Dear Hanno,

I was reading your arXiv preprint on models for singularity categories lately, and while I am still meditating on some parts of it, I also have a specific question concerning your constructions of the coderived/contraderived model structures for CDG-modules over arbitrary CDG-rings.

You write in your paper that the only description of the coacyclic and contraacyclic CDG-modules that you can give is the one provided by your Proposition 1.3.8 (of `arXiv.org/abs/1205.4473v1`). However, it appears to me that a more explicit description can be distilled from your results, certainly in the case of the coderived category, and I would hope that perhaps for the contraderived category, too.

What I want is to be able to obtain the classes of coacyclic and contraacyclic CDG-modules (in the sense of your definition and in the case of an arbitrary CDG-ring) by a kind of generation process. So I would like to have some set or series of (more or less explicit) tranformation rules by which all the co/contraacyclic CDG-modules, and only them, could be produced starting from some elementary (and also explicit) "seed".

In my theorems (which you cite) I start from the total CDG-modules of short exact sequences and then take the closure with respect to the operations of cone and (co)product. However, one can notice that there are more operations preserving the classes of co/contraacyclic CDG-modules than just these.

Let me say that a CDG-module L over a CDG-ring B is a *transfinitely iterated extension of CDG-modules K_α in the sense of inductive limit* if there exists a well-ordering of the set of indices $I = \{\alpha\}$ and an increasing filtration L_α of L by its CDG-submodules indexed by $\alpha \in I$ such that the quotient CDG-modules $L_\alpha / \bigcup_{\beta<\alpha} L_\beta$ are isomorphic to K_α and $\bigcup_{\alpha\in I} L_\alpha = L$.

Lemma 1. *Let J be a CDG-module over B such that the graded $B^\sharp$-module $J^\sharp$ is injective. Assume that a CDG-module L over B is a transfinitely iterated extension of CDG-modules K_α in the sense of inductive limit, and all the complexes* $\mathrm{Hom}_B(K_\alpha, J)$ *are acyclic. Then the complex* $\mathrm{Hom}_B(L, J)$ *is also acyclic.*

The dual definition is a bit more delicate, because filtered projective limits in the categories of modules are not always exact. I would say that a CDG-module M over a CDG-ring B is a *transfinitely iterated extension of CDG-modules N_α in the sense of projective limit* if there exists a well-ordering of the set of indices $I = \{\alpha\}$ and a projective system of CDG-modules M_α indexed by I (with closed morphisms between them) such that all the morphisms $M_\alpha \longrightarrow \varprojlim_{\beta<\alpha} M_\beta$ are surjective with the kernels N_α and M is isomorphic to $\varprojlim_{\alpha\in I} M_\alpha$.

Lemma 2. *Let P be a CDG-module over B such that the graded $B^\sharp$-module $P^\sharp$ is projective. Assume that a CDG-module M over B is a transfinitely iterated extension of CDG-modules N_α in the sense of projective limit, and all the complexes* $\mathrm{Hom}_B(P, N_\alpha)$ *are acyclic. Then the complex* $\mathrm{Hom}_B(P, M)$ *is also acyclic.*

Both Lemmas 1 and 2 follow straightforwardly from the next assertion about projective limits of complexes of abelian groups.

Fact. *Let $K_\alpha^\bullet$ be a projective system of complexes of abelian groups (with closed morphisms between them) indexed by a well-ordered set I. Assume that for all $\alpha \in I$ the morphism $K_\alpha^\bullet \longrightarrow \varprojlim_{\beta<\alpha} K_\beta^\bullet$ is surjective and its kernel is an acyclic complex. Then the complex $\varprojlim_{\alpha\in I} K_\alpha^\bullet$ is also acyclic.* □

In other words, the class of acyclic complexes of abelian groups is closed with respect to transfinitely iterated extensions in the sense of projective limit (and, of course, in the sense of inductive limit, too).

My question is whether all the co/contraacyclic CDG-modules (in the sense of your definition) can be obtained from, say, contractible ones using the operation of passage to a transfinitely iterated extension. Specifically, it seems that in the case of coacyclic CDG-modules the answer is positive, and this is essentially proven in your preprint.

Theorem. *Let B be a CDG-ring and L be a CDG-module over B such that the complex* $\mathrm{Hom}_B(L, J)$ *is acyclic for any CDG-module J over B for which the graded $B^\sharp$-module $J^\sharp$ is injective. Then L is homotopy equivalent to a transfinitely iterated extension, in the sense of inductive limit, of the cones of identity endomorphisms of certain CDG-modules over B.*

Proof. According to the proof of your Proposition 1.3.7(2), the coderived model structure is cogenerated by a set of cones of identity endomorphisms of certain CDG-modules. It remains to apply your Proposition 1.2.4 (and notice that the telescope construction allows to express direct summands in terms of a transfinitely iterated extension and a homotopy equivalence). □

Perhaps the assertion of Theorem might be strenghtened a bit, but the above seems to be what is safe to state on the basis of what you assert in you paper. (Please correct me if I am mistaken.)

Does the dual assertion for the contraderived category hold (in some form)? Let me formulate the following more precise conjecture.

Conjecture. *Let B be a CDG-ring and M be a CDG-module over B such that the complex* $\mathrm{Hom}_B(P, M)$ *is acyclic for any CDG-module J over B for which the graded $B^\sharp$-module $P^\sharp$ is injective. Then the CDG-module M can be produced from the zero CDG-module by applying the operations of the passage to a homotopy equivalent CDG-module and the passage to a transfinitely iterated extension in the sense of projective limit sufficiently many times.*

Do you think this could be deduced from the known results about model categories? Perhaps, using the techniques of your paper, or some other advanced set-theoretical techniques of the modern category theory (maybe, based on Vopenka's principle etc.)?

Part of the reason why I am interested in these questions is because, based on my experience, the definitions of co- and contraacyclicity in terms of generation processes provide a convenient and widely applicable way of proving preservation of co/contraacyclicity by various functors.

Also, it is pleasant to have a definition that one could apply to many situations without worrying in advance about the existence of a model structure or a complete cotorsion pair. E. g., to nonabelian exact categories, abelian or exact categories not having enough projective objects, etc.

So, I will be grateful for any comments you might have about these statements and questions.

Thank you and best wishes,

Leonid