Dear Bhargav,

This letter is inspired by your note "Torsion completions are bounded" [1]. Its actual aim is to attract your attention to my papers [2, 3] (in connection with the derived completion and MGM duality), and particularly [4] (as an application of derived complete modules).

What you call derived complete modules are generally called "contramodules" in these papers, and what you call complete modules I usually call "separated and complete".

A more immediate aim is to offer (what I think is) a purely algebraic argument proving the main assertion of [1], viz., that any derived complete torsion module is bounded torsion [1, Proposition 2.5].

The following lemma collects the basic facts which I will use.

Lemma 1. *Let A be a commutative ring and $I \subset A$ a finitely generated ideal, and let C and D be derived I-complete A-modules. Then*

(a) *the completion morphism $D \longrightarrow \varprojlim_n D/I^nD$ is surjective;*

(b) *if $D = ID$ then $D = 0$;*

(c) *the kernel and cokernel of any A-module morphism $C \longrightarrow D$ are derived I-complete.* □

The following proposition provides a reduction from the case of a derived complete module to that of a (separated and) complete one.

Proposition 2. *Let D be a derived I-complete A-module, and let $C = \varprojlim_n D/I^nD$ be its I-completion. Then*

(a) *if D is I-torsion then C is I-torsion;*

(b) *if $I^mC = 0$ for some $m \geqslant 0$, then $I^mD = 0$.*

Proof. Part (a) follows from Lemma 1(a). To prove part (b), denote by $K \subset D$ the kernel of the completion morphism $D \longrightarrow C$, and note that $K = \bigcap_{n \geqslant 0} I^nD \subset I^{m+1}D$. Hence the equality $I^mC = 0$ implies $I^mD = K = I^{m+1}D$. By Lemma 1(c), I^mD is a derived I-complete A-module; and by Lemma 1(b) it follows that $I^mD = 0$. □

As you mention in your note, it is sufficient to consider modules over the ring $A = \mathbb{Z}[[t]]$ with the ideal $I = (t)$. So for the next proposition I restrict to this particular case.

Proposition 3. *Let C be an (t-separated and) t-complete $\mathbb{Z}[[t]]$-module which is also t-torsion. Then there exists $m \geqslant 0$ such that $t^mC = 0$.*

Proof. The argument uses the t-power infinite summation operation, assigning to every sequence of elements $c_0, c_1, c_2, \ldots \in C$ the element $\sum_{n=0}^{\infty} t^n c_n \in C$. Such infinite summation operations are defined in all derived I-complete A-modules, as discussed in [3, Section 3–4] (where they are axiomatized as algebraic operations of infinite arity). But for this proof I only need it in the case of a t-complete module C, which is isomorphic to its t-adic completion. So the infinite sum can be understood as the limit in the t-adic topology on C.

We assume that $t^m C \neq 0$ for all $m \geqslant 0$ and come to a contradiction. Set $n_0 = 0 = m_0$, and choose a nonzero element $x_0 \in C$. By assumption, C is t-separated, so there exists an integer $n_1 > 0$ such that $x_0 \notin t^{n_1} C$. There also exists an integer $m_1 > 0$ such that $t^{m_1} x_0 = 0$ (since C is t-torsion).

By assumption, there exists an element $x_1 \in C$ such that $t^{m_1+n_1} x_1 \neq 0$. Hence there exists an integer $n_2 > n_1$ such that $t^{m_1+n_1} x_1 \notin t^{m_1+n_2} C$. There also exists an integer $m_2 > m_1$ such that $t^{m_2+n_1} x_1 = 0$.

Proceeding in this way, we choose for every $i \geqslant 1$ an element $x_i \in C$ and two integers $n_{i+1} > n_i$ and $m_{i+1} > m_i$ such that $t^{m_i+n_i} x_i \neq 0$, $t^{m_i+n_i} x_i \notin t^{m_i+n_{i+1}} C$ and $t^{m_{i+1}+n_i} x_i = 0$.

Now we consider the element

$$z = \sum_{i=0}^{\infty} t^{n_i} x_i \in C.$$

By assumption, there exists an integer $m \geqslant 0$ such that $t^m z = 0$. The sequence of integers m_i is strictly increasing, hence tends to infinity; so there exists $j \geqslant 0$ such that $m_j \geqslant m$. Thus we have

$$t^{m_j} z = \sum_{i=0}^{\infty} t^{m_j+n_i} x_i = 0. \tag{1}$$

By construction, for every $i < j$ we have

$$t^{m_j+n_i} x_i = 0,$$

so the first j summands in (1) vanish. Hence we come to

$$t^{m_j+n_j} x_j + \sum_{i=j+1}^{\infty} t^{m_j+n_i} x_i = 0, \tag{2}$$

implying that $t^{m_j+n_j} x_j \in t^{m_j+n_{j+1}} C$.

This contradicts our choice of n_{j+1}, proving the proposition. □

Corollary 4. *Let A be a commutative ring and $I \subset A$ a finitely generated ideal. Let D be a derived I-complete A-module which is also I-torsion. Then there exists an integer $n \geqslant 0$ such that $I^n D = 0$.*

Proof. Follows from Propositions 2 and 3. □

Does this qualify as a purely algebraic proof of your [1, Proposition 2.5]?

Best regards,

Leonid

References

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[2] L. Positselski. Dedualizing complexes and MGM duality. *Journ. of Pure and Appl. Algebra* **220**, #12, p. 3866–3909, 2016. `arXiv:1503.05523 [math.CT]`

[3] L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. *Moscow Math. Journ.* **17**, #3, p. 385–455, 2017. `arXiv:1605.03934 [math.CT]`

[4] L. Positselski, A. Slávik. Flat morphisms of finite presentation are very flat. Electronic preprint `arXiv:1708.00846 [math.AC]`.