Dear Kirsten, Dear Ido,

Answering the two questions in Kirsten's letter: no, I don't know how to construct (a model of) the DG-algebra $C^\bullet(G_F, \mathbb{Z}/l)$ for the absolute Galois group G_F of a field F in terms of the elements of F and the operations of addition and multiplication on them. Of course, the DG-algebra $C^\bullet(G, \mathbb{Z}/l)$ is by definition the free associative algebra generated by the quotient space $\mathbb{Z}/l(G)/(Z/l)$ of the $\mathbb{Z}/l$-vector space of locally constant $\mathbb{Z}/l$-valued functions on G by its one-dimensional subspace of constant functions (with the differential defined in terms of the convolution comultiplication on $\mathbb{Z}/l(G)$, but this is a construction based on the data of the Galois group $G = G_F$ rather than just the field with its elements.

Concerning the Massey products—let me try to explain what kind of Massey products appear in direct connection with the Koszul property, and how they are related to the more familiar elementary constructions you appear to be working with. The most relevant reference is [May66] (see also the heavier [May69]).

The connection with Koszulity was first pointed out by Priddy [Pr70, Section 8], who was discussing the homogeneous case; it becomes a bit more complicated in the nonhomogeneous augmented-adically-filtered setting relevant to the Galois theory. (I either never learned or forgot by now whatever happened to May's later work on the algebraic Eilenberg–Moore spectral sequence to which Priddy refers as "to appear".)

Let $C^\bullet = (C^*,\, d\colon C^i \to C^{i+1})$ be a DG-algebra over a field k; assume for simplicity that $C^i = 0$ for $i < 0$ and $C^0 = k$ (so in particular $d^0\colon C^0 \longrightarrow C^1$ is a zero map). The simplest construction of a Massey product starts with three elements x, y, $x \in H^1(C^\bullet)$ for which $xy = 0 = yz$ in $H^2(C^\bullet)$ and proceeds to produce an element $\langle x, y, z\rangle \in H^2(C^\bullet)$ defined up to elements of the subspace $xH^1(C^\bullet) + H^1(C^\bullet)z \subset H^2(C^\bullet)$.

Suppose that we want to extend this construction to elements of the tensor product $H^1(C^\bullet) \otimes_k H^1(C^\bullet) \otimes_k H^1(C^\bullet)$. With any three vectors x, y, $z \in H^1(C^\bullet)$ one can associate the decomposable tensor $x \otimes y \otimes z \in H^1(C^\bullet)^{\otimes 3}$; however, not every tensor is decomposable. Let $K^2 \subset H^1(C^\bullet) \otimes_k H^1(C^\bullet)$ denote the kernel of the multiplication map $m_2\colon H^1(C^\bullet) \otimes_k H^1(C^\bullet) \longrightarrow H^2(C^\bullet)$. We would like to have our triple Massey product defined on the subspace $K^2 \otimes_k H^1(C^\bullet) \cap H^1(C^\bullet) \otimes_k K^2 \subset H^1(C^\bullet)^{\otimes 3}$.

Let $B^n \subset Z^n \subset C^n$ denote the subspaces of coboundaries and cocycles, so that $H^n = H^n(C^\bullet) = Z^n/B^n$; and let $m\colon C^* \otimes_k C^* \longrightarrow C^*$ be the multiplication map. We denote the induced (conventional) multiplication on the cohomology by $m_2\colon H^*(C^\bullet) \otimes_k H^*(C^\bullet) \longrightarrow H^*(C^\bullet)$.

Given a tensor $\theta \in K^2 \otimes H^1 \cap H^1 \otimes K^2 \subset H^1 \otimes H^1 \otimes H^1$, one lifts it to a tensor $\widetilde{\theta}$ in $Z^1 \otimes Z^1 \otimes Z^1$, applies the maps of multiplication of the first two and the last two tensor factors $m^{(12)} = m \otimes \mathrm{id}$ and $m^{(23)} = \mathrm{id} \otimes m$ to obtain the element $(m^{(12)}(\widetilde{\theta}), m^{(23)}(\widetilde{\theta}))$

in $B^2 \otimes Z^1 \oplus Z^1 \otimes B^2$, lifts the latter arbitrarily to an element in $C^1 \otimes Z^1 \oplus Z^1 \otimes C^1$, and finally applies the product map m again and adds the two summands to obtain an element in C^2, which turns out to be an element of Z^2. Its image in $H^2(C^\bullet)$, denoted by $m_3(\theta)$, is the triple Massey product of our tensor θ.

What is the subspace in $H^2(C^\bullet)$ up to which the element $m_3(\theta)$ is well-defined? Let $W_l \subset H^1$ be the minimal vector subspace for which $\theta \in W_l \otimes H^1 \otimes H^1$, and let W_r be the similar minimal subspace for which $\theta \in H^1 \otimes H^1 \otimes W_r$ (hence in fact $\theta \in W_l \otimes H^1 \otimes W_r$). If one is careful, one can make the Massey product $m_3(\theta)$ well-defined up to elements of $W_l H^1 + H^1 W_r \subset H^2(C^\bullet)$. However, generally speaking, for "most" tensors $\theta \in K^2 \otimes H^1 \cap H^1 \otimes K^2$ (and certainly for "most" tensors in $H^1 \otimes H^1 \otimes H^1$) one would expect $W_l = H^1 = W_r$. So the triple Massey product that we have constructed is most simply viewed as a linear map

$$m_3\colon K^2 \otimes_k H^1(C^\bullet) \cap H^1(C^\bullet) \otimes_k K^2 \longrightarrow H^2(C^\bullet)/m_2(H^1(C^\bullet) \otimes_k H^1(C^\bullet)),$$
$$K^2 = \ker(m_2\colon H^1(C^\bullet) \otimes H^1(C^\bullet) \to H^2(C^\bullet)).$$

How is this triple Massey product construction related to the one starting from three cohomology classes $x, y, z \in H^1(C^\bullet)$ with zero pairwise products and assigning to them the cohomology class $\langle x, y, z\rangle$ in $H^2(C^\bullet)$ modulo $xH^1(C^\bullet) + H^1(C^\bullet)z$? On the one hand, a subspace $K^2 \subset H^1 \otimes H^1$ may well contain no nonzero decomposable tensors at all, while containing many nontrivial indecomposable tensors. Then there may be also many nontrivial indecomposable tensors in $K^1 \otimes H^1 \cap H^1 \otimes K^2$. So the domain of definition of the map m_3 may be essentially much wider than that of the Massey product of triples of elements $\langle x, y, z\rangle$. On the other hand, the latter, more elementary construction may produce its outputs with better precision (modulo a smaller subspace in $H^2(C^\bullet)$). Thus the map m_3 carries both more and less information about the DG-algebra $C^\bullet$ than the operation $\langle x, y, z\rangle$.

What is the tensor version of the quadruple Massey product, the map m_4? Let $K^3 \subset K^2 \otimes H^1 \cap H^1 \otimes K^2 \subset H^1(C^\bullet)^{\otimes 3}$ denote the kernel of the above map m_3. Consider the intersection of two vector subspaces $K^3 \otimes H^1 \cap H^1 \otimes K^3$ inside $H^1 \otimes H^1 \otimes H^1 \otimes H^1$. Then the desired map is

$$m_4\colon K^3 \otimes H^1(C^\bullet) \cap H^1(C^\bullet) \otimes K^3 \longrightarrow (H^2(C^\bullet)/\operatorname{im} m_2)/\operatorname{im} m_3.$$

Generally, the maps m_n are nothing but the differentials in a natural spectral sequence associated with the DG-algebra $C^\bullet$. To construct it, set $C_+^\bullet = C^\bullet/k = C^\bullet/C^0$ and consider the bar-complex

$$C_+^\bullet \longleftarrow C_+^\bullet \otimes_k C_+^\bullet \longleftarrow C_+^\bullet \otimes_k C_+^\bullet \otimes_k C_+^\bullet \longleftarrow \cdots$$

Set $D_p^q = (C_+^{*\otimes p})^q$, where the grading q on the tensor powers $C_+^{*\otimes p}$ is induced by the grading on C^*. The differential $d\colon D_p^q \longrightarrow D_p^{q+1}$ is induced by the differential on the complex $C^\bullet$, while the bar-differential $\partial\colon D_p^q \longrightarrow D_{p-1}^q$ is defined in terms of the multiplication in $C^\bullet$. As with every bicomplex, there are two spectral sequences associated with it; we are interested in the one that computes the cohomology of the differential d first, and the cohomology of the bar-differential ∂ afterwards.

One has to be a bit careful, because the bicomplex D_p^q does not satisfy the usual finiteness/boundedness conditions. So there are actually *two* ways to define its total complex: one can take either infinite direct sums or infinite products along the diagonals. The two spectral sequences associated with such a bicomplex converge to two different limits (namely, the cohomology of the two total complexes $\operatorname{Tot}^\oplus(D_\bullet^\bullet)$ and $\operatorname{Tot}^\sqcap(D_\bullet^\bullet)$). The spectral sequence we are interested in comes from the filtration by the index p, which is increasing; so it converges to the cohomology of the inductive limit of the filtration components, i. e., the direct sum total complex.

In short, we set $F_p \operatorname{Tot}^\oplus(D_\bullet^\bullet)_n = \bigoplus_{i-j=n}^{i\leqslant p}(C_+^{*\,\otimes i})^j$. Then the spectral sequence of this filtered complex has the form

$$E^1_{p,-q} = (H^*(C_+^\bullet)^{\otimes p})^q \Longrightarrow \operatorname{Tor}^{C^\bullet}_{p-q}(k,k), \qquad \partial^r_{p,-q}\colon E^r_{p,-q} \longrightarrow E^r_{p-r,-(q-r+1)},$$

where the grading q on the tensor powers $H^*(C_+^\bullet)^{\otimes p}$ is induced by the grading on $H^*(C^\bullet)$. The differential ∂^1 is induced by the conventional multiplication m_2 on the cohomology algebra $H^*(C^\bullet)$, so it acts, in particular, as

$$m_2 = \partial^1_{2,-(i+j)}\colon H^i(C^\bullet)\otimes H^j(C^\bullet) \longrightarrow H^{i+j}(C^\bullet).$$

The above triple Massey product map m_3 is the differential $\partial^2_{3,-3}\colon E^2_{3,-3} \longrightarrow E^2_{1,-2}$, which can be viewed as a partially defined multivalued linear map

$$m_3 = \partial^2_{3,-3}\colon H^1(C^\bullet)\otimes H^1(C^\bullet)\otimes H^1(C^\bullet) \dashrightarrow H^2(C^\bullet);$$

and the quadruple Massey product map m_4 is the differential $\partial^2_{4,-4}\colon E^2_{4,-4} \longrightarrow E^2_{1,-2}$,

$$m_4 = \partial^3_{4,-4}\colon H^1(C^\bullet)\otimes H^1(C^\bullet)\otimes H^1(C^\bullet)\otimes H^1(C^\bullet) \dashrightarrow H^2(C^\bullet)$$

(there may be plus/minus sign issues, which I disregard here).

Now let us consider the case when the cohomology algebra $H^1(C^\bullet)$ is generated by H^1 (as an associative algebra with the conventional multiplication m_2). Then, the map $m_2\colon H^1(C^\bullet)\otimes H^1(C^\bullet) \longrightarrow H^2(C^\bullet)$ being surjective, the above Massey product maps m_3, m_4, … vanish automatically (as their target spaces are zero). So do the similar Massey products

$$m_p = \partial^{p-1}_{p,-(j_1+\dots+j_p)}\colon H^{j_1}(C^\bullet)\otimes\dots\otimes H^{j_p}(C^\bullet) \dashrightarrow H^{j_1+\dots+j_p-p+2}(C^\bullet)$$

in the higher cohomology.

Does it mean that all the differentials $\partial^r_{p,-q}$ in our spectral sequence vanish for $r\geqslant 2$? Not necessarily. The first possibly nontrivial example would be

$$\partial^2_{4,-4}\colon H^1(C^\bullet)\otimes H^1(C^\bullet)\otimes H^1(C^\bullet)\otimes H^1(C^\bullet) \dashrightarrow H^1(C^\bullet)\otimes H^2(C^\bullet)\oplus H^2(C^\bullet)\otimes H^1(C^\bullet).$$

This is the map whose source space is actually the kernel of the differential $\partial^1_{4,-4}\colon H^1(C^\bullet)^{\otimes 4} \longrightarrow H^1(C^\bullet)^{\otimes 3}$, that is, the subspace

$$K^2\otimes H^1\otimes H^1\cap H^1\otimes K^2\otimes H^1\cap H^1\otimes H^1\otimes K^2\,\subset\, H^1(C^\bullet)^{\otimes 4}$$

and whose target space is the cokernel of the differential $\partial^1_{3,-3}\colon H^1\otimes H^1\otimes H^1\longrightarrow H^2\otimes H^1\oplus H^1\otimes H^2$, or, better to say, the middle cohomology space of the sequence

$$H^1\otimes H^1\otimes H^1\,\longrightarrow\, H^2\otimes H^1\,\oplus\, H^1\otimes H^2\,\longrightarrow\, H^3.$$

The latter cohomology space is otherwise known as the space of relations of degree 3 in the graded algebra H^*.

What does the map $\partial^2_{4,-4}$ do? Its source space can be otherwise described as the intersection

$$(K^2\otimes H^1\cap H^1\otimes K^2)\otimes H^1\,\cap\, H^1\otimes(K^2\otimes H^1\cap H^1\otimes K^2).$$

The map $(m_3\otimes\mathrm{id},\,\mathrm{id}\otimes m_3)$ acts from this subspace to the quotient space of $H^2\otimes H^1\oplus H^1\otimes H^2$ by the image of the map $(m_2\otimes\mathrm{id},\,\mathrm{id}\otimes m_3)$ coming from the direct sum of two copies of $H^1\otimes H^1\otimes H^1$. It is claimed that the map $(m_3\otimes\mathrm{id},\,\mathrm{id}\otimes m_3)$ can be naturally lifted to the quotient space of $H^2\otimes H^1\oplus H^1\otimes H^2$ by the image of only one (diagonal) copy of $H^1\otimes H^1\otimes H^1$, as one can see from the explicit constructon of m_3.

Indeed, let us restrict ourselves to decomposable tensors now (for simplicity). Let x, y, z, w be four elements in $H^1(C^\bullet)$ for which $xy=yz=zw=0$ in $H^2(C^\bullet)$. Let $\tilde x$, $\tilde y$, $\tilde z$, $\tilde w\in Z^1\subset C^1$ be some liftings of the elements x, y, z, w, and let ξ, η, and ζ be elements in C^1 for which $\tilde x\tilde y=d\xi$, $\tilde y\tilde z=d\eta$, and $\tilde z\tilde w=d\zeta$ in $B^2\subset C^2$. Then the triple Massey products are $\langle x,y,z\rangle=(\xi z+x\eta \bmod B^2)$ and $\langle y,z,w\rangle=(\eta w+y\zeta \bmod B^2)\in H^2$. When one replaces ξ, η, and ζ with $\xi'=\xi+r$, $\eta'=\eta+s$, and $\zeta'=\zeta+t$, where $r,\ s,\ t\in Z^1\subset C^1$, one obtains $\langle x,y,z\rangle'=\xi'z+x\eta'=\langle x,y,z\rangle+rz+xs$ and $\langle y,z,w\rangle'=\eta'w+y\zeta'=\langle y,z,w\rangle+sw+yt \bmod B^2$. Finally, one has

$$(\langle x,y,z\rangle'\otimes w,\,x\otimes\langle y,z,w\rangle')=(\langle x,y,z\rangle\otimes w,\,x\otimes\langle y,z,q\rangle)+((rz+xs)\otimes w,\,x\otimes(sw+yt))$$

and

$$((rz+xs)\otimes w,\ x\otimes(sw+yt))=\partial^1_{3,-3}(r\otimes z\otimes w+x\otimes s\otimes w+x\otimes y\otimes t)$$

in $H^2\otimes H^1\oplus H^1\otimes H^2$, because $zw=0=xy$ in $H^2(C^\bullet)$ by assumption.

What if the cohomology algebra $H^*(C^\bullet)$ is not only generated by H^1, but also defined by quadratic relations? There still can be nontrivial Massey operations (meaning the differentials $\partial^r_{p,-q}$ with $r\geqslant 2$), starting from

$$\partial^2_{5,-5}\colon H^1(C^\bullet)^{\otimes 5}\dashrightarrow H^2\otimes H^1\otimes H^1\,\oplus\,H^1\otimes H^2\otimes H^1\,\oplus\,H^1\otimes H^1\otimes H^2.$$

This is actually well-defined as a linear map from the source space

$$\bigcap_{i=1}^{4} H^1(C^\bullet)^{\otimes i-1} \otimes K^2 \otimes H^1(C^\bullet)^{\otimes 4-i} \subset H^1(C^\bullet)^{\otimes 5}$$

to a target space isomorphic to $\mathrm{Tor}_{3,4}^{H^*}(k,k)$. The latter Tor space is the first obstruction to Koszulity of a quadratic graded algebra H^*.

Why call these complicated polylinear maps "Massey products"? Well, they are actually invariants of the quasi-isomorphism class of the (augmented) DG-algebra $C^\bullet$. Indeed, a morphism of DG-algebras $f\colon {}'C^\bullet \longrightarrow {}''C^\bullet$ would induce a morphism of the bar-bicomplexes ${}'D^\bullet_\bullet \longrightarrow {}''D^\bullet_\bullet$, hence also a homomorphism of spectral sequences ${}'E^r_{p,-q} \longrightarrow {}''E^r_{p,-q}$. When the morphism f is a cohomology isomorphism, it induces an isomorphism ${}'E^1_{p,-q} \simeq {}''E^1_{p,-q}$ on the pages E^1 of these spectral sequences, hence also an isomorphism of all the subsequent pages $E^r_{p,-q}$ with $r \geqslant 2$. This means that the differentials ${}'\partial^r_{p,-q}$ and ${}''\partial^r_{p,-q}$ must coincide. For a formal DG-algebra $C^\bullet$, one has $\partial^r_{p,-q} = 0$ for all $r \geqslant 2$ and $p,\, q \in \mathbb{Z}$.

There still remains an unexplained notation $\mathrm{Tor}_n^{C^\bullet}(k,k)$ for the limit term of the spectral sequence $E^r_{p,-q}$ in the above exposition. This is the derived functor of tensor product of DG-modules over $C^\bullet$ defined on the (conventional) derived category of DG-modules (obtained by inverting the conventional quasi-isomorphisms, i. e., the DG-module morphisms inducing isomorphisms on the cohomology). Specializing to the case when $C^\bullet = C^\bullet(N)$ is the cochain complex of a conilpotent coaugmented coalgebra over k (such as the coalgebra $\mathbb{Z}/l(G)$ of locally constant $\mathbb{Z}/l$-valued functions on a pro-l-group G, with the convolution comultiplication over $\mathbb{Z}/l$), one has $\mathrm{Tor}_0^{C^\bullet}(k,k) = N$ and $\mathrm{Tor}_n^{C^\bullet}(k,k) = 0$ for $n \neq 0$.

For any augmented DG-algebra $C^\bullet$, the term $E^1_{p,-q}$ is simply the bar-complex of the cohomology algebra $H^*(C^\bullet)$ (and the differential $\partial^1_{p,-q}$ is the bar differential for the algebra H^*). Hence the term E^2 is easily computed as

$$E^2_{p,-q} = \mathrm{Tor}_p^{H^*}(k,k)^q,$$

where the grading p is the usual indexing of the Tor spaces and the grading q is induced by the grading of H^*. The spectral sequence $E^r_{p,-q}$ converges to

$$E^\infty_{p,-q} = \mathrm{gr}^F_p \, \mathrm{Tor}^{C^\bullet}_{p-q}(k,k)$$

in the sense of the inductive limit with respect to the increasing filtration F.

A positively graded algebra H^* is said to be Koszul if $\mathrm{Tor}_p^{H^*}(k,k)^q = 0$ for all $p \neq q$. So if when the algebra H^* is Koszul, one has $\partial^r_{p,-q} = 0$ for $r \geqslant 2$ simply "for dimension reasons": no pairs of integers (p,q) exist for which both the source and the target space of $\partial^r_{p,-q}$ would be nonvanishing. Due to the convergence of the spectral sequence, one then also has $\mathrm{Tor}_n^{C^\bullet}(k,k) = 0$ for $n \neq 0$. This is what is called "the $K(\pi,1)$ property" of $C^\bullet$.

Conversely, if $\operatorname{Tor}_n^{C^\bullet}(k,k) = 0$ for $n \neq 0$ and $\partial^r_{p,-q} = 0$ for $r \geqslant 2$, then it follows from the spectral sequence that $E^2_{p,-q} = E^\infty_{p,-q} = 0$ for $p - q \neq 0$, so the algebra $H^*(C^\bullet)$ is Koszul. This proves the main assertion promised in [Pos11, Subsection 9.11].

I hope this sheds some light on the question of the connections between vanishing Massey products and Koszul property that was raised in our conversation with Ido and the previous exchange of letters with Kirsten. The more elementary (or more delicate) notion of Massey products that has been studied in your papers is certainly also interesting generally, and in application to absolute Galois cohomology in particular. But absolute Galois groups seem to have very special properties, and in particular very special homological properties, in many respects: you look on them from one angle, and see them very special in one way, then look from another angle, and they are very special in another, seemingly unrelated, way.

I tried to formulate some of such conjectural (or, on rare occasions, provable) properties in the papers [Pos05, Pos06]. They still do not form any clear picture: the simplest illustration is, any closed subgroup H of an absolute Galois group G is also an absolute Galois group, but there is no way one could deduce quadraticity or Koszulity of $H^*(H, \mathbb{Z}/l)$ from the same condition on $H^*(G, \mathbb{Z}/l)$. So further and perhaps much stronger special properties of absolute Galois groups, uniting the presently known fragments into a coherent picture, are yet to be found.

With best wishes,

Leonid

References

[May66] J.P. May. The cohomology of augmented algebras and generalized Massey products for DGA-algebras. *Trans. Amer. Math. Soc.* **122**, p. 334–340, 1966.

[May69] J.P. May. Matrix Massey products. *Journ. of Algebra* **12**, p. 533–568, 1969.

[Pr70] S.B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.* **152**, p. 39–60, 1970.

[Pos05] L. Positselski. Koszul property and Bogomolov's conjecture. *Internat. Math. Research Notices* **2005**, #31, p. 1901–1936. Extended postpublication electronic preprint version: `arXiv:1405.0965 [math.KT]`

[Pos06] L. Positselski. Galois cohomology of certain field extensions and the divisible case of Milnor–Kato conjecture. *K-Theory* **36**, #1–2, p. 33–50, 2005. `arXiv:math.KT/0209037`

[Pos11] L. Positselski. Mixed Artin–Tate motives with finite coefficients. *Moscow Math. Journal* **11**, #2, p. 317–402, 2011. `arXiv:1006.4343 [math.KT]`