. Then $i^!\mathcal{D}^\bullet \in \mathsf{C}(Z\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is a dualizing complex on Z.*

Proof. First of all, the functor $i^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ is right adjoint to an exact functor i_*; so $i^!$ takes injectives to injectives. The rest is [19, Proposition V.2.4]. (Cf. Lemmas 4.25 and 4.11 below.) ☐

An ind-scheme is said to be *ind-semi-separated* if it can be represented by an inductive system of semi-separated schemes. It follows from Lemma 1.2(a) that any closed subscheme of an ind-semi-separated ind-scheme is ind-semi-separated (since any locally closed subscheme of a semi-separated scheme is semi-separated). Thus any ind-semi-separated ind-scheme can be represented by an inductive system of closed immersions of semi-separated schemes. Moreover, any ind-semi-separated ind-Noetherian ind-scheme can be represented by an inductive system of closed immersions of semi-separated Noetherian schemes.

Similarly, an ind-scheme is said to be *ind-separated* if it can be represented by an inductive system of separated schemes. Any closed subscheme of an ind-separated ind-scheme is ind-separated.

Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme. A *dualizing complex* $\mathscr{D}^\bullet$ on $\mathfrak{X}$ is a complex of quasi-coherent torsion sheaves, $\mathscr{D}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$, satisfying the following condition:

(iv) for every closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$, the complex $i^!\mathscr{D}^\bullet \in \mathsf{C}(Z\text{–}\mathsf{qcoh})$ is a dualizing complex on Z.

Here the functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow Z\text{–}\mathsf{qcoh}$ is applied to the complex $\mathscr{D}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ termwise (no derived functor is presumed in our notation). In view of condition (i) for $i^!\mathscr{D}^\bullet \in \mathsf{C}(Z\text{–}\mathsf{qcoh})$ and Proposition 2.17(b), it follows from condition (iv) that $\mathscr{D}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ is actually a complex of injective quasi-coherent torsion sheaves; so $\mathscr{D}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of semi-separated Noetherian schemes. Then, in view of Lemma 4.5, it suffices to check condition (iv) for the closed subschemes belonging to the inductive system $(X_\gamma)_{\gamma\in\Gamma}$; so one can assume $Z = X_\gamma$ for some $\gamma \in \Gamma$.

Remark 4.6. A more common point of view is to consider a dualizing complex on a scheme X as an object of the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$, or more specifically, of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh})$. A similar point of view on dualizing complexes on ind-schemes is also possible, but it prescribes viewing a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ as an object of the *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, as defined in Section 4.4 below. Indeed, the homotopy category of complexes of injective quasi-coherent torsion sheaves $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is equivalent to the coderived category by Corollary 4.18.

For schemes, one does not feel the difference between the derived and the coderived category in this context, because there is no such difference for complexes bounded below; see the discussion in Remark 5.3(1). The dualizing complex on an ind-scheme is often bounded above (see Remarks 5.3(3–5)), but it is usually not bounded below (unless the whole ind-scheme is finite-dimensional).

In fact, a dualizing complex $\mathscr{D}^\bullet$ on an ind-scheme $\mathfrak{X}$ can well be an *acyclic* complex, and in some very simple examples it is; see Section 11.1(7).

Example 4.7. Let $\mathfrak{X} = \text{“}\varinjlim\text{”}(X_0 \to X_1 \to X_2 \to \cdots)$ be an ind-semi-separated ind-Noetherian $\aleph_0$-ind-scheme represented by an inductive system of closed immersions of (semi-separated Noetherian) schemes indexed by the poset of nonnegative integers. Let $i_n\colon X_n \longrightarrow X_{n+1}$ denote the closed immersion morphisms in the inductive system. Suppose that we are given a dualizing complex $\mathcal{D}_n^\bullet$ on the scheme X_n for every $n \geq 0$ together with homotopy equivalences $\mathcal{D}_n^\bullet \longrightarrow i_n^!\mathcal{D}_{n+1}^\bullet$ of complexes of (injective) quasi-coherent sheaves on X_n. Then there are the related morphisms $i_{n*}\mathcal{D}_n^\bullet \longrightarrow \mathcal{D}_{n+1}^\bullet$ of complexes of quasi-coherent sheaves on X_{n+1}.

Let $k_n\colon X_n \longrightarrow \mathfrak{X}$ be the natural closed immersions. Consider the inductive system $k_{0*}\mathcal{D}_0^\bullet \longrightarrow k_{1*}\mathcal{D}_1^\bullet \longrightarrow \cdots$ of complexes of quasi-coherent torsion sheaves on the ind-scheme $\mathfrak{X}$, and put $\mathscr{D}^\bullet = \varinjlim_{n\geq 0} k_{n*}\mathcal{D}_n^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$. Then $\mathscr{D}^\bullet$ is a dualizing complex on the ind-scheme $\mathfrak{X}$.

Indeed, for every $m \geq 0$ we have $k_m^!\mathscr{D}^\bullet = \varinjlim_{n\geq 0} k_m^! k_{n*}\mathcal{D}_n^\bullet = \varinjlim_{n\geq m} i_m^! \cdots i_{n-1}^!\mathcal{D}_n^\bullet \in \mathsf{C}(X_m\text{–}\mathsf{qcoh})$. Since the class of all injective quasi-coherent sheaves on X_m is closed under direct limits (the scheme X_m being Noetherian), it follows that $k_m^!\mathscr{D}^\bullet$ is a complex of injective quasi-coherent sheaves on X_m. Now we claim that the natural morphism $\mathcal{D}_m^\bullet \longrightarrow k_m^!\mathscr{D}^\bullet$ is a homotopy equivalence of complexes of (injective) quasi-coherent sheaves on X_m. As a complex of injective quasi-coherent sheaves homotopy equivalent to a dualizing complex is also a dualizing complex, this suffices to prove the desired assertion.

The morphisms of complexes $\mathcal{D}_m^\bullet \longrightarrow i_m^! \cdots i_{n-1}^!\mathcal{D}_n^\bullet$ are homotopy equivalences by assumption. So it remains to observe that, for any sequence $\mathcal{J}_0^\bullet \longrightarrow \mathcal{J}_1^\bullet \longrightarrow \mathcal{J}_2^\bullet \longrightarrow \cdots$ of homotopy equivalences of complexes of injective quasi-coherent sheaves on a Noetherian scheme X, the natural morphism $\mathcal{J}_0^\bullet \longrightarrow \varinjlim_{n\geq 0}\mathcal{J}_n^\bullet$ is a homotopy equivalence. Indeed, the telescope short exact sequence $0 \longrightarrow \bigoplus_{n\geq 0}\mathcal{J}_n^\bullet \longrightarrow \bigoplus_{n\geq 0}\mathcal{J}_n^\bullet \longrightarrow \varinjlim_{n\geq 0}\mathcal{J}_n^\bullet \longrightarrow 0$ is a short exact sequence of complexes of injectives, so it is termwise split. Hence $\varinjlim_{n\geq 0}\mathcal{J}_n^\bullet$ is the homotopy colimit of the sequence $\mathcal{J}_0^\bullet \longrightarrow \mathcal{J}_1^\bullet \longrightarrow \mathcal{J}_2^\bullet \longrightarrow \cdots$ in the homotopy category $\mathsf{K}(X\text{–}\mathsf{qcoh})$. It remains to apply [34, Lemma 1.6.6].

Proposition 6.28 or Lemma 6.29 below, combined with Lemma 4.12, show that this construction of a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ does not depend on the arbitrary choices of the morphisms of complexes $i_{n*}\mathcal{D}_n^\bullet \longrightarrow \mathcal{D}_{n+1}^\bullet$ in the given homotopy classes of such morphisms.

Examples 4.8. (1) A morphism of ind-schemes $k\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ is said to be an *ind-closed immersion* if, for every closed subscheme $Z \subset \mathfrak{Z}$, the composition $Z \longrightarrow \mathfrak{Z} \longrightarrow \mathfrak{X}$ is a closed immersion. Let $k\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be an ind-closed immersion of reasonable ind-schemes, and let $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$. For every reasonable closed subscheme $Z \subset \mathfrak{Z}$, put $\mathscr{N}_{(Z)} = k_Z^!\mathscr{M}$, where $k_Z\colon Z \longrightarrow \mathfrak{X}$. Then the collection of quasi-coherent sheaves $\mathscr{N}_{(Z)} \in Z\text{–}\mathsf{qcoh}$ defines a quasi-coherent torsion sheaf $\mathscr{N}$ on $\mathfrak{Z}$. Put $k^!\mathscr{M} = \mathscr{N}$; this rule defines a functor $k^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$. This construction generalizes the construction of the functor $i^!$ for a closed immersion of ind-schemes $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ in Section 2.8.

(2) Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, and let $k\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be an ind-closed immersion of schemes. Let $\mathscr{D}^\bullet$ be a dualizing complex on $\mathfrak{X}$. Then it is clear from the definitions and Lemma 4.5 that $k^!\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{Z}$.

(3) In particular, let X be a semi-separated Noetherian scheme, and let $\mathfrak{X}$ be an ind-scheme endowed with an ind-closed immersion of ind-schemes $k\colon \mathfrak{X} \longrightarrow X$. This means that $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$, where $(X_\gamma)_{\gamma\in\Gamma}$ is an inductive system of closed subschemes in X. Let $\mathcal{D}^\bullet$ be a dualizing complex on X. Denoting by $k_\gamma\colon X_\gamma \longrightarrow X$ the composition $X_\gamma \longrightarrow \mathfrak{X} \longrightarrow X$, put $\mathcal{D}^\bullet_\gamma = k_\gamma^!\mathcal{D}^\bullet \in \mathsf{C}(X_\gamma\text{–}\mathsf{qcoh}_{\mathsf{inj}})$. Then, by Lemma 4.5, $\mathcal{D}^\bullet_\gamma$ is a dualizing complex on X_γ. Denoting by $i_{\gamma\delta}$ the closed immersions $X_\gamma \longrightarrow X_\delta$, we have $\mathcal{D}^\bullet_\gamma \simeq i_{\gamma\delta}^!\mathcal{D}^\bullet_\delta$ for all $\gamma < \delta \in \Gamma$. So the collection of complexes of injective quasi-coherent sheaves $\mathcal{D}^\bullet_\gamma$ on X_γ defines an complex of injective quasi-coherent torsion sheaves $\mathscr{D}^\bullet$ on $\mathfrak{X}$ (by Proposition 2.17(b)). By the definition, $\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{X}$.

4.3. **Derived categories of flat sheaves and flat pro-sheaves.** We refer to [32], [8, Section 10], or [42, Section A.7] for the definition of the *derived category* $\mathsf{D}(\mathsf{E})$ of an exact category E. The bounded below, bounded above, and bounded (on both sides) versions of the derived category are denoted, as usually, by $\mathsf{D}^+(\mathsf{E})$, $\mathsf{D}^-(\mathsf{E})$, and $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})$; these are full triangulated subcategories of the triangulated category $\mathsf{D}(\mathsf{E})$.

In particular, let X be a scheme. Then the full subcategory $X\text{–}\mathsf{flat} \subset X\text{–}\mathsf{qcoh}$ of flat quasi-coherent sheaves inherits an exact category structure from the ambient abelian category of quasi-coherent sheaves on X. So one can consider the derived category $\mathsf{D}(X\text{–}\mathsf{flat})$ alongside with the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh})$.

A complex of flat quasi-coherent sheaves which vanishes as an object of $\mathsf{D}(X\text{–}\mathsf{qcoh})$ need *not* vanish as an object of $\mathsf{D}(X\text{–}\mathsf{flat})$, generally speaking. Rather, a complex $\mathcal{F}^\bullet$ of flat quasi-coherent sheaves vanishes as an object of $\mathsf{D}(X\text{–}\mathsf{flat})$ (“is acyclic with respect to the exact category $X\text{–}\mathsf{flat}$”) if and only if $\mathcal{F}^\bullet$ is acyclic as a complex of quasi-coherent sheaves on X *and* all the quasi-coherent sheaves of cocycles of the complex $\mathcal{F}^\bullet$ are flat.

Let X be a quasi-compact semi-separated scheme. A quasi-coherent sheaf $\mathcal{P}$ on X is said to be *cotorsion* if $\operatorname{Ext}^1_{X\text{–}\mathsf{qcoh}}(\mathcal{F},\mathcal{P}) = 0$ for all flat quasi-coherent sheaves $\mathcal{F}$ on X.

Lemma 4.9. (a) *For any quasi-coherent sheaf $\mathcal{M}$ and any injective quasi-coherent sheaf $\mathcal{J}$ on a quasi-compact semi-separated scheme X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{J})$ is cotorsion.*

(b) *For any flat quasi-coherent sheaf $\mathcal{F}$ and any cotorsion quasi-coherent sheaf $\mathcal{P}$ on a quasi-compact semi-separated scheme X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})$ is cotorsion.*

(c) *For any family $(\mathcal{P}_\xi)_{\xi\in\Xi}$ of flat cotorsion quasi-coherent sheaves on a semi-separated Noetherian scheme X, the quasi-coherent sheaf $\prod_{\xi\in\Xi}\mathcal{P}_\xi$ on X is flat cotorsion. (Here the product is taken in the Grothendieck category $X\text{–}\mathsf{qcoh}$.)*

Proof. Parts (a–b): the key fact is that there are enough flat quasi-coherent sheaves on any quasi-compact semi-separated scheme, i. e., any quasi-coherent sheaf is a quotient of a flat one (see [30, Section 2.4] or [12, Lemma A.1]). Consequently, a quasi-coherent sheaf $\mathcal{Q}$ on X is cotorsion if and only if, for every short exact sequence of flat quasi-coherent sheaves $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ on X, the short sequence of abelian groups $0 \longrightarrow \mathrm{Hom}_X(\mathcal{F}'', \mathcal{Q}) \longrightarrow \mathrm{Hom}_X(\mathcal{F}, \mathcal{Q}) \longrightarrow \mathrm{Hom}_X(\mathcal{F}', \mathcal{Q}) \longrightarrow 0$ is exact. Here Hom_X is a shorthand for $\mathrm{Hom}_{X\text{–}\mathsf{qcoh}}$.

With this criterion, the assertions of both (a) and (b) follow straightforwardly from the universal property definition of $\mathcal{H}om_{X\text{-qc}}$.

Part (c): let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering of a quasi-compact semi-separated scheme X. Denote by $j_\alpha \colon U_\alpha \longrightarrow X$ the open immersion morphisms. According to [44, Lemma 4.1.12], the flat cotorsion quasi-coherent sheaves on X are precisely the direct summands of finite direct sums of the form $\bigoplus_\alpha j_{\alpha *}\mathcal{Q}_\alpha$, where $\mathcal{Q}_\alpha$ are flat cotorsion quasi-coherent sheaves on U_α. Notice that the direct image functors $j_{\alpha *}$ (being right adjoint to the inverse image functors j_α^*) preserve infinite products of quasi-coherent sheaves.

This reduces the question to the particular case of an affine scheme X. It remains to recall that, over any (commutative) ring, the class of cotorsion modules is closed under infinite products; and over a Noetherian (or coherent) ring, the class of flat modules is closed under infinite products as well. □

Lemma 4.10. *For any complexes of injective quasi-coherent sheaves $'\mathcal{J}^\bullet$ and $\mathcal{J}^\bullet$ on a semi-separated Noetherian scheme X, the complex $\mathcal{H}om_{X\text{-qc}}('\mathcal{J}^\bullet, \mathcal{J}^\bullet)$ is a complex of flat cotorsion quasi-coherent sheaves.*

Proof. The assertion follows from Lemma 4.4(d) combined with Lemma 4.9(a,c). Alternatively, one can observe that both the quasi-coherent sheaves $'\mathcal{K} = \bigoplus_{p\in\mathbb{Z}} {}'\mathcal{J}^p$ and $\mathcal{K} = \prod_{q\in\mathbb{Z}} \mathcal{J}^q$ are injective and all the terms of the complex $\mathcal{H}om_{X\text{-qc}}('\mathcal{J}^\bullet, \mathcal{J}^\bullet)$ are direct summands of the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}('\mathcal{K}, \mathcal{K})$. Then it remains to apply Lemma 4.4(d) and Lemma 4.9(a). □

Lemma 4.11. *For any dualizing complex $\mathcal{D}^\bullet$ on a semi-separated Noetherian scheme X, the natural morphism $\mathcal{O}_X \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)$ from condition (iii) in Section 4.2 is a quasi-isomorphism of complexes in the exact category X–*flat.

Proof. First of all, $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)$ is a complex of flat quasi-coherent sheaves by Lemma 4.10. Now the point is that, by the definition, any dualizing complex $\mathcal{D}^\bullet$ on X is homotopy equivalent to a bounded dualizing complex $'\mathcal{D}^\bullet$. Hence the complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)$ is homotopy equivalent to the complex $\mathcal{H}om_{X\text{-qc}}('\mathcal{D}^\bullet, '\mathcal{D}^\bullet)$, which is a bounded complex of flat quasi-coherent sheaves. Finally, a morphism of bounded (above) complexes of flat quasi-coherent sheaves is a quasi-isomorphism of complexes in X–flat if and only if it is a quasi-isomorphism of complexes in X–qcoh (because any bounded above complex of flat quasi-coherent sheaves that is acyclic as a complex of quasi-coherent sheaves has flat sheaves of cocycles). □

Lemma 4.12. *For any dualizing complex $\mathcal{D}^\bullet$ on a semi-separated Noetherian scheme X, one has $\mathrm{Hom}_{\mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})}(\mathcal{D}^\bullet, \mathcal{D}^\bullet[n]) = 0$ for all $n < 0$.*

Proof. By the adjunction property defining the quasi-coherent internal Hom, we have

$$\mathrm{Hom}_{\mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})}(\mathcal{D}^\bullet, \mathcal{D}^\bullet[n]) = \mathrm{Hom}_{\mathsf{K}(X\text{–}\mathsf{flat})}(\mathcal{O}_X, \mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)[n]).$$

Without loss of generality, we can assume the complex $\mathcal{D}^\bullet$ to be a bounded complex of injective quasi-coherent sheaves. Set $\mathcal{P}^\bullet = \mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)$; then $\mathcal{P}^\bullet$ is a bounded complex as well. All the quasi-coherent sheaves $\mathcal{P}^n$ on X are cotorsion by Lemma 4.9(a). Since there are enough flat quasi-coherent sheaves on a quasi-compact semi-separated scheme X, one has $\mathrm{Ext}^m_{X\text{–}\mathsf{qcoh}}(\mathcal{F}, \mathcal{P}) = 0$ for all $\mathcal{F} \in X\text{–}\mathsf{flat}$, all cotorsion quasi-coherent sheaves $\mathcal{P}$ on X, and all $m > 0$; in particular, in the situation at hand $\mathrm{Ext}^m_{X\text{–}\mathsf{qcoh}}(\mathcal{O}_X, \mathcal{P}^n) = 0$ for all $m > 0$ and $n < 0$. As the complex $\mathcal{P}^\bullet$ in $X\text{–}\mathsf{qcoh}$ also has vanishing cohomology sheaves in the negative cohomological degrees, it follows that $\mathrm{Hom}_{\mathsf{K}(X\text{–}\mathsf{flat})}(\mathcal{O}_X, \mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)[n]) = 0$ for $n < 0$. ☐

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme represented by an inductive system of closed immersions of schemes. The construction of Proposition 3.5 defines an exact category structure on the category of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{–}\mathsf{flat}$. Hence the related derived category $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$.

Lemma 4.13. *A complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ is acyclic (as a complex in $\mathfrak{X}\text{–}\mathsf{flat}$) if and only if, for every $\gamma \in \Gamma$, the complex of flat quasi-coherent sheaves $\mathfrak{F}^{\bullet(X_\gamma)}$ is acyclic (as a complex in $X_\gamma\text{–}\mathsf{flat}$).*

Proof. The proof is straightforward. ☐

4.4. **Coderived category of torsion sheaves.** Let E be an exact category. A complex in E is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of $\mathsf{K}(\mathsf{E})$ containing the totalizations of short exact sequences of complexes in E. Here a short sequence of complexes in E is said to be *exact* if it is termwise exact (i. e., exact in every degree), and “totalization” of a short sequence of complexes means taking the total complex of a bicomplex with three rows. The triangulated quotient category of $\mathsf{K}(\mathsf{E})$ by the thick subcategory of absolutely acyclic complexes is called the *absolute derived category* of an exact category E and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathsf{E})$.

Let E be an exact category in which infinite coproducts exist and the class of all short exact sequences is closed under coproducts (in this case, we will say that E has *exact coproducts*). Then a complex in E is said to be *coacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{K}(\mathsf{E})$ containing the totalizations of short exact sequences in $\mathsf{C}(\mathsf{E})$ and closed under coproducts. The *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$ is defined as the triangulated quotient category of the homotopy category $\mathsf{K}(\mathsf{E})$ by the thick subcategory of coacyclic complexes.

The reader is referred to [40, Section 2.1], [41, Sections 3–4], [12, Section 1.3], [44, Appendix A], [47, Section 2] for a discussion of these definitions.

An exact category E is said to have *homological dimension* $\le d$ (where $d \ge -1$ is an integer) if $\mathrm{Ext}^{d+1}_{\mathsf{E}}(E, F) = 0$ for all E, $F \in \mathsf{E}$.

Lemma 4.14. (a) *In any exact category E, any absolutely acyclic complex is acyclic; so there is a natural triangulated Verdier quotient functor $\mathsf{D}^{\mathsf{abs}}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$.*

(b) *In any exact category* E *with exact coproducts, any absolutely acyclic complex is coacyclic, and any coacyclic complex is acyclic; so there are natural triangulated Verdier quotient functors* $\mathsf{D}^{\mathsf{abs}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$.

(c) *In an exact category* E *of finite homological dimension, any acyclic complex is absolutely acyclic.*

Proof. Parts (a–b) are straightforward; part (c) is [40, Remark 2.1]. ☐

Proposition 4.15. (a) *For any exact category* E *with exact coproducts, the composition* $\mathsf{K}(\mathsf{E}_{\mathsf{inj}}) \longrightarrow \mathsf{K}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *of the inclusion functor* $\mathsf{K}(\mathsf{E}_{\mathsf{inj}}) \longrightarrow \mathsf{E}(\mathsf{E})$ *(induced by the inclusion* $\mathsf{E}_{\mathsf{inj}} \longrightarrow \mathsf{E}$*) with the Verdier quotient functor* $\mathsf{K}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *is a fully faithful triangulated functor* $\mathsf{K}(\mathsf{E}_{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$.

(b) *Let* E *be an exact category with infinite coproducts and enough injective objects. Assume that the class of all injectives* $\mathsf{E}_{\mathsf{inj}} \subset \mathsf{E}$ *is closed under coproducts in* E. *Then the triangulated functor* $\mathsf{K}(\mathsf{E}_{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *from part* (a) *is an equivalence of triangulated categories. Moreover, for any complex* $E^\bullet$ *in* E *there exists a complex* $J^\bullet$ *in* $\mathsf{E}_{\mathsf{inj}}$ *together with a morphism of complexes* $E^\bullet \longrightarrow J^\bullet$ *whose cone is a coacyclic complex in* E.

Proof. It should be noticed that in any exact category with infinite coproducts and enough injective objects the infinite coproduct functors are exact. Part (a) is [41, Section 3.5] or [44, Lemma A.1.3(a)]. Part (b) is [41, Section 3.7]; for a far-reaching generalization, see [47, Proposition 2.1]. ☐

A Grothendieck abelian category is said to be *locally Noetherian* if it has a set of generators consisting of Noetherian objects. Equivalently, a Grothendieck abelian category A is locally Noetherian if and only if every object of A is the union of its Noetherian subobjects.

Lemma 4.16. *In any locally Noetherian Grothendieck category, the class of all injective objects is closed under coproducts.*

Proof. Follows from Lemma 2.15. ☐

Proposition 4.17. *For any ind-Noetherian ind-scheme* $\mathfrak{X}$, *the category* $\mathfrak{X}\text{–}\mathsf{tors}$ *of quasi-coherent torsion sheaves on* $\mathfrak{X}$ *is a locally Noetherian Grothendieck category. The direct images of coherent sheaves from closed subschemes of* $\mathfrak{X}$ *are the Noetherian objects of* $\mathfrak{X}\text{–}\mathsf{tors}$.

Proof. The category $\mathfrak{X}\text{–}\mathsf{tors}$ is Grothendieck by Theorem 2.4, and it has a set of Noetherian generators by Lemma 2.16 (because the category of quasi-coherent sheaves on a Noetherian scheme is locally Noetherian and the coherent sheaves are its Noetherian objects). The description of the Noetherian objects in $\mathfrak{X}\text{–}\mathsf{tors}$ easily follows. ☐

Corollary 4.18. *For any ind-Noetherian ind-scheme* $\mathfrak{X}$, *the coderived category of quasi-coherent torsion sheaves is naturally equivalent to the homotopy category of injective quasi-coherent torsion sheaves,* $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Proof. Combine Proposition 4.17, Lemma 4.16, and Proposition 4.15(b). ☐

Remark 4.19. Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme. One can say that a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$ is *coherent* if there exists a closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$ and a coherent sheaf $\mathcal{M}$ on Z such that $\mathscr{M} \simeq i_*\mathcal{M}$. Then the full subcategory of coherent torsion sheaves $\mathfrak{X}\text{-}\mathsf{tcoh} \subset \mathfrak{X}\text{–}\mathsf{tors}$ is an abelian Serre subcategory; by Proposition 4.17, it is the full subcategory of Noetherian objects in the locally Noetherian category $\mathfrak{X}\text{–}\mathsf{tors}$. It follows that the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ is compactly generated and the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathfrak{X}\text{-}\mathsf{tcoh})$ is the full subcategory of compact objects in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Lemma 4.20. *Let* A *be a Grothendieck abelian category and* $\mathsf{S} \subset \mathsf{A}$ *be a class of objects, closed under quotients and containing a set of generators of* A*. Let* $J^\bullet \in \mathsf{K}(\mathsf{A}_{\mathsf{inj}})$ *be a complex of injective objects in* A *such that for every object* $S \in \mathsf{S}$ *one has* $\mathrm{Hom}_{\mathsf{K}(\mathsf{A})}(S, J^\bullet) = 0$*. Then the complex* $J^\bullet$ *is contractible.*

Proof. Using the assumption that S contains a set of generators for A, one proves that the complex $J^\bullet$ is acyclic in A. Since S is also closed under quotients, Lemma 2.15 shows that any object $K \in \mathsf{A}$ for which $\mathrm{Ext}^1_{\mathsf{A}}(S, K) = 0$ for all $S \in \mathsf{S}$ is injective. From this observation one deduces injectivity of the cocycle objects of the complex $J^\bullet$. ☐

Lemma 4.21. *Let* $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ *be a reasonable ind-scheme represented by an inductive system of closed immersions of reasonable closed subschemes, and let* $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ *be the natural closed immersions. Let* $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ *be a complex of injective quasi-coherent torsion sheaves on* $\mathfrak{X}$*. Assume that the complex of injective quasi-coherent sheaves* $i_\gamma^!\mathscr{J}^\bullet$ *on* X_γ *is contractible for every* $\gamma \in \Gamma$*. Then the complex of injective quasi-coherent torsion sheaves* $\mathscr{J}^\bullet$ *on* $\mathfrak{X}$ *is contractible as well.*

Proof. For every complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on X_γ, $\gamma \in \Gamma$, we have $\mathrm{Hom}_{\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})}(i_{\gamma*}\mathcal{M}^\bullet, \mathscr{J}^\bullet) \simeq \mathrm{Hom}_{\mathsf{K}(X_\gamma\text{–}\mathsf{qcoh})}(\mathcal{M}^\bullet, i_\gamma^!\mathscr{J}^\bullet) = 0$. In view of Lemmas 4.20 and 2.16, it follows that the complex $\mathscr{J}^\bullet$ is contractible. ☐

4.5. **The triangulated equivalence.** To begin with, we recall the triangulated equivalence for a semi-separated Noetherian scheme with a dualizing complex.

Theorem 4.22. *Let* X *be a semi-separated Noetherian scheme with a dualizing complex* $\mathcal{D}^\bullet$*. Then there is a natural equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{flat})$*, provided by mutually inverse triangulated functors* $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}^\bullet, -)\colon \mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{flat})$ *and* $\mathcal{D}^\bullet \otimes_{\mathcal{O}_X} - \colon \mathsf{D}(X\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$*.*

Proof. Notice first of all that any Noetherian scheme with a dualizing complex has finite Krull dimension [19, Corollary V.7.2]; hence the exact category $X\text{–}\mathsf{flat}$ has finite homological dimension [57, Corollaire II.3.2.7], [44, Lemma 5.4.1] (for a direct argument showing that existence of a dualizing complex implies finite projective dimension of flat modules, see [9, Proposition 1.5], [44, Corollary B.4.2], or [47, Proposition 4.3]). By Lemma 4.14, it follows that $\mathsf{D}(X\text{–}\mathsf{flat}) = \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{flat}) = \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{flat})$.

Furthermore, the equivalence $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is a particular case of Corollary 4.18.

The assertion of the theorem is a result of Murfet [31, Theorem 8.4 and Proposition 8.9]; for a different argument, which is much closer to our exposition below, see [12, Theorem 2.5] (which is stated and proved in the more complicated context of matrix factorizations). Cf. [44, Theorem 5.7.1] and [47, Theorem 4.5].

The dualizing complex $\mathcal{D}^\bullet$ on X is assumed to be a bounded complex of injectives in the above references; here we assume it to be homotopy equivalent to bounded, which is essentially the same. For a generalization to Noetherian schemes of infinite Krull dimension with pointwise dualizing complexes of infinite injective dimension, see [36, Corollary 3.10]. □

The following theorem is the main result of Section 4.

Theorem 4.23. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$. Then there is a natural equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$, provided by mutually inverse triangulated functors $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,-)\colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \longrightarrow \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$ and $\mathscr{D}^\bullet \otimes_{\mathfrak{X}} - \colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$.*

The notation $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(-,-)$ will be explained below, and the proof of Theorem 4.23 will be given below in this Section 4.5.

Lemma 4.24. *Let $f\colon Y \longrightarrow X$ be a flat morphism of schemes and $Z \subset X$ be a reasonable closed subscheme with the closed immersion $i\colon Z \longrightarrow X$. Consider the pullback diagram*

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{\;k\;} & Y \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{\;i\;} & X \end{array}$$

Put $W = Z \times_X Y$. Then there is a natural isomorphism $g^*i^! \simeq k^!f^*$ of functors $X\text{–}\mathsf{qcoh} \longrightarrow W\text{–}\mathsf{qcoh}$.

Proof. The assertion is local in X and reduces to the case of affine schemes, for which it means the following. Let $R \longrightarrow S$ be a homomorphism of commutative rings such that S is a flat R-module, and let $R \longrightarrow T$ be a surjective homomorphism of commutative rings with a finitely generated kernel ideal. Let M be an R-module. Then the natural map

$$(S \otimes_R T) \otimes_T \operatorname{Hom}_R(T, M) \simeq S \otimes_R \operatorname{Hom}_R(T, M) \\ \longrightarrow \operatorname{Hom}_R(T,\, S \otimes_R M) \simeq \operatorname{Hom}_S(S \otimes_R T,\, S \otimes_R M)$$

is an isomorphism of $(S \otimes_R T)$-modules. □

Lemma 4.25. *Let $i\colon Z \longrightarrow X$ be a closed immersion of schemes, and let $\mathcal{M}$, $\mathcal{K}$ be quasi-coherent sheaves on X. Then there is a natural morphism*

$$i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{K}) \longrightarrow \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}, i^!\mathcal{K})$$

of quasi-coherent sheaves on Z. This morphism is an isomorphism whenever the scheme X is (quasi-compact and) semi-separated, $i(Z)$ is a reasonable closed subscheme in X, and $\mathcal{K}$ is an injective quasi-coherent sheaf on X.

Proof. To prove the first assertion, it suffices to construct a natural morphism $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{K}) \longrightarrow i_*\,\mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}, i^!\mathcal{K})$ of quasi-coherent sheaves on X. Let $\mathcal{L}$ be an arbitrary quasi-coherent sheaf on X. Then morphisms $\mathcal{L} \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{K})$ correspond bijectively to morphisms $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M} \longrightarrow \mathcal{K}$, while morphisms $\mathcal{L} \longrightarrow i_*\,\mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}, i^!\mathcal{K})$ correspond bijectively to morphisms $i^*\mathcal{L}\otimes_{\mathcal{O}_Z} i^!\mathcal{M} \longrightarrow i^!\mathcal{K}$. Applying $i^!$ to a morphism $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{M} \longrightarrow \mathcal{K}$ and precomposing with the natural morphism from Lemma 3.6, one obtains a morphism $i^*\mathcal{L}\otimes_{\mathcal{O}_Z} i^!\mathcal{M} \longrightarrow i^!\mathcal{K}$.

For the second assertion, notice that any injective quasi-coherent sheaf on a quasi-compact scheme X is a direct summand of a finite direct direct sum of direct images of injective quasi-coherent sheaves from affine open subschemes of X (as there are enough injective objects of this form in X–qcoh). Let $U\subset X$ be an affine open subscheme and $j\colon U\longrightarrow X$ be the open immersion morphism. We can assume that $\mathcal{K} = j_*\mathcal{J}$, where $\mathcal{J}$ is an injective quasi-coherent sheaf on U.

Now

$$i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, j_*\mathcal{J}) \simeq i^*j_*\,\mathcal{H}om_{U\text{-qc}}(j^*\mathcal{M},\mathcal{J}) \simeq g_*k^*\,\mathcal{H}om_{U\text{-qc}}(j^*\mathcal{M},\mathcal{J}),$$

where $k\colon W = Z\times_X U \longrightarrow U$, $g\colon W\longrightarrow Z$, the first isomorphism holds by Lemma 4.2, and the second one by Lemma 3.3(a). (The assumption that X is semi-separated is used here, as we need j to be an affine morphism.)

On the other hand, we have $i^!j_*\mathcal{J}\simeq g_*k^!\mathcal{J}$ by Lemma 2.3(a) and $g^*i^!\mathcal{M}\simeq k^!j^*\mathcal{M}$ by Lemma 4.24. Hence

$$\mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}, i^!j_*\mathcal{J}) \simeq \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}, g_*k^!\mathcal{J}) \\ \simeq g_*\,\mathcal{H}om_{W\text{-qc}}(g^*i^!\mathcal{M}, k^!\mathcal{J}) \simeq g_*\,\mathcal{H}om_{W\text{-qc}}(k^!j^*\mathcal{M}, k^!\mathcal{J}).$$

This reduces the second assertion of the lemma to the particular case of an affine scheme U with a reasonable closed subscheme $k(W)\subset U$, with the quasi-coherent sheaf $j^*\mathcal{M}$ and the injective quasi-coherent sheaf $\mathcal{J}$ on U. In the affine case, the assertion means the following. Let $R\longrightarrow S$ be a surjective ring homomorphism with a finitely generated kernel ideal, let M be an R-module, and let J be an injective R-module. Then the natural morphism of S-modules

$$S\otimes_R \operatorname{Hom}_R(M,J) \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(S,M),J) \simeq \operatorname{Hom}_S(\operatorname{Hom}_R(S,M),\operatorname{Hom}_R(S,J))$$

is an isomorphism. ☐

Lemma 4.26. *Let $i\colon Z\longrightarrow X$ be a closed immersion of schemes, and let $(\mathcal{P}_\xi)_{\xi\in\Xi}$ be a family of quasi-coherent sheaves on X. Then there is a natural morphism*

$$i^*\prod\nolimits_{\xi\in\Xi}\mathcal{P}_\xi \longrightarrow \prod\nolimits_{\xi\in\Xi} i^*\mathcal{P}_\xi$$

of quasi-coherent sheaves on Z. This morphism is an isomorphism whenever the scheme X is (quasi-compact and) semi-separated, $i(Z)$ is a reasonable closed subscheme in X, and $\mathcal{P}_\xi$ are flat cotorsion quasi-coherent sheaves on X.

Proof. The first assertion holds for any functor between categories with products (in the role of i^*). The result of [44, Lemma 4.1.12], as restated in the proof of Lemma 4.9(c), reduces the second assertion to the particular case when $\mathcal{P}_\xi = j_*\mathcal{Q}_\xi$ for every $\xi \in \Xi$, where $j\colon U \longrightarrow X$ is the immersion of an affine open subscheme and $\mathcal{Q}_\xi$ are (flat cotorsion) quasi-coherent sheaves on U.

Put $W = Z \times_X U$, and denote by $k\colon W \longrightarrow U$ and $g\colon W \longrightarrow Z$ the natural morphisms. In view of Lemma 3.3, we have

$$i^* \prod\nolimits_{\xi\in\Xi} j_*\mathcal{Q}_\xi \simeq i^*j_* \prod\nolimits_{\xi\in\Xi} \mathcal{Q}_\xi \simeq g_*k^* \prod\nolimits_{\xi\in\Xi} \mathcal{Q}_\xi,$$

while

$$\prod\nolimits_{\xi\in\Xi} i^*j_*\mathcal{Q}_\xi \simeq \prod\nolimits_{\xi\in\Xi} g_*k^*\mathcal{Q}_\xi \simeq g_* \prod\nolimits_{\Xi\in\Xi} k^*\mathcal{Q}_\xi.$$

Now we have reduced the second assertion of the lemma to the particular case of an affine scheme U with a reasonable closed subscheme $k(W) \subset U$ and the family of (flat cotorsion) quasi-coherent sheaves $\mathcal{Q}_\xi$ on U. In the affine case, the assertion means the following. Let $R \longrightarrow S$ be a surjective ring homomorphism with a finitely generated kernel ideal, and let $(Q_\xi)_{\xi\in\Xi}$ be a family of (flat cotorsion) R-modules. Then the natural morphism of S-modules

$$S \otimes_R \prod\nolimits_{\xi\in\Xi} Q_\xi \longrightarrow \prod\nolimits_{\xi\in\Xi} S \otimes_R Q_\xi$$

is an isomorphism. The assumption that the R-modules Q_ξ are flat cotorsion is not needed here; it was only used in order to reduce the question to the affine case. □

Lemma 4.27. *Let $i\colon Z \longrightarrow X$ be a closed immersion of schemes, and let $\mathcal{M}^\bullet$, $\mathcal{K}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh})$ be complexes of quasi-coherent sheaves on X. Then there is a natural morphism*

$$i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^\bullet, \mathcal{K}^\bullet) \longrightarrow \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}^\bullet, i^!\mathcal{K}^\bullet)$$

of complexes of quasi-coherent sheaves on Z. This morphism is an isomorphism whenever the scheme X is (quasi-compact and) semi-separated, $i(Z)$ is a reasonable closed subscheme in X, and $\mathcal{K}^\bullet$ is a complex of injective quasi-coherent sheaves on X.

Proof. For every $n \in \mathbb{Z}$, the degree n component of the desired morphism of complexes of quasi-coherent sheaves on Z is the composition

$$\begin{aligned}(3)\quad i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^\bullet, \mathcal{K}^\bullet)^n &= i^* \prod\nolimits_{q-p=n} \mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q) \\ &\longrightarrow \prod\nolimits_{q-p=n} i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q) \\ &\longrightarrow \prod\nolimits_{q-p=n} \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}^p, i^!\mathcal{K}^q) = \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}^\bullet, i^!\mathcal{K}^\bullet)^n\end{aligned}$$

of the morphisms induced by the natural morphisms from Lemmas 4.26 and 4.25.

Assuming that X is semi-separated, $i(Z)$ is reasonable in X, and $\mathcal{K}^\bullet$ is a complex of injectives, the morphisms $i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q) \longrightarrow \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M}^p, i^!\mathcal{K}^q)$ are isomorphisms by Lemma 4.25. Assuming additionally that X is a Noetherian scheme and $\mathcal{M}^\bullet$ is also a complex of injectives, the quasi-coherent sheaves

$\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q)$ are flat cotorsion by Lemmas 4.4(d) and 4.9(a). Hence the map $i^* \prod_{q-p=n} \mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q) \longrightarrow \prod_{q-p=n} i^* \mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p, \mathcal{K}^q)$ is an isomorphism by Lemma 4.26, and we are done.

In the general case of the assumptions of the lemma, put $\mathcal{N} = \bigoplus_{p\in\mathbb{Z}} \mathcal{M}^p$ and $\mathcal{L} = \prod_{q\in\mathbb{Z}} \mathcal{K}^q$. Then $\mathcal{N}$ is a quasi-coherent sheaf and $\mathcal{L}$ is an injective quasi-coherent sheaf on X. The functor $i^!\colon X\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ preserves both the infinite products (as a right adjoint functor) and coproducts (in fact, it even preserves direct limits, since Z is a reasonable closed subscheme in X). By Lemma 4.25, the natural morphism $i^* \mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{L}) \longrightarrow \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{N}, i^!\mathcal{L})$ is an isomorphism of quasi-coherent sheaves on Z. In every degree $n \in \mathbb{Z}$, the morphism (3) is a direct summand of this isomorphism, hence also an isomorphism.

Yet another approach is to notice that the proof of Lemma 4.26 is actually applicable to any family of quasi-coherent sheaves $\mathcal{P}_\xi$ each of which is a direct summand of a finite direct sum of direct images of quasi-coherent sheaves from affine open subschemes in a fixed finite affine open covering $X = \bigcup_\alpha U_\alpha$. Then it remains to observe that, for any quasi-coherent sheaf $\mathcal{M}$ and any injective quasi-coherent sheaf $\mathcal{K}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{K})$ is a direct summand of such a direct sum of direct images (since the quasi-coherent sheaf $\mathcal{K}$ is). $\square$

Let $\mathfrak{X}$ be a reasonable ind-semi-separated ind-scheme. Let $\mathscr{E}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$, and let $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Then the complex of pro-quasi-coherent pro-sheaves $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet) \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ is constructed as follows.

For every reasonable closed subscheme $Z \subset \mathfrak{X}$, put $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet)^{(Z)} = \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{E}^\bullet, i^!\mathscr{J}^\bullet) = \mathcal{H}om_{Z\text{-qc}}(\mathscr{E}^\bullet_{(Z)}, \mathscr{J}^\bullet_{(Z)})$, where $i\colon Z \longrightarrow X$ is the closed immersion morphism. According to Lemma 4.27, for every pair of reasonable closed subschemes Y, $Z \subset \mathfrak{X}$ such that $Z \subset Y$, we have $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet)^{(Z)} \simeq i_{ZY}^* \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet)^{(Y)}$, as required in the definition of a pro-quasi-coherent pro-sheaf (where $i_{ZY}\colon Z \longrightarrow Y$ is the closed immersion). This explains the meaning of the notation in Theorem 4.23.

Proof of Theorem 4.23. The equivalence $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \simeq \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is provided by Corollary 4.18.

The tensor product functor $\otimes_\mathfrak{X}\colon \mathfrak{X}\text{–}\mathsf{flat} \times \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ was constructed in Sections 3.2 and 3.4. Here we switch the two arguments and write $\otimes_\mathfrak{X}\colon \mathfrak{X}\text{–}\mathsf{tors} \times \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$. Given a complex of quasi-coherent torsion sheaves $\mathscr{E}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ and a complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$, the complex of quasi-coherent torsion sheaves $\mathscr{E}^\bullet \otimes_\mathfrak{X} \mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ is constructed by taking coproducts along the diagonals of the bicomplex of quasi-coherent torsion sheaves $\mathscr{E}^p \otimes_\mathfrak{X} \mathfrak{F}^q$, $p, q \in \mathbb{Z}$.

This construction defines a functor $\mathscr{E}^\bullet \otimes_\mathfrak{X} -\,\colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$, which obviously descends to a triangulated functor between the homotopy categories $\mathscr{E}^\bullet \otimes_\mathfrak{X} -\,\colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$. By Proposition 3.7, for any complex $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ and any closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$, we have $i^!(\mathscr{E}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet) \simeq i^!\mathscr{E}^\bullet \otimes_{\mathcal{O}_Z} i^*\mathfrak{F}^\bullet$.

Now let us assume that $\mathscr{E}^\bullet$ is a complex of injective quasi-coherent torsion sheaves, $\mathscr{E}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$. By Lemma 4.4(b), the complex $i^!\mathscr{E}^\bullet \otimes_{\mathcal{O}_Z} i^*\mathfrak{F}^\bullet$ is a complex of injective quasi-coherent sheaves on Z (recall that injectivity of quasi-coherent sheaves on a Noetherian scheme is preserved by coproducts). By Proposition 2.17(b), it follows that $\mathscr{E}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. We have constructed a triangulated functor

$$\mathscr{E}^\bullet \otimes_{\mathfrak{X}} - : \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}). \tag{4}$$

Let us check that the latter functor induces a well-defined triangulated functor

$$\mathscr{E}^\bullet \otimes_{\mathfrak{X}} - : \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}). \tag{5}$$

For this purpose, we need to show that the complex $\mathscr{E}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is contractible whenever a complex $\mathfrak{F}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat})$ is acyclic with respect to the exact category $\mathfrak{X}$–flat. By Lemma 4.21, it suffices to check that the complex $i^!(\mathscr{E}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet) \simeq i^!\mathscr{E}^\bullet \otimes_{\mathcal{O}_Z} i^*\mathfrak{F}^\bullet \in \mathsf{K}(Z\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is contractible. Here $i^!\mathscr{E}^\bullet$ is a complex of injective quasi-coherent sheaves on Z and $i^*\mathfrak{F}^\bullet$ is an acyclic complex in the exact category Z–flat.

The latter assertion is essentially a part of Theorem 4.22. One can say that any acyclic complex in Z–flat is coacyclic, and it is easy to see that the tensor product of a coacyclic complex in Z–flat with any complex in Z–qcoh is coacyclic; a coacyclic complex of injectives is contractible. Alternatively, one can assume that $i^!\mathscr{E}^\bullet$ is homotopy equivalent to a bounded complex in Z–$\mathsf{qcoh}_{\mathsf{inj}}$; on any Noetherian scheme Z, it is clear that the tensor product of an acyclic complex in Z–flat with a bounded complex of injectives is contractible. For quite general results in this direction, see Lemmas 5.1(d) and 5.2(c) below.

On the other hand, for any complex $\mathscr{E}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$, the construction preceding this proof provides a functor $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, -)\colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \longrightarrow \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$, which obviously descends to a triangulated functor between the homotopy categories $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, -)\colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{pro})$. By construction, for any complex $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ and any closed subscheme $Z \subset X$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$, we have $i^*\,\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet) = \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{E}^\bullet, i^!\mathscr{J}^\bullet)$.

Once again, assume that $\mathscr{E}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Then, by Lemma 4.10, the complex $\mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{E}^\bullet, i^!\mathscr{J}^\bullet)$ is a complex of flat quasi-coherent sheaves on Z. Hence $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \mathscr{J}^\bullet)$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. We have constructed a triangulated functor

$$\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, -)\colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}). \tag{6}$$

Composing the latter functor with the canonical triangulated Verdier quotient functor $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$, we obtain a triangulated functor

$$\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, -)\colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) \longrightarrow \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}). \tag{7}$$

It is straightforward to see that the functor (6) is right adjoint to the functor (4). Hence the functor (7) is right adjoint to the functor (5).

It remains to show that the functors (5) and (7) are mutually inverse equivalences when $\mathscr{E}^\bullet = \mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{X}$. For this purpose, we need to check that the adjunction morphisms are isomorphisms.

Let $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. Then the adjunction $\mathfrak{F}^\bullet \longrightarrow \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet)$ is a natural morphism in $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$; we need to show that it is an isomorphism in $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$. By Lemma 4.13, it suffices to check that $i^*\mathfrak{F}^\bullet \longrightarrow i^*\,\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet)$ is an isomorphism in $\mathsf{D}(Z\text{–}\mathsf{flat})$. According to the discussion above, we have isomorphisms of complexes

$$i^*\,\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet) \simeq \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet,\ i^!(\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet)) \\ \simeq \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet,\ i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} i^*\mathfrak{F}^\bullet).$$

So we have to show that the natural morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet,\ i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{F}^\bullet)$ is an isomorphism in $\mathsf{D}(Z\text{–}\mathsf{flat})$ for every complex $\mathcal{F}^\bullet$ of flat quasi-coherent sheaves on Z.

By the definitions of a dualizing complex on $\mathfrak{X}$ and on Z (see Section 4.2), the complex of injective quasi-coherent sheaves $i^!\mathscr{D}^\bullet$ on Z is homotopy equivalent to a bounded complex of injective quasi-coherent sheaves $\mathcal{D}_Z^\bullet$ (which is also a dualizing complex on Z). So the complex of flat quasi-coherent sheaves $\mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet, i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{F}^\bullet)$ on Z is homotopy equivalent to the complex of flat quasi-coherent sheaves $\mathcal{H}om_{Z\text{-qc}}(\mathcal{D}_Z^\bullet,\ \mathcal{D}_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{F}^\bullet)$. Finally, the assertion that the natural morphism of complexes of flat quasi-coherent sheaves $\mathcal{F}^\bullet \longrightarrow \mathcal{H}om_{Z\text{-qc}}(\mathcal{D}_Z^\bullet,\ \mathcal{D}_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{F}^\bullet)$ is an isomorphism in $\mathsf{D}(Z\text{–}\mathsf{flat})$ is essentially a part of Theorem 4.22 for the semi-separated Noetherian scheme Z.

Similarly, let $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Then the adjunction $\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{J}^\bullet) \longrightarrow \mathscr{J}^\bullet$ is a natural morphism in $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$; we have to show that it is a homotopy equivalence. By Lemma 4.21, it suffices to check that $i^!(\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{J}^\bullet)) \longrightarrow i^!\mathscr{J}^\bullet$ is a homotopy equivalence of complexes in $Z\text{–}\mathsf{qcoh}_{\mathsf{inj}}$. According to the discussion above, we have isomorphisms of complexes

$$i^!(\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{J}^\bullet)) \simeq i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} i^*\,\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,\ \mathscr{J}^\bullet) \\ \simeq i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet, i^!\mathscr{J}^\bullet).$$

So we have to show that the natural morphism $i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet, \mathcal{J}^\bullet) \longrightarrow \mathcal{J}^\bullet$ is a homotopy equivalence of complexes in $Z\text{–}\mathsf{qcoh}_{\mathsf{inj}}$ for every complex $\mathcal{J}^\bullet$ of injective quasi-coherent sheaves on Z.

As above, the complex of injective quasi-coherent sheaves $i^!\mathscr{D}^\bullet \otimes_{\mathcal{O}_Z} \mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{D}^\bullet, \mathcal{J}^\bullet)$ on Z is homotopy equivalent to the complex of injective quasi-coherent sheaves $\mathcal{D}_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{H}om_{Z\text{-qc}}(\mathcal{D}_Z^\bullet, \mathcal{J}^\bullet)$. Once again, the assertion that the natural morphism of complexes of injective quasi-coherent sheaves $\mathcal{D}_Z^\bullet \otimes_{\mathcal{O}_Z} \mathcal{H}om_{Z\text{-qc}}(\mathcal{D}_Z^\bullet, \mathcal{J}^\bullet) \longrightarrow \mathcal{J}^\bullet$ is a homotopy equivalence is essentially a part of Theorem 4.22. □

5. The Cotensor Product

The triangulated tensor structure on the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ of quasi-coherent sheaves on a Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet$ was introduced in [31, Propositions 6.2, 8.10, and B.6] and studied in [12, Section B.2.5], where it was denoted by $\square_{\mathcal{D}^\bullet}$ and called the *cotensor product* of complexes of quasi-coherent sheaves on X over the dualizing complex $\mathcal{D}^\bullet$. The aim of this section is to generalize this construction to complexes of quasi-coherent torsion sheaves on an ind-Noetherian ind-scheme with a dualizing complex, and explain the connection with the cotensor product of complexes of comodules over a cocommutative coalgebra.

5.1. **Construction of cotensor product.** We start with a lemma about tensor products of complexes of quasi-coherent sheaves on a scheme.

Lemma 5.1. *Let $\mathcal{M}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh})$ be a complex of quasi-coherent sheaves and $\mathcal{F}^\bullet$, $G^\bullet \in \mathsf{C}(X\text{–}\mathsf{flat})$ be two complexes of flat quasi-coherent sheaves on a scheme X.*

(a) *If the complex $\mathcal{F}^\bullet$ is acyclic in $X\text{–}\mathsf{flat}$, then the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ is acyclic in $X\text{–}\mathsf{qcoh}$.*

(b) *If the complex $\mathcal{F}^\bullet$ is acyclic in $X\text{–}\mathsf{flat}$, then the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet$ is acyclic in $X\text{–}\mathsf{flat}$.*

(c) *If the complex $\mathcal{M}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$, then the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$.*

(d) *If the complex $\mathcal{F}^\bullet$ is acyclic in $X\text{–}\mathsf{flat}$ and the scheme X is Noetherian, then the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$.*

Proof. The assertions (a–b) are essentially local and reduce to the case of an affine scheme X. In this context, both the assertions are explained by the observation that over an (arbitrary associative) ring R, an acyclic (in $R\text{–}\mathsf{mod}$) complex of flat modules is homotopy flat if and only if it has flat modules of cocycles.

Specifically, part (a) is provable by representing $\mathcal{M}^\bullet$ as a direct limit of bounded complexes (with a silly truncation on the left and a canonical truncation on the right). This reduces the question to the case of a one-term complex $\mathcal{M}^\bullet = \mathcal{M}$, which is obvious. To prove part (b), it suffices to check that the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}$ is exact in $X\text{–}\mathsf{qcoh}$ for every sheaf $\mathcal{N} \in X\text{–}\mathsf{qcoh}$. This follows from part (a) applied to the complexes $\mathcal{F}^\bullet$ and $\mathcal{M}^\bullet = \mathcal{G}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}$.

The assertions (c–d) are essentially local, too (assuming that X is either quasi-compact and semi-separated or else Noetherian). But this needs to be explained (we postpone this discussion to Section A.2 of the appendix; cf. [12, Remark 1.3]). Part (c) is straightforward: it suffices to observe that the functor $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} -$ takes short exact sequences of complexes in $X\text{–}\mathsf{qcoh}$ to short exact sequences of complexes in $X\text{–}\mathsf{qcoh}$ and preserves coproducts of complexes in $X\text{–}\mathsf{qcoh}$.

Part (d): by Corolary 4.18, there exists a complex of injective quasi-coherent sheaves $\mathcal{J}^\bullet$ on X endowed with a morphism of complexes $\mathcal{M}^\bullet \longrightarrow \mathcal{J}^\bullet$ whose cone $\mathcal{N}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$. By part (c), the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$. It remains to check that the complex $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{J}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$; in fact, this complex is contractible. This is the result of [35, Corollary 9.7(ii)] (see [35, Theorem 8.6] for context) and [31, Lemma 8.2].

Let us spell out some details, following [35, 31]. By Lemma 4.4(b), $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{J}^\bullet$ is a complex of injective quasi-coherent sheaves on X. A complex of injective objects is coacyclic if and only if it is contractible, and if and only if its objects of cocycles are injective. Injectivity of a quasi-coherent sheaf on a Noetherian scheme is a local property; hence the question is local and reduces to affine schemes.

Let $F^\bullet$ be a complex of flat modules over a Noetherian commutative ring R and $J^\bullet$ be a complex of injective R-modules. Assume that the complex $F^\bullet$ is acyclic in $R\text{–}\mathsf{flat}$. Let L be a finitely generated (equivalently, finitely presentable) R-module. Then we have a natural isomorphism of complexes of R-modules $\operatorname{Hom}_R(L,\, F^\bullet \otimes_R J^\bullet) \simeq F^\bullet \otimes_R \operatorname{Hom}_R(L, J^\bullet)$. By part (a), the complex $F^\bullet \otimes_R \operatorname{Hom}_R(L, J^\bullet)$ is acyclic. By Lemma 4.20, acyclicity of the complexes $\operatorname{Hom}_R(L,\, F^\bullet \otimes_R J^\bullet)$ for all finitely generated R-modules L implies contractibility of the complex $F^\bullet \otimes_R J^\bullet$. ☐

Let $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ be two complexes of pro-quasi-coherent pro-sheaves on an ind-scheme $\mathfrak{X}$. Then the complex $\mathfrak{P}^\bullet \otimes^{\mathfrak{X}} \mathfrak{Q}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ is constructed by totalizing the bicomplex $\mathfrak{P}^p \otimes^{\mathfrak{X}} \mathfrak{Q}^q$, $p, q \in \mathbb{Z}$, by taking infinite coproducts in $\mathfrak{X}\text{–}\mathsf{pro}$ along the diagonals of the bicomplex. Since the full subcategory $\mathfrak{X}\text{–}\mathsf{flat}$ is closed under coproducts in $\mathfrak{X}\text{–}\mathsf{pro}$ (see Section 3.5), the tensor product of two complexes in $\mathfrak{X}\text{–}\mathsf{flat}$ is a complex in $\mathfrak{X}\text{–}\mathsf{flat}$,

$$\otimes^{\mathfrak{X}} \colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}). \tag{8}$$

Similarly, let $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ be a complex of pro-quasi-coherent pro-sheaves and $\mathscr{M}^\bullet \in \mathsf{C}(X\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on a reasonable ind-scheme $\mathfrak{X}$. Then the complex $\mathfrak{P}^\bullet \otimes_{\mathfrak{X}} \mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ is constructed by totalizing the bicomplex $\mathfrak{P}^p \otimes_{\mathfrak{X}} \mathscr{M}^q$, $p, q \in \mathbb{Z}$, by taking infinite coproducts in $\mathfrak{X}\text{–}\mathsf{tors}$ along the diagonals of the bicomplex (as in the construction of the tensor product functor in the proof of Theorem 4.23). In particular, we have

$$\otimes_{\mathfrak{X}} \colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}). \tag{9}$$

As the tensor product on $\mathfrak{X}\text{–}\mathsf{pro}$ and the action of $\mathfrak{X}\text{–}\mathsf{pro}$ in $\mathfrak{X}\text{–}\mathsf{tors}$ preserve coproducts in both the categories (see Section 3.5), the above tensor products of complexes are associative. So $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ is a tensor category, $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \subset \mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$ is a tensor subcategory, and $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ is a module category over $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{pro})$. Hence, in particular, $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ is a module category over $\mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$.

The tensor product functors (8–9) obviously descend to the homotopy categories, providing tensor product functors

$$\otimes^{\mathfrak{X}} \colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}), \tag{10}$$

$$\otimes_{\mathfrak{X}} \colon \mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}), \tag{11}$$

making $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat})$ a tensor triangulated category and $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$ a triangulated module category over $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{flat})$.

Lemma 5.2. (a) *Let $\mathfrak{X}$ be an ind-scheme and $\mathfrak{F}^\bullet$, $\mathfrak{G}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ be two complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. Assume that the complex $\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{X}\text{–}\mathsf{flat}$. Then the complex $\mathfrak{F}^\bullet \otimes^{\mathfrak{X}} \mathfrak{G}^\bullet$ is acyclic in $\mathfrak{X}\text{–}\mathsf{flat}$.*

(b) *Let $\mathfrak{X}$ be a reasonable ind-scheme, $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$, and $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. Assume that the complex $\mathscr{M}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. Then the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{M}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$.*

(c) *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$, and $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. Assume that the complex $\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{X}\text{–}\mathsf{flat}$. Then the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{M}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$.*

Proof. Part (a) follows from Lemmas 4.13 and 5.1(b). Part (b) holds, because the functor $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} -$ takes short exact sequences of complexes in $\mathfrak{X}\text{–}\mathsf{tors}$ to short exact sequences of complexes in $\mathfrak{X}\text{–}\mathsf{tors}$ and preserves coproducts of complexes in $\mathfrak{X}\text{–}\mathsf{tors}$. Similarly one shows that the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{M}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$ whenever the complex $\mathfrak{F}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{flat}$ and $\mathscr{M}^\bullet$ is an arbitrary complex in $\mathfrak{X}\text{–}\mathsf{tors}$.

Part (c): by Corollary 4.18, there exists a complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ together with a morphism of complexes $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ whose cone $\mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. By part (b), the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. It remains to check that the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{J}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. In fact, we will show that this complex is contractible.

Indeed, according to the arguments in the beginning of the proof of Theorem 4.23 in Section 4.5, $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{J}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Furthermore, for any closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$, Proposition 3.7 provides a natural isomorphism of complexes of quasi-coherent sheaves $i^!(\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{J}^\bullet) \simeq i^*\mathfrak{F}^\bullet \otimes_{\mathcal{O}_Z} i^!\mathscr{J}^\bullet$ on Z. In the situation at hand, the complex $i^*\mathfrak{F}^\bullet$ is acyclic in $Z\text{–}\mathsf{flat}$, while $i^!\mathscr{J}^\bullet$ is a complex of injective quasi-coherent sheaves on Z; hence, by (the proof of) Lemma 5.1(d), the complex of injective quasi-coherent sheaves $i^*\mathfrak{F}^\bullet \otimes_{\mathcal{O}_Z} i^!\mathscr{J}^\bullet$ on Z is contractible. By Lemma 4.21, it follows that the complex $\mathfrak{F}^\bullet \otimes_{\mathfrak{X}} \mathscr{J}^\bullet$ in $\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$ is contractible, too. ☐

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme. It is clear from Lemma 5.2 that the tensor product functors (10–11) descend to the derived and coderived categories, providing tensor product functors

$$\otimes^{\mathfrak{X}}\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}), \tag{12}$$

$$\otimes_{\mathfrak{X}}\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}). \tag{13}$$

So $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$ is a tensor triangulated category and $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ is a triangulated module category over $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$.

Now let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$. Then the triangulated equivalence $\mathscr{D}^\bullet \otimes_{\mathfrak{X}} -\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ from Theorem 4.23 is an equivalence of module categories over $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$. This follows from the associativity of the tensor products,

$$(\mathfrak{F}^\bullet \otimes^{\mathfrak{X}} \mathfrak{G}^\bullet) \otimes_{\mathfrak{X}} \mathscr{D}^\bullet \simeq \mathfrak{F}^\bullet \otimes_{\mathfrak{X}} (\mathfrak{G}^\bullet \otimes_{\mathfrak{X}} \mathscr{D}^\bullet)$$

for all complexes of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ on $\mathfrak{X}$.

Using the triangulated equivalence $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, we transfer the tensor structure of the category $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$ to the category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. The resulting functor

$$\square_{\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}), \tag{14}$$

defining a tensor triangulated category structure on $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, is called the *cotensor product* of complexes of quasi-coherent torsion sheaves on $\mathfrak{X}$ over the dualizing complex $\mathscr{D}^\bullet$. Explicitly, we have

$$\mathscr{M}^\bullet \,\square_{\mathscr{D}^\bullet}\, \mathscr{N}^\bullet = \mathscr{D}^\bullet \otimes_{\mathfrak{X}} (\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{K}^\bullet) \otimes^{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet)) \\ \simeq \mathscr{M}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet)$$

in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ for any complexes $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$ endowed with morphisms with coacyclic cones $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$ into complexes $\mathscr{K}^\bullet$ and $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$. The dualizing complex $\mathscr{D}^\bullet \in \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ is the unit object of the tensor structure $\square_{\mathscr{D}^\bullet}$ on $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, since $\mathscr{D}^\bullet$ corresponds to the unit object $\mathfrak{O}_{\mathfrak{X}} \in \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})$ under the equivalence of categories $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Remarks 5.3. The cotensor product functor $\square_{\mathscr{D}^\bullet}$ (14) is "similar to a right derived functor" in the following sense (see Theorem 6.35 below for a much more specific assertion).

(1) Given an additive category A, let us denote by $\mathsf{K}^{\le 0}(\mathsf{A})$ and $\mathsf{K}^{\ge 0}(\mathsf{A}) \subset \mathsf{K}(\mathsf{A})$ the full subcategories in the homotopy category consisting of complexes concentrated in the nonpositive and nonnegative cohomological degrees, respectively. For an abelian category A, the notation $\mathsf{D}^{\le 0}(\mathsf{A})$ and $\mathsf{D}^{\ge 0}(\mathsf{A})$ is understood similarly. When A is abelian, a complex $A^\bullet \in \mathsf{D}^+(\mathsf{A})$ is said to have *injective dimension* ≤ 0 if $\mathrm{Hom}_{\mathsf{D}^+(\mathsf{A})}(C^\bullet, A^\bullet) = 0$ for all bounded complexes $C^\bullet \in \mathsf{D}^{\ge 0}(\mathsf{A})$. When A is abelian with enough injective objects, a bounded below complex in A has injective dimension ≤ 0 if and only if it is quasi-isomorphic to a bounded complex in $\mathsf{K}^{\le 0}(\mathsf{A}_{\mathsf{inj}})$.

For any abelian category A with exact functors of infinite direct sum, the full subcategories of nonpositively and nonnegatively situated complexes $\mathsf{D}^{\mathsf{co},\le 0}(\mathsf{A})$ and $\mathsf{D}^{\mathsf{co},\ge 0}(\mathsf{A})$ form a t-structure (of the derived type) on the coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{A})$ [40, Remark 4.1], [55, Proposition 5.5]. Furthermore, if there are enough injective objects in A, then for any complex $A^\bullet \in \mathsf{K}^{\ge 0}(\mathsf{A})$ there is a complex $J^\bullet \in \mathsf{K}^{\ge 0}(\mathsf{A}_{\mathsf{inj}})$ together with a quasi-isomorphism $A^\bullet \longrightarrow J^\bullet$ of complexes in A. Since any bounded below acyclic complex is coacyclic [40, Lemma 2.1] (cf. [44, Lemma A.1.2(a)]), the morphism of complexes $A^\bullet \longrightarrow J^\bullet$ is an isomorphism in $\mathsf{D}^{\mathsf{co}}(\mathsf{A})$. So we have equivalences of categories $\mathsf{K}^{\ge 0}(\mathsf{A}_{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{co},\ge 0}(\mathsf{A}) \simeq \mathsf{D}^{\ge 0}(\mathsf{A})$.

(2) In the context of the exposition above in this section, assume that the dualizing complex $\mathscr{D}^\bullet$ is concentrated in the nonpositive cohomological degrees, $\mathscr{D}^\bullet \in \mathsf{K}^{\le 0}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$. For any complexes $\mathscr{M}^\bullet$, $\mathscr{N}^\bullet \in \mathsf{K}^{\ge 0}(\mathfrak{X}\text{–}\mathsf{tors})$, one can choose a complex $\mathscr{J}^\bullet \in \mathsf{K}^{\ge 0}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ together with a morphism with (co)acyclic cone $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$.

Then one has $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet) \in \mathsf{K}^{\ge 0}(\mathfrak{X}\text{–flat})$ and $\mathscr{M}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet) \in \mathsf{K}^{\ge 0}(\mathfrak{X}\text{–tors})$. Thus the functor $\square_{\mathscr{D}^\bullet}$ (14) restricts to a functor

$$\square_{\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{co},\ge 0}(\mathfrak{X}\text{–tors}) \times \mathsf{D}^{\mathsf{co},\ge 0}(\mathfrak{X}\text{–tors}) \longrightarrow \mathsf{D}^{\mathsf{co},\ge 0}(\mathfrak{X}\text{–tors}).$$

(3) Let us explain why the assumption that $\mathscr{D}^\bullet \in \mathsf{K}^{\le 0}(\mathfrak{X}\text{–tors}_{\mathsf{inj}})$ is mild and reasonable. Suppose that the ind-scheme $\mathfrak{X}$ is of ind-finite type over a Noetherian scheme S with a dualizing complex $\mathcal{D}^\bullet$. Without loss of generality, one can assume $\mathcal{D}^\bullet$ to be a bounded complex of injective quasi-coherent sheaves on S; shifting if necessary, one can assume further that $\mathcal{D}^\bullet \in \mathsf{K}^{\le 0}(S\text{–qcoh}_{\mathsf{inj}})$. Let $Z \subset \mathfrak{X}$ be a closed subscheme; so we have a morphism of finite type $Z \longrightarrow S$.

The idea is to use the extraordinary inverse image functor in order to lift $\mathcal{D}^\bullet$ to a dualizing complex $\mathcal{D}^\bullet_Z$ on Z. Given a morphism of finite type between Noetherian schemes $f\colon Y \longrightarrow X$, the relevant functor is denoted by $f^!$ in [19] and Deligne's appendix to [19] (where it was first constructed, under mild assumptions on f); in the terminology and notation of [44], it is called the *extraordinary inverse image functor in the sense of Deligne* and denoted by $f^+\colon \mathsf{D}^+(X\text{–qcoh}) \longrightarrow \mathsf{D}^+(Y\text{–qcoh})$.

First of all one observes that the derived direct image functor $\mathbb{R}f_*\colon \mathsf{D}^+(Y\text{–qcoh}) \longrightarrow \mathsf{D}^+(X\text{–qcoh})$ has a right adjoint functor; this functor is called the *extraordinary inverse image functor in the sense of Neeman* and denoted by $f^!$ in [44]. Both the functor $\mathbb{R}f_*$ and its right adjoint $f^!$ are also well-defined on the unbounded derived categories and the coderived categories. In particular, for a closed immersion $i\colon Z \longrightarrow Y$, the triangulated functor $i^! = \mathbb{R}i^!\colon \mathsf{D}^+(Y\text{–qcoh}) \longrightarrow \mathsf{D}^+(Z\text{–qcoh})$ is simply the right derived functor of the left exact functor $i^!\colon Y\text{–qcoh} \longrightarrow Z\text{–qcoh}$.

The functor f^+ is essentially characterized by three properties: for a composable pair of morphisms f and g, one has $(fg)^+ \simeq g^+f^+$; for an open immersion g, one has $g^+ = g^*$; for a proper morphism f, one has $f^+ = f^!$. In particular, for a closed immersion i one has $i^+ = \mathbb{R}i^!$. For a smooth morphism f, the functor f^+ only differs from f^* by a shift and a twist [19, Chapter III].

The functor f^+ takes dualizing complexes to dualizing complexes [19, Proposition V.2.4, Theorem V.8.3 and Remark in Section V.8]. More precisely, in our terminology one can say that, given a morphism $f\colon Y \longrightarrow X$ and a dualizing complex $\mathcal{D}^\bullet_X$ on X, the object $f^+\mathcal{D}^\bullet_X \in \mathsf{D}^+(Y\text{–qcoh})$ is quasi-isomorphic to a dualizing complex (of injective quasi-coherent sheaves) $\mathcal{D}^\bullet_Y$ on Y.

Concerning an unbounded version of the functor f^+, it turns out to be well-defined as a functor between the coderived categories $f^+\colon \mathsf{D}^{\mathsf{co}}(X\text{–qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Y\text{–qcoh})$, but *not* as a functor between the conventional unbounded derived categories $\mathsf{D}(X\text{–qcoh})$ and $\mathsf{D}(Y\text{–qcoh})$ [16], [44, Introduction and Section 5.16].

In the recent overview [38], the notation $f^\times$ is used for the functor which we denote by $f^!$, and the notation $f^!$ is used for the functor which we denote by f^+.

(4) The key observation for our purposes is that the functor f^+ takes complexes of injective dimension ≤ 0 to complexes of injective dimension ≤ 0. Indeed, the restriction to an open subscheme in a Noetherian scheme preserves injective dimension, since it preserves injectivity. It remains to see that the right adjoint functor $f^!$ to the

functor $\mathbb{R}f_*$ preserves injective dimension. Indeed, if $\mathcal{N}^\bullet \in \mathsf{D}^{\ge 0}(Y\text{–}\mathsf{qcoh})$ is a bounded complex and $\mathcal{K}^\bullet \in \mathsf{K}^{\le 0}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is a bounded complex of injective dimension ≤ 0, then

$$\mathrm{Hom}_{\mathsf{D}^+(Y\text{–}\mathsf{qcoh})}(\mathcal{N}^\bullet, f^!\mathcal{K}^\bullet) \simeq \mathrm{Hom}_{\mathsf{D}^+(X\text{–}\mathsf{qcoh})}(\mathbb{R}f_*\mathcal{N}^\bullet, \mathcal{K}^\bullet) = 0$$

since $\mathbb{R}f_*\mathcal{N}^\bullet \in \mathsf{D}^{\ge 0}(X\text{–}\mathsf{qcoh})$.

(5) Returning to the situation at hand, we have a closed subscheme $Z \subset \mathfrak{X}$; denote by $p_Z\colon Z \longrightarrow S$ the related morphism, and let $\mathcal{D}_Z^\bullet$ be a dualizing complex (of injective quasi-coherent sheaves) on Z quasi-isomorphic to $p_Z^+\mathcal{D}^\bullet \in \mathsf{D}^+(Z\text{–}\mathsf{qcoh})$. The argument above allows to have $\mathcal{D}_Z^\bullet \in \mathsf{K}^{\le 0}(Z\text{–}\mathsf{qcoh}_{\mathsf{inj}})$. For any pair of closed subschemes $Z \subset Y \subset \mathcal{D}^\bullet$ with the related open immersion $i\colon Z \longrightarrow Y$, the complex $i^!\mathcal{D}_Y^\bullet \in \mathsf{K}^{\le 0}(Z\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is naturally homotopy equivalent to $\mathcal{D}_Z^\bullet$, since $p_Z = p_Y i$ and $i^+ \simeq \mathbb{R}i^!$. In this sense, the dualizing complexes $\mathcal{D}_Z^\bullet$ on closed subschemes $Z \subset \mathfrak{X}$ agree with each other up to homotopy equivalence.

Assuming additionally that $\mathfrak{X}$ is an $\aleph_0$-ind-scheme, one can apply the construction of Example 4.7 in order to produce a dualizing complex $\mathscr{D}^\bullet \in \mathsf{K}^{\le 0}(\mathfrak{X}\text{–}\mathsf{tors})$ on $\mathfrak{X}$ out of the dualizing complexes $\mathcal{D}_Z^\bullet$ on the closed subschemes $Z \subset \mathfrak{X}$.

5.2. **Ind-Artinian examples.** The following example explains the terminology “coderived category” and “cotensor product”.

Examples 5.4. (1) Let $\mathscr{C}$ be a coassociative, counital coalgebra over a field $\Bbbk$. Let $\mathscr{M}$ be a right $\mathscr{C}$-comodule and $\mathscr{N}$ be a left $\mathscr{C}$-comodule. Then the *cotensor product* $\mathscr{M} \,\square_{\mathscr{C}}\, \mathscr{N}$ is the $\Bbbk$-vector space constructed as the kernel of the difference of the natural pair of maps

$$\mathscr{M} \otimes_{\Bbbk} \mathscr{N} \rightrightarrows \mathscr{M} \otimes_{\Bbbk} \mathscr{C} \otimes_{\Bbbk} \mathscr{N}.$$

Here one map $\mathscr{M} \otimes_{\Bbbk} \mathscr{N} \longrightarrow \mathscr{M} \otimes_{\Bbbk} \mathscr{C} \otimes_{\Bbbk} \mathscr{N}$ is induced by the right coaction map $\mathscr{M} \longrightarrow \mathscr{M} \otimes_{\Bbbk} \mathscr{C}$ and the other one by the left coaction map $\mathscr{N} \longrightarrow \mathscr{C} \otimes_{\Bbbk} \mathscr{N}$.

(2) When the coalgebra $\mathscr{C}$ is cocommutative, there is no difference between left and right $\mathscr{C}$-comodules. Moreover, the cotensor product $\mathscr{M} \,\square_{\mathscr{C}}\, \mathscr{N}$ of two $\mathscr{C}$-comodules $\mathscr{M}$ and $\mathscr{N}$ has a natural $\mathscr{C}$-comodule structure in this case. The cotensor product operation $\square_{\mathscr{C}}$ makes the abelian category of $\mathscr{C}$-comodules $\mathscr{C}\text{–}\mathsf{comod}$ an associative, commutative, and unital tensor category with the unit object $\mathscr{C} \in \mathscr{C}\text{–}\mathsf{comod}$.

(3) For any coalgebra $\mathscr{C}$ as in (1), the categories $\mathscr{C}\text{–}\mathsf{comod}$ and $\mathsf{comod}\text{–}\mathscr{C}$ of left and right $\mathscr{C}$-comodules are locally Noetherian (in fact, locally finite) Grothendieck abelian categories, so Proposition 4.15 with Lemma 4.16 are applicable. Hence the coderived categories of $\mathscr{C}$-comodules are equivalent to the respective homotopy categories of (complexes of) injective comodules, $\mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod}) \simeq \mathsf{K}(\mathscr{C}\text{–}\mathsf{comod}_{\mathsf{inj}})$ and $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}\mathscr{C}) \simeq \mathsf{K}(\mathsf{comod}_{\mathsf{inj}}\text{–}\mathscr{C})$.

The left $\mathscr{C}$-comodule $\mathscr{C}$ is an injective cogenerator of $\mathscr{C}\text{–}\mathsf{comod}$; moreover, a $\mathscr{C}$-comodule is injective if and only if it is a direct summand of a direct sum of copies of the $\mathscr{C}$-comodule $\mathscr{C}$. For a given left $\mathscr{C}$-comodule $\mathscr{J}$, the coproduct functor $-\,\square_{\mathscr{C}}\,\mathscr{J}\colon \mathsf{comod}\text{–}\mathscr{C} \longrightarrow \Bbbk\text{–}\mathsf{vect}$ is exact if and only if $\mathscr{J}$ is an injective left $\mathscr{C}$-comodule (and similarly for a right $\mathscr{C}$-comodule). Here $\Bbbk\text{–}\mathsf{vect}$ denotes the category of $\Bbbk$-vector spaces.

The functor of cotensor product of complexes of comodules

$$\square_{\mathscr{C}}\colon \mathsf{C}(\mathsf{comod}\text{–}\mathscr{C}) \times \mathsf{C}(\mathscr{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{C}(\Bbbk\text{–}\mathsf{vect})$$

is constructed in the obvious way (totalizing the bicomplex of cotensor products by taking direct sums along the diagonals). To define the right derived functor of cotensor product

$$\square_{\mathscr{C}}^{\mathbb{R}}\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}\mathscr{C}) \times \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(\Bbbk\text{–}\mathsf{vect}),$$

suppose that we are given a complex of right $\mathscr{C}$-comodules $\mathscr{M}^\bullet$ and a complex of left $\mathscr{C}$-comodules $\mathscr{N}^\bullet$. Let $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ be a morphism in $\mathsf{C}(\mathsf{comod}\text{–}\mathscr{C})$ from $\mathscr{M}^\bullet$ to a complex of injective right $\mathscr{C}$-comodules $\mathscr{K}^\bullet$ such that the cone of $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ is coacyclic in $\mathsf{comod}\text{–}\mathscr{C}$. Let $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$ be a similar resolution of the complex of left $\mathscr{C}$-comodules $\mathscr{N}^\bullet$; so $\mathscr{J}^\bullet \in \mathsf{C}(\mathscr{C}\text{–}\mathsf{comod}_{\mathsf{inj}})$ and the cone is coacyclic in $\mathscr{C}\text{–}\mathsf{comod}$. Then the derived cotensor product is defined as the object

$$\mathscr{M}^\bullet \mathbin{\square_{\mathscr{C}}^{\mathbb{R}}} \mathscr{N}^\bullet = \mathscr{K}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{J}^\bullet \simeq \mathscr{M}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{J}^\bullet \simeq \mathscr{K}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{N}^\bullet \in \mathsf{D}(\Bbbk\text{–}\mathsf{vect}).$$

We refer to [40, Section 0.2] for a discussion.

(4) For a cocommutative coalgebra $\mathscr{C}$ as in (2), the functor of cotensor product of complexes of comodules

$$\square_{\mathscr{C}}\colon \mathsf{C}(\mathscr{C}\text{–}\mathsf{comod}) \times \mathsf{C}(\mathscr{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{C}(\mathscr{C}\text{–}\mathsf{comod})$$

is constructed, once again, by totalizing the complex of cotensor products by taking coproducts along the diagonals (notice that the forgetful functor $\mathscr{C}\text{–}\mathsf{comod} \longrightarrow \Bbbk\text{–}\mathsf{vect}$ preserves coproducts). The construction of the right derived functor of cotensor product

$$\square_{\mathscr{C}}^{\mathbb{R}}\colon \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod}) \times \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod})$$

is similar to the one in (3). In the same notation and the same conditions on the resolutions $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$, we put

$$\mathscr{M}^\bullet \mathbin{\square_{\mathscr{C}}^{\mathbb{R}}} \mathscr{N}^\bullet = \mathscr{K}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{J}^\bullet \simeq \mathscr{M}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{J}^\bullet \simeq \mathscr{K}^\bullet \mathbin{\square_{\mathscr{C}}} \mathscr{N}^\bullet \in \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod}).$$

The derived tensor product operation $\square_{\mathscr{C}}^{\mathbb{R}}$ makes $\mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod})$ an associative, commutative, and unital tensor triangulated category.

(5) For a cocommutative coalgebra $\mathscr{C}$, let $\mathfrak{X} = \operatorname{Spi}\mathscr{C}^*$ be the ind-Artinian ind-scheme corresponding to $\mathscr{C}$ as per the construction from Example 1.5(2) and the discussion in Example 4.1(2). According to Section 2.4(4), the abelian category $\mathfrak{X}\text{–}\mathsf{tors}$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is equivalent to the abelian category of $\mathscr{C}$-comodules $\mathscr{C}\text{–}\mathsf{comod}$.

Notice that any Artinian scheme admits a dualizing injective quasi-coherent sheaf, i. e., a dualizing complex which is a one-term complex of injective quasi-coherent sheaves. In particular, for any finite-dimensional cocommutative coalgebra $\mathcal{E}$ over $\Bbbk$, the quasi-coherent sheaf on $\operatorname{Spec}\mathcal{E}^*$ corresponding to the injective $\mathcal{E}^*$-module $\mathcal{E}$ is a dualizing complex on $\operatorname{Spec}\mathcal{E}^*$. It follows that the injective quasi-coherent torsion sheaf on $\operatorname{Spi}\mathscr{C}^*$ corresponding to the injective $\mathscr{C}$-comodule $\mathscr{C}$ is a dualizing complex on $\operatorname{Spi}\mathscr{C}^*$.

Following the proof of Theorem 4.23 specialized to the particular case of a one-term dualizing complex of injectives $\mathscr{D}^\bullet = \mathscr{C}$, one can see that there is an equivalence of additive categories of injective quasi-coherent torsion sheaves and flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$, provided by the mutually inverse functors $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{C}, -)$ and $\mathscr{C} \otimes_{\mathfrak{X}} -$,

$$(15) \qquad \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{C}, -) \colon \mathfrak{X}\text{–tors}_{\mathsf{inj}} \simeq \mathfrak{X}\text{–flat} : \mathscr{C} \otimes_{\mathfrak{X}} -.$$

Moreover, this equivalence transforms short exact sequences in $\mathfrak{X}$–flat into split short exact sequences in $\mathfrak{X}\text{–tors}_{\mathsf{inj}}$; so the exact category structure on $\mathfrak{X}$–flat is split.

(6) We refer to [40, Section 0.2.4] or [45, Section 1] for an introductory discussion of the category of left contramodules $\mathscr{C}$–contra over a coassociative coalgebra $\mathscr{C}$. Contramodules over a coalgebra $\mathscr{C}$ are the same thing as contramodules over the topological ring $\mathfrak{R} = \mathscr{C}^*$ [43, Section 1.10], [45, Sections 2.1 and 2.3].

The category $\mathscr{C}$–contra is abelian with enough projective objects. Denoting by $\mathscr{C}\text{–contra}_{\mathsf{proj}} \subset \mathscr{C}\text{–contra}$ the full subcategory of projective $\mathscr{C}$-contramodules, one has a natural equivalence of additive categories [40, Section 0.2.6], [45, Sections 1.2 and 3.1], [41, Sections 5.1–5.2]

$$(16) \qquad \mathrm{Hom}_{\mathscr{C}}(\mathscr{C}, -) \colon \mathscr{C}\text{–comod}_{\mathsf{inj}} \simeq \mathscr{C}\text{–contra}_{\mathsf{proj}} : \mathscr{C} \odot_{\mathscr{C}} -.$$

Here $\mathrm{Hom}_{\mathscr{C}} = \mathrm{Hom}_{\mathscr{C}\text{–comod}}$ denotes the $\Bbbk$-vector space of morphisms in the abelian category $\mathscr{C}$–comod, while $\odot_{\mathscr{C}}$ is the *contratensor product* functor.

For a cocommutative coalgebra $\mathscr{C}$, it is not difficult to construct a pair of adjoint functors between the additive categories $\mathfrak{X}$–pro and $\mathscr{C}$–contra (cf. Examples 3.8). For a finite-dimensional subcoalgebra $\mathcal{E} \subset \mathscr{C}$, let $X_{\mathcal{E}} \subset \mathfrak{X}$ denote the closed subscheme $\operatorname{Spec} \mathcal{E}^* \subset \operatorname{Spi} \mathscr{C}^*$. For a finite-dimensional coalgebra $\mathcal{E}$, the category of $\mathcal{E}$-contramodules is naturally equivalent to the category of $\mathcal{E}^*$-modules.

The left adjoint functor $\mathscr{C}\text{–contra} \longrightarrow \mathfrak{X}\text{–pro}$ assigns to a $\mathscr{C}$-contramodule $\mathfrak{G}$ the pro-quasi-coherent pro-sheaf $\mathfrak{P}$ whose component $\mathfrak{P}^{(X_{\mathcal{E}})}$ is the $\mathcal{E}^*$-module produced as the maximal quotient contramodule of $\mathfrak{G}$ whose $\mathscr{C}$-contramodule structure comes from an $\mathcal{E}$-contramodule structure. The right adjoint functor $\mathfrak{X}\text{–pro} \longrightarrow \mathscr{C}\text{–contra}$ assigns to a pro-quasi-coherent pro-sheaf $\mathfrak{P}$ the projective limit $\varprojlim_{\mathcal{E} \subset \mathscr{C}} \mathfrak{P}^{(X_{\mathcal{E}})}$.

Furthermore, it is well-known that all flat modules over an Artinian ring are projective. Moreover, all (contra)flat contramodules over a coalgebra over a field are projective [40, Sections 0.2.9 and A.3]. The functor $\mathscr{C}\text{–contra} \longrightarrow \mathfrak{X}\text{–pro}$ restricts to a functor $\mathscr{C}\text{–contra}_{\mathsf{proj}} \longrightarrow \mathfrak{X}\text{–flat}$, which is obviously fully faithful.

Comparing the two equivalences of additive categories (15) and (16) and taking into account the equivalence $\mathfrak{X}\text{–tors}_{\mathsf{inj}} \simeq \mathscr{C}\text{–comod}_{\mathsf{inj}}$ induced by the equivalence $\mathfrak{X}\text{–tors} \simeq \mathscr{C}\text{–comod}$, one can see that the functor $\mathscr{C}\text{–contra}_{\mathsf{proj}} \longrightarrow \mathfrak{X}\text{–flat}$ is an equivalence of additive categories. Moreover, the projective limit functor $\mathfrak{X}\text{–pro} \longrightarrow \mathscr{C}\text{–contra}$ restricts to a functor $\mathfrak{X}\text{–flat} \longrightarrow \mathscr{C}\text{–contra}_{\mathsf{proj}}$, providing the inverse equivalence.

Similarly to Example 3.8(3), the equivalences of categories $\mathfrak{X}\text{–tors} \simeq \mathscr{C}\text{–comod}$ and $\mathfrak{X}\text{–flat} \simeq \mathscr{C}\text{–contra}_{\mathsf{proj}}$ transform the tensor product functor $\otimes_{\mathfrak{X}} \colon \mathfrak{X}\text{–tors} \times \mathfrak{X}\text{–flat} \longrightarrow \mathfrak{X}\text{–tors}$ into the contratensor product functor $\odot_{\mathscr{C}} \colon \mathscr{C}\text{–comod} \times \mathscr{C}\text{–contra} \longrightarrow \mathscr{C}\text{–comod}$ restricted to $\mathscr{C}\text{–contra}_{\mathsf{proj}} \subset \mathscr{C}\text{–contra}$.

(7) Let us explain why the equivalence of coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{co}}(\mathscr{C}\text{–}\mathsf{comod})$ induced by the equivalence of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathscr{C}\text{–}\mathsf{comod}$ transforms the cotensor product functor $\square_{\mathscr{D}^\bullet}$ (14) from Section 5.1 for the dualizing complex $\mathscr{D}^\bullet = \mathscr{C}$ on the ind-scheme $\mathfrak{X} = \operatorname{Spi}\mathscr{C}^*$ into the right derived cotensor product functor $\square_{\mathscr{C}}^{\mathbb{R}}$ from Example 5.4(4).

Let $\mathcal{E} \subset \mathscr{C}$ be a finite-dimensional subcoalgebra and $i_{\mathcal{E}}\colon X_{\mathcal{E}} \longrightarrow \mathfrak{X}$ be the related immersion of the closed subscheme into the ind-scheme. Then the functor $i_{\mathcal{E}}^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow X_{\mathcal{E}}\text{–}\mathsf{qcoh}$ corresponds, under the equivalences of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathscr{C}\text{–}\mathsf{comod}$ and $X_{\mathcal{E}}\text{–}\mathsf{qcoh} \simeq \mathcal{E}\text{–}\mathsf{comod}$, to the functor $\mathscr{C}\text{–}\mathsf{comod} \longrightarrow \mathcal{E}\text{–}\mathsf{comod}$ assigning to an $\mathscr{C}$-comodule $\mathscr{M}$ its maximal subcomodule $\mathscr{M}_{(\mathcal{E})} \subset \mathscr{M}$ whose $\mathscr{C}$-comodule structure comes from an $\mathcal{E}$-comodule structure. The $\mathcal{E}$-comodule $\mathscr{M}_{(\mathcal{E})}$ can be computed as the cotensor product

$$\mathscr{M}_{(\mathcal{E})} \simeq \mathcal{E} \square_{\mathscr{C}} \mathscr{M}.$$

Consequently, for any two $\mathscr{C}$-comodules $\mathscr{M}$ and $\mathscr{N}$ one has

$$(\mathscr{M} \square_{\mathscr{C}} \mathscr{N})_{(\mathcal{E})} \simeq \mathcal{E} \square_{\mathscr{C}} \mathscr{M} \square_{\mathscr{C}} \mathscr{N} \simeq (\mathcal{E} \square_{\mathscr{C}} \mathscr{M}) \square_{\mathcal{E}} (\mathcal{E} \square_{\mathscr{C}} \mathscr{N}) \simeq \mathscr{M}_{(\mathcal{E})} \square_{\mathcal{E}} \mathscr{N}_{(\mathcal{E})}.$$

Together with the computations in the proof of Theorem 4.23, this reduces the question to the case of a finite-dimensional coalgebra $\mathcal{E}$ and related Artinian scheme $X_{\mathcal{E}} = \operatorname{Spec}\mathcal{E}^*$, for which it means the following.

Let $\mathcal{M}$ be an $\mathcal{E}$-comodule and $\mathcal{J}$ be an injective $\mathcal{E}$-comodule (we recall once again that an $\mathcal{E}$-comodule is the same thing as an $\mathcal{E}^*$-module). Then there are natural isomorphisms of $\mathcal{E}$-comodules

$$\mathcal{M} \square_{\mathcal{E}} \mathcal{J} \simeq \mathcal{M} \square_{\mathcal{E}} (\mathcal{E} \otimes_{\mathcal{E}^*} \operatorname{Hom}_{\mathcal{E}^*}(\mathcal{E}, \mathcal{J})) \\ \simeq (\mathcal{M} \square_{\mathcal{E}} \mathcal{E}) \otimes_{\mathcal{E}^*} \operatorname{Hom}_{\mathcal{E}^*}(\mathcal{E}, \mathcal{J}) \simeq \mathcal{M} \otimes_{\mathcal{E}^*} \operatorname{Hom}_{\mathcal{E}^*}(\mathcal{E}, \mathcal{J}).$$

Here the injective $\mathcal{E}$-comodule $\mathcal{E}$ corresponds to the dualizing (one-term) complex $\mathcal{D}^\bullet = i_{\mathcal{E}}^!\mathscr{D}^\bullet$ on the scheme $X_{\mathcal{E}}$ under the equivalence of categories $X_{\mathcal{E}}\text{–}\mathsf{qcoh} \simeq \mathcal{E}\text{–}\mathsf{comod}$, where $\mathscr{D}^\bullet \in \mathfrak{X}\text{–}\mathsf{tors}$ corresponds to $\mathscr{C} \in \mathscr{C}\text{–}\mathsf{comod}$ under $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathscr{C}\text{–}\mathsf{comod}$, as per the discussion in (5). Notice that the $\mathcal{E}^*$-module $\operatorname{Hom}_{\mathcal{E}^*}(\mathcal{E}, \mathcal{J})$ is projective for any injective $\mathcal{E}^*$-module $\mathcal{J}$.

(8) Alternatively, one can avoid the reduction to finite-dimensional coalgebras when establishing the comparison between $\square_{\mathscr{D}^\bullet}$ and $\square_{\mathscr{C}}^{\mathbb{R}}$, by using $\mathscr{C}$-contramodules and the discussion of the equivalence $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathscr{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ in (6).

In this context, the desired comparison is expressed by the natural isomorphisms of $\mathscr{C}$-comodules

$$\mathscr{M} \square_{\mathscr{C}} \mathscr{J} \simeq \mathscr{M} \square_{\mathscr{C}} (\mathscr{C} \odot_{\mathscr{C}} \operatorname{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{J})) \\ \simeq (\mathscr{M} \square_{\mathscr{C}} \mathscr{C}) \odot_{\mathscr{C}} \operatorname{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{J}) \simeq \mathscr{M} \odot_{\mathscr{C}} \operatorname{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{J}),$$

which hold for any $\mathscr{C}$-comodule $\mathscr{M}$ and any injective $\mathscr{C}$-comodule $\mathscr{J}$. We refer to [40, Proposition 5.2.1] or [45, Proposition 3.1.1] for a discussion on this kind of associativity isomorphisms connecting the cotensor and contratensor products.

Our next aim is to discuss the *torsion product* functor $\operatorname{Tor}_1^R(-,-)$ for modules over a Dedekind domain R. Let Q denote the field of quotients of R. The motivating example for us is the case of torsion abelian groups, when $R = \mathbb{Z}$ and $Q = \mathbb{Q}$. We refer to the book [25, Section V.6] or the overview [22] for the explicit construction and discussion of the torsion products of abelian groups (see also the classical paper [39]). Let us start the discussion in a more general context of a Noetherian ring of Krull dimension 1 before specializing to Dedekind domains.

Examples 5.5. (1) Let R be a Noetherian commutative ring of Krull dimension ≤ 1, and let $S \subset R$ denote the complement to the union of all the nonmaximal prime ideals in R. Notice that all the nonmaximal prime ideals in R are minimal, hence there is only a finite number of nonmaximal prime ideals; so an ideal $I \subset R$ does not intersect S if and only if it is contained in one of the nonmaximal prime ideals. If I does intersect S, then R/I is an Artinian ring. We refer to [49, Section 13] for a more detailed discussion of this setting.

Let Γ denote the directed poset of all ideals in R intersecting S, with respect to the reverse inclusion order. To every ideal $I \in \Gamma$, assign the Artinian scheme $X_I = \operatorname{Spec} R/I$. Whenever $I'' \subset I'$ are two ideals in R, with I'' intersecting S, there is a unique (surjective) ring homomorphism $R/I'' \longrightarrow R/I'$ forming a commutative triangle diagram with the natural projections $R \longrightarrow R/I''$ and $R \longrightarrow R/I'$. Let $X_{I'} \longrightarrow X_{I''}$ be the related closed immersion of Artinian schemes. The inductive system of schemes $(X_I)_{I\in\Gamma}$ represents an ind-Artinian ind-scheme $\mathfrak{X}$. The ind-scheme $\mathfrak{X}$ comes together with an ind-closed immersion of ind-schemes $\mathfrak{X} \longrightarrow \operatorname{Spec} R$ (in the sense of Examples 4.8).

(2) In the context of (1), let $\mathfrak{R}$ be the topological ring $\mathfrak{R} = \varprojlim_{I\in\Gamma} R/I$, with the topology of projective limit of discrete rings R/I. The topological ring $\mathfrak{R}$ can be computed as the topological product $\mathfrak{R} = \prod_{m\subset R} \widehat{R}_m$, where m ranges over the maximal ideals of R and $\widehat{R}_m$ is the completion of the local ring R_m. The complete local ring $\widehat{R}_m$ is endowed with the m-adic topology and the product $\prod_{m\subset R} \widehat{R}_m$ is endowed with the product topology. Then the ind-scheme $\mathfrak{X} = \text{“}\varinjlim\text{”}_{I\in\Gamma} X_I$ from (1) can be described as $\mathfrak{X} = \operatorname{Spi}\mathfrak{R}$, in the notation of Example 1.6(1).

An R-module M is said to be *S-torsion* if for every $b \in M$ there exists $s \in S$ such that $sb = 0$ in M. Denote by $Q = S^{-1}R$ the localization of the ring R at the multiplicative subset S. An R-module M is S-torsion if and only if $Q \otimes_R M = 0$. Denote by $R\text{–}\mathsf{tors} \subset R\text{–}\mathsf{mod}$ the full subcategory of S-torsion R-modules; clearly, $R\text{–}\mathsf{tors}$ is a locally Noetherian (in fact, locally finite) Grothendieck abelian category, which is closed under subobjects, quotients, and extensions as a full subcategory in $R\text{–}\mathsf{mod}$. The category of S-torsion R-modules is naturally equivalent to the category of discrete $\mathfrak{R}$-modules and to the category of quasi-coherent torsion sheaves on $\mathfrak{X}$, that is $R\text{–}\mathsf{tors} \simeq \mathfrak{R}\text{–}\mathsf{discr} \simeq \mathfrak{X}\text{–}\mathsf{tors}$ (cf. Section 2.4(6)).

The ind-scheme $\mathfrak{X}$ is the disjoint union (coproduct) of the ind-schemes $\operatorname{Spi}\widehat{R}_m$ over the maximal ideals $m \subset R$. The topological ring $\widehat{R}_m$ has a countable base of neighborhoods of zero, so the category of flat pro-quasi-coherent pro-sheaves on $\operatorname{Spi}\widehat{R}_m$ is equivalent to the category of flat $\widehat{R}_m$-contramodules by Example 3.8(2). Using the result of [52, Lemma 7.1(b)] providing an equivalence between the abelian category of $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$ and the Cartesian product of the abelian categories of $\widehat{R}_m$-contramodules $\widehat{R}_m\text{–}\mathsf{contra}$, one can conclude that the category of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ is equivalent to the category of flat $\mathfrak{R}$-contramodules, $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{flat}$. The construction of this equivalence is similar to the one in Example 3.8(1). In fact, by [52, Corollary 8.4 or Theorem 10.1(vi)⇒(iii)], all flat $\mathfrak{R}$-contramodules are projective, $\mathfrak{R}\text{–}\mathsf{flat} = \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$.

(3) The following description of the category $\mathfrak{X}\text{–}\mathsf{flat}$ may be more instructive. An R-module C is said to be *S-reduced* if it has no submodules in which all the elements of S act by invertible operators; equivalently in our context, this means that $\mathrm{Hom}_R(Q, C) = 0$ [49, Theorem 13.8(a)]. An R-module C is said to be *S-weakly cotorsion* if $\mathrm{Ext}^1_R(Q, C) = 0$; equivalently in our context, this means that C is *cotorsion*, that is $\mathrm{Ext}^1_R(F, C) = 0$ for all flat R-modules F [49, Theorem 13.9(b)].

Reduced cotorsion abelian groups were called "co-torsion" in [18], and R-modules C satisfying $\mathrm{Hom}_R(Q, C) = 0 = \mathrm{Ext}^1_R(Q, C)$ (for a domain R and the multiplicative subset $S = R \setminus \{0\}$) were called "cotorsion" in [27]. R-modules C satisfying $\mathrm{Hom}_R(Q, C) = 0 = \mathrm{Ext}^1_R(Q, C)$ are called "S-contramodules" in [50, 5].

We claim that the category of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{–}\mathsf{flat}$ is naturally equivalent to the category of flat S-reduced cotorsion R-modules. The equivalence assigns to a flat pro-quasi-cohererent pro-sheaf $\mathfrak{F} \in \mathfrak{X}\text{–}\mathsf{flat}$ the R-module $\varprojlim_{I\in\Gamma} \mathfrak{F}^{(X_I)}(X_I)$. Conversely, to a flat S-reduced cotorsion R-module F, the flat pro-quasi-coherent pro-sheaf $\mathfrak{F}$ with the components $\mathfrak{F}^{(X_I)} \in X_I\text{–}\mathsf{flat}$ corresponding to the flat R/I-modules F/IF is assigned.

Indeed, by the result of [49, Corollary 13.13(b)] or [5, Corollary 6.17], taken together with [43, Theorem B.1.1] and [52, Lemma 7.1(b)], the category S-reduced cotorsion R-modules is abelian and equivalent to the category of $\mathfrak{R}$-contramodules (the equivalence being provided by the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$). It remains to notice that an $\widehat{R}_m$-contramodule is flat, or equivalently, projective (as a contramodule) if and only if it is a flat R-module [49, Corollary 10.3(a) or Theorem 10.5], [43, Corollary B.8.2].

The equivalence of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq R\text{–}\mathsf{tors} \subset R\text{–}\mathsf{mod}$ and the fully faithful functor $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{flat} \longrightarrow R\text{–}\mathsf{mod}$ identify the tensor product functor $\otimes_{\mathfrak{X}} \colon \mathfrak{X}\text{–}\mathsf{tors} \times \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat}$ with the restriction of the tensor product functor $\otimes_R \colon R\text{–}\mathsf{tors} \times R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{tors}$ to the full subcategory of flat S-reduced cotorsion R-modules in the second argument.

(4) Choosing a dualizing complex $\mathcal{D}^\bullet$ on $\operatorname{Spec} R$, one can use the construction of Example 4.8(3) to obtain a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$. This construction allows to produce a dualizing complex which would be a two-term complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$.

Here is how one can construct a dualizing one-term complex on $\mathfrak{X}$. Let $E \in R\text{–}\mathsf{mod}$ be the direct sum of injective envelopes of the simple R-modules R/m, where m ranges over the maximal ideals of R (one copy of each). Then E is an injective S-torsion R-module, $E \in R\text{–}\mathsf{tors}_{\mathsf{inj}}$ (notice that an S-torsion R-module is injective in $R\text{–}\mathsf{mod}$ if and only if it is injective in $R\text{–}\mathsf{tors}$, since R is Noetherian). Let $\mathscr{E} \in \mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$ be an injective quasi-coherent torsion sheaf corresponding to E under the equivalence $R\text{–}\mathsf{tors} \simeq \mathfrak{X}\text{–}\mathsf{tors}$. Then $\mathscr{D}^\bullet = \mathscr{E}$ is a one-term dualizing complex on $\mathfrak{X}$.

Indeed, let $i_I \colon X_I \longrightarrow \mathfrak{X}$ denote the natural closed immersion. Then the quasi-coherent sheaf $i_I^!\mathscr{E}$ on X_I corresponds to an injective R/I-module which is the direct sum of the injective envelopes of the simple modules over the Artinian ring R/I. Since, by Matlis duality [26, Theorem 3.7], the injective envelope of the residue field of a local Artinian ring is a dualizing complex, $i_I^!\mathscr{E}$ is a dualizing complex on X_I.

(5) Similarly to Example 5.4(5), the proof of Theorem 4.23 specialized to the case of a one-term dualizing complex $\mathscr{D}^\bullet = \mathscr{E}$ shows that there is an equivalence of additive (split exact) categories of injective quasi-coherent torsion sheaves and flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$,

$$\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E},-)\colon \mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}} \simeq \mathfrak{X}\text{–}\mathsf{flat} :\mathscr{E}\otimes_{\mathfrak{X}} -. \tag{17}$$

In view of the above interpretation of the categories $\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$ and $\mathfrak{X}\text{–}\mathsf{flat}$ as full subcategories in $R\text{–}\mathsf{mod}$, this means an equivalence between the additive categories of injective S-torsion R-modules and flat S-reduced cotorsion R-modules, provided by the mutually inverse functors $\operatorname{Hom}_R(E,-)$ and $E\otimes_R -$.

Let us *warn* the reader that this equivalence of additive subcategories in $R\text{–}\mathsf{mod}$ is different from the Matlis equivalence between the full subcategories of S-divisible S-torsion R-modules and S-torsionfree S-reduced cotorsion R-modules [27, Theorem 3.4], [50, Corollary 5.2], which is given by a different pair of adjoint functors. The equivalence (17) is induced by a *dualizing* torsion module/complex E, while the Matlis equivalence is induced by a *dedualizing* compex $R \longrightarrow Q$ (see [46, Introduction and Remark 4.10] for a discussion of dualizing and dedualizing complexes). For a Dedekind domain R, the two equivalences are the same.

Let R be a Dedekind domain. In the context of Examples 5.5, we have $S = R\setminus\{0\}$; so $Q = S^{-1}R$ is the field of quotients of R. An R-module M is said to be *torsion* if for every $m \in M$ there exists $r \in R$, $r \neq 0$, such that $rm = 0$ in M; as above, we denote by $R\text{–}\mathsf{tors} \subset R\text{–}\mathsf{mod}$ the full subcategory of torsion R-modules.

Notice that $Q\otimes_R \operatorname{Tor}_1^R(M,N) \simeq \operatorname{Tor}_1^R(Q\otimes_R M,\, N) = 0$ for all R-modules M and N, so the R-module $\operatorname{Tor}_1^R(M,N)$ is always torsion. Partly following the notation in [22], we put $M \circledT_{Q/R} N = \operatorname{Tor}_1^R(M,N)$ for all torsion R-modules M and N; so

$$\circledT_{Q/R} = \operatorname{Tor}_1^R(-,-)\colon R\text{–}\mathsf{tors}\times R\text{–}\mathsf{tors} \longrightarrow R\text{–}\mathsf{tors}.$$

The subindex Q/R in our notation $\circledT_{Q/R}$ is explained by the observation that there are natural isomorphisms $Q/R \circledT_{Q/R} M \simeq M \simeq M \circledT_{Q/R} Q/R$ for all torsion R-modules M.

Notice that $\operatorname{Tor}_2^R(M,N) = 0$ for all R-modules M and N (since R is a Dedekind domain). It follows that $\circledT_{Q/R}$ is a left exact functor. Furthermore, associativity of the derived tensor product functor $\otimes_R^{\mathbb{L}}\colon \mathsf{D}(R\text{–}\mathsf{mod})\times\mathsf{D}(R\text{–}\mathsf{mod})\longrightarrow\mathsf{D}(R\text{–}\mathsf{mod})$ implies associativity of the functor $\circledT_{Q/R}$, as

$$L\circledT_{Q/R}(M\circledT_{Q/R}N)\simeq H^{-2}(L\otimes_R^{\mathbb{L}}M\otimes_R^{\mathbb{L}}N)\simeq(L\circledT_{Q/R}M)\circledT_{Q/R}N$$

for all torsion R-modules L, M, and N. So the category R–tors with the functor $\circledT_{Q/R}$ is an associative, commutative, and unital tensor category with the unit object Q/R.

Example 5.6. Let $\Bbbk$ be a field and $R=\Bbbk[x]_{(x)}$ be the localization of the ring of polynomials $\Bbbk[x]$ in one variable x at the prime ideal $(x)=x\Bbbk[x]\subset\Bbbk[x]$. So the field of rational functions $Q=\Bbbk(x)=R[x^{-1}]$ can be obtained by inverting the single element $x\in R$. Let $\mathscr{C}$ be the coalgebra over $\Bbbk$ whose dual topological algebra is $\mathscr{C}^*=\Bbbk[[x]]$, as mentioned in Examples 1.5. Then the category of torsion R-modules is naturally equivalent to the category of $\mathscr{C}$-comodules, $R\text{–}\mathsf{tors}\simeq\mathscr{C}\text{–}\mathsf{comod}$. This equivalence of abelian categories identifies the tensor structure $\circledT_{Q/R}$ on R–tors with the tensor structure $\square_{\mathscr{C}}$ on $\mathscr{C}$–comod.

Let R be a Dedekind domain. Then the homological dimension of the abelian category R–mod is equal to 1, hence the homological dimension of the abelian category R–tors is equal to 1 as well. One easily concludes that the unbounded derived category $\mathsf{D}(R\text{–}\mathsf{tors})$ is equivalent to the homotopy category of complexes of injecitves $\mathsf{K}(R\text{–}\mathsf{tors}_{\mathsf{inj}})$. Similarly, the derived category $\mathsf{D}(R\text{–}\mathsf{mod})$ is equivalent to the homotopy category $\mathsf{K}(R\text{–}\mathsf{mod}_{\mathsf{inj}})$. It follows that $\mathsf{D}(R\text{–}\mathsf{tors})$ is a full subcategory in $\mathsf{D}(R\text{–}\mathsf{mod})$ (see [50, Theorem 6.6(a)] for a more general result).

The torsion R-module Q/R is an injective cogenerator of R–tors; moreover, a torsion R-module is injective if and only if it is a direct summand of a direct sum of copies of Q/R. In fact, the injective R-module Q/R is a direct sum of injective envelopes of the simple R-modules R/m, where m ranges over the maximal ideals of R (one copy of each); so in the context of Example 5.5(4) one can take $E=Q/R$ for a Dedekind domain‘R. For a given torsion R-module J, the torsion product functor $J\circledT_{Q/R}-\colon R\text{–}\mathsf{tors}\longrightarrow R\text{–}\mathsf{tors}$ is exact if and only if J is injective.

The functor of torsion product of complexes of torsion modules $\circledT_{Q/R}\colon\mathsf{C}(R\text{–}\mathsf{tors})\times\mathsf{C}(R\text{–}\mathsf{tors})\longrightarrow\mathsf{C}(R\text{–}\mathsf{tors})$ is constructed in the obvious way (using the totalization by taking the direct sums along the diagonals of the bicomplex). To define the right derived functor of torsion product

$$\circledT_{Q/R}^{\mathbb{R}}\colon\mathsf{D}(R\text{–}\mathsf{tors})\times\mathsf{D}(R\text{–}\mathsf{tors})\longrightarrow\mathsf{D}(R\text{–}\mathsf{tors}),$$

suppose that we are given two complexes of torsion R-modules $M^\bullet$ and $N^\bullet$. Let $K^\bullet$ and $J^\bullet$ be complexes of torsion R-modules endowed with quasi-isomorphisms of complexes of torsion R-modules $M^\bullet\longrightarrow K^\bullet$ and $N^\bullet\longrightarrow J^\bullet$. Then the derived torsion product is defined as the object

$$M^\bullet\circledT_{Q/R}^{\mathbb{R}}N^\bullet=K^\bullet\circledT_{Q/R}J^\bullet\simeq M^\bullet\circledT_{Q/R}J^\bullet\simeq K^\bullet\circledT_{Q/R}N^\bullet\in\mathsf{D}(R\text{–}\mathsf{tors}),$$

similarly to Example 5.4(4).

The right derived torsion product functor agrees with the left derived tensor product: restricting the derived tensor product functor $\otimes_R^{\mathbb{L}}\colon\mathsf{D}(R\text{–}\mathsf{mod})\times\mathsf{D}(R\text{–}\mathsf{mod})\longrightarrow$

$\mathsf{D}(R\text{–}\mathsf{mod})$ to the full subcategory $\mathsf{D}(R\text{–}\mathsf{tors}) \times \mathsf{D}(R\text{–}\mathsf{tors}) \subset \mathsf{D}(R\text{–}\mathsf{mod}) \times \mathsf{D}(R\text{–}\mathsf{mod})$, one obtains the functor $\circledT^{\mathbb{R}}_{Q/R}\colon \mathsf{D}(R\text{–}\mathsf{tors}) \times \mathsf{D}(R\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}(R\text{–}\mathsf{tors})$. This comparison holds, essentially, because one has $M \otimes_R J = 0$ for any torsion R-module M and any injective torsion R-module J.

Proposition 5.7. *Let R be a Dedekind domain and $\mathfrak{X} = \text{“}\varinjlim\text{”}_{I\in\Gamma} X_I = \operatorname{Spi}\varprojlim_{I\in\Gamma} R/I$ be the related ind-Artinian ind-scheme from Examples 5.5. (Here Γ is the poset of all nonzero ideals in R in the reverse inclusion order, and $X_I = \operatorname{Spec} R/I$.) Let $\mathscr{D}^\bullet = \mathscr{E}$ be the one-term dualizing complex of $\mathfrak{X}$ corresponding to the injective torsion R-module $E = Q/R$. Then the equivalence of (co)derived categories $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{tors}) \simeq \mathsf{D}(R\text{–}\mathsf{tors})$ induced by the equivalence of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq R\text{–}\mathsf{tors}$ transforms the cotensor product functor $\square_{\mathscr{D}^\bullet}$* (14) *from Section 5.1 into the right derived torsion product functor $\circledT^{\mathbb{R}}_{Q/R}$.*

Proof. One can argue similarly to Example 5.4(7), reducing the question to the case of an Artinian scheme R/I, but we prefer to spell out an argument in the spirit of Example 5.4(8), working with special classes of R-modules and the ind-scheme $\mathfrak{X}$ as a whole. In this context, the desired comparison is expressed by the composition of the natural isomorphisms of torsion R-modules

$$\begin{aligned} M \circledT_{Q/R} J &\simeq M \circledT_{Q/R} (Q/R \otimes_R \operatorname{Hom}_R(Q/R, J)) \\ &\simeq (M \circledT_{Q/R} Q/R) \otimes_R \operatorname{Hom}_R(Q/R, J) \simeq M \otimes_R \operatorname{Hom}_R(Q/R, J), \end{aligned}$$

which hold for any torsion R-module M and any injective torsion R-module J.

Here the natural isomorphism $J \simeq Q/R \otimes_R \operatorname{Hom}_R(Q/R, J)$ for a divisible torsion R-module J is due to Harrison [18, Proposition 2.1] and Matlis [27, Theorem 3.4], while the middle (associativity) isomorphism is provided by part (b) of the next lemma. ☐

Lemma 5.8. *Let R be a Dedekind domain. Then, for any torsion R-modules M and E, and any R-module P, there is a natural homomorphism of torsion R-modules*

$$(M \circledT_{Q/R} E) \otimes_R P \longrightarrow M \circledT_{Q/R} (E \otimes_R P), \tag{18}$$

which is an isomorphism whenever either (a) *M is injective, or* (b) *P is flat.*

Proof. This result is analogous to [45, Proposition 3.1.1] (cf. Example 5.6); it is also a particular case of [43, Lemma 1.7.2(a)]. Denote the left-hand side of (18) by $l(M, P)$ and the right-hand side by $r(M, P)$.

There is an obvious isomorphism (18) when $P = F$ is a free R-module with a fixed set of free generators (since the functor $\circledT_{Q/R}$ preserves direct sums). It is straightforward to check that this isomorphism is functorial with respect to arbitrary morphisms of free R-modules F.

Now let P be the cokernel of a morphism of free R-modules $f\colon F' \longrightarrow F''$. Then there is a natural isomorphism $\operatorname{coker}(l(M, f)) \simeq l(M, P)$ and a natural morphism $\operatorname{coker}(r(M, f)) \longrightarrow r(M, P)$. As we already have a natural isomorphism of morphisms $l(M, f) \simeq r(M, f)$, the desired morphism $l(M, P) \longrightarrow r(M, P)$ is obtained. Its functoriality is again straightforward.

Now we can prove part (a). Since M is injective and therefore the functor $M \circledT_{Q/R} -$ is exact, the natural morphism $\operatorname{coker}(r(M,f)) \longrightarrow r(M,P)$ is an isomorphism. Hence $l(M,P) \longrightarrow r(M,P)$ is an isomorphism.

Let M be the kernel of a morphism of injective torsion R-modules $g\colon K' \longrightarrow K''$. Then there is a natural morphism $l(M,P) \longrightarrow \ker(l(g,P))$ and a natural isomorphism $r(M,P) \simeq \ker(r(g,P))$ (since the functor $\circledT_{Q/R}$ is left exact).

Now we can prove part (b). Since P is flat, the morphism $l(M,P) \longrightarrow \ker(l(g,P))$ is also an isomorphism. As we already know from part (a) that the natural morphism of morphisms $l(g,P) \longrightarrow r(g,P)$ is an isomorphism, it follows that the morphism of torsion modules $l(M,P) \longrightarrow r(M,P)$ is an isomorphism. □

Example 5.9. Quite generally, let $\mathfrak{X}$ be an ind-Artinian ind-scheme, and let $\mathfrak{R}$ be the related pro-Artinian topological commutative ring such that $\mathfrak{X} = \operatorname{Spi}\mathfrak{R}$, as per Example 4.1(3). According to Section 2.4(6), the abelian category $\mathfrak{X}$–tors is equivalent to the abelian category of discrete $\mathfrak{R}$-modules $\mathfrak{R}$–discr.

The abelian category $\mathfrak{R}$–discr does not have a natural injective cogenerator which would correspond to a one-term dualizing complex on $\mathfrak{X}$. Such an injective discrete $\mathfrak{R}$-module, namely, the direct sum of injective envelopes of all the simple objects in $\mathfrak{R}$–discr, does exist, but it is only defined up to a nonunique isomorphism.

In the memoir [43, Section 1], the category of $\mathfrak{R}$-*comodules* $\mathfrak{R}$–comod is defined in such a way that it comes endowed with a natural injective cogenerator $\mathcal{C}(\mathfrak{R})$, similar to the injective cogenerator $\mathscr{C}$ of the category of comodules over a cocommutative coalgebra $\mathscr{C}$, as in Examples 5.4. Accordingly, there is a naturally defined cotensor product functor $\square_{\mathfrak{R}}\colon \mathfrak{R}\text{–}\mathsf{comod} \times \mathfrak{R}\text{–}\mathsf{comod} \longrightarrow \mathfrak{R}\text{–}\mathsf{comod}$, making $\mathfrak{R}$–comod a tansor category with the unit object $\mathcal{C}(\mathfrak{R})$.

The choice of an injective object $\mathscr{C} \in \mathfrak{R}\text{–}\mathsf{discr}$ isomorphic to a direct sum of injective cogenerators of simple discrete modules, induces an equivalence of categories $\mathfrak{R}\text{–}\mathsf{discr} \simeq \mathfrak{R}\text{–}\mathsf{comod}$ taking $\mathscr{C} \in \mathfrak{R}\text{–}\mathsf{discr}$ to $\mathcal{C} \in \mathfrak{R}\text{–}\mathsf{comod}$. So $\mathfrak{R}$–discr becomes a tensor category with the unit object $\mathscr{C}$; the cotensor product operation $\square_{\mathfrak{R}} = \square_{\mathscr{C}}$ defining this tensor structure is described in [43, Section 1.9]. The injective object $\mathscr{C} \in \mathfrak{R}\text{–}\mathsf{discr}$ corresponds to a one-term dualizing complex on $\mathfrak{X}$.

Similarly to Example 5.4(6), the category $\mathfrak{X}$–flat can be naturally identified with the category of flat, or which is the same, projective contramodules over the topological ring $\mathfrak{R}$, that is $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{flat} = \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$ (see [52, Section 2] for the definition of a flat $\mathfrak{R}$-contramodule and [43, Lemma 1.9.1(a)] or [52, Corollary 8.4 or Theorem 10.1(vi)⇒(iii)] for a proof that all flat contramodules are projective over a pro-Artinian commutative topological ring $\mathfrak{R}$). In particular, the exact category structure on $\mathfrak{X}$–flat is split.

Having chosen a one-term dualizing complex $\mathscr{C} \in \mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$, one obtains an equivalence of additive categories as in Example 5.4(5),

$$\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{C}, -)\colon \mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}} \simeq \mathfrak{X}\text{–}\mathsf{flat} :\mathscr{C} \otimes_{\mathfrak{X}} -,$$

which corresponds to the equivalence of additive categories $\mathfrak{R}\text{–}\mathsf{comod}_{\mathsf{inj}} \simeq \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$ [43, Proposition 1.5.1] under the identifications $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathfrak{R}\text{–}\mathsf{comod}$ and $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$.

Similarly to Example 5.4(4), one constructs the right derived functor of cotensor product of discrete $\mathfrak{R}$-modules

$$\square^{\mathbb{R}}_{\mathscr{C}}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr}).$$

The cotensor product functor $\square_{\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ (14) from Section 5.1 for $\mathscr{D}^\bullet = \mathscr{C}$ is transformed into the functor $\square^{\mathbb{R}}_{\mathscr{C}}$ by the equivalence of coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{R}\text{–}\mathsf{discr})$ induced by the equivalence of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathfrak{R}\text{–}\mathsf{discr}$.

Example 5.10. For an ind-affine ind-Noetherian $\aleph_0$-ind-scheme with a dualizing complex $\mathscr{D}^\bullet$, the construciton of the cotensor product functor $\square_{\mathscr{D}^\bullet}$ in Section 5.1 agrees with the one in [44, Section D.3], as one can see by comparing the two constructions in light of the discussion in Examples 3.8.

6. Ind-Schemes of Ind-Finite Type and the !-Tensor Product

Throughout this section, $\Bbbk$ denotes a fixed ground field. Given two ind-schemes $\mathfrak{X}'$ and $\mathfrak{X}''$ (or two schemes X' and X'') over $\Bbbk$, we denote the fibered product $\mathfrak{X}' \times_{\operatorname{Spec}\Bbbk} \mathfrak{X}''$ (or $X' \times_{\operatorname{Spec}\Bbbk} X''$) simply by $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ (or $X' \times_\Bbbk X''$) for brevity. (See Sections 1.1–1.2 for a discussion of fibered products of ind-schemes.)

Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over the field $\Bbbk$. The aim of this section is to describe the cotensor product functor $\square_{\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$, for a suitable choice of the dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$, as the derived !-restriction to the diagonal $\Delta_\mathfrak{X}\colon \mathfrak{X} \longrightarrow \mathfrak{X} \times_\Bbbk \mathfrak{X}$ of the external tensor product on $\mathfrak{X} \times_\Bbbk \mathfrak{X}$ of the two given complexes of quasi-coherent torsion sheaves on $\mathfrak{X}$.

6.1. External tensor product of quasi-coherent sheaves.

Let X' and X'' be two schemes over $\Bbbk$. Consider the Cartesian product $X' \times_\Bbbk X''$, and let $p'\colon X' \times_\Bbbk X'' \longrightarrow X'$ and $p''\colon X' \times_\Bbbk X'' \longrightarrow X''$ denote the natural projections.

Let $\mathcal{M}'$ be a quasi-coherent sheaf over X' and $\mathcal{M}'$ be a quasi-coherent sheaf over X''. Then the *external tensor product* $\mathcal{M}' \boxtimes_\Bbbk \mathcal{M}''$ of the quasi-coherent sheaves $\mathcal{M}'$ and $\mathcal{M}''$ is a quasi-coherent sheaf on $X' \times_\Bbbk X''$ defined by the formula

$$\mathcal{M}' \boxtimes_\Bbbk \mathcal{M}'' = p'^*\mathcal{M}' \otimes_{\mathcal{O}_{X' \times_\Bbbk X''}} p''^*\mathcal{M}''.$$

Lemma 6.1. *Let $\mathcal{F}'$ be a flat quasi-coherent sheaf over X' and $\mathcal{F}''$ be a flat quasi-coherent sheaf over X''. Then $\mathcal{F}' \boxtimes_\Bbbk \mathcal{F}''$ is a flat quasi-coherent sheaf over $X' \times_\Bbbk X''$.*

Proof. Follows from immediately from the definition of the external tensor product $\mathcal{F}' \boxtimes_\Bbbk \mathcal{F}''$ and the facts that the inverse images p'^*, p''^* and the tensor product $\otimes_{\mathcal{O}_{X' \times_\Bbbk X''}}$ preserve flatness of quasi-coherent sheaves. □

Lemma 6.2. *The external tensor product functor*

$$\boxtimes_{\Bbbk}\colon X'\text{–}\mathsf{qcoh}\times X''\text{–}\mathsf{qcoh}\longrightarrow (X'\times_{\Bbbk}X'')\text{–}\mathsf{qcoh}$$

is exact and preserves coproducts (hence all colimits) in each of its arguments.

Proof. The assertion is local in both X' and X'', so it reduces to the case of affine schemes, for which it means the following. Let R' and R'' be two commutative $\Bbbk$-algebras. Let M' be an R'-module and M'' be an R''-module. Then the functor assigning to M' and M'' the $(R'\otimes_{\Bbbk}R'')$-module

$$((R'\otimes_{\Bbbk}R'')\otimes_{R'}M')\otimes_{R'\otimes_{\Bbbk}R''}((R'\otimes_{\Bbbk}R'')\otimes_{R''}M'') \\ \simeq (R''\otimes_{\Bbbk}M')\otimes_{R'\otimes_{\Bbbk}R''}(R'\otimes_{\Bbbk}M'')\simeq M'\otimes_{\Bbbk}M''$$

is exact in each of the arguments. The preservation of coproducts is obvious. □

Lemma 6.3. *Let $f'\colon Y'\longrightarrow X'$ and $f''\colon Y''\longrightarrow X''$ be two morphisms of schemes over $\Bbbk$, and let $f=f'\times_{\Bbbk}f''\colon Y'\times_{\Bbbk}Y''\longrightarrow X'\times_{\Bbbk}X''$ be the induced morphism of the Cartesian products. Let $\mathcal{M}'$ be a quasi-coherent sheaf on X' and $\mathcal{M}''$ be a quasi-coherent sheaf on X''. Then there is a natural isomorphism*

$$f^*(\mathcal{M}'\boxtimes_{\Bbbk}\mathcal{M}'')\simeq f'^*\mathcal{M}'\boxtimes_{\Bbbk}f''^*\mathcal{M}''$$

of quasi-coherent sheaves on $Y'\times_{\Bbbk}Y''$.

Proof. Let $p^{(s)}\colon X'\times_{\Bbbk}X''\longrightarrow X^{(s)}$ and $q^{(s)}\colon Y'\times_{\Bbbk}Y''\longrightarrow \operatorname{Spec}Y^{(s)}$, $s=1,\,2$, be the natural morphisms. Then one has

$$f^*(\mathcal{M}'\boxtimes_{\Bbbk}\mathcal{M}'')=f^*(p'^*\mathcal{M}'\otimes_{\mathcal{O}_{X'\times_{\Bbbk}X''}}p''^*\mathcal{M}'')\simeq f^*p'^*\mathcal{M}'\otimes_{\mathcal{O}_{Y'\times_{\Bbbk}Y''}}f^*p''^*\mathcal{M}'' \\ \simeq q'^*f'^*\mathcal{M}'\otimes_{\mathcal{O}_{Y'\times_{\Bbbk}Y''}}q''^*f''^*\mathcal{M}''=f'^*\mathcal{M}'\boxtimes_{\Bbbk}f''^*\mathcal{M}'',$$

since $p'f=f'q'$ and $p''f=f''q''$. □

Lemma 6.4. *Let $f'\colon Y'\longrightarrow X'$ and $f''\colon Y''\longrightarrow X''$ be two affine morphisms of schemes over $\Bbbk$, and let $f=f'\times_{\Bbbk}f''\colon Y'\times_{\Bbbk}Y''\longrightarrow X'\times_{\Bbbk}X''$ be the induced morphism of the Cartesian products. Let $\mathcal{N}'$ be a quasi-coherent sheaf on Y' and $\mathcal{N}''$ be a quasi-coherent sheaf on Y''. Then there is a natural isomorphism*

$$f_*(\mathcal{N}'\boxtimes_{\Bbbk}\mathcal{N}'')\simeq f'_*\mathcal{N}'\boxtimes_{\Bbbk}f''_*\mathcal{N}''$$

of quasi-coherent sheaves on $X'\times_{\Bbbk}X''$.

Proof. The assertion is essentially local in X' and X'', so it reduces to the case of affine schemes, for which it means the following very tautological observation (cf. the computation in the proof of Lemma 6.2). Let $R'\longrightarrow S'$ and $R''\longrightarrow S''$ be two homomorphisms of commutative $\Bbbk$-algebras. Let N' be an S'-module and N'' be an S''-module; then $N'\otimes_{\Bbbk}N''$ is an $(S'\otimes_{\Bbbk}S'')$-module. Consider the underlying R'-module of N' and the underlying R''-module of N''; then the tensor product $N'\otimes_{\Bbbk}N''$ acquires the structure of an $(R'\otimes_{\Bbbk}R'')$-module. The claim is that the latter $(R'\otimes_{\Bbbk}R'')$-module structure $N'\otimes_{\Bbbk}N''$ underlies the former $(S'\otimes_{\Bbbk}S'')$-module structure with respect to the ring homomorphism $R'\otimes_{\Bbbk}R''\longrightarrow S'\otimes_{\Bbbk}S''$ (i. e., the two $(R'\otimes_{\Bbbk}R'')$-module structures on $N'\otimes_{\Bbbk}N''$ coincide). □

Lemma 6.5. *Let Z' be a reasonable closed subscheme in a scheme X' over $\Bbbk$ and Z' be a reasonable closed subscheme in a scheme Z'' over $\Bbbk$. Then $Z'\times_\Bbbk Z''$ is a reasonable closed subscheme in the scheme $X'\times_\Bbbk X''$.*

Proof. To deduce the assertion from Lemma 2.1, decompose the closed immersion $Z'\times_\Bbbk Z''\longrightarrow X'\times_\Bbbk X''$ as $Z'\times_\Bbbk Z''\longrightarrow Z'\times_\Bbbk X''\longrightarrow X'\times_\Bbbk X''$ and notice that $Z'\times_\Bbbk Z''=(Z'\times_\Bbbk X'')\times_{X''}Z''$. ☐

Lemma 6.6. *Let Z' be a reasonable closed subscheme in a scheme X' over $\Bbbk$ and Z' be a reasonable closed subscheme in a scheme Z'' over $\Bbbk$. Let $i'\colon Z'\longrightarrow X'$ and $i''\colon Z''\longrightarrow X''$ be the closed immersion morphisms, and let $i=i'\times_\Bbbk i''\colon Z'\times_\Bbbk Z''\longrightarrow X'\times_\Bbbk X''$ be the induced closed immersion of the Cartesian products. Let $\mathcal{M}'$ be a quasi-coherent sheaf on X' and $\mathcal{M}''$ be a quasi-coherent sheaf on X''. Then there is a natural isomorphism*

$$i^!(\mathcal{M}'\boxtimes_\Bbbk\mathcal{M}'')\simeq i'^!\mathcal{M}'\boxtimes_\Bbbk i''^!\mathcal{M}''$$

of quasi-coherent sheaves on $Z'\times_\Bbbk Z''$.

Proof. The assertion is essentially local in X' and X'', so it reduces to affine schemes, for which it means the following (cf. the computation in the proof of Lemma 6.2). Let $R'\longrightarrow S'$ and $R''\longrightarrow S''$ be two surjective homomorphisms of commutative $\Bbbk$-algebras (with finitely generated kernel ideals). Let M' be an R'-module and M'' be an R''-module. Then there is a natural isomorphism of $(S'\otimes_\Bbbk S'')$-modules

$$\operatorname{Hom}_{R'\otimes_\Bbbk R''}(S'\otimes_\Bbbk S'',\,M'\otimes_\Bbbk M'')\simeq\operatorname{Hom}_{R'}(S',M')\otimes_\Bbbk\operatorname{Hom}_{R''}(S'',M'').$$

To obtain the latter isomorphism, one can notice firstly that $\operatorname{Hom}_{R'\otimes_\Bbbk R''}(S'\otimes_\Bbbk R'',\,M'\otimes_\Bbbk M'')\simeq\operatorname{Hom}_{R'}(S',\,M'\otimes_\Bbbk M'')\simeq\operatorname{Hom}_{R'}(S',M')\otimes_\Bbbk M''$ and similarly $\operatorname{Hom}_{R'\otimes_\Bbbk R''}(R'\otimes_\Bbbk S'',\,M'\otimes_\Bbbk M'')\simeq M'\otimes_\Bbbk\operatorname{Hom}_{R''}(S'',M'')$. ☐

Let $\mathcal{M}'^\bullet$ be a complex of quasi-coherent sheaves on X' and $\mathcal{M}''^\bullet$ be a complex of quasi-coherent sheaves on X''. Then the complex $\mathcal{M}'^\bullet\boxtimes_\Bbbk\mathcal{M}''^\bullet$ of quasi-coherent sheaves on $X'\times_\Bbbk X''$ is constructed by totalizing the bicomplex of external tensor products using infinite coproducts along the diagonals.

Lemma 6.7. (a) *Let $\mathcal{M}'^\bullet$ be a complex of quasi-coherent sheaves on a scheme X' and $\mathcal{M}''^\bullet$ be a complex of quasi-coherent sheaves on a scheme X'' over $\Bbbk$. Assume that the complex $\mathcal{M}'^\bullet$ is acyclic (in X'–qcoh). Then the complex $\mathcal{M}'^\bullet\boxtimes_\Bbbk\mathcal{M}''^\bullet$ is acyclic (in $(X'\times_\Bbbk X'')$–qcoh).*

(b) *Let $\mathcal{F}'^\bullet$ be a complex of flat quasi-coherent sheaves on a scheme X' and $\mathcal{F}''^\bullet$ be a complex of flat quasi-coherent sheaves on a scheme X'' over $\Bbbk$. Assume that the complex $\mathcal{F}'^\bullet$ is acyclic in X'–flat. Then the complex $\mathcal{F}'^\bullet\boxtimes_\Bbbk\mathcal{F}''^\bullet$ is acyclic in $(X'\times_\Bbbk X'')$–flat.*

Proof. Both the assertions (a) and (b) are local in X' and X'', so they reduce to the case of affine schemes. Then part (a) follows from the fact that the tensor product of an acyclic complex of $\Bbbk$-vector spaces with an arbitrary complex of $\Bbbk$-vector spaces is an acyclic complex. Part (b) can be proved directly using the definition of the external tensor product of quasi-coherent sheaves as the tensor product of inverse images with respect to the projection maps: the inverse image under any morphism of schemes is exact as a functor between the exact categories of flat quasi-coherent sheaves, and it remains to refer to Lemma 5.1(b). $\square$

Let X be a scheme over $\Bbbk$, and let $\mathcal{M}'$ and $\mathcal{M}''$ be two quasi-coherent sheaves on X. Then $\mathcal{M}'\boxtimes_\Bbbk\mathcal{M}''$ is a quasi-coherent sheaf on $X\times_\Bbbk X$. Denote by $\Delta_X\colon X\longrightarrow X\times_\Bbbk X$ the diagonal morphism (defined by the property that its compositions with both the projections $X\times_\Bbbk X\rightrightarrows X$ are the identity morphisms).

Lemma 6.8. *For any two quasi-coherent sheaves $\mathcal{M}'$ and $\mathcal{M}''$ on a scheme X over $\Bbbk$, there is a natural isomorphism*

$$\mathcal{M}'\otimes_{\mathcal{O}_X}\mathcal{M}''\simeq\Delta_X^*(\mathcal{M}'\boxtimes_\Bbbk\mathcal{M}'')$$

of quasi-coherent sheaves on X.

Proof. The assertion is essentially local in X, so it reduces to the case of an affine scheme, for which it means the following. Let R be a commutative $\Bbbk$-algebra, and let M' and M'' be two R-modules. Then there is a natural isomorphism of R-modules

$$M'\otimes_R M''\simeq R\otimes_{R\otimes_\Bbbk R}(M'\otimes_\Bbbk M''),$$

where the diagonal ring homomorphism $R\otimes_\Bbbk R\longrightarrow R$ endows R with an $(R\otimes_\Bbbk R)$-module structure.

Alternatively, denoting by p' and $p''\colon X\times_\Bbbk X\longrightarrow X$ the natural projections, one computes

$$\Delta_X^*(\mathcal{M}'\boxtimes_\Bbbk\mathcal{M}'')=\Delta_X^*(p'^*\mathcal{M}'\otimes_{\mathcal{O}_{X\times_\Bbbk X}}p''^*\mathcal{M}'')$$
$$\simeq\Delta_X^*p'^*\mathcal{M}'\otimes_{\mathcal{O}_X}\Delta_X^*p''^*\mathcal{M}''\simeq(p'\Delta_X)^*\mathcal{M}'\otimes_{\mathcal{O}_X}(p''\Delta_X)^*\mathcal{M}''\simeq\mathcal{M}'\otimes_{\mathcal{O}_X}\mathcal{M}''$$

using the fact that the tensor products of quasi-coherent sheaves are preserved by the inverse images. $\square$

6.2. **External tensor product of pro-sheaves.** Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes over $\Bbbk$. Let $\mathfrak{P}'$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}'$ and $\mathfrak{P}''$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}''$. For every pair of closed subschemes $Z'\subset\mathfrak{X}'$ and $Z''\subset\mathfrak{X}''$ put $\mathfrak{Q}^{(Z'\times_\Bbbk Z'')}=\mathfrak{P}'^{(Z')}\boxtimes_\Bbbk\mathfrak{P}''^{(Z'')}\in(Z'\times_\Bbbk Z'')$–$\mathsf{qcoh}$. Then it follows from Lemma 6.3 that the collection of quasi-coherent sheaves $\mathfrak{Q}^{(Z'\times_\Bbbk Z'')}$ on the closed subschemes $Z'\times_\Bbbk Z''\subset\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$ defines a pro-quasi-coherent pro-sheaf $\mathfrak{Q}$ on the ind-scheme $\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$.

Put $\mathfrak{P}'\boxtimes_\Bbbk\mathfrak{P}''=\mathfrak{Q}$. This construction defines the functor of *external tensor product of pro-quasi-coherent pro-sheaves*

$$\boxtimes_\Bbbk\colon\mathfrak{X}'\text{–}\mathsf{pro}\times\mathfrak{X}''\text{–}\mathsf{pro}\longrightarrow(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{pro}. \tag{19}$$

It is clear from Lemma 6.1 that the external tensor product of two flat pro-quasi-coherent pro-sheaves is a flat pro-quasi-coherent pro-sheaf,

$$\boxtimes_\Bbbk\colon\mathfrak{X}'\text{–}\mathsf{flat}\times\mathfrak{X}''\text{–}\mathsf{flat}\longrightarrow(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{flat}. \tag{20}$$

Furthermore, it follows from Lemma 6.2 that the functor (19) preserves colimits in each of its argument, while the functor (20) is exact (as a functor between exact categories) and preserves direct limits in each of its arguments.

Let $\mathfrak{P}'^\bullet$ be a complex of pro-quasi-coherent pro-sheaves on $\mathfrak{X}'$ and $\mathfrak{P}''^\bullet$ be a complex of pro-quasi-coherent pro-sheaves on $\mathfrak{X}''$. Then the complex $\mathfrak{P}'^\bullet \boxtimes_\Bbbk \mathfrak{P}''^\bullet$ of pro-quasi-coherent pro-sheaves on $\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$ is constructed by totalizing the bicomplex of external tensor products using infinite coproducts along the diagonals.

Lemma 6.9. *Let $\mathfrak{F}'^\bullet$ be a complex of flat pro-quasi-coherent pro-sheaves on an ind-scheme $\mathfrak{X}'$ and $\mathfrak{F}''^\bullet$ be a complex of flat pro-quasi-coherent pro-sheaves on an ind-scheme $\mathfrak{X}''$ over $\Bbbk$. Assume that the complex $\mathfrak{F}'^\bullet$ is acyclic in $\mathfrak{X}'$–flat. Then the complex $\mathfrak{F}'^\bullet \boxtimes_\Bbbk \mathfrak{F}''^\bullet$ is acyclic in $(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')$–$\mathsf{flat}$.*

Proof. The result of Lemma 4.13 reduces the question to the case of schemes (rather than ind-schemes), for which we have Lemma 6.7(b). ☐

It follows from Lemma 6.9 that the external tensor product is well-defined as a functor between the derived categories of flat pro-quasi-coherent pro-sheaves,

$$\boxtimes_\Bbbk\colon \mathsf{D}(\mathfrak{X}'\text{–}\mathsf{flat})\times\mathsf{D}(\mathfrak{X}''\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}((\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{flat}). \tag{21}$$

The next three lemmas do not depend on any flatness conditions.

Lemma 6.10. *Let $f'\colon\mathfrak{Y}'\longrightarrow\mathfrak{X}'$ and $f''\colon\mathfrak{Y}''\longrightarrow\mathfrak{X}''$ be two morphisms of ind-schemes over $\Bbbk$, and let $f=f'\times_\Bbbk f''\colon \mathfrak{Y}'\times_\Bbbk\mathfrak{Y}''\longrightarrow\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$ be the induced morphism of the Cartesian products. Let $\mathfrak{P}'$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}'$ and $\mathfrak{P}''$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}''$. Then there is a natural isomorphism*

$$f^*(\mathfrak{P}'\boxtimes_\Bbbk\mathfrak{P}'')\simeq f'^*\mathfrak{P}'\boxtimes_\Bbbk f''^*\mathfrak{P}''$$

of pro-quasi-coherent pro-sheaves on $\mathfrak{Y}'\times_\Bbbk\mathfrak{Y}''$.

Proof. Follows immediately from the definition of the external tensor product of pro-quasi-coherent pro-sheaves (above), the definition of the inverse image of pro-quasi-coherent pro-sheaves (see Section 3.3), and Lemma 6.3. ☐

Lemma 6.11. *Let $f'\colon\mathfrak{Y}'\longrightarrow\mathfrak{X}'$ and $f''\colon\mathfrak{Y}''\longrightarrow\mathfrak{X}''$ be two affine morphisms of ind-schemes over $\Bbbk$, and let $f=f'\times_\Bbbk f''\colon \mathfrak{Y}'\times_\Bbbk\mathfrak{Y}''\longrightarrow\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$ be the induced morphism of the Cartesian products. Let $\mathfrak{Q}'$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}'$ and $\mathfrak{Q}''$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}''$. Then there is a natural isomorphism*

$$f_*(\mathfrak{Q}'\boxtimes_\Bbbk\mathfrak{Q}'')\simeq f'_*\mathfrak{Q}'\boxtimes_\Bbbk f''_*\mathfrak{Q}''$$

of pro-quasi-coherent pro-sheaves on $\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$.

Proof. Follows from the definition of the external tensor product of pro-quasi-coherent pro-sheaves, the definition of the direct image of pro-quasi-coherent pro-sheaves (see Section 3.3), and Lemma 6.4. ☐

Let $\mathfrak{X}$ be an ind-scheme over $\Bbbk$, and let $\mathfrak{P}'$ and $\mathfrak{P}''$ be two pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. Then $\mathfrak{P}'\boxtimes_\Bbbk\mathfrak{P}''$ is a pro-quasi-coherent pro-sheaf on $\mathfrak{X}\times_\Bbbk\mathfrak{X}$. Let $\Delta_\mathfrak{X}\colon\mathfrak{X}\longrightarrow\mathfrak{X}\times_\Bbbk\mathfrak{X}$ denote the diagonal morphism of ind-schemes (defined by the property that its compositions with both the projections $\mathfrak{X}\times_{\Bbbk}\mathfrak{X}\rightrightarrows\mathfrak{X}$ are the identity morphisms).

Lemma 6.12. *For any two pro-quasi-coherent pro-sheaves $\mathfrak{P}'$ and $\mathfrak{P}''$ on an ind-scheme $\mathfrak{X}$ over $\Bbbk$, there is a natural isomorphism*

$$\mathfrak{P}'\otimes^{\mathfrak{X}}\mathfrak{P}'' \simeq \Delta_{\mathfrak{X}}^*(\mathfrak{P}'\boxtimes_{\Bbbk}\mathfrak{P}'')$$

of pro-quasi-coherent pro-sheaves on $\mathfrak{X}$.

Proof. Follows from Lemma 6.8 and the definitions of the functors $\otimes^{\mathfrak{X}}$ (see Section 3.1), $\Delta_{\mathfrak{X}}^*$ (see Section 3.3), and $\boxtimes_{\Bbbk}$. ☐

6.3. **External tensor product of torsion sheaves.** Let $\mathfrak{X}'=\text{“}\varinjlim\text{”}_{\gamma'\in\Gamma'}X'_{\gamma'}$ and $\mathfrak{X}''=\text{“}\varinjlim\text{”}_{\gamma''\in\Gamma''}X''_{\gamma''}$ be two reasonable ind-schemes over $\Bbbk$, represented by inductive systems of closed immersions of reasonable closed subschemes. Then $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''=\text{“}\varinjlim\text{”}_{(\gamma',\gamma'')\in\Gamma'\times\Gamma''}X'_{\gamma'}\times_{\Bbbk}X''_{\gamma''}$ is a representation of the ind-scheme $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$ by an inductive system of closed immersions of reasonable closed subschemes (by Lemma 6.5).

Let $\mathscr{M}'$ be a quasi-coherent torsion sheaf on $\mathfrak{X}'$ and $\mathscr{M}''$ be a quasi-coherent torsion sheaf on $\mathfrak{X}''$. For every pair of reasonable closed subschemes $Z'\subset\mathfrak{X}'$ and $Z''\subset\mathfrak{X}''$ put $\mathscr{L}_{(Z'\times_{\Bbbk}Z'')}=\mathscr{M}'_{(Z')}\boxtimes_{\Bbbk}\mathscr{M}''_{(Z'')}\in(Z'\times_{\Bbbk}Z'')\text{–}\mathsf{qcoh}$. Then it is clear from Lemma 6.6 that the collection of quasi-coherent sheaves $\mathscr{L}_{(Z'\times_{\Bbbk}Z'')}$ on the reasonable closed subschemes $Z'\times_{\Bbbk}Z''\subset\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$ defines a quasi-coherent torsion sheaf $\mathscr{L}$ on $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$.

Put $\mathscr{M}'\boxtimes_{\Bbbk}\mathscr{M}''=\mathscr{L}$. This construction defines the functor of *external tensor product of quasi-coherent torsion sheaves*

$$\boxtimes_{\Bbbk}\colon\mathfrak{X}'\text{–}\mathsf{tors}\times\mathfrak{X}''\text{–}\mathsf{tors}\longrightarrow(\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}'')\text{–}\mathsf{tors}. \tag{22}$$

Lemma 6.13. *Let $f'\colon\mathfrak{Y}'\longrightarrow\mathfrak{X}'$ and $f''\colon\mathfrak{Y}''\longrightarrow\mathfrak{X}''$ be two affine morphisms of reasonable ind-schemes over $\Bbbk$, and let $f=f'\times_{\Bbbk}f''\colon\mathfrak{Y}'\times_{\Bbbk}\mathfrak{Y}''\longrightarrow\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$ be the induced morphism of the Cartesian products. Let $\mathscr{N}'$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}'$ and $\mathscr{N}''$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}''$. Then there is a natural isomorphism*

$$f_*(\mathscr{N}'\boxtimes_{\Bbbk}\mathscr{N}'')\simeq f'_*\mathscr{N}'\boxtimes_{\Bbbk}f''_*\mathscr{N}''$$

of quasi-coherent torsion sheaves on $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$.

Proof. Follows immediately from the definition of the external tensor product of quasi-coherent torsion sheaves (above), the definition of the direct image of quasi-coherent torsion sheaves (see Section 2.6), and Lemma 6.4. ☐

Similarly to the construction above, one defines the functor of external tensor product of Γ-systems

$$\boxtimes_{\Bbbk}\colon(\mathfrak{X}',\Gamma')\text{–}\mathsf{syst}\times(\mathfrak{X}',\Gamma'')\text{–}\mathsf{syst}\longrightarrow(\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}'',\ \Gamma'\times\Gamma'')\text{–}\mathsf{syst} \tag{23}$$

by setting $(\mathbb{M}'\boxtimes_{\Bbbk}\mathbb{M}'')_{(\gamma'\times\gamma'')}=\mathbb{M}'_{(\gamma')}\boxtimes_{\Bbbk}\mathbb{M}''_{(\gamma'')}$ for any Γ'-system $\mathbb{M}'$ on $\mathfrak{X}'$, any Γ''-system $\mathbb{M}''$ on $\mathfrak{X}''$, and any two indices $\gamma'\in\Gamma'$, $\gamma''\in\Gamma''$.

Lemma 6.14. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be reasonable ind-schemes over $\Bbbk$. Then*

(a) *the functor of external tensor product* $\boxtimes_\Bbbk\colon \mathfrak{X}'\text{–}\mathsf{tors}\times\mathfrak{X}''\text{–}\mathsf{tors}\longrightarrow(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{tors}$ *preserves direct limits (and in particular, coproducts) in each of the arguments;*

(b) *the functors of external tensor product of Γ-systems* (23) *and of quasi-coherent torsion sheaves* (22) *form a commutative square diagram with the functors* $\mathbb{M}^{(s)}\longmapsto\mathbb{M}^{(s)+}\colon(\mathfrak{X}^{(s)},\Gamma^{(s)})\text{–}\mathsf{syst}\longrightarrow\mathfrak{X}^{(s)}\text{–}\mathsf{tors}$, $s=1$, 2, *and* $\mathbb{L}\longmapsto\mathbb{L}^+\colon(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'',\Gamma'\times\Gamma'')\text{–}\mathsf{syst}\longrightarrow(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{tors}$.

Proof. Part (a) follows from the description of direct limits of quasi-coherent torsion sheaves in Section 2.5 together with Lemma 6.2. Part (b) holds because the functors $(-)^+$ are constructed in terms of direct images (with respect to closed immersions of reasonable closed subschemes into ind-schemes) and direct limits of quasi-coherent torsion sheaves (see Section 2.7). The external tensor products commute with the direct images by Lemma 6.13 and preserve direct limits by part (a). □

Lemma 6.15. *For for any reasonable ind-schemes $\mathfrak{X}'$ and $\mathfrak{X}''$ over $\Bbbk$, the functor of external tensor product* $\boxtimes_\Bbbk\colon\mathfrak{X}'\text{–}\mathsf{tors}\times\mathfrak{X}''\text{–}\mathsf{tors}\longrightarrow(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{tors}$ *is exact in each of its arguments.*

Proof. Exactness of the functor of external tensor product of Γ-systems (23) follows immediately from Lemma 6.2. To deduce exactness of the functor of external tensor product of quasi-coherent torsion sheaves (22), it remains to recall that the functors $(-)^+$ represent the abelian categories of quasi-coherent torsion sheaves as quotient categories of the abelian categories of Γ-systems by some Serre subcategories (see the proof of Proposition 2.8) and use Lemma 6.14(b). □

Lemma 6.16. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be reasonable ind-schemes over $\Bbbk$, and let $f'\colon\mathfrak{Y}'\longrightarrow\mathfrak{X}'$ and $f''\colon\mathfrak{Y}''\longrightarrow\mathfrak{X}''$ be morphisms of ind-schemes which are "representable by schemes". Let $f=f'\times_\Bbbk f''\colon\mathfrak{Y}'\times_\Bbbk\mathfrak{Y}''\longrightarrow\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$ be the induced morphism of the Cartesian products. Let $\mathscr{M}'$ be a quasi-coherent torsion sheaf on $\mathfrak{X}'$ and $\mathscr{M}''$ be a quasi-coherent torsion sheaf on $\mathfrak{X}''$. Then there is a natural isomorphism*

$$f^*(\mathscr{M}'\boxtimes_\Bbbk\mathscr{M}'')\simeq f'^*\mathscr{M}'\boxtimes_\Bbbk f''^*\mathscr{M}''$$

of quasi-coherent torsion sheaves on $\mathfrak{Y}'\times_\Bbbk\mathfrak{Y}''$.

Proof. Follows from the definition of the inverse image of quasi-coherent torsion sheaves (see Section 2.8) and Lemmas 6.4 and 6.14(b). □

Let us say that a closed immersion of ind-schemes $i\colon\mathfrak{Z}\longrightarrow\mathfrak{X}$ is *reasonable* if, for any scheme T and a morphism of schemes $T\longrightarrow\mathfrak{X}$, the morphism of schemes $\mathfrak{Z}\times_\mathfrak{X}T\longrightarrow T$ is the closed immersion of a reasonable closed subscheme in T. If $\mathfrak{X}=\text{“}\varinjlim\text{”}_{\gamma\in\Gamma}X_\gamma$ is a representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes, then a closed immersion of ind-schemes $i\colon\mathfrak{Z}\longrightarrow\mathfrak{X}$ is reasonable if and only if, for every $\gamma\in\Gamma$, the fibered product $\mathfrak{Z}\times_\mathfrak{X}X_\gamma$ is a reasonable closed subscheme in X_γ (use Lemma 2.1(a)). It is clear from Lemma 6.5 that the Cartesian product of reasonable closed immersions of ind-schemes over $\Bbbk$ is a reasonable closed immersion.

Lemma 6.17. *Let $i'\colon \mathfrak{Z}' \longrightarrow \mathfrak{X}'$ and $i''\colon \mathfrak{Z}'' \longrightarrow \mathfrak{X}''$ be reasonable closed immersions of reasonable ind-schemes over $\Bbbk$, and let $i = i' \times_\Bbbk i''\colon \mathfrak{Z}' \times_\Bbbk \mathfrak{Z}'' \longrightarrow \mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ be the induced reasonable closed immersion of the Cartesian products. Let $\mathscr{M}'$ be a quasi-coherent torsion sheaf on $\mathfrak{X}'$ and $\mathscr{M}''$ be a quasi-coherent torsion sheaf on $\mathfrak{X}''$. Then there is a natural isomorphism*

$$i^!(\mathscr{M}' \boxtimes_\Bbbk \mathscr{M}'') \simeq i'^!\mathscr{M}' \boxtimes_\Bbbk i''^!\mathscr{M}''$$

of quasi-coherent torsion sheaves on $\mathfrak{Z}' \times_\Bbbk \mathfrak{Z}''$.

Proof. Follows immediately from the definition of the external tensor product of quasi-coherent torsion sheaves, the definition of the functor $i^!$ for a closed immersion of ind-schemes i (see Section 2.8), and Lemma 6.6. ☐

Let $\mathscr{M}'^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $\mathscr{M}''^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}''$. Then the complex $\mathscr{M}'^\bullet \boxtimes_\Bbbk \mathscr{M}''^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ is constructed by totalizing the bicomplex of external tensor products using infinite coproducts along the diagonals.

Lemma 6.18. *Let $\mathscr{M}'^\bullet$ be a complex of quasi-coherent torsion sheaves on a reasonable ind-scheme $\mathfrak{X}'$ and $\mathscr{M}''^\bullet$ be a complex of quasi-coherent torsion sheaves on a reasonable ind-scheme $\mathfrak{X}''$ over $\Bbbk$. Assume that the complex $\mathscr{M}'^\bullet$ is coacyclic (as a complex in $\mathfrak{X}'\text{–}\mathsf{tors}$). Then the complex $\mathscr{M}'^\bullet \boxtimes_\Bbbk \mathscr{M}''^\bullet$ is coacyclic (as a complex in $(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')\text{–}\mathsf{tors}$).*

Proof. Follows from Lemmas 6.14(a) and 6.15. ☐

It is clear from Lemma 6.18 that the external tensor product is well-defined as a functor between the coderived categories of quasi-coherent torsion sheaves,

$$\boxtimes_\Bbbk\colon \mathsf{D^{co}}(\mathfrak{X}'\text{–}\mathsf{tors}) \times \mathsf{D^{co}}(\mathfrak{X}''\text{–}\mathsf{tors}) \longrightarrow \mathsf{D^{co}}((\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')\text{–}\mathsf{tors}). \tag{24}$$

6.4. **Derived restriction with supports.** Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme and $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of ind-schemes (then $\mathfrak{Z}$ is also an ind-Noetherian ind-scheme). The functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ was defined in Section 2.8.

According to Corollary 4.18, the inclusion of the full subcategory of injective quasi-coherent torsion sheaves $\mathfrak{X}\text{–}\mathsf{tors_{inj}} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ induces a triangulated equivalence $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors_{inj}}) \simeq \mathsf{D^{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. The right derived functor

$$\mathbb{R}i^!\colon \mathsf{D^{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{Z}\text{–}\mathsf{tors})$$

is constructed by applying the functor $i^!$ to complexes of injective quasi-coherent torsion sheaves on $\mathfrak{X}$.

Notice that the right derived functor $\mathbb{R}i^!$ preserves coproducts. Indeed, the underived functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ preserves coproducts for any reasonable closed immersion of reasonable ind-schemes $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$; and coproducts of injective quasi-coherent torsion sheaves are injective on an ind-Noetherian ind-scheme (by Lemma 4.16 and Proposition 4.17).

Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes of ind-finite type over $\Bbbk$ (then $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ is also an ind-scheme of ind-finite type). Let $i'\colon \mathfrak{Z}' \longrightarrow \mathfrak{X}'$ and $i''\colon \mathfrak{Z}'' \longrightarrow \mathfrak{X}''$ be closed immersions of ind-schemes; denote by $i = i' \times_{\Bbbk} i'' \colon \mathfrak{Z}' \times_{\Bbbk} \mathfrak{Z}'' \longrightarrow \mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$ the induced closed immersion of the Cartesian products. The aim of this Section 6.4 is to prove the following proposition (for another comparable result, see Proposition 10.14 below).

Proposition 6.19. *For any complexes of quasi-coherent torsion sheaves $\mathscr{M}'^{\bullet}$ on $\mathfrak{X}'$ and $\mathscr{M}''^{\bullet}$ on $\mathfrak{X}''$, there is a natural isomorphism*

$$\mathbb{R}i^!(\mathscr{M}'^{\bullet} \boxtimes_{\Bbbk} \mathscr{M}''^{\bullet}) \simeq \mathbb{R}i'^!(\mathscr{M}'^{\bullet}) \boxtimes_{\Bbbk} \mathbb{R}i''^!(\mathscr{M}''^{\bullet})$$

in the coderived category $\mathsf{D}^{\mathsf{co}}((\mathfrak{Z}' \times_{\Bbbk} \mathfrak{Z}'')\text{–}\mathsf{tors})$.

The proof of Proposition 6.19 requires some work because the external tensor product of two injective quasi-coherent (torsion) sheaves is usually *not* an injective quasi-coherent (torsion) sheaf. In fact, the assertion of the proposition follows almost immediately from the next lemma (together with Lemma 6.17).

Lemma 6.20. *Let $\mathscr{J}'^{\bullet}$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $\mathscr{J}''^{\bullet}$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}''$. Let $r\colon \mathscr{J}'^{\bullet} \boxtimes_{\Bbbk} \mathscr{J}''^{\bullet} \longrightarrow \mathscr{K}^{\bullet}$ be a morphism of complexes of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$ such that $\mathscr{K}^{\bullet}$ is a complex of injective quasi-coherent torsion sheaves and the cone of r is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$. Then the induced morphism of complexes of quasi-coherent torsion sheaves on $\mathfrak{Z}' \times_{\Bbbk} \mathfrak{Z}''$*

$$i^!(r)\colon i^!(\mathscr{J}'^{\bullet} \boxtimes_{\Bbbk} \mathscr{J}''^{\bullet}) \longrightarrow i^!\mathscr{K}^{\bullet}$$

has coacyclic cone.

The proof of Lemma 6.20 will be given below near the end of Section 6.4.

Lemma 6.21. *Let R' and R'' be Noetherian commutative $\Bbbk$-algebras, M' be a finitely generated R'-module, M'' be a finitely generated R''-module, N' be an R'-module, and N'' be an R''-module. Then for every $n \geq 0$ there is a natural isomorphism of $\Bbbk$-vector spaces*

$$\operatorname{Ext}^n_{R' \otimes_{\Bbbk} R''}(M' \otimes_{\Bbbk} M'',\ N' \otimes_{\Bbbk} N'') \simeq \bigoplus\nolimits_{p+q=n} \operatorname{Ext}^p_{R'}(M', N') \otimes_{\Bbbk} \operatorname{Ext}^q_{R''}(M'', N'').$$

In particular, for any injective R'-module J' and any injective R''-module J'' one has $\operatorname{Ext}^n_{R' \otimes_{\Bbbk} R''}(M' \otimes_{\Bbbk} M'',\ J' \otimes_{\Bbbk} J'') = 0$ for all $n > 0$.

Proof. The assumption of commutativity of the rings R' and R'' is actually not needed. One starts with the observation that, for any finitely generated projective R'-module P' and any finitely generated projective R''-module P'' there is a natural isomorphism of Hom spaces

$$\operatorname{Hom}_{R' \otimes_{\Bbbk} R''}(P' \otimes_{\Bbbk} P'',\ N' \otimes_{\Bbbk} N'') \simeq \operatorname{Hom}_{R'}(P', N') \otimes_{\Bbbk} \operatorname{Hom}_{R''}(P'', N'').$$

Now let $P'_{\bullet} \longrightarrow M'$ be a resolution of M' by finitely generated projective R'-modules and $P''_{\bullet} \longrightarrow M''$ be a resolution of M'' by finitely generated projective R''-modules. Then the tensor product of two complexes $P'_{\bullet} \otimes_{\Bbbk} P''_{\bullet}$ is a resolution of $M' \otimes_{\Bbbk} M''$ by (finitely generated) projective $(R' \otimes_{\Bbbk} R'')$-modules. It remains to compute

$$\operatorname{Hom}_{R' \otimes_{\Bbbk} R''}(P'_{\bullet} \otimes_{\Bbbk} P''_{\bullet},\ N' \otimes_{\Bbbk} N'') \simeq \operatorname{Hom}_{R'}(P'_{\bullet}, N') \otimes_{\Bbbk} \operatorname{Hom}_{R''}(P''_{\bullet}, N'')$$

and recall that for any two complexes of $\Bbbk$-vector spaces $C'^{\bullet}$ and $C''^{\bullet}$ one has $H^n(C'^{\bullet}\otimes_{\Bbbk} C''^{\bullet})\simeq\bigoplus_{p+q=n} H^p(C'^{\bullet})\otimes_{\Bbbk} H^q(C''^{\bullet})$. ☐

Lemma 6.22. *Let X' and X'' be schemes of finite type over $\Bbbk$, and let $i'\colon Z'\longrightarrow X'$ and $i''\colon Z''\longrightarrow X''$ be closed immersions of schemes. Denote by $i\colon Z'\times_{\Bbbk} Z''\longrightarrow X'\times_{\Bbbk} X''$ the induced closed immersion of the Cartesian products. Let $\mathcal{J}'$ be an injective quasi-coherent sheaf on X' and $\mathcal{J}''$ be an injective quasi-coherent sheaf on X''. Let $\mathcal{J}'\boxtimes_{\Bbbk}\mathcal{J}''\longrightarrow\mathcal{K}^{\bullet}$ be an injective resolution of the quasi-coherent sheaf $\mathcal{J}'\boxtimes_{\Bbbk}\mathcal{J}''$ on the scheme $X'\times_{\Bbbk} X''$. Then one has*

$$H^0(i^!\mathcal{K}^{\bullet})\simeq i'^!\mathcal{J}'\boxtimes_{\Bbbk} i''^!\mathcal{J}'' \quad \text{and} \quad H^n(i^!\mathcal{K}^{\bullet})=0 \ \text{ for } n>0.$$

Proof. To compute $H^0(i^!\mathcal{K}^{\bullet})$, it suffices to observe that the functor $i^!$ is left exact (as a right adjoint), so $H^0(i^!\mathcal{K}^{\bullet})\simeq i^!(\mathcal{J}'\boxtimes_{\Bbbk}\mathcal{J}'')\simeq i'^!\mathcal{J}'\boxtimes_{\Bbbk} i''^!\mathcal{J}''$ by Lemma 6.6. The vanishing assertion in local in X' and X'' (notice that all the schemes involved are Noetherian, so injectivity of a quasi-coherent sheaf is a local property), so it reduces to the case of affine schemes, for which it means the following.

Let $R'\longrightarrow S'$ and $R''\longrightarrow S''$ be surjective homomorphisms of finitely generated commutative $\Bbbk$-algebras, and let J' and J'' be injective modules over R' and R'', respectively. Let $K^{\bullet}$ be an injective resolution of the $(R'\otimes_{\Bbbk} R'')$-module $J'\otimes_{\Bbbk} J''$. Then the complex $\operatorname{Hom}_{R'\otimes_{\Bbbk} R''}(S'\otimes_{\Bbbk} S'',K^{\bullet})$ has vanishing cohomology in the positive cohomological degrees. This is a particular case of Lemma 6.21. ☐

Lemma 6.23. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes of ind-finite type over $\Bbbk$, and let $Z'\subset\mathfrak{X}'$ and $Z''\subset\mathfrak{X}''$ be closed subschemes with the closed immersion morphisms $Z'\longrightarrow\mathfrak{X}'$ and $Z''\longrightarrow\mathfrak{X}''$. Denote by $i\colon Z'\times_{\Bbbk} Z''\longrightarrow\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$ the induced closed immersion of the Cartesian products. Let $\mathscr{J}'$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}'$ and $\mathscr{J}''$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}''$. Let $\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}''\longrightarrow\mathscr{K}^{\bullet}$ be an injective resolution of the quasi-coherent torsion sheaf $\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}''$ on the ind-scheme $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$. Then one has*

$$H^0(i^!\mathscr{K}^{\bullet})\simeq i'^!\mathscr{J}'\boxtimes_{\Bbbk} i''^!\mathscr{J}'' \quad \text{and} \quad H^n(i^!\mathscr{K}^{\bullet})=0 \ \text{ for } n>0.$$

Proof. The computation of H^0 is similar to the one in Lemma 6.22 (use Lemma 6.17). To prove the higher cohomology vanishing, choose inductive systems of closed immersions of schemes of finite type over $\Bbbk$ representing the ind-schemes $\mathfrak{X}'$ and $\mathfrak{X}''$, and consider the related inductive system representing the ind-scheme $\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$, as in the beginning of Section 6.3. Put $\mathfrak{X}=\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''$ and $\Gamma=\Gamma'\times\Gamma''$. For any biindex $\gamma=(\gamma',\gamma'')\in\Gamma$, put $X_\gamma=X'_{\gamma'}\times X''_{\gamma''}$. We can always assume that there exist $\gamma'_0\in\Gamma'$ and $\gamma''_0\in\Gamma''$ such as $Z'=X'_{\gamma'_0}\subset\mathfrak{X}'$ and $Z''=X''_{\gamma''_0}\subset\mathfrak{X}''$.

Our aim is to show that the exact sequence of quasi-coherent torsion sheaves $0\longrightarrow\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}''\longrightarrow\mathscr{K}^0\longrightarrow\mathscr{K}^1\longrightarrow\mathscr{K}^2\longrightarrow\cdots$ on $\mathfrak{X}$ remains exact after applying the functor $\mathscr{M}\longmapsto\mathscr{M}|_\Gamma\colon\mathfrak{X}\text{–tors}\longrightarrow(\mathfrak{X},\Gamma)\text{–syst}$. Notice that, by the definitions of the external tensor products, we have $(\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'')|_\Gamma=\mathscr{J}'|_{\Gamma'}\boxtimes_{\Bbbk}\mathscr{J}''|_{\Gamma''}$.

Denote by $\mathbb{M}^1$ the cokernel of the morphism of Γ-systems $(\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'')|_\Gamma\longrightarrow\mathscr{K}^0|_\Gamma$. Let $\gamma=(\gamma',\gamma'')\le\delta=(\delta',\delta'')$ be two biindices (where $\gamma'\le\delta'\in\Gamma'$ and $\gamma''\le\delta''\in\Gamma''$).

Denote the related transition maps in the inductive systems by $i'_{\gamma'\delta'}\colon X'_{\gamma'} \longrightarrow X'_{\delta'}$ and $i''_{\gamma''\delta''}\colon X''_{\gamma''} \longrightarrow X''_{\delta''}$, and put $i_{\gamma\delta} = i'_{\gamma'\delta'} \times_{\Bbbk} i''_{\gamma''\delta''}\colon X_\gamma \longrightarrow X_\delta$.

By Proposition 2.17(a), the quasi-coherent sheaves $\mathscr{K}^n_{(X_\delta)}$ on the scheme X_δ are injective for all $n \ge 0$. By Lemma 6.22, the short exact sequence $0 \longrightarrow (\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'')_{(X_\delta)} \longrightarrow \mathscr{K}^0_{(X_\delta)} \longrightarrow \mathbb{M}^1_{(\delta)} \longrightarrow 0$ of quasi-coherent sheaves on X_δ remains exact after applying the functor $i^!_{\gamma\delta}$. Hence the structure map $\mathbb{M}^1_{(\gamma)} \longrightarrow i^!_{\gamma\delta}\mathbb{M}^1_{(\delta)}$ in the Γ-system $\mathbb{M}^1$ is an isomorphism (of quasi-coherent sheaves on X_γ).

As this holds for all $\gamma \le \delta \in \Gamma$, we can conclude that the collection of quasi-coherent sheaves $\mathscr{M}^1_{(X_\gamma)} = \mathbb{M}^1_{(\gamma)}$ defines a quasi-coherent torsion sheaf $\mathscr{M}^1$ on $\mathfrak{X}$. So we have $\mathbb{M}^1 = \mathscr{M}^1|_\Gamma$ and $\mathscr{M}^1 = \mathbb{M}^{1+}$. In other words, this means that the adjunction morphism $\mathbb{M}^1 \longrightarrow \mathbb{M}^{1+}|_\Gamma$ is an isomorphism of Γ-systems. Notice that the quasi-coherent torsion sheaf $\mathbb{M}^{1+}$ on $\mathfrak{X}$ is, by the definition, the cokernel of the monomorphism of quasi-coherent torsion sheaves $\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'' \longrightarrow \mathscr{K}^0$. We have shown that the short exact sequence of quasi-coherent torsion sheaves $0 \longrightarrow \mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'' \longrightarrow \mathscr{K}^0 \longrightarrow \mathbb{M}^{1+} \longrightarrow 0$ on $\mathfrak{X}$ remains exact after applying the functor $\mathscr{M} \longmapsto \mathscr{M}|_\Gamma$.

Denote by $\mathbb{M}^2$ the cokernel of the morphism of Γ-systems $\mathscr{K}^0|_\Gamma \longrightarrow \mathscr{K}^1|_\Gamma$; as we have seen, this is the same thing as the cokernel of the morphism of Γ-systems $\mathbb{M}^1 \longrightarrow \mathscr{K}^1|_\Gamma$. By Lemma 6.22, the exact sequence $0 \longrightarrow (\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'')_{(X_\delta)} \longrightarrow \mathscr{K}^0_{(X_\delta)} \longrightarrow \mathscr{K}^1_{(X_\delta)} \longrightarrow \mathbb{M}^2_{(\delta)} \longrightarrow 0$ of quasi-coherent sheaves on X_δ remains exact after applying the functor $i^!_{\gamma\delta}$. Hence the structure map $\mathbb{M}^2_{(\gamma)} \longrightarrow i^!_{\gamma\delta}\mathbb{M}^2_{(\delta)}$ in the Γ-system $\mathbb{M}^2$ is an isomorphism. As this holds for all $\gamma \le \delta \in \Gamma$, we can conclude that the adjunction morphism $\mathbb{M}^2 \longrightarrow \mathbb{M}^{2+}|_\Gamma$ is an isomorphism of Γ-systems.

Notice that the quasi-coherent torsion sheaf $\mathbb{M}^{2+}$ on $\mathfrak{X}$ is, by the definition, the cokernel of the morphism of quasi-coherent torsion sheaves $\mathscr{K}^0 \longrightarrow \mathscr{K}^1$. We have shown that the exact sequence of quasi-coherent torsion sheaves $0 \longrightarrow \mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'' \longrightarrow \mathscr{K}^0 \longrightarrow \mathscr{K}^1 \longrightarrow \mathbb{M}^{2+} \longrightarrow 0$ on $\mathfrak{X}$ remains exact after applying the functor $\mathscr{M} \longmapsto \mathscr{M}|_\Gamma$. Proceeding in this way, we prove the desired preservation of exactness by induction in the cohomological degree. □

Lemma 6.24. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes of ind-finite type over $\Bbbk$, and let $i'\colon \mathfrak{Z}' \longrightarrow \mathfrak{X}'$ and $i''\colon \mathfrak{Z}'' \longrightarrow \mathfrak{X}''$ be closed immersions of ind-schemes. Denote by $i\colon \mathfrak{Z}' \times_{\Bbbk} \mathfrak{Z}'' \longrightarrow \mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$ the induced closed immersion of the Cartesian products. Let $\mathscr{J}'$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}'$ and $\mathscr{J}''$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}''$. Let $\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}'' \longrightarrow \mathscr{K}^\bullet$ be an injective resolution of the quasi-coherent torsion sheaf $\mathscr{J}'\boxtimes_{\Bbbk}\mathscr{J}''$ on the ind-scheme $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$. Then one has*

$$H^0(i^!\mathscr{K}^\bullet) \simeq i'^!\mathscr{J}'\boxtimes_{\Bbbk} i''^!\mathscr{J}'' \quad \text{and} \quad H^n(i^!\mathscr{K}^\bullet) = 0 \ \text{ for } n > 0.$$

Proof. The computation of H^0 is similar to the one in Lemmas 6.22 and 6.23. The functor $i^!$ is left exact as a right adjoint, and it remains to use Lemma 6.17. To prove the vanishing assertion, choose closed subschemes $Z' \subset \mathfrak{Z}'$ and $Z'' \subset \mathfrak{Z}''$ with the closed immersion morphisms $k'\colon Z' \longrightarrow \mathfrak{Z}'$ and $k''\colon Z'' \longrightarrow \mathfrak{Z}''$. Denote by $k = k' \times_{\Bbbk} k''\colon Z' \times_{\Bbbk} Z'' \longrightarrow \mathfrak{Z}' \times_{\Bbbk} \mathfrak{Z}''$ the induced closed immersion of the Cartesian products. Then by Lemma 6.23 we have $H^n(k^!i^!\mathscr{K}^\bullet) = 0$ for $n > 0$, and it follows that $H^n(i^!\mathscr{K}^\bullet) = 0$ for $n > 0$ as well. □

Lemma 6.25. *In the notation of Lemma 6.24, let $(\mathscr{J}'_\theta)_{\theta\in\Theta}$ be a family of injective quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $(\mathscr{J}''_\theta)_{\theta\in\Theta}$ be a family of injective quasi-coherent torsion sheaves on $\mathfrak{X}''$, indexed by the same set Θ. Let $\coprod_{\theta\in\Theta}\mathscr{J}'_\theta\boxtimes_\Bbbk\mathscr{J}''_\theta \longrightarrow \mathscr{K}^\bullet$ be an injective resolution of the coproduct of quasi-coherent torsion sheaves $\mathscr{J}'_\theta\boxtimes_\Bbbk \mathscr{J}''_\theta$ on the ind-scheme $\mathfrak{X}'\times_\Bbbk\mathfrak{X}''$. Then one has*

$$H^0(i^!\mathscr{K}^\bullet) \simeq \coprod\nolimits_{\theta\in\Theta} i'^!\mathscr{J}'_\theta\boxtimes_\Bbbk i''^!\mathscr{J}''_\theta \quad \textit{and} \quad H^n(i^!\mathscr{K}^\bullet) = 0 \ \ \textit{for } n > 0.$$

Proof. Essentially, the claim is that the right derived functor

$$\mathbb{R}i^!\colon \mathsf{D^{co}}((\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')\text{–}\mathsf{tors}) \longrightarrow \mathsf{D^{co}}((\mathfrak{Z}'\times_\Bbbk\mathfrak{Z}'')\text{–}\mathsf{tors})$$

preserves coproducts (as mentioned above in the beginning of Section 6.4). This reduces the question to Lemma 6.24. □

Proof of Lemma 6.20. Given two complexes $\mathscr{J}'^\bullet$ and $\mathscr{J}''^\bullet$, a related complex $\mathscr{K}^\bullet$ is defined uniquely up to a homotopy equivalence (by Proposition 4.15(a)); so it suffices to prove the assertion of the lemma for one specific choice of the complex $\mathscr{K}^\bullet$. We will use the complex $\mathscr{K}^\bullet$ provided by the construction on which the proof of Proposition 4.15(b) is based.

Let $\mathscr{J}'^\bullet\boxtimes_\Bbbk\mathscr{J}''^\bullet \longrightarrow \mathscr{L}^{0,\bullet}$ be a monomorphism of complexes in $(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')$–$\mathsf{tors}$ such that $\mathscr{L}^{0,\bullet}$ is a complex of injective quasi-coherent torsion sheaves. Denote by $\mathscr{M}^{1,\bullet}$ the cokernel of this morphism of complexes, and let $\mathscr{M}^{1,\bullet}\longrightarrow\mathscr{L}^{1,\bullet}$ be a monomorphism of complexes in which $\mathscr{L}^{1,\bullet}$ is a complex of injectives. Proceeding in this way, we construct a bounded below complex of complexes of injective quasi-coherent torsion sheaves $\mathscr{L}^{\bullet,\bullet}$ together with a quasi-isomorphism $\mathscr{J}'^\bullet\boxtimes_\Bbbk\mathscr{J}''^\bullet \longrightarrow \mathscr{L}^{\bullet,\bullet}$ of complexes of complexes in $(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')$–$\mathsf{tors}$. Let us emphasize that the notation $\mathscr{J}'^\bullet\boxtimes_\Bbbk\mathscr{J}''^\bullet$ here stands for the total complex of the bicomplex of tensor products, while $\mathscr{L}^{\bullet,\bullet}$ is a bicomplex (not totalized yet). In every cohomological degree n, the complex $\mathscr{L}^{\bullet,n}$ is an injective resolution of the quasi-coherent torsion sheaf $(\mathscr{J}'^\bullet\boxtimes_\Bbbk\mathscr{J}''^\bullet)^n$. The complex $\mathscr{K}^\bullet$ is then constructed by totalizing the bicomplex $\mathscr{L}^{\bullet,\bullet}$ using infinite coproducts along the diagonals.

Recall that the functor $i^!\colon(\mathfrak{X}'\times_\Bbbk\mathfrak{X}'')$–$\mathsf{tors}\longrightarrow(\mathfrak{Z}'\times_\Bbbk\mathfrak{Z}'')$–$\mathsf{tors}$ preserves coproducts. In every cohomological degree n, applying $i^!$ to the complex $0\longrightarrow(\mathscr{J}'^\bullet\boxtimes_\Bbbk\mathscr{J}''^\bullet)^n\longrightarrow \mathscr{L}^{0,n}\longrightarrow\mathscr{L}^{1,n}\longrightarrow\cdots$ produces an acyclic complex in $(\mathfrak{Z}'\times_\Bbbk\mathfrak{Z}'')$–$\mathsf{tors}$ by Lemma 6.25. It remains to point out that the coproduct totalization of an acyclic bounded below complex of complexes is a coacyclic complex [40, Lemma 2.1]. □

Proof of Proposition 6.19. It is relevant that the external tensor product is well-defined as a functor between the coderived categories (by Lemma 6.18). Choose a complex of injective quasi-coherent torsion sheaves $\mathscr{J}'^\bullet$ on $\mathfrak{X}'$ and a complex of injective quasi-coherent torsion sheaves $\mathscr{J}''^\bullet$ on $\mathfrak{X}''$ together with morphisms $\mathscr{M}'^\bullet\longrightarrow\mathscr{J}'^\bullet$ and $\mathscr{M}''^\bullet\longrightarrow\mathscr{J}''^\bullet$ with coacyclic cones. Then it remains to apply Lemma 6.20 and take Lemma 6.17 into account. □

6.5. **Rigid dualizing complexes.** The notion of a rigid dualizing complex was introduced originally in [64, Definition 8.1] and studied in [66, 4] and many other papers (see also [65] and [59]). Without going into details, we will formulate a simple definition of a rigid dualizing complex on an ind-scheme of ind-finite type over a field $\Bbbk$ in the form suitable for our purposes.

Lemma 6.26. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-semi-separated ind-schemes of ind-finite type over $\Bbbk$. Let $\mathscr{D}'^\bullet$ be a dualizing complex on $\mathfrak{X}'$ and $\mathscr{D}''^\bullet$ be a dualizing complex on $\mathfrak{X}''$. Let $\mathscr{E}^\bullet$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ endowed with a morphism of complexes $\mathscr{D}'^\bullet \boxtimes_\Bbbk \mathscr{D}''^\bullet \longrightarrow \mathscr{E}^\bullet$ with coacyclic cone. Then $\mathscr{E}^\bullet$ is a dualizing complex on $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$.*

Proof. Proposition 6.19 (for closed subschemes $Z' \subset \mathfrak{X}'$ and $Z'' \subset \mathfrak{X}''$) reduces the question to schemes, for which it is straightforward. □

Let $\mathfrak{X}$ be an ind-scheme over $\Bbbk$. Denote by $\Delta\colon \mathfrak{X} \longrightarrow \mathfrak{X}\times_\Bbbk\mathfrak{X}$ the diagonal morphism. Then the ind-scheme $\mathfrak{X}$ is ind-separated (as defined in Section 4.2) if and only if the morphism Δ is a closed immersion of ind-schemes.

Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over $\Bbbk$. An object $\mathscr{E}^\bullet \in \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ in the coderived category of quasi-coherent torsion sheaves on $\mathfrak{X}$ is said to be *rigid* if it is endowed with an isomorphism $\mathscr{E}^\bullet \simeq \mathbb{R}\Delta^!(\mathscr{E}^\bullet \boxtimes_\Bbbk \mathscr{E}^\bullet)$ (called the *rigidifying isomorphism*) in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. A dualizing complex $\mathscr{D}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ on $\mathfrak{X}$ is said to be *rigid* if it is rigid (i. e., has been endowed with a rigidifying isomorphism) as an object of $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Our aim is to explain how to produce a rigid dualizing complex on an $\aleph_0$-ind-scheme from rigid dualizing complexes on schemes, in the spirit of Example 4.7. For this purpose, we need to start with some preliminary work.

Lemma 6.27. *Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be an ind-Noetherian ind-scheme represented by an inductive system of closed immersions of (Noetherian) schemes. Denote by $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ the closed immersion morphisms. Let $f\colon \mathscr{M}^\bullet \longrightarrow \mathscr{N}^\bullet$ be a morphism in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. Then the morphism f is an isomorphism if and only if the morphism $\mathbb{R}i_\gamma^!(f)\colon \mathbb{R}i_\gamma^!\mathscr{M}^\bullet \longrightarrow \mathbb{R}i_\gamma^!\mathscr{N}^\bullet$ is an isomorphism in $\mathsf{D}^{\mathsf{co}}(X_\gamma\text{–}\mathsf{qcoh})$ for every $\gamma \in \Gamma$.*

Proof. It suffices to show that $\mathbb{R}i_\gamma^!\mathscr{L}^\bullet = 0$ for $\mathscr{L}^\bullet \in \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ and all $\gamma \in \Gamma$ implies $\mathscr{L}^\bullet = 0$ in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. Indeed, by Corollary 4.18, there exists a complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ such that $\mathscr{L}^\bullet \simeq \mathscr{J}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$. Then it remains to apply Lemma 4.21. □

Proposition 6.28. *Let $\mathfrak{X} = \text{“}\varinjlim\text{”}(X_0 \to X_1 \to X_2 \to \cdots)$ be an ind-Noetherian $\aleph_0$-ind-scheme represented by an inductive system of closed immersions of (Noetherian) schemes indexed by the poset of nonnegative integers. Let $i_n\colon X_n \longrightarrow X_{n+1}$ and $k_n\colon X_n \longrightarrow \mathfrak{X}$, $n \ge 0$, denote the closed immersion morphisms. Let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet \in \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ be two objects in the coderived category of quasi-coherent torsion sheaves. Suppose that, for every $n \ge 0$, we are given a morphism $f_n\colon \mathbb{R}k_n^!\mathscr{M}^\bullet \longrightarrow \mathbb{R}k_n^!\mathscr{N}^\bullet$ in*

$\mathsf{D^{co}}(X_n\text{–}\mathsf{qcoh})$ *such that, for every* $n \ge 0$, *the morphisms* f_n *and* $\mathbb{R}i_n^!(f_{n+1})$ *form a commutative square diagram with the natural isomorphisms* $\mathbb{R}k_n^!\mathscr{M}^\bullet \simeq \mathbb{R}i_n\mathbb{R}k_{n+1}^!\mathscr{M}^\bullet$ *and* $\mathbb{R}k_n^!\mathscr{N}^\bullet \simeq \mathbb{R}i_n\mathbb{R}k_{n+1}^!\mathscr{N}^\bullet$. *Then*

(a) *there exists a morphism* $f\colon \mathscr{M}^\bullet \longrightarrow \mathscr{N}^\bullet$ *in* $\mathsf{D^{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ *such that* $f_n = \mathbb{R}k_n^!(f)$ *for every* $n \ge 0$;

(b) *assuming additionally that* $\operatorname{Hom}_{\mathsf{D^{co}}(X_n\text{–}\mathsf{qcoh})}(\mathbb{R}k_n^!\mathscr{M}^\bullet, \mathbb{R}k_n^!\mathscr{N}^\bullet[-1]) = 0$ *for all* $n \ge 0$, *the morphism* f *with these properties is also unique.*

The proof of the proposition is based on the next lemma.

Lemma 6.29. *In the assumptions and notation of the proposition, let* $\mathscr{L}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$ *be a complex of quasi-coherent torsion sheaves and* $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors_{inj}})$ *be a complex of injective quasi-coherent torsion sheaves on* $\mathfrak{X}$. *Suppose that, for every* $n \ge 0$, *we are given a morphism* $g_n\colon k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet$ *in* $\mathsf{K}(X_n\text{–}\mathsf{qcoh})$ *such that, for every* $n \ge 0$, *the morphisms* g_n *and* $i_n^!(g_{n+1})$ *form a commutative square diagram with the natural isomorphisms* $k_n^!\mathscr{L}^\bullet \simeq i_n^!k_{n+1}^!\mathscr{L}^\bullet$ *and* $k_n^!\mathscr{J}^\bullet \simeq i_nk_{n+1}^!\mathscr{J}^\bullet$. *Then*

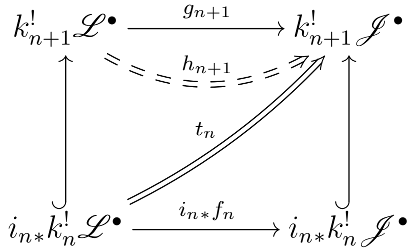
(a) *there exists a morphism* $g\colon \mathscr{L}^\bullet \longrightarrow \mathscr{J}^\bullet$ *in* $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$ *such that* $g_n = k_n^!(g)$ *in* $\mathsf{K}(X_n\text{–}\mathsf{qcoh})$ *for every* $n \ge 0$;

(b) *assuming additionally that* $\operatorname{Hom}_{\mathsf{K}(X_n\text{–}\mathsf{qcoh})}(k_n^!\mathscr{L}^\bullet, k_n^!\mathscr{J}^\bullet[-1]) = 0$ *for all* $n \ge 0$, *the morphism* g *with these properties is also unique.*

Proof. Let $\mathscr{L}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ and $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors_{inj}})$ be our two complexes. The lemma claims that, given a system of morphisms $g_n\colon k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet$ which is compatible up to chain homotopy, one can replace every morphism g_n by a homotopic morphism $f_n\colon k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet$ in $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$ such that the morphisms f_n are compatible with each other in the strict sense, i. e., the square diagrams become commutative in the abelian category of complexes $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$.

Part (a): proceeding by induction in $n \ge 0$, suppose that we have constructed chain homotopies $h_m\colon k_m^!\mathscr{L}^\bullet \longrightarrow k_m^!\mathscr{J}^\bullet[-1]$ for all $m \le n$ such that the morphisms $f_m = g_m + d(h_m)$ satisfy $f_m = i_m^!(f_{m+1})$ in $\mathsf{C}(X_m\text{–}\mathsf{qcoh})$ for all $m < n$. For $n = 0$ (the induction base), it suffices to take $h_0 = 0$. Let us construct h_{n+1}.

By assumption, we have $f_n = i_n^!g_{n+1}$ in $\mathsf{K}(X_n\text{–}\mathsf{qcoh})$; in other words, there exists a chain homotopy $u_n\colon k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet[-1]$ such that $f_n = i_n^!g_{n+1} + d(u_n)$ in $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$. Consider the map $i_{n*}u_n\colon i_{n*}k_n^!\mathscr{L}^\bullet \longrightarrow i_{n*}k_n^!\mathscr{J}^\bullet[-1]$ and compose it with the adjunction morphism $i_{n*}k_n^!\mathscr{J}^\bullet[-1] = i_{n*}i_n^!k_{n+1}^!\mathscr{J}^\bullet[-1] \longrightarrow k_{n+1}^!\mathscr{J}^\bullet[-1]$. The resulting map $t_n\colon i_{n*}k_n^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{J}^\bullet[-1]$ is a morphism of graded objects in $X_{n+1}\text{–}\mathsf{qcoh}$, where the target $k_{n+1}^!\mathscr{J}^\bullet[-1]$ is a graded object in $X_{n+1}\text{–}\mathsf{qcoh_{inj}}$.



Now the adjunction morphism $i_{n*}k_n^!\mathscr{L}^\bullet = i_{n*}i_n^!k_{n+1}^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{L}^\bullet$ is a monomorphism, so it makes $i_{n*}k_n^!\mathscr{L}^\bullet$ a graded subobject (in fact, a subcomplex) in $k_{n+1}^!\mathscr{L}^\bullet$. By injectivity of $k_{n+1}^!\mathscr{J}^\bullet$, the map t_n can be extended to a morphism of graded objects $h_{n+1}\colon k_{n+1}^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{J}^\bullet[-1]$ in X_{n+1}–qcoh, which provides the desired chain homotopy for which the morphisms of complexes $f_{n+1} = g_{n+1} + d(h_{n+1})\colon k_{n+1}^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{J}^\bullet$ satisfy $f_n = i_n^! f_{n+1}$ in $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$.

The proof part (b) is similar. Suppose that we are given a morphism $g\colon \mathscr{L}^\bullet \longrightarrow \mathscr{J}^\bullet$ in $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors})$ such that $k_n^!(g) = 0$ in the homotopy category $\mathsf{K}(X_n\text{–}\mathsf{qcoh})$ for every $n \geq 0$. Then there exist chain homotopies $h_n\colon k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet[-1]$ such that $k_n^!(g) = d(h_n)$ in the category of complexes $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$. The difference $h_n - i_n^!(h_{n+1})$ satisfies $d(h_n - i_n^!(h_{n+1})) = d(h_n) - i_n^!(d(h_{n+1})) = k_n^!(g) - i_n^!k_{n+1}^!(g) = 0$; so $h_n - i_n^!(h_{n+1})$ is a morphism $k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet[-1]$ in the category of complexes $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$. By the assumption of part (b), this morphism must be homotopic to zero. So the chain homotopies h_n "agree with each other up to a chain homotopy of the next degree".

Arguing as in part (a) and using the assumption that $k_{n+1}^!\mathscr{J}^\bullet$ are complexes of injective quasi-coherent sheaves on X_{n+1}, one proceeds by induction in n, constructing chain homotopies $r_{n+1}\colon k_{n+1}^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{J}^\bullet[-2]$ such that the chain homotopies $q_{n+1} = h_{n+1} + d(r_{n+1})$ satisfy $q_n = i_n^! q_{n+1}$ in $\mathsf{C}(X_n\text{–}\mathsf{qcoh})$ for all $n \geq 0$. The point is that the discrepancy between q_n and $i_n^! h_{n+1}$ can be lifted from a chain homotopy $k_n^!\mathscr{L}^\bullet \longrightarrow k_n^!\mathscr{J}^\bullet[-2]$ to a chain homotopy $k_{n+1}^!\mathscr{L}^\bullet \longrightarrow k_{n+1}^!\mathscr{J}^\bullet[-2]$. ☐

Proof of Proposition 6.28. Use Corollary 4.18 in order to find two complexes of injective quasi-coherent torsion sheaves $\mathscr{L}^\bullet$ and $\mathscr{J}^\bullet$ on $\mathfrak{X}$ together with a morphism $\mathscr{M}^\bullet \longrightarrow \mathscr{L}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$ with coacyclic cones. Furthermore, notice that, for any scheme X and any complexes $\mathcal{L}^\bullet \in \mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ and $\mathcal{J}^\bullet \in \mathsf{K}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$, the natural morphism of Hom groups $\operatorname{Hom}_{\mathsf{K}(X\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{J}^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{J}^\bullet)$ is an isomorphism. The latter assertion holds by Proposition 4.15(a) (a slightly more precise version of Proposition 4.15(a), which is also valid in any abelian/exact category with exact coproducts, tells that the same holds for any $\mathcal{L}^\bullet \in \mathsf{K}(X\text{–}\mathsf{qcoh})$; but we still need $\mathscr{L}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ in order to have $\mathbb{R}k_n^!\mathscr{L}^\bullet = k_n^!\mathscr{L}^\bullet$). These observations reduce the assertions of the proposition to those of Lemma 6.29. ☐

Examples 6.30. (0) We refer to Remark 5.3(3) for the discussion of the extraordinary inverse image functors and the notation $f^+\colon \mathsf{D}^+(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{qcoh})$ for a morphism of finite type between Noetherian schemes $f\colon Y \longrightarrow X$.

The functor f^+ commutes with external tensor products. In particular, given a morphism of finite type $p\colon X \longrightarrow \operatorname{Spec}\Bbbk$, consider the morphism $p\times_\Bbbk p\colon X\times_\Bbbk X \longrightarrow \operatorname{Spec}\Bbbk$, and denote simply by $\Bbbk$ the quasi-coherent sheaf on $\operatorname{Spec}\Bbbk$ corresponding to the $\Bbbk$-vector space $\Bbbk$. Then there is a natural isomorphism $(p\times_\Bbbk p)^+(\Bbbk) \simeq p^+(\Bbbk)\boxtimes_\Bbbk p^+(\Bbbk)$ in $\mathsf{D}^+((X\times_\Bbbk X)\text{–}\mathsf{qcoh})$. In fact, $p^+(\Bbbk)$ is a complex of quasi-coherent sheaves with bounded cohomology sheaves, which are coherent sheaves on X, and the complex $p^+(\Bbbk)$ also has finite injective dimension.

Assume that the scheme X is separated, and choose a bounded complex $\mathcal{D}^\bullet \in \mathsf{K}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ of injective quasi-coherent sheaves on X quasi-isomorphic to $p^+(\Bbbk)$. Then $\mathcal{D}^\bullet$ is a dualizing complex on X. Moreover, $\mathcal{D}_n^\bullet$ is a *rigid* dualizing complex, i. e., denoting the diagonal morphism by $\Delta_X\colon X \longrightarrow X \times_\Bbbk X$, there is a natural isomorphism $\mathcal{D}^\bullet \simeq \mathbb{R}\Delta_X^!(\mathcal{D}^\bullet \boxtimes_\Bbbk \mathcal{D}^\bullet)$ in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}^+(X\text{–}\mathsf{qcoh}) \subset \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$.

(1) Let $\mathfrak{X} = \text{“}\varinjlim\text{”}(X_0 \to X_1 \to X_2 \to \cdots)$ be an ind-separated $\aleph_0$-ind-scheme of ind-finite type over $\Bbbk$, represented by an inductive system of closed immersions of schemes indexed by the nonnegative integers. For every $n \ge 0$, denote by $p_n\colon X_n \longrightarrow \operatorname{Spec}\Bbbk$ the structure morphism of the scheme X_n over $\Bbbk$, and let $\mathcal{D}_n^\bullet \in \mathsf{K}^{\mathsf{b}}(X_n\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ denote a bounded complex of injective quasi-coherent sheaves on X_n quasi-isomorphic to $p_n^+(\Bbbk)$. Then $\mathcal{D}_n^\bullet$ is a rigid dualizing complex on X_n.

In the above notation $i_n\colon X_n \longrightarrow X_{n+1}$ for the transition morphisms in the inductive system of schemes, there are also natural isomorphisms $p_n^+(\Bbbk) \simeq \mathbb{R}i_n^! p_{n+1}^+(\Bbbk)$ in $\mathsf{D}^+(X_n\text{–}\mathsf{qcoh})$, which lead to natural homotopy equivalences $\mathcal{D}_n^\bullet \simeq i_n^!\mathcal{D}_{n+1}^\bullet$ in $\mathsf{K}^{\mathsf{b}}(X_n\text{–}\mathsf{qcoh}_{\mathsf{inj}})$. Using the construction of Example 4.7, we produce a dualizing complex $\mathscr{D}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ together with natural homotopy equivalences $\mathcal{D}_n^\bullet \simeq k_n^!\mathscr{D}^\bullet$ (where $k_n\colon X_n \longrightarrow \mathfrak{X}$ are the closed immersions).

(2) For every $n \ge 0$, choose a bounded complex of injective quasi-coherent sheaves $\mathcal{E}_n^\bullet$ on $X_n \times_\Bbbk X_n$ quasi-isomorphic to $(p_n \times_\Bbbk p_n)^+(\Bbbk)$. So $\mathcal{E}_n^\bullet$ is a dualizing complex on $X_n \times_\Bbbk X_n$ naturally isomorphic to the complex $\mathcal{D}_n^\bullet \boxtimes_\Bbbk \mathcal{D}_n^\bullet$ in the derived category $\mathsf{D}^{\mathsf{b}}((X_n \times_\Bbbk X_n)\text{–}\mathsf{qcoh})$. We also have natural homotopy equivalences $\Delta_n^!(\mathcal{E}_n^\bullet) \simeq \mathcal{D}_n^\bullet$ in $\mathsf{K}^{\mathsf{b}}(X_n\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ (where $\Delta_n\colon X_n \longrightarrow X_n \times_\Bbbk X_n$ is the diagonal morphism) and $\mathcal{E}_n^\bullet \simeq (i_n \times_\Bbbk i_n)^!\mathcal{E}_{n+1}^\bullet$ in $\mathsf{K}^{\mathsf{b}}((X_n \times_\Bbbk X_n)\text{–}\mathsf{qcoh}_{\mathsf{inj}})$.

Using the construction of Example 4.7, we can produce a dualizing complex $\mathscr{E}^\bullet \in \mathsf{K}((\mathfrak{X} \times_\Bbbk \mathfrak{X})\text{–}\mathsf{tors}_{\mathsf{inj}})$ together with natural homotopy equivalences $\mathcal{E}_n^\bullet \simeq (k_n \times_\Bbbk k_n)^!\mathscr{E}^\bullet$. Now Lemma 6.29(a) allows to extend these data to a morphism $\mathscr{D}^\bullet \longrightarrow \Delta^!\mathscr{E}^\bullet$ in $\mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ (where $\Delta\colon \mathfrak{X} \longrightarrow \mathfrak{X} \times_\Bbbk \mathfrak{X}$ is the diagonal). Using the same lemma, we also obtain a morphism $\mathscr{D}^\bullet \boxtimes_\Bbbk \mathscr{D}^\bullet \longrightarrow \mathscr{E}^\bullet$ in $\mathsf{K}((\mathfrak{X} \times_\Bbbk \mathfrak{X})\text{–}\mathsf{tors})$. Moreover, Lemma 6.29(b) combined with Lemma 4.12 show that the relevant morphisms in the homotopy categories of quasi-coherent torsion sheaves are uniquely defined.

Alternatively, one can use Proposition 6.28(a) together with Proposition 6.19. This allows to avoid mentioning the specific complexes $\mathcal{E}_n^\bullet$ and $\mathscr{E}^\bullet$, constructing directly a morphism $\mathscr{D}^\bullet \longrightarrow \mathbb{R}\Delta^!(\mathscr{D}^\bullet \boxtimes_\Bbbk \mathscr{D}^\bullet)$ in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ instead. This morphism in the coderived category is uniquely defined by Proposition 6.28(b) combined with Lemma 4.12.

By Lemma 6.27, both the morphisms $\mathscr{D}^\bullet \longrightarrow \Delta^!\mathscr{E}^\bullet$ and $\mathscr{D}^\bullet \boxtimes_\Bbbk \mathscr{D}^\bullet \longrightarrow \mathscr{E}^\bullet$ are isomorphisms in the respective coderived categories $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ and $\mathsf{D}^{\mathsf{co}}((\mathfrak{X} \times_\Bbbk \mathfrak{X})\text{–}\mathsf{tors})$. Thus $\mathscr{D}^\bullet \in \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is a rigid dualizing complex on $\mathfrak{X}$.

6.6. **Covariant duality commutes with external tensor products.** Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. For any complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ on

$\mathfrak{X}$, put

$$\Phi_{\mathscr{M}^\bullet}(\mathfrak{F}^\bullet) = \mathscr{M}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–tors}).$$

According to formula (13) from Section 5.1, the functor $\Phi_{\mathscr{M}^\bullet}$ induces a well-defined triangulated functor from the derived category of flat pro-quasi-coherent pro-sheaves to the coderived category of quasi-coherent torsion sheaves,

$$\Phi_{\mathscr{M}^\bullet}\colon \mathsf{D}(\mathfrak{X}\text{–flat}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–tors}).$$

Furthermore, any morphism $\mathscr{M}^\bullet \longrightarrow \mathscr{N}^\bullet$ in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–tors})$ induces a morphism of functors $\Phi_{\mathscr{M}^\bullet} \longrightarrow \Phi_{\mathscr{N}^\bullet}$, and any isomorphism $\mathscr{M}^\bullet \simeq \mathscr{N}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–tors})$ induces an isomorphism of triangulated functors $\Phi_{\mathscr{M}^\bullet} \simeq \Phi_{\mathscr{N}^\bullet}$.

Lemma 6.31. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be reasonable ind-schemes over $\Bbbk$. Let $\mathscr{M}'$ and $\mathscr{M}''$ be quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $\mathfrak{X}''$, and let $\mathfrak{P}'$ and $\mathfrak{P}''$ be pro-quasi-coherent pro-sheaves on $\mathfrak{X}'$ and $\mathfrak{X}''$, respectively. Then there is a natural isomorphism*

$$(\mathscr{M}' \boxtimes_{\Bbbk} \mathscr{M}'') \otimes_{\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''} (\mathfrak{P}' \boxtimes_{\Bbbk} \mathfrak{P}'') \simeq (\mathscr{M}' \otimes_{\mathfrak{X}'} \mathfrak{P}') \boxtimes_{\Bbbk} (\mathscr{M}'' \otimes_{\mathfrak{X}''} \mathfrak{P}'')$$

of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$.

Proof. Let $\mathfrak{X}' = \text{“}\varinjlim\text{”}_{\gamma'\in\Gamma'} X'_{\gamma'}$ and $\mathfrak{X}'' = \text{“}\varinjlim\text{”}_{\gamma''\in\Gamma''} X''_{\gamma''}$ be representations of $\mathfrak{X}'$ and $\mathfrak{X}''$ by inductive systems of closed immersions of reasonable closed subschemes. Then $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}'' = \text{“}\varinjlim\text{”}_{(\gamma',\gamma'')\in\Gamma'\times\Gamma''} X'_{\gamma'} \times_{\Bbbk} X''_{\gamma''}$ is a similar representation of the ind-scheme $\mathfrak{X} \times_{\Bbbk} \mathfrak{X}''$ (see Section 6.3). To prove the lemma, one first observes that a similar isomorphism obviously holds for Γ-systems in place of quasi-coherent torsion sheaves. Let $\mathbb{M}'$ be a Γ'-system on $\mathfrak{X}'$ and $\mathbb{M}''$ be a Γ''-system on $\mathfrak{X}''$. Then the external tensor product $\mathbb{M}' \boxtimes_{\Bbbk} \mathbb{M}''$, as defined in Section 6.3, is a $(\Gamma' \times \Gamma'')$-system on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$. One has a natural isomorphism of $(\Gamma' \times \Gamma'')$-systems

$$(\mathfrak{P}' \boxtimes_{\Bbbk} \mathfrak{P}'') \otimes_{\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}''} (\mathbb{M}' \boxtimes_{\Bbbk} \mathbb{M}'') \simeq (\mathfrak{P}' \otimes_{\mathfrak{X}'} \mathbb{M}') \boxtimes_{\Bbbk} (\mathfrak{P}'' \otimes_{\mathfrak{X}''} \mathbb{M}'').$$

In order to deduce the desired isomorphism of quasi-coherent torsion sheaves, it remains to recall the definition of the functor $\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–pro} \times \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{X}\text{–tors}$ in Section 3.2 and use Lemma 6.14(b). ☐

Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes of ind-finite type over $\Bbbk$, and let $\mathscr{M}'^\bullet$ and $\mathscr{M}''^\bullet$ be complexes of quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $\mathfrak{X}''$. Then it follows from Lemma 6.31 that, for any complexes flat pro-quasi-coherent pro-sheaves $\mathfrak{P}'^\bullet$ and $\mathfrak{P}''^\bullet$ on $\mathfrak{X}'$ and $\mathfrak{X}''$, the natural isomorphism

$$\Phi_{\mathscr{M}'^\bullet\boxtimes_{\Bbbk}\mathscr{M}''^\bullet}(\mathfrak{P}'^\bullet \boxtimes_{\Bbbk} \mathfrak{P}''^\bullet) \simeq \Phi_{\mathscr{M}'^\bullet}(\mathfrak{P}'^\bullet) \boxtimes_{\Bbbk} \Phi_{\mathscr{M}''^\bullet}(\mathfrak{P}''^\bullet)$$

holds in the category of complexes of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$.

Corollary 6.32. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-semi-separated ind-schemes of ind-finite type over $\Bbbk$. Let $\mathscr{D}'^\bullet$ and $\mathscr{D}''^\bullet$ be dualizing complexes on $\mathfrak{X}'$ and $\mathfrak{X}''$, respectively, and let $\mathscr{E}^\bullet$ be the related dualizing complex on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$, as in Lemma 6.26. Then the triangulated equivalences $\mathsf{D}(\mathfrak{X}'\text{–flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}''\text{–tors})$, $\mathsf{D}(\mathfrak{X}''\text{–flat}) \simeq \mathsf{D}(\mathfrak{X}''\text{–tors})$, and $\mathsf{D}((\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}'')\text{–flat}) \simeq \mathsf{D}((\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}'')\text{–tors})$ from Theorem 4.23, induced by the dualizing complexes $\mathscr{D}'^\bullet$, $\mathscr{D}''^\bullet$, and $\mathscr{E}^\bullet$, form a commutative square diagram with the external tensor product functors $\boxtimes_{\Bbbk}$* (21) *and* (24) *from Sections 6.2–6.3.*

Proof. Follows immediately from the preceding discussion. □

6.7. **The cotensor product as the !-tensor product.** The definition of a reasonable closed immersion of ind-schemes, which is used in the second assertion of the following lemma, was given in Section 6.3.

Lemma 6.33. *Let $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of reasonable ind-schemes. Let $\mathfrak{F}$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ and $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$. Then there is a natural morphism of quasi-coherent torsion sheaves on $\mathfrak{Z}$*

$$i^*\mathfrak{F} \otimes_{\mathfrak{Z}} i^!\mathscr{M} \longrightarrow i^!(\mathfrak{F} \otimes_{\mathfrak{X}} \mathscr{M}),$$

which is an isomorphism whenever i is a reasonable closed immersion and $\mathfrak{F}$ is a flat pro-quasi-coherent pro-sheaf on $\mathfrak{X}$.

Proof. This is a generalization of Lemma 3.6 and Proposition 3.7. To construct the desired morphism, let $\mathscr{L}$ be an arbitrary quasi-coherent torsion sheaf on $\mathfrak{Z}$, and let $\mathscr{L} \longrightarrow i^*\mathfrak{F} \otimes_{\mathfrak{Z}} i^!\mathscr{M}$ be a morphism in $\mathfrak{Z}$–tors. Applying the direct image functor i_*, we produce a morphism $i_*\mathscr{L} \longrightarrow i_*(i^*\mathfrak{F} \otimes_{\mathfrak{Z}} i^!\mathscr{M}) \simeq \mathfrak{F} \otimes_{\mathfrak{X}} i_*i^!\mathscr{M}$ in $\mathfrak{X}$–tors (where the isomorphism holds by Lemma 8.2 below). Composing with the morphism $\mathfrak{F} \otimes_{\mathfrak{X}} i_*i^!\mathscr{M} \longrightarrow \mathfrak{F} \otimes_{\mathfrak{X}} \mathscr{M}$ induced by the adjunction morphism $i_*i^!\mathscr{M} \longrightarrow \mathscr{M}$, we obtain a morphism $i_*\mathscr{L} \longrightarrow \mathfrak{F} \otimes_X \mathscr{M}$ in $\mathfrak{X}$–tors, which corresponds by adjunction to a morphism $\mathscr{L} \longrightarrow i^!(\mathfrak{F} \otimes_{\mathfrak{X}} \mathscr{M})$ in $\mathfrak{Z}$–tors.

To prove isomorphism assertion, we let $Z \subset \mathfrak{Z}$ be a reasonable closed subscheme with the closed immersion morphism $k\colon Z \longrightarrow \mathfrak{Z}$, and apply the functor $k^!$ to the morphism in question. Notice that $ik(Z)$ is a reasonable closed subscheme in $\mathfrak{X}$. By Proposition 3.7 applied to the closed subscheme $Z \subset \mathfrak{Z}$, we have

$$k^!(i^*\mathfrak{F} \otimes_{\mathfrak{Z}} i^!\mathscr{M}) \simeq k^*i^*\mathfrak{F} \otimes_{\mathcal{O}_Z} k^!i^!\mathscr{M}.$$

Using the same proposition applied to the closed subscheme $ik(Z) \subset \mathfrak{X}$, we compute

$$k^!i^!(\mathfrak{F} \otimes_{\mathfrak{X}} \mathscr{M}) \simeq (ik)^!(\mathfrak{F} \otimes_{\mathfrak{X}} \mathscr{M}) \simeq (ik)^*\mathfrak{F} \otimes_{\mathcal{O}_Z} (ik)^!\mathscr{M} \simeq k^*i^*\mathfrak{F} \otimes_{\mathcal{O}_Z} k^!i^!\mathscr{M},$$

so the functor $k^!$ transforms the morphism in question into an isomorphism. As this holds for every reasonable closed subscheme $Z \subset \mathfrak{Z}$, the assertion follows. □

Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes. Then the inverse image functor $f^*\colon \mathfrak{X}$–$\mathsf{flat} \longrightarrow \mathfrak{Y}$–$\mathsf{flat}$ is exact (see Section 3.4), so it induces a triangulated functor between the derived categories

$$f^*\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{flat}).$$

Proposition 6.34. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian scheme and $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of ind-schemes. Let $\mathscr{D}^\bullet$ be a dualizing complex on $\mathfrak{X}$; then $i^!\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{Z}$ (cf. Example 4.8(2)). Then the triangulated equivalences $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ and $\mathsf{D}(\mathfrak{Z}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{Z}\text{–}\mathsf{tors})$ from Theorem 4.23, induced by the dualizing complexes $\mathscr{D}^\bullet$ and $i^!\mathscr{D}^\bullet$, transform the inverse image functor*

$$i^*\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Z}\text{–}\mathsf{flat})$$

into the right derived functor

$$\mathbb{R}i^!\colon \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{Z}\text{–}\mathsf{tors})$$

from Section 6.4.

Proof. Follows from Lemma 6.33 together with the fact that $\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$ for every complex $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ (which was explained in the proof of Theorem 4.23). □

The following theorem is the main result of Section 6.

Theorem 6.35. *Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over $\mathbb{k}$, and let $\mathscr{D}^\bullet$ be a rigid dualizing complex on $\mathfrak{X}$ (in the sense of Section 6.5). Let $\Delta\colon \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathbb{k}} \mathfrak{X}$ be the diagonal morphism. Then for any two complexes of quasi-coherent torsion sheaves $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ on $\mathfrak{X}$ there is a natural isomorphism*

$$\mathscr{M}^\bullet \,\square_{\mathscr{D}^\bullet}\, \mathscr{N}^\bullet \simeq \mathbb{R}\Delta^!(\mathscr{M}^\bullet \boxtimes_{\mathbb{k}} \mathscr{N}^\bullet) \tag{25}$$

in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Proof. Notice the natural isomorphism of complexes of (flat) pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \otimes^{\mathfrak{X}} \mathfrak{G}^\bullet \simeq \Delta^*(\mathfrak{F}^\bullet \boxtimes_{\mathbb{k}} \mathfrak{G}^\bullet)$ for all complexes of (flat) pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ on $\mathfrak{X}$ (see Lemma 6.12). By Corollary 6.32 (applied to the ind-schemes $\mathfrak{X}' = \mathfrak{X} = \mathfrak{X}''$) and Proposition 6.34 (applied to the closed immersion $\Delta\colon \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathbb{k}} \mathfrak{X}$), it follows that the triangulated equivalence $\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ induced by $\mathscr{D}^\bullet$ transforms the tensor product functor (12) from Section 5.1 into the !-tensor product in the right-hand side of (25). But this coincides with the definition of the left-hand side of (25) given in Section 5.1 (see (14)). □

As a byproduct of Theorem 6.35, we see that a rigid dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ is unique up to a natural isomorphism in the assumptions of the theorem (because it can be recovered as the unit object of the tensor structure on $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ given by the right-hand side of (25)).

7. $\mathfrak{X}$-Flat Pro-Quasi-Coherent Pro-Sheaves on $\mathfrak{Y}$

In this section we consider a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Eventually we will assume that $\mathfrak{X}$ is ind-semi-separated, ind-Noetherian, and endowed with a dualizing complex $\mathscr{D}^\bullet$.

7.1. **Semiderived category of torsion sheaves.** Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes which is "representable by schemes" in the sense of Section 1.3. Assume that the ind-scheme $\mathfrak{X}$ is reasonable; following the discussion in Section 2.1, the ind-scheme $\mathfrak{Y}$ is then reasonable as well.

Moreover, let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of reasonable closed subschemes. Put $Y_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$; then Y_γ are reasonable closed subschemes in $\mathfrak{Y}$, and $\mathfrak{Y} = \varinjlim_{\gamma\in\Gamma} Y_\gamma$.

By Theorem 2.4, we have Grothendieck abelian categories of quasi-coherent torsion sheaves $\mathfrak{Y}$–tors and $\mathfrak{X}$–tors. Furthermore, there is the direct image functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ constructed in Section 2.6. According to Lemma 2.10(b), the functor f_* has a left adjoint functor $f^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$.

Furthermore, following the discussion in Section 2.8, there is also a pair of adjoint functors of direct and inverse images of Γ-systems on $\mathfrak{X}$ and $\mathfrak{Y}$, with the inverse image functor $f^*\colon (\mathfrak{X},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst}$ left adjoint to the direct image functor $f_*\colon (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–}\mathsf{syst}$.

Lemma 7.1. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of reasonable ind-schemes which is "representable by schemes". Then*

(a) *the functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ preserves direct limits (and in particular, coproducts);*

(b) *the functors of direct image of Γ-systems $f_*\colon (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–}\mathsf{syst}$ and of quasi-coherent torsion sheaves $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ form a commutative square diagram with the functors $\mathbb{N} \longmapsto \mathbb{N}^+\colon (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ and $\mathbb{M} \longmapsto \mathbb{M}^+\colon (\mathfrak{X},\Gamma)\text{–}\mathsf{syst} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$.*

Proof. Part (a) follows from the description of direct limits of quasi-coherent torsion sheaves in Section 2.5 and the description of direct images in Section 2.6, together with the fact that the direct image functors $f_{\gamma *}\colon Y_\gamma\text{–}\mathsf{qcoh} \longrightarrow X_\gamma\text{–}\mathsf{qcoh}$ for the morphisms of (concentrated) schemes $f_\gamma\colon Y_\gamma \longrightarrow X_\gamma$ preserve direct limits. To prove part (b), one observes that the functors $(-)^+$ are constructed in terms of the functors of direct image (with respect to the closed immersions $Y_\gamma \longrightarrow \mathfrak{Y}$ and $X_\gamma \longrightarrow \mathfrak{X}$) and direct limit in $\mathfrak{Y}$–tors and $\mathfrak{X}$–tors (see Section 2.7). Direct images obviously commute with direct images, and they preserve direct limits by part (a). □

Lemma 7.2. *For any affine morphism of reasonable ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the direct image functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is exact and faithful.*

Proof. The faithfulness assertion follows immediately from the fact that the direct image functors of quasi-coherent sheaves $f_{\gamma *}\colon Y_\gamma\text{–}\mathsf{qcoh} \longrightarrow X_\gamma\text{–}\mathsf{qcoh}$ are faithful for affine morphisms of schemes $f_\gamma\colon Y_\gamma \longrightarrow X_\gamma$. To check exactness, notice that exactness of the functor of direct image of Γ-systems $f_*\colon (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{X},\Gamma)\text{–}\mathsf{syst}$ follows immediately from exactness of the functors $f_{\gamma *}$ for affine morphisms of schemes f_γ. To deduce exactness of the functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$, it remains to recall that the functors $(-)^+$ represent the abelian categories of quasi-coherent torsion sheaves as quotient categories of the abelian categories of Γ-systems by some Serre subcategories (see the proof of Proposition 2.8) and use Lemma 7.1(b). □

Lemma 7.3. *For any flat morphism of reasonable ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the inverse image functor $f^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ is exact.*

Proof. The functor of inverse image of Γ-systems $f^*\colon (\mathfrak{X},\Gamma)\text{–}\mathsf{syst} \longrightarrow (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst}$ is exact, because the functors of inverse image of quasi-coherent sheaves $f_\gamma^*\colon X_\gamma\text{–}\mathsf{qcoh} \longrightarrow Y_\gamma\text{–}\mathsf{qcoh}$ are exact for flat morphisms of schemes f_γ. Exactness of the functor $f^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ now follows from the construction of this functor in Section 2.8, similarly to the proof of Lemma 7.2. □

Remark 7.4. The argument above is sufficient to prove Lemma 7.3, but in fact one can say more. The functor $f^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ was constructed in a relatively complicated way, using Γ-systems and the functors $(-)^+$, in Section 2.8 in order to include not necessarily flat morphisms f.

For a flat morphism of reasonable ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ and a quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$, one can simply put $\mathscr{N}_{(Y_\gamma)} = f_\gamma^*\mathscr{M}_{(X_\gamma)} \in Y_\gamma\text{–}\mathsf{qcoh}$ for all $\gamma \in \Gamma$. Then it follows from Lemma 4.24 that there are natural isomorphisms of quasi-coherent sheaves $k_{\gamma\delta}^!\mathscr{N}_{(Y_\delta)} = k_{\gamma\delta}^! f_\delta^*\mathscr{M}_{(X_\delta)} \simeq f_\gamma^* i_{\gamma\delta}^!\mathscr{M}_{(X_\delta)} \simeq f_\gamma^*\mathscr{M}_{(X_\gamma)} = \mathscr{N}_{(Y_\gamma)}$ for all $\gamma < \delta \in \Gamma$, where $i_{\gamma\delta}\colon X_\gamma \longrightarrow X_\delta$ and $k_{\gamma\delta}\colon Y_\gamma \longrightarrow Y_\delta$ are the closed immersions. So the collection of quasi-coherent sheaves $(\mathscr{N}_{(Y_\gamma)})_{\gamma\in\Gamma}$ defines a quasi-coherent torsion sheaf $\mathscr{N}$ on $\mathfrak{Y}$. It is easy to see that $\mathscr{N} = f^*\mathscr{M}$.

Lemma 7.5. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Let $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$ and $\boldsymbol{\mathfrak{Q}}$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$. Then there is a natural isomorphism*

$$\mathscr{M} \otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}} \simeq f_*(f^*\mathscr{M} \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}),$$

of quasi-coherent torsion sheaves on $\mathfrak{X}$.

Proof. Similarly to Lemma 2.2, the natural morphism

$$\mathscr{M} \otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}} \longrightarrow f_*(f^*\mathscr{M} \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}) \qquad (26)$$

is adjoint to the composition $f^*(\mathscr{M} \otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}}) \simeq f^*\mathscr{M} \otimes_{\mathfrak{Y}} f^*f_*\boldsymbol{\mathfrak{Q}} \longrightarrow f^*\mathscr{M} \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}$ of the isomorphism $f^*(\mathscr{M} \otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}}) \simeq f^*\mathscr{M} \otimes_{\mathfrak{Y}} f^*f_*\boldsymbol{\mathfrak{Q}}$ provided by Lemma 3.4 and the morphism $f^*\mathscr{M} \otimes_{\mathfrak{Y}} f^*f_*\boldsymbol{\mathfrak{Q}} \longrightarrow f^*\mathscr{M} \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}$ induced by the adjunction morphism $f^*f_*\boldsymbol{\mathfrak{Q}} \longrightarrow \boldsymbol{\mathfrak{Q}}$. Similarly one constructs, for any Γ-system $\mathbb{M}$ on $\mathfrak{X}$, a natural morphism of Γ-systems $\mathbb{M}\otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}} \longrightarrow f_*(f^*\mathbb{M}\otimes_{\mathfrak{Y}}\boldsymbol{\mathfrak{Q}})$, which is an isomorphism essentially by Lemma 2.2.

To show that (26) is an isomorphism, one computes

$$\begin{aligned}\mathscr{M} \otimes_{\mathfrak{X}} f_*\boldsymbol{\mathfrak{Q}} = (\mathscr{M}|_\Gamma \otimes_{\mathfrak{X}} f^*\boldsymbol{\mathfrak{Q}})^+ &\simeq (f_*(f^*(\mathscr{M}|_\Gamma) \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}))^+ \\ &\simeq f_*((f^*(\mathscr{M}|_\Gamma) \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}})^+) \simeq f_*((f^*(\mathscr{M}|_\Gamma))^+ \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}) \\ &\simeq f_*(f^*((\mathscr{M}|_\Gamma)^+) \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}}) \simeq f_*(f^*\mathscr{M} \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{Q}})\end{aligned}$$

using the definitions of the functors $\otimes_{\mathfrak{X}}\colon \mathfrak{X}\text{–}\mathsf{pro}\times\mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ and $\otimes_{\mathfrak{Y}}\colon \mathfrak{Y}\text{–}\mathsf{pro}\times \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$, and also Lemmas 2.11 and 7.1(b). The point is that both the direct and inverse image functors, as well as the tensor products in question, commute with the functors $(-)^+$. $\square$

Let $\pi\colon \boldsymbol{\mathfrak{Y}} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. The $\boldsymbol{\mathfrak{Y}}/\mathfrak{X}$-*semiderived category* (or more precisely, *semicoderived category*) $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\boldsymbol{\mathfrak{Y}}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\boldsymbol{\mathfrak{Y}}$ is defined as the triangulated quotient category of the homotopy category $\mathsf{K}(\boldsymbol{\mathfrak{Y}}\text{–}\mathsf{tors})$ by the thick subcategory of all complexes of quasi-coherent torsion sheaves $\boldsymbol{\mathscr{M}}^\bullet$ on $\boldsymbol{\mathfrak{Y}}$ such that *the complex of quasi-coherent torsion sheaves $\pi_*\boldsymbol{\mathscr{M}}^\bullet$ on $\mathfrak{X}$ is coacyclic* (in the sense of Section 4.4).

Notice that the functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ takes coacyclic complexes to coacyclic complexes (by Lemmas 7.1(a) and 7.2), and a complex $\mathscr{M}^\bullet$ in $\mathfrak{Y}\text{–}\mathsf{tors}$ is acyclic if and only if the complex $\pi_*\mathscr{M}^\bullet$ is acyclic in $\mathfrak{X}\text{–}\mathsf{tors}$ (also by Lemma 7.2). Hence $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is an intermediate Verdier quotient category between the derived category $\mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors})$ and the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{Y}\text{–}\mathsf{tors})$, i. e., there are natural Verdier quotient functors

$$\mathsf{D}^{\mathsf{co}}(\mathfrak{Y}\text{–}\mathsf{tors}) \twoheadrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \twoheadrightarrow \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

Let us say that a quasi-coherent torsion sheaf $\mathscr{K}$ on $\mathfrak{Y}$ is $\mathfrak{X}$*-injective* if the quasi-coherent torsion sheaf $\pi_*\mathscr{K}$ on $\mathfrak{X}$ is injective. We will denote the full subcategory of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$ by $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}} \subset \mathfrak{Y}\text{–}\mathsf{tors}$.

In particular, given an affine morphism of schemes $f\colon \mathbf{Y} \longrightarrow X$, a quasi-coherent sheaf $\boldsymbol{\mathcal{K}}$ on $\mathbf{Y}$ is said to be X*-injective* if the quasi-coherent sheaf $f_*\boldsymbol{\mathcal{K}}$ on X is injective. The full subcategory of X-injective quasi-coherent sheaves is denoted by $\mathbf{Y}\text{–}\mathsf{qcoh}_{X\text{-}\mathsf{inj}} \subset \mathbf{Y}\text{–}\mathsf{qcoh}$.

Lemma 7.6. (a) *For any affine morphism of reasonable ind-schemes* $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$*, the full subcategory* $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}$ *is closed under extensions and cokernels of monomorphisms in* $\mathfrak{Y}\text{–}\mathsf{tors}$*.*

(b) *If* $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ *is a flat affine morphism of reasonable ind-schemes, then any injective quasi-coherent torsion sheaf on* $\mathfrak{Y}$ *is* $\mathfrak{X}$*-injective, that is* $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathsf{inj}} \subset \mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}$*.*

Proof. Part (a) holds because the functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is exact and the full subcategory $\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}} \subset \mathfrak{X}\text{–}\mathsf{tors}$ is closed under extensions and cokernels of monos. Part (b) claims that the functor π_* preserves injectives; this is so because π_* is right adjoint to the functor π^*, which is exact by Lemma 7.3. ☐

The assertions of Lemma 7.6 can be expressed by saying that $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}$ is a *coresolving subcategory* in $\mathfrak{Y}\text{–}\mathsf{tors}$. In particular, being a full subcategory closed under extensions, $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}$ inherits an exact category structure from the abelian category $\mathfrak{Y}\text{–}\mathsf{tors}$. So one can form the derived category $\mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}})$ (cf. [47, paragraphs preceding Theorem 5.2] for a discussion).

Lemma 7.7. *Let* $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ *be an affine morphism of reasonable ind-schemes. Then a complex* $\mathscr{K}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}})$ *is acyclic in* $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}$ *if and only if the complex* $\pi_*\mathscr{K}^\bullet$ *in* $\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$ *is contractible, and if and only if the complex* $\pi_*\mathscr{K}^\bullet$ *is coacyclic in* $\mathfrak{X}\text{–}\mathsf{tors}$*.*

Proof. The first assertion holds because the functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is exact and faithful, and a complex of injective objects in $\mathfrak{X}\text{–}\mathsf{tors}$ is contractible if and only if its cocycle objects are injective. Furthermore, a complex of injectives is contractible if and only if it is coacyclic (by Proposition 4.15(a)). ☐

The next proposition is a generalization of Corollary 4.18. It should be also compared to [47, Theorems 5.1(a) and 5.2(a)].

Proposition 7.8. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme and $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Then the inclusion of exact categories $\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}} \longrightarrow \mathfrak{Y}\text{–tors}$ induces an equivalence of triangulated categories $\mathsf{D}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$.*

Proof. It follows from Lemma 7.7 that the triangulated functor $\mathsf{D}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$ induced by the inclusion $\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}} \longrightarrow \mathfrak{Y}\text{–tors}$ is well-defined. Moreover, by [41, Lemma 1.6(b)], in order to show that this triangulated functor is an equivalence it suffices to check that for any complex $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–tors})$ there exists a complex $\mathscr{K}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}})$ together with a morphism of complexes $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ whose cone becomes coacyclic after applying π_*.

In fact, one can even make the cone of $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ coacyclic in $\mathfrak{Y}\text{–tors}$. One only needs to observe that, since the direct image functor π_* preserves coproducts and the class of injective objects in $\mathfrak{X}\text{–tors}$ is closed under coproducts (as $\mathfrak{X}$ is ind-Noetherian; see Lemma 4.16 and Proposition 4.17), the full subcategory $\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}} \subset \mathfrak{X}\text{–tors}$ is closed under coproducts.

Besides, there are enough injective objects in a Grothendieck category $\mathfrak{Y}\text{–tors}$ and all of them belong to $\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}$; so any complex $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–tors})$ admits a termwise monic morphism into a complex $\mathscr{J}^{0,\bullet} \in \mathsf{C}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}})$. The cokernel $\mathscr{J}^{0,\bullet}/\mathscr{M}^\bullet$, in turn, can be embedded into a complex $\mathscr{J}^{1,\bullet} \in \mathsf{C}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}})$, etc. Totalizing the bicomplex $\mathscr{J}^{\bullet,\bullet}$ by taking countable coproducts along the diagonals, one produces the desired complex $\mathscr{K}^\bullet$ together with a morphism $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$, whose cone is coacyclic by [40, Lemma 2.1]. We refer to [41, proof of Theorem 3.7] or [44, proof of Proposition A.3.1(b)] for further details of this argument. □

7.2. **Pro-sheaves flat over the base.** Let $f\colon \mathbf{Y} \longrightarrow X$ be an affine morphism of schemes. A quasi-coherent sheaf $\mathcal{G}$ on $\mathbf{Y}$ is said to be X*-flat* (or *flat over* X) if the quasi-coherent sheaf $f_*\mathcal{G}$ on X is flat. We will denote the full subcategory of X-flat quasi-coherent sheaves on $\mathbf{Y}$ by $\mathbf{Y}_X\text{–flat} \subset \mathbf{Y}\text{–qcoh}$.

Lemma 7.9. *Let $f\colon \mathbf{Y} \longrightarrow X$ be an affine morphism of schemes. Then, for any flat quasi-coherent sheaf $\mathcal{F}$ on $\mathbf{Y}$ and any X-flat quasi-coherent sheaf $\mathcal{G}$ on $\mathbf{Y}$, the quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_\mathbf{Y}} \mathcal{G}$ on $\mathbf{Y}$ is X-flat.*

Proof. The assertion is local in X, so it reduces to the case of affine schemes, for which it means the following. Let $R \longrightarrow \mathbf{S}$ be a homomorphism of commutative rings, let F be a flat $\mathbf{S}$-module, and let G be an R-flat $\mathbf{S}$-module (i. e., an $\mathbf{S}$-module whose underlying R-module is flat). Then $F \otimes_\mathbf{S} G$ is an R-flat $\mathbf{S}$-module. □

Clearly, the full subcategory $\mathbf{Y}_X\text{–flat}$ is closed under extensions in the abelian category $\mathbf{Y}\text{–qcoh}$ (since the functor f_* is exact and the full subcategory $X\text{–flat} \subset X\text{–qcoh}$ is closed under extensions); so it inherits an exact category structure. The full subcategory $\mathbf{Y}_X\text{–flat}$ is closed under direct limits in $\mathbf{Y}\text{–qcoh}$ (because the functor f_* preserves direct limits). When the morphism f is (affine and) flat, any flat quasi-coherent sheaf on $\mathbf{Y}$ is X-flat, as the functor f_* takes flat quasi-coherent sheaves on $\mathbf{Y}$ to flat quasi-coherent sheaves on X; so $\mathbf{Y}\text{–flat} \subset \mathbf{Y}_X\text{–flat} \subset \mathbf{Y}\text{–qcoh}$.

For a flat affine morphism of schemes $f\colon \mathbf{Y} \longrightarrow X$, the equivalence of categories from Lemma 3.11 restricts to an equivalence between the category $\mathbf{Y}_X\text{–}\mathsf{flat}$ of X-flat quasi-coherent sheaves on $\mathbf{Y}$ and the category of module objects over the algebra object $f_*\mathcal{O}_{\mathbf{Y}}$ in the tensor category $X\text{–}\mathsf{flat}$. This is an equivalence of exact categories (with the exact structure on the category of module objects over $f_*\mathcal{O}_{\mathbf{Y}}$ in $X\text{–}\mathsf{flat}$ coming from the exact structure on $X\text{–}\mathsf{flat}$).

Lemma 7.10. *Let $f\colon \mathbf{Y} \longrightarrow X$ be an affine morphism and $h\colon Z \longrightarrow X$ be a morphism of schemes. Consider the pullback diagram*

$$\begin{array}{ccc} Z \times_X \mathbf{Y} & \xrightarrow{\ k\ } & \mathbf{Y} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{\ h\ } & X \end{array}$$

and put $\mathbf{W} = Z \times_X \mathbf{Y}$. Then

(a) *for any X-flat quasi-coherent sheaf $\boldsymbol{\mathcal{G}}$ on $\mathbf{Y}$, the quasi-coherent sheaf $k^*\boldsymbol{\mathcal{G}}$ on $\mathbf{W}$ is Z-flat;*

(b) *the functor $k^*\colon \mathbf{Y}_X\text{–}\mathsf{flat} \longrightarrow \mathbf{W}_Z\text{–}\mathsf{flat}$ takes short exact sequences to short exact sequences;*

(c) *assuming that h is a flat affine morphism, for any Z-flat quasi-coherent sheaf $\boldsymbol{\mathcal{H}}$ on $\mathbf{W}$, the quasi-coherent sheaf $k_*\boldsymbol{\mathcal{H}}$ on $\mathbf{Y}$ is X-flat.*

Proof. Parts (a–b): both the assertions are local in X and Z, so they reduce to the case of affine schemes, for which they mean the following. Let $R \longrightarrow \mathbf{S}$ and $R \longrightarrow T$ be two morphisms of commutative rings. Let $\mathbf{N}$ be an $\mathbf{S}$-module which is flat over R; then the $(T \otimes_R \mathbf{S})$-module $(T \otimes_R \mathbf{S}) \otimes_{\mathbf{S}} \mathbf{N}$ is flat over T. Let $0 \longrightarrow \mathbf{L} \longrightarrow \mathbf{M} \longrightarrow \mathbf{N} \longrightarrow 0$ be a short exact sequence of R-flat $\mathbf{S}$-modules; then $0 \longrightarrow (T \otimes_R \mathbf{S}) \otimes_{\mathbf{S}} \mathbf{L} \longrightarrow (T \otimes_R \mathbf{S}) \otimes_{\mathbf{S}} \mathbf{M} \longrightarrow (T \otimes_R \mathbf{S}) \otimes_{\mathbf{S}} \mathbf{N} \longrightarrow 0$ is a short exact sequence of T-flat $(T \otimes_R \mathbf{S})$-modules. Alternatively, part (a) follows from Lemma 3.3(a) and the fact that the functor $h^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ takes flat quasi-coherent sheaves on X to flat quasi-coherent sheaves on Z. Part (c): the assertion follows from the fact that the functor $h_*\colon Z\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ takes flat quasi-coherent sheaves on Z to flat quasi-coherent sheaves on X (for a flat affine morphism h). ☐

Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. We refer to Section 3.3 for the construction of the functors of inverse and direct image of pro-quasi-coherent pro-sheaves, $\pi^*\colon \mathfrak{X}\text{–}\mathsf{pro} \longrightarrow \mathfrak{Y}\text{–}\mathsf{pro}$ and $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$.

A pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ is said to be $\mathfrak{X}$-*flat* if the pro-quasi-coherent pro-sheaf $\pi_*\mathfrak{G}$ on $\mathfrak{X}$ is flat in the sense of Section 3.4, that is $\pi_*\mathfrak{G} \in \mathfrak{X}\text{–}\mathsf{flat} \subset \mathfrak{X}\text{–}\mathsf{pro}$. Explicitly, this means that, for every closed subscheme $Z \subset \mathfrak{X}$, denoting by π_Z the related morphism $\mathbf{W} = Z \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow Z$, the quasi-coherent sheaf $\mathfrak{G}^{(\mathbf{W})}$ on $\mathbf{W}$ is Z-flat. Given a representatation $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma \in \Gamma} X_\gamma$ of the ind-scheme $\mathfrak{X}$ by an inductive system of closed immersions $(X_\gamma)_{\gamma \in \Gamma}$, it suffices to check the latter condition for the closed subschemes $Z = X_\gamma$ (cf. Lemma 7.10(a)). We will denote the full subcategory of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves by $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \subset \mathfrak{Y}\text{–}\mathsf{pro}$.

Lemma 7.11. *Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Then, for any flat pro-quasi-coherent pro-sheaf $\mathfrak{F}$ on $\mathfrak{Y}$ and any $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$, the pro-quasi-coherent pro-sheaf $\mathfrak{F} \otimes^{\mathfrak{Y}} \mathfrak{G}$ on $\mathfrak{Y}$ is $\mathfrak{X}$-flat.*

Proof. Follows from Lemma 7.9 and the construction of the tensor product functor $\otimes^{\mathfrak{Y}}$ in Section 3.1. □

The full subcategory $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ is closed under direct limits (in particular, coproducts) in $\mathfrak{Y}$–pro. When the morphism π is affine and flat, the functor $\pi_*\colon \mathfrak{Y}$–$\mathsf{pro} \longrightarrow \mathfrak{X}$–$\mathsf{pro}$ takes flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ to flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ (see Section 3.4); so $\mathfrak{Y}$–$\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \subset \mathfrak{Y}$–$\mathsf{pro}$.

Let $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$ be a short sequence of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. We say that this is an (*admissible*) short *exact* sequence in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ if, for every closed subscheme $Z \subset \mathfrak{X}$ and the related closed subscheme $\mathbf{W} = Z \times_{\mathfrak{X}} \mathfrak{Y} \subset \mathfrak{Y}$, the sequence of quasi-coherent sheaves $0 \longrightarrow \mathfrak{F}^{(\mathbf{W})} \longrightarrow \mathfrak{G}^{(\mathbf{W})} \longrightarrow \mathfrak{H}^{(\mathbf{W})} \longrightarrow 0$ is exact in the abelian category $\mathbf{W}$–qcoh. It suffices to check this condition for the closed subschemes $Z = X_\gamma \subset X$, $\gamma \in \Gamma$, belonging any fixed representation $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ of the ind-scheme $\mathfrak{X}$ by an inductive system of closed immersions of schemes.

Proposition 7.12. *For any affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the category $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, endowed with the class of admissible short exact sequences as defined above, is an exact category.*

Proof. The argument is similar to the proof of Proposition 3.5 and based on Lemma 7.10(a–b). It is helpful to notice that a pro-quasi-coherent pro-sheaf $\mathfrak{P}$ on $\mathfrak{Y}$ can be defined as a collection of quasi-coherent sheaves $\mathfrak{P}^{(\mathbf{W})} \in \mathbf{W}$–$\mathsf{qcoh}$ on the closed subschemes $\mathbf{W} \subset \mathfrak{Y}$ of the form $\mathbf{W} = Z \times_{\mathfrak{X}} \mathfrak{Y}$ (where Z ranges over the closed subschemes in X), endowed with the (iso)morphisms and satisfying the compatibilies listed in items (i–iv) of Section 3.1. □

Given an affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, where the ind-scheme $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ is represented by an inductive system of closed immersions $(X_\gamma)_{(\gamma\in\Gamma)}$, we put $\mathbf{Y}_\gamma = X_\gamma \times_X \mathfrak{Y}$. Then $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} \mathbf{Y}_\gamma$ is a representation of $\mathfrak{Y}$ by an inductive system of closed immersions.

Lemma 7.13. *A complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{G}^\bullet$ on $\mathfrak{Y}$ is acyclic (as a complex in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$) if and only if, for every $\gamma \in \Gamma$, the complex of X_γ-flat quasi-coherent sheaves $\mathfrak{G}^{\bullet(X_\gamma)}$ on $\mathbf{Y}_\gamma$ is acyclic (as a complex in $(\mathbf{Y}_\gamma)_{X_\gamma}$–$\mathsf{flat}$).*

Proof. This is a generalization of Lemma 4.13, provable in the same way. The “only if” assertion is obvious. To prove the “if”, one needs to observe that the functor assigning to an acyclic complex of quasi-coherent sheaves its sheaves of cocycles can be expressed as a kind of cokernel in the category of quasi-coherent sheaves, and as such, commutes with inverse images (whenever the latter preserve acyclicity of a given complex). Then one needs also to use Lemma 7.10(a–b). □

For any affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the direct image functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$ takes the full subcategory $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \subset \mathfrak{Y}\text{–}\mathsf{pro}$ into the full subcategory $\mathfrak{X}\text{–}\mathsf{flat} \subset \mathfrak{X}\text{–}\mathsf{pro}$. The resulting direct image functor

$$\pi_*\colon \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat}$$

is an exact functor between exact categories (i. e, it takes short exact sequences to short exact sequences). This follows immediately from the observation that the direct image functor $\pi_{Z*}\colon \mathbf{W}_Z\text{–}\mathsf{flat} \longrightarrow Z\text{–}\mathsf{flat}$ for the affine morphism $\pi_Z\colon \mathbf{W} \longrightarrow Z$ is exact (which is true because the functor $\pi_{Z*}\colon \mathbf{W}\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ is exact).

For a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the inclusion functor $\mathfrak{Y}\text{–}\mathsf{flat} \longrightarrow \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ is exact; in fact, a short sequence in $\mathfrak{Y}\text{–}\mathsf{flat}$ is exact if and only if it is exact in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Furthermore, there is an exact inverse image functor

$$\pi^*\colon \mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{Y}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat},$$

which is left adoint to π_*.

Lemma 7.14. *Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Then a complex $\mathfrak{G}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ if and only if the complex $\pi_*\mathfrak{G}^\bullet$ is acyclic in $\mathfrak{X}\text{–}\mathsf{flat}$.*

Proof. Follows from Lemmas 4.13 and 7.13 together with the fact that, for an affine morphism of schemes $f\colon \mathbf{Y} \longrightarrow X$, a complex $\mathcal{G}^\bullet$ in $\mathbf{Y}_X\text{–}\mathsf{flat}$ is acyclic if and only if the complex $f_*\mathcal{G}^\bullet$ is acyclic in $X\text{–}\mathsf{flat}$. The latter assertion holds because the functor $f_*\colon \mathbf{Y}\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ is exact and faithful. □

For a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the equivalence of categories from Proposition 3.13(a) restricts to an equivalence between the category $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ and the category of module objects over the algebra object $\pi_*\mathfrak{O}_{\mathfrak{Y}}$ in the tensor category $\mathfrak{X}\text{–}\mathsf{flat}$. This is an equivalence of exact categories (with the exact structure on the category of module objects over $\pi_*\mathfrak{O}_{\mathfrak{Y}}$ in $\mathfrak{X}\text{–}\mathsf{flat}$ coming from the exact structure on $\mathfrak{X}\text{–}\mathsf{flat}$).

7.3. **The triangulated equivalence.** The following theorem, generalizing Theorem 4.23, is the main result of Section 7.

Theorem 7.15. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Then there is a natural equivalence of triangulated categories $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$, provided by mutually inverse triangulated functors $\mathfrak{Hom}_{\mathfrak{Y}\text{-}\mathsf{qc}}(\pi^*\mathscr{D}^\bullet, -)\colon \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ and $\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} -\,\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}})$.*

The notation $\mathfrak{Hom}_{\mathfrak{Y}\text{-}\mathsf{qc}}(-,-)$ will be explained below, and the proof of Theorem 7.15 will be given below in this Section 7.3. The next two lemmas are not needed for this proof and are included here mostly for completeness of the exposition and to help the reader feel more comfortable (they will be useful, however, in Section 9.8). The subsequent three lemmas play a more important role, and among them Lemma 7.20 is essential.

Lemma 7.16. *Let $f\colon \mathbf{Y} \longrightarrow X$ be a flat affine morphism of schemes and $Z \subset X$ be a reasonable closed subscheme with the closed immersion morphism $i\colon Z \longrightarrow X$. Consider the pullback diagram*

$$\begin{array}{ccc} Z \times_X \mathbf{Y} & \xrightarrow{k} & \mathbf{Y} \\ \big\downarrow g & & \big\downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

and put $\mathbf{W} = Z \times_X \mathbf{Y}$. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\boldsymbol{\mathcal{G}}$ be an X-flat quasi-coherent sheaf on $\mathbf{Y}$; put $\boldsymbol{\mathcal{N}} = f^*\mathcal{M} \in \mathbf{Y}$–$\mathsf{qcoh}$. Then the natural morphism of quasi-coherent sheaves on $\mathbf{W}$

$$k^*\boldsymbol{\mathcal{G}} \otimes_{\mathcal{O}_\mathbf{W}} k^!\boldsymbol{\mathcal{N}} \longrightarrow k^!(\boldsymbol{\mathcal{G}} \otimes_{\mathcal{O}_\mathbf{Y}} \boldsymbol{\mathcal{N}})$$

from Lemma 3.6 is an isomorphism.

Proof. The assertion is local in X, so it reduces to the case of affine schemes, for which it means the following. Let $R \longrightarrow \mathbf{S}$ be a homomorphism of commutative rings such that $\mathbf{S}$ is a flat R-module and $R \longrightarrow T$ be a surjective homomorphism of commutative rings with a finitely generated kernel ideal. Let M be an R-module and $\mathbf{G}$ be an R-flat $\mathbf{S}$-module. Then the natural homomorphism of $(T \otimes_R \mathbf{S})$-modules

$$\begin{aligned} &((T \otimes_R \mathbf{S}) \otimes_\mathbf{S} \mathbf{G}) \otimes_{(T\otimes_R\mathbf{S})} \operatorname{Hom}_\mathbf{S}(T \otimes_R \mathbf{S},\ \mathbf{S} \otimes_R M) \\ &\qquad = \mathbf{G} \otimes_\mathbf{S} \operatorname{Hom}_\mathbf{S}(T \otimes_R \mathbf{S},\ \mathbf{S} \otimes_R M) = \mathbf{G} \otimes_\mathbf{S} \operatorname{Hom}_R(T,\ \mathbf{S} \otimes_R M) \\ &\qquad\qquad \longrightarrow \operatorname{Hom}_R(T,\ \mathbf{G} \otimes_R M) = \operatorname{Hom}_\mathbf{S}(T \otimes_R \mathbf{S},\ \mathbf{G} \otimes_\mathbf{S} (\mathbf{S} \otimes_R M)) \end{aligned}$$

is an isomorphism, because both the maps

$$\begin{aligned} \mathbf{G} \otimes_\mathbf{S} \operatorname{Hom}_R(T,\ \mathbf{S} \otimes_R M) &\longleftarrow \mathbf{G} \otimes_\mathbf{S} (\mathbf{S} \otimes_R \operatorname{Hom}_R(T, M)) \\ &= \mathbf{G} \otimes_R \operatorname{Hom}_R(T, M) \longrightarrow \operatorname{Hom}_R(T,\ \mathbf{G} \otimes_R M) \end{aligned}$$

are isomorphisms (cf. Lemma 4.24 for the first arrow). ☐

Lemma 7.17. *Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of reasonable ind-schemes and $Z \subset \mathfrak{X}$ be a reasonable closed subscheme with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$. Put $\mathbf{W} = Z \times_\mathfrak{X} \mathfrak{Y}$ and denote by $k\colon \mathbf{W} \longrightarrow \mathfrak{Y}$ the natural closed immersion. Let $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$ and $\mathfrak{G}$ be an $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$; put $\mathscr{N} = \pi^*\mathscr{M} \in \mathfrak{Y}$–$\mathsf{tors}$. Then there is a natural isomorphism $k^!(\mathfrak{G} \otimes_\mathfrak{Y} \mathscr{N}) \simeq k^*\mathfrak{G} \otimes_{\mathcal{O}_\mathbf{W}} k^!\mathscr{N} = \mathfrak{G}^{(\mathbf{W})} \otimes_{\mathcal{O}_\mathbf{W}} \mathscr{N}_{(\mathbf{W})}$ in $\mathbf{W}$–qcoh.*

Proof. The argument is similar to the proof of Proposition 3.7 and uses Lemma 7.16. ☐

Lemma 7.18. *Let $f\colon \mathbf{Y} \longrightarrow X$ be a flat affine morphism of semi-separated schemes and $Z \subset X$ be a reasonable closed subscheme with the closed immersion morphism*

$i\colon Z \longrightarrow X$. *Consider the pullback diagram*

$$\begin{array}{ccc} Z\times_X \mathbf{Y} & \xrightarrow{\ k\ } & \mathbf{Y} \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{\ i\ } & X \end{array} \tag{27}$$

and put $\mathbf{W} = Z\times_X\mathbf{Y}$. *Let* $\mathcal{M}$ *be a quasi-coherent sheaf on* X *and* $\boldsymbol{\mathcal{K}}$ *be an* X*-injective quasi-coherent sheaf on* $\mathbf{Y}$*; put* $\boldsymbol{\mathcal{N}} = f^*\mathcal{M} \in \mathbf{Y}\text{–}\mathsf{qcoh}$. *Then the natural morphism of quasi-coherent sheaves on* $\mathbf{W}$

$$k^*\,\mathcal{H}om_{\mathbf{Y}\text{-qc}}(\boldsymbol{\mathcal{N}},\boldsymbol{\mathcal{K}}) \longrightarrow \mathcal{H}om_{\mathbf{W}\text{-qc}}(k^!\boldsymbol{\mathcal{N}},k^!\boldsymbol{\mathcal{K}})$$

from Lemma 4.25 is an isomorphism.

Proof. Since the direct image functor $g_*\colon \mathbf{W}\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ is exact and faithful, it suffices to check that the morphism in question becomes an isomorphism after applying g_*. We have

$$g_*k^*\,\mathcal{H}om_{\mathbf{Y}\text{-qc}}(f^*\mathcal{M},\boldsymbol{\mathcal{K}}) \simeq i^*f_*\,\mathcal{H}om_{\mathbf{Y}\text{-qc}}(f^*\mathcal{M},\boldsymbol{\mathcal{K}}) \simeq i^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M},f_*\boldsymbol{\mathcal{K}})$$

by Lemmas 3.3(a) and 4.2. On the other hand,

$$g_*\,\mathcal{H}om_{\mathbf{W}\text{-qc}}(k^!f^*\mathcal{M},k^!\boldsymbol{\mathcal{K}}) \simeq g_*\,\mathcal{H}om_{\mathbf{W}\text{-qc}}(g^*i^!\mathcal{M},k^!\boldsymbol{\mathcal{K}})$$
$$\simeq \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M},g_*k^!\boldsymbol{\mathcal{K}}) \simeq \mathcal{H}om_{Z\text{-qc}}(i^!\mathcal{M},i^!f_*\boldsymbol{\mathcal{K}})$$

by Lemmas 4.24, 4.2, and 2.3(a). The assertion now follows from Lemma 4.25, as the quasi-coherent sheaf $f_*\boldsymbol{\mathcal{K}}$ on X is injective by assumption. ☐

Lemma 7.19. *Let* $f\colon \mathbf{Y} \longrightarrow X$ *be an affine morphism of semi-separated schemes and* $Z \subset X$ *be a reasonable closed subscheme with the closed immersion morphism* $i\colon Z \longrightarrow X$. *Consider the pullback diagram* (27) *with* $\mathbf{W} = Z\times_X \mathbf{Y}$. *Let* $(\boldsymbol{\mathcal{Q}}_\xi)_{\xi\in\Xi}$ *be a family of quasi-coherent sheaves on* $\mathbf{Y}$ *such that the quasi-coherent sheaves* $f_*\boldsymbol{\mathcal{Q}}$ *on* X *are flat cotorsion. Then the natural morphism of quasi-coherent sheaves on* $\mathbf{W}$

$$k^*\prod\nolimits_{\xi\in\Xi}\boldsymbol{\mathcal{Q}}_\xi \longrightarrow \prod\nolimits_{\xi\in\Xi}k^*\boldsymbol{\mathcal{Q}}_\xi$$

from Lemma 4.26 is an isomorphism.

Proof. As in the previous proof, it suffices to show that the morphism in question becomes an isomorphism after applying g_*. Since the direct image functors f_* and g_*, being right adjoints, preserve infinite products of quasi-coherent sheaves, and $g_*k^* \simeq i^*f_*$ by Lemma 3.3(a), the desired assertion follows from Lemma 4.26. ☐

Lemma 7.20. *Let* $f\colon \mathbf{Y} \longrightarrow X$ *be an affine morphism of semi-separated schemes and* $Z \subset X$ *be a reasonable closed subscheme with the closed immersion morphism* $i\colon Z \longrightarrow X$. *Consider the pullback diagram* (27) *with* $\mathbf{W} = Z\times_X \mathbf{Y}$. *Let* $\mathcal{M}^\bullet$ *be a complex of quasi-coherent sheaves on* X *and* $\boldsymbol{\mathcal{K}}^\bullet$ *be a complex of* X*-injective quasi-coherent sheaves on* $\mathbf{Y}$*; put* $\boldsymbol{\mathcal{N}}^\bullet = f^*\mathcal{M}^\bullet \in \mathsf{C}(\mathbf{Y}\text{–}\mathsf{qcoh})$. *Then the natural morphism of complexes of quasi-coherent sheaves on* $\mathbf{W}$

$$k^*\,\mathcal{H}om_{\mathbf{Y}\text{-qc}}(\boldsymbol{\mathcal{N}}^\bullet,\boldsymbol{\mathcal{K}}^\bullet) \longrightarrow \mathcal{H}om_{\mathbf{W}\text{-qc}}(k^!\boldsymbol{\mathcal{N}}^\bullet,k^!\boldsymbol{\mathcal{K}}^\bullet)$$

from Lemma 4.27 is an isomorphism.

Proof. This is provable similarly to Lemma 7.18, using Lemma 4.3 in order to reduce the assertion to Lemma 4.27. Alternatively, a direct proof, similar to the proof of Lemma 4.27 and based on the result of Lemma 7.18, is possible; in particular, assuming additionally that X is Noetherian and $\mathcal{M}^\bullet$ is a complex of injective quasi-coherent sheaves on X, one can use Lemma 7.19. (For this purpose, one observes that $f_*\mathcal{H}om_{\mathbf{Y}\text{-qc}}(f^*\mathcal{M}^p,\mathcal{K}^q)\simeq\mathcal{H}om_{X\text{-qc}}(\mathcal{M}^p,f_*\mathcal{K}^q)$ are flat cotorsion quasi-coherent sheaves on X by Lemmas 4.4(d) and 4.9(a).) $\square$

Let $\pi\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be a flat affine morphism of reasonable ind-semi-separated ind-schemes. Let $\mathscr{E}^\bullet\in\mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$, and let $\mathscr{K}^\bullet\in\mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}})$ be a complex of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then the complex $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\mathscr{K}^\bullet)\in\mathsf{C}(\mathfrak{Y}\text{–}\mathsf{pro})$ of pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is constructed as follows.

For every reasonable closed subscheme $Z\subset\mathfrak{X}$, consider the pullback diagram

$$\begin{array}{ccc} Z\times_{\mathfrak{X}}\mathfrak{Y} & \xrightarrow{\ k\ } & \mathfrak{Y} \\ \big\downarrow{\scriptstyle \pi_Z} & & \big\downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\ i\ } & \mathfrak{X} \end{array}$$

so $\mathbf{W}=Z\times_{\mathfrak{X}}\mathfrak{Y}$ is a reasonable closed subscheme in $\mathfrak{Y}$. Put

$$\begin{aligned}\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\mathscr{K}^\bullet)^{(\mathbf{W})}&=\mathcal{H}om_{\mathbf{W}\text{-qc}}(k^!\pi^*\mathscr{E}^\bullet,k^!\mathscr{K}^\bullet)\\ &\simeq\mathcal{H}om_{\mathbf{W}\text{-qc}}(\pi_Z^*i^!\mathscr{E}^\bullet,k^!\mathscr{K}^\bullet)=\mathcal{H}om_{\mathbf{W}\text{-qc}}(\pi_Z^*\mathscr{E}^\bullet_{(Z)},\mathscr{K}^\bullet_{(\mathbf{W})}),\end{aligned}$$

where the middle isomorphism holds by Remark 7.4. According to Lemma 7.20, for every pair of reasonable closed subschemes $Z'\subset Z''\subset\mathfrak{X}$ and the related closed subschemes $\mathbf{W}'\subset\mathbf{W}''\subset\mathfrak{Y}$, $\mathbf{W}^{(s)}=Z^{(s)}\times_{\mathfrak{X}}\mathfrak{Y}$ with the natural closed immersion $k_{\mathbf{W}'\mathbf{W}''}\colon\mathbf{W}'\longrightarrow\mathbf{W}''$, we have

$$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\mathscr{K}^\bullet)^{(\mathbf{W}')}\simeq k_{\mathbf{W}'\mathbf{W}''}^*\,\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\mathscr{K}^\bullet)^{(\mathbf{W}'')}$$

as required for the construction of a pro-quasi-coherent pro-sheaf. This explains the meaning of the notation in Theorem 7.15.

Lemma 7.21. *Let $\pi\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be a flat affine morphism of reasonable ind-semi-separated ind-schemes. Let $\mathscr{E}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$, and let $\mathscr{K}^\bullet$ be a complex of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then there is a natural isomorphism*

$$\pi_*\,\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\mathscr{K}^\bullet)\simeq\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet,\pi_*\mathscr{K}^\bullet)$$

of complexes of pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ (where the functor $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}$ in the right-hand side was defined before the proof of Theorem 4.23 in Section 4.5).

Proof. In the notation above, we have

$$\pi_{Z*}\,\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet,\boldsymbol{\mathscr{K}}^\bullet)^{(\mathbf{W})}\simeq\pi_{Z*}\,\mathcal{H}om_{\mathbf{W}\text{-qc}}(\pi_Z^*i^!\mathscr{E}^\bullet,k^!\boldsymbol{\mathscr{K}}^\bullet)$$
$$\simeq\mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{E}^\bullet,\pi_{Z*}k^!\boldsymbol{\mathscr{K}}^\bullet)=\mathcal{H}om_{Z\text{-qc}}(i^!\mathscr{E}^\bullet,i^!\pi_*\boldsymbol{\mathscr{K}}^\bullet)=\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet,\pi_*\boldsymbol{\mathscr{K}}^\bullet)^{(Z)},$$

where the second isomorphism is provided by Lemma 4.3, while the isomorphism $\pi_{Z*}k^!\simeq i^!\pi_*$ holds by the construction of the functor $\pi_*\colon\mathfrak{Y}\text{–tors}\longrightarrow\mathfrak{X}\text{–tors}$ in Section 2.6. Having this computation done, it remains to recall the construction of the functor $\pi_*\colon\mathfrak{Y}\text{–pro}\longrightarrow\mathfrak{X}\text{–pro}$ from Section 3.3. □

Proof of Theorem 7.15. The argument follows the idea of the proof of [47, Theorem 5.6], which is a module version.

The equivalence $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})\simeq\mathsf{D}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}})$ is provided by Proposition 7.8.

The tensor product functor $\otimes_{\mathfrak{Y}}\colon\mathfrak{Y}\text{–tors}\times\mathfrak{Y}\text{–pro}\longrightarrow\mathfrak{Y}\text{–tors}$ was constructed in Section 3.2. Here we restrict it to the full subcategory of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}\subset\mathfrak{Y}\text{–pro}$, obtaining a functor $\otimes_{\mathfrak{Y}}\colon\mathfrak{Y}\text{–tors}\times\mathfrak{Y}_{\mathfrak{X}}\text{–flat}\longrightarrow\mathfrak{Y}\text{–tors}$. This functor is extended to complexes similarly to the construction in (the proof of) Theorem 4.23, using the coproduct totalization of bicomplexes. Given a complex of quasi-coherent torsion sheaves $\boldsymbol{\mathscr{E}}^\bullet$ on $\mathfrak{Y}$, we obtain the functor

$$\boldsymbol{\mathscr{E}}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\longrightarrow\mathsf{C}(\mathfrak{Y}\text{–tors}),$$

which obviously descends to a triangulated functor between the homotopy categories $\boldsymbol{\mathscr{E}}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{K}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\longrightarrow\mathsf{K}(\mathfrak{Y}\text{–tors})$.

Now let us assume that $\boldsymbol{\mathscr{E}}^\bullet=\pi^*\mathscr{E}^\bullet$, where $\mathscr{E}^\bullet\in\mathsf{C}(\mathfrak{X}\text{–tors}_{\mathsf{inj}})$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Let $\mathfrak{G}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. By Lemma 7.5, we have

$$\pi_*(\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet)\simeq\mathscr{E}^\bullet\otimes_{\mathfrak{X}}\pi_*\mathfrak{G}^\bullet$$

(recall that the functor π_* preserves coproducts by Lemma 7.1(a)). Following the proof of Theorem 4.23, $\mathscr{E}^\bullet\otimes_{\mathfrak{X}}\pi_*\mathfrak{G}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$ (since $\pi_*\mathfrak{G}^\bullet$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$). So $\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet$ is a complex of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. We have constructed a triangulated functor

(28) $$\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{K}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\longrightarrow\mathsf{K}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}).$$

Let us check that the latter functor induces a well-defined triangulated functor

(29) $$\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\longrightarrow\mathsf{D}(\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}).$$

We need to show that the complex $\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet$ is acyclic with respect to $\mathfrak{Y}\text{–tors}_{\mathfrak{X}\text{-inj}}$ whenever a complex $\mathfrak{G}^\bullet$ is acyclic with respect to $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$. By Lemma 7.7, it suffices to check that the complex $\pi_*(\pi^*\mathscr{E}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet)\simeq\mathscr{E}^\bullet\otimes_{\mathfrak{X}}\pi_*\mathfrak{G}^\bullet$ in $\mathfrak{X}\text{–tors}_{\mathsf{inj}}$ is contractible. By Lemma 7.14, the complex $\pi_*\mathfrak{G}^\bullet$ is acyclic in $\mathfrak{X}\text{–flat}$. Now the functor (29) is well-defined because the functor (5) from (the proof of) Theorem 4.23 is well-defined.

On the other hand, for any complex $\mathscr{E}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$, the construction before this proof provides a functor

$$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, -)\colon \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{pro}),$$

which obviously descends to a triangulated functor between the homotopy categories $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, -)\colon \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{pro})$.

Assume that $\mathscr{E}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$, and let $\mathscr{K}^\bullet$ be a complex of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then, according to the proof of Theorem 4.23, $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \pi_*\mathscr{K}^\bullet)$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. By Lemma 7.21, it follows that $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, \mathscr{K}^\bullet)$ is a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. We have constructed a triangulated functor

(30) $$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, -)\colon \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{K}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}).$$

Let us check that the latter functor induces a well-defined triangulated functor

(31) $$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, -)\colon \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}).$$

We need to show that the complex $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, \mathscr{K}^\bullet)$ is acyclic with respect to $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ whenever a complex $\mathscr{K}^\bullet$ is acyclic with respect to $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}$. By Lemma 7.14, it suffices to check that the complex $\pi_*\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{E}^\bullet, \mathscr{K}^\bullet) \simeq \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{E}^\bullet, \pi_*\mathscr{K}^\bullet)$ is acyclic in $\mathfrak{X}\text{–}\mathsf{flat}$. By Lemma 7.7, the complex $\pi_*\mathscr{K}^\bullet$ is contractible in $\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}$, and the assertion follows.

It is straightforward to see that the functor (30) is right adjoint to the functor (28). Hence the functor (31) is right adjoint to the functor (29). It remains to show that the functors (29) and (31) are mutually inverse equivalences when $\mathscr{E}^\bullet = \mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{X}$. For this purpose, it suffices to check that the adjunction morphisms are isomorphisms.

Now, similarly to the proof of [47, Theorem 5.6], we consider the direct image ("forgetful") functors

$$\pi_*\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}),$$
$$\pi_*\colon \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}),$$

and

$$\pi_*\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}),$$

which are well-defined, and moreover, conservative (by the definition of the semiderived category or) by Lemmas 7.7 and 7.14.

As we have already seen in the above discussion, by Lemmas 7.5 and 7.21, the direct image functors transform the functors (29) and (31) into the functors (5) and (7) from (the proof of) Theorem 4.23. In other words, there are commutative diagrams of triangulated functors

$$\begin{array}{ccccc}
\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) & \xrightarrow{\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-} & \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}) & = & \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \\
\downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi_*} \\
\mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) & \xrightarrow[\mathscr{D}^\bullet\otimes_{\mathfrak{X}}-]{} & \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) & = & \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})
\end{array}$$

and

$$\begin{array}{ccccc}
\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) & = & \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}) & \xrightarrow{\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet,-)} & \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \\
\downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi_*} & & \downarrow{\scriptstyle\pi_*} \\
\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) & = & \mathsf{K}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}}) & \xrightarrow[\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,-)]{} & \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat})
\end{array}$$

The direct image functors also transform the adjunction morphisms for the pair of adjoint functors $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-$ and $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet,-)$ into the adjunction morphisms for the pair of adjoint functors $\mathscr{D}^\bullet\otimes_{\mathfrak{X}}-$ and $\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet,-)$. Since the latter pair of adjunction morphisms are isomorphisms in the respective derived/homotopy categories by Theorem 4.23, and the direct image functors are conservative, the former pair of adjunction morphisms are isomorphisms in the derived categories of the respective exact categories, too. $\square$

8. The Semitensor Product

The aim of this section is to construct the semitensor product operation

$$\lozenge_{\pi^*\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})\times\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})\longrightarrow\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$$

on the $\mathfrak{Y}/\mathfrak{X}$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$, making $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ a tensor triangulated category. The inverse image $\pi^*\mathscr{D}^\bullet$ on $\mathfrak{Y}$ of the dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ is the unit object of this tensor structure.

We follow the approach of [47, Section 6], with suitable modifications. We also explain how to correct a small mistake in the exposition in [47, Section 6].

8.1. Underived tensor products in the relative context.

We start with a sequence of lemmas extending Lemma 5.2 to the relative situation.

Lemma 8.1. *Let $\pi\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be an affine morphism of ind-schemes. Let $\mathfrak{F}^\bullet\in\mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ and $\mathfrak{G}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which is acyclic as a complex in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Then the complex $\pi^*\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is also acyclic as a complex in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$.*

Proof. First of all, $\pi^*\mathfrak{F}^\bullet$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ according to the discussion in Section 3.4, hence $\pi^*\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet$ is a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ by Lemma 7.11. (Notice that the full subcategory of

$\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves is closed under direct limits, and in particular, under coproducts in $\mathfrak{Y}$–pro, as one can see from the discussion in Section 3.5.)

By Lemma 7.14, in order to show that the complex $\pi^*\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \boldsymbol{\mathfrak{G}}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$, it suffices to check that the complex $\pi_*(\pi^*\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \boldsymbol{\mathfrak{G}}^\bullet)$ is acyclic in $\mathfrak{X}$–flat. By the projection formula (2) from Section 3.3, we have $\pi_*(\pi^*\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \boldsymbol{\mathfrak{G}}^\bullet) \simeq \mathfrak{F}^\bullet \otimes^{\mathfrak{X}} \pi_*\boldsymbol{\mathfrak{G}}^\bullet$. Again by Lemma 7.14, the complex $\pi_*\boldsymbol{\mathfrak{G}}^\bullet$ is acyclic in $\mathfrak{X}$–flat. It remains to refer to Lemma 5.2(a). □

Similarly one can prove that the complex $\pi^*\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \boldsymbol{\mathfrak{G}}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ whenever a complex $\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{X}$–flat and $\boldsymbol{\mathfrak{G}}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat})$ is an arbitrary complex.

The next lemma is another version of projection formula for the action of pro-quasi-coherent pro-sheaves in quasi-coherent torsion sheaves (cf. Lemma 7.5).

Lemma 8.2. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Let $\mathfrak{P}$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ and $\mathcal{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}$. Then there is a natural isomorphism*

$$\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathcal{N} \simeq f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N})$$

of quasi-coherent torsion sheaves on $\mathfrak{X}$.

Proof. The argument is similar to (but simpler than) the proof of Lemma 7.5. The natural morphism

$$\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathcal{N} \longrightarrow f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}) \tag{32}$$

is adjoint to the composition $f^*(\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathcal{N}) \simeq f^*\mathfrak{P} \otimes_{\mathfrak{Y}} f^*f_*\mathcal{N} \longrightarrow f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}$ of the isomorphism $f^*(\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathcal{N}) \simeq f^*\mathfrak{P} \otimes_{\mathfrak{Y}} f^*f_*\mathcal{N}$ provided by Lemma 3.4 and the morphism $f^*\mathfrak{P} \otimes_{\mathfrak{Y}} f^*f_*\mathcal{N} \longrightarrow f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}$ induced by the adjunction morphism $f^*f_*\mathcal{N} \longrightarrow \mathcal{N}$.

In the notation of Section 7.1, we consider Γ-systems of quasi-coherent sheaves on $\mathfrak{X}$ and on $\mathfrak{Y}$. Similarly to the above one constructs, for any Γ-system $\mathbb{N}$ on $\mathfrak{Y}$, a natural morphism of Γ-systems $\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathbb{N} \longrightarrow f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathbb{N})$, which is an isomorphism essentially by Lemma 2.2.

To show that (32) is an isomorphism, it remains to compute

$$\begin{aligned}\mathfrak{P} \otimes_{\mathfrak{X}} f_*\mathcal{N} \simeq \mathfrak{P} \otimes_{\mathfrak{X}} f_*((\mathcal{N}|_\Gamma)^+) \simeq \mathfrak{P} \otimes_{\mathfrak{X}} (f_*(\mathcal{N}|_\Gamma))^+ \\ \simeq (\mathfrak{P} \otimes_{\mathfrak{X}} f_*(\mathcal{N}|_\Gamma))^+ \simeq (f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}|_\Gamma))^+ \simeq f_*((f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}|_\Gamma)^+) \\ \simeq f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} (\mathcal{N}_\Gamma)^+) \simeq f_*(f^*\mathfrak{P} \otimes_{\mathfrak{Y}} \mathcal{N}),\end{aligned}$$

using the definitions of the functors $\otimes_{\mathfrak{X}}\colon \mathfrak{X}$–$\mathsf{pro} \times \mathfrak{X}$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$ and $\otimes_{\mathfrak{Y}}\colon \mathfrak{Y}$–$\mathsf{pro} \times \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{Y}$–$\mathsf{tors}$, and also Lemma 7.1(b). The point is that the direct image and tensor product functors in question commute with the functors $(-)^+$. □

Lemma 8.3. *Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Let $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{X}$–$\mathsf{flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ and $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$ such that the complex of quasi-coherent torsion sheaves $\pi_*\boldsymbol{\mathcal{N}}^\bullet$ on $\mathfrak{X}$ is coacyclic. Then the complex $\pi^*\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ also has the property that its direct image $\pi_*(\pi^*\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{N}}^\bullet)$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{X}$.*

Proof. By Lemma 8.2, we have $\pi_*(\pi^*\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{N}^\bullet) \simeq \mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \pi_*\mathscr{N}^\bullet$ (recall that the direct image functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ preserves coproducts by Lemma 7.1(a)). So it remains to refer to Lemma 5.2(b). □

Similarly one can prove, assuming that $\mathfrak{X}$ is an ind-Noetherian ind-scheme and using Lemma 5.2(c), that $\pi_*(\pi^*\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{N}^\bullet)$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{X}$ whenever $\mathfrak{F}^\bullet$ is an acyclic complex in $\mathfrak{X}\text{–}\mathsf{flat}$ and $\mathscr{N}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ is an arbitrary complex.

Lemma 8.4. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Let $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$ and $\mathfrak{G}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which is acyclic as a complex in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Then the complex $\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}} \pi^*\mathscr{M}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ has the property that its direct image $\pi_*(\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}} \pi^*\mathscr{M}^\bullet)$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{X}$.*

Proof. By Lemma 7.5, we have $\pi_*(\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}} \pi^*\mathscr{M}^\bullet) \simeq \pi_*\mathfrak{G}^\bullet \otimes_{\mathfrak{X}} \mathscr{M}^\bullet$, so it remains to refer to Lemma 5.2(c). □

Similarly one can prove, using Lemma 5.2(b), that $\pi_*(\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}} \pi^*\mathscr{M}^\bullet)$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{X}$ whenever $\mathscr{M}^\bullet$ is a coacyclic complex in $\mathfrak{X}\text{–}\mathsf{tors}$ and $\mathfrak{G}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ is an arbitrary complex.

Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Consider the functor of tensor product composed with inverse image

$$\pi^*(-) \otimes^{\mathfrak{Y}} - \,\colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}).$$

It follows from Lemma 8.1 and the subsequent discussion that this functor descends to a triangulated functor of two arguments

$$\pi^*(-) \otimes^{\mathfrak{Y}} - \,\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}). \tag{33}$$

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of schemes. Consider the two functors of tensor product composed with inverse image

$$\pi^*(-) \otimes_{\mathfrak{Y}} - \,\colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}),$$
$$- \otimes_{\mathfrak{Y}} \pi^*(-)\colon \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

It follows from Lemmas 8.3–8.4 and the discussion that these functors descend to triangulated functors of two arguments

$$\pi^*(-) \otimes_{\mathfrak{Y}} - \,\colon \mathsf{D}(\mathfrak{X}\text{–}\mathsf{flat}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}), \tag{34}$$

$$- \otimes_{\mathfrak{Y}} \pi^*(-)\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}). \tag{35}$$

8.2. **Relatively homotopy flat resolutions.** Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Recall once again that, for any flat pro-quasi-coherent pro-sheaf $\mathfrak{F}$ on $\mathfrak{Y}$ and any $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf $\mathfrak{Q}$ on $\mathfrak{Y}$, the pro-quasi-coherent pro-sheaf $\mathfrak{F} \otimes^{\mathfrak{Y}} \mathfrak{Q}$ on $\mathfrak{Y}$ is $\mathfrak{X}$-flat (see Lemma 7.11).

We will say that a complex $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is *relatively homotopy flat* if the following two conditions hold:

(i) for any complex $\mathfrak{Q}^\bullet \in \mathsf{C}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$ which is acyclic in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$, the complex $\mathfrak{F}^\bullet \otimes^\mathfrak{Y} \mathfrak{Q}^\bullet$ is acyclic in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$;

(ii) for any complex $\mathscr{N}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ such that the complex $\pi_*\mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$, the complex $\pi_*(\mathfrak{F}^\bullet \otimes_\mathfrak{Y} \mathscr{N}^\bullet)$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$.

Lemma 8.5. *Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Then*

(a) *the relatively homotopy flat complexes form a full triangulated subcategory closed under coproducts in $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{flat})$;*

(b) *for any complex $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$, the complex $\pi^*\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is relatively homotopy flat.*

Proof. Part (a): clearly, any complex in $\mathfrak{Y}\text{–}\mathsf{flat}$ which is homotopy equivalent to a relatively homotopy flat complex is also relatively homotopy flat. Furthermore, for any complex $\mathfrak{Q}^\bullet \in \mathsf{K}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$, the functor $-\otimes^\mathfrak{Y} \mathfrak{Q}^\bullet\colon \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$ is a triangulated functor preserving coproducts (see Section 3.5), and the class of all short exact sequences (hence the class of all acyclic complexes) in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$ is closed under coproducts. Similarly, for any complex $\mathscr{N}^\bullet \in \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$, the functor $-\otimes_\mathfrak{Y}\mathscr{N}^\bullet\colon \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{flat}) \longrightarrow \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ is a triangulated functor preserving coproducts, and the class all complexes in $\mathfrak{Y}\text{–}\mathsf{tors}$ whose direct images are coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$ is closed under coproducts (by Lemma 7.1(a)). Therefore, the class of all relatively homotopy flat complexes is closed under shifts, cones, and coproducts in $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{flat})$.

Part (b) follows immediately from Lemmas 8.1 and 8.3. □

The next two abstract category-theoretic lemmas are well-known.

Lemma 8.6. *Let B be an additive category with countable coproducts, and let $\cdots \longrightarrow B_2^\bullet \longrightarrow B_1^\bullet \longrightarrow B_0^\bullet \longrightarrow 0$ be a bounded above complex of complexes in B. Let $T^\bullet$ be the total complex of the bicomplex $B_\bullet^\bullet$ constructed by taking infinite coproducts along the diagonals. Then the complex $T^\bullet \in \mathsf{K}(\mathsf{B})$ is homotopy equivalent to a complex which can be obtained from the complexes $B_n^\bullet$, $n \geq 0$, using the operations of shift, cone, and countable coproduct.*

Proof. Denote by $C_n^\bullet$ the total complex of the finite complex of complexes $B_n^\bullet \longrightarrow B_{n-1}^\bullet \longrightarrow \cdots \longrightarrow B_1^\bullet \longrightarrow B_0^\bullet$. Then there is a natural termwise split monomorphism of complexes $C_n^\bullet \longrightarrow T^\bullet$ for every $n \geq 0$, and the complex $T^\bullet$ is the direct limit of the sequence of its termwise split subcomplexes $C_n^\bullet$. Hence the telescope sequence $0 \longrightarrow \coprod_{n\geq 0} C_n^\bullet \longrightarrow \coprod_{n\geq 0} C_n^\bullet \longrightarrow T^\bullet \longrightarrow 0$ is a termwise split short exact sequence of complexes in B. It follows that the complex $T^\bullet$ is homotopy equivalent to the cone of the morphism of complexes $\coprod_{n\geq 0} C_n^\bullet \longrightarrow \coprod_{n\geq 0} C_n^\bullet$. It remains to observe that every complex $C_n^\bullet$ can be obtained from the complexes $B_0^\bullet$, …, $B_n^\bullet$ using the operations of shift and cone finitely many times. □

Lemma 8.7. *Let A and B be additive categories, $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be a functor, and $G\colon \mathsf{B} \longrightarrow \mathsf{A}$ be a functor right adjoint to F. Denote the adjunction morphisms by*

$\mu\colon FG \longrightarrow \mathrm{Id}_{\mathsf{B}}$ *and* $\eta\colon \mathrm{Id}_{\mathsf{A}} \longrightarrow GF$. *Then, for any object* $B \in \mathsf{B}$, *there is a "bar-complex" in* B

$$\cdots \longrightarrow B_n = (FG)^{n+1}(B) \longrightarrow \cdots \longrightarrow FGFG(B) \longrightarrow FG(B) \longrightarrow B \longrightarrow 0$$

whose differential $d_n\colon B_n \longrightarrow B_{n-1}$ *is the alternating sum of* $n+1$ *morphisms*

$$d_n = \sum\nolimits_{i=0}^{n}(-1)^i\,(FG)^i\mu(FG)^{n-i}(B)\colon (FG)^{n+1}(B) \longrightarrow (FG)^n(B).$$

Applying the functor G *to the above complex produces a contractible complex in* A *with the contracting homotopy* $h_n = \eta(GF)^nG(B)\colon (GF)^nG(B) \longrightarrow (GF)^{n+1}G(B)$.

Proof. The proof is a straightforward computation using the identity $(G\mu)\circ(\eta G) = \mathrm{id}_{GFG}$. ☐

Proposition 8.8. *Let* $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ *be a flat affine morphism of reasonable ind-schemes. Then for any complex* $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ *of* $\mathfrak{X}$*-flat pro-quasi-coherent pro-sheaves on* $\mathfrak{Y}$ *there exists a relatively homotopy flat complex* $\mathfrak{F}^\bullet$ *of flat pro-quasi-coherent pro-sheaves on* $\mathfrak{Y}$ *together with a morphism of complexes* $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ *whose cone is acyclic in* $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$.

Proof. Notice first of all that $\mathfrak{Y}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, since the morphism π assumed to be flat. In the context of Lemma 8.7, put $\mathsf{A} = \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$, $\mathsf{B} = \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$, $F = \pi^*\colon \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat}) \longrightarrow \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat}) \subset \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$, and $G = \pi_*\colon \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{C}(\mathfrak{X}\text{–}\mathsf{flat})$. Applying the construction of the lemma to the object $B = \mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) = \mathsf{B}$, we obtain a bicomplex

$$\cdots \longrightarrow (\pi^*\pi_*)^{n+1}(\mathfrak{P}^\bullet) \longrightarrow \cdots \longrightarrow \pi^*\pi_*\pi^*\pi_*\mathfrak{P}^\bullet \longrightarrow \pi^*\pi_*\mathfrak{P}^\bullet \longrightarrow \mathfrak{P}^\bullet \longrightarrow 0$$

in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. The differentials are the alternating sums of the maps induced by the adjunction morphism $\pi^*\pi_* \longrightarrow \mathrm{Id}$.

Let $\mathfrak{F}^\bullet$ be the total complex of the truncated bicomplex $\cdots \longrightarrow \pi^*\pi_*\pi^*\pi_*\mathfrak{P}^\bullet \longrightarrow \pi^*\pi_*\mathfrak{P}^\bullet \longrightarrow 0$, constructed by taking infinite coproducts along the diagonals. By the last assertion of Lemma 8.7, the cone $\mathfrak{H}^\bullet$ of the morphism $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ is a complex in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ which becomes contractible after applying the direct image functor $\pi_*\colon \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \longrightarrow \mathfrak{X}\text{–}\mathsf{flat}$. By Lemma 7.14, it follows that the complex $\mathfrak{H}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, as desired.

On the other hand, by Lemma 8.6, the complex $\mathfrak{F}^\bullet$ can be obtained from the complexes $(\pi^*\pi_*)^{n+1}(\mathfrak{P}^\bullet)$, $n \geq 0$, using the operations of shift, cone, countable coproduct, and the passage to a homotopy equivalent complex. The complex $\pi_*(\pi^*\pi_*)^n(\mathfrak{P}^\bullet)$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$; hence, by Lemma 8.5(b), the complex $(\pi^*\pi_*)^{n+1}(\mathfrak{P}^\bullet)$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. According to Lemma 8.5(a), it follows that the complex $\mathfrak{F}^\bullet$ is relatively homotopy flat. ☐

Now we have to work out the torsion sheaf side of the story. We will say that a complex $\mathscr{G}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is *homotopy* $\mathfrak{Y}/\mathfrak{X}$*-flat* if, for any complex $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ which is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, the complex $\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ has the property that its direct image $\pi_*(\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors.

Lemma 8.9. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Then*

(a) *the homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complexes form a full triangulated subcategory closed under coproducts in $\mathsf{K}(\mathfrak{Y}$–$\mathsf{tors})$;*

(b) *for any complex $\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{X}$–$\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{X}$, the complex $\pi^*\mathscr{M}^\bullet \in \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is homotopy $\mathfrak{Y}/\mathfrak{X}$-flat.*

Proof. Part (a): clearly, any complex in $\mathfrak{Y}$–tors which is homotopy equivalent to a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat compex is also homotopy $\mathfrak{Y}/\mathfrak{X}$-flat. Furthermore, for any complex $\mathfrak{P}^\bullet \in \mathsf{K}(\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat})$, the functor $\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} - : \mathsf{K}(\mathfrak{Y}$–$\mathsf{tors}) \longrightarrow \mathsf{K}(\mathfrak{Y}$–$\mathsf{tors})$ is a triangulated functor preserving coproducts, and the class of all complexes in $\mathfrak{Y}$–tors whose direct images are coacyclic in $\mathfrak{X}$–tors is closed under coproducts. Therefore, the class of all homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complexes is closed under shifts, cones, and coproducts in $\mathsf{K}(\mathfrak{Y}$–$\mathsf{tors})$. These arguments do not need the assumption that $\mathfrak{X}$ is ind-Noetherian yet. Part (b) follows immediately from Lemma 8.4 (which depends on the ind-Noetherianity assumption). □

Proposition 8.10. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Then for any complex $\mathscr{N}^\bullet \in \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ there exists a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex $\mathscr{G}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ together with a morphism of complexes $\mathscr{G}^\bullet \longrightarrow \mathscr{N}^\bullet$ whose cone has the property that its direct image is coacyclic in $\mathfrak{X}$–tors.*

Proof. In the context of Lemma 8.7, put $\mathsf{A} = \mathsf{C}(\mathfrak{X}$–$\mathsf{tors})$, $\mathsf{B} = \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors})$, $F = \pi^*\colon \mathsf{C}(\mathfrak{X}$–$\mathsf{tors}) \longrightarrow \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors})$, and $G = \pi_*\colon \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors}) \longrightarrow \mathsf{C}(\mathfrak{X}$–$\mathsf{tors})$. Applying the construction of the lemma to the object $B = \mathscr{N}^\bullet \in \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors}) = \mathsf{B}$, we obtain a bicomplex

$$\cdots \longrightarrow (\pi^*\pi_*)^{n+1}(\mathscr{N}^\bullet) \longrightarrow \cdots \longrightarrow \pi^*\pi_*\pi^*\pi_*\mathscr{N}^\bullet \longrightarrow \pi^*\pi_*\mathscr{N}^\bullet \longrightarrow \mathscr{N}^\bullet \longrightarrow 0$$

in $\mathfrak{Y}$–tors. The differentials are the alternating sums of the maps induced by the adjunction morphism $\pi^*\pi_* \longrightarrow \mathrm{Id}$.

Let $\mathscr{G}^\bullet$ be the total complex of the truncated bicomplex $\cdots \longrightarrow \pi^*\pi_*\pi^*\pi_*\mathscr{N}^\bullet \longrightarrow \pi^*\pi_*\mathscr{N}^\bullet \longrightarrow 0$, constructed by taking infinite coproducts along the diagonals. By the last assertion of Lemma 8.7, the cone of the morphism $\mathscr{G}^\bullet \longrightarrow \mathscr{N}^\bullet$ is a complex in $\mathfrak{Y}$–tors which becomes contractible (hence coacyclic) after applying the direct image functor $\pi_*\colon \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{X}$–$\mathsf{tors}$.

On the other hand, by Lemma 8.6, the complex $\mathscr{G}^\bullet$ can be obtained from the complexes $(\pi^*\pi_*)^{n+1}(\mathscr{N}^\bullet)$, $n \geq 0$, using the operations of shift, cone, countable coproduct, and the passage to a homotopy equivalent complex. By Lemma 8.9(b), the complex $(\pi^*\pi_*)^{n+1}(\mathscr{N}^\bullet)$ is a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. According to Lemma 8.9(a), it follows that the complex $\mathscr{G}^\bullet$ is homotopy $\mathfrak{Y}/\mathfrak{X}$-flat. □

8.3. **Left derived tensor products for pro-sheaves flat over a base.** Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of reasonable ind-schemes. The left derived functor of tensor product of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$

$$\otimes^{\mathfrak{Y},\mathbb{L}}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \tag{36}$$

is constructed in the following way.

Let $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ be two complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Using Proposition 8.8, choose two morphisms of complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ and $\mathfrak{G}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ such that the cones of both morphisms are acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, and both the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ are relatively homotopy flat complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Then, by condition (i) in the definition of a relatively homotopy flat complex, both the induced morphisms

$$\mathfrak{P}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet \longleftarrow \mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet \longrightarrow \mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{Q}^\bullet$$

have cones acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. So we put

$$\mathfrak{P}^\bullet \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{Q}^\bullet = \mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet \simeq \mathfrak{P}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet \simeq \mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{Q}^\bullet \in \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}).$$

Using the definition of a relatively homotopy flat complex again, the complex $\mathfrak{P}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ whenever the complex $\mathfrak{P}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, and the complex $\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{Q}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ whenever the complex $\mathfrak{Q}^\bullet$ is. So the derived functor $\otimes^{\mathfrak{Y},\mathbb{L}}$ is well-defined. We refer to [40, Lemma 2.7] for an abstract formulation of this kind of construction of balanced derived functors of two arguments (which is applicable in a much more general context of two-sided derived functors).

Remark 8.11. If one is only interested in the derived functor $\otimes^{\mathfrak{Y},\mathbb{L}}$ defined above (and *not* in the derived functor $\otimes^{\mathbb{L}}_{\mathfrak{Y}}$, which we will define immediately below), then one can harmlessly drop condition (ii) from the definition of a relatively homotopy flat complex. Then the assumption that the ind-schemes $\mathfrak{X}$ and $\mathfrak{Y}$ are reasonable is not needed in the above construction.

Remark 8.12. Let us emphasize that the underived tensor product $\mathfrak{P} \otimes^{\mathfrak{Y}} \mathfrak{Q}$ of two $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{P}$ and $\mathfrak{Q}$ on $\mathfrak{Y}$ need *not* be $\mathfrak{X}$-flat. It is only the derived tensor product $\mathfrak{P}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{Q}^\bullet$ of two complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet$ that is well-defined as an object of the derived category of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$. This is the reason why we had to assume our relatively homotopy flat complexes to be complexes of flat (and not just $\mathfrak{X}$-flat) pro-quasi-coherent pro-sheaves in the definition given in Section 8.2.

This subtlety was overlooked in the exposition in [47, Section 6]. In the context of [47], a commutative ring homomorphism $A \longrightarrow R$ was considered, with the assumption that R is a flat A-module. Then the tensor product of two A-flat R-modules, taken over R, need not be A-flat. It is only the tensor product of an R-flat R-module and an A-flat R-module that is always A-flat. To correct the mistake, one needs to include the assumption of termwise flatness over R into the definition of a "relatively homotopy R-flat complex" in the proof of [47, Proposition 6.3] and the formulation of [47, Lemma 6.4].

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. The left derived functor of tensor product of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves and quasi-coherent torsion sheaves on $\mathfrak{Y}$

$$\otimes_{\mathfrak{Y}}^{\mathbb{L}}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{D}_{\mathfrak{X}}^{\mathsf{si}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}_{\mathfrak{X}}^{\mathsf{si}}(\mathfrak{Y}\text{–}\mathsf{tors}) \tag{37}$$

is constructed in the following way.

Let $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves and $\mathscr{N}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Using Proposition 8.8, choose a morphism of complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ whose cone is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$, while $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Using Proposition 8.10, choose a morphism of complexes of quasi-coherent torsion sheaves $\mathscr{G}^\bullet \longrightarrow \mathscr{N}^\bullet$ whose cone has the property that its direct image is coacyclic in $\mathfrak{X}$–tors, while $\mathscr{G}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ is a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then, by condition (ii) in the definition of a relatively homotopy flat complex, and by the definition of a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex, both the induced morphisms

$$\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet \longleftarrow \mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet \longrightarrow \mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{N}^\bullet$$

have cones whose direct images are coacyclic in $\mathfrak{X}$–tors. So we put

$$\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}}^{\mathbb{L}} \mathscr{N}^\bullet = \mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet \simeq \mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet \simeq \mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{N}^\bullet \in \mathsf{D}_{\mathfrak{X}}^{\mathsf{si}}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

Using the definition of a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex again, the complex $\pi_*(\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–flat whenever the complex $\mathfrak{P}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$. Using condition (ii) from the definition of a relatively homotopy flat complex, the complex $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{N}^\bullet)$ is coacyclic in $\mathfrak{X}$–flat whenever the complex $\pi_*\mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}$–flat. Thus the derived functor $\otimes_{\mathfrak{Y}}^{\mathbb{L}}$ is well-defined. This construction of a derived functor of two arguments is also a particular case of [40, Lemma 2.7].

The derived functor $\otimes^{\mathfrak{Y},\mathbb{L}}$ (36) defines an (associative, commutative, and unital) tensor triangulated category structure on the derived category $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ of the exact category of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. The "pro-structure pro-sheaf" $\mathfrak{O}_{\mathfrak{Y}} \in \mathfrak{Y}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \subset \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ is the unit object. (Notice that the one-term complex $\mathfrak{O}_{\mathfrak{Y}}$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$.)

The derived functor $\otimes_{\mathfrak{Y}}^{\mathbb{L}}$ (37) defines a structure of triangulated module category over the tensor triangulated category $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ on the $\mathfrak{Y}/\mathfrak{X}$-semiderived category $\mathsf{D}_{\mathfrak{X}}^{\mathsf{si}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$.

8.4. **Construction of semitensor product.** In this section, $\mathfrak{X}$ is an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$, and $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is a flat affine morphism of ind-schemes.

The following lemma may help the reader feel more comfortable.

Lemma 8.13. (a) *In the assumptions above, condition* (ii) *from the definition of a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves in Section 8.2 implies condition* (i)*.*

(b) *In the same assumptions, let* $\mathfrak{F}^\bullet$ *be a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on* $\mathfrak{Y}$*. Then the complex* $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet$ *of quasi-coherent torsion sheaves on* $\mathfrak{Y}$ *is homotopy* $\mathfrak{Y}/\mathfrak{X}$*-flat.*

Proof. Both the assertions are based on the associativity property of the action of the tensor category $\mathsf{C}(\mathfrak{Y}\text{–pro})$ in its module category $\mathsf{C}(\mathfrak{Y}\text{–tors})$.

Part (a): let $\mathfrak{G}^\bullet\in\mathsf{C}(\mathfrak{Y}\text{–flat})$ be a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ satisfying (ii), and let $\mathfrak{P}^\bullet\in\mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$. Then, according to the proof of Theorem 7.15 in Section 7.3, the complex of quasi-coherent torsion sheaves $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet$ has the property that its direct image $\pi_*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$ is coacyclic (in fact, contractible) in $\mathfrak{X}\text{–tors}$. By condition (ii), it follows that the complex

$$(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet\simeq\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}(\mathfrak{P}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet)$$

has the same property, i. e., the compex $\pi_*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet)$ is coacyclic in $\mathfrak{X}\text{–tors}$. Now the assertion of Theorem 7.15 implies that the complex $\mathfrak{P}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$, since the corresponding object vanishes in $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$.

Part (b): let $\mathfrak{Q}^\bullet\in\mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})$ a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$. Then we have

$$(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet)\otimes_{\mathfrak{Y}}\mathfrak{Q}^\bullet\simeq\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}(\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{Q}^\bullet)$$

in $\mathsf{C}(\mathfrak{Y}\text{–tors})$. By condition (i), the complex $\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{Q}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$. According to the proof of Theorem 7.15, it follows that the complex $\pi_*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{Q}^\bullet)$ is coacyclic in $\mathfrak{X}\text{–tors}$, as desired. ☐

The triangulated equivalence $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\longrightarrow\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$ is an equivalence of module categories over the tensor category $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})$. Indeed, let $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet$ be two complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, and let $\mathfrak{F}^\bullet\longrightarrow\mathfrak{P}^\bullet$ and $\mathfrak{G}^\bullet\longrightarrow\mathfrak{Q}^\bullet$ be two morphisms of complexes with the cones acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–flat}$ and relatively homotopy flat complexes of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ on $\mathfrak{Y}$. Then the desired natural isomorphism

$$(\mathfrak{P}^\bullet\otimes^{\mathfrak{Y},\mathbb{L}}\mathfrak{Q}^\bullet)\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet\;\simeq\;\mathfrak{P}^\bullet\otimes_{\mathfrak{Y}}^{\mathbb{L}}(\mathfrak{Q}^\bullet\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet)$$

in the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$ is represented by any one of the associativity isomorphisms

$$(\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{Q}^\bullet)\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet\;\simeq\;\mathfrak{F}^\bullet\otimes_{\mathfrak{Y}}(\mathfrak{Q}^\bullet\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet),$$
$$(\mathfrak{P}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet)\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet\;\simeq\;\mathfrak{P}^\bullet\otimes_{\mathfrak{Y}}(\mathfrak{G}^\bullet\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet),$$

or

$$(\mathfrak{F}^\bullet\otimes^{\mathfrak{Y}}\mathfrak{G}^\bullet)\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet\;\simeq\;\mathfrak{F}^\bullet\otimes_{\mathfrak{Y}}(\mathfrak{G}^\bullet\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet)$$

in the category of complexes $\mathsf{C}(\mathfrak{Y}\text{–tors})$. Notice that the complex $\mathfrak{G}^\bullet\otimes_{\mathfrak{Y}}\pi^*\mathscr{D}^\bullet$ is homotopy $\mathfrak{Y}/\mathfrak{X}$-flat by Lemma 8.13(b).

Using the triangulated equivalence $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–tors})$, we transfer the tensor structrue of the category $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})$ to the category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{X}\text{–tors})$. The resulting functor

$$\lozenge_{\pi^*\mathscr{D}^\bullet}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}), \tag{38}$$

defining a tensor triangulated category structure on the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$, is called the *semitensor product* of complexes of quasi-coherent torsion sheaves on $\mathfrak{Y}$ over the inverse image $\pi^*\mathscr{D}^\bullet$ of the dualizing complex $\mathscr{D}^\bullet$. The object $\pi^*\mathscr{D}^\bullet \in \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is the unit object of the tensor structure $\lozenge_{\pi^*\mathscr{D}^\bullet}$ on $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$, since $\pi^*\mathscr{D}^\bullet$ corresponds to the unit object $\mathfrak{O}_{\mathfrak{Y}} \in \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ under the equivalence of categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$.

Explicitly, let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet \in \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ be two complexes endowed with morphisms $\mathscr{M}^\bullet \longrightarrow \mathscr{K}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$ into complexes $\mathscr{K}^\bullet$ and $\mathscr{J}^\bullet \in \mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}})$ with cones whose direct images are coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. Then one has

$$\mathscr{M}^\bullet \lozenge_{\pi^*\mathscr{D}^\bullet} \mathscr{N}^\bullet = \pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} (\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{K}^\bullet) \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{J}^\bullet))$$
$$\simeq \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{K}^\bullet) \otimes^{\mathbb{L}}_{\mathfrak{Y}} \mathscr{N}^\bullet \simeq \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{J}^\bullet) \otimes^{\mathbb{L}}_{\mathfrak{Y}} \mathscr{M}^\bullet.$$

Let $\mathscr{F}^\bullet \longrightarrow \mathscr{M}^\bullet$ and $\mathscr{G}^\bullet \longrightarrow \mathscr{N}^\bullet$ be two morphisms in $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ whose cones' direct images are coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$, while $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complexes of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then the semiderived category object $\mathscr{M}^\bullet \lozenge_{\pi^*\mathscr{D}^\bullet} \mathscr{N}^\bullet \in \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is represented by any one of the two complexes

$$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{K}^\bullet) \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet \quad \text{or} \quad \mathscr{F}^\bullet \otimes_{\mathfrak{Y}} \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathscr{J}^\bullet)$$

of quasi-coherent torsion sheaves on $\mathfrak{Y}$.

8.5. **The ind-Artinian base example.** In this section we discuss flat affine morphisms of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is an ind-Artinian ind-scheme of ind-finite type over a field (as in Examples 5.4).

(1) Let $\mathscr{C}$ be a coassociative coalgebra over a field $\Bbbk$. A (*semiassociative, semiunital*) *semialgebra* $\mathscr{S}$ over $\mathscr{C}$ is defined as an associative, unital algebra object in the (associative, unital, noncommutative) tensor category of *$\mathscr{C}$-$\mathscr{C}$-bicomodules.*

Let $\mathscr{S}$ be a semialgebra over $\mathscr{C}$. A *left semimodule* over $\mathscr{S}$ is defined as a module object in the module category of left $\mathscr{C}$-comodules over the algebra object $\mathscr{S}$ in the tensor category of $\mathscr{C}$-$\mathscr{C}$-bicomodules. *Right semimodules* are defined similarly [40, Sections 0.3.2 and 1.3.1], [45, Section 2.6].

The category of left $\mathscr{S}$-semimodules, which we will denote by $\mathscr{S}\text{–}\mathsf{simod}$, is abelian whenever $\mathscr{S}$ is an injective right $\mathscr{C}$-comodule. In this case, $\mathscr{S}\text{–}\mathsf{simod}$ is a Grothendieck abelian category. The category of right $\mathscr{S}$-semimodules is denoted by $\mathsf{simod}\text{–}\mathscr{S}$.

(2) We are interested in *semicommutative* semialgebras, which are a particular case of (1). Let $\mathscr{C}$ be a cocommutative coalgebra over $\Bbbk$. Then, following Example 5.4(2), the category of $\mathscr{C}$-comodules $\mathscr{C}\text{–}\mathsf{comod}$ is an associative, commutative, and unital tensor category with respect to the cotensor product operation $\square_{\mathscr{C}}$. In fact, $\mathscr{C}\text{–}\mathsf{comod}$ is a tensor subcategory in the tensor category of $\mathscr{C}$-$\mathscr{C}$-bicomodules (consisting of those bicomodules in which the left and right $\mathscr{C}$-coactions agree).

A *semicommutative semialgebra* $\boldsymbol{\mathcal{S}}$ over $\mathcal{C}$ is defined as a commutative (associative, and unital) algebra object in the tensor category $\mathcal{C}$–comod. In other words, $\boldsymbol{\mathcal{S}}$ is an $\mathcal{C}$-comodule endowed with a *semiunit map* $\mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$ and a *semimultiplication map* $\boldsymbol{\mathcal{S}} \,\square_{\mathcal{C}}\, \boldsymbol{\mathcal{S}} \longrightarrow \boldsymbol{\mathcal{S}}$. Both the maps must be $\mathcal{C}$-comodule morphisms, and the usual associativity, commutativity, and unitality equations should be satisfied.

Over a semicommutative semialgebra $\boldsymbol{\mathcal{S}}$, there is no difference between left and right semimodules.

(3) More specifically, we are interested in *$\mathcal{C}$-injective* semicommutative semialgebras $\boldsymbol{\mathcal{S}}$ over $\mathcal{C}$. So the underlying $\mathcal{C}$-comodule of $\boldsymbol{\mathcal{S}}$ is assumed to be injective. Injective $\mathcal{C}$-comodules form a tensor subcategory in $\mathcal{C}$–comod.

For a cocommutative coalgebra $\mathcal{C}$, the category of $\mathcal{C}$-contramodules is also naturally an (associative, commutative, and unital) tensor category with respect to the operation of *contramodule tensor product* $\otimes^{\mathcal{C}} = \otimes^{\mathcal{C}^*}$ [43, Section 1.6]. The free $\mathcal{C}$-contramodule with one generator $\mathcal{C}^*$ is the unit object. The full subcategory of projective $\mathcal{C}$-contramodules $\mathcal{C}$–$\mathsf{contra}_{\mathsf{proj}} \subset \mathcal{C}$–$\mathsf{contra}$ is a tensor subcategory.

The equivalence between the additive categories of injective $\mathcal{C}$-comodules and projective $\mathcal{C}$-contramodules $\mathcal{C}$–$\mathsf{comod}_{\mathsf{inj}} \simeq \mathcal{C}$–$\mathsf{contra}_{\mathsf{proj}}$ (see formula (16) in Example 5.4(6)) is an equivalence of tensor categories.

(4) Let $\mathfrak{X} = \operatorname{Spi} \mathcal{C}^*$ be the ind-Artinian ind-scheme corresponding to the coalgebra $\mathcal{C}$, as in Examples 1.5(2), 4.1(2), and 5.4(5). According to Example 5.4(6), there is an equivalence of additive categories $\mathfrak{X}$–$\mathsf{flat} \simeq \mathcal{C}$–$\mathsf{contra}_{\mathsf{proj}}$. Moreover, similarly to Example 3.8(4) (cf. Example 5.9), this is an equivalence of tensor categories.

We are interested in transformations of commutative algebra objects under the equivalences of tensor categories above. Let $\boldsymbol{\mathcal{S}}$ be a $\mathcal{C}$-injective semicommutative semialgebra over $\mathcal{C}$, let $\boldsymbol{\mathfrak{S}}$ be the corresponding commutative algebra object in the tensor category $\mathcal{C}$–$\mathsf{contra}_{\mathsf{proj}}$, and let $\boldsymbol{\mathfrak{A}}$ be the corresponding commutative algebra object in the tensor category $\mathfrak{X}$–flat. Then the anti-equivalence of categories from Proposition 3.12 assigns to $\boldsymbol{\mathfrak{A}}$ an ind-scheme $\boldsymbol{\mathfrak{Y}} = \operatorname{Spi}_{\mathfrak{X}} \boldsymbol{\mathfrak{A}}$ together with a flat affine morphism of ind-schemes $\pi\colon \boldsymbol{\mathfrak{Y}} \longrightarrow \mathfrak{X}$. Let us describe the ind-scheme $\boldsymbol{\mathfrak{Y}}$ and the morphism π more explicitly.

For any algebra object $\mathfrak{S}$ in the tensor category $\mathcal{C}$–contra, precomposing the multiplication morphism $\mathfrak{S} \otimes^{\mathcal{C}^*} \mathfrak{S} \longrightarrow \mathfrak{S}$ with the natural map $\mathfrak{S} \otimes_{\mathcal{C}^*} \mathfrak{S} \longrightarrow \mathfrak{S} \otimes^{\mathcal{C}^*} \mathfrak{S}$ allows to define the underlying $\mathcal{C}^*$-algebra structure on $\mathfrak{S}$. Moreover, any projective $\mathcal{C}$-contramodule $\mathfrak{F}$ has a natural underlying structure of a complete, separated topological module over a topological ring $\mathcal{C}^*$; for a free $\mathcal{C}$-contramodule $\mathfrak{F} = \operatorname{Hom}_{\Bbbk}(\mathcal{C}, V)$ spanned by a $\Bbbk$-vector space V, this is the usual topology on the Hom space of infinite-dimensional vector spaces. For an algebra object $\mathfrak{S}$ in the tensor category $\mathcal{C}$–$\mathsf{contra}_{\mathsf{proj}}$, this topology makes $\mathfrak{S}$ a complete, separated topological ring with a base of neighborhoods of zero formed by open (two-sided) ideals. The unit morphism of $\mathfrak{S}$ provides a continuous ring homomorphism $\mathcal{C}^* \longrightarrow \mathfrak{S}$.

In the situation at hand, $\boldsymbol{\mathfrak{S}} = \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}, \boldsymbol{\mathcal{S}}) = \operatorname{Hom}_{\mathcal{C}^*}(\mathcal{C}, \boldsymbol{\mathcal{S}})$ is the $\Bbbk$-vector space of $\mathcal{C}$-comodule (or equivalently, $\mathcal{C}^*$-module) homomorphisms $\mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$. The topology on $\mathfrak{S}$ is the *finite topology* of the Hom module: the annihilators of finite-dimensional subspaces (equivalently, finite-dimensional subcoalgebras) in $\mathcal{C}$ form a base of neighborhoods of zero in $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\boldsymbol{\mathcal{S}})$. The semiunit map $\mathcal{C} \longrightarrow \boldsymbol{\mathcal{S}}$ induces the unit map $\mathcal{C}^* \simeq \mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\mathcal{C}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\boldsymbol{\mathcal{S}})$, which is a continuous ring homomorphism $\mathcal{C}^* \longrightarrow \mathfrak{S}$. Then one has $\mathfrak{Y} = \mathrm{Spi}\,\mathfrak{S}$ in the notation of Example 1.6(1); the flat affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ corresponds to the homomorphism of topological rings $\mathcal{C}^* \longrightarrow \mathfrak{S}$.

For any cocommutative coalgebra $\mathcal{C}$ over $\Bbbk$ and the corresponding ind-Artinian ind-scheme $\mathfrak{X} = \mathrm{Spi}\,\mathcal{C}^*$, we have constructed a natural anti-equivalence between the category of $\mathcal{C}$-injective semicommutative semalgebras $\boldsymbol{\mathcal{S}}$ over $\mathcal{C}$ and the category of ind-schemes $\mathfrak{Y}$ endowed with a flat affine morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$.

(5) We keep the notation of (4). According to Proposition 3.13(b), the abelian category $\mathfrak{Y}\text{–}\mathsf{tors}$ is equivalent to the category of module objects in the module category $\mathfrak{X}\text{–}\mathsf{tors}$ over the algebra object $\mathfrak{A}$ in the tensor category $\mathfrak{X}\text{–}\mathsf{flat}$. Following Section 2.4(4), we have an equivalence of categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathcal{C}\text{–}\mathsf{comod}$. Hence the category $\mathfrak{Y}\text{–}\mathsf{tors}$ is equivalent to the category of module objects in the module category $\mathcal{C}\text{–}\mathsf{comod}$ over the algebra object $\mathfrak{S}$ in the tensor category $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$. Here the action of $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ in $\mathcal{C}\text{–}\mathsf{comod}$ is given by the contratensor product functor $\odot_{\mathcal{C}}$ (see the last paragraph of Example 5.4(6)).

The equivalence of tensor categories $\mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}} \simeq \mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ transforms the action of $\mathcal{C}\text{–}\mathsf{comod}_{\mathsf{inj}}$ in $\mathcal{C}\text{–}\mathsf{comod}$ (by the cotensor product) into the action of $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ in $\mathcal{C}\text{–}\mathsf{comod}$ (by the contratensor product). This is clear from the associativity isomorphism connecting the cotensor and contratensor products [40, Proposition 5.2.1], [45, Proposition 3.1.1], which was already mentioned in Example 5.4(8). Taken together, the constructions above combine into a natural equivalence between the abelian category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ and the abelian category of $\boldsymbol{\mathcal{S}}$-semimodules, $\mathfrak{Y}\text{–}\mathsf{tors} \simeq \boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}$.

A quasi-coherent torsion sheaf on $\mathfrak{Y}$ is $\mathfrak{X}$-injective if and only if the corresponding $\boldsymbol{\mathcal{S}}$-semimodule is injective *as a $\mathcal{C}$-comodule.* We will denote the full subcategory of semimodules whose underlying comodules are injective by $\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}_{\mathcal{C}\text{-}\mathsf{inj}} \subset \boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}$. So the equivalence of abelian categories $\mathfrak{Y}\text{–}\mathsf{tors} \simeq \boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}$ restricts to an equivalence of full subcategories $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}} \simeq \boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}_{\mathcal{C}\text{-}\mathsf{inj}}$. The latter one is obviously an equivalence of exact categories (with the exact category structures inherited from the ambient abelian categories).

(6) According to Proposition 3.13(a) and the discussion in Section 7.2, the exact category $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ is equivalent to the exact category of module objects over the algebra object $\mathfrak{A}$ in the tensor category $\mathfrak{X}\text{–}\mathsf{flat}$. As the tensor category $\mathfrak{X}\text{–}\mathsf{flat}$ is equivalent to the tensor category $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ by (4), it follows that the exact category $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ is equivalent to the exact category of module objects over the algebra object $\mathfrak{S}$ in the tensor category $\mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$. Here the exact structure on $\mathfrak{X}\text{–}\mathsf{flat} \simeq \mathcal{C}\text{–}\mathsf{contra}_{\mathsf{proj}}$ is split, but the exact structure on the category of module objects is not; rather, a short sequence of module objects is exact if and only if it becomes split exact after the module structures are forgotten.

One can see that specifying a module structure over $\mathfrak{S}$ on a given projective $\mathscr{C}$-contramodule $\mathfrak{F}$ is equivalent to specifying a $\mathscr{S}$-*semicontramodule* structure on $\mathfrak{F}$ (see [40, Sections 0.3.5 and 3.3.1] or [45, Section 2.6] for the definition). So the exact category $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ is equivalent to the exact category of $\mathscr{C}$-projective $\mathscr{S}$-semicontramodules. Semicontramodules over an $\mathscr{C}$-injective semialgebra $\mathscr{S}$ form an abelian category $\mathscr{S}$–sicntr; we will denote the full subcategory of $\mathscr{C}$-injective semicontramodules by $\mathscr{S}$–$\mathsf{sicntr}_{\mathscr{C}\text{-}\mathsf{proj}} \subset \mathscr{S}$–$\mathsf{sicntr}$. The exact structure on $\mathscr{S}$–$\mathsf{sicntr}_{\mathscr{C}\text{-}\mathsf{proj}}$ is inherited from the abelian category $\mathscr{S}$–sicntr. So we have an equivalence of exact categories $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \simeq \mathscr{S}$–$\mathsf{sicntr}_{\mathscr{C}\text{-}\mathsf{proj}}$.

(7) It is explained in [54, Section 10.3] that the datum of an $\mathscr{S}$-semicontramodule structure on a given vector space (or $\mathscr{C}$-contramodule) is equivalent to the datum of a contramodule structure over the topological ring $\mathfrak{S}$, that is $\mathscr{S}$–$\mathsf{sicntr} \simeq \mathfrak{S}$–$\mathsf{contra}$. It follows that the exact category $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ is equivalent to the exact category of $\mathfrak{S}$-contramodules which are projective *as contramodules over the topological ring* $\mathscr{C}^*$, i. e., $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \simeq \mathfrak{S}$–$\mathsf{contra}_{\mathscr{C}^*\text{-}\mathsf{proj}}$. The latter equivalence restricts to an equivalence between the category of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ and the category of flat $\mathfrak{S}$-contramodules (in the sense of [52, Section 2]), $\mathfrak{Y}$–$\mathsf{flat} \simeq \mathfrak{S}$–$\mathsf{flat}$.

Furthermore, the datum of an $\mathscr{S}$-semimodule structure on a given vector space (or $\mathscr{C}$-comodule) is equivalent to the datum of a discrete $\mathfrak{S}$-module structure, $\mathscr{S}$–$\mathsf{simod} \simeq \mathfrak{S}$–$\mathsf{discr}$ [54, Remark 10.9]. Consequently, we have a natural equivalence of abelian categories $\mathfrak{Y}$–$\mathsf{tors} \simeq \mathfrak{S}$–$\mathsf{discr}$.

The equivalences of categories $\mathfrak{Y}$–$\mathsf{flat} \simeq \mathfrak{S}$–$\mathsf{flat}$ and $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \simeq \mathfrak{S}$–$\mathsf{contra}_{\mathscr{C}^*\text{-}\mathsf{proj}}$ from (7) transform the tensor product functor $\otimes^{\mathfrak{Y}} \colon \mathfrak{Y}$–$\mathsf{flat} \times \mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \longrightarrow \mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ into the contramodule tensor product functor $\otimes^{\mathfrak{S}} \colon \mathfrak{S}$–$\mathsf{contra} \times \mathfrak{S}$–$\mathsf{contra} \longrightarrow \mathfrak{S}$–$\mathsf{contra}$ (as defined in [43, Section 1.6]), restricted to the full subcategories $\mathfrak{S}$–$\mathsf{flat} \subset \mathfrak{S}$–$\mathsf{contra}_{\mathscr{C}^*\text{-}\mathsf{proj}} \subset \mathfrak{S}$–$\mathsf{contra}$. The equivalences of categories $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \simeq \mathfrak{S}$–$\mathsf{contra}_{\mathscr{C}^*\text{-}\mathsf{proj}}$ and $\mathfrak{Y}$–$\mathsf{tors} \simeq \mathfrak{S}$–$\mathsf{discr}$ from (7) transform the tensor product functor $\otimes_{\mathfrak{Y}} \colon \mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \times \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{Y}$–$\mathsf{tors}$ into the contratensor product functor $\odot_{\mathfrak{S}} \colon \mathfrak{S}$–$\mathsf{contra} \times \mathfrak{S}$–$\mathsf{discr} \longrightarrow \mathfrak{S}$–$\mathsf{discr}$ restricted to $\mathfrak{S}$–$\mathsf{contra}_{\mathscr{C}^*\text{-}\mathsf{proj}} \subset \mathfrak{S}$–$\mathsf{contra}$ (see the discussion and references in Example 3.8(3)).

The equivalences of categories $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \simeq \mathscr{S}$–$\mathsf{sicntr}_{\mathscr{C}\text{-}\mathsf{proj}}$ and $\mathfrak{Y}$–$\mathsf{tors} \simeq \mathscr{S}$–$\mathsf{simod}$ from (6) and (5) transform the tensor product functor $\otimes_{\mathfrak{Y}} \colon \mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat} \times \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{Y}$–$\mathsf{tors}$ into the functor of *contratensor product of semimodules and semicontramodules* $\circledcirc_{\mathscr{S}} \colon \mathscr{S}$–$\mathsf{sicntr} \times \mathscr{S}$–$\mathsf{simod} \longrightarrow \mathscr{S}$–$\mathsf{simod}$ (constructed in [40, Sections 0.3.7 and 6.1]), restricted to the full subcategory $\mathscr{S}$–$\mathsf{sicntr}_{\mathscr{C}\text{-}\mathsf{proj}} \subset \mathscr{S}$–$\mathsf{sicntr}$.

(8) As explained in Example 5.4(5), the $\mathscr{C}$-comodule $\mathscr{C}$ corresponds to a one-term dualizing complex of injective quasi-coherent torsion sheaves $\mathscr{D}^\bullet = \mathscr{C}$ on the ind-Artinian ind-scheme $\mathfrak{X} = \operatorname{Spi} \mathscr{C}^*$.

Following the proof of Theorem 7.15 specialized to the particular case of a one-term dualizing complex of injectives $\mathscr{D}^\bullet = \mathscr{C}$ on $\mathfrak{X}$, one can see that there is an equivalence between the exact categories of $\mathfrak{X}$-injective quasi-coherent torsion sheaves and $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, provided by the mutually inverse functors $\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{C},-)$ and $\pi^*\mathscr{C}\otimes_{\mathfrak{Y}}-$,

$$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{C},-)\colon \mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}} \simeq \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} :\pi^*\mathscr{C}\otimes_{\mathfrak{Y}}-. \tag{39}$$

The equivalence of exact categories $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}} \simeq \boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}_{\mathscr{C}\text{-inj}}$ and $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat} \simeq \boldsymbol{\mathscr{S}}\text{–}\mathsf{sicntr}_{\mathscr{C}\text{-proj}}$ from items (5) and (6) form a commutative square diagram with the equivalence (39) and the equivalence of exact categories

$$\mathrm{Hom}_{\boldsymbol{\mathscr{S}}}(\boldsymbol{\mathscr{S}},-)\colon \boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}_{\mathscr{C}\text{-inj}} \simeq \boldsymbol{\mathscr{S}}\text{–}\mathsf{sicntr}_{\mathscr{C}\text{-proj}} :\boldsymbol{\mathscr{S}}\circledcirc_{\boldsymbol{\mathscr{S}}}-, \tag{40}$$

which was constructed in [40, Sections 0.3.7 and 6.2] and discussed in [45, Section 3.5]. Here $\mathrm{Hom}_{\boldsymbol{\mathscr{S}}} = \mathrm{Hom}_{\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}}$ denoted the vector space of morphisms in the abelian category of $\boldsymbol{\mathscr{S}}$-semimodules, while $\circledcirc_{\boldsymbol{\mathscr{S}}}$ is the contratensor product functor mentioned in (7) above. Notice that the quasi-coherent torsion sheaf $\pi^*\mathscr{C} = \pi^*\mathscr{D}^\bullet$ on $\mathfrak{Y}$ corresponds to the $\boldsymbol{\mathscr{S}}$-semimodule $\boldsymbol{\mathscr{S}}$ under the equivalence of categories $\mathfrak{Y}\text{–}\mathsf{tors} \simeq \boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}$ (or $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}} \simeq \boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}_{\mathscr{C}\text{-inj}}$).

(9) Given a semiassociative semialgebra $\boldsymbol{\mathscr{S}}$ over a coassociative coalgebra $\mathscr{C}$, the *semitensor product* $\boldsymbol{\mathscr{M}}\lozenge_{\boldsymbol{\mathscr{S}}}\boldsymbol{\mathscr{N}}$ of a right $\boldsymbol{\mathscr{S}}$-semimodule $\boldsymbol{\mathscr{M}}$ and a left $\boldsymbol{\mathscr{S}}$-semimodule $\boldsymbol{\mathscr{N}}$ is the $\Bbbk$-vector space constructed as the cokernel of the difference of the natural pair of maps

$$\boldsymbol{\mathscr{M}}\,\square_{\mathscr{C}}\,\boldsymbol{\mathscr{S}}\,\square_{\mathscr{C}}\,\boldsymbol{\mathscr{N}} \rightrightarrows \boldsymbol{\mathscr{M}}\,\square_{\mathscr{C}}\,\boldsymbol{\mathscr{N}}.$$

Here one map is induced by the right semiaction map $\boldsymbol{\mathscr{M}}\,\square_{\mathscr{C}}\,\boldsymbol{\mathscr{S}}\longrightarrow\boldsymbol{\mathscr{M}}$ and the other one by the left semiaction map $\boldsymbol{\mathscr{S}}\,\square_{\mathscr{C}}\,\boldsymbol{\mathscr{N}}\longrightarrow\boldsymbol{\mathscr{N}}$ [40, Sections 0.3.2 and 1.4.1–2] (cf. the definition of the cotensor product $\square_{\mathscr{C}}$ in Example 5.4(1) above).

Let $\boldsymbol{\mathscr{S}}$ be a semicommutative semialgebra over a cocommutative coalgebra $\mathscr{C}$; assume that $\boldsymbol{\mathscr{S}}$ is an injective $\mathscr{C}$-comodule. Then the semitensor product $\boldsymbol{\mathscr{M}}\lozenge_{\boldsymbol{\mathscr{S}}}\boldsymbol{\mathscr{N}}$ of two $\boldsymbol{\mathscr{S}}$-semimodules $\boldsymbol{\mathscr{M}}$ and $\boldsymbol{\mathscr{N}}$ has a natural $\boldsymbol{\mathscr{S}}$-semimodule structure [40, Section 1.4.4]. The semitensor product operation $\lozenge_{\boldsymbol{\mathscr{S}}}$ on the category $\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}$ is commutative and unital; the $\boldsymbol{\mathscr{S}}$-semimodule $\boldsymbol{\mathscr{S}}$ is the unit object.

However, one needs to impose some additional assumptions in order to make sure that the semitensor product is associative. The semitensor product of any three $\mathscr{C}$-injective $\boldsymbol{\mathscr{S}}$-semimodules is associative [40, Proposition 1.4.4(a)], but the full subcategory $\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}_{\mathscr{C}\text{-inj}}\subset\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}$ is *not* preserved by $\lozenge_{\boldsymbol{\mathscr{S}}}$, generally speaking. The full subcategory of so-called *semiflat* $\boldsymbol{\mathscr{S}}$-semimodules (defined in [40, Section 1.4.2]) is a commutative, associative, and unital tensor category with respect to the semitensor product over $\boldsymbol{\mathscr{S}}$.

(10) For a semiassociative semialgebra $\boldsymbol{\mathscr{S}}$ over a coassociative coalgebra $\mathscr{C}$, the double-sided derived functor of semitensor product

$$\mathrm{SemiTor}_{\boldsymbol{\mathscr{S}}}\colon \mathsf{D}^{\mathsf{si}}(\mathsf{simod}\text{–}\boldsymbol{\mathscr{S}})\times\mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod})\longrightarrow\mathsf{D}(\Bbbk\text{–}\mathsf{vect})$$

is constructed in [40, Section 2.7]. Here $\mathsf{D}^{\mathsf{si}}(\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod}) = \mathsf{D}^{\mathsf{si}}_{\mathscr{C}}(\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod})$ is the semiderived (or the "semicoderived") category of left $\boldsymbol{\mathscr{S}}$-semimodules *relative to* $\mathscr{C}$, i. e., the triangulated quotient category of the homotopy category $\mathsf{K}(\boldsymbol{\mathscr{S}}\text{–}\mathsf{simod})$ by the thick subcategory of complexes that are coacyclic as complexes of $\mathscr{C}$-comodules. The semiderived category $\mathsf{D}^{\mathsf{si}}(\mathsf{simod}\text{–}\boldsymbol{\mathscr{S}}) = \mathsf{D}^{\mathsf{si}}_{\mathscr{C}}(\mathsf{simod}\text{–}\boldsymbol{\mathscr{S}})$ is defined similarly.

A construction of the double-sided derived functor of semitensor product of *bisemimodules*, taking values in a semiderived category of bisemimodules, can be found in [40, Section 2.9]. As one can use (strongly semiflat complexes of) semiflat bisemimodules in the construction of this derived functor, the double-sided derived functor of semitensor product of bisemimodules is associative.

Similarly, for a semicommutative semialgebra $\boldsymbol{\mathcal{S}}$ over a cocommutative coalgebra $\mathcal{C}$, one can construct the double-sided derived functor of semitensor product

$$\Diamond_{\boldsymbol{\mathcal{S}}}^{\mathbb{D}} \colon \mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}) \times \mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}) \longrightarrow \mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}), \tag{41}$$

which defines a structure of associative, commutative, and unital tensor category on the triangulated category $\mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod})$. The one-term complex of $\boldsymbol{\mathcal{S}}$-semimodules $\boldsymbol{\mathcal{S}}$ is the unit object.

The equivalence of abelian categories $\mathfrak{Y}\text{–}\mathsf{tors} \simeq \boldsymbol{\mathcal{S}}\text{–}\mathsf{simod}$ from item (5) forms a commutative square diagram with the equivalence of abelian categories $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathcal{C}\text{–}\mathsf{comod}$, the direct image functor $\pi_* \colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$, and the forgetful functor $\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod} \longrightarrow \mathcal{C}\text{–}\mathsf{comod}$. Therefore, a triangulated equivalence of the semiderived categories $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod})$ is induced.

The result of [40, Corollary 6.6(b)] together with the discussion in items (7–8) above shows that the triangulated equivalence $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D^{si}}(\boldsymbol{\mathcal{S}}\text{–}\mathsf{simod})$ transforms the semitensor product functor $\Diamond_{\pi^*\mathcal{D}^\bullet}$ (38) from Section 8.4 for the dualizing complex $\mathcal{D}^\bullet = \mathcal{C}$ on $\mathfrak{X}$ into the double-sided derived functor $\Diamond_{\boldsymbol{\mathcal{S}}}^{\mathbb{D}}$ (41).

9. Flat Affine Ind-Schemes over Ind-Schemes of Ind-Finite Type

In this section, as in Section 6, $\Bbbk$ denotes a fixed ground field. Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over $\Bbbk$, and let $\pi \colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of schemes. Consider the diagonal morphism $\Delta_{\mathfrak{Y}} \colon \mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$; the morphism $\Delta_{\mathfrak{Y}}$ factorizes naturally into the composition

$$\mathfrak{Y} \xrightarrow{\ \delta\ } \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y} \xrightarrow{\ \eta\ } \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}.$$

We denote the two morphisms involved by $\delta = \delta_{\mathfrak{Y}/\mathfrak{X}} \colon \mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ and $\eta = \eta_{\mathfrak{Y}/\mathfrak{X}} \colon \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$.

Let $\mathcal{D}^\bullet$ be a rigid dualizing complex on $\mathfrak{X}$ (as defined in Section 6.5). The aim of this section is to describe the semitensor product functor $\Diamond_{\pi^*\mathcal{D}^\bullet} \colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ as the composition of the left derived $*$-restriction and the right derived !-restriction of the external tensor product on $\mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$ to the closed immersions $\delta_{\mathfrak{Y}/\mathfrak{X}}$ and $\eta_{\mathfrak{Y}/\mathfrak{X}}$, resprectively; that is

$$\boldsymbol{\mathcal{M}}^\bullet \Diamond_{\pi^*\mathcal{D}^\bullet} \boldsymbol{\mathcal{N}}^\bullet \;=\; \mathbb{L}\delta^* \, \mathbb{R}\eta^! (\boldsymbol{\mathcal{M}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathcal{N}}^\bullet)$$

for any two complexes of quasi-coherent torsion sheaves $\boldsymbol{\mathcal{M}}^\bullet$ and $\boldsymbol{\mathcal{N}}^\bullet$ on $\mathfrak{Y}$.

9.1. **Derived inverse image of pro-sheaves.** Suppose that we are given a commutative square diagram of morphisms of ind-schemes

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\ g\ } & \mathfrak{Y} \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \mathfrak{Z} & \xrightarrow{\ f\ } & \mathfrak{X} \end{array} \tag{42}$$

Assume that the morphisms π and ρ are flat and affine, and the ind-scheme $\mathfrak{X}$ is ind-Noetherian. The aim of this Section 9.1 is to construct the left derived functor of inverse image

$$\mathbb{L}g^*\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{W}_{\mathfrak{Z}}\text{–}\mathsf{flat}) \tag{43}$$

acting from the derived category of the exact category of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ to the derived category of the exact category of $\mathfrak{Z}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$.

Lemma 9.1. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Let $\mathfrak{F}^\bullet$ be a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ (as defined in Section 8.2). Assume that the complex $\mathfrak{F}^\bullet$ is acyclic in the exact category $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$. Then, for any complex of quasi-coherent torsion sheaves $\boldsymbol{\mathcal{M}}^\bullet$ on $\mathfrak{Y}$, the complex of quasi-coherent torsion sheaves $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{M}}^\bullet)$ on $\mathfrak{X}$ is coacyclic.*

Proof. This assertion is implicit in the construction of the derived functor $\otimes_{\mathfrak{Y}}^{\mathbb{L}}$ (37) in Section 8.3. Explicitly, by Proposition 8.10 there exists a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex $\boldsymbol{\mathcal{G}}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ together with a morphism of complexes $\boldsymbol{\mathcal{G}}^\bullet \longrightarrow \boldsymbol{\mathcal{M}}^\bullet$ whose cone $\boldsymbol{\mathcal{N}}^\bullet$ has the property that its direct image under π is coacyclic in $\mathfrak{X}$–tors. Then the complex $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{G}}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors, since the complex $\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ and the complex $\boldsymbol{\mathcal{G}}^\bullet$ is homotopy $\mathfrak{Y}/\mathfrak{X}$-flat. Furthermore, the complex $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{N}}^\bullet)$ is also coacyclic in $\mathfrak{X}$–tors, since $\mathfrak{F}^\bullet$ be a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ and the complex $\pi_*(\boldsymbol{\mathcal{N}}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors (see condition (ii) in Section 8.2). It follows that the complex $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathcal{M}}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors. ☐

Proposition 9.2. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Let $\mathfrak{F}^\bullet$ be a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Assume that the complex $\mathfrak{F}^\bullet$ is acyclic in the exact category $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Then the complex $\mathfrak{F}^\bullet$ is also acyclic in the exact category $\mathfrak{Y}$–flat of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$.*

Proof. Let $X \subset \mathfrak{X}$ be a closed subscheme with the closed immersion morphism $i\colon X \longrightarrow \mathfrak{X}$. Put $\mathbf{Y} = X \times_{\mathfrak{X}} \mathfrak{Y}$, and denote by $k\colon \mathbf{Y} \longrightarrow \mathfrak{Y}$ the natural closed immersion. In view of Lemma 4.13, it suffices to show that the complex $k^*\mathfrak{F}^\bullet$ is acyclic in $\mathbf{Y}$–flat. For this purpose, we will show that the complex of quasi-coherent sheaves $k^*\mathfrak{F}^\bullet \otimes_{\mathcal{O}_{\mathbf{Y}}} \boldsymbol{\mathcal{N}}$ on $\mathbf{Y}$ is acyclic for any quasi-coherent sheaf $\boldsymbol{\mathcal{N}}$ on $\mathbf{Y}$.

Since $\mathbf{Y}$ is a scheme, we can consider $\boldsymbol{\mathcal{N}}$ as a quasi-coherent torsion sheaf on $\mathbf{Y}$; then $k^*\mathfrak{F}^\bullet \otimes_{\mathcal{O}_\mathbf{Y}} \boldsymbol{\mathcal{N}} = k^*\mathfrak{F}^\bullet \otimes_\mathbf{Y} \boldsymbol{\mathcal{N}}$ is viewed as a complex of quasi-coherent torsion sheaves on $\mathbf{Y}$. As the functor $k_*\colon \mathbf{Y}\text{–}\mathsf{qcoh} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ is exact and faithful (see Lemma 2.16(a)), acyclicity of this complex is equivalent to acyclicity of the complex $k_*(k^*\mathfrak{F}^\bullet \otimes_\mathbf{Y} \boldsymbol{\mathcal{N}})$ in $\mathfrak{Y}\text{–}\mathsf{tors}$. By Lemma 8.2, we have an isomorphism in $\mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$

$$k_*(k^*\mathfrak{F}^\bullet \otimes_\mathbf{Y} \boldsymbol{\mathcal{N}}) \simeq \mathfrak{F}^\bullet \otimes_\mathfrak{Y} k_*\boldsymbol{\mathcal{N}}.$$

By Lemma 9.1, the complex $\pi_*(\mathfrak{F}^\bullet \otimes_\mathfrak{Y} k_*\boldsymbol{\mathcal{N}})$ is coacyclic, hence acyclic, in $\mathfrak{X}\text{–}\mathsf{tors}$. As the functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ is exact and faithful by Lemma 7.2, it follows that the complex $\mathfrak{F}^\bullet \otimes_\mathfrak{Y} k_*\boldsymbol{\mathcal{N}}$ is acyclic in $\mathfrak{Y}\text{–}\mathsf{tors}$. ☐

The following corollary plays the key role.

Corollary 9.3. *In the context of diagram* (42)*, let $\mathfrak{F}^\bullet$ be a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Assume that the complex $\mathfrak{F}^\bullet$ is acyclic in the exact category $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$. Then the complex $g^*\mathfrak{F}^\bullet$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$ is acyclic in the exact category $\mathfrak{W}_\mathfrak{Z}\text{–}\mathsf{flat}$.*

Proof. By Proposition 9.2, the complex $\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{Y}\text{–}\mathsf{flat}$. As the direct image functor $g^*\colon \mathfrak{Y}\text{–}\mathsf{flat} \longrightarrow \mathfrak{W}\text{–}\mathsf{flat}$ is exact, it follows immediately that the complex $g^*\mathfrak{F}^\bullet$ is acyclic in $\mathfrak{W}\text{–}\mathsf{flat}$, hence also in $\mathfrak{W}_\mathfrak{Z}\text{–}\mathsf{flat}$. ☐

Using Proposition 8.8 and Corollary 9.3, the left derived functor (43) can be constructed following the general approach of a "derived functor in the sense of Deligne" [11, 1.2.1–2], [40, Lemma 6.5.2].

Let $\mathfrak{P}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. By Proposition 8.8, there exists a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ on $\mathfrak{Y}$ together with a morphism of complexes $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ whose cone is acyclic in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$. Put

$$\mathbb{L}g^*(\mathfrak{P}^\bullet) = g^*(\mathfrak{F}^\bullet) \in \mathsf{D}(\mathfrak{W}_\mathfrak{Z}\text{–}\mathsf{flat}).$$

Notice that $g^*(\mathfrak{F}^\bullet)$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$ (see Section 3.4); hence it is also a complex of $\mathfrak{Z}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$, as the morphism ρ is flat and affine by assumption (as per the discussion in Section 7.2).

Let $a\colon \mathfrak{P}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ be a morphism of complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, and let $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ and $\mathfrak{G}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ be two morphisms in $\mathsf{C}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$ with the cones acyclic in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$ such that both the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ are relatively homotopy flat complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. In order to construct the induced morphism

$$\mathbb{L}g^*(a)\colon g^*(\mathfrak{F}^\bullet) \longrightarrow g^*(\mathfrak{G}^\bullet)$$

in $\mathsf{D}(\mathfrak{W}_\mathfrak{Z}\text{–}\mathsf{flat})$, choose a complex $\mathfrak{R}^\bullet$ in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$ together with morphisms $\mathfrak{R}^\bullet \longrightarrow \mathfrak{F}^\bullet$ and $\mathfrak{R}^\bullet \longrightarrow \mathfrak{G}^\bullet$ in $\mathsf{C}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$ such that the diagram $\mathfrak{R}^\bullet \longrightarrow \mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ and $\mathfrak{R}^\bullet \longrightarrow \mathfrak{G}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ is commutative in $\mathsf{K}(\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat})$ and the cone of the morphism $\mathfrak{R}^\bullet \longrightarrow \mathfrak{F}^\bullet$ is acyclic in $\mathfrak{Y}_\mathfrak{X}\text{–}\mathsf{flat}$. Using Proposition 8.8, choose a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{H}^\bullet$ on $\mathfrak{Y}$ together with a morphism of complexes $\mathfrak{H}^\bullet \longrightarrow \mathfrak{R}^\bullet$ whose cone is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$.

Then the cone $\mathfrak{S}^\bullet$ of the composition $s\colon \mathfrak{H}^\bullet \longrightarrow \mathfrak{R}^\bullet \longrightarrow \mathfrak{F}^\bullet$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$. By Corollary 9.3, the complex $g^*(\mathfrak{S}^\bullet)$ is acyclic in $\mathfrak{W}_3$–flat. Denote by b the composition $\mathfrak{H}^\bullet \longrightarrow \mathfrak{R}^\bullet \longrightarrow \mathfrak{G}^\bullet$. Now the fraction formed by the morphism $g^*(b)\colon g^*(\mathfrak{H}^\bullet) \longrightarrow g^*(\mathfrak{G}^\bullet)$ and the isomorphism $g^*(s)\colon g^*(\mathfrak{H}^\bullet) \longrightarrow g^*(\mathfrak{F}^\bullet)$ represents the desired morphism $\mathbb{L}g^*(a)\colon g^*(\mathfrak{F}^\bullet) \longrightarrow g^*(\mathfrak{G}^\bullet)$ in $\mathsf{D}(\mathfrak{W}_3$–$\mathsf{flat})$.

9.2. **Derived inverse image of torsion sheaves.** Suppose that we are given a commutative triangle diagram of morphisms of ind-schemes

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\ \ g\ \ } & \mathfrak{Y} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathfrak{X} & \end{array} \tag{44}$$

Assume that the morphisms π and ρ are flat and affine, and the ind-scheme $\mathfrak{X}$ is ind-Noetherian. The aim of Section 9.2 is to construct the left derived functor of inverse image

$$\mathbb{L}g^*\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors}) \tag{45}$$

acting from the $\mathfrak{Y}/\mathfrak{X}$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ to the $\mathfrak{W}/\mathfrak{X}$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{W}$.

Lemma 9.4. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Let $\mathscr{G}^\bullet$ be a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$ (as defined in Section 8.2). Assume that the complex $\pi_*(\mathscr{G}^\bullet)$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is coacyclic. Then, for any complex of $\mathfrak{Y}/\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{P}^\bullet$ on $\mathfrak{Y}$, the complex of quasi-coherent torsion sheaves $\pi_*(\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ on $\mathfrak{X}$ is coacyclic.*

Proof. Similarly to Lemma 9.1, this assertion is implicit in the construction of the derived functor $\otimes^{\mathbb{L}}_{\mathfrak{Y}}$ (37) in Section 8.3. Explicitly, by Proposition 8.8 there exists a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ on $\mathfrak{Y}$ together with a morphism of complexes $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ whose cone $\mathfrak{Q}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$. Then the complex $\pi_*(\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors, since $\mathfrak{F}^\bullet$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ and the complex $\pi_*(\mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors. Furthermore, the complex $\pi_*(\mathfrak{Q}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors, since the complex $\mathfrak{Q}^\bullet$ is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ and $\mathscr{G}^\bullet$ is a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. It follows that the complex $\pi_*(\mathfrak{P}^\bullet \otimes_{\mathfrak{Y}} \mathscr{G}^\bullet)$ is coacyclic in $\mathfrak{X}$–tors. ☐

The next proposition is the key technical assertion.

Proposition 9.5. *In the context of the diagram* (44)*, let $\mathscr{F}^\bullet$ be a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Assume that the complex $\pi_*(\mathscr{F}^\bullet)$*

of quasi-coherent torsion sheaves on $\mathfrak{X}$ is coacyclic. Then the complex $\rho_*g^*(\mathscr{F}^\bullet)$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is coacyclic, too.

Proof. Let us show that the morphism of ind-schemes $g\colon \mathfrak{W} \longrightarrow \mathfrak{Y}$ is affine (whenever the morphisms π and $\rho = \pi g$ are). First of all, the morphism g is "representable by schemes" (since the morphisms π and $\rho = \pi g$ are). Indeed, let $X \subset \mathfrak{X}$ be a closed subscheme. Then $\mathbf{Y} = X \times_{\mathfrak{X}} \mathfrak{Y}$ is a closed subscheme in $\mathfrak{Y}$ (and any closed subscheme in $\mathfrak{Y}$ is contained in a closed subscheme of this form). Therefore, $\mathbf{W} = \mathbf{Y} \times_{\mathfrak{Y}} \mathfrak{W} = X \times_{\mathfrak{X}} \mathfrak{W}$ is a closed subscheme in $\mathfrak{W}$.

We know that the morphisms of schemes $\mathbf{Y} \longrightarrow X$ and $\mathbf{W} \longrightarrow X$ are affine, and we have to show that the morphism $\mathbf{W} \longrightarrow \mathbf{Y}$ is. Let $X = \bigcup_\alpha U_\alpha$ be an affine open covering of X. Then $\mathbf{Y} = \bigcup_\alpha (U_\alpha \times_X \mathbf{Y})$ is an affine open covering of $\mathbf{Y}$; and the schemes $(U_\alpha \times_X \mathbf{Y}) \times_{\mathbf{Y}} \mathbf{W} = U_\alpha \times_X \mathbf{W}$ are affine (cf. [20, Tag 01SG]).

Now we have $\rho_*g^*(\mathscr{F}^\bullet) \simeq \pi_*g_*g^*(\mathscr{F}^\bullet)$, since $\rho = \pi g$. Furthermore, $g^*\mathscr{F}^\bullet \simeq \mathfrak{O}_{\mathfrak{W}} \otimes_{\mathfrak{W}} g^*\mathscr{F}^\bullet$, where $\mathfrak{O}_{\mathfrak{W}} \in \mathfrak{W}\text{–}\mathsf{flat}$ is the "pro-structure pro-sheaf" on $\mathfrak{W}$. By Lemma 7.5,

$$g_*g^*(\mathscr{F}^\bullet) \simeq g_*(\mathfrak{O}_{\mathfrak{W}} \otimes_{\mathfrak{W}} g^*\mathscr{F}^\bullet) \simeq g_*(\mathfrak{O}_{\mathfrak{W}}) \otimes_{\mathfrak{Y}} \mathscr{F}^\bullet.$$

Notice that the pro-quasi-coherent pro-sheaf $g_*(\mathfrak{O}_{\mathfrak{W}})$ on $\mathfrak{Y}$ is $\mathfrak{X}$-flat, because $\pi_*g_*(\mathfrak{O}_{\mathfrak{W}}) \simeq \rho_*(\mathfrak{O}_{\mathfrak{W}})$ and the morphism of ind-schemes $\rho\colon \mathfrak{W} \longrightarrow \mathfrak{X}$ is flat by assumption, hence the direct image functor $\rho_*\colon \mathfrak{W}\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$ takes flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$ to flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. Applying Lemma 9.4, we conclude that the complex $\rho_*g^*(\mathscr{F}^\bullet) \simeq \pi_*(g_*(\mathfrak{O}_{\mathfrak{W}}) \otimes_{\mathfrak{Y}} \mathscr{F}^\bullet)$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. ☐

Similarly to Section 9.1, we use Propositions 8.10 and 9.5 in order to construct the left derived functor (45) following the general approach of [11, 1.2.1–2] and [40, Lemma 6.5.2].

Let $\mathscr{M}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. By Proposition 8.10, there exists a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves $\mathscr{F}^\bullet$ on $\mathfrak{Y}$ together with a morphism of complexes $\mathscr{F}^\bullet \longrightarrow \mathscr{M}^\bullet$ whose cone has the property that its direct image is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. Put

$$\mathbb{L}g^*(\mathscr{M}^\bullet) = g^*(\mathscr{F}^\bullet) \in \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors}).$$

The action of the functor $\mathbb{L}g^*$ on morphisms in $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is constructed in the same way as in Section 9.1 (we omit the obvious details).

9.3. **Derived restriction with supports in the relative context.** Suppose that we are given a pullback diagram of morphisms of ind-schemes (so $\mathfrak{W} = \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$)

$$(46) \qquad \begin{array}{ccc} \mathfrak{W} & \xrightarrow{k} & \mathfrak{Y} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathfrak{Z} & \xrightarrow{i} & \mathfrak{X} \end{array}$$

Assume that the morphism π (hence also the morphism ρ) is flat and affine, the morphism i (hence also the morphism k) is a closed immersion, and the ind-scheme

$\mathfrak{X}$ (hence also the ind-scheme $\mathfrak{Z}$) is ind-Noetherian. The aim of Section 9.3 is to construct the right derived functor

$$\mathbb{R}k^!\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors}) \tag{47}$$

acting from the $\mathfrak{Y}/\mathfrak{X}$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ to the $\mathfrak{W}/\mathfrak{Z}$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{W}$.

Lemma 9.6. *In the context of the diagram* (46)*, there is a natural isomorphism* $i^!\pi_* \simeq \rho_*k^!$ *of functors* $\mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$.

Proof. Weaker assumptions than stated above are sufficient for the validity of this lemma, which is an ind-scheme version of Lemma 2.3(a). It suffices to assume that $\mathfrak{W} = \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$, the morphism π (hence also ρ) is "representable by schemes", and i (hence also k) is a reasonable closed immersion.

Indeed, let $\mathcal{N}$ be a quasi-coherent torsion sheaf on $\mathfrak{Y}$, and let $Z \subset \mathfrak{Z}$ be a reasonable closed subscheme. Put $\mathbf{W} = Z \times_{\mathfrak{Z}} \mathfrak{W}$; so $\mathbf{W}$ is a reasonable closed subscheme in $\mathfrak{W}$. The scheme Z can be also viewed as a reasonable closed subscheme in $\mathfrak{X}$, embedded via i; and the scheme $\mathbf{W}$ can be viewed as a reasonable closed subscheme in $\mathfrak{Y}$, embedded via k. Denote by $\rho_Z\colon \mathbf{W} \longrightarrow Z$ the natural morphism. In order to obtain the desired isomorphism $i^!\pi_*\mathcal{N} \simeq \rho_*k^!\mathcal{N}$ in $\mathfrak{Z}\text{–}\mathsf{tors}$, one constructs a compatible system of isomorphisms in $Z\text{–}\mathsf{qcoh}$

$$(i^!\pi_*\mathcal{N})_{(Z)} \simeq (\pi_*\mathcal{N})_{(Z)} \simeq \rho_{Z*}(\mathcal{N}_{(\mathbf{W})}) \simeq \rho_{Z*}((k^!\mathcal{N})_{(\mathbf{W})}) \simeq (\rho_*k^!\mathcal{N})_{(Z)}$$

for all the reasonable closed subschemes $Z \subset \mathfrak{Z}$. (See the definition of the direct image functor for quasi-coherent torsion sheaves in Section 2.6.) ☐

By Proposition 7.8, we have natural equivalences of triangulated categories $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}})$ and $\mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{W}\text{–}\mathsf{tors}_{\mathfrak{Z}\text{-inj}})$. We will use these triangulated equivalences in order to construct the right derived functor (47).

Lemma 9.7. *In the context of the diagram* (46)*, the functor* $k^!\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{W}\text{–}\mathsf{tors}$ *restricts to an exact functor between exact categories* $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-inj}} \longrightarrow \mathfrak{W}\text{–}\mathsf{tors}_{\mathfrak{Z}\text{-inj}}$.

Proof. The functor $i^!\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$, being right adjoint to an exact functor $i_*\colon \mathfrak{Z}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$, takes injective objects to injective objects. In view of Lemma 9.6, it follows that the functor $k^!\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{W}\text{–}\mathsf{tors}$ takes $\mathfrak{X}$-injective quasi-coherent torsion sheaves to $\mathfrak{Z}$-injective ones.

Let $0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$ be a short exact sequence of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then $0 \longrightarrow \pi_*\mathcal{L} \longrightarrow \pi_*\mathcal{M} \longrightarrow \pi_*\mathcal{N} \longrightarrow 0$ is a split short exact sequence of quasi-coherent torsion sheaves on $\mathfrak{X}$. Hence $0 \longrightarrow i^!\pi^*\mathcal{L} \longrightarrow i^*\pi_*\mathcal{M} \longrightarrow i^!\pi_*\mathcal{N} \longrightarrow 0$ is a split short exact sequence of quasi-coherent torsion sheaves on $\mathfrak{Z}$. As $\rho_*k^! \simeq i^!\pi_*$, it follows that $0 \longrightarrow k^!\mathcal{L} \longrightarrow k^!\mathcal{M} \longrightarrow k^!\mathcal{N} \longrightarrow 0$ is a short exact sequence of $\mathfrak{Z}$-injective quasi-coherent torsion sheaves on $\mathfrak{W}$ (because the functor $\rho_*\colon \mathfrak{W}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Z}\text{–}\mathsf{tors}$ is exact and faithful by Lemma 7.2). ☐

In view of Lemma 9.7, the functor $k^!\colon \mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}} \longrightarrow \mathfrak{W}\text{–}\mathsf{tors}_{\mathfrak{Z}\text{-}\mathsf{inj}}$ induces a well-defined triangulated functor between the derived categories of the two exact categories,

$$k^!\colon \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}}) \longrightarrow \mathsf{D}(\mathfrak{W}\text{–}\mathsf{tors}_{\mathfrak{Z}\text{-}\mathsf{inj}}).$$

Using the triangulated equivalences $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathfrak{X}\text{-}\mathsf{inj}})$ and $\mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors}) \simeq \mathsf{D}(\mathfrak{W}\text{–}\mathsf{tors}_{\mathfrak{Z}\text{-}\mathsf{inj}})$, we obtain the desired right derived functor (47). This construction is also a particular case of the construction of [11, 1.2.1–2] or [40, Lemma 6.5.2], which was used in Sections 9.1–9.2.

9.4. **Composition of derived inverse images of pro-sheaves.** Suppose that we are given a composable pair of commutative square diagrams of morphisms of ind-schemes

$$\begin{array}{ccccc} \mathfrak{V} & \xrightarrow{\ h\ } & \mathfrak{W} & \xrightarrow{\ g\ } & \mathfrak{Y} \\ {\scriptstyle\sigma}\downarrow & & {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathfrak{U} & \xrightarrow{\ t\ } & \mathfrak{Z} & \xrightarrow{\ f\ } & \mathfrak{X} \end{array} \tag{48}$$

Assume that the morphisms π, ρ, and σ are flat and affine, and the ind-schemes $\mathfrak{X}$ and $\mathfrak{Z}$ are ind-Noetherian. Consider the composition of left derived functors (43) constructed in Section 9.1

$$\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \xrightarrow{\mathbb{L}g^*} \mathsf{D}(\mathfrak{W}_{\mathfrak{Z}}\text{–}\mathsf{flat}) \xrightarrow{\mathbb{L}h^*} \mathsf{D}(\mathfrak{V}_{\mathfrak{U}}\text{–}\mathsf{flat}).$$

Proposition 9.8. *There is a natural isomorphism $\mathbb{L}h^*\circ\mathbb{L}g^* \simeq \mathbb{L}(gh)^*$ of triangulated functors $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{V}_{\mathfrak{U}}\text{–}\mathsf{flat})$.*

Proof. The related underived isomorphism is obvious: clearly, one has $h^*g^*\mathfrak{P}^\bullet \simeq (gh)^*\mathfrak{P}^\bullet$ in $\mathsf{C}(\mathfrak{V}\text{–}\mathsf{pro})$ for any complex of pro-quasi-coherent pro-sheaves $\mathfrak{P}^\bullet$ on $\mathfrak{Y}$.

The best possible argument for the proof of the composition of derived functors being isomorphic to the derived functor of the composition would be to show that the functor g^* takes complexes adjusted to $\mathbb{L}g^*$ and $\mathbb{L}(gh)^*$ to complexes adjusted to $\mathbb{L}h^*$. In our context, this would mean showing that, for any relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ on $\mathfrak{Y}$ (relative to the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$), the complex of flat pro-quasi-coherent pro-sheaves $g^*(\mathfrak{F}^\bullet)$ on $\mathfrak{W}$ is also relatively homotopy flat (relative to the morphism $\rho\colon \mathfrak{W} \longrightarrow \mathfrak{Z}$).

However, we do *not* know how to prove this preservation of relative homotopy flatness in full generality. Instead, we will show that, given a complex $\mathfrak{P}^\bullet$ in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, a relatively homotopy flat complex $\mathfrak{F}^\bullet$ endowed with a morphism $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ with the cone acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ *can be chosed in such a way that* the complex of flat pro-quasi-coherent pro-sheaves $g^*(\mathfrak{F}^\bullet)$ on $\mathfrak{W}$ is a relatively homotopy flat complex.

Let us recall the construction of the relatively homotopy flat resolutions from Proposition 8.8. All the complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which can be obtained from the inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ using the operations of cone, infinite coproduct, and the passage to a homotopy equivalent complex, are relatively homotopy flat (by Lemma 8.5); and any complex $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ admits a relatively homotopy flat resolution of this form.

Assume that the complex $\mathfrak{F}^\bullet$ is homotopy equivalent to a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ obtained from inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ using the operations of cone and infinite coproduct. Then the complex $g^*(\mathfrak{F}^\bullet)$ is homotopy equivalent to a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{W}$ similarly obtained from the inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Z}$. This follows from the natural isomorphism $g^*\pi^* \simeq \rho^* f^*$ of functors $\mathfrak{X}\text{–}\mathsf{flat} \longrightarrow \mathfrak{W}\text{–}\mathsf{flat}$ and the commutation of inverse images of (flat) pro-quasi-coherent pro-sheaves with the coproducts. ☐

9.5. **External tensor products in the relative context.** Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-schemes over $\Bbbk$, and let $\pi'\colon \mathfrak{Y}' \longrightarrow \mathfrak{X}'$ and $\pi''\colon \mathfrak{Y}'' \longrightarrow \mathfrak{X}''$ be affine morphisms of schemes. Let $\pi' \times_\Bbbk \pi''\colon \mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'' \longrightarrow \mathfrak{X}' \times_\Bbbk \mathfrak{X}''$ be the induced morphism of the Cartesian products. Clearly, $\pi' \times_\Bbbk \pi''$ is an affine morphism of ind-schemes.

The functor of external tensor product of pro-quasi-coherent pro-sheaves (19)

$$\boxtimes_\Bbbk\colon \mathfrak{Y}'\text{–}\mathsf{pro} \times \mathfrak{Y}''\text{–}\mathsf{pro} \longrightarrow (\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')\text{–}\mathsf{pro}$$

was constructed in Section 6.2.

Lemma 9.9. *Let $\mathfrak{Q}'$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}'$ and $\mathfrak{Q}''$ be a pro-quasi-coherent pro-sheaf on $\mathfrak{Y}''$. Then there is a natural isomorphism*

$$(\pi' \times_\Bbbk \pi'')_*(\mathfrak{Q}' \boxtimes_\Bbbk \mathfrak{Q}'') \simeq \pi'_*\mathfrak{Q}' \boxtimes_\Bbbk \pi''_*\mathfrak{Q}''$$

of pro-quasi-coherent pro-sheaves on $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$.

Proof. Follows from Lemma 6.4. ☐

Lemma 9.10. (a) *Let $\mathfrak{G}'$ be an $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}'$ and $\mathfrak{G}''$ be an $\mathfrak{X}''$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}''$. Then $\mathfrak{G}' \boxtimes_\Bbbk \mathfrak{G}''$ is an $(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}''$.*

(b) *The external tensor product functor*

$$(49) \qquad \boxtimes_\Bbbk\colon \mathfrak{Y}'_{\mathfrak{X}'}\text{–}\mathsf{flat} \times \mathfrak{Y}''_{\mathfrak{X}''}\text{–}\mathsf{flat} \longrightarrow (\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')_{(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')}\text{–}\mathsf{flat}$$

is exact (as a functor between exact categories) and preserves direct limits (in particular, coproducts).

Proof. Part (a) follows from Lemma 9.9 and formula (20). Part (b) follows directly from the definitions of the exact structures involved and Lemma 6.2. ☐

Lemma 9.11. *Let $\mathfrak{G}'^\bullet$ be a complex of $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}'$ and $\mathfrak{G}''^\bullet$ be a complex of $\mathfrak{X}''$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}''$. Assume that the complex $\mathfrak{G}'^\bullet$ is acyclic in $\mathfrak{Y}'_{\mathfrak{X}'}\text{–}\mathsf{flat}$. Then the complex $\mathfrak{G}'^\bullet \boxtimes_\Bbbk \mathfrak{G}''^\bullet$ is acyclic in $(\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')_{(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')}\text{–}\mathsf{flat}$.*

Proof. The results of Lemmas 7.14 and 9.9 reduce the question to Lemma 6.9. ☐

It follows from Lemma 9.11 that the external tensor product of pro-quasi-coherent pro-sheaves which are flat over the base is well-defined as a functor between the derived categories of the respective exact categories,

$$(50) \qquad \boxtimes_\Bbbk\colon \mathsf{D}(\mathfrak{Y}'_{\mathfrak{X}'}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{Y}''_{\mathfrak{X}''}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}((\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')_{(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')}\text{–}\mathsf{flat}).$$

Now assume additionally that the ind-schemes $\mathfrak{X}'$ and $\mathfrak{X}''$ are reasonable (then so are the ind-schemes $\mathfrak{Y}'$ and $\mathfrak{Y}''$). The functor of external tensor product of quasi-coherent torsion sheaves (22)

$$\boxtimes_{\Bbbk}\colon \mathfrak{Y}'\text{–tors} \times \mathfrak{Y}''\text{–tors} \longrightarrow (\mathfrak{Y}' \times_{\Bbbk} \mathfrak{Y}'')\text{–tors}$$

was constructed in Section 6.3.

Lemma 9.12. *Let $\mathcal{N}'^{\bullet}$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}'$ and $\mathcal{N}''^{\bullet}$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}''$. Assume that the complex $\pi'_*(\mathcal{N}'^{\bullet})$ of quasi-coherent torsion sheaves on $\mathfrak{X}'$ is coacyclic. Then the complex $(\pi' \times_{\Bbbk} \pi'')_*(\mathcal{N}'^{\bullet} \boxtimes_{\Bbbk} \mathcal{N}''^{\bullet})$ of quasi-coherent torsion sheaves on $\mathfrak{X}' \times_{\Bbbk} \mathfrak{X}''$ is coacyclic, too.*

Proof. Follows immediately from Lemmas 6.13 and 6.18. □

It is clear from Lemma 9.12 that the external tensor product is well-defined as a functor between the semiderived categories of quasi-coherent torsion sheaves,

$$\boxtimes_{\Bbbk}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}'\text{–tors}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}''}(\mathfrak{Y}''\text{–tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{(\mathfrak{X}'\times_{\Bbbk}\mathfrak{X}'')}((\mathfrak{Y}' \times_{\Bbbk} \mathfrak{Y}'')\text{–tors}). \tag{51}$$

We have not used the assumption of flatness of morphisms π' and π'' in this Section 9.5 yet, but it is worth noticing that the morphism of ind-schemes $\pi' \times_{\Bbbk} \pi''$ is flat whenever the morphisms π' and π'' are. This follows from the similar property of morphisms of schemes over $\Bbbk$.

9.6. **Derived tensor product of pro-sheaves as derived restriction to the diagonal.** Let $\mathfrak{X}$ be an ind-scheme of ind-finite type over $\Bbbk$, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Then the following commutative square diagram of morphisms of ind-schemes

$$\begin{array}{ccc}
\mathfrak{Y} & \xrightarrow{\ \Delta_{\mathfrak{Y}}\ } & \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi\times_{\Bbbk}\pi} \\
\mathfrak{X} & \xrightarrow{\ \Delta_{\mathfrak{X}}\ } & \mathfrak{X} \times_{\Bbbk} \mathfrak{X}
\end{array}$$

is a particular case of the diagram (42) from Section 9.1.

Let $\mathfrak{P}^{\bullet}$ and $\mathfrak{Q}^{\bullet}$ be two complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Our aim is to construct a natural isomorphism

$$\mathfrak{P}^{\bullet} \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{Q}^{\bullet} \simeq \mathbb{L}\Delta_{\mathfrak{Y}}^{*}(\mathfrak{P}^{\bullet} \boxtimes_{\Bbbk} \mathfrak{Q}^{\bullet}) \tag{52}$$

in the derived category $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–flat})$. Here the derived functor of tensor product $\otimes^{\mathfrak{Y},\mathbb{L}}$ (36) was constructed in Section 8.3, the functor of external tensor product $\boxtimes_{\Bbbk}$ was discussed in Section 9.5, and the derived functor of inverse image $\mathbb{L}\Delta_{\mathfrak{Y}}^{*}$ was defined in Section 9.1. Recall that the isomorphism of underived functors $\mathfrak{P}^{\bullet} \otimes^{\mathfrak{Y}} \mathfrak{Q}^{\bullet} \simeq \Delta_{\mathfrak{Y}}^{*}(\mathfrak{P}^{\bullet} \boxtimes_{\Bbbk} \mathfrak{Q}^{\bullet})$ is provided by Lemma 6.12. The following proposition shows that the derived functors agree.

Proposition 9.13. *For any two complexes* $\mathfrak{P}^\bullet$ *and* $\mathfrak{Q}^\bullet \in \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$*, a natural isomorphism of left derived functors* (52) *holds in* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$.

Proof. Let $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ be two relatively homotopy flat complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ endowed with two morphisms of complexes $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ and $\mathfrak{G}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ with the cones acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Then the external tensor product $\mathfrak{F}^\bullet \boxtimes_{\Bbbk} \mathfrak{G}^\bullet$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$ by formula (20) from Section 6.2, and the cone of the morphism $\mathfrak{F}^\bullet \boxtimes_{\Bbbk} \mathfrak{G}^\bullet \longrightarrow \mathfrak{P}^\bullet \boxtimes_{\Bbbk} \mathfrak{Q}^\bullet$ is acyclic in the exact category $(\mathfrak{Y} \times_{\Bbbk} \mathfrak{Y})_{(\mathfrak{X}\times_{\Bbbk}\mathfrak{X})}\text{–}\mathsf{flat}$ by Lemma 9.11.

A problem similar to the one in the proof of Proposition 9.8 arises here. The best possible argument for a proof of the desired isomorphism would be to show that the complex $\mathfrak{F}^\bullet \boxtimes_{\Bbbk} \mathfrak{G}^\bullet$ is a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$ (relative to the morphism $\pi \times_{\Bbbk} \pi \colon \mathfrak{Y} \times_{\Bbbk} \mathfrak{Y} \longrightarrow \mathfrak{X} \times_{\Bbbk} \mathfrak{X}$). Then the isomorphism of the derived functors would follow immediately from (their constructions and) the isomorphism of the underived ones. However, we do *not* know how to prove this relative homotopy flatness in full generality. Instead, we will show that the relatively homotopy flat resolutions $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ of any two given complexes $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet$ *can be chosen in such a way that* the external tensor product $\mathfrak{F}^\bullet \boxtimes_{\Bbbk} \mathfrak{G}^\bullet$ is a relatively homotopy flat complex.

Once again we recall the construction of the relatively homotopy flat resolutions from Proposition 8.8. All the complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ which can be obtained from the inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ using the operations of cone, infinite coproduct, and the passage to a homotopy equivalent complex, are relatively homotopy flat; and any complex $\mathfrak{P}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ admits a relatively homotopy flat resolution of this form.

Now assume that both the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ are homotopy equivalent to complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ obtained from the inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ using the operations of cone and infinite coproduct. Then the complex $\mathfrak{F}^\bullet \boxtimes_{\Bbbk} \mathfrak{G}^\bullet$ is homotopy equivalent to a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y} \times_{\Bbbk} \mathfrak{Y}$ similarly obtained from the inverse images of complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X} \times_{\Bbbk} \mathfrak{X}$. This follows from the commutation of external tensor products with the inverse images (Lemma 6.10), the commutation of external tensor products with the coproducts (see Section 6.2 and Lemma 9.10(b)), and the preservation of flatness by the external tensor products of pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ (formula (20)). ☐

9.7. **Semiderived equivalence and change of fiber.** Let us return to the setting of the commutative triangle diagram of morphisms of ind-schemes (44) from Section 9.2

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{\ g\ } & \mathfrak{Y} \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathfrak{X} & \end{array}$$

where the morphisms π and ρ are flat and affine, and the ind-scheme $\mathfrak{X}$ is ind-Noetherian. Assume further that $\mathfrak{X}$ is ind-semi-separated and endowed with a dualizing complex $\mathscr{D}^\bullet$. Then Theorem 7.15 provides triangulated equivalences

$$\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}):\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet,-)$$

and

$$\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}-:\mathsf{D}(\mathfrak{W}_{\mathfrak{X}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors}):\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{W}\text{-qc}}(\rho^*\mathscr{D}^\bullet,-).$$

Here the notation $\mathbb{R}\,\mathfrak{Hom}$ stands for the fact that the functors $\mathfrak{Hom}$ should be applied to complexes of $\mathfrak{X}$-*injective* quasi-coherent torsion sheaves (while the functors $\otimes_{\mathfrak{Y}}$ and $\otimes_{\mathfrak{W}}$ are applied to arbitrary complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves).

Proposition 9.14. *In the context above, the triangulated equivalences* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ *and* $\mathsf{D}(\mathfrak{W}_{\mathfrak{X}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors})$ *from Theorem 7.15 transform the left derived functor* $\mathbb{L}g^*\colon\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\longrightarrow\mathsf{D}(\mathfrak{W}_{\mathfrak{X}}\text{–}\mathsf{flat})$ (43) *from Section 9.1 into the left derived functor* $\mathbb{L}g^*\colon\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})\longrightarrow\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors})$ (45) *from Section 9.2.*

Proof. Let $\mathfrak{P}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. We have to construct a natural isomorphism

$$\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}\mathbb{L}g^*(\mathfrak{P}^\bullet)\simeq\mathbb{L}g^*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$$

in the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors})$. Notice first of all that the related isomorphism for underived functors

$$\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}g^*(\mathfrak{P}^\bullet)\simeq g^*\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}g^*(\mathfrak{P}^\bullet)\simeq g^*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$$

holds because $\rho^*\mathscr{D}^\bullet\simeq g^*\pi^*\mathscr{D}^\bullet$ (since $\rho=\pi g$) and by Lemma 3.4. (It was explained in the proof of Proposition 9.5 that the morphism g is "representable by schemes"—in fact, affine.)

To prove the desired isomorphism for derived functors, replace the complex $\mathfrak{P}^\bullet$ with a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ on $\mathfrak{Y}$ endowed with a morphism of complexes $\mathfrak{F}^\bullet\longrightarrow\mathfrak{P}^\bullet$ with the cone acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Then it remains to recall that the complex $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is homotopy $\mathfrak{Y}/\mathfrak{X}$-flat by Lemma 8.13(b). So we have

$$\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}\mathbb{L}g^*(\mathfrak{P}^\bullet)=\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}g^*(\mathfrak{F}^\bullet)\simeq g^*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet)=\mathbb{L}g^*(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$$

in $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors})$. Here the rightmost equality holds because the morphism of complexes of quasi-coherent torsion sheaves $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet\longrightarrow\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet$ on $\mathfrak{Y}$ has a cone whose direct image under π is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. Indeed, the functor $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-$ takes complexes acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ to complexes with the direct image coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$, according to the proof of Theorem 7.15. The notation $\rho^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}\mathbb{L}g^*(\mathfrak{P}^\bullet)$ with a derived category object $\mathbb{L}g^*(\mathfrak{P}^\bullet)$ is well-defined for the same reason. □

9.8. **Semiderived equivalence and base change.** Now we return to the setting of a pullback diagram of morphisms of ind-schemes similar to (46) from Section 9.3 (so $\mathfrak{W} = \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$).

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{k} & \mathfrak{Y} \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathfrak{Z} & \xrightarrow{i} & \mathfrak{X} \end{array}$$

We start with an ind-scheme version of Lemma 4.24.

Lemma 9.15. *In the diagram above, assume that i is a reasonable closed immersion of reasonable ind-schemes and π is a flat morphism. Then there is a natural isomorphism $\rho^* i^! \simeq k^! \pi^*$ of functors $\mathfrak{X}$–$\mathsf{tors} \longrightarrow \mathfrak{W}$–$\mathsf{tors}$.*

Proof. The (essentially obvious) argument is based on Remark 7.4 (which, in turn, is based on Lemma 4.24). We use the notation similar to the proof of Lemma 9.6. Let $Z \subset \mathfrak{Z}$ be a reasonable closed subscheme; then Z can be also viewed as a reasonable closed subscheme in $\mathfrak{X}$, embedded via i. Put $\mathbf{W} = Z \times_{\mathfrak{Z}} \mathfrak{W}$; then $\mathbf{W}$ is a reasonable closed subscheme in $\mathfrak{W}$, and the scheme $\mathbf{W}$ can be also viewed as a reasonable closed subscheme in $\mathfrak{Y}$, embedded via k. Denote by $\rho_Z \colon \mathbf{W} \longrightarrow Z$ the natural morphism. For any quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$, the desired isomorphism of quasi-coherent torsion sheaves $\rho^* i^! \mathscr{M} \simeq k^! \pi^* \mathscr{M}$ on $\mathfrak{W}$ is provided by the compatible system of isomorphisms

$$(\rho^* i^! \mathscr{M})_{(\mathbf{W})} \simeq \rho_Z^*((i^! \mathscr{M})_{(Z)}) \simeq \rho_Z^*(\mathscr{M}_{(Z)}) \simeq (\pi^* \mathscr{M})_{(\mathbf{W})} \simeq (k^! \pi^* \mathscr{M})_{(\mathbf{W})},$$

of quasi-coherent sheaves on W. Here the first and third isomorphisms hold by Remark 7.4, while the second and third ones are the definition of $i^!$ and $k^!$. □

The next lemma is a generalization of Lemma 7.17.

Lemma 9.16. *In the diagram above, assume that i is a reasonable closed immersion of reasonable ind-schemes and π is a flat affine morphism. Let $\mathscr{M}$ be a quasi-coherent torsion sheaf on $\mathfrak{X}$ and $\mathfrak{G}$ be an $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$; put $\mathscr{N} = \pi^* \mathscr{M} \in \mathfrak{Y}$–$\mathsf{tors}$. Then there is a natural isomorphism*

$$k^!(\mathfrak{G} \otimes_{\mathfrak{Y}} \mathscr{N}) \simeq k^* \mathfrak{G} \otimes_{\mathfrak{W}} k^! \mathscr{N}$$

of quasi-coherent torsion sheaves on $\mathfrak{W}$.

Proof. In the notation $Z \subset \mathfrak{Z}$, $\mathbf{W} = Z \times_{\mathfrak{Z}} \mathfrak{W} \subset \mathfrak{W}$, and $\rho_Z \colon \mathbf{W} \longrightarrow Z$ from the proofs of Lemmas 9.6 and 9.15, we compute

$$(k^!(\mathfrak{G} \otimes_{\mathfrak{Y}} \mathscr{N}))_{(\mathbf{W})} \simeq (\mathfrak{G} \otimes_{\mathfrak{Y}} \mathscr{N})_{(\mathbf{W})} \simeq \mathfrak{G}^{(\mathbf{W})} \otimes_{\mathcal{O}_{\mathbf{W}}} \mathscr{N}_{(\mathbf{W})} \simeq (k^* \mathfrak{G})^{(\mathbf{W})} \otimes_{\mathcal{O}_{\mathbf{W}}} (k^! \mathscr{N})_{(\mathbf{W})}$$

using Lemma 7.17 for the middle isomorphism. Hence the collection of quasi-coherent sheaves $(k^* \mathfrak{G})^{(\mathbf{W})} \otimes_{\mathcal{O}_{\mathbf{W}}} (k^! \mathscr{N})_{(\mathbf{W})}$ on the reasonable closed subschemes $\mathbf{W} \subset \mathfrak{W}$ defines a quasi-coherent torsion sheaf on $\mathfrak{W}$, and it follows easily that this quasi-coherent torsion sheaf is naturally isomorphic to the tensor product $k^* \mathfrak{G} \otimes_{\mathfrak{W}} k^! \mathscr{N}$. □

Now we assume, in the diagram above, that π is a flat affine morphism, i is a closed immersion, and the ind-scheme $\mathfrak{X}$ is ind-semi-separated, ind-Noetherian, and endowed with a dualizing complex $\mathscr{D}^\bullet$. Then $i^!\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{Z}$ (cf. Example 4.8(2)). So Theorem 7.15 provides triangulated equivalences

$$\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}-:\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\,\simeq\,\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}):\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet,-)$$

and

$$\rho^*i^!\mathscr{D}^\bullet\otimes_{\mathfrak{W}}-:\mathsf{D}(\mathfrak{W}_{\mathfrak{Z}}\text{–}\mathsf{flat})\,\simeq\,\mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors}):\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{W}\text{-qc}}(\rho^*i^!\mathscr{D}^\bullet,-).$$

Proposition 9.17. *In the context above, the triangulated equivalences* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ *and* $\mathsf{D}(\mathfrak{W}_{\mathfrak{Z}}\text{–}\mathsf{flat})\simeq\mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors})$ *from Theorem 7.15 transform the left derived functor* $\mathbb{L}k^*\colon\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\longrightarrow\mathsf{D}(\mathfrak{W}_{\mathfrak{X}}\text{–}\mathsf{flat})$ (43) *from Section 9.1 into the right derived functor* $\mathbb{R}k^!\colon\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})\longrightarrow\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{W}\text{–}\mathsf{tors})$ (47) *from Section 9.3.*

Proof. Let $\mathfrak{P}^\bullet$ be a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. We have to construct a natural isomorphism

$$\rho^*i^!\mathscr{D}^\bullet\otimes_{\mathfrak{W}}\mathbb{L}k^*(\mathfrak{P}^\bullet)\,\simeq\,\mathbb{R}k^!(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$$

in the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{Z}}(\mathfrak{W}\text{–}\mathsf{tors})$. The related isomorphism for underived functors

$$\rho^*i^!\mathscr{D}^\bullet\otimes_{\mathfrak{W}}k^*(\mathfrak{P}^\bullet)\,\simeq\,k^!\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{W}}k^*(\mathfrak{P}^\bullet)\,\simeq\,k^!(\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet)$$

holds by Lemmas 9.15 and 9.16.

To prove the desired isomorphism for derived functors, replace the complex $\mathfrak{P}^\bullet$ with a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet$ endowed with a morphism of complexes $\mathfrak{F}^\bullet\longrightarrow\mathfrak{P}^\bullet$ which is an isomorphism in $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$. Then it remains to recall that, according to the proof of Theorem 7.15, the complex $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{F}^\bullet$ (as well as the complex $\pi^*\mathscr{D}^\bullet\otimes_{\mathfrak{Y}}\mathfrak{P}^\bullet$) is a complex of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. ☐

9.9. **Semiderived equivalence and external tensor product.** This Section 9.9 is a relative version of Section 6.6. Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme and $\pi\colon\mathfrak{Y}\longrightarrow\mathfrak{X}$ be an affine morphism of schemes. Let $\mathscr{M}^\bullet\in\mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors})$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. For any complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{G}^\bullet$ on $\mathfrak{Y}$, put

$$\mathbf{\Phi}_{\mathscr{M}^\bullet}(\mathfrak{G}^\bullet)=\pi^*(\mathscr{M}^\bullet)\otimes_{\mathfrak{Y}}\mathfrak{G}^\bullet\,\in\,\mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

According to formula (35) from Section 8.1, the functor $\mathbf{\Phi}_{\mathscr{M}^\bullet}$ induces a well-defined triangulated triangulated functor

$$\mathbf{\Phi}_{\mathscr{M}^\bullet}\colon\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})\longrightarrow\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

Furthermore, any morphism $\mathscr{M}^\bullet\longrightarrow\mathscr{N}^\bullet$ in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$ induces a morphism of triangulated functors $\mathbf{\Phi}_{\mathscr{M}^\bullet}\longrightarrow\mathbf{\Phi}_{\mathscr{N}^\bullet}$, which is an isomorphism of functors whenever the morphism $\mathscr{M}^\bullet\longrightarrow\mathscr{N}^\bullet$ is an isomorphism in $\mathsf{D}^{\mathsf{co}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Proposition 9.18. *Let $\mathfrak{X}'$ and $\mathfrak{X}''$ be ind-semi-separated ind-schemes of ind-finite type over $\Bbbk$, and let $\pi'\colon \mathfrak{Y}' \longrightarrow \mathfrak{X}'$ and $\pi''\colon \mathfrak{Y}'' \longrightarrow \mathfrak{X}''$ be flat affine morphisms of ind-schemes. Consider the induced morphism of the Cartesian products $\pi = \pi' \times_\Bbbk \pi''\colon \mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'' \longrightarrow \mathfrak{X}' \times_\Bbbk \mathfrak{X}''$. Let $\mathscr{D}'^\bullet$ and $\mathscr{D}''^\bullet$ be dualizing complexes on $\mathfrak{X}'$ and $\mathfrak{X}''$, respectively, and let $\mathscr{E}^\bullet$ be the related dualizing complex on $\mathfrak{X}' \times_\Bbbk \mathfrak{X}''$, as in Lemma 6.26. Then the triangulated equivalences*

$$\pi'^*\mathscr{D}'^\bullet \otimes_{\mathfrak{Y}'} - :\ \mathsf{D}(\mathfrak{Y}'_{\mathfrak{X}'}\text{–}\mathsf{flat}) \ \simeq\ \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}'\text{–}\mathsf{tors}),$$
$$\pi''^*\mathscr{D}''^\bullet \otimes_{\mathfrak{Y}''} - :\ \mathsf{D}(\mathfrak{Y}''_{\mathfrak{X}''}\text{–}\mathsf{flat}) \ \simeq\ \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}''}(\mathfrak{Y}''\text{–}\mathsf{tors}),$$

and

$$\pi^*\mathscr{E}^\bullet \otimes_{(\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')} - :\ \mathsf{D}((\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')_{(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')}\text{–}\mathsf{flat}) \ \simeq\ \mathsf{D}_{(\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')}((\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}'')\text{–}\mathsf{tors})$$

from Theorem 7.15 form a commutative square diagram with the external tensor product functors $\boxtimes_\Bbbk$ (50) *and* (51) *from Section 9.5.*

Proof. Let $\mathscr{M}'^\bullet$ and $\mathscr{M}''^\bullet$ be complexes of quasi-coherent torsion sheaves on $\mathfrak{X}'$ and $\mathfrak{X}''$. Then it follows from Lemmas 6.16 and 6.31 that, for any complex of $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaves $\mathfrak{Q}'^\bullet$ on $\mathfrak{Y}'$ and any complex of $\mathfrak{X}''$-flat pro-quasi-coherent pro-sheaves $\mathfrak{Q}''^\bullet$ on $\mathfrak{Y}''$, there is a natural isomorphism

$$\mathbf{\Phi}_{\mathscr{M}'^\bullet \boxtimes_\Bbbk \mathscr{M}''^\bullet}(\mathfrak{Q}'^\bullet \boxtimes_\Bbbk \mathfrak{Q}''^\bullet) \ \simeq\ \mathbf{\Phi}_{\mathscr{M}'^\bullet}(\mathfrak{Q}'^\bullet) \boxtimes_\Bbbk \mathbf{\Phi}_{\mathscr{M}''^\bullet}(\mathfrak{Q}''^\bullet)$$

in the category of complexes of quasi-coherent torsion sheaves on $\mathfrak{Y}' \times_\Bbbk \mathfrak{Y}''$. It remains to take $\mathscr{M}'^\bullet = \mathscr{D}'^\bullet$ and $\mathscr{M}''^\bullet = \mathscr{D}''^\bullet$, and observe that the isomorphism $\mathscr{D}'^\bullet \boxtimes_\Bbbk \mathscr{D}''^\bullet \longrightarrow \mathscr{E}^\bullet$ in the coderived category $\mathsf{D}^{\mathsf{co}}((\mathfrak{X}' \times_\Bbbk \mathfrak{X}'')\text{–}\mathsf{tors})$ induces an isomorphism of triangulated functors $\mathbf{\Phi}_{\mathscr{D}'^\bullet \boxtimes_\Bbbk \mathscr{D}''^\bullet} \longrightarrow \mathbf{\Phi}_{\mathscr{E}^\bullet}$, as per the discussion above. $\square$

9.10. **The semitensor product computed.** Let us recall the setting and notation from the introductory paragraphs of Section 9. Let $\mathfrak{X}$ be an ind-separated ind-scheme of ind-finite type over a field $\Bbbk$, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of schemes. Then the diagonal morphism $\Delta_\mathfrak{Y}\colon \mathfrak{Y} \longrightarrow \mathfrak{Y} \times_\Bbbk \mathfrak{Y}$ decomposes as

$$\mathfrak{Y} \xrightarrow{\delta_{\mathfrak{Y}/\mathfrak{X}}} \mathfrak{Y} \times_\mathfrak{X} \mathfrak{Y} \xrightarrow{\eta_{\mathfrak{Y}/\mathfrak{X}}} \mathfrak{Y} \times_\Bbbk \mathfrak{Y}.$$

Both $\delta = \delta_{\mathfrak{Y}/\mathfrak{X}}$ and $\eta = \eta_{\mathfrak{Y}/\mathfrak{X}}$ are closed immersions of ind-schemes (see [20, Tags 01S7, 01KU(1), 01KR] for scheme versions of these assertions).

In fact, there is a commutative square diagram (a particular case of (42))

(53)
$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\ \ \Delta_\mathfrak{Y}\ \ } & \mathfrak{Y} \times_\Bbbk \mathfrak{Y} \\ & \searrow^{\pi} & \big\downarrow{\scriptstyle \pi \times_\Bbbk \pi} \\ & \mathfrak{X} \xrightarrow{\ \Delta_\mathfrak{X}\ } & \mathfrak{X} \times_\Bbbk \mathfrak{X} \end{array}$$

which is composed of a commutative triangle diagram (a particular case of (44))

(54)
$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\ \delta_{\mathfrak{Y}/\mathfrak{X}}\ } & \mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{Y} \\ & {\scriptstyle\pi}\searrow & \Big\downarrow{\scriptstyle \pi\times_{\mathfrak{X}}\pi} \\ & & \mathfrak{X} \end{array}$$

and a pullback diagram (a particular case of (46))

(55)
$$\begin{array}{ccc} \mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{Y} & \xrightarrow{\ \eta_{\mathfrak{Y}/\mathfrak{X}}\ } & \mathfrak{Y}\times_{\Bbbk}\mathfrak{Y} \\ \Big\downarrow{\scriptstyle \pi\times_{\mathfrak{X}}\pi} & & \Big\downarrow{\scriptstyle \pi\times_{\Bbbk}\pi} \\ \mathfrak{X} & \xrightarrow{\ \Delta_{\mathfrak{X}}\ } & \mathfrak{X}\times_{\Bbbk}\mathfrak{X} \end{array}$$

The morphism $\Delta_{\mathfrak{X}}$ is a closed immersion of ind-schemes. The morphisms of ind-schemes $\pi\times_{\Bbbk}\pi$ and $\pi\times_{\mathfrak{X}}\pi$ are flat and affine.

Theorem 9.19. *In the context above, let $\mathscr{D}^\bullet$ be a rigid dualizing complex on the ind-scheme $\mathfrak{X}$ (in the sense of Section 6.5). Then, for any two complexes of quasi-coherent torsion sheaves $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet \in \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$, there is a natural isomorphism*

$$\mathcal{M}^\bullet \lozenge_{\pi^*\mathscr{D}^\bullet} \mathcal{N}^\bullet \;\simeq\; \mathbb{L}\delta^*\,\mathbb{R}\eta^!(\mathcal{M}^\bullet \boxtimes_{\Bbbk} \mathcal{N}^\bullet) \tag{56}$$

in the $\mathfrak{Y}/\mathfrak{X}$-semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Here the semitensor product functor $\lozenge_{\pi^*\mathscr{D}^\bullet}$ was defined in formula (38) *in Section 8.4. The external tensor product functor $\boxtimes_{\Bbbk}$ was defined in formula* (51) *in Section 9.5, the right derived functor $\mathbb{R}\eta^!$ was defined in Section 9.3, and the left derived functor $\mathbb{L}\delta^*$ was defined in Section 9.2.*

Proof. Let $\mathcal{K}^\bullet$ and $\mathcal{J}^\bullet$ be two complexes of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$ endowed with morphisms of complexes $\mathcal{M}^\bullet \longrightarrow \mathcal{K}^\bullet$ and $\mathcal{N}^\bullet \longrightarrow \mathcal{J}^\bullet$ with the cones whose direct images under π are coacyclic in $\mathfrak{X}$–tors. Then $\mathfrak{F}^\bullet = \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathcal{K}^\bullet)$ and $\mathfrak{G}^\bullet = \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet, \mathcal{J}^\bullet)$ are two complexes in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$ corresponding to $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$, respectively, under the equivalence of categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ from Theorem 7.15; so natural isomorphisms $\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \mathfrak{F}^\bullet \longrightarrow \mathcal{K}^\bullet \longleftarrow \mathcal{M}^\bullet$ and $\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \mathfrak{G}^\bullet \longrightarrow \mathcal{J}^\bullet \longleftarrow \mathcal{N}^\bullet$ exist in $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$. By the definition, we have

$$\mathcal{M}^\bullet \lozenge_{\pi^*\mathscr{D}^\bullet} \mathcal{N}^\bullet = \pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} (\mathfrak{F}^\bullet \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{G}^\bullet).$$

Let $\mathscr{E}^\bullet$ be a (dualizing) complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}\times_{\Bbbk}\mathfrak{X}$ endowed with a morphism of complexes $\mathscr{D}^\bullet \boxtimes_{\Bbbk} \mathscr{D}^\bullet \longrightarrow \mathscr{E}^\bullet$ with the cone coacyclic in $(\mathfrak{X}\times_{\Bbbk}\mathfrak{X})$–$\mathsf{tors}$. Since $\mathscr{D}^\bullet$ is assumed to be a rigid dualizing complex, we have a homotopy equivalence $\mathscr{D}^\bullet \simeq \Delta_{\mathfrak{X}}^!\mathscr{E}^\bullet$ of complexes in $\mathfrak{X}$–$\mathsf{tors}_{\mathsf{inj}}$. Now we compute

$$\begin{aligned}
&\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} (\boldsymbol{\mathfrak{F}}^\bullet \otimes^{\mathfrak{Y},\mathbb{L}} \boldsymbol{\mathfrak{G}}^\bullet) \overset{9.13}{\simeq} \pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \mathbb{L}\Delta_{\mathfrak{Y}}^*(\boldsymbol{\mathfrak{F}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathfrak{G}}^\bullet) \\
&\overset{9.8}{\simeq} \pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \mathbb{L}\delta^*\,\mathbb{L}\eta^*(\boldsymbol{\mathfrak{F}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathfrak{G}}^\bullet) \overset{9.14}{\simeq} \mathbb{L}\delta^*\big((\pi\times_{\mathfrak{X}}\pi)^*(\mathscr{D}^\bullet) \otimes_{(\mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{Y})} \mathbb{L}\eta^*(\boldsymbol{\mathfrak{F}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathfrak{G}}^\bullet)\big) \\
&\overset{9.17}{\simeq} \mathbb{L}\delta^*\,\mathbb{R}\eta^!\big((\pi\times_{\Bbbk}\pi)^*(\mathscr{E}^\bullet) \otimes_{(\mathfrak{Y}\times_{\Bbbk}\mathfrak{Y})} (\boldsymbol{\mathfrak{F}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathfrak{G}}^\bullet)\big) \\
&\overset{9.18}{\simeq} \mathbb{L}\delta^*\,\mathbb{R}\eta^!\big((\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{F}}^\bullet) \boxtimes_{\Bbbk} (\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} \boldsymbol{\mathfrak{G}}^\bullet)\big) \overset{9.12}{\simeq} \mathbb{L}\delta^*\,\mathbb{R}\eta^!(\boldsymbol{\mathscr{M}}^\bullet \boxtimes_{\Bbbk} \boldsymbol{\mathscr{N}}^\bullet),
\end{aligned}$$

where the numbers over the isomorphism signs indicate the relevant propositions and lemma where the natural isomorphisms are established. □

10. Invariance under Postcomposition with a Smooth Morphism

Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, and let $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be a smooth affine morphism of finite type. Let $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ denote the composition $\pi = \tau\pi'$. Let $\mathscr{D}^\bullet$ be a dualizing complex on $\mathfrak{X}$; then $\mathscr{D}'^\bullet = \tau^*\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{X}'$.

The aim of this section is to show that the constructions of Sections 7–8, including the semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ and the semitensor product operation on it, are preserved by the passage from the flat affine morphism $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ to the flat affine morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$.

10.1. **Weakly smooth morphisms.** We refer to [20, Tag 01V4] for a discussion of smooth morphisms of schemes. For the purposes of this section, a slightly weaker condition is suffient; we call it *weak smoothness.* Essentially, a morphism of schemes is said to be weakly smooth if it is flat with regular fibers of bounded Krull dimension.

Let X be a scheme and $x \in X$ be a point. Denote by $\kappa_X(x)$ the residue field of the point x on X. Abusing notation, we will denote simply by x the one-point scheme $\operatorname{Spec}\kappa_X(x)$. Then we have a natural morphism of schemes $x \longrightarrow X$.

Let $f\colon Y \longrightarrow X$ be a morphism of schemes and $x \in X$ be a point. Then the scheme $Y_x = x \times_X Y$ is called the *fiber* of f over x. Given an integer $d \geq 0$, we will say that the morphism f is *weakly smooth of relative dimension* $\leq d$ if f is flat *and* for every point $x \in X$ the fiber Y_x is a regular Noetherian scheme of Krull dimension $\leq d$.

By [20, Tags 01VB and 00TT], any smooth morphism of schemes is weakly smooth. According to [20, Tag 01V8], any weakly smooth morphism of finite type between Noetherian schemes over a field of characteristic 0 is smooth. This is not true in finite characteristic because of nonseparability issues (an inseparable finite field extension is the simplest example of a nonsmooth flat morphism with regular fibers).

Notice that weak smoothness is *not* preserved by base change, generally speaking (e. g., base changes of inseparable finite field extensions can have nilpotent elements in the fibers). However, some base changes do preserve it. Let us say that a morphism of schemes $Z \longrightarrow X$ *does not extend the residue fields* if for every point $z \in Z$ and its image $x \in X$ the induced field extension $\kappa_X(x) \longrightarrow \kappa_Z(z)$ is an isomorphism.

In particular, locally closed immersions of schemes do not extend the residue fields. Clearly, if a morphism $Y \longrightarrow X$ is weakly smooth of relative dimension $\le d$ and a morphism $Z \longrightarrow X$ does not extend the residue fields, then the morphism $Z \times_X Y \longrightarrow Z$ is also weakly smooth of relative dimension $\le d$.

Any closed immersion of schemes is injective as a map of the underlying sets of points. So any strict ind-scheme $\mathfrak{X}$, represented by an inductive system of closed immersions of schemes $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma \in \Gamma} X_\gamma$, gives rise to an inductive system of injective maps of the underlying sets. The inductive limit of this inductive system of sets is called the *underlying set of points of* $\mathfrak{X}$. Given a point $x \in \mathfrak{X}$, the residue field $\kappa = \kappa_{\mathfrak{X}}(x)$ is well-defined, because the closed immersions $X_\gamma \longrightarrow X_\delta$, $\gamma < \delta \in \Gamma$, do not extend the residue fields (so one can take any $\gamma \in \Gamma$ such that $x \in X_\gamma \subset \mathfrak{X}$ and put $\kappa_{\mathfrak{X}}(x) = \kappa_{X_\gamma}(x)$). Denoting the scheme $\operatorname{Spec} \kappa_{\mathfrak{X}}(x)$ simply by x, we have a morphism of ind-schemes $x \longrightarrow \mathfrak{X}$.

Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes with is “representable by schemes”. Then, for every point $x \in X$, the fiber $x \times_{\mathfrak{X}} \mathfrak{Y}$ is a scheme. As above, we will say that the morphism f is *weakly smooth of relative dimension* $\le d$ if f is flat and for every point $x \in X$ the scheme $x \times_{\mathfrak{X}} \mathfrak{Y}$ is Noetherian and regular of Krull dimension $\le d$. Clearly, the morphism of ind-schemes f is weakly smooth of relative dimension $\le d$ if and only if, for every $\gamma \in \Gamma$, the morphism of schemes $f_\gamma : Y_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow X_\gamma$ is weakly smooth of relative dimension $\le d$.

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme and $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of ind-schemes. One says that f is a *morphism of finite type* if for every Noetherian scheme T and every morphism of ind-schemes $T \longrightarrow \mathfrak{X}$ the fibered product $T \times_{\mathfrak{X}} \mathfrak{Y}$ is a scheme *and* the morphism of schemes $T \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow T$ is of finite type. It suffices to check these conditions for the closed subschemes $T = X_\gamma$ appearing in a given representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes.

10.2. **Flat and injective dimension under weakly smooth morphisms.** Let $\mathcal{M}$ be a quasi-coherent sheaf on a scheme X, and $d \ge 0$ be an integer. One says that the *injective dimension* of $\mathcal{M}$ does not exceed d if there exists an exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \cdots \longrightarrow \mathcal{J}^d \longrightarrow 0$ of quasi-coherent sheaves on X with injective quasi-coherent sheaves $\mathcal{J}^i$. On a (locally) Noetherian scheme X, injectivity of quasi-coherent sheaves is a local property [19, Proposition II.7.17 and Theorem II.7.18], hence so is the injective dimension: the injective dimension of $\mathcal{M}$ is equal to the supremum of the injective dimensions of the $\mathcal{O}_X(U_\alpha)$-modules $\mathcal{M}(U_\alpha)$, where $X = \bigcup_\alpha U_\alpha$ is any given affine open covering of the scheme X.

Let $\mathcal{M}$ be a quasi-coherent sheaf on a quasi-compact, semi-separated scheme X. According to [30, Section 2.4] or [12, Lemma A.1], every quasi-coherent sheaf on X is a quotient sheaf of a flat quasi-coherent sheaf. One says that the *flat dimension* of $\mathcal{M}$ does not exceed d if there exists an exact sequence $0 \longrightarrow \mathcal{F}_d \longrightarrow \mathcal{F}_{d-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{M} \longrightarrow 0$ of quasi-coherent sheaves on X with flat quasi-coherent sheaves $\mathcal{F}_i$. Since flatness of quasi-coherent sheaves is a local property, so is the flat dimension: in the same notation as above, the flat dimension of $\mathcal{M}$ is equal to the supremum of the flat dimensions of the $\mathcal{O}_X(U_\alpha)$-modules $\mathcal{M}(U_\alpha)$.

The following relative form of Hilbert's syzygy theorem is essentially well-known.

Proposition 10.1. *Let R be an associative ring and $S = R[x_1, \dots, x_d]$ be the ring of polynomials in d variables with the coefficients in R. Then, for any S-module N:*

(a) *the flat dimension of N as an S-module does not exceed d plus the flat dimension of N as an R-module;*

(b) *the projective dimension of N as an S-module does not exceed d plus the projective dimension of N as an R-module;*

(c) *the injective dimension of N as an S-module does not exceed d plus the injective dimension of N as an R-module.*

Proof. Parts (b) and (c) follow directly from [13, spectral sequence (I) in Section 2 and Theorem 6 in Section 4]. Parts (a) and (b) are provable by straightforward induction in d using [29, Proposition 7.5.2]. ☐

The aim of this Section 10.2 is to prove the following partial generalization of Proposition 10.1(a,c).

Proposition 10.2. *Let X be a Noetherian scheme and $f\colon Y \longrightarrow X$ be an affine morphism of schemes. Assume that the morphism f is weakly smooth of relative dimension $\le d$. Then, for any quasi-coherent sheaf $\mathcal{N}$ on Y:*

(a) *assuming that the scheme X is semi-separated, the flat dimension of $\mathcal{N}$ does not exceed d plus the flat dimension of the quasi-coherent sheaf $f_*\mathcal{N}$ on X;*

(b) *assuming that f is a morphism of finite type, the injective dimension of $\mathcal{N}$ does not exceed d plus the injective dimension of the quasi-coherent sheaf $f_*\mathcal{N}$ on X.*

For any commutative ring R, an element $a \in R$, and an R-module M, we denote by ${}_aM = \ker(M \overset{a}{\longrightarrow} M)$ the submodule of all elements annihilated by a in M. So ${}_aM$ is a module over the ring R/aR.

Lemma 10.3. *Let R be a commutative ring and $a \in R$ be an element. Then*

(a) *for every flat R-module F, the R/aR-module F/aF is flat;*

(b) *for every injective R-module J, the R/aR-module ${}_aJ$ is injective.*

Proof. For any ring homomorphism $R \longrightarrow T$, the functor $F \longmapsto T \otimes_R F$ takes flat R-modules to flat T-modules, and the functor $J \longmapsto \operatorname{Hom}_R(T, J)$ takes injective R-modules to injective T-modules. It remains to apply these observations to the ring homomorphism $R \longrightarrow R/aR = T$ in order to deduce the assertions of the lemma. ☐

Lemma 10.4. *Let $R \longrightarrow S$ be a homomorphism of commutative rings such that S is a flat R-module. Let $a \in R$ be an element. Then*

(a) *for any flat resolution $F_\bullet$ of an R-flat S-module G, the complex $F_\bullet/aF_\bullet$ is a flat resolution of an (R/aR-flat) S/aS-module G/aG;*

(b) *for any injective resolution $J^\bullet$ of an R-injective S-module K, the complex ${}_aJ^\bullet$ is an injective resolution of an (R/aR-injective) S/aS-module ${}_aK$.*

Proof. Let us prove part (b). Since S is a flat R-module, the underlying R-module of any injective S-module is injective. So, viewed as a complex of R-modules,

$J^\bullet$ is an injective resolution of an injective R-module K. Hence the complex ${}_aJ^\bullet = \operatorname{Hom}_R(R/aR, J^\bullet)$ is exact, i. e., it is a resolution of the R-module ${}_aK$. The R/aR-module ${}_aK$ is injective by Lemma 10.3(b), and the terms of the complex ${}_aJ^\bullet$ are injective S/aS-modules by the same lemma. □

Lemma 10.5. *Let S be a commutative ring and $a \in S$ be a nonzero-dividing (regular) element.*

(a) *Let M and N be two S-modules for which the maps $M \xrightarrow{a} M$ and $N \xrightarrow{a} N$ are injective. Then there is a distinguished triangle*

$$M \otimes_{\S}^{\mathbb{L}} N \xrightarrow{\ a\ } M \otimes_S^{\mathbb{L}} N \longrightarrow M/aM \otimes_{S/aS}^{\mathbb{L}} N/aN \longrightarrow M \otimes_S^{\mathbb{L}} N[1]$$

in the derived category of S-modules.

(b) *Let M and K be two S-modules such that the map $M \xrightarrow{a} M$ is injective and the map $K \xrightarrow{a} K$ is surjective. Then there is a distinguished triangle*

$$\mathbb{R}\operatorname{Hom}_S(M,K)[-1] \longrightarrow \mathbb{R}\operatorname{Hom}_{S/aS}(M/aM,\ {}_aK) \longrightarrow \mathbb{R}\operatorname{Hom}_S(M,K) \xrightarrow{\ a\ } \mathbb{R}\operatorname{Hom}_S(M,K)$$

in the derived category of S-modules.

Proof. Let us prove part (b). Firstly, there is a distinguished triangle $M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow M[1]$ in $\mathsf{D}(S\text{–}\mathsf{mod})$. Applying $\mathbb{R}\operatorname{Hom}_S(-,K)$, we obtain

$$\mathbb{R}\operatorname{Hom}_S(M,K)[-1] \longrightarrow \mathbb{R}\operatorname{Hom}_S(M/aM,K) \longrightarrow \mathbb{R}\operatorname{Hom}_S(M,K) \xrightarrow{\ a\ } \mathbb{R}\operatorname{Hom}_S(M,K).$$

It remains to construct an isomorphism

$$\mathbb{R}\operatorname{Hom}_S(M/aM,K) \simeq \mathbb{R}\operatorname{Hom}_{S/aS}(M/aM,\ {}_aK)$$

in $\mathsf{D}(S\text{–}\mathsf{mod})$. For this purpose, choose an injective resolution $J^\bullet$ of the S-module K. Notice that, for any injective S-module J, the map $J \xrightarrow{a} J$ is surjective, because it can be obtained by applying the functor $\operatorname{Hom}_S(-,J)$ to an injective S-module morphism $S \xrightarrow{a} S$. So $J^\bullet$ is a resolution of an a-divisible S-module K by a-divisible S-modules. It follows that the complex ${}_aJ^\bullet$ is exact, i. e., it is a resolution of the S-module ${}_aK$. By Lemma 10.3(b), ${}_aJ^\bullet$ is an injective resolution of the S/aS-module ${}_aK$. Finally, we use the isomorphism of complexes of S-modules (in fact, S/aS-modules) $\operatorname{Hom}_S(M/aM, J^\bullet) \simeq \operatorname{Hom}_{S/aS}(M/aM,\ {}_aJ^\bullet)$. □

Proof of Proposition 10.2(a). A result bearing some similarity with, but still quite different from our assertion can be found in [3, Lemma 2.7].

The question is local in X, so it reduces to affine schemes, for which it means the following. Let $R \longrightarrow S$ be a homomorphism of commutative rings such that the ring R is Noetherian, the R-module S is flat, and for every prime ideal $\mathfrak{p} \subset R$, the fiber ring $\kappa_R(\mathfrak{p}) \otimes_R S$ is Noetherian and regular of Krull dimension $\le d$. Here $\kappa_R(\mathfrak{p}) = R_\mathfrak{p}/R_\mathfrak{p}\mathfrak{p}$ denotes the residue field of the prime ideal $\mathfrak{p}$ in R. Then, for any

S-module N, the flat dimension of the S-module N does not exceed d plus the flat dimension of the R-module N.

Since S is a flat R-module, all flat S-modules are also flat as R-modules. Let $F_\bullet$ be a flat resolution of the S-module N; for every $n \geq 0$, denote by $\Omega^n N$ the cokernel of the S-module morphism $F_{n+1} \longrightarrow F_n$ (so, in particular, $\Omega^0 N = N$). Suppose that the flat dimension e of the R-module N is finite. Then $G = \Omega^e N$ is a flat R-module. It remains to show that the flat dimension of the S-module G does not exceed d; then it will follow that $\Omega^d G = \Omega^{d+e} N$ is a flat S-module, so the flat dimension of the S-module N does not exceed $d + e$. Thus, in order to prove the desired assertion, it suffices to consider the case of an R-flat S-module G.

The argument proceeds by Noetherian induction in the ring R. So we assume that the desired assertion holds of all the quotient rings of R by nonzero ideals. Let G be an R-flat S-module. We consider two cases separately.

Case I. Suppose that R has zero-divisors. Let a and $b \in R$ be a pair of nonzero elements for which $ab = 0$.

By Lemma 10.3(a), G/aG is an R/aR-flat S/aS-module and G/bG is an R/bR-flat S/bS-module. Furthermore, the ring S/aS is a flat R/aR-module, and it is clear that the fiber rings of the ring homomorphism $R/aR \longrightarrow S/aS$ are regular of Krull dimension $\leq d$ (and similarly for $R/bR \longrightarrow S/bS$). By the assumption of Noetherian induction, the flat dimensions of the S/aS-module G/aG and the S/bS-module G/bG do not exceed d.

We have to show that $\mathrm{Tor}^S_{d+1}(M, G) = 0$ for all S-modules M. Consider the short exact sequence of S-modules $0 \longrightarrow aM \longrightarrow M \longrightarrow M/aM \longrightarrow 0$. The R-module aM is annihilated by b and the R-module M/aM is annihilated by a. So the problem reduces to R-modules M for which either $aM = 0$ or $bM = 0$.

Suppose that $aM = 0$. Let $F_\bullet$ be a flat resolution of the S-module G; then, by Lemma 10.4(a), $F_\bullet/aF_\bullet$ is a flat resolution of the S/aS-module G/aG. Hence $\mathrm{Tor}^S_i(M, G) \simeq \mathrm{Tor}^{S/aS}_i(M, G/aG) = 0$ for $i > d$, as desired.

Case II. Suppose that R is an integral domain, and denote by Q the field of fractions of R. For any S-module M, we have $Q \otimes_R \mathrm{Tor}^S_i(M, G) \simeq \mathrm{Tor}^{Q\otimes_R S}_i(Q \otimes_R M, Q \otimes_R G)$ for all $i \geq 0$. By assumption, $Q \otimes_R S$ is a regular Noetherian ring of Krull dimension $\leq d$, so the global dimension of $Q \otimes_R S$ does not exceed d. Hence $Q \otimes_R \mathrm{Tor}^S_i(M, G) = 0$ for all $i > d$, and it follows that $\mathrm{Tor}^S_i(M, G)$ is a torsion R-module.

Let $a \in R$ be a nonzero element. In order to show that $\mathrm{Tor}^S_i(M, G) = 0$ for $i > d$, it suffices to prove that there are no nonzero elements annihilated by a in $\mathrm{Tor}^S_i(M, G)$.

Let $0 \longrightarrow \Omega M \longrightarrow F \longrightarrow M \longrightarrow 0$ be a short exact sequence of S-modules with a flat S-module F. If $d \geq 1$, then we have $\mathrm{Tor}^S_{d+1}(M, G) \simeq \mathrm{Tor}^S_d(\Omega M, G)$. When $d = 0$, the S-module $\mathrm{Tor}^S_1(M, G)$ is the kernel of the morphism $\Omega M \otimes_S G \longrightarrow F \otimes_S G$. Since F is a flat S-module and G is a flat R-module, the R-module $F \otimes_S G$ is flat; in particular, it contains no nonzero elements annihilated by a. So the submodules of elements annihilated by a in the S-modules $\mathrm{Tor}^S_1(M, G)$ and $\Omega M \otimes_S G$ are naturally isomorphic. In both cases, it remains to show that there are no nonzero elements annihilated by a in $\mathrm{Tor}^S_d(\Omega M, G)$.

Both the R-modules ΩM and G contain no nonzero elements annihilated by a (since ΩM is a submodule in a flat R-module F). The same applies to the R-module S. By Lemma 10.5(a), we have a distinguished triangle in $\mathsf{D}(S\text{–}\mathsf{mod})$

$$\Omega M \otimes_S^{\mathbb{L}} G \xrightarrow{\ a\ } \Omega M \otimes_S^{\mathbb{L}} G \longrightarrow (\Omega M/a\Omega M) \otimes_{S/aS}^{\mathbb{L}} G/aG \longrightarrow \Omega M \otimes_S^{\mathbb{L}} G[1].$$

It follows from the related long exact sequence of cohomology modules that $\mathrm{Tor}_{d+1}^{S/aS}(\Omega M/a\Omega M, G/aG) \neq 0$ whenever there are any nonzero elements annihilated by a in $\mathrm{Tor}_d^S(\Omega M, G)$.

Finally, similarly to Case I, G/aG is an R/aR-flat S/aS-module, the ring S/aS is a flat R/aR-module, and the fiber rings of the ring homomorphism $R/aR \longrightarrow S/aS$ are regular of Krull dimension $\leq d$. By the assumption of Noetherian induction, the flat dimension of the S/aS-module G/aG does not exceed d, hence $\mathrm{Tor}_{d+1}^{S/aS}(\Omega M/a\Omega M, G/aG) = 0$ and we are done. □

For any element a in a ring R, we denote by $R[a^{-1}]$ the localization of the ring R at the multiplicative subset $\{1, a, a^2, a^3, \dots\} \subset R$. For any R-module E, we put $E[a^{-1}] = R[a^{-1}] \otimes_R E$.

Lemma 10.6 (Grothendieck's generic freeness)**.** *Let R be a Noetherian commutative integral domain, S be a finitely generated commutative R-algebra, and M be a finitely generated S-module. Then there exists a nonzero element $a \in R$ such that $M[a^{-1}]$ is a free $R[a^{-1}]$-module.*

Proof. This is [17, Lemme 6.9.2] or [28, Theorem 24.1]. □

Lemma 10.7. *Let S be a commutative ring and $a \in S$ be an element. Let M and K be S-modules such that $\mathrm{Ext}_S^1(S[a^{-1}], K) = 0$. Then for every $i \geq 0$ there is a natural surjective S-module map*

$$\mathrm{Ext}^i_{S[a^{-1}]}(M[a^{-1}], \mathrm{Hom}_S(S[a^{-1}], K)) \longrightarrow \mathrm{Hom}_S(S[a^{-1}], \mathrm{Ext}^i_S(M, K)).$$

Proof. The key observation is that the projective dimension of the S-module $S[a^{-1}]$ cannot exceed 1 (see [49, proof of Lemma 2.1] or [50, Lemma 1.9]). Therefore, our assumption implies that $\mathbb{R}\mathrm{Hom}_S(S[a^{-1}], K) = \mathrm{Hom}_S(S[a^{-1}], K)$. Furthermore, one clearly has

$$\mathbb{R}\mathrm{Hom}_{S[a^{-1}]}(M[a^{-1}], \mathbb{R}\mathrm{Hom}_S(S[a^{-1}], K)) \simeq \mathbb{R}\mathrm{Hom}_S(S[a^{-1}], \mathbb{R}\mathrm{Hom}_S(M, K)).$$

Finally, since $\mathbb{R}\mathrm{Hom}_S(S[a^{-1}], -)$ is a derived functor of homological dimension ≤ 1, for any complex of S-modules $C^\bullet$ and integer $i \in \mathbb{Z}$ there is a natural short exact sequence of S-modules

$$0 \longrightarrow \mathrm{Ext}^1_S(S[a^{-1}], H^{i-1}(C^\bullet)) \\ \longrightarrow H^i\mathbb{R}\mathrm{Hom}_S(S[a^{-1}], C^\bullet) \longrightarrow \mathrm{Hom}_S(S[a^{-1}], H^i(C^\bullet)) \longrightarrow 0.$$

Taking $C^\bullet = \mathbb{R}\operatorname{Hom}_S(M, K)$ and combining these observations, we obtain, in the situation at hand, a natural short exact sequence of S-modules

$$0 \longrightarrow \operatorname{Ext}^1_S(S[a^{-1}], \operatorname{Ext}^{i-1}_S(M, K)) \longrightarrow \operatorname{Ext}^i_{S[a^{-1}]}(M[a^{-1}], \operatorname{Hom}_S(S[a^{-1}], K)) \\ \longrightarrow \operatorname{Hom}_S(S[a^{-1}], \operatorname{Ext}^i_S(M, K)) \longrightarrow 0.$$

□

Lemma 10.8. *Let S be a commutative ring, $a \in S$ be an element, and E be an S-module. Suppose that the map $a\colon E \longrightarrow E$ is surjective. Then the map $\operatorname{Hom}_S(S[a^{-1}], E) \longrightarrow E$ induced by the localization map $S \longrightarrow S[a^{-1}]$ is surjective. In particular, if $\operatorname{Hom}_S(S[a^{-1}], E) = 0$, then $E = 0$.*

Proof. Given an element $e \in E$, put $e_0 = e$, and for every $n \geq 1$ choose an element $e_n \in E$ such that $ae_n = e_{n-1}$. Then the sequence of elements $e_0, e_1, e_2, \ldots \in E$ defines the desired S-module morphism $S[a^{-1}] \longrightarrow E$. □

Proof of Proposition 10.2(b). The question is local in X, so it reduces to affine schemes, for which it means the following. Let $R \longrightarrow S$ be a homomorphism of Noetherian commutative rings such that S is a finitely generated R-algebra, the R-module S is flat, and and for every prime ideal $\mathfrak{p} \subset R$, the fiber ring $\kappa_R(\mathfrak{p}) \otimes_R S$ is regular of Krull dimension $\leq d$. Then for any S-module N, the injective dimension of the S-module N does not exceed d plus the injective dimension of the R-module N.

Since S is a flat R-module, all injective S-modules are also injective as R-modules. Arguing similarly to the proof of part (a) above, one reduces the question to the case of an R-injective S-module K, for which one has to prove that its injective dimension as an S-module does not exceed d.

As in part (a), the argument proceeds by Noetherian induction in R (so we assume that the desired assertion holds for all the quotient rings of R by nonzero ideals).

Case I. Suppose that R has zero-divisors. Let a and $b \in R$ be a pair of nonzero elements for which $ab = 0$. By Lemma 10.3(b), ${}_aK$ is an R/aR-injective S/aS-module and ${}_bK$ is an R/bR-injective S/bS-module. By the assumption of Noetherian induction, the flat dimensions of the S/aS-module ${}_aK$ and the S/bS-module ${}_bK$ do not exceed d.

We have to show that $\operatorname{Ext}^{d+1}_S(M, K) = 0$ for all S-modules M. Using the short exact sequence of S-modules $0 \longrightarrow aM \longrightarrow M \longrightarrow M/aM \longrightarrow 0$, the question is reduced to R-modules M for which either $aM = 0$ or $bM = 0$.

Suppose that $aM = 0$. Let $J^\bullet$ be an injective resolution of the S-module K; then, by Lemma 10.4(b), ${}_aJ^\bullet$ is an injective resolution of the S/aS-module ${}_aK$. Hence $\operatorname{Ext}^i_S(M, K) \simeq \operatorname{Ext}^i_{S/aS}(M, {}_aK) = 0$ for $i > d$, as desired.

Case II. Suppose that R is an integral domain. It suffices to show that $\operatorname{Ext}^{d+1}_S(M, K) = 0$ for all finitely generated S-modules M. By Lemma 10.6, there exists a nonzero element $a \in R$ such that the $R[a^{-1}]$-module $M[a^{-1}]$ is free. According to the assertion of part (a), it follows that the flat dimension of the $S[a^{-1}]$-module $M[a^{-1}]$ does not exceed d. Since $S[a^{-1}]$ is a Noetherian ring and $M[a^{-1}]$ is a finitely generated $S[a^{-1}]$-module, the projective dimension of the

$S[a^{-1}]$-module $M[a^{-1}]$ is equal to its flat dimension (as all finitely presented flat modules are projective). Thus the projective dimension also does not exceed d.

We have shown that $\operatorname{Ext}^i_{S[a^{-1}]}(M[a^{-1}], L) = 0$ for all $S[a^{-1}]$-modules L and all $i > d$. Let us apply this observation to the $S[a^{-1}]$-module $L = \operatorname{Hom}_S(S[a^{-1}], K)$. Notice that $\operatorname{Ext}^1_S(S[a^{-1}], K) \simeq \operatorname{Ext}^1_R(R[a^{-1}], K) = 0$, since K is an injective R-module. Using Lemma 10.7, we can conclude from $\operatorname{Ext}^i_{S[a^{-1}]}(M[a^{-1}], \operatorname{Hom}_S(S[a^{-1}], K)) = 0$ that $\operatorname{Hom}_S(S[a^{-1}], \operatorname{Ext}^i_S(M, K)) = 0$ for $i > d$.

In view of Lemma 10.8, it now suffices to show that the map $a\colon \operatorname{Ext}^{d+1}_S(M, K) \longrightarrow \operatorname{Ext}^{d+1}_S(M, K)$ is surjective, i. e., the S-module $\operatorname{Ext}^{d+1}_S(M, K)$ is a-divisible. From this point on, the argument again proceeds similarly (or rather, dually) to the proof of part (a).

Let $0 \longrightarrow \Omega M \longrightarrow P \longrightarrow M \longrightarrow 0$ be a short exact sequence of S-modules with a projective S-module P. If $d \geq 1$, then the we have $\operatorname{Ext}^{d+1}_S(M, K) \simeq \operatorname{Ext}^d_S(\Omega M, K)$. When $d = 0$, the S-module $\operatorname{Ext}^1_S(M, K)$ is the cokernel of the morphism $\operatorname{Hom}_S(P, K) \longrightarrow \operatorname{Hom}_S(\Omega M, K)$. Since P is a projective S-module and K is an injective R-module, the R-module $\operatorname{Hom}_S(P, K)$ is injective; in particular, it is a-divisible (as a is a nonzero-divisor in R). So the quotient modules of $\operatorname{Ext}^1_S(M, K)$ and $\operatorname{Hom}_S(\Omega M, K)$ by the action of a are naturally isomorphic; in particular, the S-module $\operatorname{Ext}^1_S(M, K)$ is a-divisible if and only if the S-module $\operatorname{Hom}_S(\Omega M, K)$ is. Thus, in both cases $d \geq 1$ or $d = 0$, it remains to show that the S-module $\operatorname{Ext}^d_S(\Omega M, K)$ is a-divisible.

Both the R-modules S and ΩM contain no nonzero elements annihilated by a (since S is a flat R-module and ΩM is a submodule of a projective S-module), while the R-module K is a-divisible (since it is injective). By Lemma 10.5(b), we have a distinguished triangle in $\mathsf{D}(S\text{–}\mathsf{mod})$

$$\mathbb{R}\operatorname{Hom}_S(\Omega M, K)[-1] \longrightarrow \mathbb{R}\operatorname{Hom}_{S/aS}(\Omega M/a\Omega M,\ {}_aK) \longrightarrow \mathbb{R}\operatorname{Hom}_S(\Omega M, K) \xrightarrow{\ a\ } \mathbb{R}\operatorname{Hom}_S(\Omega M, K).$$

It follows from the related long exact sequence of cohomology modules that $\operatorname{Ext}^{d+1}_{S/aS}(\Omega M/a\Omega M,\ {}_aK) \neq 0$ whenever the map $a\colon \operatorname{Ext}^d_S(\Omega M, K) \longrightarrow \operatorname{Ext}^d_S(\Omega M, K)$ is not surjective.

Finally, similarly to the proof of part (a), the assumption of Noetherian induction is applicable to the ring homomorphism $R/aR \longrightarrow S/aS$ and the S/aS-module ${}_aK$, which is R/aR-injective by Lemma 10.3(b). Hence $\operatorname{Ext}^{d+1}_{S/aS}(\Omega M/a\Omega M,\ {}_aK) = 0$ and we are done. □

10.3. **Preservation of the derived category of pro-sheaves.** Let E be an exact category. Assume that the additive category E is *weakly idempotent complete*, i. e., it contains the kernels of its split epimorphisms, or equivalently, the cokernels of its split monomorphisms. A full subcategory $\mathsf{F} \subset \mathsf{E}$ is said to be *resolving* if the following conditions are satisfied:

(i) F is closed under extensions in E, i. e., for any admissible short exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ in E with $E', E'' \in \mathsf{F}$ one has $E \in \mathsf{F}$;

(ii) F is closed under the kernels of admissible epimorphisms in E, i. e., for any admissible short exact sequence $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ in E with E, $E'' \in \mathsf{F}$ one has $E' \in \mathsf{F}$;

(iii) for any object $E \in \mathsf{E}$ there exists an object $F \in \mathsf{F}$ together with an admissible epimorphism $F \longrightarrow E$ in E.

Let $\mathsf{F} \subset \mathsf{E}$ be a resolving subcategory and $d \geq 0$ be an integer. One says that the F-*resolution dimension* of an object $E \in \mathsf{E}$ does not exceed d if there exists an exact sequence $0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0$ in E with $F_i \in \mathsf{F}$. According to [61, Proposition 2.3(1)] or [44, Corollary A.5.2], the F-resolution dimension of an object $E \in \mathsf{E}$ does not depend on the choice of a resolution.

Since a resolving subcategory $\mathsf{F} \subset \mathsf{E}$ is closed under extensions by (i), it inherits an exact category structure from the ambient exact category F.

Proposition 10.9. *Let E be a weakly idempotent-complete exact category and $\mathsf{F} \subset \mathsf{E}$ be a resolving subcategory such that, for a certain finite integer $d \geq 0$, the F-resolution dimensions of all the objects in E do not exceed d. Then the triangulated functor between the derived categories $\mathsf{D}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{E})$ induced by the exact inclusion of exact categories $\mathsf{F} \longrightarrow \mathsf{E}$ is an equivalence of triangulated categories.*

Proof. This is a part of [44, Proposition A.5.6]. □

Let $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be flat affine morphisms of ind-semi-separated ind-schemes. Put $\pi = \tau\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Recall the notation $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ for the exact category of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ (as defined in Section 7.2).

Lemma 10.10. (a) *The exact category $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$ is a resolving full subcategory in the exact category $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. The exact category structure of $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$ is inherited from the exact category structure of $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$.*

(b) *Assume additionally that the ind-scheme $\mathfrak{X}$ is ind-Noetherian and the morphism $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ is weakly smooth of relative dimension $\leq d$ (in the sense of Section 10.1) for some finite integer d. Then the resolution dimension of any object of the exact category $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ with respect to the resolving subcategory $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$ does not exceed d (in other words, the $\mathfrak{X}'$-flat dimension of any $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$ does not exceed d).*

Proof. Part (a): let $\mathfrak{G}$ be an $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$; so $\pi'_*\mathfrak{G}$ is a flat pro-quasi-coherent pro-sheaf on $\mathfrak{X}'$. Since $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ is a flat affine morphism, the functor $\tau_*\colon \mathfrak{X}'\text{–}\mathsf{pro} \longrightarrow \mathfrak{X}\text{–}\mathsf{pro}$ takes flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}'$ to flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$. Hence $\pi_*\mathfrak{G} = \tau_*\pi'_*\mathfrak{G}$ is a flat pro-quasi-coherent pro-sheaf on $\mathfrak{X}$; so the pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ is $\mathfrak{X}$-flat. This proves the inclusion $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of ind-schemes. Put $X'_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{X}'$ and $\mathbf{Y}_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$; then $\mathfrak{X}' = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X'_\gamma$ and $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} \mathbf{Y}_\gamma$ are similar representations of $\mathfrak{X}'$ and $\mathfrak{Y}$. We have flat affine morphisms of schemes $\mathbf{Y}_\gamma \longrightarrow X'_\gamma \longrightarrow X_\gamma$.

A pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ is $\mathfrak{X}$-flat (resp., $\mathfrak{X}'$-flat) if and only if the quasi-coherent sheaf $\mathfrak{G}^{(\mathbf{Y}_\gamma)}$ on $\mathbf{Y}_\gamma$ is X_γ-flat (resp., X'_γ-flat) for every $\gamma \in \Gamma$. Furthermore, a short sequence $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$ is exact in $\mathfrak{Y}_\mathfrak{X}\text{–flat}$ (resp., in $\mathfrak{Y}_{\mathfrak{X}'\text{–flat}}$) if and only if the short sequence $0 \longrightarrow \mathfrak{F}^{(\mathbf{Y}_\gamma)} \longrightarrow \mathfrak{G}^{(\mathbf{Y}_\gamma)} \longrightarrow \mathfrak{H}^{(\mathbf{Y}_\gamma)} \longrightarrow 0$ is exact in $(\mathbf{Y}_\gamma)_{X_\gamma}\text{–flat}$ (resp., in $(\mathbf{Y}_\gamma)_{X'_\gamma}\text{–flat}$) for every $\gamma \in \Gamma$.

The full subcategory $(\mathbf{Y}_\gamma)_{X'_\gamma}\text{–flat} \subset (\mathbf{Y}_\gamma)_{X_\gamma}\text{–flat}$ is closed under extensions and the kernels of admissible epimorphisms, because the full subcategory $X'_\gamma\text{–flat} \subset X'_\gamma\text{–qcoh}$ is. It follows that the full subcategory $\mathfrak{Y}_{\mathfrak{X}'}\text{–flat} \subset \mathfrak{Y}_\mathfrak{X}\text{–flat}$ is closed under extensions and the kernels of admissible epimorphisms, too. The exact category structure on $(\mathbf{Y}_\gamma)_{X'_\gamma}\text{–flat}$ is inherited from that on $(\mathbf{Y}_\gamma)_{X_\gamma}\text{–flat}$ (since both are inherited from the abelian category $\mathbf{Y}\text{–qcoh}$). It follows that the exact category structure of $\mathfrak{Y}_{\mathfrak{X}'}\text{–flat}$ is inherited from $\mathfrak{Y}_\mathfrak{X}\text{–flat}$.

We still have not used the assumption that the morphism π' (hence also π) is flat; now we need to use it in order to construct an admissible epimorphism onto any $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$ from an $\mathfrak{X}'$-flat one. Indeed, let $\mathfrak{G}$ be an $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$. Then the adjunction morphism $\pi^*\pi_*\mathfrak{G} \longrightarrow \mathfrak{G}$ is an admissible epimorphism in $\mathfrak{Y}_\mathfrak{X}\text{–flat}$ (cf. Lemma 8.7), and the pro-quasi-coherent pro-sheaf $\pi^*\pi_*\mathfrak{G}$ on $\mathfrak{Y}$ is even flat, hence $\mathfrak{X}'$-flat.

Part (b): let $\mathfrak{G}$ be an $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$, and let $0 \longrightarrow \mathfrak{F}_d \longrightarrow \mathfrak{F}_{d-1} \longrightarrow \cdots \longrightarrow \mathfrak{F}_0 \longrightarrow \mathfrak{G} \longrightarrow 0$ be an exact sequence in $\mathfrak{Y}_\mathfrak{X}\text{–flat}$ with $\mathfrak{F}_i \in \mathfrak{Y}_{\mathfrak{X}'}\text{–flat}$ for all $0 \le i < d$. We need to show that $\mathfrak{F}_d \in \mathfrak{Y}_{\mathfrak{X}'}\text{–flat}$. In the notation above, it suffices to check that $\mathfrak{F}_d^{(\mathbf{Y}_\gamma)} \in (\mathbf{Y}_\gamma)_{X'_\gamma}\text{–flat}$. We know that $\mathfrak{F}_i^{(\mathbf{Y}_\gamma)} \in (\mathbf{Y}_\gamma)_{X'_\gamma}\text{–flat}$ for $0 \le i < d$ and $\mathfrak{G}^{(\mathbf{Y}_\gamma)} \in (\mathbf{Y}_\gamma)_{X_\gamma}\text{–flat}$.

Introduce the notation $\tau_\gamma\colon X'_\gamma \longrightarrow X_\gamma$, $\pi'_\gamma\colon \mathbf{Y}_\gamma \longrightarrow X'_\gamma$, and $\pi_\gamma\colon \mathbf{Y}_\gamma \longrightarrow X_\gamma$ for the relevant affine morphisms of schemes. We need to show that $\pi'_{\gamma*}\mathfrak{F}_d^{(\mathbf{Y}_\gamma)} \in X'_\gamma\text{–flat}$. Put $\mathcal{F}_i = \pi'_{\gamma*}\mathfrak{F}_i^{(\mathbf{Y}_\gamma)}$ and $\mathcal{G} = \pi'_{\gamma*}\mathfrak{G}^{(\mathbf{Y}_\gamma)} \in X'_\gamma\text{–qcoh}$. Then we have an exact sequence $0 \longrightarrow \mathcal{F}_d \longrightarrow \mathcal{F}_{d-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{G} \longrightarrow 0$ of quasi-coherent sheaves on X'_γ. We know that $\mathcal{F}_i \in X'_\gamma\text{–flat}$ for $0 \le i < d$ and $\tau_{\gamma*}\mathcal{G} = \tau_{\gamma*}\pi'_{\gamma*}\mathfrak{G}^{(\mathbf{Y}_\gamma)} = \pi_{\gamma*}\mathfrak{G}^{(\mathbf{Y}_\gamma)} \in X_\gamma\text{–flat}$.

Applying Proposition 10.2(a) to the quasi-coherent sheaf $\mathcal{G}$ on X'_γ and the morphism of schemes $\tau_\gamma\colon X'_\gamma \longrightarrow X_\gamma$, which is affine and weakly smooth of relative dimension $\le d$ by assumptions, while the scheme X_γ is Noetherian and semi-separated, we conclude that $\mathcal{F}_d \in X'_\gamma\text{–flat}$. □

Corollary 10.11. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism which is weakly smooth of relative dimension $\le d$, and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism of ind-schemes. Put $\pi = \tau\pi'$. Then the exact inclusion of exact categories $\mathfrak{Y}_{\mathfrak{X}'}\text{–flat} \longrightarrow \mathfrak{Y}_\mathfrak{X}\text{–flat}$ induces a triangulated equivalence between the derived categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–flat}) \simeq \mathsf{D}(\mathfrak{Y}_\mathfrak{X}\text{–flat})$.*

Proof. Follows from Lemma 10.10 and Proposition 10.9. □

10.4. **Preservation of the semiderived category of torsion sheaves.** Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, and let $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism of ind-schemes. Assume that the morphism τ is of finite type and weakly smooth of relative dimension $\leq d$ (in the sense of Section 10.1) for some finite integer d. Consider the direct image functor $\tau_*\colon \mathfrak{X}'\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$.

Proposition 10.12. *A complex of quasi-coherent torsion sheaves $\mathscr{M}^\bullet$ on $\mathfrak{X}'$ is coacyclic if and only if the complex of quasi-coherent torsion sheaves $\tau_*\mathscr{M}^\bullet$ on $\mathfrak{X}$ is coacyclic.*

Proof. It is clear from Lemmas 7.1(a) and 7.2 that the functor τ_* (for any affine morphism of reasonable ind-schemes τ) takes coacyclic complexes to coacyclic complexes.

To prove the converse, assume that the complex $\tau_*\mathscr{M}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. The ind-scheme $\mathfrak{X}'$ is ind-Noetherian, so Corollary 4.18 is applicable and there exists a complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet$ on $\mathfrak{X}'$ together with a morphism of complexes $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ with a coacyclic cone. Then the cone of the morphism $\tau_*\mathscr{M}^\bullet \longrightarrow \tau_*\mathscr{J}^\bullet$ is also coacyclic, as we have already seen. Hence the complex $\tau_*\mathscr{J}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$, and it remains to show that the complex $\mathscr{J}^\bullet$ is coacyclic in $\mathfrak{X}'\text{–}\mathsf{tors}$.

Furthermore, for a flat morphism of reasonable ind-schemes $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ the functor $\tau_*\colon \mathfrak{X}'\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ takes injectives to injectives, since its left adjoint functor $\tau^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}'\text{–}\mathsf{tors}$ is exact (by Lemma 7.3). So $\tau_*\mathscr{J}^\bullet$ is a coacyclic complex of injectives in $\mathfrak{X}\text{–}\mathsf{tors}$, and it follows that $\tau_*\mathscr{J}^\bullet$ is a contractible complex.

Therefore, for any closed subscheme $Z \subset \mathfrak{X}$, the complex $(\tau_*\mathscr{J}^\bullet)_{(Z)}$ of quasi-coherent sheaves on Z is a contractible complex of injectives, too. Put $Z' = Z \times_{\mathfrak{X}} \mathfrak{X}'$ and denote by $\tau_Z\colon Z' \longrightarrow Z$ the natural morphism; then we have $(\tau_*\mathscr{J}^\bullet)_{(Z)} = \tau_{Z*}\mathscr{J}^\bullet_{(Z')}$. Now $\mathscr{J}^\bullet_{(Z')}$ is a complex of injective quasi-coherent sheaves on Z'. Since the complex $\tau_{Z*}\mathscr{J}^\bullet_{(Z')}$ is acyclic and the functor $\tau_{Z*}\colon Z'\text{–}\mathsf{qcoh} \longrightarrow Z\text{–}\mathsf{qcoh}$ is exact and faithful (the morphism τ_Z being affine), it follows that $\mathscr{J}^\bullet_{(Z')}$ is an acyclic complex.

Denote by $\mathscr{K}^n \in Z'\text{–}\mathsf{qcoh}$ the quasi-coherent sheaves of cocycles of the acyclic complex $\mathscr{J}^\bullet_{(Z')}$. Then $\tau_{Z*}\mathscr{K}^n \in Z\text{–}\mathsf{qcoh}$ are the quasi-coherent sheaves of cocycles of the contractible complex of injective quasi-coherent sheaves $\tau_{Z*}\mathscr{J}^\bullet_{(Z')}$ on Z. Hence the quasi-coherent sheaves $\tau_{Z*}\mathscr{K}^n$ on Z are injective.

By assumptions, the morphism of schemes $\tau_Z\colon Z' \longrightarrow Z$ is affine, weakly smooth of relative dimension $\leq d$, and of finite type, while Z is a Noetherian scheme. Applying Proposition 10.2(b) to the morphism τ and each of the quasi-coherent sheaves $\mathscr{K}^n$ on Z', we see that they have finite injective dimensions in $Z'\text{–}\mathsf{qcoh}$. As these are the objects of cocycles of an acyclic complex of injectives $\mathscr{J}^\bullet_{(Z')}$, we can conclude that the quasi-coherent sheaves $\mathscr{K}^n$ are injective and the complex $\mathscr{J}^\bullet_{(Z')}$ is contractible.

Finally, Lemma 4.21 tells that the complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet$ on $\mathfrak{X}'$ is contractible. ☐

Corollary 10.13. *Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme, $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism of finite type which is weakly smooth of relative dimension $\leq d$, and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism of ind-schemes. Put $\pi = \tau\pi'$. Then the $\mathfrak{Y}/\mathfrak{X}'$-semiderived category of quasi-coherent torsion sheaves on $\mathfrak{Y}$ coincides with the $\mathfrak{Y}/\mathfrak{X}$-semiderived category, $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) = \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$.*

Proof. Let $\mathscr{N}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then we have an isomorphism of complexes $\pi_*\mathscr{N}^\bullet \simeq \tau_*\pi'_*\mathscr{N}^\bullet$ in $\mathfrak{X}$–tors. By Proposition 10.12, it follows that the complex $\pi'_*\mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}'$–tors if and only if the complex $\pi_*\mathscr{N}^\bullet$ is coacyclic in $\mathfrak{X}$–tors. □

10.5. **Derived restriction with supports commutes with flat pullback.** Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat morphism of ind-Noetherian ind-schemes, and let $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$ be a closed immersion of ind-schemes. Consider the pullback diagram (so $\mathfrak{W} = \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$)

$$\begin{array}{ccc} \mathfrak{W} & \xrightarrow{k} & \mathfrak{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{Z} & \xrightarrow{i} & \mathfrak{X} \end{array}$$

Since the morphisms i and k are closed immersions, the ind-schemes $\mathfrak{Z}$ and $\mathfrak{W}$ are also ind-Noetherian.

The right derived functor $\mathbb{R}i^!\colon \mathsf{D^{co}}(\mathfrak{X}\text{–tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{Z}\text{–tors})$ was defined in Section 6.4, and similarly there is the right derived functor $\mathbb{R}k^!\colon \mathsf{D^{co}}(\mathfrak{Y}\text{–tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{W}\text{–tors})$.

The morphisms of ind-schemes f and g are flat, so the inverse image functors $f^*\colon \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{Y}\text{–tors}$ and $g^*\colon \mathfrak{Z}\text{–tors} \longrightarrow \mathfrak{W}\text{–tors}$ are exact by Lemma 7.3. The functors f^* and g^* are also left adjoints by Lemma 2.10(b), so they preserve coproducts. Hence the functors f^* and g^* take coacyclic complexes to coacyclic complexes, and consequently induce well-defined inverse image functors between the coderived categories,

$$f^*\colon \mathsf{D^{co}}(\mathfrak{X}\text{–tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{Y}\text{–tors})$$

and similarly $g^*\colon \mathsf{D^{co}}(\mathfrak{Z}\text{–tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{Y}\text{–tors})$.

The aim of this Section 10.5 is to prove the following proposition (which is to be compared with Proposition 6.19).

Proposition 10.14. *There is a natural isomorphism $g^*\circ\mathbb{R}i^! \simeq \mathbb{R}k^!\circ f^*$ of triangulated functors $\mathsf{D^{co}}(\mathfrak{X}\text{–tors}) \longrightarrow \mathsf{D^{co}}(\mathfrak{W}\text{–tors})$.*

The underived version of the natural isomorphism from Proposition 10.14 is provided by Lemma 9.15. Given the underived version, the assertion of the proposition follows almost immediately from the next lemma.

Lemma 10.15. *Let $\mathscr{J}^\bullet$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$, and let $r\colon f^*\mathscr{J}^\bullet \longrightarrow \mathscr{K}^\bullet$ be a morphism of complexes of quasi-coherent torsion sheaves on $\mathfrak{Y}$ such $\mathscr{K}^\bullet$ is a complex of injective quasi-coherent torsion sheaves and the cone of r is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then the induced morphism of complexes of quasi-coherent torsion sheaves on $\mathfrak{W}$*

$$k^!(r)\colon k^!f^*\mathscr{J}^\bullet \longrightarrow k^!\mathscr{K}^\bullet$$

has coacyclic cone.

The proof of Lemma 10.15 will be given below near the end of Section 10.5.

Lemma 10.16. *Let R be a Noetherian commutative ring and $R \longrightarrow S$ be a homomorphism of commutative rings such that S is a flat R-module. Let M be a finitely generated R-module and N be an R-module. Then for every $n \geq 0$ there is a natural isomorphism of S-modules*

$$\mathrm{Ext}^n_S(S \otimes_R M,\ S \otimes_R N) \simeq S \otimes_R \mathrm{Ext}^n_R(M, N).$$

In particular, for any injective R-module J one has $\mathrm{Ext}^n_S(S \otimes_R M,\ S \otimes_R J) = 0$ for all $n > 0$.

Proof. The assumption of commutativity of S can be actually dropped (then one obtains S-S-bimodule isomorphisms). We observe that for any finitely generated R-module P there are natural isomorphisms of S-modules $\mathrm{Hom}_S(S \otimes_R P,\ S \otimes_R N) \simeq \mathrm{Hom}_R(P,\ S \otimes_R N) \simeq S \otimes_R \mathrm{Hom}_R(P, N)$. Now let $P_\bullet \longrightarrow M$ be a resolution of M by finitely generated projective R-modules. The $S \otimes_R P_\bullet \longrightarrow S \otimes_R M$ is a resolution of $S \otimes_R M$ by finitely generated projective S-modules. The isomorphism of complexes of S-modules $\mathrm{Hom}_S(S \otimes_R P_\bullet,\ S \otimes_R N) \simeq S \otimes_R \mathrm{Hom}_R(P_\bullet, N)$ induces the desired isomorphism for the Ext modules. ☐

Lemma 10.17. *Let $f\colon Y \longrightarrow X$ be a flat morphism of Noetherian schemes, and let $i\colon Z \longrightarrow X$ be a closed immersion of schemes. Put $W = Z \times_X Y$, and denote by $g\colon W \longrightarrow Z$ and $k\colon W \longrightarrow Y$ the natural morphisms. Let $\mathcal{J}$ be an injective quasi-coherent sheaf on X and $f^*\mathcal{J} \longrightarrow \mathcal{K}^\bullet$ be an injective resolution of the quasi-coherent sheaf $f^*\mathcal{J}$ on Y. Then one has*

$$H^0(k^!\mathcal{K}^\bullet) \simeq g^*i^!\mathcal{J} \quad \textit{and} \quad H^n(k^!\mathcal{K}^\bullet) = 0 \ \ \textit{for } n > 0.$$

Proof. To compute $H^0(k^!\mathcal{K}^\bullet)$, one can observe that the functor $k^!$ is left exact (as a right adjoint), so $H^0(k^!\mathcal{K}^\bullet) \simeq k^!f^*\mathcal{J} \simeq g^*i^!\mathcal{J}$ by Lemma 4.24. The cohomology vanishing assertion is local in X and in Y (since injectivity of a quasi-coherent sheaf on a Noetherian scheme is a local property), so it reduces to the case of affine schemes, for which it means the following.

Let $R \longrightarrow S$ be a homomorphism of Noetherian commutative rings such that S is a flat R-module, and let $R \longrightarrow T$ be a surjective homomorphism of commutative rings. Let J be an injective R-module, and let $K^\bullet$ be an injective resolution of the S-module $S \otimes_R J$. Then the complex $\mathrm{Hom}_S(S \otimes_R T,\ K^\bullet)$ has vanishing cohomology in the positive cohomological degrees. This is a particular case of Lemma 10.16. ☐

Lemma 10.18. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat morphism of ind-Noetherian ind-schemes, and let $Z \subset \mathfrak{X}$ be a closed subscheme with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$. Put $W = Z \times_{\mathfrak{X}} \mathfrak{Y}$ (so W is a closed subscheme in $\mathfrak{Y}$), and denote by $f_Z\colon W \longrightarrow Z$ and $k\colon W \longrightarrow \mathfrak{Y}$ the natural morphisms. Let $\mathscr{J}$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}$ and $f^*\mathscr{J} \longrightarrow \mathscr{K}^\bullet$ be an injective resolution of the quasi-coherent torsion sheaf $f^*\mathscr{J}$ on $\mathfrak{Y}$. Then one has*

$$H^0(k^!\mathscr{K}^\bullet) \simeq f_Z^*i^!\mathscr{J} \quad \textit{and} \quad H^n(k^!\mathscr{K}^\bullet) = 0 \ \ \textit{for } n > 0.$$

Proof. The computation of H^0 is similar to the one in Lemma 10.17 (use Remark 7.4). To prove the higher cohomology vanishing, choose an inductive system of closed immersions of schemes $(X_\gamma)_{\gamma\in\Gamma}$ representing the ind-scheme $\mathfrak{X}$, so $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$. We can always assume that there exists $\gamma_0 \in \Gamma$ such that $Z = X_{\gamma_0}$. Put $Y_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$; then $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} Y_\gamma$ is a representation of $\mathfrak{Y}$ by an inductive system of closed immersions of schemes. Let $f_\gamma\colon Y_\gamma \longrightarrow X_\gamma$ denote the natural morphism.

Our aim is to show that the exact sequence of quasi-coherent torsion sheaves $0 \longrightarrow f^*\mathscr{J} \longrightarrow \mathscr{K}^0 \longrightarrow \mathscr{K}^1 \longrightarrow \mathscr{K}^2 \longrightarrow \cdots$ remains exact after applying the functor $\mathscr{N} \longmapsto \mathscr{N}|_\Gamma\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst}$. Notice that, by Remark 7.4, we have $(f^*\mathscr{J})|_\Gamma \simeq f^*(\mathscr{J}|_\Gamma)$, or in other words, $(f^*\mathscr{J})_{(Y_\gamma)} = f_\gamma^*\mathscr{J}_{(X_\gamma)}$.

Denote by $\mathbb{N}^1 \in (\mathfrak{Y},\Gamma)\text{–}\mathsf{syst}$ the cokernel of the morphism of Γ-systems $(f^*\mathscr{J})|_\Gamma \longrightarrow \mathscr{K}^0|_\Gamma$. Let $\gamma < \delta \in \Gamma$ be a pair of indices. Denote the related transition maps in the inductive systems by $i_{\gamma\delta}\colon X_\gamma \longrightarrow X_\delta$ and $k_{\gamma\delta}\colon Y_\gamma \longrightarrow Y_\delta$.

The quasi-coherent sheaves $\mathscr{K}^n_{(Y_\delta)}$ on the scheme Y_δ are injective for all $n \ge 0$, since the quasi-coherent torsion sheaves $\mathscr{K}^n$ on the ind-scheme $\mathfrak{Y}$ are injective. By Lemma 10.17, the short exact sequence $0 \longrightarrow f_\delta^*\mathscr{J}_{(X_\delta)} \longrightarrow \mathscr{K}^0_{(Y_\delta)} \longrightarrow \mathbb{N}^1_{(\delta)} \longrightarrow 0$ remains exact after applying the functor $k_{\gamma\delta}^!$. Hence the structure map $\mathbb{N}^1_{(\gamma)} \longrightarrow k_{\gamma\delta}^!\mathbb{N}^1_{(\delta)}$ in the Γ-system $\mathbb{N}^1$ is an isomorphism (of quasi-coherent sheaves on Y_γ).

As this holds for all $\gamma < \delta \in \Gamma$, we can conclude that the collection of quasi-coherent sheaves $\mathscr{N}^1_{(Y_\gamma)} = \mathbb{N}^1_{(\gamma)}$, $\gamma \in \Gamma$ defines a quasi-coherent torsion sheaf $\mathscr{N}^1$ on $\mathfrak{Y}$. So we have $\mathbb{N}^1 = \mathscr{N}^1|_\Gamma$ and $\mathscr{N}^1 = \mathbb{N}^{1+}$; in other words, this means that the adjunction morphism $\mathbb{N}^1 \longrightarrow \mathbb{N}^{1+}|_\Gamma$ is an isomorphism of Γ-systems. Notice that the quasi-coherent torsion sheaf $\mathbb{N}^{1+}$ on $\mathfrak{Y}$ is, by the definition, the cokernel of the monomorphism of quasi-coherent torsion sheaves $f^*\mathscr{J} \longrightarrow \mathscr{K}^0$. We have shown that the short exact sequence of quasi-coherent torsion sheaves $0 \longrightarrow f^*\mathscr{J} \longrightarrow \mathscr{K}^0 \longrightarrow \mathbb{N}^{1+} \longrightarrow 0$ on $\mathfrak{Y}$ remains exact after applying the functor $\mathscr{N} \longmapsto \mathscr{N}|_\Gamma$.

The argument finishes similarly to the proof of Lemma 6.23, proceeding step by step up the cohomological degree and using Lemma 10.17 on every step. ☐

Lemma 10.19. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat morphism of ind-Noetherian ind-schemes, and let $i\colon \mathfrak{Z} \subset \mathfrak{X}$ be a closed immersion of schemes. Put $\mathfrak{W} = \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$, and denote by $g\colon \mathfrak{W} \longrightarrow \mathfrak{Z}$ and $k\colon \mathfrak{W} \longrightarrow \mathfrak{Y}$ the natural morphisms. Let $\mathscr{J}$ be an injective quasi-coherent torsion sheaf on $\mathfrak{X}$ and $f^*\mathscr{J} \longrightarrow \mathscr{K}^\bullet$ be an injective resolution of the quasi-coherent torsion sheaf $f^*\mathscr{J}$ on $\mathfrak{Y}$. Then one has*

$$H^0(k^!\mathscr{K}^\bullet) \simeq g^*i^!\mathscr{J} \quad \text{and} \quad H^n(k^!\mathscr{K}^\bullet) = 0 \ \text{ for } n > 0.$$

Proof. The computation of H^0 is similar to the one in Lemmas 10.17 and 10.18. The functor $k^!$ is left exact as a right adjoint, and it remains to use Lemma 9.15. To prove the vanishing assertion, choose a closed subscheme $Z \subset \mathfrak{Z}$ with the closed immersion morphism $j\colon Z \longrightarrow \mathfrak{Z}$. Put $W = Z \times_{\mathfrak{Z}} \mathfrak{W} = Z \times_{\mathfrak{X}} \mathfrak{Y}$, and denote by $l\colon W \longrightarrow \mathfrak{W}$ the natural closed immersion. Then by Lemma 10.18 we have $H^n(l^!k^!\mathscr{K}^\bullet) = 0$ for $n > 0$, and it follows that $H^n(k^!\mathscr{K}^\bullet) = 0$ for $n > 0$ as well. ☐

Proof of Lemma 10.15. The argument is similar to the proof of Lemma 6.20. Given a complex $\mathscr{J}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{X}$, the related complex $\mathscr{K}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is defined uniquely up to a homotopy equivalence (by Proposition 4.15(a)); so it suffices to prove the assertion of the lemma for one specific choice of the complex $\mathscr{K}^\bullet$. We will use the complex $\mathscr{K}^\bullet$ provided by the construction on which the proof of Proposition 4.15(b) is based.

Let $f^*\mathscr{J}^\bullet \longrightarrow \mathscr{L}^{0,\bullet}$ be a monomorphism of complexes in $\mathfrak{Y}$–tors such that $\mathscr{L}^{0,\bullet}$ is a complex of injective quasi-coherent torsion sheaves. Denote by $\mathscr{N}^{1,\bullet}$ the cokernel of this morphism of complexes, and let $\mathscr{N}^{1,\bullet} \longrightarrow \mathscr{L}^{1,\bullet}$ be a monomorphism of complexes in which $\mathscr{L}^{1,\bullet}$ is a complex of injectives. Proceeding in this way, we construct a bounded below complex of complexes of injective quasi-coherent torsion sheaves $\mathscr{L}^{\bullet,\bullet}$ together with a quasi-isomorphism $f^*\mathscr{J}^\bullet \longrightarrow \mathscr{L}^{\bullet,\bullet}$ of complexes of complexes in $\mathfrak{Y}$–tors. In every cohomological degree n, the complex $\mathscr{L}^{\bullet,n}$ is an injective resolution of the quasi-coherent torsion sheaf $f^*\mathscr{J}^n$ on $\mathfrak{Y}$. The complex $\mathscr{K}^\bullet$ is then constructed by totalizing the bicomplex $\mathscr{L}^{\bullet,\bullet}$ using infinite coproducts along the diagonals.

Recall that the functor $k^!\colon \mathfrak{Y}$–$\mathsf{tors} \longrightarrow \mathfrak{W}$–$\mathsf{tors}$ preserves coproducts. In every cohomological degree n, applying $k^!$ to the complex $0 \longrightarrow f^*\mathscr{J}^n \longrightarrow \mathscr{L}^{0,n} \longrightarrow \mathscr{L}^{1,n} \longrightarrow \cdots$ produces an acyclic complex in $\mathfrak{W}$–tors by Lemma 10.19. It remains to use the fact that the coproduct totalization of an acyclic bounded below complex of complexes is a coacyclic complex [40, Lemma 2.1]. ☐

Proof of Proposition 10.14. Let $\mathscr{M}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. Choose a complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet$ on $\mathfrak{X}$ together with a morphism $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ with a coacyclic cone. Then the cone of the morphism $f^*\mathscr{M}^\bullet \longrightarrow f^*\mathscr{J}^\bullet$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Choose a complex of injective quasi-coherent torsion sheaves $\mathscr{K}^\bullet$ on $\mathfrak{Y}$ together with a morphism $f^*\mathscr{J}^\bullet \longrightarrow \mathscr{K}^\bullet$ with a coacyclic cone. By Lemma 10.15, the cone of the morphism $k^!f^*\mathscr{J}^\bullet \longrightarrow k^!\mathscr{K}^\bullet$ is a coacyclic complex of quasi-coherent torsion sheaves on $\mathfrak{W}$. Thus the complex $k^!f^*\mathscr{J}^\bullet$ represents the object $\mathbb{R}k^! \circ f^*(\mathscr{M}^\bullet)$ in the coderived category $\mathsf{D}^{\mathsf{co}}(\mathfrak{W}\text{–}\mathsf{tors})$.

On the other hand, the complex $i^!\mathscr{J}^\bullet$ represents the object $\mathbb{R}i^!(\mathscr{M}^\bullet) \in \mathsf{D}^{\mathsf{co}}(\mathfrak{Z}\text{–}\mathsf{tors})$, hence the complex $g^*i^!\mathscr{J}^\bullet$ represents the object $g^* \circ \mathbb{R}i^!(\mathscr{M}^\bullet) \in \mathsf{D}^{\mathsf{co}}(\mathfrak{W}\text{–}\mathsf{tors})$. It remains to recall the isomorphism $g^*i^!\mathscr{J}^\bullet \simeq k^!f^*\mathscr{J}^\bullet$ of complexes of quasi-coherent torsion sheaves on $\mathfrak{W}$ provided by Lemma 9.15. ☐

10.6. **Preservation of the semiderived equivalence.** Let $\tau\colon X' \longrightarrow X$ be a morphism of semi-separated Noetherian schemes. Assume that the morphism τ is of finite type and weakly smooth of relative dimension $\le d$. Consider the inverse image functor $\tau^*\colon X$–$\mathsf{qcoh} \longrightarrow X'$–$\mathsf{qcoh}$.

Lemma 10.20. *Let $\mathcal{D}^\bullet$ be a dualizing complex of quasi-coherent sheaves on X. Choose a complex of injective quasi-coherent sheaves $\mathcal{D}'^\bullet$ on X' together with a morphism of complexes $\tau^*\mathcal{D}^\bullet \longrightarrow \mathcal{D}'^\bullet$ with a coacyclic cone. Then $\mathcal{D}'^\bullet$ is a dualizing complex of quasi-coherent sheaves on X'.*

Proof. It is helpful to keep in mind that a bounded below complex is coacyclic if and only if it is acyclic, and any (unbounded) coacyclic complex of injectives is contractible. Up to the homotopy equivalence, one can assume both $\mathcal{D}^\bullet$ and $\mathcal{D}'^\bullet$ to be bounded below, and then it suffices that the cone be acyclic.

The assertion is local in both X and X', so it reduces to the case of affine schemes, for which it consists of three independent observations corresponding to the three conditions (i–iii) in the definition of a dualizing complex in Section 4.2. In each case, we consider a homomorphism of Noetherian commutative rings $R \longrightarrow S$ such that S is a flat R-module.

(1) Assume that S is a finitely generated R-algebra and for every prime ideal $\mathfrak{p} \subset R$, the fiber ring $\kappa(\mathfrak{p})\otimes_R S$ is regular of Krull dimension $\le d$. If $D^\bullet$ is a bounded complex of injective R-modules, then the complex of S-modules $S \otimes_R D^\bullet$ is quasi-isomorphic to a bounded complex of injective S-modules.

Indeed, it suffices to show that, for every injective R-module J, the S-module $S \otimes_R J$ has finite injective dimension. Notice that the R-module $S \otimes_R J$ is injective (since the R-module S is flat and R is a Noetherian ring); hence the assertion follows from Proposition 10.2(b).

In fact, according to [14, Theorem 1] (see also [15, Corollary 1]), it suffices to assume the fiber rings to be Gorenstein (of bounded Krull dimension) rather than regular, and the assumption that S is a finitely generated R-algebra can be dropped. The assumptions on the morphism τ before the formulation of the lemma can be weakened accordingly.

(2) If $D^\bullet$ is a complex of R-modules with finitely generated cohomology R-modules, then $S \otimes_R D^\bullet$ is a complex of S-modules with finitely generated cohomology S-modules. This is obvious for a flat R-algebra S.

(3) If $D^\bullet$ is a bounded complex of R-modules with finitely generated cohomology R-modules and the homothety map $R \longrightarrow \mathbb{R}\operatorname{Hom}_R(D^\bullet, D^\bullet)$ is a quasi-isomorphism, then the homothety map $S \longrightarrow \mathbb{R}\operatorname{Hom}_S(S \otimes_R D^\bullet,\, S \otimes_R D^\bullet)$ is a quasi-isomorphism, too.

More generally, if $M^\bullet$ is a bounded above complex of R-modules with finitely generated cohomology modules and $N^\bullet$ is a bounded below complex of R-modules, then there is a natural isomorphism $\mathbb{R}\operatorname{Hom}_S(S\otimes_R M^\bullet,\, S\otimes_R N^\bullet) \simeq S\otimes_R \mathbb{R}\operatorname{Hom}_R(M^\bullet, N^\bullet)$ in the derived category of S-modules. This is a straightforward generalization of Lemma 10.16; one just needs to replace $M^\bullet$ with a quasi-isomorphic bounded above complex of finitely generated projective R-modules. ☐

Let $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an morphism of ind-semi-separated ind-Noetherian ind-schemes. Assume that the morphism τ is of finite type and weakly smooth of relative dimension $\le d$. Consider the inverse image functor $\tau^*\colon \mathfrak{X}\text{–tors} \longrightarrow \mathfrak{X}'\text{–tors}$.

Lemma 10.21. *Let $\mathscr{D}^\bullet$ be a dualizing complex of quasi-coherent torsion sheaves on $\mathfrak{X}$. Choose a complex of injective quasi-coherent torsion sheaves $\mathscr{D}'^\bullet$ on $\mathfrak{X}'$ together with a morphism of complexes $\tau^*\mathscr{D}^\bullet \longrightarrow \mathscr{D}'^\bullet$ with a coacyclic cone. Then $\mathscr{D}'^\bullet$ is a dualizing complex of quasi-coherent torsion sheaves on $\mathfrak{X}'$.*

Proof. Let $Z \subset \mathfrak{X}$ be a closed subscheme; put $Z' = Z \times_{\mathfrak{X}} \mathfrak{X}'$. Let $i\colon Z \longrightarrow \mathfrak{X}$ and $k\colon Z' \longrightarrow \mathfrak{X}'$ denote the closed immersion morphisms, and let $\tau_Z\colon Z' \longrightarrow Z$ be the natural morphism (which is of finite type and weakly smooth of relative dimension $\le d$). By Proposition 10.14, we have a natural isomorphism $\tau_Z^* i^! \mathscr{D}^\bullet =$

$\tau_Z^*\mathbb{R}i^!\mathscr{D}^\bullet \simeq \mathbb{R}k^!\tau^*\mathscr{D}^\bullet = k^!\mathscr{D}'^\bullet$ in $\mathsf{D^{co}}(Z'\text{–}\mathsf{qcoh})$. Since $\mathscr{D}^\bullet$ is a dualizing complex on $\mathfrak{X}'$, the complex of quasi-coherent sheaves $i^!\mathscr{D}^\bullet$ is a dualizing complex on Z. Now Lemma 10.20 applied to the morphism of schemes τ_Z tells that $k^!\mathscr{D}'^\bullet$ is a dualizing complex on Z', which by the definition means that $\mathscr{D}'^\bullet$ is a dualizing complex on $\mathfrak{X}'$ (see condition (iv) in Section 4.2 and the discussion after it). ☐

Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism of finite type which is weakly smooth of relative dimension $\le d$, and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism of ind-schemes. Put $\pi = \tau\pi'$.

Let $\mathscr{D}^\bullet$ be a dualizing complex on $\mathfrak{X}$ and $\mathscr{D}'^\bullet$ be the related dualizing complex on $\mathfrak{X}'$, as per the rule of Lemma 10.21. Then Theorem 7.15 provides triangulated equivalences

$$\pi^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} - : \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$$

and

$$\pi'^*\mathscr{D}'^\bullet \otimes_{\mathfrak{Y}} - : \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

Proposition 10.22. *In the context above, the triangulated equivalences* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ *and* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ *from Theorem 7.15 form a commutative square diagram with the triangulated equivalences* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \simeq \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ *and* $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ *from Corollaries 10.11 and 10.13.*

Proof. Notice the natural isomorphism $\pi^*\mathscr{D}^\bullet \simeq \pi'^*\tau^*\mathscr{D}^\bullet$ of complexes of quasi-coherent torsion sheaves on $\mathfrak{Y}$. We are given a morphism $\tau^*\mathscr{D}^\bullet \longrightarrow \mathscr{D}'^\bullet$ of complexes of quasi-coherent torsion sheaves on $\mathfrak{X}'$, whose cone is coacyclic in $\mathfrak{X}'\text{–}\mathsf{tors}$. According to the discussion in the beginning of Section 9.9, based on the existence of a well-defined functor (35) from Section 8.1, the triangulated functors $\mathbf{\Phi}_{\tau^*\mathscr{D}^\bullet} = \pi'^*\tau^*\mathscr{D}^\bullet \otimes_{\mathfrak{Y}} - : \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ and $\mathbf{\Phi}_{\mathscr{D}'^\bullet} = \pi'^*\mathscr{D}'^\bullet \otimes_{\mathfrak{Y}} - : \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ are naturally isomorphic. ☐

10.7. **Preservation of the semitensor product.** Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism of finite type which is weakly smooth of relative dimension $\le d$, and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism of ind-schemes. Put $\pi = \tau\pi'$.

A construction from Section 8.3 (see formula (36)) defines the left derived tensor product functors

$$\otimes^{\mathfrak{Y},\mathbb{L}} = \otimes^{\mathfrak{Y}/\mathfrak{X},\mathbb{L}}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$$

and

$$\otimes^{\mathfrak{Y},\mathbb{L}} = \otimes^{\mathfrak{Y}/\mathfrak{X}',\mathbb{L}}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \times \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \longrightarrow \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}),$$

endowing the derived categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ and $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat})$ with tensor triangulated category structures.

Proposition 10.23. *In the context above, the triangulated equivalence* $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \simeq \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ *from Corollary 10.11 is an equivalence of tensor triangulated categories.*

Proof. We just have to show that the two constructions of derived functors of tensor product agree. The definition of a relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ was given in Section 8.2. To distinguish the two settings in the situation at hand, let us speak about *π-relatively homotopy flat complexes* and *π'-relatively homotopy flat complexes.*

The key observation is that every π-relatively homotopy flat complex is π'-relatively homotopy flat. Indeed, we have $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Furthermore, it follows from Corollary 10.11 that a complex in $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$ is acyclic if and only if it is acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$. Hence, given a complex $\mathfrak{F}^\bullet$ in $\mathfrak{Y}\text{–}\mathsf{flat}$, condition (i) from Section 8.2 holds for $\mathfrak{F}^\bullet$ with respect to the morphism $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ whenever it holds with respect to the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Concerning condition (ii), it is clear from Proposition 10.12 that this condition holds for a given complex $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ with respect to the morphism π' if and only if it holds with respect to the morphism π.

Now let $\mathfrak{P}^\bullet$ and $\mathfrak{Q}^\bullet \in \mathsf{C}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat})$ be two complexes of $\mathfrak{X}'$-flat (hence also $\mathfrak{X}$–flat) pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Using Proposition 8.8, choose two morphisms of complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ and $\mathfrak{G}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ such that the cones of both morphisms are acyclic in $\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$, and both the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{flat})$ are π-relatively homotopy flat complexes of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Then both the morphisms can be also viewed as morphisms of complexes of $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, the cones of both the morphisms are also acyclic in $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$, and both the complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet$ are also π'-relatively homotopy flat. Thus the tensor product $\mathfrak{F}^\bullet \otimes^{\mathfrak{Y}} \mathfrak{G}^\bullet$ computes both the derived functors $\mathfrak{P}^\bullet \otimes^{\mathfrak{Y}/\mathfrak{X},\mathbb{L}} \mathfrak{Q}^\bullet$ and $\mathfrak{P}^\bullet \otimes^{\mathfrak{Y}/\mathfrak{X}',\mathbb{L}} \mathfrak{Q}^\bullet$. ☐

Furthermore, another construction from Section 8.3 (see formula (37)) defines the left derived tensor product functors

$$\otimes^{\mathbb{L}}_{\mathfrak{Y}} = \otimes^{\mathbb{L}}_{\mathfrak{Y}/\mathfrak{X}}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$$

and

$$\otimes^{\mathbb{L}}_{\mathfrak{Y}} = \otimes^{\mathbb{L}}_{\mathfrak{Y}/\mathfrak{X}'}\colon \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$$

endowing the semiderived categories $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ and $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ with triangulated module category structures over the tensor categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ and $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat})$, respectively.

Proposition 10.24. *In the context above, the triangulated equivalences $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}) \simeq \mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ and $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ from Corollaries 10.11 and 10.13 preserve the module category structures on $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ and $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ over the tensor categories $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat})$ and $\mathsf{D}(\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat})$.*

Proof. The argument is similar to the proof of Proposition 10.23. We just have to show that the two constructions of derived functors of tensor product agree. The definition of a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$ was given in Section 8.2; similarly one defines the homotopy $\mathfrak{Y}/\mathfrak{X}'$-flat complexes.

The key observation is that every homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex in $\mathfrak{Y}\text{–}\mathsf{tors}$ is $\mathfrak{Y}/\mathfrak{X}'\text{–}\mathsf{flat}$. This holds because $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat} \subset \mathfrak{Y}_{\mathfrak{X}}\text{–}\mathsf{flat}$ a complex in $\mathfrak{Y}_{\mathfrak{X}'}\text{–}\mathsf{flat}$ is acyclic if and only if it is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$, while a complex in $\mathfrak{X}'$–tors is coacyclic if and only if its direct image is coacyclic in $\mathfrak{X}$–tors.

Now let $\mathfrak{P}^\bullet$ be a complex of $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaves and $\mathcal{N}^\bullet$ be a complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Using Proposition 8.8, choose a morphism of complexes of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{F}^\bullet \longrightarrow \mathfrak{P}^\bullet$ whose cone is acyclic in $\mathfrak{Y}_{\mathfrak{X}}$–$\mathsf{flat}$, while $\mathfrak{F}^\bullet \in \mathsf{C}(\mathfrak{Y}$–$\mathsf{flat})$ is a π-relatively homotopy flat complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$. Then the same morphism can be also viewed as a morphism of complexes of $\mathfrak{X}'$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$, its cone is also acyclic in $\mathfrak{Y}_{\mathfrak{X}'}$–$\mathsf{flat}$, and the complex $\mathfrak{F}^\bullet$ is also π'-relatively homotopy flat (as pointed out in the proof of Proposition 10.23).

Using Proposition 8.10, choose a morphism of complexes of quasi-coherent torsion sheaves $\mathcal{G}^\bullet \longrightarrow \mathcal{N}^\bullet$ whose cone has the property that its direct image is coacyclic in $\mathfrak{X}$–tors, while $\mathcal{G}^\bullet \in \mathsf{C}(\mathfrak{Y}$–$\mathsf{tors})$ is a homotopy $\mathfrak{Y}/\mathfrak{X}$-flat complex of quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then the direct image of the cone is also coacyclic in $\mathfrak{X}'$–tors, and $\mathcal{G}^\bullet$ is also a homotopy $\mathfrak{Y}/\mathfrak{X}'$-flat complex. Thus the tensor product $\mathfrak{F}^\bullet \otimes_{\mathfrak{Y}} \mathcal{G}^\bullet$ computes both the derived functors $\mathfrak{P}^\bullet \otimes^{\mathbb{L}}_{\mathfrak{Y}/\mathfrak{X}} \mathcal{N}^\bullet$ and $\mathfrak{P}^\bullet \otimes^{\mathbb{L}}_{\mathfrak{Y}/\mathfrak{X}'} \mathcal{N}^\bullet$. ☐

Theorem 10.25. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\tau\colon \mathfrak{X}' \longrightarrow \mathfrak{X}$ be an affine morphism of finite type which is weakly smooth of relative dimension $\le d$, and $\pi'\colon \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a flat affine morphism of ind-schemes. Put $\pi = \tau\pi'$. Let $\mathcal{D}^\bullet$ be a dualizing complex on $\mathfrak{X}$ and $\mathcal{D}'^\bullet$ be the related dualizing complex on $\mathfrak{X}'$, as per the rule of Lemma 10.21. Then the triangulated equivalence $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}$–$\mathsf{tors}) \simeq \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}$–$\mathsf{tors})$ from Corollary 10.13 is an equivalence of tensor triangulated categories with the semitensor product operations*

$$\lozenge_{\pi'^*\mathcal{D}'^\bullet}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$$

and

$$\lozenge_{\pi^*\mathcal{D}^\bullet}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \times \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}).$$

as in formula (38) *from Section 8.4.*

Proof. Follows from Proposition 10.22 together with Proposition 10.23 or 10.24. ☐

11. Some Infinite-Dimensional Geometric Examples

In this section we discuss several examples illustrating the nature of infinite-dimensional algebro-geometric objects for which the constructions and results of Sections 7–10 are designed. All the examples below in this section will be those of flat affine morphisms of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is an ind-separated ind-scheme of ind-finite type over a field $\Bbbk$.

11.1. **The Tate affine space example.** The following example, while geometrically very simple (a trivial bundle), has unusual and attractive invariance properties. The idea (following [7, Example 7.11.2(ii)] or [58, Example 1.3(1)]) is to consider vector spaces as a kind of affine schemes or, as it may happen, ind-affine ind-schemes.

(0) Let X be a scheme over a field $\Bbbk$. Then, in the algebro-geometric parlance, by a "$\Bbbk$-point on X" one means a morphism $\operatorname{Spec}\Bbbk \longrightarrow X$ of schemes over $\Bbbk$ (i. e., a section of the structure morphism $X \longrightarrow \operatorname{Spec}\Bbbk$). More generally, given a commutative $\Bbbk$-algebra K, by a "K-point on X" one means a morphism $\operatorname{Spec} K \longrightarrow X$ of schemes over $\Bbbk$ (i. e., a morphism forming a commutative triangle diagram with the structure morphisms $\operatorname{Spec} K \longrightarrow \operatorname{Spec}\Bbbk$ and $X \longrightarrow \operatorname{Spec}\Bbbk$).

Similarly one can speak of $\Bbbk$-points or K-points on an ind-scheme $\mathfrak{X}$ over $\Bbbk$ (considering morphisms of ind-schemes in lieu of the morphisms of schemes). An ind-scheme $\mathfrak{X}$ over $\Bbbk$ is determined by its "functor of points", assigning to a commutative $\Bbbk$-algebra K the set $\mathfrak{X}(K)$ of all K-points on $\mathfrak{X}$ and to every homomorphism of commutative $\Bbbk$-algebras $K' \longrightarrow K''$ the induced map $\mathfrak{X}(K') \longrightarrow \mathfrak{X}(K'')$ (cf. the discussion in the first paragraph of Section 1.2).

(1) Let V be a finite-dimensional $\Bbbk$-vector space. We would like to assign a $\Bbbk$-scheme X_V to V in such a way that $\Bbbk$-points on X would correspond bijectively to the elements of V. More generally, for a commutative $\Bbbk$-algebra K, the K-points on X will correspond bijectively to elements of the tensor product $K \otimes_\Bbbk V$.

Here is the (obvious) construction. Consider the dual vector space $V^* = \operatorname{Hom}_\Bbbk(V, k)$, and consider the symmetric algebra $\operatorname{Sym}_\Bbbk(V^*)$. By the definition, $\operatorname{Sym}_\Bbbk(V^*)$ is the commutative $\Bbbk$-algebra freely generated by the $\Bbbk$-vector space V^*. Put $X_V = \operatorname{Spec}\operatorname{Sym}_\Bbbk(V^*)$.

(2) Let V be an infinite-dimensional $\Bbbk$-vector space. Then there is *no* natural construction of a $\Bbbk$-scheme whose $\Bbbk$-points would correspond to the elements of V. More precisely, the functor assigning to an affine scheme $\operatorname{Spec} K$ over $\Bbbk$ the set of all elements of $K \otimes_\Bbbk V$ is *not* representable by a scheme over K. However, it is representable by an ind-affine ind-scheme $\mathfrak{X}_V$ of ind-finite type over $\Bbbk$.

Denote by Γ the directed poset of all finite-dimensional vector subspaces $U \subset V$, ordered by inclusion. Then the ind-scheme $\mathfrak{X}_V$ is given by the Γ-indexed inductive system of the affine schemes X_U constructed in (1), that is $\mathfrak{X}_V = \text{"}\varinjlim\text{"}_{U\in\Gamma} X_U$.

(3) Let V be a *linearly compact* $\Bbbk$-vector space, i. e., a complete, separated topological $\Bbbk$-vector space in which open vector subspaces of finite codimension form a base of neighborhoods of zero. Denote by V^* the vector space of all continuous linear maps $V \longrightarrow \Bbbk$. Then V^* is an (infinite-dimensional) discrete vector space; the vector space V can be recovered as the dual vector space $V = \operatorname{Hom}_\Bbbk(V^*, k)$ to the discrete vector space V^*, with the natural topology on such dual space.

Consider the symmetric algebra $\operatorname{Sym}_\Bbbk(V^*)$ (which can be defined as the commutative $\Bbbk$-algebra freely generated by V^*, or as the direct limit of the symmetric algebras of finite-dimensional subspaces of V^*). Put $Y_V = \operatorname{Spec}\operatorname{Sym}_\Bbbk(V^*)$. This is the infinite-dimensional affine scheme corresponding to a linearly compact $\Bbbk$-vector space. For any commutative $\Bbbk$-algebra K, the set of K-points on Y_V is naturally bijective to the set of all $\Bbbk$-linear maps $V^* \longrightarrow K$.

(4) Let V be a *locally linearly compact* (or *Tate*) $\Bbbk$-vector space, i. e., a (complete, separated) topological vector space admitting a linearly compact open subspace. Then the corresponding ind-affine ind-scheme $\mathfrak{Y}_V$ (*not* of ind-finite type) over $\Bbbk$ is constructed as follows.

Denote by Γ the directed poset of all linearly compact open subspaces $U \subset V$, ordered by inclusion. Then the ind-scheme $\mathfrak{Y}_V$ is defined as $\mathfrak{Y}_V = \text{“}\varinjlim\text{”}_{U\in\Gamma} Y_U$. Here, for any pair of linearly compact open subspaces $U' \subset U'' \subset V$, the inclusion map $U' \longrightarrow U''$ induces a surjective map of the dual discrete vector spaces $U''^* \longrightarrow U'^*$. This linear map, in turn induces a surjective homomorphism of commutative $\Bbbk$-algebras $\mathrm{Sym}_\Bbbk(U''^*) \longrightarrow \mathrm{Sym}_\Bbbk(U'^*)$, which corresponds to the natural closed immersion of affine schemes $Y_{U'} \longrightarrow Y_{U''}$ appearing in the inductive system.

(5) Now let V be a locally linearly compact $\Bbbk$-vector space and $W \subset V$ be a fixed linearly compact open subspace. Then the quotient space V/W is discrete in the induced topology. Put $\mathfrak{X} = \mathfrak{X}_{V/W}$ and $\mathfrak{Y} = \mathfrak{Y}_V$, in the notation of (2) and (4).

Then $\mathfrak{X}$ is an ind-affine ind-scheme of ind-finite type over $\Bbbk$; hence, in particular, $\mathfrak{X}$ is ind-semi-separated and ind-Noetherian. The surjective linear map $V \longrightarrow V/W$ induces a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. The fibers of π over the $\Bbbk$-points of $\mathfrak{X}$ are infinite-dimensional affine schemes isomorphic to Y_W.

(6) Finally, let us construct a dualizing complex on the ind-Noetherian ind-scheme $\mathfrak{X} = \mathfrak{X}_{V/W}$. Let $U \subset V/W$ be a finite-dimensional vector subspace. Then $\mathrm{Sym}_\Bbbk(U)$ is a regular commutative ring of Krull dimension $\dim_\Bbbk U$ (in fact, the ring of polynomials in $\dim_\Bbbk U$ variables over $\Bbbk$). Choose a (finite, if one wishes) injective resolution $E_U^\bullet$ of the free $\mathrm{Sym}_\Bbbk(U)$-module $\mathrm{Sym}_\Bbbk(U)$, and let $\mathcal{E}_U^\bullet$ be the corresponding complex of injective quasi-coherent sheaves on X_U. So $\mathcal{E}_U^\bullet$ is an injective resolution of the structure sheaf $\mathcal{O}_{X_U}$ in the category $X_U\text{–}\mathsf{qcoh}$.

Put $\mathcal{D}_U^\bullet = \mathcal{E}_U^\bullet \otimes_\Bbbk \Lambda_\Bbbk^{\dim_\Bbbk U}(U^*)[\dim_\Bbbk U]$. Here $\Lambda_\Bbbk^{\dim_\Bbbk U}(U^*)[\dim_\Bbbk U]$ is a complex of $\Bbbk$-vector spaces whose only term is the one-dimensional top exterior power of the vector space U^*, placed in the cohomological degree $-\dim_\Bbbk U$. Then $\mathcal{D}_U^\bullet$ is a dualizing complex on X_U. Moreover, for any pair of finite-dimensional subspaces $U' \subset U'' \subset V/W$ and the related closed immersion of affine schemes $i_{U'U''}\colon X_{U'} \longrightarrow X_{U''}$, there is a natural homotopy equivalence $\mathcal{D}_{U'}^\bullet \simeq i_{U'U''}^! \mathcal{D}_{U''}^\bullet$ of complexes of injective quasi-coherent sheaves on $X_{U'}$.

It remains to glue the system of dualizing complexes $\mathcal{D}_U^\bullet$ on the schemes X_U into a dualizing complex (of injective quasi-coherent torsion sheaves) $\mathscr{D}^\bullet$ on the ind-scheme $\mathfrak{X}_{V/W}$. For this purpose, we assume that V/W is a vector space of at most countable dimension over $\Bbbk$ (so $\mathfrak{X} = \mathfrak{X}_{V/W}$ and $\mathfrak{Y} = \mathfrak{Y}_V$ are $\aleph_0$-ind-schemes). Then the construction of Example 4.7 does the job.

(7) Notice that, in the context of (6), the complex $\mathcal{D}_U^\bullet$ has its only cohomology sheaf situated in the negative cohomological degree $-\dim_\Bbbk U$. Assuming that V/W is countably infinite-dimensional vector space and looking into the construction of

Example 4.7 keeping exactness of the functors of direct image (under a closed immersion) and direct limit in mind, one can see that $\mathscr{D}^\bullet$ is an *acyclic* complex of quasi-coherent torsion sheaves on $\mathfrak{X}_{V/W}$. The only cohomology sheaf just runs to the cohomological degree $-\infty$ and disappears in the direct limit as $\dim_{\Bbbk} U$ grows to infinity. Hence $\pi^*\mathscr{D}^\bullet$ is an acyclic complex of quasi-coherent torsion sheaves on $\mathfrak{Y}_V$. These acyclic complexes, representing quite nontrivial objects in the coderived and semiderived categories, are the unit objects of the respective tensor structures given by the cotensor and semitensor product operations!

(8) At last, let us discuss the invariance properties of the constructions above with respect to replacing a linearly compact open subspace $W \subset V$ with another linearly compact open subspace.

Let W' and $W'' \subset V$ be two linearly compact open subspaces. Then $W = W' + W'' \subset V$ is a linearly compact open subspace as well. The quotient spaces W/W' and W/W'' are finite-dimensional, so the morphisms of ind-schemes of ind-finite type $\mathfrak{X}' = \mathfrak{X}_{V/W'} \longrightarrow \mathfrak{X}_{V/W} = \mathfrak{X}$ and $\mathfrak{X}'' = \mathfrak{X}_{V/W''} \longrightarrow \mathfrak{X}_{V/W} = \mathfrak{X}$ induced by the surjective linear maps of discrete vector spaces $V/W' \longrightarrow V/W$ and $V/W'' \longrightarrow V/W$ are affine and weakly smooth (in fact, smooth) of finite relative dimension. Hence Corollary 10.13 tells that the class of morphisms in $\mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors})$ or $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ which are inverted in order to construct the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors})$ coincides with the class of morphisms of complexes which are inverted in order to construct the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}''}(\mathfrak{Y}\text{–}\mathsf{tors})$. In other words, the semiderived category of quasi-coherent torsion sheaves on the ind-scheme $\mathfrak{Y} = \mathfrak{Y}_V$ is determined by the locally linearly compact topological vector space V, and *does not depend on the choice of a linearly compact open subspace* $W \subset V$.

(9) The invariance property of the semitensor product operation $\lozenge_{\pi^*\mathscr{D}^\bullet}$ on $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is only slightly more complicated. The *relative dimension* $\dim W'/W'' \in \mathbb{Z}$ is defined as the difference $\dim_{\Bbbk} \widetilde{W}/W'' - \dim_{\Bbbk} \widetilde{W}/W' = \dim_{\Bbbk} W'/\overline{W} - \dim_{\Bbbk} W''/\overline{W}$, where $\widetilde{W}$ and $\overline{W} \subset V$ are arbitrary linearly compact open subspaces such that $\overline{W} \subset W' \cap W''$ and $W' + W'' \subset \widetilde{W}$. The *relative determinant* $\det W'/W'' \in \Bbbk\text{–}\mathsf{vect}$ is a one-dimensional $\Bbbk$-vector space defined as $\det V'/W'' = \det_{\Bbbk}(\widetilde{W}/W'') \otimes_{\Bbbk} \det_{\Bbbk}(\widetilde{W}/W')^* = \det_{\Bbbk}(W'/\overline{W}) \otimes_{\Bbbk} (\det_{\Bbbk} W''/\overline{W})^*$, where $\det_{\Bbbk}(U) = \Lambda_{\Bbbk}^{\dim_{\Bbbk} U}(U)$ for a finite-dimensional $\Bbbk$-vector space U. Here the equality signs mean natural isomorphisms.

Then the functor $(\det W'/W'')[\dim W'/W''] \otimes_{\Bbbk} \mathrm{Id}\colon \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}'}(\mathfrak{Y}\text{–}\mathsf{tors}) \longrightarrow \mathsf{D}^{\mathsf{si}}_{\mathfrak{X}''}(\mathfrak{Y}\text{–}\mathsf{tors})$ is a tensor triangulated equivalence between the two tensor triangulated categories, with the tensor structures given by the constructions above. In particular, this functor takes the unit object to the unit object. Here $(\det W'/W'')[\dim W'/W'']$ is a complex of $\Bbbk$-vector spaces in which the one-dimensional vector space $\det W'/W''$ sits in the cohomological degree $-\dim W'/W''$ (and the components in all the other cohomological degrees vanish). This is the conclusion one obtains from Theorem 10.25.

Example 11.1. Let us spell out in coordinate notation an important particular case of the above example. Let $\Bbbk((t))$ be the $\Bbbk$-vector space of formal Laurent power series in a variable t over the field $\Bbbk$, endowed with the usual topology in which the vector subspaces $t^n\Bbbk[[t]] \subset \Bbbk((t))$, $n \in \mathbb{Z}$, form a base of neighborhoods of zero. Then $V = \Bbbk((t))$ is a locally linearly compact topological vector space over $\Bbbk$.

A generic element of $\Bbbk((t))$ has the form $f(t) = \sum_{n=-N}^{\infty} x_n t^n$, where $N \in \mathbb{Z}$ and $x_n \in \Bbbk$ for all $n \in \mathbb{Z}$. So $x_n \colon V = \Bbbk((t)) \longrightarrow \Bbbk$ are continuous linear functions; we will consider them as coordinates on the ind-scheme $\mathfrak{Y}_V$.

Let $W \subset V$ be the linearly compact open subspace $W = \Bbbk[[t]] \subset \Bbbk((t))$. Then the $\aleph_0$-ind-scheme $\mathfrak{X} = \mathfrak{X}_{V/W}$ can be described, in terms of the anti-equivalence of categories from Example 1.6(2), as $\mathfrak{X}_{V/W} = \operatorname{Spi}\mathfrak{A}$, where $\mathfrak{A}$ is the topological commutative ring $\mathfrak{A} = \varprojlim_{n>0} \Bbbk[x_{-n}, \dots, x_{-1}]$ with the topology of projective limit of the discrete polynomial rings $A_n = \Bbbk[x_{-n}, \dots, x_{-1}]$. Here the transition map $A_{n+1} \longrightarrow A_n$ in the projective system takes x_{-n-1} to 0 and x_{-i} to x_{-i} for $i \le n$.

The $\aleph_0$-ind-scheme $\mathfrak{Y} = \mathfrak{Y}_V$ can be similarly described, in terms of the same anti-equivalence of categories, as $\mathfrak{Y}_V = \operatorname{Spi}\mathfrak{R}$, where $\mathfrak{R}$ is the topological commutative ring $\mathfrak{R} = \varprojlim_{n>0} \Bbbk[x_{-n}, \dots, x_{-1}, x_0, x_1, \dots]$ with the topology of projective limit of the discrete ring of polynomials in infinitely many variables $R_n = \Bbbk[x_{-n}, \dots, x_{-1}, x_0, x_1, \dots]$. The transition map $R_{n+1} \longrightarrow R_n$ in the projective system of rings takes x_{-n-1} to 0 and x_i to x_i for $i \ge -n$. The flat affine morphism of ind-schemes $\pi \colon \mathfrak{Y}_V \longrightarrow \mathfrak{X}_{V/W}$ corresponds to the natural injective continuous ring homomorphism $\mathfrak{A} \longrightarrow \mathfrak{R}$, which can be obtained as the projective limit of the natural subring inclusions $A_n \longrightarrow R_n$.

Following the discussion in Section 2.4(6), the Grothendieck abelian category $\mathfrak{X}\text{–}\mathsf{tors}$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is equivalent to the category $\mathfrak{A}\text{–}\mathsf{discr}$ of discrete $\mathfrak{A}$-modules. The abelian category $\mathfrak{A}\text{–}\mathsf{discr}$ can be simply described in explicit terms as the category of modules $\mathscr{M}$ over the ring of polynomials in infinitely many variables $A = \Bbbk[\dots, x_{-3}, x_{-2}, x_{-1}]$ having the property that for every $b \in \mathscr{M}$ there exists $n > 0$ such that $x_{-i}b = 0$ for all $i > n$. The Grothendieck abelian category $\mathfrak{Y}\text{–}\mathsf{tors}$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is equivalent to the category $\mathfrak{R}\text{–}\mathsf{discr}$ of discrete $\mathfrak{R}$-modules, which explicitly means modules $\mathscr{N}$ over the ring of polynomials in doubly infinitely many variables $R = \Bbbk[\dots, x_{-2}, x_{-1}, x_0, x_1, \dots]$ with the property that for every $b \in \mathscr{N}$ there exists $n > 0$ such that $x_{-i}b = 0$ for all $i > n$ (while no condition is imposed on x_ib for $i \ge 0$).

So the obvious forgetful functor $R\text{–}\mathsf{mod} \longrightarrow A\text{–}\mathsf{mod}$ induced by the subring inclusion $A \longrightarrow R$ takes discrete $\mathfrak{R}$-modules to discrete $\mathfrak{A}$-modules. The forgetful functor $\mathfrak{R}\text{–}\mathsf{discr} \longrightarrow \mathfrak{A}\text{–}\mathsf{discr}$ corresponds, under the above equivalences between the sheaf and module categories, to the direct image functor $\pi_* \colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$, in terms of which the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is defined.

Finally, the dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ as per the construction in (6) is a complex of injective discrete modules over $\mathfrak{A}$ (concentrated, if one wishes, in the nonpositive cohomological degrees; cf. Remarks 5.3(3–5)) with the following property. For every $n > 0$, the subcomplex $\mathcal{D}^\bullet_n$ of all elements annihilated by $\dots$, x_{-n-3}, x_{-n-2}, x_{-n-1} in $\mathscr{D}^\bullet$ is a complex of injective A_n-modules homotopy equivalent to an injective resolution of the free A_n-module with one generator $\Bbbk[x_{-n}, \dots, x_{-1}]\, dx_{-n} \wedge \dots \wedge dx_{-1}$ shifted cohomologically by $[n]$.

As explained in (8–9), the results of Section 10 imply that the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is preserved by all the continuous *linear* coordinate changes in the topological vector space $V = \Bbbk((t))$. This means all the bijective continuous $\Bbbk$-linear maps $\Bbbk((t)) \longrightarrow \Bbbk((t))$ with continuous inverse maps. Moreover, the semitensor product operation on $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ is also preserved by such coordinate changes, up to dimensional cohomological shifts and determinantal twists.

Question 11.2. In the context of Example 11.1, one can write $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors}) = \mathsf{D}^{\mathsf{si}}_{\mathfrak{A}}(\mathfrak{R}\text{–}\mathsf{discr})$, referring to the equivalences of abelian categories $\mathfrak{Y}\text{–}\mathsf{tors} \simeq \mathfrak{R}\text{–}\mathsf{discr}$ and $\mathfrak{X}\text{–}\mathsf{tors} \simeq \mathfrak{A}\text{–}\mathsf{discr}$. Is the semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{A}}(\mathfrak{R}\text{–}\mathsf{discr})$ preserved by arbitrary continuous *polynomial* coordinate changes, i. e., all the automorphisms of the topological ring $\mathfrak{R}$?

11.2. **Cotangent bundle to discrete projective space.** Let V be an infinite-dimensional discrete $\Bbbk$-vector space. Then the *projectivization* of V is an ind-scheme $\mathfrak{P}(V)$ of ind-finite type over $\Bbbk$, defined informally as the space of all one-dimensional vector subspaces in V (cf. [58, Example 1.3(2)]). Any one-dimensional vector subspace $L \in V$ corresponds to a $\Bbbk$-point $l\colon \operatorname{Spec}\Bbbk \longrightarrow \mathfrak{P}(V)$.

The tangent space to $\mathfrak{P}(V)$ at the point l can be computed as $T_l\mathfrak{P}(V) = \operatorname{Hom}_\Bbbk(L, V/L)$; so it is a discrete $\Bbbk$-vector space (as one would expect). Accordingly, the cotangent space $T^*_l\mathfrak{P}(V) = \operatorname{Hom}_\Bbbk(V/L, L)$ is a linearly compact $\Bbbk$-vector space. Following Section 11.1(3), there is an infinite-dimensional affine scheme corresponding to $T^*_l\mathfrak{P}(V)$, described as $Y_{T^*_l\mathfrak{P}(V)} = \operatorname{Spec}\operatorname{Sym}_\Bbbk(T_l\mathfrak{P}(V))$. So, denoting by $\mathfrak{Y}$ the total space of the cotangent bundle to $\mathfrak{X} = \mathfrak{P}(V)$, one would expect the fibration $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ to be a flat affine morphism of ind-schemes. In order to show that this is indeed the case, let us explain how to formalize this informal discussion.

(1) Let U be a finite-dimensional $\Bbbk$-vector space. Then the projectivization $P(U)$ is defined as the projective spectrum of the graded ring $\operatorname{Sym}_\Bbbk(U^*)$, i. e., $P(U) = \operatorname{Proj}\operatorname{Sym}_\Bbbk(U^*)$ (where the grading on the commutative $\Bbbk$-algebra $\operatorname{Sym}_\Bbbk(U^*)$ is defined by the rule that the elements of $U^* \subset \operatorname{Sym}_\Bbbk(U^*)$ have degree 1). This means that the scheme points of $P(U)$ correspond bijectively to homogeneous prime ideals in the graded ring $R = \operatorname{Sym}_\Bbbk(U^*)$ not containing (in other words, different from) the ideal $\bigoplus_{n>0} R_n \subset R$ of all elements of positive degree.

For every element $r \in R_1$, the subset in $\operatorname{Proj} R$ consisting of all homogeneous prime ideals not containing r is an affine open subscheme in $\operatorname{Proj} R$ naturally isomorphic to $\operatorname{Spec} R[r^{-1}]_0$, where $R[r^{-1}]$ is the $\mathbb{Z}$-graded ring obtained by inverting the element $r \in R_1$ and $R[r^{-1}]_0 \subset R[r^{-1}]$ is the subring of all elements of degree 0.

Let U'' be a finite-dimensional $\Bbbk$-vector space and $U' \subset U''$ be a vector subspace. Then the surjective $\Bbbk$-linear map $U''^* \longrightarrow U'^*$ induces a surjective homomorphism of graded rings $R'' = \operatorname{Sym}_\Bbbk(U''^*) \longrightarrow \operatorname{Sym}_\Bbbk(U'^*) = R'$. Hence the induced closed immersion of projective spectra $P(U') = \operatorname{Proj} R' \longrightarrow \operatorname{Proj} R'' = P(U'')$.

(2) Let M be a graded module over the graded ring $R = \operatorname{Sym}_\Bbbk(U^*)$. Then a quasi-coherent sheaf $\widetilde{M}$ over $P(U) = \operatorname{Proj} R$ is assigned to M in the following way. For every element $r \in R_1$, the restriction of $\widetilde{M}$ to the affine open subscheme

$\operatorname{Spec} R[r^{-1}]_0 \subset \operatorname{Proj} R$ is the quasi-coherent sheaf over $\operatorname{Spec} R[r^{-1}]_0$ corresponding to the $R[r^{-1}]_0$-module $M[r^{-1}]_0$. Here $M[r^{-1}] = R[r^{-1}] \otimes_R M$ and $M[r^{-1}]_0$ is the degree 0 component of the $\mathbb{Z}$-graded module $M[r^{-1}]$.

The functor $M \longmapsto \widetilde{M}$ from the abelian category of graded R-modules to the abelian category of quasi-coherent sheaves on $\operatorname{Proj} R$ is exact and preserves coproducts; it is also a tensor functor between the two tensor categories. In particular, the functor $M \longmapsto \widetilde{M}$ takes flat graded R-modules F to flat quasi-coherent sheaves $\widetilde{F}$ on $\operatorname{Proj} R$. One has $\widetilde{M} = 0$ if and only if, for every $b \in M$, there exists $n \geq 1$ such that $R_i b = 0$ in M for all $i \geq n$. The functor $M \longmapsto \widetilde{M}$ also takes finitely generated graded R-modules to coherent sheaves on $P(U) = \operatorname{Proj} R$.

For any graded R-module M and any integer $n \in \mathbb{Z}$, denote by $M(n)$ the graded R-module with the components $M(n)_i = M_{n+i}$ (and the same action of R as in M). The *tautological line bundle* $\mathcal{O}_{P(V)}(-1)$ on $P(V)$ is defined informally by the rule that the line L is the fiber of $\mathcal{O}_{P(V)}(-1)$ over a $\mathbb{k}$-point $l\colon \operatorname{Spec} \mathbb{k} \longrightarrow P(V)$ corresponding to a one-dimensional $\mathbb{k}$-vector subspace $L \subset V$. More formally, the quasi-coherent sheaf $\mathcal{O}_{P(V)}(n)$ on $P(V)$ (for any $n \in \mathbb{Z}$) corresponds to the graded R-module $R(n)$. Accordingly, one has $\widetilde{M(n)} = \widetilde{M}(n)$ for any graded R-module M, where $\mathcal{M}(n) = \mathcal{O}_{P(V)}(n) \otimes_{\mathcal{O}_{P(V)}} \mathcal{M}$ for any quasi-coherent sheaf $\mathcal{M}$ on $P(V)$.

Let $U' \subset U''$ be a vector subspace in a finite-dimensional $\mathbb{k}$-vector space, and let $R'' \longrightarrow R'$ be the related surjective morphism of graded rings, as in (1). Denote by $i_{U'U''}\colon P(U') \longrightarrow P(U'')$ the related closed immersion of projective spaces over $\mathbb{k}$. Let M'' be a graded R''-module and $\widetilde{M''}$ be the related quasi-coherent sheaf on $P(U'')$. Then the quasi-coherent sheaf $i_{U'U''}^*\widetilde{M''}$ on $P(U')$ corresponds to the graded R'-module $R' \otimes_{R''} M''$.

(3) Let V be a discrete $\mathbb{k}$-vector space. Denote by Γ the directed poset of all finite-dimensional vector subspaces $U \subset V$, ordered by inclusion. The *projectivization* of V is defined as the ind-scheme $\mathfrak{P}(V) = \text{“}\varinjlim\text{”}_{U\in\Gamma} P(U)$, where the transition maps $i_{U'U''}\colon P(U') \longrightarrow P(U'')$ are the ones defined above in (1–2).

Our aim is to construct a flat pro-quasi-coherent pro-sheaf $\mathfrak{T}$ on $\mathfrak{P}(V)$ corresponding to the tangent bundle. So, for any $\mathbb{k}$-point $l\colon \operatorname{Spec} \mathbb{k} \longrightarrow \mathfrak{P}(V)$ and the related one-dimensional vector subspace $L \subset V$, the fiber of $\mathfrak{T}$ over l should be the discrete $\mathbb{k}$-vector space $\operatorname{Hom}_{\mathbb{k}}(L, V/L) \simeq (L^* \otimes_{\mathbb{k}} V)/\mathbb{k}$.

Let $U \subset V$ be a finite-dimensional vector subspace. We start with constructing the restriction of $\mathfrak{T}$ onto the closed subscheme $P(U) \subset \mathfrak{P}(V)$. Denoting by $i_U\colon P(U) \longrightarrow \mathfrak{P}(V)$ the closed immersion morphism, we would like to construct the quasi-coherent sheaf $\mathcal{T}_U = i_U^*\mathfrak{T}$ on the scheme $P(U)$.

The infinite-dimensional vector bundle $V(1)$ on $P(U)$ corresponds to the graded module $V \otimes_{\mathbb{k}} R(1)$ over the graded ring $R = \operatorname{Sym}_{\mathbb{k}}(U^*)$. The degree -1 component of this graded module is the vector space V, and the degree 0 component of the vector space $V \otimes_{\mathbb{k}} U^*$. The latter vector space contains a canonical element $e \in V \otimes_{\mathbb{k}} U^* \simeq \operatorname{Hom}_{\mathbb{k}}(U, V)$ corresponding to the identity map $U \longrightarrow V$.

Let $f_U\colon R \longrightarrow V \otimes_{\Bbbk} R(1)$ be the graded R-module map taking the free generator $1 \in R$ to the element $e \in V \otimes_{\Bbbk} U^*$. Denote by $T_U = \operatorname{coker}(f_U)$ the cokernel of f_U taken in the category of graded R-modules. By definition, the quasi-coherent sheaf $\mathcal{T}_U$ on $P(U) = \operatorname{Proj} R$ corresponds to the graded R-module T_U.

(4) The graded R-module T_U is *not* flat, but it is "flat up to torsion R-modules", in a suitable sense; so the quasi-coherent sheaf $\mathcal{T}_U$ on $P(U)$ is flat. More precisely, the map $f_U\colon R \longrightarrow V \otimes_{\Bbbk} R(1)$ factorizes as the composition $R \longrightarrow U \otimes_{\Bbbk} R(1) \longrightarrow V \otimes_{\Bbbk} R(1)$, where the split monomorphism of free graded R-modules $U \otimes_{\Bbbk} R(1) \longrightarrow V \otimes_{\Bbbk} R(1)$ is induced by the inclusion of $\Bbbk$-vector spaces $U \longrightarrow V$. The cokernel of the morphism $R \longrightarrow U \otimes_{\Bbbk} R(1)$ is finitely generated graded R-module; the corresponding coherent sheaf on $P(U)$ is the locally free sheaf corresponding to the tangent bundle to $P(U)$.

To see algebraically that the cokernel of the graded R-module morphism $R \longrightarrow U \otimes_{\Bbbk} R(1)$ corresponds to a locally free coherent (or at least, a flat quasi-coherent) sheaf on $P(U)$, one can consider the Koszul complex

$$0 \longrightarrow R \longrightarrow U \otimes_{\Bbbk} R(1) \longrightarrow \Lambda^2_{\Bbbk}(U) \otimes_{\Bbbk} R(2) \longrightarrow \cdots \longrightarrow \Lambda^{\dim_{\Bbbk} U}(U) \otimes_{\Bbbk} R(\dim_{\Bbbk} U),$$

where $\Lambda^n_{\Bbbk}(U)$ are the exterior powers of the $\Bbbk$-vector space U. This complex is a free graded resolution of the graded R-module $\Lambda^{\dim_{\Bbbk} U}(U)(\dim_{\Bbbk} U)$, which is a one-dimensional $\Bbbk$-vector space viewed as a graded R-module concentrated in the single degree $-\dim_{\Bbbk} U$. The exact functor $M \longmapsto \widetilde{M}$ annihilates this graded R-module, so this functor takes the Koszul complex to an exact finite complex of locally free coherent sheaves on $P(U)$. Accordingly, all the graded R-modules of cycles and boundaries of the Koszul complex are also taken to locally free coherent sheaves on $P(U)$ by the functor $M \longmapsto \widetilde{M}$.

(5) Let $U' \subset U'' \subset V$ be two finite-dimensional subspaces in our discrete $\Bbbk$-vector space V. Then the functor $R' \otimes_{R''} -$ takes the graded R''-module morphism $f_{U''}$ to the graded R'-module morphism $f_{U'}$. Hence we have a natural isomorphism of graded R'-modules $T_{U'} \simeq R' \otimes_{R''} T_{U''}$, and consequently a natural isomorphism $\mathcal{T}_{U'} \simeq i^*_{U'U''}\mathcal{T}_{U''}$ of flat quasi-coherent sheaves on $P(U')$.

Now the rule $\mathfrak{T}^{(P(U))} = \mathcal{T}_U$ for all $U \in \Gamma$ defines the desired flat pro-quasi-coherent pro-sheaf $\mathfrak{T}$ on the ind-scheme $\mathfrak{P}(V)$.

(6) Similarly to the symmetric algebra of a vector space, one can define the symmetric algebra of a module over a commutative ring S. Given an S-module N, the symmetric algebra $\operatorname{Sym}_S(N)$ can be defined as the commutative S-algebra freely generated by the S-module N. When F is a free S-module, $\operatorname{Sym}_S(F)$ is a free S-module, too; and the (nonadditive) functor $N \longmapsto \operatorname{Sym}_S(N)$ preserves direct limits; so when F is a flat S-module, $\operatorname{Sym}_S(F)$ is a flat S-algebra.

Furthermore, given a quasi-coherent sheaf $\mathcal{N}$ on a scheme Z, one defines the quasi-coherent commutative algebra $\mathcal{S}ym_Z(\mathcal{N})$ on Z (in the sense of Section 3.6) by the rule $\mathcal{S}ym_Z(\mathcal{N})(W) = \operatorname{Sym}_{\mathcal{O}(W)}(\mathcal{N}(W))$ for all the affine open subschemes $W \subset Z$. Clearly, $\mathcal{S}ym_Z(\mathcal{F})$ is a flat quasi-coherent commutative algebra on Z whenever $\mathcal{F}$ is a flat quasi-coherent sheaf on Z. For every morphism of schemes $f\colon Z' \longrightarrow Z''$ and a quasi-coherent sheaf $\mathcal{N}''$ on Z'', one has a natural isomorphism $\mathcal{S}ym_{Z'}(f^*\mathcal{N}'') \simeq f^*\,\mathcal{S}ym_{Z''}(\mathcal{N}'')$ of quasi-coherent algebras on Z'.

(7) In the context of (3–5), we put $\mathcal{A}_U = \mathcal{S}ym_{P(U)}\,\mathcal{T}_U$ for every finite-dimensional vector subspace U in the given discrete $\Bbbk$-vector space V. So $\mathcal{A}_U$ is a flat quasi-coherent algebra on the projective space $P(U)$. Then the rule $\mathfrak{A}^{(P(U))} = \mathcal{A}_U$ defines a flat pro-quasi-coherent commutative algebra $\mathfrak{A}$ on the ind-scheme $\mathfrak{X} = \mathfrak{P}(V)$. The ind-scheme $\mathfrak{Y}$ together with the flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ corresponds to the pro-quasi-coherent algebra $\mathfrak{A}$ on $\mathfrak{X}$ via the construction of Proposition 3.12. This is the desired flat affine morphism of ind-schemes corresponding to the cotangent bundle on the ind-projective space $\mathfrak{P}(V)$.

(8) Assuming that the dimension of V is at most countable, one can construct a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X} = \mathfrak{P}(V)$ following the approach of Remarks 5.3(3–5). Notice that, similarly to Section 11.1(7), the dualizing complex $\mathscr{D}^\bullet$ is an *acyclic* complex of quasi-coherent torsion sheaves on $\mathfrak{X}$ whenever V is a $\Bbbk$-vector space of (countably) infinite dimension.

11.3. **Universal fibration of quadratic cones in linearly compact vector space.** Let W be an infinite-dimensional discrete $\Bbbk$-vector space and W^* be the dual linearly compact $\Bbbk$-vector space. Then the elements of the vector space $\mathrm{Sym}^2_\Bbbk(W)$ correspond to continuous quadratic functions $q\colon W^* \longrightarrow \Bbbk$. It is worth noticing that any such quadratic function actually factorizes through a finite-dimensional discrete quotient vector space of W^*.

The zero locus Y_q of any nonzero continuous quadratic function $q\colon W^* \longrightarrow \Bbbk$ is an infinite dimensional affine scheme, or more specifically a closed subscheme in the affine scheme Y_{W^*} corresponding to the linearly compact topological vector space W^* under the construction of Section 11.1(3). This closed subscheme is an infinite-dimensional quadratic cone. Such quadratic cones $Y_q \subset Y_{W^*}$ are parametrized by nonzero continuous quadratic functions $q\colon W^* \longrightarrow \Bbbk$ viewed up to a multiplication by a scalar from $\Bbbk$. In other words, the space of parameters of the quadratic cones in W^* is the ind-scheme $\mathfrak{P}(V)$ from Section 11.2(3), where the infinite-dimensional discrete $\Bbbk$-vector space V is constructed as $V = \mathrm{Sym}^2_\Bbbk(W)$.

This informal discussion suggests that there should be a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X} = \mathfrak{P}(V)$ whose fibers over the $\Bbbk$-points of $\mathfrak{P}(V)$ are the quadratic cones Y_q. The aim of this section is to spell out a precise construction of the ind-scheme $\mathfrak{Y}$ and the morphism π.

(0) Let us first return to the discussion of the ind-scheme $\mathfrak{P}(V)$ from Section 11.2(3). The approach hinted at in Section 11.1(0) suggests to describe schemes and ind-schemes by their "functors of points", i. e., the functors they represent on the category of affine schemes. For our present purposes, let us restrict ourselves to *field extensions* $\Bbbk \longrightarrow K$. One easly observes that, for any such field extension, the set $\mathfrak{X}(K)$ of K-points in $\mathfrak{X} = \mathfrak{P}(V)$ is naturally bijective to the set of all one-dimensional vector subspaces $L \subset K \otimes_\Bbbk V$ in the K-vector space $K \otimes_\Bbbk V$. (The

case of an arbitrary commutative $\Bbbk$-algebra K is considerably more complicated; see [58, Example 1.3(2)].)

(1) Let W be a discrete $\Bbbk$-vector space. The discrete vector space $\mathrm{Sym}^2_\Bbbk(W)$ can be defined as the degree 2 component of the graded $\Bbbk$-algebra $\mathrm{Sym}_\Bbbk(W)$, that is, the vector subspace in the $\Bbbk$-algebra $\mathrm{Sym}_\Bbbk(W)$ spanned by the products $w'w''$ with w', $w'' \in W$. Put $V = \mathrm{Sym}^2_\Bbbk(W)$.

Let $U \subset V$ be a finite-dimensional vector subspace. Consider the graded ring $R = \mathrm{Sym}_\Bbbk(U^*)$. Then the tensor product $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ is a bigraded commutative R-algebra. For the purposes of applying the functor $M \longmapsto \widetilde{M}$ (see Section 11.2(2)) we will consider the grading on $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ induced by the grading of R. Then the functor $M \longmapsto \widetilde{M}$ takes the graded R-algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ to the quasi-coherent algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk \mathcal{O}_{P(U)}$ on the scheme $P(U)$. This quasi-coherent algebra corresponds to the affine morphism of schemes $Y_{W^*} \times_\Bbbk P(U) \longrightarrow P(U)$, where $Y_{W^*} = \mathrm{Spec}\,\mathrm{Sym}_\Bbbk(W)$ (as in Section 11.1(3)).

(2) Consider the free graded module $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R(-1)$ over the graded algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$. Applying the functor $M \longmapsto \widetilde{M}$ to this graded module produces the quasi-coherent module $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk \mathcal{O}_{P(U)}(-1)$ over the quasi-coherent algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk \mathcal{O}_{P(U)}$ on $P(U)$.

The free graded module $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R(-1)$ over the graded algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ is spanned by the element 1 sitting in degree 1 (in the grading induced by the grading of R). The degree 0 component of the graded algebra $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ is the algebra $\mathrm{Sym}_\Bbbk(W)$, and the degree 1 component is the vector space $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk U^* \simeq \mathrm{Hom}_\Bbbk(U, \mathrm{Sym}_\Bbbk(W))$. The natural injective $\Bbbk$-linear map $U \longrightarrow V \simeq \mathrm{Sym}^2_\Bbbk(W) \longrightarrow \mathrm{Sym}_\Bbbk(W)$ defines a canonical element $e \in \mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk U^*$.

Let $f_U \colon \mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R(-1) \longrightarrow \mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ be the morphism of graded R-modules taking the generator $1 \in \mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R(-1)$ to the element $e \in \mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk U^*$. The morphism f_U is injective because the ring $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R \simeq \mathrm{Sym}_\Bbbk(W \oplus U^*)$ has no zero-divisors. Denote by C_U the cokernel of f_U taken in the category of graded R-modules. So C_U is naturally a graded R-algebra (namely, the quotient algebra of $\mathrm{Sym}_\Bbbk(W) \otimes_\Bbbk R$ by the ideal generated by the homogeneous element e). By the definition, the quasi-coherent algebra $\mathcal{C}_U$ on $P(U) = \mathrm{Proj}\, R$ is obtained by applying the functor $M \longmapsto \widetilde{M}$ to the graded R-algebra C_U.

Our next aim is to show that the quasi-coherent sheaf $\mathcal{C}_U$ on $P(U)$ is flat.

Lemma 11.3. *Let $f \colon \mathcal{G} \longrightarrow \mathcal{F}$ be a monomorphism of locally free coherent sheaves on a Noetherian scheme X. Assume that, for every field K and any morphism of schemes $i \colon \mathrm{Spec}\, K \longrightarrow X$, the morphism of coherent sheaves on $\mathrm{Spec}\, K$ (i. e., of K-vector spaces) $i^*f \colon i^*\mathcal{G} \longrightarrow i^*\mathcal{F}$ is injective. Then the cokernel $\mathcal{F}/f(\mathcal{G})$ of the monomorphism f is a locally free sheaf on X.*

Proof. This lemma is well-known, so we restrict ourselves to pointing out that it remains true for flat quasi-coherent sheaves in lieu of locally free coherent ones. If $f \colon \mathcal{G} \longrightarrow \mathcal{F}$ is a monomorphism of flat quasi-coherent sheaves on a Noetherian scheme X and, for every $i \colon \mathrm{Spec}\, K \longrightarrow X$ as in the lemma, the morphism $i^*f \colon i^*\mathcal{G} \longrightarrow i^*\mathcal{F}$

is injective, then $\mathcal{F}/f(\mathcal{G})$ is a flat quasi-coherent sheaf on X. This is (a particular case of) the result of [10, Remark 2.3], which is based on [2, Proposition 5.3.F]; it also follows from [10, Theorem 1.1]. $\square$

(3) One can start from observing that, for any finite-dimensional subspace $U \subset \mathrm{Sym}^2_{\Bbbk}(W)$, there exists a finite-dimensional subspace $\overline{W} \subset W$ such that $U \subset \mathrm{Sym}^2_{\Bbbk}(\overline{W})$. The map f_U is the direct limit of the similar maps $\bar{f}_U$ with the vector space W replaced by its finite-dimensional subspaces $\overline{W}$ satisfying this inclusion. Furthermore, the morphism of free graded R-modules $\bar{f}_U\colon \mathrm{Sym}_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R(-1) \longrightarrow \mathrm{Sym}_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R$ is the direct sum of the morphisms of finitely generated free graded R-modules $\mathrm{Sym}^n_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R(-1) \longrightarrow \mathrm{Sym}^{n+2}_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R$, where $n = 0, 1, 2, \dots$ These observations make Lemma 11.3 applicable as it is stated (for locally free coherent sheaves), and one does not even need the more general version of it suggested in the proof.

Let K be a field and $l\colon \mathrm{Spec}\, K \longrightarrow P(U)$ be a morphism of schemes. Then the composition $\mathrm{Spec}\, K \longrightarrow P(U) \longrightarrow \mathrm{Spec}\, \Bbbk$ makes $\Bbbk$ a subfield in K. As mentioned above in (0), the morphism l corresponds to a one-dimensional vector subspace $L \subset K \otimes_{\Bbbk} U \subset K \otimes_{\Bbbk} \mathrm{Sym}^2_{\Bbbk}(\overline{W}) \simeq \mathrm{Sym}^2_K(K \otimes_{\Bbbk} \overline{W}) \subset K \otimes_{\Bbbk} V$. So any nonzero vector $q \in L$ defines a quadratic function $q\colon (K \otimes_{\Bbbk} W)^* \longrightarrow K$, which factorizes as $(K \otimes_{\Bbbk} W)^* \twoheadrightarrow (K \otimes_{\Bbbk} \overline{W})^* \xrightarrow{\bar{q}} K$. Applying the functor $M \longmapsto \widetilde{M}$ and then the functor i^* to the morphism of graded R-modules $\bar{f}_U\colon \mathrm{Sym}_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R(-1) \longrightarrow \mathrm{Sym}_{\Bbbk}(\overline{W}) \otimes_{\Bbbk} R$, one obtains the morphism of K-vector spaces $\mathrm{Sym}_K(K \otimes_{\Bbbk} \overline{W}) \otimes_K L \longrightarrow \mathrm{Sym}_K(K \otimes_{\Bbbk} \overline{W})$ taking a tensor $a \otimes_K \bar{q}$ with $a \in \mathrm{Sym}^n_K(K \otimes_{\Bbbk} \overline{W})$ and $\bar{q} \in L \subset \mathrm{Sym}^2_K(K \otimes_{\Bbbk} \overline{W})$ to the vector $a\bar{q} \in \mathrm{Sym}^{n+2}_K(K \otimes_{\Bbbk} \overline{W})$ (where the multiplication is performed in the symmetric algebra $\mathrm{Sym}_K(K \otimes_{\Bbbk} \overline{W})$). This map of K-vector spaces is injective.

(4) According to Lemma 11.3, it follows that the quasi-coherent algebra $\mathcal{C}_U$ on the projective space $P(U)$ is flat. Now let $U' \subset U'' \subset V$ be two finite-dimensional subspaces in the $\Bbbk$-vector space $V = \mathrm{Sym}^2_{\Bbbk}(W)$. Put $R' = \mathrm{Sym}_{\Bbbk}(U'^*)$ and $R'' = \mathrm{Sym}_{\Bbbk}(U''^*)$. As explained in Section 11.2(1–2), we have a surjective morphism of graded rings $R'' \longrightarrow R'$ inducing a closed immersion of the projective spectra $i_{U'U''}\colon P(U') \longrightarrow P(U'')$.

Similarly to Section 11.2(5), the functor $R' \otimes_{R''} -$ takes the morphism of graded R''-modules $f_{U''}$ to the morphism of graded R'-modules $f_{U'}$. Hence we have a natural isomorphism of graded R'-modules, and in fact of graded commutative R'-algebras, $C_{U'} \simeq R' \otimes_{R''} C_{U''}$, and consequently a natural isomorphism $\mathcal{C}_{U'} \simeq i^*_{U'U''}\mathcal{C}_{U''}$ of flat quasi-coherent commutative algebras on $P(U')$.

Finally, the rule $\mathfrak{C}^{(P(U))} = \mathcal{C}_U$ defines a flat pro-quasi-coherent commutative algebra $\mathfrak{C}$ on the ind-scheme $\mathfrak{X} = \mathfrak{P}(V)$. The ind-scheme $\mathfrak{Y}$ together with the flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ corresponds to the pro-quasi-coherent algebra $\mathfrak{C}$ on $\mathfrak{X}$ via the construction of Proposition 3.12. This is the desired flat affine morphism of ind-schemes corresponding to the $\mathfrak{P}(V)$-parametric family of quadratic cones in the linearly compact topological vector space W^*.

(5) To make the setting for possible application of the constructions and results of Sections 7–9 complete, it remains to specify a dualizing complex on the ind-scheme $\mathfrak{X} = \mathfrak{P}(V)$. It was mentioned in Section 11.2(8) how this can be done (assuming the dimension of the $\Bbbk$-vector space W, hence also V, is at most countable).

Appendix. The Semiderived Category for a Nonaffine Morphism

Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\mathfrak{Y}$ be an ind-semi-separated ind-scheme, and $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a *nonaffine* flat morphism of ind-schemes. The aim of this appendix is to spell out a definition of the $\mathfrak{Y}/\mathfrak{X}$-semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ in this context.

A.1. **Becker's coderived category.** In this appendix, we will use a different approach to the definition of the coderived category than in the main body of the paper (cf. Section 4.4). The relevant references for Becker's coderived category are [23, 37, 6, 62, 56].

Let E be an exact category with enough injective objects. Notice that in any such exact category the infinite coproduct functors are exact (if the coproducts exist). A complex $E^\bullet$ in E is said to be *Becker coacyclic* (or "coacyclic in the sense of Becker") if, for any complex of injective objects $J^\bullet$ in E, the complex of morphisms $\mathrm{Hom}_{\mathsf{E}}(E^\bullet, J^\bullet)$ is acyclic (as a complex of abelian groups).

Lemma A.4. (a) *The totalization of any short exact sequence of complexes in* E *is a Becker coacyclic complex.*

(b) *The coproduct of any family of Becker coacyclic complexes, if it exists in* $\mathsf{K}(\mathsf{E})$*, is a Becker coacyclic complex.*

(c) *Consequently, if the infinite coproducts exist in* E*, then any coacyclic complex in the sense of Section 4.4 is also coacyclic in the sense of Becker.*

Proof. Parts (a–b) are (a straightforward generalization of) [56, Lemma 9.1]; they are closely related to Proposition 4.15(a). Part (c) follows immediately from (a–b). ☐

Lemma A.5. *If the category* E *is abelian with the abelian exact structure, then any coacyclic complex in* E *is acyclic.*

Proof. A complex $E^\bullet$ in E is acyclic if and only if the complex of abelian groups $\mathrm{Hom}_{\mathsf{E}}(E^\bullet, J)$ is acyclic for any injective object $J \in \mathsf{E}$ (viewed as a one-term complex of injective objects). ☐

The *Becker coderived category* $\mathsf{D}^{\mathsf{bco}}(\mathsf{E})$ is defined as the triangulated quotient category of the homotopy category $\mathsf{K}(\mathsf{E})$ by the thick subcategory of Becker coacyclic complexes. It is clear from Lemma A.4(c) that Becker's coderived category of an exact category E with infinite coproducts and enough injectives is (at worst) a triangulated quotient category of the coderived category in the sense of Section 4.4. So there is a triangulated Verdier quotient functor $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{E}^{\mathsf{bco}}(\mathsf{E})$ forming a commutative triangle diagram with the triangulated Verdier quotient functors $\mathsf{K}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$

and $\mathsf{K}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathsf{E})$. When E is abelian, it follows from Lemma A.5 that there is also a triangulated Verdier quotient functor $\mathsf{D}^{\mathsf{bco}}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$.

Proposition A.6. *Let E be an exact category with infinite coproducts and enough injective objects such that the full subcategory of injective objects $\mathsf{E}_{\mathsf{proj}}$ is preserved by the infinite coproducts in E (e. g., this holds for any locally Noetherian Grothendieck abelian category with the abelian exact structure). Then the canonical functor $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathsf{E})$ is a triangulated equivalence.*

Proof. Follows from Proposition 4.15(b). The conditions on the exact category E can be relaxed a bit; see condition $(*)$ in [41, Section 3.7] or (even more generally) the results of [44, Section A.6]. ☐

In is an *open problem* whether the functor $\mathsf{D}^{\mathsf{bco}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ is a triangulated equivalence for every Grothendieck abelian category E (with the abelian exact structure), or even for the category of modules over an arbitrary ring. See, e. g., [51, Example 2.5(3)] for a discussion. The advantage of Becker's coderived category, though, is that it is known to work well for all Grothendieck abelian categories.

Theorem A.7. *Let A be a Grothendieck abelian category and $\mathsf{A}_{\mathsf{inj}} \subset \mathsf{A}$ be its full subcategory of injective objects. Then the composition $\mathsf{K}(\mathsf{A}_{\mathsf{inj}}) \longrightarrow \mathsf{K}(\mathsf{A}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathsf{A})$ of the inclusion functor $\mathsf{K}(\mathsf{A}_{\mathsf{inj}}) \longrightarrow \mathsf{K}(\mathsf{A})$ and the Verdier quotient functor $\mathsf{K}(\mathsf{A}) \longrightarrow \mathsf{D}^{\mathsf{bco}}(\mathsf{A})$ is a triangulated equivalence $\mathsf{K}(\mathsf{A}_{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{bco}}(\mathsf{A})$.*

Proof. This result can be found in [24, Corollary 5.13], [37, Theorem 3.13], or [56, Corollary 9.4]. ☐

Lemma A.8. *Let A and B be Grothendieck abelian categories, and let $F\colon \mathsf{A} \longrightarrow \mathsf{B}$ be an exact functor which has a right adjoint (equivalently, F is exact and preserves coproducts). Then the functor F takes Becker coacyclic complexes in A to Becker coacyclic complexes in B.*

Remark A.9. An analogue of Lemma A.8 holds for coacyclic complexes in the sense of Section 4.4 under weaker assumptions: any exact functor preserving coproducts, acting between exact categories with exact coproducts, preserves coacyclicity. This assertion, following immediately from the definitions, was mentioned and used many times throughout the main body of this paper.

Proof of Lemma A.8. It is a particular case of the Special Adjoint Functor Theorem that a functor between cocomplete abelian categories having sets of generators is a left adjoint if and only if it preserves colimits (equivalently, is right exact and preserves coproducts). This explains the equivalent reformulation of the lemma's assumptions in the parentheses. Now let $G\colon \mathsf{B} \longrightarrow \mathsf{A}$ be the right adjoint functor to F. Since the functor F is exact, the functor G takes injectives to injectives. Let $A^\bullet$ be a complex in A and $J^\bullet$ be a complex of injective objects in B. Then the isomorphism of complexes of abelian groups $\mathrm{Hom}_{\mathsf{B}}(F(A^\bullet), J^\bullet) \simeq \mathrm{Hom}_{\mathsf{A}}(A^\bullet, G(J^\bullet))$ shows that the complex $F(A^\bullet)$ is coacyclic in B whenever a complex $A^\bullet$ is coacyclic in A. ☐

A.2. **Locality of coacyclity on schemes.** Let $\mathfrak{X}$ be an reasonable ind-scheme. Then the category $\mathfrak{X}$–tors of quasi-coherent torsion sheaves on $\mathfrak{X}$ is a Grothendieck abelian category (by Theorem 2.4). So it makes sense to speak about Becker coacyclic complexes in $\mathfrak{X}$–tors and the Becker coderived category $\mathsf{D}^{\mathsf{bco}}(\mathfrak{X}\text{–}\mathsf{tors})$.

Lemma A.10. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat morphism of reasonable ind-schemes. Then the inverse image functor $f^*\colon \mathfrak{X}\text{–}\mathsf{tors} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ takes Becker coacyclic complexes in $\mathfrak{X}$–tors to Becker coacyclic complexes in $\mathfrak{Y}$–tors.*

Proof. The functor f^* is exact by Lemma 7.3 and has a right adjoint by Lemma 2.10(b), so it remains to apply Lemma A.8. ☐

Lemma A.11. *Let $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be an affine morphism of reasonable ind-schemes. Then the direct image functor $f_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ takes Becker coacyclic complexes in $\mathfrak{Y}$–tors to Becker coacyclic complexes in $\mathfrak{X}$–tors.*

Proof. The functor f_* is exact by Lemma 7.2 and preserves coproducts by Lemma 7.1(a), so Lemma A.8 is applicable. ☐

The ideas of the formulations and proofs of the next two lemmas can be found in [12, Remark 1.3] (where a more complicated setting of quasi-coherent curved DG-modules is considered).

Lemma A.12. *Let $X = \bigcup_\alpha U_\alpha$ be an open covering of a Noetherian scheme X. Let $j_\alpha\colon U_\alpha \longrightarrow X$ denote the open immersion morphisms. Then a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on X is Becker coacyclic if and only if, for every α, the complex of quasi-coherent sheaves $j_\alpha^*\mathcal{M}^\bullet$ on U_α is Becker coacyclic.*

Proof. In fact, by Proposition A.6, there is no difference between the Becker coacyclicity and coacyclicity in the sense of Section 4.4 in the assumptions of this lemma. The functors j_α^* preserve coacyclicity by Lemma A.10; so the "only if" assertion is clear.

To prove the "if", one says that either by Proposition 4.15(b) or by Theorem A.7 there exists a complex of injective quasi-coherent sheaves $\mathcal{J}^\bullet$ on X together with a morphism of complexes $\mathcal{M}^\bullet \longrightarrow \mathcal{J}^\bullet$ with a coacyclic cone. By Lemma A.10, the cones of the morphisms $j_\alpha^*\mathcal{M}^\bullet \longrightarrow j_\alpha^*\mathcal{J}^\bullet$ are coacyclic as well. If the complexes $j_\alpha^*\mathcal{M}^\bullet$ are coacyclic, then it follows that so are the complexes $j_\alpha^*\mathcal{J}^\bullet$.

On a Noetherian scheme, injectivity of quasi-coherent sheaves is a local property; so $j_\alpha^*\mathcal{J}^\bullet$ is a complex of injective quasi-coherent sheaves on U_α. Any coacyclic complex of injectives is contractible; so $j_\alpha^*\mathcal{J}^\bullet$ is a contractible complex. Finally, a complex of injective objects is contractible if and only if it is acyclic and its objects of cocycles are injective. As injectivity of sheaves and acyclicity of complexes are local properties on X, we can conclude that the complex $\mathcal{J}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh}_{\mathsf{inj}})$ is contractible. It follows that the complex $\mathcal{M}^\bullet \in \mathsf{C}(X\text{–}\mathsf{qcoh})$ is coacyclic. ☐

Proposition A.13. *Let $X = \bigcup_\alpha U_\alpha$ be an open covering of a quasi-compact semi-separated scheme X. Let $j_\alpha\colon U_\alpha \longrightarrow X$ denote the open immersion morphisms. Then a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on X is Becker coacyclic if and only if, for every α, the complex of quasi-coherent sheaves $j_\alpha^*\mathcal{M}^\bullet$ on U_α is Becker coacyclic.*

Proof. Similarly to the previous lemma, the functors of restriction to open subschemes preserve coacyclicity by Lemma A.10. So the "only if" assertion is obvious. Moreover, refining the covering if necessary and using the quasi-compactness, we can assume that $X = \bigcup_\alpha U_\alpha$ is a finite affine open covering of X.

Choose a linear order on the set of indices α, and for any subset of indices $\alpha_1 < \cdots < \alpha_k$ denote by $j_{\alpha_1,\dots,\alpha_k}\colon \bigcap_{s=1}^k U_{\alpha_k} \longrightarrow X$ the open immersion of the intersection of the open subschemes $U_1, \dots, U_k$ in X. Since the scheme X is semi-separated, the open subscheme $\bigcap_{s=1}^k U_{\alpha_k} \subset X$ is affine for all $k > 0$, and the open immersion of any affine open subscheme into X is an affine morphism of schemes; so the morphism $j_{\alpha_1,\dots,\alpha_k}$ is affine for all $k \geq 0$.

For any quasi-coherent sheaf $\mathcal{N}$ on X, the Čech complex

$$0 \longrightarrow \mathcal{N} \longrightarrow \bigoplus\nolimits_\alpha j_{\alpha*}j_\alpha^*\mathcal{N} \longrightarrow \bigoplus\nolimits_{\alpha<\beta} j_{\alpha,\beta\,*}\, j_{\alpha,\beta}^*\mathcal{N} \longrightarrow \cdots \longrightarrow 0$$

is an acyclic finite complex of quasi-coherent sheaves on X (to show the acyclicity, one observes that the restriction of the Čech complex to each of the open subschemes $U_\alpha \subset X$ is contractible). Therefore, given a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on X, we have an acyclic finite complex of complexes

$$(57)\quad 0 \longrightarrow \mathcal{M}^\bullet \longrightarrow \bigoplus\nolimits_\alpha j_{\alpha*}j_\alpha^*\mathcal{M}^\bullet \longrightarrow \bigoplus\nolimits_{\alpha<\beta} j_{\alpha,\beta\,*}\, j_{\alpha,\beta}^*\mathcal{M}^\bullet \longrightarrow \cdots \longrightarrow 0.$$

Now, if the complex $j_\alpha^*\mathcal{M}^\bullet$ is coacyclic for every α, then the complex $j_{\alpha_1,\dots,\alpha_k}^*\mathcal{M}^\bullet$ is coacyclic for all $\alpha_1 < \cdots < \alpha_k$ with $k > 0$ (as the restriction to an open subscheme preserves coacyclicity). Then, by Lemma A.11, the complex $j_{\alpha_1,\dots,\alpha_k *}\, j_{\alpha_1,\dots,\alpha_k}^*\mathcal{M}^\bullet$ is a coacyclic complex of quasi-coherent sheaves on X.

Finally, the total complex of the bicomplex (57) is coacyclic by Lemma A.4(a). Since the Becker coacyclic complexes form a full triangulated subcategory in $\mathsf{K}(X\text{–}\mathsf{qcoh})$, we can conclude that the complex $\mathcal{M}^\bullet$ is Becker coacyclic. ☐

Remark A.14. All the results of this Section A.2 are also valid for the coacyclicity in the sense of Section 4.4 in lieu of the coacyclicity in the sense of Becker (cf. Remark A.9). The advantage of Becker's coderived categories for the purposes of the present appendix is that Theorem A.7 is available for them (specifically, in application to the categories $\mathfrak{Y}$–tors for non-ind-Noetherian ind-schemes $\mathfrak{Y}$).

A.3. **The semiderived category for a nonaffine morphism of schemes.** In this section, unlike in the rest of the paper, we do *not* automatically assume all the schemes to be concentrated (i. e., quasi-compact and semi-separated). We start with a series of lemmas before proceeding to the key definition.

Lemma A.15. *Let* A *be an abelian category with countable coproducts and enough injective objects, and let* $M_0^\bullet \longrightarrow M_1^\bullet \longrightarrow M_2^\bullet \longrightarrow \cdots$ *be an inductive system of complexes in* A, *indexed by the poset of nonnegative integers. Assume that the complex* $M_n^\bullet$ *is Becker coacyclic in* A *for every* $n \geq 0$. *Then the complex* $\varinjlim_{n\geq0} M_n^\bullet$ *is Becker coacyclic in* A *as well.*

Proof. The complex $\coprod_{n\ge0} M_n^\bullet$ is Becker coacyclic in A by Lemma A.4(b). The total complex of the short exact sequence of complexes $0 \longrightarrow \coprod_{n\ge0} M_n^\bullet \longrightarrow \coprod_{n\ge0} M_n^\bullet \longrightarrow \varinjlim_{n\ge0} M_n^\bullet$ is Becker coacyclic by Lemma A.4(a). Since the Becker coacyclic complexes form a full triangulated subcategory in $\mathsf{K}(\mathsf{A})$, it follows that the complex $\varinjlim_{n\ge0} M_n^\bullet$ is also Becker coacyclic. $\square$

Let $X = \operatorname{Spec} R$ be an affine scheme. The *principal affine open subschemes* in X are the open subschemes $\operatorname{Spec} R[r^{-1}] \subset R$ corresponding to the morphisms $R \longrightarrow R[r^{-1}]$ of localization by multiplicative subsets generated by a single element in R. The principal affine open subschemes form a base of neighborhoods of zero in $\operatorname{Spec} R$, and the intersection of any two principal affine open subschemes is a principal affine open subscheme, $\operatorname{Spec} R[r_1^{-1}] \times_{\operatorname{Spec} R} \operatorname{Spec} R[r_2^{-1}] = \operatorname{Spec} R[(r_1r_2)^{-1}]$. Furthermore, for any morphism of affine schemes $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$, the full preimage of any principal affine open subscheme in $\operatorname{Spec} R$ is a principal affine open subscheme in $\operatorname{Spec} S$.

Lemma A.16. *Let $R \longrightarrow S$ be a morphism of commutative rings and $N^\bullet$ be a complex of S-modules. Let $s \in S$ be an element. Assume that $N^\bullet$ is Becker coacyclic as a complex of R-modules. Then $S[s^{-1}] \otimes_S N^\bullet$ is also Becker coacyclic as a complex of R-modules.*

Proof. The complex of R-modules $S[s^{-1}] \otimes_S N^\bullet$ can be described as the direct limit of the sequence of morphisms of complexes of R-modules $N^\bullet \xrightarrow{s} N^\bullet \xrightarrow{s} N^\bullet \longrightarrow \cdots$, so Lemma A.15 is applicable. $\square$

Lemma A.17. *Let $\mathbf{Y} = \bigcup_\alpha \mathbf{V}_\alpha$ be an affine scheme covered by principal affine open subschemes. Denote by $j_\alpha\colon \mathbf{V}_\alpha \longrightarrow \mathbf{Y}$ the open immersion morphisms. Let $f\colon \mathbf{Y} \longrightarrow X$ be a morphism of affine schemes, and let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$. Then the complex $f_*\boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent sheaves on X is Becker coacyclic if and only if, for every α, the complex $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent sheaves on X is Becker coacyclic.*

Proof. If the complex $f_*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in $X\text{–}\mathsf{qcoh}$, then Lemma A.16 tells that the complex $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in $X\text{–}\mathsf{qcoh}$ for every α.

Conversely, using quasi-compactness of the affine scheme $\mathbf{Y}$, we can assume that the set of indices α is finite. Choose a linear order on these indices, and for any subset of indices $\alpha_1 < \cdots < \alpha_k$ denote by $j_{\alpha_1,\dots,\alpha_k}\colon \bigcap_{s=1}^k \mathbf{V}_{\alpha_k} \longrightarrow \mathbf{Y}$ the open immersion. Assume that the complex $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in $X\text{–}\mathsf{qcoh}$ for every α. Then Lemma A.16 tells that the complex $f_*j_{\alpha_1,\dots,\alpha_k*}\,j_{\alpha_1,\dots,\alpha_k}^*\boldsymbol{\mathcal{N}}^\bullet$ is coacyclic in $X\text{–}\mathsf{qcoh}$ for all $\alpha_1 < \ldots < \alpha_k$ and $k > 0$ (since $\bigcap_{s=1}^k \mathbf{V}_{\alpha_k}$ is a principal affine open subscheme in $\mathbf{V}_{\alpha_1}$).

Consider the Čech complex of complexes (57) for the complex of quasi-coherent sheaves $\boldsymbol{\mathcal{N}}^\bullet$ on the affine scheme $\mathbf{Y}$ with its open covering by the affine open subschemes $\mathbf{V}_\alpha \subset \mathbf{Y}$:

$$(58)\quad 0 \longrightarrow \boldsymbol{\mathcal{N}}^\bullet \longrightarrow \bigoplus\nolimits_\alpha j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \bigoplus\nolimits_{\alpha<\beta} j_{\alpha,\beta\,*}\,j_{\alpha,\beta}^*\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \cdots \longrightarrow 0.$$

The direct image functor f_* for the morphism of affine schemes $f\colon \mathbf{Y} \longrightarrow X$ is exact, so it takes the exact complex of complexes (58) in $\mathbf{Y}$–qcoh to an exact complex of complexes in X–qcoh:

$$(59) \qquad 0 \longrightarrow f_*\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \bigoplus\nolimits_\alpha f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \bigoplus\nolimits_{\alpha<\beta} f_*j_{\alpha,\beta\,*}\,j_{\alpha,\beta}^*\boldsymbol{\mathcal{N}}^\bullet \longrightarrow \cdots \longrightarrow 0.$$

Finally, the total complex of the bicomplex (59) is Becker coacyclic in X–qcoh by Lemma A.4(a), and we have seen that all the terms of this complex of complexes except possibly the leftmost one are Becker coacyclic. It follows that the leftmost term $f_*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in X–qcoh, too. □

Lemma A.18. *Let $\mathbf{Y} = \bigcup_\alpha \mathbf{V}_\alpha$ be an affine scheme covered by (not necessarily principal) affine open subschemes. Denote by $j_\alpha\colon \mathbf{V}_\alpha \longrightarrow \mathbf{Y}$ the open immersion morphisms. Let $f\colon \mathbf{Y} \longrightarrow X$ be a morphism of affine schemes, and let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$. Then the complex $f_*\boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent sheaves on X is Becker coacyclic if and only if, for every α, the complex $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent sheaves on X is Becker coacyclic.*

Proof. As the principal affine open subschemes form a base of the topology of $\mathbf{Y}$, every affine open subscheme $\mathbf{V}_\alpha \subset \mathbf{Y}$ in our covering can be represented as a (finite) union $\mathbf{V}_\alpha = \bigcup_\theta \mathbf{W}_{\alpha,\theta}$ of some principal affine open subschemes $\mathbf{W}_{\alpha,\theta} \subset \mathbf{Y}$. If $\mathbf{W}_{\alpha,\theta} \subset \mathbf{V}_\alpha \subset \mathbf{Y}$ are affine open subschemes in an affine scheme and $\mathbf{W}_{\alpha,\theta}$ is a principal affine open subscheme in $\mathbf{Y}$, then $\mathbf{W}_{\alpha,\theta}$ is also a principal affine open subscheme in $\mathbf{V}_\alpha$.

Denote by $k_{\alpha,\theta}\colon \mathbf{W}_{\alpha,\theta} \longrightarrow \mathbf{V}_\alpha$ the open immersion morphisms, and put $l_{\alpha,\theta} = j_\alpha k_{\alpha,\theta}\colon \mathbf{W}_{\alpha,\theta} \longrightarrow \mathbf{Y}$. Now if the complex $f_*\boldsymbol{\mathcal{N}}$ is Becker coacyclic in X–qcoh, then the complexes $f_*l_{\alpha,\theta\,*}\,l_{\alpha,\theta}^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh by Lemma A.17 (as $\mathbf{W}_{\alpha,\theta}$ are principal affine open subschemes in $\mathbf{Y}$). Since $\mathbf{V}_\alpha = \bigcup_\theta \mathbf{W}_{\alpha,\theta}$ is a covering of an affine scheme by its principle affine open subschemes, Lemma A.17 implies that the complex $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in X–qcoh.

Conversely, if the complexes $f_*j_{\alpha*}j_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh, then so are the complexes $f_*l_{\alpha,\theta\,*}\,l_{\alpha,\theta}^*\boldsymbol{\mathcal{N}}^\bullet$ (by the same Lemma A.17, as $\mathbf{W}_{\alpha,\theta}$ are principal affine open subschemes in $\mathbf{V}_\alpha$). Since $\mathbf{Y} = \bigcup_{\alpha,\theta} \mathbf{W}_{\alpha,\theta}$ is a covering of an affine scheme by its principal affine open subschemes, yet another application of Lemma A.17 allows to conclude that the complex $f_*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in X–qcoh. □

Lemma A.19. *Let $\mathbf{Y}$ be a scheme, and let $\bigcup_\beta \mathbf{V}_\beta = \mathbf{Y} = \bigcup_\gamma \mathbf{W}_\gamma$ be two affine open coverings of $\mathbf{Y}$. Denote by $j_\beta\colon \mathbf{V}_\beta \longrightarrow \mathbf{Y}$ and $k_\gamma\colon \mathbf{W}_\gamma \longrightarrow \mathbf{Y}$ the open immersion morphisms. Let X be an affine scheme, $f\colon \mathbf{Y} \longrightarrow X$ be a morphism of schemes, and let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$. Then the complexes of quasi-coherent sheaves $f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{N}}^\bullet$ on X are Becker coacyclic for all β if and only if the complexes of quasi-coherent sheaves $f_*k_{\gamma*}k_\gamma^*\boldsymbol{\mathcal{N}}^\bullet$ on X are Becker coacycl for all γ.*

Proof. For every pair of indices β and γ choose an affine open covering $\mathbf{V}_\beta \cap \mathbf{W}_\gamma = \bigcup_\theta \mathbf{U}_{\beta,\gamma,\theta}$ of the open subscheme $\mathbf{V}_\beta \cap \mathbf{W}_\gamma \subset \mathbf{Y}$. Then $X = \bigcup_{\beta,\gamma,\theta} \mathbf{U}_{\beta,\gamma,\theta}$ is an affine open covering of the scheme X, for every β, γ, θ one has $\mathbf{U}_{\beta,\gamma,\theta} \subset \mathbf{V}_\beta$ and $\mathbf{U}_{\beta,\gamma,\theta} \subset \mathbf{W}_\gamma$, for every β one has $\mathbf{V}_\beta = \bigcup_{\gamma,\theta} \mathbf{U}_{\beta,\gamma,\theta}$, and for every γ one has $\mathbf{W}_\gamma = \bigcup_{\beta,\theta} \mathbf{U}_{\beta,\gamma,\theta}$.

Denote by $l_{\beta,\gamma,\theta}\colon \mathbf{U}_{\beta,\gamma,\theta} \longrightarrow \mathbf{Y}$ the open immersion morphisms. Assume that the complexes $f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh for all β. Then, by Lemma A.18 applied to the affine open subscheme $\mathbf{U}_{\beta,\gamma,\theta}$ in the affine scheme $\mathbf{V}_\beta$, the complexes $f_*l_{\beta,\gamma,\theta\,*}\,l^*_{\beta,\gamma,\theta}\boldsymbol{\mathcal{N}}^\bullet$ are coacyclic in X–qcoh for all β, γ, θ. By the same Lemma A.18 applied to the affine open covering $\mathbf{W}_\gamma = \bigcup_{\beta,\theta}\mathbf{U}_{\beta,\gamma,\theta}$ of the affine scheme $\mathbf{W}_\gamma$, it follows that the complexes $f_*k_{\gamma*}k_\gamma^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh for all γ. $\square$

Let X be a semi-separated scheme. Then, for any affine scheme U, any morphism of schemes $U \longrightarrow X$ is affine. For any two affine schemes U and $\mathbf{V}$ and any morphisms of schemes $U \longrightarrow X$ and $\mathbf{V} \longrightarrow X$, the scheme $U \times_X \mathbf{V}$ is affine.

Lemma A.20. *Let $f\colon \mathbf{Y} \longrightarrow X$ be an affine morphism of quasi-compact semi-separated schemes. Let $X = \bigcup_\alpha U_\alpha$ be an affine open covering of X. Put $\mathbf{V}_\alpha = U_\alpha \times_X \mathbf{Y}$; then $\mathbf{Y} = \bigcup_\alpha \mathbf{V}_\alpha$ is an affine open covering of $\mathbf{Y}$. Consider the pullback diagram*

$$\begin{array}{ccc} \mathbf{V}_\alpha & \xrightarrow{k_\alpha} & \mathbf{Y} \\ {\scriptstyle f_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ U_\alpha & \xrightarrow{j_\alpha} & X \end{array}$$

Let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$. Then the following conditions are equivalent:

(a) *the complex $f_*\boldsymbol{\mathcal{N}}^\bullet$ is Becker coacyclic in X–qcoh;*
(b) *the complexes $f_*k_{\alpha*}k_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh for all α;*
(c) *the complexes $f_{\alpha*}k_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in U_α–qcoh for all α.*

Proof. (a)$\iff$(c) In view of the natural isomorphism $j_\alpha^*f_*\boldsymbol{\mathcal{N}}^\bullet \simeq f_{\alpha*}k_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$ of complexes of quasi-coherent sheaves on U_α, the assertion follows from Proposition A.13.

(b)$\iff$(c) We have $f_*k_{\alpha*}k_\alpha^*\boldsymbol{\mathcal{N}}^\bullet \simeq j_{\alpha*}f_{\alpha*}k_\alpha^*\boldsymbol{\mathcal{N}}^\bullet$, since $fk_\alpha = j_\alpha f_\alpha$. It remains to observe that, for any affine open immersion $j\colon U \longrightarrow X$, a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ on U is Becker coacyclic if and only if the complex of quasi-coherent sheaves $j_*\mathcal{M}^\bullet$ on X is Becker coacyclic. This follows from Lemmas A.10 and A.11. $\square$

Proposition A.21. *Let X be a quasi-compact semi-separated scheme, and let $f\colon \mathbf{Y} \longrightarrow X$ be a morphism of schemes. Let $X = \bigcup_\alpha U_\alpha$ be an affine open covering of X, and let $\bigcup_\beta \mathbf{V}_\beta = \mathbf{Y} = \bigcup_\gamma \mathbf{W}_\gamma$ be two open coverings of $\mathbf{Y}$ such that the compositions $\mathbf{V}_\beta \longrightarrow \mathbf{Y} \longrightarrow X$ and $\mathbf{W}_\gamma \longrightarrow \mathbf{Y} \longrightarrow X$ are affine morphisms.*

Denote by $j_\alpha\colon U_\alpha \longrightarrow X$, $k_\beta\colon \mathbf{V}_\beta \longrightarrow \mathbf{Y}$, and $l_\gamma\colon \mathbf{W}_\gamma \longrightarrow \mathbf{Y}$ the open immersion morphisms. Furthermore, put $\mathbf{S}_{\alpha,\beta} = U_\alpha \times_X \mathbf{V}_\beta$ and $\mathbf{T}_{\alpha,\gamma} = U_\alpha \times_X \mathbf{W}_\gamma$, and denote by $g_{\alpha,\beta}\colon \mathbf{S}_{\alpha,\beta} \longrightarrow \mathbf{Y}$ and $h_{\alpha,\gamma}\colon \mathbf{T}_{\alpha,\gamma} \longrightarrow \mathbf{Y}$ the natural open immersions, and by $f'_{\alpha,\beta}\colon \mathbf{S}_{\alpha,\beta} \longrightarrow U_\alpha$ and $f''_{\alpha,\gamma}\colon \mathbf{T}_{\alpha,\gamma} \longrightarrow U_\alpha$ the natural morphisms of affine schemes.

Let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$. Then the following conditions are equivalent:

(a) *the complexes $f_*k_{\beta*}k_\beta^*\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh for all β;*
(b) *the complexes $f_*g_{\alpha,\beta\,*}\,g^*_{\alpha,\beta}\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in X–qcoh for all α and β;*
(c) *the complexes $f'_{\alpha,\beta\,*}\,g^*_{\alpha,\beta}\boldsymbol{\mathcal{N}}^\bullet$ are Becker coacyclic in U_α–qcoh for all α and β;*

(d) *the complexes* $f''_{\alpha,\gamma\,*}\, h^*_{\alpha,\gamma}\boldsymbol{\mathcal{N}}^\bullet$ *are Becker coacyclic in* U_α–qcoh *for all* α *and* γ*;*
(e) *the complexes* $f_*h_{\alpha,\gamma\,*}\, h^*_{\alpha,\gamma}\boldsymbol{\mathcal{N}}^\bullet$ *are Becker coacyclic in* X–qcoh *for all* α *and* γ*;*
(f) *the complexes* $f_*l_{\gamma*}l^*_\gamma\boldsymbol{\mathcal{N}}^\bullet$ *are Becker coacyclic in* X–qcoh *for all* γ*.*

Proof. Conditions (a–c) are equivalent to each other by Lemma A.20 applied to the affine morphism of schemes $fk_\beta\colon \mathbf{V}_\beta \longrightarrow X$ and the complex of quasi-coherent sheaves $k^*_\beta\boldsymbol{\mathcal{N}}^\bullet$ on $\mathbf{V}_\beta$. Similarly, conditions (d–f) are equivalent to each other by Lemma A.20 applied to the affine morphism of schemes $fl_\gamma\colon \mathbf{W}_\gamma \longrightarrow X$ and the complex of quasi-coherent sheaves $l^*_\gamma\boldsymbol{\mathcal{N}}^\bullet$ on $\mathbf{W}_\gamma$. Finally, the equivalence (c) $\Longleftrightarrow$ (d) is provided by Lemma A.19 applied to the morphism of schemes $U_\alpha\times_X\mathbf{Y}\longrightarrow U_\alpha$ into the affine scheme U_α, the restriction of the complex of quasi-coherent sheaves $\boldsymbol{\mathcal{N}}^\bullet$ on $\mathbf{Y}$ to the open subscheme $U_\alpha\times_X\mathbf{Y}\subset\mathbf{Y}$, and the two affine coverings $\bigcup_\beta \mathbf{S}_{\alpha,\beta} = U_\alpha\times_X\mathbf{Y} = \bigcup_\gamma \mathbf{T}_{\alpha,\gamma}$ of the scheme $U_\alpha\times_X\mathbf{Y}$. ☐

Now we can formulate the promised definition. Let $f\colon \mathbf{Y}\longrightarrow X$ be a morphism of schemes; assume that the scheme X is quasi-compact and semi-separated. Let $\boldsymbol{\mathcal{N}}^\bullet$ be a complex of quasi-coherent sheaves on $\mathbf{Y}$.

Choose a covering of the scheme $\mathbf{Y}$ by open subschemes $\mathbf{V}_\beta\subset\mathbf{Y}$ such that the compositions $\mathbf{V}_\beta\longrightarrow\mathbf{Y}\longrightarrow X$ are affine morphisms of schemes (for example, any affine open covering of $\mathbf{Y}$ satisfies this condition). Denote by $k_\beta\colon\mathbf{V}_\beta\longrightarrow\mathbf{Y}$ the open immersion morphisms.

We will say that the complex $\boldsymbol{\mathcal{N}}^\bullet$ is *semiacyclic* (or more precisely $\mathbf{Y}/X$*-semiacyclic*) if, for every index β, the complex $f_*k_{\beta*}k^*_\beta\boldsymbol{\mathcal{N}}^\bullet$ of quasi-coherent sheaves on X (that is, the direct image to X of the restriction of $\boldsymbol{\mathcal{N}}^\bullet$ to $\mathbf{V}_\beta$) is Becker coacyclic in X–qcoh. According to Proposition A.21 (a) $\Leftrightarrow$ (f), this property does not depend on the choice of an open covering $\mathbf{Y}=\bigcup_\beta\mathbf{V}_\beta$.

Remark A.22. The reader should be *warned* that our terminology is misleading. The semiacyclicity of a complex in $\mathbf{Y}$–qcoh is by design an intermediate property between the acyclicity and the Becker coacyclicity. Any Becker coacyclic complex in $\mathbf{Y}$–qcoh is $\mathbf{Y}/X$-semiacyclic (by Lemmas A.10 and A.11).

On the other hand, any $\mathbf{Y}/X$-semiacyclic complex $\boldsymbol{\mathcal{N}}^\bullet$ in $\mathbf{Y}$–qcoh is acyclic. Iindeed, it suffices to check that the restriction of $\boldsymbol{\mathcal{N}}^\bullet$ to $\mathbf{V}_\beta$ is acyclic for every β. Notice that any Becker coacyclic complex in X–qcoh is acyclic by Lemma A.5. Now acyclicity of the complex $f_*k_{\beta*}k^*_\beta\boldsymbol{\mathcal{N}}^\bullet$ in X–qcoh implies acyclicity of the complex $k^*_\beta\boldsymbol{\mathcal{N}}^\bullet$ in $\mathbf{V}_\beta$–qcoh, since the direct image functor $f_*k_{\beta*}=(fk_\beta)_*\colon\mathbf{V}_\beta$–$\mathsf{qcoh}\longrightarrow X$–$\mathsf{qcoh}$ for an affine morphism of schemes $fk_\beta\colon\mathbf{V}_\beta\longrightarrow X$ is exact and faithful.

So the semiacyclicity is a *stronger* property than the acyclicity.

Remark A.23. Similarly to Section A.2 (cf. Remark A.14) all the results of this Section A.3 are equally valid for the coacyclicity in the sense of Section 4.4 in lieu of coacyclicity in the sense of Becker. Moreover, when the scheme X is Noetherian, the two coacyclicity notions involved are equivalent to each other by Proposition A.6.

Finally, we can define the *semiderived category* (or the $\mathbf{Y}/X$*-semiderived category*) $\mathsf{D}^{\mathsf{si}}_X(\mathbf{Y}$–$\mathsf{qcoh})$ of quasi-coherent sheaves on $\mathbf{Y}$ as the triangulated quotient category of the homotopy category $\mathsf{K}(\mathbf{Y}\text{–}\mathsf{qcoh})$ by the thick subcategory of $\mathbf{Y}/X$-semiacyclic complexes. In view of the previous remark, this definition agrees with the definition in Section 7.1 *assuming that X is an semi-separated Noetherian scheme.*

Indeed, the definition in Section 7.1 (specialized from ind-schemes to schemes) presumes the morphism $f\colon \mathbf{Y} \longrightarrow X$ to be affine. In this case, one can use the open covering of $\mathbf{Y}$ consisting of a single open subscheme $\mathbf{V} = \mathbf{Y}$ for the purposes of the definition above in this section. Then it is clear that the two definitions are the same.

A.4. **Direct images of restrictions of injective sheaves.** The aim of this section is to prove the following technical lemma.

Lemma A.24. *Let X be a Noetherian scheme, Y be a quasi-compact semi-separated scheme, and $f\colon Y \longrightarrow X$ be a flat morphism of schemes. Let $V \subset Y$ be an open subscheme with the open immersion morphism $k\colon V \longrightarrow Y$. Assume that the composition $fk\colon V \longrightarrow X$ is an affine morphism of schemes. Let $\mathcal{J}$ be an injective quasi-coherent sheaf on Y. Then the quasi-coherent sheaf $f_*k_*k^*\mathcal{J}$ on X is injective.*

We will deduce Lemma A.24 from the next proposition.

Proposition A.25. *Let X be a Noetherian scheme, Y be a quasi-compact semi-separated scheme, and $f\colon Y \longrightarrow X$ be a flat morphism of schemes. Let $\mathcal{J}$ be an injective quasi-coherent sheaf on Y and $\mathcal{F}$ be a flat quasi-coherent sheaf on Y. Then the quasi-coherent sheaf $f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{J})$ on X is injective.*

The following particular cases of Proposition A.25 are easy. If $\mathcal{F} = \mathcal{O}_Y$, then the assertion of the proposition holds because the direct image functor f_* for a flat morphism of schemes $f\colon Y \longrightarrow X$ preserves injectivity of quasi-coherent sheaves. In this case, there is no need to assume that the scheme X is Noetherian. If $X = Y$ and $f = \mathrm{id}$ is the identity morphism, then the result reduces to Lemma 4.4(b) (for which the Noetherianity assumption is essential).

Lemma A.26. *Let $f\colon Y \longrightarrow X$ a flat morphism of affine schemes, where the affine scheme X in Noetherian. Let $\mathcal{J}$ be an injective quasi-coherent sheaf on Y and $\mathcal{F}$ be a flat quasi-coherent sheaf on Y. Then the quasi-coherent sheaf $f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{J})$ on X is injective.*

Proof. In algebraic language, the assertion means the following. Let $R \longrightarrow S$ be a homomorphism of commutative rings such that S is a flat R-module. Assume that the ring R is Noetherian. Let J be an injective S-module and F be a flat S-module. Then the R-module $F \otimes_S J$ is injective.

Indeed, it suffices to observe that F is a (filtered) direct limit of finitely generated free S-modules, J is an injective R-module (since S is a flat R-module and J is an injective S-module), and the class of all injective R-modules is closed under direct limits in $R\text{–}\mathsf{mod}$ (since R is a Noetherian ring). ☐

Lemma A.27. *Let X be a Noetherian scheme, Y be an affine scheme, and $f\colon Y \longrightarrow X$ be a flat morphism of schemes. Assume that there exists an affine open subscheme $U \subset X$ such that the morphism f factorizes as $Y \longrightarrow U \longrightarrow X$. Let $\mathcal{J}$ be an* *injective quasi-coherent sheaf on Y and $\mathcal{F}$ be a flat quasi-coherent sheaf on Y. Then the quasi-coherent sheaf $f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{J})$ on X is injective.*

Proof. Denote the morphisms involved by $g\colon Y \longrightarrow U$ and $h\colon U \longrightarrow X$. Applying Lemma A.26 to the flat morphism of affine schemes $g\colon Y \longrightarrow U$ with a Noetherian scheme U, we see that the quasi-coherent sheaf $g_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{J})$ on U is injective. It remains to say that the direct image functor $h_*\colon U\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ preserves injectivity, as an open immersion h is a flat morphism. □

Proof of Proposition A.25. Let $Y = \bigcup_\alpha W_\alpha$ be a finite affine open covering of the scheme Y. Denote by $l_\alpha\colon W_\alpha \longrightarrow Y$ the open immersions.

The key observation is that any injective quasi-coherent sheaf $\mathcal{J}$ on Y is a direct summand of a direct sum $\bigoplus_\alpha l_{\alpha *}\mathcal{K}_\alpha$, where $\mathcal{K}_\alpha$ are some injective quasi-coherent sheaves on W_α. Indeed, one easily observes that there are enough injective quasi-coherent sheaves of this particular form, that is, any quasi-coherent sheaf $\mathcal{M}$ on Y is a subobject of a quasi-coherent sheaf of the form $\bigoplus_\alpha l_{\alpha *}\mathcal{K}_\alpha$, where $\mathcal{K}_\alpha \in W_\alpha\text{–}\mathsf{qcoh}_{\mathsf{inj}}$. (It suffices to choose injective quasi-coherent sheaves $\mathcal{K}_\alpha$ in such a way that $l_\alpha^*\mathcal{M}$ is a subobject of $\mathcal{K}_\alpha$ for every α.)

As we are free to choose our finite affine open covering $Y = \bigcup_\alpha W_\alpha$ of the scheme Y, we can make the affine open subschemes $W_\alpha \subset Y$ as small as we wish. Specifically, we can assume that for every α there exists an affine open subscheme $U_\alpha \subset X$ such that $f(W_\alpha) \subset U_\alpha$.

Hence the question reduces to the following. We can assume that $\mathcal{J} = l_*\mathcal{K}$, where $l_*\colon W \longrightarrow Y$ is the immersion of an affine open subscheme and $\mathcal{K} \in W\text{–}\mathsf{qcoh}_{\mathsf{inj}}$. Moreover, we can have $f(W) \subset U$ for some affine open subscheme $U \subset X$. In this context, we have to prove that, for any flat quasi-coherent sheaf $\mathcal{F}$ on Y, the quasi-coherent sheaf $f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} l_*\mathcal{K})$ on X is injective.

By Lemma 2.2, we have $\mathcal{F}\otimes_{\mathcal{O}_Y} l_*\mathcal{K} \simeq l_*(l^*\mathcal{F}\otimes_{\mathcal{O}_W}\mathcal{K})$ in $Y\text{–}\mathsf{qcoh}$ (as the morphism l is affine, since the scheme Y is semi-separated by assumption). Hence $f_*(\mathcal{F}\otimes_{\mathcal{O}_Y} l_*\mathcal{K}) \simeq f_*l_*(l^*\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{K})$ in $X\text{–}\mathsf{qcoh}$.

It remains to apply Lemma A.27 to the flat morphism of schemes $fl\colon W \longrightarrow X$, the injective quasi-coherent sheaf $\mathcal{K}$ on W, and the flat quasi-coherent sheaf $l^*\mathcal{F}$ on W. Here X is a Noetherian scheme, W is an affine scheme, and the morphism fl factorizes as $W \longrightarrow U \longrightarrow X$ for an affine open subscheme $U \subset X$. □

Proof of Lemma A.24. We have the assumption that the morphism $fk\colon V \longrightarrow X$ is affine. Let us deduce from this assumption that the morphism $k\colon V \longrightarrow Y$ is affine.

Let $W \subset Y$ be an affine open subscheme. We have to show that $W \times_Y V$ is an affine scheme. Indeed, $W \times_X V$ is an affine scheme, since W is an affine scheme and $V \longrightarrow X$ is an affine morphism (so $W \times_X V \longrightarrow W$ is an affine morphism as a base change of an affine morphism). As $W \times_Y V = Y \times_{Y\times Y} (W \times_X V)$ and the morphism $Y \longrightarrow Y \times Y = Y \times_{\operatorname{Spec}\mathbb{Z}} Y$ is affine (the scheme Y being semi-separated by assumption), it follows that $W \times_Y V$ is an affine scheme.

Finally, by Lemma 2.2 we have $k_*k^*\mathcal{J} \simeq k_*(\mathcal{O}_V \otimes_{\mathcal{O}_V} k^*\mathcal{J}) \simeq k_*\mathcal{O}_V \otimes_{\mathcal{O}_Y} \mathcal{J}$ in $Y\text{–}\mathsf{qcoh}$. Hence $f_*k_*k^*\mathcal{J} \simeq f_*(k_*\mathcal{O}_V \otimes_{\mathcal{O}_Y} \mathcal{J})$ in $X\text{–}\mathsf{qcoh}$. The quasi-coherent sheaf

$\mathcal{F} = k_*\mathcal{O}_V$ on Y is flat, as the direct image with respect to a flat affine morphism of schemes takes flat quasi-coherent sheaves to flat quasi-coherent sheaves. Thus Proposition A.25 is applicable.

Notice that we have never used the assumption that V is an open subscheme in Y. It suffices that $k\colon V \longrightarrow Y$ be a flat morphism. $\square$

A.5. **The semiderived category for a morphism of ind-schemes.** The definition of the semiderived category in this section resembles the ones in [43, Section 4.3] (where the context is very different). We start with a consistency lemma involving only schemes.

Lemma A.28. *Let $i\colon Z \longrightarrow X$ be a closed immersion of semi-separated Noetherian schemes, and let $f\colon \mathbf{Y} \longrightarrow X$ be a flat morphism of quasi-compact semi-separated schemes. Consider the pullback diagram (so $\mathbf{W} = Z \times_X \mathbf{Y}$)*

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{k} & \mathbf{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

Let $\boldsymbol{\mathcal{J}}^\bullet$ be a $\mathbf{Y}/X$-semiacyclic complex of injective quasi-coherent sheaves on $\mathbf{Y}$. Then $k^!\boldsymbol{\mathcal{J}}^\bullet$ is a $\mathbf{W}/Z$-semiacyclic complex of (injective) quasi-coherent sheaves on $\mathbf{W}$.

Proof. Let $\mathbf{Y} = \bigcup_\beta \mathbf{V}_\beta$ be an open covering of the scheme $\mathbf{Y}$ such that the compositions $\mathbf{V}_\beta \longrightarrow \mathbf{Y} \longrightarrow X$ are affine morphisms of schemes. Put $\mathbf{T}_\beta = \mathbf{V}_\beta \times_{\mathbf{Y}} \mathbf{W} = \mathbf{V}_\beta \times_X Z$. Then $\mathbf{W} = \bigcup_\beta \mathbf{T}_\beta$ is an open covering of the scheme $\mathbf{W}$ such that the compositions $\mathbf{T}_\beta \longrightarrow \mathbf{W} \longrightarrow Z$ are affine morphisms of schemes. Denote by $j_\beta\colon \mathbf{V}_\beta \longrightarrow \mathbf{Y}$ and $l_\beta\colon \mathbf{T}_\beta \longrightarrow \mathbf{W}$ the open immersion morphisms.

By the definition, the condition that $\boldsymbol{\mathcal{J}}^\bullet$ is a $\mathbf{Y}/X$-semiacyclic complex of quasi-coherent sheaves on $\mathbf{Y}$ means that the complex of quasi-coherent sheaves $f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ on X is Becker coacyclic for every β. Since $\boldsymbol{\mathcal{J}}^\bullet$ is a complex of injective quasi-coherent sheaves on $\mathbf{Y}$, Lemma A.24 implies that $f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ is a complex of injective quasi-coherent sheaves on X. Any Becker coacyclic complex of injectives is contractible. We have shown that $f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ is a contractible complex of (injective) quasi-coherent sheaves on X for every β.

Consider two pullback diagrams

$$\begin{array}{ccc} \mathbf{T}_\beta & \xrightarrow{k_\beta} & \mathbf{V}_\beta \\ {\scriptstyle l_\beta}\downarrow & & \downarrow{\scriptstyle j_\beta} \\ \mathbf{W} & \xrightarrow{k} & \mathbf{Y} \end{array} \qquad \begin{array}{ccc} \mathbf{T}_\beta & \xrightarrow{k_\beta} & \mathbf{V}_\beta \\ {\scriptstyle gl_\beta}\downarrow & & \downarrow{\scriptstyle fj_\beta} \\ Z & \xrightarrow{i} & X \end{array}$$

By Lemma 4.24 applied to the leftmost diagram (taking into account Lemma 2.1(a)), we have $l_\beta^*k^!\boldsymbol{\mathcal{J}}^\bullet \simeq k_\beta^!j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ in $\mathsf{C}(\mathbf{T}_\beta\text{–}\mathsf{qcoh})$. By Lemma 2.3(a) applied to the rightmost diagram, we have $g_*l_{\beta*}k_\beta^!j_\beta^*\boldsymbol{\mathcal{J}}^\bullet \simeq i^!f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ in $\mathsf{C}(Z\text{–}\mathsf{qcoh})$. Combining these isomorphisms together, we obtain $g_*l_{\beta*}l_\beta^*k^!\boldsymbol{\mathcal{J}}^\bullet \simeq g_*l_{\beta*}k_\beta^!j_\beta^*\boldsymbol{\mathcal{J}}^\bullet \simeq i^!f_*j_{\beta*}j_\beta^*\boldsymbol{\mathcal{J}}^\bullet$ in $\mathsf{C}(Z\text{–}\mathsf{qcoh})$.

Since $f_*j_{\beta*}j_\beta^*\mathcal{J}^\bullet$ is a contractible complex of (injective) quasi-coherent sheaves on X, the complex $i^!f_*j_{\beta*}j_\beta^*\mathcal{J}^\bullet$ is a contractible complex of (injective) quasi-coherent sheaves on Z. Thus the complex $g_*l_{\beta*}l_\beta^*k^!\mathcal{J}^\bullet$ is a contractible, hence Becker coacyclic, complex of (injective) quasi-coherent sheaves on Z. By the definition, this means that $k^!\mathcal{J}^\bullet$ is a $\mathbf{W}/Z$-semiacyclic complex of quasi-coherent sheaves on $\mathbf{W}$. ☐

Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme, $\mathfrak{Y}$ be an (ind-quasi-compact) ind-semi-separated ind-scheme, and $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat (but not necessarily affine) morphism of ind-schemes. Our aim is to define the $\mathfrak{Y}/\mathfrak{X}$-semiderived category $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathbf{Y}$.

Firstly, let $\mathscr{J}^\bullet$ be a complex of injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. We will say that $\mathscr{J}^\bullet$ is a $\mathfrak{Y}/\mathfrak{X}$-*semiacyclic complex* if, for every closed subscheme $Z \subset \mathfrak{X}$ with the closed immersion morphism $i\colon Z \longrightarrow \mathfrak{X}$ and the related closed subscheme $\mathbf{W} = Z \times_{\mathfrak{X}} \mathfrak{Y} \subset \mathfrak{Y}$ with the closed immersion morphism $k\colon \mathbf{W} \longrightarrow \mathfrak{Y}$, the complex $k^!\mathscr{J}^\bullet$ of injective quasi-coherent sheaves on $\mathbf{W}$ is $\mathbf{W}/Z$-semiacyclic in the sense of the definition in Section A.3.

As we have seen in the proof of Lemma A.28, this means that, for every open subscheme $\mathbf{V} \subset \mathbf{W}$ such that the composition $\mathbf{V} \longrightarrow \mathbf{W} \longrightarrow Z$ is an affine morphism, the complex of (injective) quasi-coherent sheaves $\pi_{Z*}j_*j^*k^!\mathscr{J}^\bullet$ on Z must be contractible. Here the notation for morphisms is $\pi_Z\colon \mathbf{W} \longrightarrow Z$ and $j\colon \mathbf{V} \longrightarrow \mathbf{W}$. It suffices to check this condition for open subschemes $\mathbf{V} \subset \mathbf{W}$ belonging to a chosen covering of $\mathbf{W}$ by open subschemes that are affine over Z.

Furthermore, let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be some chosen representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes. Then Lemma A.28 tells that it suffices to check the $\mathbf{W}/Z$-semiacyclicity condition above for the closed subschemes $Z = X_\gamma \subset \mathfrak{X}$, where $\gamma \in \Gamma$.

Now we want to explain what it means for a complex of not necessarily injective quasi-coherent torsion sheaves $\mathscr{N}^\bullet$ on $\mathfrak{Y}$ to be $\mathfrak{Y}/\mathfrak{X}$-semiacyclic. Here we are going to use Theorem A.7 (so it is important that we are working with the Becker coderived categories). By Theorem A.7, there exists a morphism of complexes of quasi-coherent torsion sheaves $\mathscr{N}^\bullet \longrightarrow \mathscr{J}^\bullet$ on $\mathfrak{Y}$ whose cone is Becker coacyclic in $\mathfrak{Y}\text{–}\mathsf{tors}$, while $\mathscr{J}^\bullet$ is a complex of injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. The complex $\mathscr{N}^\bullet$ in $\mathfrak{Y}\text{–}\mathsf{tors}$ is said to be $\mathfrak{Y}/\mathfrak{X}$-*semiacyclic* if the complex $\mathscr{J}^\bullet$ in $\mathfrak{Y}\text{–}\mathsf{tors}_{\mathsf{inj}}$ is $\mathfrak{Y}/\mathfrak{X}$-semiacyclic in the sense of the previous definition (which is applicable to the complexes of injectives only).

In other words, the full subcategory of $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complexes in $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ is, by the definition, the minimal thick subcategory in $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ containing *both* the Becker coacyclic complexes and the $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complexes of injective quasi-coherent torsion sheaves on $\mathfrak{Y}$.

Remark A.29. Similarly to Remark A.22, we observe that any $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complex in $\mathfrak{Y}\text{–}\mathsf{tors}$ is acyclic. Indeed, all the Becker coacyclic complexes in $\mathfrak{Y}\text{–}\mathsf{tors}$ are acyclic by Lemma A.5. Now let $\mathscr{J}^\bullet$ be a $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complex of injective quasi-coherent torsion sheaves on $\mathfrak{Y}$. Then, by Remark A.22, the ind-scheme $\mathfrak{Y}$ can be represented by an inductive system of closed immersions of schemes $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} \mathbf{Y}_\gamma$ such that, denoting by $k_\gamma\colon \mathbf{Y}_\gamma \longrightarrow \mathfrak{Y}$ the closed immersion morphisms, the complexes of quasi-coherent sheaves $k_\gamma^!\mathscr{J}^\bullet$ on $\mathbf{Y}_\gamma$ are ($\mathbf{Y}_\gamma/X_\gamma$-semiacyclic, hence) acyclic for all γ. As the direct image functors $k_{\gamma*}\colon \mathbf{Y}_\gamma\text{–}\mathsf{qcoh} \longrightarrow \mathfrak{Y}\text{–}\mathsf{tors}$ are exact, it follows that the complex $\mathscr{J}^\bullet = \varinjlim_{\gamma\in\Gamma} k_{\gamma*}k_\gamma^!\mathscr{J}^\bullet$ is acyclic in $\mathfrak{Y}\text{–}\mathsf{tors}$.

The *semiderived category* (or the $\mathfrak{Y}/\mathfrak{X}$-semiderived category) $\mathsf{D}^{\mathsf{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{–}\mathsf{tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is defined as the triangulated quotient category of the homotopy category $\mathsf{K}(\mathfrak{Y}\text{–}\mathsf{tors})$ by the thick subcategory of $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complexes. The following lemma tells that this definition agrees with the one in Section 7.1 when both are applicable.

Lemma A.30. *Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme and $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Then a complex $\mathscr{N}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ is $\mathfrak{Y}/\mathfrak{X}$-semiacyclic if and only if the complex $\pi_*\mathscr{N}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is coacyclic.*

Proof. First of all, let us mention once again that there is no difference between the two notions of coacyclicity for complexes of quasi-coherent torsion sheaves on an ind-Noetherian ind-scheme $\mathfrak{X}$ (by Proposition A.6). Furthermore, the direct image functor π_* takes Becker coacyclic complexes in $\mathfrak{Y}\text{–}\mathsf{tors}$ to coacyclic complexes in $\mathfrak{X}\text{–}\mathsf{tors}$ by Lemma A.11. Hence it suffices to consider the case of a complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{Y}\text{–}\mathsf{tors}_{\mathsf{inj}})$.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a representation of $\mathfrak{X}$ by an inductive system of closed immersions of schemes. Put $\mathbf{Y}_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$; then $\mathfrak{Y} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} \mathbf{Y}_\gamma$ is a representation of $\mathfrak{Y}$ by an inductive system of closed immersions of schemes. Denote by $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ and $k_\gamma\colon \mathbf{Y}_\gamma \longrightarrow \mathfrak{Y}$ the closed immersion morphisms, and by $\pi_\gamma\colon \mathbf{Y}_\gamma \longrightarrow X_\gamma$ the natural flat affine morphisms of schemes. By the definition of the functor $\pi_*\colon \mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow \mathfrak{X}\text{–}\mathsf{tors}$ (see Section 2.6), we have a natural isomorphism $i_\gamma^!\pi_* \simeq \pi_{\gamma*}k_\gamma^!$ of functors $\mathfrak{Y}\text{–}\mathsf{tors} \longrightarrow X_\gamma\text{–}\mathsf{qcoh}$ for every $\gamma\in\Gamma$.

Assume that the complex $\pi_*\mathscr{J}^\bullet$ is coacyclic in $\mathfrak{X}\text{–}\mathsf{tors}$. By Lemma 7.6(b), the functor π_* takes injective objects to injective objects; so $\pi_*\mathscr{J}^\bullet$ is a complex of injectives in $\mathfrak{X}\text{–}\mathsf{tors}$. Hence the complex $\pi_*\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is contractible. It follows that the complex $i_\gamma^!\pi_*\mathscr{J}^\bullet$ is a contractible complex of injective quasi-coherent sheaves on X_γ. Therefore, so is the complex $\pi_{\gamma*}k_\gamma^!\mathscr{J}^\bullet \in \mathsf{C}(X_\gamma\text{–}\mathsf{qcoh}_{\mathsf{inj}})$; in particular, the complex $\pi_{\gamma*}k_\gamma^!\mathscr{J}^\bullet$ is coacyclic. Following the discussion of $\mathbf{Y}/X$-semiacyclicity in the end of Section A.3, this means that the complex $k_\gamma^!\mathscr{J}^\bullet \in \mathsf{C}(\mathbf{Y}_\gamma\text{–}\mathsf{qcoh})$ is $\mathbf{Y}_\gamma/X_\gamma$-semiacyclic. By the definition, we can conclude that the complex $\mathscr{J}^\bullet \in \mathsf{C}(\mathfrak{Y}/\mathfrak{X}\text{–}\mathsf{tors}_{\mathsf{inj}})$ is $\mathfrak{Y}/\mathfrak{X}$-semiacyclic.

Conversely, assume that the complex $\mathscr{J}^\bullet$ is $\mathfrak{Y}/\mathfrak{X}$-semiacyclic. Then the complex $\pi_{\gamma*}k_\gamma^!\mathscr{J}^\bullet$ is a (Becker) coacyclic complex of injective objects in $X_\gamma\text{–}\mathsf{qcoh}$, so it is a contractible complex of injective objects. Hence the complex $i_\gamma^!\pi_*\mathscr{J}^\bullet$ is contractible in $X_\gamma\text{–}\mathsf{qcoh}_{\mathsf{inj}}$. We also know from the previous paragraph that $\pi_*\mathscr{J}^\bullet$ is a complex of injective objects in $\mathfrak{X}$–tors. Using Lemma 4.21, we conclude that $\pi_* \mathscr{J}^\bullet$ is a contractible (hence coacyclic) complex in $\mathfrak{X}$–tors. ☐

Remark A.31. One can think of this appendix as pointing a direction for possible generalization of the results of Sections 7–10 to nonaffine morphisms of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, but this a long way. To begin with, it is not obvious (and needs to be checked) that the full triangulated subcategory of $\mathfrak{Y}/\mathfrak{X}$-semiacyclic complexes in $\mathsf{K}(\mathfrak{Y}$–tors) (in the sense of the definition in this appendix) is closed under coproducts. The difficulty arises from the fact that the full subcategory of injective objects in $\mathfrak{Y}$–tors is not closed under coproducts. Perhaps more importantly, it is not clear how to extend the constructions of resolutions in Section 8.2 to nonaffine morphisms π, or what to replace them with in the nonaffine case. So the semi-infinite algebraic geometry of quasi-coherent torsion sheaves for nonaffine morphisms of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ remains a challenge.

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