Dear Hanno,

It is me who should be apologizing for late answer! I was a bit overwhelmed with various duties and events, as it oftentimes happens to me these days.

Thank you for your comment concerning the formulation of Theorem in my letter. What I had in mind were, of course, the CDG-modules in the image of the functor G^+ rather than the cones of identity endomorphisms. I should have said "contractible CDG-modules" instead of "the cones of identity endomorphisms of CDG-modules" (as all CDG-modules belonging to the image of G^+ are contractible).

In fact, the difference between the two formulations does not seem to be that essential: every contractible CDG-module is a direct summand of the cone of the identity endomorphism of a certain CDG-module, namely, of itself (as it is a direct summand of the cone of the zero endomorphism of itself and the cones of homotopic closed morphisms are isomorphic). So my original formulation of the theorem may be correct, too, after all (or am I missing something again?)

Concerning my conjecture about contraacyclic DG-modules, you point out various difficulties and ambiguities aring from insufficiency of the conventional set theory axioms. This may be a matter of taste, but I tend to trust those set theorists who say that the constructibility axiom is "wrong" and the "large large cardinals" (such as the measurables and above) are "right".

In other words, I am willing to accept Vopěnka's principle (at least, whenever there are reasonably convincing arguments that the ZFC axioms do not resolve the questions, as seems to be the case here). Do the problems of the kind that you describe in your letter seem to persist even if one assumes large cardinals?

Speaking of various ways to define classes of objects (occuring in semiorthogonal decompositions, model structures, etc.), I would distinguish the following two. On the one hand, one can specify a (simply described) generating subset/subclass ("seed") and a family of transformation rules for generation. On the other hand, one can choose a functor from the category under consideration to a less complicated category—typically, some kind of forgetful functor—and consider the full preimage of a simply described class of objects in the target category.

Let me proceed to discuss the examples that I have in mind one by one. All of these will be abelian model structures; moreover, all of them will be, in fact, either projective or injective abelian model structures.

In what you call the standard projective model structure on the category of DG-modules over a DG-ring A (Proposition 1.3.5(1) of your paper [1]), the class of cofibrant objects is generated by the DG-module A over A using the operations of shift, cone, infinite direct sum, and the passage to a homotopy equivalent DG-module inside the larger class of DG-modules with projective underlying graded $A^{\#}$-modules. Alternatively, one can say that a DG-module is cofibrant if and only if it is a direct summand of (or homotopy equivalent to) a transfinitely iterated extension of the DG-modules $A[i]$ (in the sense of inductive limit).

The class of weakly trivial DG-modules, on the other hand, can be described as the full preimage of the class of acyclic (= coacyclic = contraacyclic = absolutely acyclic = ...) complexes of abelian groups with respect to the natural forgetful functor acting from the category of DG-modules over A to the category of complexes of abelian groups (= DG-modules over $\mathbb{Z}$).

In the standard injective model structure on the same category of DG-modules over A [1, Proposition 1.3.5(2)], the class of fibrant objects is generated by the DG-module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ using the operations of shift, cone, infinite product, and the passage to a homotopy equivalent DG-module inside the larger class of DG-modules with injective underlying graded $A^{\#}$-modules (Theorem 8.1(b) in my memoir [3]).

One can also say that a DG-module is fibrant if and only if it is a direct summand of (or homotopy equivalent to) a transfinitely iterated extension of the DG-modules $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})[i]$ (in the sense of projective limit). The latter two characterizations can be obtained from the construction of a homotopy injective resolution (fibrant replacement) of a DG-module over A as the (product) totalization of a kind of DG-module cobar resolution in [3, proof of Theorem 1.5].

The class class of weakly trivial DG-modules is the same in the standard projective model structure, so the same description applies.

Furthermore, let $C = (C^{\#}, d)$ be a DG-coalgebra over a field k. Then the above two standard model category structures on DG-modules have their analogues for DG-comodules and DG-contramodules over C, with the similar descriptions of the classes of objects involved (at least, under Vopěnka's principle, in the case of the comodules). More precisely, the category of DG-contramodules carries a "standard projective model structure" and the category of DG-comodules has a "standard injective model structure" [3, Remark 8.2].

In the category of DG-contramodules over C, the class of cofibrant objects is generated by the DG-contramodule $\mathrm{Hom}_k(C, k)$ using the operations of shift, cone, infinite direct sum, and the passage to a homotopy equivalent DG-contramodule inside the larger class of DG-contramodules with projective underlying graded $C^{\#}$-contramodules. This description is obtained in the proof of [3, Theorem 2.4(b)] given in [3, Section 5.5]; the argument is based on Neeman and Krause's theory of well-generated triangulated categories.

In the category of DG-comodules over C, the class of fibrant objects is generated by the DG-comodule C over C using the operations of shift, cone, infinite product, and the passage to a homotopy equivalent DG-comodule inside the larger class of DG-comodules with injective underlying graded $C^{\#}$-comodules. This is proven in the new "note added three years later" at the end of Section 5.5 of the recent post-publication arXiv version of [3]; the argument is based on the results of Casacuberta–Gutiérrez–Rosický [2, Theorem 2.4] and Vopěnka's principle.

The classes of weakly trivial DG-comodules and DG-contramodules are described as the full preimages of the classes of acyclic (= contractible) complexes of k-vector spaces with respect to the forgetful functors from the categories of DG-comodules and DG-contramodules to the category of complexes of vector spaces.

In both cases, the constructions of the model structures are based on obtaining the related semiorthogonal decompositions of the homotopy categories of DG-comodules and DG-contramodules first. The homotopy category of DG-contramodules over C has a semiorthogonal decomposition formed by the triangulated subcategory generated by the DG-contramodule $\mathrm{Hom}_k(C, k)$ using infinite direct sums and the triangulated subcategory of acyclic DG-contramodules. The homotopy category of DG-comodules over C has a semiorthogonal decomposition formed by the triangulated subcategory of acyclic DG-comodules and the triangulated subcategory generated by the DG-comodule C over C using infinite products (under Vopěnka's principle).

Without Vopěnka's principle, one still has the "standard injective" model structure on the category of DG-comodules, but there seems to be no explicit description of the class of fibrant objects (except by using the universal quantifier).

All the preceding examples were those of "theories" (model structures, derived categories) "of the first kind"; now I pass to the ones "of the second kind". Firstly, let $C = (C^\#, d, h)$ be a CDG-coalgebra over k. Then there is the contraderived model structure on the category of CDG-contramodules over C and the coderived model structure on the category of CDG-comodules over C [3, Theorems 4.4 and 8.2].

In the category of CDG-contramodules over C, the class of cofibrant objects is defined as the full preimage of the class of projective graded $C^\#$-contramodules with respect to the functor of forgetting the differential $P \longmapsto P^\#$ acting from the category of CDG-contramodules over C to the category of graded contramodules over $C^\#$. The class of weakly trivial objects is generated by the totalizations of short exact sequences of CDG-contramodules using the operations of cone, infinite product, and the passage to (a direct summand or) a homotopy equivalent CDG-contramodule. Alternatively, one can say that a DG-contramodule is weakly trivial if and only if it is a direct summand of (or homotopy equivalent to) a transfinitely iterated extension of contractible CDG-contramodules (in the sense of projective limit).

In the category of CDG-comodules over C, the class of fibrant objects is defined as the full preimage of the class of injective graded $C^\#$-comodules with respect to the forgetful functor $M \longmapsto M^\#$ acting from the category of CDG-comodules over C to the category of graded $C^\#$-comodules. The class of weakly trivial objects is generated by the totalizations of short exact sequences of CDG-comodules using the operations of cone, infinite direct sum, and the passage to (a direct summand or) a homotopy equivalent CDG-comodule. Alternatively, one can say that a CDG-comodule is weakly trivial if and only if it is a direct summand of a transfinitely iterated extension of contractible CDG-comodules (in the sense of inductive limit).

Now let $B = (B^\#, d, h)$ be a CDG-ring. Then, according to your theorem [1, Proposition 1.3.6], the category of CDG-modules over B has the contraderived and coderived model category structures.

In the contraderived model structure, the class of cofibrant objects is defined as the full preimage of the class of projective graded $B^\#$-modules with respect to the functor of forgetting the differential $K \longmapsto K^\#$ acting from the category of CDG-modules over B to the category of graded modules over $B^\#$. In the coderived model structure, the class of fibrant objects is defined as the full preimage of the class of injective graded $B^{\#}$-modules with respect to the same forgetful functor.

In the assumption of my condition $(**)$ on the graded ring $B^{\#}$, the class of weakly trivial objects in the contraderived model structure is generated by the totalizations of short exact sequences of CDG-modules over B using the operations of cone, infinite product, and the passage to (a direct summand or) a homotopy equivalent CDG-module. In the assumption of my condition $(*)$ on $B^{\#}$, the class of weakly trivial objects in the coderived model structure is generated by the totalizations of short exact sequences of CDG-modules using the operations of cone, infinite direct sum, and the passage to (a direct summand or) a homotopy equivalent CDG-module [3, Remark 8.3].

Finally, according to the proof of your [1, Proposition 1.3.6(2)], for any CDG-ring B a CDG-comodule over B is weakly trivial in the coderived model structure if and only if it is (a direct summand of, or) homotopy equivalent to a transfinitely iterated extension of contractible CDG-modules (in the sense of inductive limit).

In every one of these examples, of course, one can define each of the two classes of objects forming the abelian model structure or cotorsion pair in terms of the universal quantifier running over the complementary other class. But each of them also has an independent description in one of the two forms that I mentioned in the beginning: either it is generated by something rather simple using some class of operations, or it is the full preimage of something rather simple with respect to a forgetful functor.

Or, strictly speaking, sometimes both components are needed, when I am saying that one can pass to any homotopy equivalent differential module object inside a larger class of differential objects whose underlying graded objects are projective or injective. Still in these cases one can sense that the generation procedure is somehow the more important aspect of the definition.

With the latter reservation in mind, one can notice that in the model structures of the first kind ("standard projective", "standard injective"), the class of weakly trivial objects is defined as the full preimage with respect to a forgetful functor, and the class of fibrant or cofibrant objects is constructed by a generation procedure. In the model structures of the second kind ("contraderived", "coderived"), the class of weakly trivial objects is constructed by a generation procedure and the class of (co)fibrant objects is defined as the full preimage with respect to a forgetful functor. This is, basically, the essence of the distinction between the two kinds of derived categories/model structures.

Now, if a class of objects is constructed from a certain "seed" using a generation procedure, it may be naturally true that the universal quantifier running over this class of objects and defining the complementary class can be restricted to the "seed". If the seed consists of a single object, the condition of orthogonality to this object may be expressible as belonging to the full preimage of something simple with respect to a forgetful functor which is closely related to the Hom or Ext^1 functor from/into this particular object. This is what happens in the above examples of model structures of the first kind. The special object in question is the DG-module A over A, or the DG-module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ over A, or the DG-contramodule $\mathrm{Hom}_k(C, k)$ over C, or the DG-comodule C over C.

Having a generation process with the seed consisting of, basically, all objects of the "size" bounded by a particular cardinality (in a certain class) is not as illuminating, in my view. Proving assertions about modules of bounded cardinality is not generally any easier than proving assertions about arbitrary modules, except if one intends to use the set-theoretical existence proof techniques.

The applications of such techniques that I like the most, however, are those which lead to an equivalence of explicitly defined categories or a semiorthogonal decomposition/model structure with explicit classes of objects (defined generally either as the full preimages or in terms of a simple "seed" and transformaion rules, or perhaps as a combination of the two). The "standard projective" and "standard injective" model structures on the categories of DG-contramodules and DG-comodules, as discussed above, provide a nice example of such explicit equivalences of categories or explicitly described model structures obtained using set-theoretical methods.

Let me try to point out a possible approach to proving my conjecture about the contraderived model structure (formulated in my first letter). It is based on the same theorem of Casacuberta–Gutiérrez–Rosický (and therefore also presumes Vopěnka's principle).

Let $B = (B^{\#}, d, h)$ be a CDG-ring. Then the category $\mathsf{Hot}(B\text{–}\mathsf{mod})$ of left CDG-modules over B with the morphisms up to chain homotopy can be obtained as the homotopy category of a certain stable model structure on the category $Z^0(B\text{–}\mathsf{mod})$ of CDG-modules and closed morphisms between them (which you denote by B-Mod).

This is not an abelian model structure on the abelian category $Z^0(B\text{–}\mathsf{mod})$, but rather an exact model structure for the exact category structure on $Z^0(B\text{–}\mathsf{mod})$ in which a short sequence is exact if it is split exact as a short sequence of graded $B^{\#}$-modules. This model category structure on $Z^0(B\text{–}\mathsf{mod})$ has chain homotopy equivalences as weak equivalences, maps that are split injective over $B^{\#}$ as cofibrations, and maps that are split surjective over $B^{\#}$ as fibrations. This seems to be a pretty well-known construction.

So $\mathsf{Hot}(B\text{–}\mathsf{mod})$ is the homotopy category of a stable model category structure on a Grothendieck abelian category, which is, consequently, a locally presentable category. This model category structure is probably not cofibrantly generated, but the assertion of [2, Theorem 2.4] does not seem to require the cofibrant generation assumption. Under Vopěnka's principle, this theorem should allow one to claim that every triangulated subcategory closed under infinite products in $\mathsf{Hot}(B\text{–}\mathsf{mod})$ is reflective, i. e., taken together with its left orthogonal complement forms a semiorthogonal decomposition of $\mathsf{Hot}(B\text{–}\mathsf{mod})$.

It only remains to describe the left orthogonal complement to the triangulated subcategory of contraacyclic CDG-modules in $\mathsf{Hot}(B\text{–}\mathsf{mod})$ (in whatever definition of the latter subcategory one prefers—it can be either the definition from [3] based on taking closure under the infinite products, or the definition from my letter using transfinitely iterated extensions).

Let P be a CDG-module over B. Assume that the complex $\mathrm{Hom}_B(P,E)$ is acyclic whenever E is the totalization of any short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ of CDG-modules over B. Can one prove that P is homotopy equivalent to a CDG-module whose underlying graded $B^{\#}$-module is projective? Or can one perhaps prove the same conclusion about P if the complex $\mathrm{Hom}_B(P,E)$ is acyclic whenever E is a transfinitely iterated extension of contractible CDG-modules over B (in the sense of projective limit)?

In view of [2, Theorem 2.4], this would seem to suffice to prove the conjecture from my first letter (or even a stronger result about the contraderived category as defined originally in [3]).

Notice that this approach apparently is *not* applicable to the coderived category, as the results from the Casacuberta–Gutiérrez–Rosický paper about triangulated subcategories closed under infinite direct sums in the homotopy categories of stable model categories [2, Theorem 3.9] require the model category to be combinatorial (i. e., a cofibrantly generated model structure on a locally presentable category).

So, what do you think now? Are you still skeptical?

Thank you and best wishes,

Leonid

References

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