REMARKS ON EXACT AND TRIANGULATED CATEGORIES

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1. Preliminaries

1.1. Additive categories. A *preadditive category* $\mathcal{A}$ is a category whose sets of morphisms $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ are endowed with the structures of abelian groups and the compositions maps are biadditive. One can show that if an object X of a preadditive category $\mathcal{A}$ is the product of a finite set of objects $X_1, \dots, X_n$, then it is also the coproduct of this set of objects, and vice versa. In this case the object X is called the *direct sum* of $X_1, \dots, X_n$ and denoted by $X_1 \oplus \dots \oplus X_n$. In particular, for $n = 0$ the above statement means that an object X is a universal initial object if and only if it is a universal final object, and, actually, if and only if $\mathrm{id}_X = 0$; in this case X is called a *zero object* and denoted simply by $X = 0$.

An *additive category* is a preadditive category in which the (co)product of any finite set of objects exists. Equivalently, one can define an additive category as an abstract category in which all the finite products and coproducts exist, the initial object is isomorphic to the final object, and the natural morphism from the coproduct of any finite set of objects to their product (which is defined using the initial-final object) in an isomorphism. In particular, the preadditive category structure on an additive category can be recovered from the abstract category structure.

If $\mathcal{A}$ and $\mathcal{B}$ are preadditive categories, then a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is called an *additive functor* if it preserves the group structures on the sets of morphisms. Any additive functor preserves finite (co)products, and, moreover, any functor $\mathcal{A} \longrightarrow \mathcal{B}$ with this property is additive provided that $\mathcal{A}$ is an additive category.

The category of functors $\mathcal{F}un(\mathcal{C}, \mathcal{A})$ from any small category $\mathcal{C}$ to a (pre)additive category $\mathcal{A}$ is naturally a (pre)additive category again. Furthermore, if $\mathcal{C}$ is a preadditive category, then there is a full (pre)additive subcategory of additive functors $\mathcal{F}un_{\mathrm{ad}}(\mathcal{C}, \mathcal{A}) \subset \mathcal{F}un(\mathcal{C}, \mathcal{A})$. For example, the category of morphisms in an additive category is additive, the category of complexes in an additive category is additive, etc.

1.2. Saturation axioms. Here we follow the terminology of A. Neeman's paper [7].

An additive category $\mathcal{A}$ is called *semi-saturated* if any pair of its morphisms $p\colon X \longrightarrow Y$ and $i\colon Y \longrightarrow X$ such that $p \circ i = \mathrm{id}_Y$ comes from an isomorphism $X \simeq Y \oplus Z$ for some object $Z \in \mathcal{A}$. An additive category is called *saturated* (*Karoubian*, *pseudo-abelian*) if any its morphism $e\colon X \longrightarrow X$ such that $e \circ e = e$ is the projection on a direct summand of a decomposition $X \simeq Y \oplus Z$.

For any additive category $\mathcal{A}$, there exists a semi-saturated additive category $\mathcal{A}^{\mathrm{ss}}$ (called the *semi-saturation* of $\mathcal{A}$) together with a fully faithful functor $\mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{ss}}$ such that for any semi-saturated additive category $\mathcal{B}$ the composition with this functor defines an equivalence of categories $\mathcal{F}un_{\mathrm{ad}}(\mathcal{A}^{\mathrm{ss}}, \mathcal{B}) \longrightarrow \mathcal{F}un_{\mathrm{ad}}(\mathcal{A}, \mathcal{B})$. In other words, any additive functor $\mathcal{A} \longrightarrow \mathcal{B}$ can be extended to a functor $\mathcal{A}^{\mathrm{ss}} \longrightarrow \mathcal{B}$ and the latter is defined uniquely up to a unique isomorphism. Analogously, for any additive category $\mathcal{A}$ there exists a saturated additive category $\mathcal{A}^{\mathrm{sat}}$ (the *saturation* of $\mathcal{A}$) together with a fully faithful functor $\mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{sat}}$ such that for any saturated category $\mathcal{B}$ the composition defines an equivalence $\mathcal{F}un_{\mathrm{ad}}(\mathcal{A}^{\mathrm{sat}}, \mathcal{B}) \longrightarrow \mathcal{F}un_{\mathrm{ad}}(\mathcal{A}, \mathcal{B})$.

More specifically, the objects of $\mathcal{A}^{\mathrm{ss}}$ can be defined as the quadruples (X, Y, p, i); the objects of $\mathcal{A}^{\mathrm{sat}}$ are represented by the couples (X, e). One can figure out easily how to construct the groups of morphisms. For example, the category of projective modules over a ring is the saturation of the category of free modules.

Let $\mathcal{A}$ be a full subcategory of an additive category $\mathcal{B}$. Then the *semi-saturated closure* $\mathcal{A}^{\mathrm{ss}}_{\mathcal{B}}$ of the subcategory $\mathcal{A}$ in $\mathcal{B}$ is the full subcategory of $\mathcal{B}$ whose objects are all the objects $Z \in \mathcal{B}$ for which there exists an object $Y \in \mathcal{A}$ such that the object $X \oplus Y$ is also isomorphic to an object from $\mathcal{A}$. The *saturated closure* $\mathcal{A}^{\mathrm{sat}}_{\mathcal{B}}$ of a subcategory $\mathcal{A}$ is defined in the same way except that one allows $Y \in \mathcal{B}$.

Remarks: The following statements describe the relations of the definitions of this subsection with various notions introduced below. An additive category admitting either the kernels or the cokernels of all morphisms is saturated; in particular, any abelian category is saturated. Any triangulated category is semi-saturated: one can take $Z = \mathrm{Cone}(i) = \mathrm{Cone}(p)[-1]$. A typical example of a nonsaturated triangulated category is the derived category of bounded complexes of vector spaces with zero Euler characteristic. A triangulated subcategory of a triangulated category is thick if and only if it is saturation-closed (see subsection 1.7). The saturation of a triangulated category is naturally a triangulated category again.

1.3. Kernels and cokernels. A morphism $k\colon K \longrightarrow X$ in a preadditive category $\mathcal{A}$ is called the *kernel* of a morphism $f\colon X \longrightarrow Y$ if for any object $Z \in \mathcal{A}$ the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, K) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Z, Y)$$

is exact; in this case we write $K = \mathrm{Ker}\, f$ and $k = \ker f$. Analogously, a morphism $c\colon Y \longrightarrow C$ is called the *cokernel* of a morphism $f\colon X \longrightarrow Y$ if for any object $Z \in \mathcal{A}$ there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(C, Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Z);$$

the notation: $C = \mathrm{Coker}\, f$ and $c = \mathrm{coker}\, f$.

A morphism f is called *injective* if one has $\mathrm{Ker}\, f = 0$ and *surjective* if $\mathrm{Coker}\, f = 0$. It is easy to see that the composition of two injective (surjective) morphisms is an injective (surjective) morphism again. A right divisor g of an injective morphism fg is injective; a left divisor f of a surjective morphism fg is surjective.

We will say that a morphism f is *a kernel* if there exists a morphism g such that $f = \ker g$ and that f is *a cokernel* if there is g such that $f = \operatorname{coker} g$. Clearly, any kernel is injective, any cokernel is surjective, and any surjective kernel or injective cokernel is an isomorphism. Furthermore, one can see that if a morphism f is a kernel and its cokernel exists, then $f = \ker(\operatorname{coker} f)$ and, analogously, if a morphism f is a cokernel and its kernel exists, then $f = \operatorname{coker}(\ker f)$.

1.4. Abelian categories. An additive category $\mathcal{A}$ is called *abelian* [3] if all its morphisms have kernels and cokernels and, furthermore, the natural morphism

$$\operatorname{Coim} f = \operatorname{Coker}(\ker f) \longrightarrow \operatorname{Im} f = \operatorname{Ker}(\operatorname{coker} f)$$

is an isomorphism for any morphism f. In particular, the latter condition implies that any morphism which is simultaneously injective and surjective is an isomorphism. This property is *not sufficient* for the definition, though, because the morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ can be neither injective nor surjective in general (a counterexample is given in subsection 1.5). Nevertheless, it follows from Proposition 1.5 that an additive category is abelian if and only if all its morphisms have kernels and cokernels, any injective morphism is a kernel, and any surjective morphism is a cokernel.

A sequence of objects and morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an abelian category $\mathcal{A}$ is called an *exact triple* if one has $f = \ker g$ and $g = \operatorname{coker} f$. An additive functor between abelian categories $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is called *left exact* if it commutes with the kernels of all morphisms and right exact if it commutes with the cokernels. A functor is called *exact* if it preserves exact triples, or, equivalently, if it is left exact and right exact. A full subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$ is called an *abelian subcategory* if the category $\mathcal{A}$ is abelian and the embedding functor $\mathcal{A} \longrightarrow \mathcal{B}$ is exact.

The category of functors $\mathcal{F}un(\mathcal{C}, \mathcal{A})$ from any small category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is an abelian category. Specifically, the (co)kernel of a morphism $F \longrightarrow G$ in the category $\mathcal{F}un(\mathcal{C}, \mathcal{A})$ is the functor sending an object $X \in \mathcal{C}$ to the (co)kernel of the morphism $F(X) \longrightarrow G(X)$. If $\mathcal{C}$ is a preadditive category, then the category of additive functors $\mathcal{F}un_{\mathrm{ad}}(\mathcal{C}, \mathcal{A})$ is an abelian subcategory of $\mathcal{F}un(\mathcal{C}, \mathcal{A})$.

1.5. Nonabelian additive categories. Throughout this paper, we will sometimes use the following notation for computations in additive categories: the composition of several (previously defined) morphisms, say $X \longrightarrow Y$, $Y \longrightarrow Z$, and $Z \longrightarrow T$ will be denoted by $[X \to Y \to Z \to T]$. Then an equation like $[X \to Y \to Z] = [X \to T \to Z]$ means that the morphisms form a commutative square, etc.

Besides, we will say that a morphism $X \longrightarrow Z$ *factors through* a given morphism $X \longrightarrow Y$ or $Y \longrightarrow Z$ if there exists a (most often unique) morphism $Y \longrightarrow Z$ or $X \longrightarrow Y$, respectively, such that $[X \to Y \to Z] = [X \to Z]$.

Proposition. *Let $\mathcal{A}$ be an additive category in which all morphisms have kernels and cokernels. Then the morphism* $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ *is surjective for any morphism* f *in the category* $\mathcal{A}$ *if and only if the composition of any two kernels is a kernel in* $\mathcal{A}$ *and if and only if a right divisor of any kernel is a kernel. The dual result relates injectivity of the morphisms* $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ *with the properties of cokernels.*

Proof: Since the morphism $X \longrightarrow \operatorname{Coim} f$ is a cokernel, the morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is surjective if and only if the morphism $X \longrightarrow \operatorname{Im} f$ is surjective. Let us show that this is the case if the composition of two kernels is a kernel. Let $c\colon Y \longrightarrow C$ be the cokernel of the morphism f; by the definition, the morphism $\operatorname{im} f\colon \operatorname{Im} f \longrightarrow Y$ is the kernel of c. Let $d\colon \operatorname{Im} f \longrightarrow D$ be the cokernel of the morphism $X \longrightarrow \operatorname{Im} f$ and $l\colon L \longrightarrow \operatorname{Im} f$ be the kernel of the morphism d. By the assumption, the composition $(\operatorname{im} f) \circ l\colon L \longrightarrow Y$ should be a kernel. Since the morphism $X \longrightarrow \operatorname{Im} f$ factors through l and the composition $L \longrightarrow Y \longrightarrow C$ is zero, it is easy to deduce that c is the cokernel of the morphism $L \longrightarrow Y$. Now it turns out that both the morphisms $\operatorname{Im} f \longrightarrow Y$ and $L \longrightarrow Y$ are the kernels of the morphism $Y \longrightarrow C$, hence the morphism $L \longrightarrow \operatorname{Im} f$ is an isomorphism and $d = 0$.

Conversely, let us show that the composition f of two kernels $K \longrightarrow L$ and $L \longrightarrow M$ is a kernel whenever the morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is surjective. The morphism f is equal to the composition $K \longrightarrow \operatorname{Im} f \longrightarrow M$; by the assumption, the morphism $K \longrightarrow \operatorname{Im} f$ is surjective. Since the morphism $\operatorname{Im} f \longrightarrow M$ is a kernel, it suffices to check that the morphism $K \longrightarrow \operatorname{Im} f$ is an isomorphism. The latter is a particular case of the following general statement, which is not difficult to prove: a surjective right divisor of a composition of several kernels is an isomorphism.

Now let us show that a right divisor of a kernel is a kernel provided that the composition of two kernels is a kernel. Let $f\colon K \longrightarrow Y$ and $Y \longrightarrow Z$ be a pair of morphisms whose composition $K \longrightarrow Z$ is the kernel of a morphism $Z \longrightarrow D$. Let $c\colon Y \longrightarrow C$ be the cokernel of the morphism $K \longrightarrow Y$ and $\operatorname{im} f\colon \operatorname{Im} f \longrightarrow Y$ be kernel of c; then the morphism $K \longrightarrow Y$ factors through $\operatorname{im} f$. Since we have $[K \to Y \to Z \to D] = 0$, it is clear that the composition $Y \longrightarrow Z \longrightarrow D$ factors through c and, consequently, $[\operatorname{Im} f \to Y \to Z \to D] = 0$. Since the morphism $K \longrightarrow Z$ is the kernel of the morphism $Z \longrightarrow D$, there exists a morphism $\operatorname{Im} f \longrightarrow K$ such that $[\operatorname{Im} f \to Y \to Z] = [\operatorname{Im} f \to K \to Y \to Z]$. We have constructed morphisms between K and $\operatorname{Im} f$ in both directions; since the morphism $K \longrightarrow Z$ is injective and $[K \to \operatorname{Im} f \to K \to Y \to Z] = [K \to \operatorname{Im} f \to Y \to Z] = [K \to Y \to Z]$, it follows that $[K \to \operatorname{Im} f \to K] = \mathrm{id}_K$. Therefore, the morphism $K \longrightarrow \operatorname{Im} f$ is the embedding of a direct summand, hence it is a kernel and the morphism $f\colon K \longrightarrow Y$ is the composition of two kernels $K \longrightarrow \operatorname{Im} f$ and $\operatorname{Im} f \longrightarrow Y$.

Finally, assume that a right divisor of a kernel is a kernel. Let a morphism $K \longrightarrow L$ be the kernel of a morphism $c\colon L \longrightarrow C$ and a morphism $l\colon L \longrightarrow M$ be the kernel of a morphism $d\colon M \longrightarrow D$. Consider the anti-diagonal morphism $(c, -l)\colon L \longrightarrow C \oplus M$; since its composition with the projection $C \oplus M \longrightarrow M$ is a kernel, the morphism

$(c,-l)$ should be the kernel of a certain morphism $(i,e)\colon C\oplus M \longrightarrow E$. Since the morphism $l\colon L\longrightarrow M$ is injective, it is easy to see that the morphism $i\colon C\longrightarrow E$ is injective, too. Now one can check that the composition $K\longrightarrow L\longrightarrow M$ is the kernel of the diagonal morphism $(d,e)\colon M\longrightarrow D\oplus E$. $\square$

Example: The following counterexample shows that the morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ in an additive category admitting the kernels and cokernels of all morphisms does not have to be surjective. Let $\mathcal{B}$ be the abelian category of 3-term sequences of vector spaces and morphisms $V^{(1)}\longrightarrow V^{(2)}\longrightarrow V^{(3)}$ (the composition can be nonzero). There are 6 indecomposable objects in this category; we will denote them by E_1, E_2, E_3, E_{12}, E_{23}, E_{123}, where $\dim E_J^{(i)}=1$ for $i\in J$ and 0 otherwise. Let $\mathcal{A}\subset\mathcal{B}$ be the full additive subcategory whose objects are the direct sums of all the indecomposables except E_{12} and g be the morphism $E_3\longrightarrow E_{123}$. Then one has $\operatorname{Coker}_{\mathcal{B}} g=E_{12}$, $\operatorname{Coker}_{\mathcal{A}} g=E_1$ and $\operatorname{Im}_{\mathcal{A}} g=E_{23}$, while $\operatorname{Coim}_{\mathcal{A}} g=E_3$ and therefore $\operatorname{Coker}_{\mathcal{A}}(\operatorname{Coim}_{\mathcal{A}} g\longrightarrow \operatorname{Im}_{\mathcal{A}} g)=E_2\neq 0$. Equivalently, the composition g of two kernels $E_3\longrightarrow E_{23}$ and $E_{23}\longrightarrow E_{123}$ is not a kernel in the category $\mathcal{A}$. On the other hand, the morphism $\operatorname{Coim}_{\mathcal{A}} f\longrightarrow \operatorname{Im}_{\mathcal{A}} f$ is always injective, because a morphism in the category $\mathcal{A}$ is a cokernel if and only if it is surjective, so the composition of two cokernels is a cokernel.

1.6. Triangulated categories. A *triangulated category* $\mathcal{D}$ is an additive category endowed with an invertible functor $\mathcal{D}\longrightarrow\mathcal{D}$, which is called the *shift* and denoted together with all its integral degrees by $X\longmapsto X[n]$ for any $n\in\mathbb{Z}$, and a class of *distinguished triangles* of objects and morphisms

$$X\longrightarrow Y\longrightarrow Z\longrightarrow X[1]$$

satisfying a set of axioms Tr1–Tr4 introduced in [11]. The object Z is called a *cone* of the morphism $f\colon X\longrightarrow Y$ and denoted by $Z=\operatorname{Cone}(f)$; a cone is defined uniquely up to a nonunique isomorphism. The middle term Y of a distinguished triangle is sometimes said to be an *extension* of the objects X and Z.

The *octahedron axiom* Tr4 describes a cone of the composition of any two morphisms: one has $\operatorname{Cone}(g\circ f)\simeq\operatorname{Cone}(\operatorname{Cone}(g)[-1]\longrightarrow\operatorname{Cone}(f))$, or, in other words, a cone of the composition $g\circ f$ is an extension of cones of the morphisms f and g.

An additive functor between triangulated categories $F\colon\mathcal{D}_1\longrightarrow\mathcal{D}_2$ is called *exact* if it commutes with the shifts and sends distinguished triangles to distinguished triangles. A morphism of exact functors is a morphism of functors which commutes with the shift. A full subcategory $\mathcal{B}$ of a triangulated category $\mathcal{D}$ is called a *triangulated subcategory* if there exists a (unique) triangulated category structure on the category $\mathcal{B}$ such that the embedding functor $\mathcal{B}\longrightarrow\mathcal{D}$ is exact.

1.7. Localizations. Let $\mathcal{C}$ be a small category and $S\subset\operatorname{Mor}\mathcal{C}$ be a set of its morphisms. Then it is not difficult to see that there exists a category $S^{-1}\mathcal{C}$ such that for any category $\mathcal{C}'$ the category $\mathcal{F}un(S^{-1}\mathcal{C},\mathcal{C}')$ is isomorphic to the full subcategory of $\mathcal{F}un(\mathcal{C}, \mathcal{C}')$ whose objects are all the functors F such that the morphism $F(s)$ is invertible for any $s \in S$. Specifically, the category $S^{-1}\mathcal{C}$ has the same objects as the category $\mathcal{C}$ and the morphisms are freely generated by the morphisms of $\mathcal{C}$ together with the formal inverse morphisms to the morphisms from S. The category $S^{-1}\mathcal{C}$ is called the *localization* of the category $\mathcal{C}$ with respect to the set of morphisms S and the natural functor $l_{\mathcal{C},S}\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ is called the *localization functor*.

Let $\mathcal{D}$ be a triangulated category and $\mathcal{B} \subset \mathcal{D}$ be a triangulated subcategory. Then the *quotient category* $\mathcal{D}/\mathcal{B}$ is defined as the localization $S_{\mathcal{B}}^{-1}\mathcal{D}$ of the category $\mathcal{D}$ by the class $S = S_{\mathcal{B}}$ of all morphisms f such that $\mathrm{Cone}(f) \in \mathcal{B}$. The class $S_{\mathcal{B}}$ satisfies a set of conditions listed in [11] which allow to describe the category $S_{\mathcal{B}}^{-1}\mathcal{D}$ more explicitly. Namely, for any two objects X, $Y \in \mathcal{D}$ the set of morphisms $\mathrm{Hom}_{S^{-1}\mathcal{D}}(X, Y)$ is isomorphic to the set of all formal quotients f/s of morphisms $s\colon X' \longrightarrow X$ and $f\colon X' \longrightarrow Y$ in the category $\mathcal{D}$ such that $s \in S$ modulo the following equivalence relation: the quotient f/s of morphisms $s\colon X' \longrightarrow X$ and $f\colon X' \longrightarrow Y$ is equivalent to the quotient g/t of morphisms $t\colon X'' \longrightarrow Y$ and $g\colon X'' \longrightarrow Y$ if there exist two morphisms $p\colon X''' \longrightarrow X'$ and $q\colon X''' \longrightarrow X''$ such that $sp = tq \in S$ and $fp = gq$.

Using this construction, one can check that the category $\mathcal{D}/\mathcal{B}$ is additive and admits a uniquely defined triangulated category structure such that the localization functor $l_{\mathcal{D},\mathcal{B}}\colon \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{B}$ is exact; a triangle in the category $\mathcal{D}/\mathcal{B}$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle from $\mathcal{D}$. Since a morphism in a triangulated category is an isomophism if and only if it has zero cone, the following universal property of the quotient category easily follows: for any triangulated category $\mathcal{D}'$, the category of exact functors $\mathcal{D} \longrightarrow \mathcal{D}'$ sending all the objects of the subcategory $\mathcal{B}$ to zero objects of the category $\mathcal{D}'$ is equivalent to the category of all exact functors $\mathcal{D}/\mathcal{B} \longrightarrow \mathcal{D}'$. Conversely, the image $l_{\mathcal{D},\mathcal{B}}(X)$ of an object $X \in \mathcal{D}$ is a zero object of the category $\mathcal{D}/\mathcal{B}$ if and only if the object X belongs to the saturated closure $\mathcal{B}_{\mathcal{D}}^{\mathrm{sat}}$ of the subcategory $\mathcal{B} \subset \mathcal{D}$. Therefore, the categories $\mathcal{D}/\mathcal{B}$ and $\mathcal{D}/\mathcal{B}_{\mathcal{D}}^{\mathrm{sat}}$ are always naturally isomorphic. A triangulated subcategory $\mathcal{B} \subset \mathcal{D}$ is called a *thick subcategory* if one has $\mathcal{B} = \mathcal{B}_{\mathcal{D}}^{\mathrm{sat}}$.

2. Exact Categories

2.1. Axioms. Let $\mathcal{E}$ be an additive category endowed with a class of *admissible triples* of objects and morphisms

$$\mathcal{T}_{\mathcal{E}} = \{X' \longrightarrow X \longrightarrow X''\}, \qquad X',\ X,\ X'' \in \mathcal{E}.$$

A morphism $X \longrightarrow Y$ in the category $\mathcal{E}$ will be called an *admissible monomorphism* if it can be embedded into an admissible triple $X \longrightarrow Y \longrightarrow Z$ and an *admissible epimorphism* if there exists an admissible triple $T \longrightarrow X \longrightarrow Y$. An additive category $\mathcal{E}$ together with a class of admissible triples $\mathcal{T}_{\mathcal{E}}$ is called an *exact category* if it satisfies the following axioms Ex0–Ex3:

Ex0: The zero triple $0 \longrightarrow 0 \longrightarrow 0$ is admissible. Any triple isomorphic to an admissible triple is admissible.

Ex1: For any admissible triple $X' \longrightarrow X \longrightarrow X''$ and any object $Z \in \mathcal{E}$ there are exact sequences

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{E}}(Z, X') \longrightarrow \mathrm{Hom}_{\mathcal{E}}(Z, X) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(Z, X'')$$
$$0 \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X'', Z) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, Z) \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X', Z)$$

In other words, the morphisms $X' \longrightarrow X$ and $X \longrightarrow X''$ are each other's kernel and cokernel.

Ex2: Let $X' \longrightarrow X \longrightarrow X''$ be an admissible triple. Then (a) for any morphism $X' \longrightarrow Y'$ there exists a commutative diagram

$$(*) \qquad \begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \\ Y' \arrow[r] & Y \arrow[ur] & \end{tikzcd}$$

with an admissible triple $Y' \longrightarrow Y \longrightarrow X''$ and, analogously, (b) for any morphism $Y'' \longrightarrow X''$ there exists a commutative diagram

$$(**) \qquad \begin{tikzcd} X' \arrow[r] \arrow[dr] & X \arrow[r] & X'' \\ & Y \arrow[u] \arrow[r] & Y'' \arrow[u] \end{tikzcd}$$

with an admissible triple $X' \longrightarrow Y \longrightarrow Y''$.

Ex3: (a) The composition of any two admissible monomorphisms is an admissible monomorphism. (b) The composition of any two admissible epimorphisms is an admissible epimorphism.

This set of axioms is essentially a subsystem of the set of conditions proposed by Quillen in [9]. It will be shown in the next subsection that Quillen's definition is in fact equivalent to the above one; in particular, the third "obscure" axiom from Quillen's list follows from the first two. Moreover, the conditions Ex3 (a) and Ex3 (b) are equivalent modulo Ex0–Ex2 (see [5]).

We will also show that for any class of admissible triples $\mathcal{T}_{\mathcal{E}}$ satisfying Ex0-Ex2 the diagrams $(*)$ and $(**)$ in Ex2 are defined uniquely up to a unique isomorphism by the original triple $X' \longrightarrow X \longrightarrow X''$ and the morphism $X' \longrightarrow Y'$ or $Y'' \longrightarrow X''$, respectively; the object Y in the diagram $(*)$ is necessarily the fibered coproduct of the morphisms $X' \longrightarrow Y'$ and $X' \longrightarrow X$ and the object Y in the diagram $(**)$ is the fibered product of the morphisms $Y'' \longrightarrow X''$ and $X \longrightarrow X''$. In other words, the axiom Ex2 is equivalent modulo Ex0–Ex1 to the following axiom Ex2$'$:

Ex2′: (a) For any admissible monomorphism $X' \longrightarrow X$ and any morphism $X' \longrightarrow Y'$ there exists a fibered coproduct $Y = Y' \sqcup_{X'} X$ in the category $\mathcal{E}$ and the natural morphism $Y' \longrightarrow Y$ is an admissible monomorphism.
(b) For any admissible epimorphism $X \longrightarrow X''$ and any morphism $Y'' \longrightarrow X''$ there exists a fibered product $Y = Y'' \sqcap_{X''} X$ in the category $\mathcal{E}$ and the natural morphism $Y \longrightarrow Y''$ is an admissible epimorphism.

Finally, the axiom Ex2 together with the semi-saturation condition on the category $\mathcal{E}$ is equivalent modulo Ex0–Ex1 to the following condition Ex2″:

Ex2″: (a) A right divisor of an admissible monomorphism is an admissible monomorphism.
(b) A left divisor of an admissible epimorphism is an admissible epimorphism.
(c) If in the following commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & X_1 \arrow[r] \arrow[d] & X'' \\ & X_2 \arrow[ur] & \end{tikzcd}$$

both triples $X' \longrightarrow X_1 \longrightarrow X''$ and $X' \longrightarrow X_2 \longrightarrow X''$ are admissible, then the morphism $X_1 \longrightarrow X_2$ is an isomorphism.

2.2. Equivalences. This subsection contains the proofs of the statements formulated above. In what follows we assume that we are given an additive category $\mathcal{E}$ endowed with a class of admissible triples satisfying the axioms Ex0–Ex1.

Proposition 1. *It follows from the condition Ex2(a) that if in the diagram* $(*)$ *the triple* $Y' \longrightarrow Y \longrightarrow X''$ *is admissible and the morphism* $X \longrightarrow X''$ *is the cokernel of the morphism* $X' \longrightarrow X$*, then the object* Y *is the fibered coproduct* $Y = Y' \sqcup_{X'} X$ *in the category* $\mathcal{E}$*. Analogously, it follows from Ex2(b) that if in the diagram* $(**)$ *the triple* $X' \longrightarrow Y \longrightarrow Y''$ *is admissible and the morphism* $X' \longrightarrow X$ *is the kernel of the morphism* $X \longrightarrow X''$*, then the object* Y *is the fibered product* $Y = Y'' \sqcap_{X''} X$*.*

Proof: We have to show that for any object $Z \in \mathcal{E}$ there is a bijective correspondence between the set of all morphisms $Y \xrightarrow{f} Z$ and the set of all pairs of morphisms $Y' \xrightarrow{g} Z$ and $X \xrightarrow{h} Z$ which form a commutative diagram with $X' \longrightarrow Y'$ and $X' \longrightarrow X$. Suppose a morphism $Y \xrightarrow{f} Z$ is annihilated by the compositions with $Y' \longrightarrow Y$ and $X \longrightarrow Y$. Then it follows by Ex1 that it factors through $Y \longrightarrow X''$ and therefore it is zero since the morphism $X \longrightarrow X''$ is assumed to be a cokernel.

Now let us assume that we are given a compatible pair of morphisms $Y' \xrightarrow{g} Z$ and $X \xrightarrow{h} Z$. Applying Ex2 to the admissible triple $Y' \longrightarrow Y \longrightarrow X''$ and the morphism $Y' \xrightarrow{g} Z$, we obtain an admissible triple $Z \longrightarrow T \longrightarrow X''$ together with the commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \\
Y' \arrow[r] \arrow[d, "g"'] & Y \arrow[d] \arrow[ur] & \\
Z \arrow[r] & T \arrow[uur] &
\end{tikzcd}$$

Consider the morphism $\chi\colon X \longrightarrow T$ which is equal to the difference of two compositions $\chi = [X \to Y \to T] - [X \xrightarrow{h} Z \to T]$. We have $[X' \to X \xrightarrow{\chi} T] = 0$, since $[X' \to X \to Y \to T] = [X' \to Y' \xrightarrow{g} Z \to T]$ and $[X' \to Y' \xrightarrow{g} Z] = [X' \to Y \xrightarrow{h} Z]$. Therefore, the morphism χ factors to the composition $\chi = [X \to X'' \xrightarrow{\phi} T]$ with some morphism $X'' \xrightarrow{\phi} T$. The composition $X'' \xrightarrow{\phi} T \longrightarrow X''$ is the identity, since $[X \to X'' \xrightarrow{\phi} T \to X''] = [X \xrightarrow{\chi} T \to X''] = [X \to Y \to T \to X''] - [X \xrightarrow{h} Z \to T \to X''] = [X \to X'']$ and the morphism $X \longrightarrow X''$ is surjective. It follows that the morphism $\mathrm{id}_T - [T \to X'' \xrightarrow{\phi} T]$ annihilates $T \longrightarrow X''$ and, therefore, factors to the composition $[T \xrightarrow{\psi} Z \to T]$ with some morphism $T \xrightarrow{\psi} Z$.

Let us check that the morphism $f = [Y \to T \xrightarrow{\psi} Z]$ is the desired one. We have $[Y \xrightarrow{f} Z \to T] = [Y \to T \xrightarrow{\psi} Z \to T] = [Y \to T] - [Y \to T \to X'' \xrightarrow{\phi} T] = [Y \to T] - [Y \to X'' \xrightarrow{\phi} T]$. Now $[X \to Y \xrightarrow{f} \to Z \to T] = [X \to Y \to T] - [X \to Y \to X'' \xrightarrow{\phi} T] = [X \to Y \to T] - [X \to X'' \xrightarrow{\phi} T] = [X \to Y \to T] - [X \xrightarrow{\chi} T] = [X \xrightarrow{h} Z \to T]$, which implies $[X \to Y \xrightarrow{f} Z] = [X \xrightarrow{h} Z]$. In the same way, $[Y' \to Y \xrightarrow{f} Z \to T] = [Y' \to Y \to T] - [Y' \to Y \to X'' \xrightarrow{\phi} T] = [Y' \to Y \to T] = [Y' \xrightarrow{g} Z \to T]$ and $[Y' \to Y \xrightarrow{f} Z] = [Y' \xrightarrow{g} Z]$. $\square$

Proposition 2. *Assume that the condition Ex2 is satisfied. Then if in the diagram $(*)$ the triple $X' \longrightarrow X \longrightarrow X''$ is admissible and the morphism $Y' \longrightarrow Y$ is the kernel of the morphism $Y \longrightarrow X''$, then the triple $Y' \longrightarrow Y \longrightarrow X''$ is admissible, too. Analogously, if in the diagram $(**)$ the triple $X' \longrightarrow X \longrightarrow X''$ is admissible and the morphism $Y \longrightarrow Y''$ is the cokernel of the morphism $X' \longrightarrow Y$, then the triple $X' \longrightarrow Y \longrightarrow Y''$ is admissible.*

Proof: Applying Ex2 to the admissible triple $X' \longrightarrow X \longrightarrow X''$ and the morphism $X' \longrightarrow Y'$, we obtain an admissible triple $Y' \longrightarrow T \longrightarrow X''$. According to Proposition 1, we have $T = Y' \sqcup_{X'} X$. Therefore, there exists a unique morphism $T \xrightarrow{\phi} Y$ commuting with the morphisms from Y' and from X. Let us show that it commutes with the morphisms to X'', too.

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & X'' \\
Y' \arrow[r] \arrow[dr] & T \arrow[d, "\phi"'] \arrow[ur] & \\
 & Y \arrow[uur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r] \arrow[dr] & Y \arrow[r] & X'' \\
 & T \arrow[u, "\phi"'] \arrow[ur] &
\end{tikzcd}$$

We have $[X \to T \to Y \to X''] = [X \to Y \to X''] = [X \to X''] = [X \to T \to X'']$ and $Y' \to T \to Y \to X''] = [Y' \to Y \to X''] = 0 = [Y' \to T \to X'']$, which implies the desired $[T \to Y \to X''] = [T \to X'']$ by the property of a fibered coproduct. Therefore, the whole diagram to the left is commutative.

Consider the diagram to the right as a diagram of type (**) with the rightmost morphism reduced to the identity. Since the triple $Y' \longrightarrow T \longrightarrow X''$ is admissible and the morphism $Y' \longrightarrow Y$ is the kernel of the morphism $Y \longrightarrow X''$, we can apply the second statement of Proposition 1 and conclude that $T = X'' \sqcap_{X''} Y = Y$. $\square$

Proposition 3. *Assume that the condition Ex2 is satisfied. Then if in the diagram (*) both triples $X' \longrightarrow X \longrightarrow X''$ and $Y' \longrightarrow Y \longrightarrow X''$ are admissible, then the triple $X' \xrightarrow{(-1,1)} Y' \oplus X \xrightarrow{(1,1)} Y$ is admissible, too. If in the diagram (**) both triples $X' \longrightarrow X \longrightarrow X''$ and $X' \longrightarrow Y \longrightarrow Y''$ are admissible, then the triple $Y \xrightarrow{(1,1)} Y'' \oplus X \xrightarrow{(-1,1)} X''$ is admissible.*

Proof: It is easy to check that the following diagram is commutative:

$$\begin{tikzcd}
X' \arrow[r] \arrow[dr, "(-1,1)"'] & X \arrow[r] & X'' \\
 & Y' \oplus X \arrow[u, "(0,1)"'] \arrow[r, "(1,1)"] & Y \arrow[u]
\end{tikzcd}$$

Since the object Y is the fibered coproduct $Y' \sqcup_{X'} X$, the morphism $Y' \oplus X \xrightarrow{(1,1)} Y$ is the cokernel of the morphism $X' \xrightarrow{(-1,1)} Y' \oplus X$. By the second statement of Proposition 2, the triple $X' \longrightarrow Y' \oplus X \longrightarrow Y$ is admissible. $\square$

Corollary 1. *The conditions Ex2(a) and Ex2′(a) are equivalent. Analogously, the conditions Ex2(b) and Ex2′(b) are equivalent.*

Proof: It follows from Proposition 1 that Ex2(a) implies Ex2′(a). To prove the converse, it suffices to notice that if there exists a fibered coproduct $Y = Y' \sqcup_{X'} X$ of two morphisms $X' \longrightarrow Y$ and $X' \longrightarrow Y'$ and the morphism $X \longrightarrow X''$ is the cokernel of the morphism $X' \longrightarrow X$, then there exists an only morphism $Y \longrightarrow X''$

such that $[X \to Y \to X''] = [X \to X'']$ and $[Y' \to Y \to X''] = 0$, and, moreover, this morphism $Y \longrightarrow X''$ is the cokernel of the morphism $Y' \longrightarrow Y$. $\square$

Corollary 2. *The system of axioms Ex0–Ex3 is equivalent to Quillen's set of axioms for exact categories.*

Proof: The first axiom of Quillen's coincides with the union of our Ex0 and Ex1, except that it also claims that all the split triples $X' \longrightarrow X' \oplus X'' \longrightarrow X''$ are admissible. The latter follows immediately from Ex0 and Ex2$'$. The second Quillen's axiom is the union of our Ex2$'$ and Ex3. Finally, the third "obscure" axiom from Quillen's list claims that a right divisor of an admissible monomorphism is an admissible monomorphism provided that it admits a cokernel, and a left divisor of an admissible epimorphism is an admissible epimorphism whenever it admits a kernel. This statement is essentially our Proposition 2. $\square$

Corollary 3. *If the conditions Ex0–Ex1 are satisfied, then the condition Ex2$''$ holds if and only if the category $\mathcal{E}$ is semi-saturated and the condition Ex2 holds.* $\square$

Example 1: The following counterexample shows that the condition Ex2$''$(c) is indeed necessary. Let $\mathcal{A}$ be the additive category whose objects are morphisms of vector spaces $f\colon V'' \longrightarrow V'$ endowed with a subspace $V \subset \operatorname{Im} f$ and let $\mathcal{T}_{\mathcal{A}}$ be the class of all triples for which Ex1 holds. One can check that this class of admissible triples satisfies Ex0, Ex2$''$(a-b) and Ex3, but not Ex2$''$(c).

Note that if two classes of triples $\mathcal{T}'_{\mathcal{E}}$ and $\mathcal{T}''_{\mathcal{E}}$ satisfy Ex0–Ex2, then their union $\mathcal{T}'_{\mathcal{E}} \cup \mathcal{T}''_{\mathcal{E}}$ also does. In particular, one has to use the axiom Ex3 in order to prove that the direct sum of two admissible triples is an admissible triple. The next counterexample shows that even if one assumes the latter property, the conditions Ex0–Ex2 still wouldn't imply Ex3.

Example 2: Let $\mathcal{B}$ be the abelian category of 3-term sequences of vector spaces and morphisms considered in Example 1.5. Let $\mathcal{T}_{\mathcal{B}}$ be the class of all exact triples $X' \longrightarrow X \longrightarrow X''$ in the abelian category $\mathcal{B}$ such that for any morphism $X' \longrightarrow E_3$ the triple $E_{23} \longrightarrow Y \longrightarrow X''$ induced from the original triple by the morphism $X' \longrightarrow E_3 \longrightarrow E_{23}$ is split. Then it is not difficult to check that the class $\mathcal{T}_{\mathcal{B}}$ satisfies Ex0–Ex2, but the composition of two admissible monomorphisms $E_3 \longrightarrow E_{23}$ and $E_{23} \longrightarrow E_{123}$ is not an admissible monomorphism.

2.3. Examples. Various examples of exact categories are listed below.

(1) The trivial exact category structure: for any additive category $\mathcal{A}$, the class of all split triples $X' \longrightarrow X' \oplus X'' \longrightarrow X''$ satisfies the axioms of an exact category.

(2) The canonical exact category structure on an abelian category: the class of all exact triples in any abelian category satisfies the axioms Ex0–Ex3.

(3) More generally, let $\mathcal{A}$ be an additive category in which any morphism has a kernel and a cokernel. Then the class of all triples satisfying Ex1 defines an exact

category structure on the category $\mathcal{A}$ if and only if both the condition Ex2$''$(c) and the equivalent conditions of Proposition 1.5 hold.

(4) Even more generally, one can show that any semi-saturated additive category $\mathcal{A}$ admits a maximal exact category structure, that is a class of triples satisfying Ex0–Ex3 and containing any other such class of triples. It follows that for any two exact category structures $\mathcal{T}'_\mathcal{A}$ and $\mathcal{T}''_\mathcal{A}$ there exists a minimal exact category structure $\mathcal{T}_\mathcal{A}$ containing both $\mathcal{T}'_\mathcal{A}$ and $\mathcal{T}''_\mathcal{A}$.

(5) Let $(\mathcal{A},\mathcal{T}_\mathcal{A})$ be an exact category and $\mathcal{E}\subset\mathcal{A}$ be a full additive subcategory. Then the class $\mathcal{T}_\mathcal{E}$ of all triples in the category $\mathcal{E}$ which belong to $\mathcal{T}_\mathcal{A}$ defines an exact category structure on $\mathcal{E}$ provided that either of the following two conditions holds: (a) the subcategory $\mathcal{E}$ contains the middle term X of any admissible triple $X'\longrightarrow X\longrightarrow X''$ in the category $\mathcal{A}$ whenever it contains both terms X' and X''; or (b) the subcategory $\mathcal{E}$ contains all the terms of an admissible triple in the category $\mathcal{A}$ whenever it contains the middle term and one of the other terms.

(6) Let $(\mathcal{A},\mathcal{T}_\mathcal{A})$ and $(\mathcal{B},\mathcal{T}_\mathcal{B})$ be exact categories and $F\colon\mathcal{A}\longrightarrow\mathcal{B}$ be an additive functor sending triples from $\mathcal{T}_\mathcal{A}$ to triples satisfying Ex1 in the category $\mathcal{B}$. Then the class of triples $\mathcal{T}_\mathcal{A}\cap F^{-1}(\mathcal{T}_\mathcal{B})$ defines a new exact category structure on the category $\mathcal{A}$.

(7) The category of additive functors $\mathcal{F}un_{\mathrm{ad}}(\mathcal{A},\mathcal{E})$ from a preadditive category $\mathcal{A}$ to an exact category $\mathcal{E}$ has a natural exact category structure. Namely, a triple $F'\longrightarrow F\longrightarrow F''$ is called admissible if the triple $F'(X)\longrightarrow F(X)\longrightarrow F''(X)$ is admissible in the category $\mathcal{E}$ for any object $X\in\mathcal{A}$.

(8) A *filtered object* in an exact category $\mathcal{E}$ is a collection of objects $\mathrm{gr}^{a,b}X\in\mathcal{E}$, $a\leqslant b\in\mathbb{Z}$, and admissible triples $\mathrm{gr}^{b+1,c}X\longrightarrow\mathrm{gr}^{a,c}X\longrightarrow\mathrm{gr}^{a,b}X$ satisfying the obvious compatibility conditions. A triple of filtered objects $X'\longrightarrow X\longrightarrow X''$ is called admissible if all the triples $\mathrm{gr}^{a,b}X'\longrightarrow\mathrm{gr}^{a,b}X\longrightarrow\mathrm{gr}^{a,b}X''$ are admissible in the category $\mathcal{E}$, or, equivalently, all the triples $\mathrm{gr}^aX'\longrightarrow\mathrm{gr}^aX\longrightarrow\mathrm{gr}^aX''$ are admissible, where one denotes $\mathrm{gr}^aX=\mathrm{gr}^{a,a}X$. A *finitely filtered object* is a filtered object X such that one has $\mathrm{gr}^aX=0$ for all but a finite number of indices a. Then the category of (finitely) filtered objects in an exact category is an exact category again.

(9) The category of locally free sheaves on a scheme has a natural structure of exact category which is induced from the canonical exact category structure on the abelian category of quasi-coherent sheaves by the rule (5a).

(10) The category of topological modules (Hausdorff or not) over a topological ring is an exact category with respect to the maximal class of admissible triples described in (3). For example, the categories of Banach spaces, locally compact abelian groups, and topological modules over a topological group all have natural exact category structures.

2.4. The embedding theorem. An additive category $\mathcal{E}$ endowed with a class of admissible triples $\mathcal{T}_\mathcal{E}$ is an exact category if and only if there exists an abelian category $\mathcal{A}$ and a fully faithful functor $\rho\colon\mathcal{E}\longrightarrow\mathcal{A}$ such that the subcategory $\rho(\mathcal{E})\subset\mathcal{A}$

is closed under extensions and a triple $X' \longrightarrow X \longrightarrow X''$ in the category $\mathcal{E}$ belongs to $\mathcal{T}_{\mathcal{E}}$ if and only if its image $\rho(X') \longrightarrow \rho(X) \longrightarrow \rho(X'')$ is an exact triple in the abelian category $\mathcal{A}$. In other words, the exact category structure on $\mathcal{E}$ is induced from the canonical exact category structure on the abelian category $\mathcal{A}$ in the sense of the above example (5a). This is the statement of an embedding theorem first formulated by Quillen in [9] and proven in detail in several places, including [6] and [10].

In fact, for any exact category there are *two* natural embeddings to abelian categories satisfying all the properties mentioned; the two embeddings differ by the categorical duality. They are constructed in the following way. Consider the abelian category $\mathcal{F}un_{\mathrm{ad}}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}b)$ of contravariant additive functors from $\mathcal{E}$ to the category of abelian groups $\mathcal{A}b$. A functor $F\colon \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ is called *left exact* if for any admissible triple $X' \longrightarrow X \longrightarrow X''$ in the category $\mathcal{E}$ the sequence of abelian groups

$$0 \longrightarrow F(X'') \longrightarrow F(X) \longrightarrow F(X')$$

is exact. Let us denote by $\mathcal{A}'(\mathcal{E}) \subset \mathcal{F}un_{\mathrm{ad}}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}b)$ the full additive subcategory of left exact functors and by $\rho'_{\mathcal{E}}\colon \mathcal{E} \longrightarrow \mathcal{A}'(\mathcal{E})$ the additive functor sending an object $X \in \mathcal{E}$ to the representable contravariant functor $\mathrm{Hom}_{\mathcal{E}}(-, X)$.

The category $\mathcal{A}'(\mathcal{E})$ is abelian. This result is an additive version of the sheafification theory: arbitrary additive functors play the role of presheaves and the left exact functors are the sheaves. Note that the category $\mathcal{A}'(\mathcal{E})$ is not an abelian subcategory of the category $\mathcal{F}un_{\mathrm{ad}}(\mathcal{E}^{\mathrm{op}}, \mathcal{A}b)$: the embedding functor is only left exact. The proof is based on the following general result: if $\mathcal{A} \subset \mathcal{B}$ is a full additive subcategory of an abelian category and the embedding functor admits an left adjoint functor $p\colon \mathcal{B} \longrightarrow \mathcal{A}$ such that the composition $\mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}$ is left exact, then the category $\mathcal{A}$ is abelian, the functor p is exact, the embedding functor is left exact, and the category $\mathcal{A}$ is isomorphic to the quotient category of the category $\mathcal{B}$ by the Serre subcategory of all objects annihilated by p. The functor p is the additive sheafification functor; its construction is the main part of the proof.

The second embedding $\rho''_{\mathcal{E}}\colon \mathcal{E} \longrightarrow \mathcal{A}''(\mathcal{E})$ is defined by the covariant representable functors; the category $\mathcal{A}''(\mathcal{E})$ is the full subcategory of $\mathcal{F}un_{\mathrm{ad}}(\mathcal{E}, \mathcal{A}b)^{\mathrm{op}}$ whose objects are all the left exact covariant functors. For example, if $\mathcal{E}$ is the category of finitely generated free (or projective) left modules over a ring R endowed with the trivial exact category structure, then the functor $\rho'_{\mathcal{E}}$ is the obvious embedding into the abelian category of all left R-modules and the functor $\rho''_{\mathcal{E}}$ is the embedding into the abelian category opposite to the category of all right R-modules given by the rule $\rho''(M) = \mathrm{Hom}_R(M, R)^{\mathrm{op}}$. The following result demonstrates the difference between the two canonical embeddings $\rho'_{\mathcal{E}}$ and $\rho''_{\mathcal{E}}$.

Proposition 1. *For any morphism f in an exact category $\mathcal{E}$, the morphism $\rho'_{\mathcal{E}}(f)$ is injective if and only if the morphism f is injective. If the category $\mathcal{E}$ is semi-saturated, then the morphism $\rho'_{\mathcal{E}}(f)$ is surjective if and only if the morphism f is an admissible epimorphism. The dual statements hold for the embedding $\rho''_{\mathcal{E}}$.*

Proof: The first statement is obvious; the second one follows from Proposition 2 and Corollary 2.2.3. $\square$

Proposition 2. *For any objects $A \in \mathcal{A}'(\mathcal{E})$ and $X \in \mathcal{E}$ and any surjective morphism $A \longrightarrow \rho'_{\mathcal{E}}(X)$ in the category $\mathcal{A}'(\mathcal{E})$ there exists an admissible epimorphism $Y \longrightarrow X$ in the category $\mathcal{E}$ and a morphism $\rho'_{\mathcal{E}}(Y) \longrightarrow A$ such that one has $[\rho'_{\mathcal{E}}(Y) \to A \to \rho'_{\mathcal{E}}(X)] = [\rho'_{\mathcal{E}}(Y) \to \rho'_{\mathcal{E}}(X)]$.*

Proof: See [10]. $\square$

3. Exact Subcategories of Triangulated Categories

3.1. Derived Categories. The classical construction of the derived category of an abelian category can be extended to exact categories in a very natural way. For example, the derived category corresponding to the trivial exact category structure on any additive category is simply the homotopy category of this additive category.

The definition of the homotopy category $\mathcal{K}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is well-known. The objects of $\mathcal{K}(\mathcal{A})$ are complexes $X^\bullet = (X^i,\, i \in \mathbb{Z};\, d_X^i\colon X^i \longrightarrow X^{i+1})$ in the category $\mathcal{A}$ and the morphisms from $X^\bullet$ to $Y^\bullet$ are morphisms of graded objects $f\colon X \longrightarrow Y$ of degree 0 such that $d_D f = f d_C$ modulo those of the form $f = d_Y h + h d_X$, where $h\colon X \longrightarrow Y$ is a morphism of graded modules of degree -1. The category $\mathcal{K}(\mathcal{A})$ is a triangulated category with the shift functor defined by the rule $X[1]^i = X^{i+1},\ d_{X[1]} = -d_X$. A triangle is called distinguished if it is isomorphic to a triangle of the form $X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow C^\bullet(f) \longrightarrow X^\bullet[1]$, where $C^i(f) = Y^i \oplus X^{i+1}$ and $d_{C(f)}(y,x) = (d_Y(y) + f(x),\, -d_X(x))$.

The full triangulated subcategories $\mathcal{K}^+(\mathcal{A})$, $\mathcal{K}^-(\mathcal{A})$, and $\mathcal{K}^b(\mathcal{A}) \subset \mathcal{K}(\mathcal{A})$ consist of all complexes $X^\bullet$ such that $X^i = 0$ for all $i \ll 0$, $X^i = 0$ for all $i \gg 0$, or $X^i = 0$ for all $|i| \gg 0$, respectively.

Definition 1. A complex $A^\bullet \in \mathcal{K}(\mathcal{E})$ in an exact category $\mathcal{E}$ is called *acyclic* if there exists a sequence of objects $Z^i \in \mathcal{E}$ and admissible triples $Z^i \longrightarrow A^i \longrightarrow Z^{i+1}$ such that $d_A^i = [A^i \to Z^{i+1} \to A^{i+1}]$. A morphism of complexes $f\colon X^\bullet \longrightarrow Y^\bullet$ in $\mathcal{K}(\mathcal{E})$ is called a *quasi-isomorphism* if the complex $C^\bullet(f)$ is acyclic.

Let $\mathcal{A}c(\mathcal{E}) \subset \mathcal{K}(\mathcal{E})$ denote the full subcategory of acyclic complexes and $\mathcal{A}c^*(\mathcal{E}) = \mathcal{K}^*(\mathcal{E}) \cap \mathcal{A}c(\mathcal{E})$ for $* = +, -,$ or b.

Proposition 1. *A complex bounded from above $X^\bullet \in \mathcal{K}^-(\mathcal{E})$ in a semi-saturated exact category $\mathcal{E}$ is acyclic if and only if the complex $\rho'_{\mathcal{E}}(X^\bullet)$ is acyclic in the abelian category $\mathcal{A}'(\mathcal{E})$. Analogously, a complex $X^\bullet \in \mathcal{K}^+(\mathcal{E})$ in a semi-saturated exact category $\mathcal{E}$ is acyclic if and only if the compex $\rho''_{\mathcal{E}}(X^\bullet)$ is acyclic.*

Proof: This follows easily by induction from Proposition 2.4.1. $\square$

Proposition 2. *For any exact category $\mathcal{E}$, the subcategory of acyclic complexes $\mathcal{A}c(\mathcal{E}) \subset \mathcal{K}(\mathcal{E})$ is a triangulated subcategory, as well as the subcategories $\mathcal{A}c^*(\mathcal{E}) \subset$*

$\mathcal{K}^*(\mathcal{E})$. *For any* $* = +, -,$ *or* b, *the subcategory* $\mathcal{A}c^*(\mathcal{E}) \subset \mathcal{K}^*(\mathcal{E})$ *is a thick subcategory if and only if the category* $\mathcal{E}$ *is semi-saturated. The category* $\mathcal{A}c(\mathcal{E}) \subset \mathcal{K}(\mathcal{E})$ *is a thick subcategory if and only if the category* $\mathcal{E}$ *is saturated.*

Proof: See [7]. $\square$

Definition 2. The *derived categories* of a small exact category $\mathcal{E}$ are defined as the quotient categories $\mathcal{D}(\mathcal{E}) = \mathcal{K}(\mathcal{E})/\mathcal{A}c(\mathcal{E})$ and $\mathcal{D}^*(\mathcal{E}) = \mathcal{K}^*(\mathcal{E})/\mathcal{A}c^*(\mathcal{E})$ for any $* = +, -,$ or b.

Proposition 3. *For any exact category* $\mathcal{E}$, *the functor* $\mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^b(\mathcal{E}^{\mathrm{ss}})$ *and the functors* $\mathcal{D}^*(\mathcal{E}) \longrightarrow \mathcal{D}^*(\mathcal{E}^{\mathrm{sat}})$ *for* $* = +, -,$ *or empty are equivalences of triangulated categories.* $\square$

The following useful criterion can be found in [11].

Proposition 4. *Let* $\mathcal{D}$ *be a triangulated category and* $\mathcal{D}'$, $\mathcal{B} \subset \mathcal{D}$ *be two triangulated subcategories. Then the natural functor between the quotient categories* $\mathcal{D}'/(\mathcal{B} \cap \mathcal{D}') \longrightarrow \mathcal{D}/\mathcal{B}$ *is fully faithful provided that either of the following two dual conditions hold: (a) any morphism from an object of* $\mathcal{D}'$ *to an object of* $\mathcal{B}$ *factors through an object of* $\mathcal{B} \cap \mathcal{D}'$, *or (b) any morphism from an object of* $\mathcal{B}$ *to an object of* $\mathcal{D}'$ *factors through an object of* $\mathcal{B} \cap \mathcal{D}'$. $\square$

And the next result follows straightforwardly from the definition.

Proposition 5. *Let* $X^\bullet \in \mathcal{K}(\mathcal{E})$ *be a complex in an exact category* $\mathcal{E}$ *and* $A^\bullet \in \mathcal{K}(\mathcal{E})$ *be an acyclic complex. Let* $f\colon X^\bullet \longrightarrow A^\bullet$ *be a morphism of complexes. Suppose that* $X^i = 0$ *for all* $i > n$. *Then there exists an acyclic complex* $B^\bullet$ *and a morphism of complexes* $B^\bullet \longrightarrow A^\bullet$ *such that* $B^i = 0$ *for all* $i > n$, *the morphism* $B^i \longrightarrow A^i$ *is an isomorphism for all* $i < n$, *and the morphism* f *factors through* $B^\bullet$. *The same is true if* $X^i = 0$ *for all* $i < n$, *except that* $B^i = 0$ *for all* $i < n-1$ *and the morphism* $B^i \longrightarrow A^i$ *is an isomorphism for all* $i > n-1$. $\square$

Corollary 1. *For any small exact category* $\mathcal{E}$ *and any* $* = +, -,$ *or* b, *the natural functor* $\mathcal{D}^*(\mathcal{E}) \longrightarrow \mathcal{D}(\mathcal{E})$ *is fully faithful.* $\square$

Corollary 2. *For any exact category* $\mathcal{E}$, *the functors* $\mathcal{D}^-(\mathcal{E}) \longrightarrow \mathcal{D}^-(\mathcal{A}'(\mathcal{E}))$ *and* $\mathcal{D}^+(\mathcal{E}) \longrightarrow \mathcal{D}^+(\mathcal{A}''(\mathcal{E}))$ *induced by the canonical embeddings* $\rho'\colon \mathcal{E} \longrightarrow \mathcal{A}'(\mathcal{E})$ *and* $\rho''\colon \mathcal{E} \longrightarrow \mathcal{A}''(\mathcal{E})$ *are fully faithful.*

Proof: This follows from Proposition 2.4.2 and its dual version. $\square$

3.2. Ext groups. Let $\mathcal{E}$ be a small exact category. By the definition, we put $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{E})}(X, Y[n])$. It follows from Corollary 3.1.1 that this definition is equivalent to the one using any bounded derived category $\mathcal{D}^*(\mathcal{E})$.

The composition of morphisms in the derived category defines multiplication maps $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) \times \mathrm{Ext}^m_{\mathcal{E}}(Y,Z) \longrightarrow \mathrm{Ext}^{n+m}_{\mathcal{E}}(X,Z)$.

Proposition 1. *In any exact category* $\mathcal{E}$, *the Ext groups with negative numbers are zero,* $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) = 0$ *for* $n < 0$. *The natural morphism* $\mathrm{Hom}_{\mathcal{E}}(X,Y) \longrightarrow \mathrm{Ext}^0_{\mathcal{E}}(X,Y)$ *is an isomorphism; in other words, the functor* $\mathcal{E} \longrightarrow \mathcal{D}(\mathcal{E})$ *is fully faithful.*

Proposition 2. *The Ext spaces with positive numbers $n > 0$ in an exact category $\mathcal{E}$ are computed by the following Yoneda's construction. Consider the set $\mathrm{Yon}^n_{\mathcal{E}}(X,Y)$ of all acyclic complexes $A \in \mathcal{K}^b(\mathcal{E})$ such that $A^i = 0$ for all $i < 0$ and $i > n+1$, $A^0 = Y$, and $A^{n+1} = X$. Two elements A and $B \in \mathrm{Yon}^n_{\mathcal{E}}(X,Y)$ are said to be equivalent if there exists a third element $C \in \mathrm{Yon}^n_{\mathcal{E}}(X,Y)$ and two morphisms of complexes $C \longrightarrow A$ and $C \longrightarrow B$, both inducing identity morphisms on the terms Y in degree 0 and X in degree $n+1$. This is indeed an equivalence relation and the quotient set of the set of Yoneda extensions $\mathrm{Yon}^n_{\mathcal{E}}(X,Y)$ modulo this equivalence relation is canonically bijective to $\mathrm{Ext}^n_{\mathcal{E}}(X,Y)$. The multiplication on the Ext groups corresponds to the obvious composition operation on the Yoneda extensions.*

Proof: Both above Propositions follow straightforwardly from the construction of triangulated quotient category described in subsection 1.7 and Proposition 3.1.5. $\square$

Corollary. *For any element $\xi \in \mathrm{Ext}^n_{\mathcal{E}}(X,Y)$ with $n \geqslant 2$ in any exact category $\mathcal{E}$ there exists a sequence of objects $Z_0, \ldots, Z_n$ in the category $\mathcal{E}$ with $Z_0 = Y$ and $Z_n = X$ and elements $\xi_i \in \mathrm{Ext}^1_{\mathcal{E}}(Z_i, Z_{i-1})$ for all $1 \leqslant i \leqslant n$ such that the element ξ is equal to the composition $\xi_1 \circ \cdots \circ \xi_n$.* $\square$

Another interesting exlicit construction of the higher Ext spaces in an exact category is provided by Corollary 3.3.1 below.

Proposition 3. *For any small exact category $\mathcal{E}$, the canonical embeddings $\rho' \colon \mathcal{E} \longrightarrow \mathcal{A}'(\mathcal{E})$ and $\rho'' \colon \mathcal{E} \longrightarrow \mathcal{A}''(\mathcal{E})$ induce isomorphisms of the Ext groups $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}'(\mathcal{E})}(X,Y)$ and $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{A}''(\mathcal{E})}(X,Y)$ for all $X, Y \in \mathcal{E}$.*

Proof: This follows Corollary 3.1.2 or directly from Proposition 2.4.2. $\square$

3.3. Exact subcategories. A full subcategory $\mathcal{A} \subset \mathcal{D}$ in a triangulated category $\mathcal{D}$ is called *closed under extensions* if it contains the middle term Y of any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ with $X, Z \in \mathcal{D}$. Note that $\mathcal{A}$ is a triangulated subcategory if and only if it is closed under extensions and $\mathcal{A}[1] = \mathcal{A}$.

Proposition 1. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{E} \subset \mathcal{D}$ be a full subcategory which is closed under extensions and such that $\mathrm{Hom}^n_{\mathcal{D}}(X,Y) = 0$ for any $X, Y \in \mathcal{D}$ and $n < 0$, or even only for $n = -1$. Then there is a natural exact category structure on the category $\mathcal{E}$ in which a triple $X \longrightarrow Y \longrightarrow Z$ is admissible if and only if there exists a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in the category $\mathcal{D}$. The morphism $Z \longrightarrow X[1]$ making the triangle distinguished is unique if it exists.*

Proof: The condition Ex0 is obviously satisfied; Ex1 follows immediately from the long exact sequence of homomorphism groups in a triangulated category. One has to apply the octahedron axiom in order to check Ex2 and Ex3. The last statement holds for any sequence $X \longrightarrow Y \longrightarrow Z$ in a triangulated category $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(X, Z[-1]) = 0$. $\square$

We will simply say that a full subcategory $\mathcal{E} \subset \mathcal{D}$ is an *exact subcategory* if it is closed under extensions and $\mathrm{Hom}^n_{\mathcal{D}}(\mathcal{E},\mathcal{E}) = 0$ for $n < 0$, presuming the exact category structure defined in Proposition 1.

Let $\mathcal{E}\subset\mathcal{D}$ be a small exact subcategory in a triangulated category. Consider the following graded ring of higher Hom spaces:

$$\mathcal{H}(\mathcal{E},\mathcal{D})=\bigoplus\nolimits_{n=0}^{\infty}\mathcal{H}^n(\mathcal{E},\mathcal{D});\qquad \mathcal{H}^n(\mathcal{E},\mathcal{D})=\bigoplus\nolimits_{X,Y\in\mathcal{E}}\operatorname{Hom}_{\mathcal{D}}(X,Y[n])$$

Let $\mathcal{H}(\mathcal{E})=\mathcal{H}(\mathcal{E},\mathcal{D}(\mathcal{E}))$ denote the graded ring of Ext groups in an exact category $\mathcal{E}$.

Proposition 2. *For any small exact subcategory $\mathcal{E}\subset\mathcal{D}$ in a triangulated category $\mathcal{D}$, the graded ring $\mathcal{H}=\mathcal{H}(\mathcal{E},\mathcal{D})$ has the following property: the multiplication map $\mathcal{H}^1\otimes_{\mathcal{H}^0}\mathcal{H}^n\longrightarrow\mathcal{H}^{n+1}$ is injective.*

Proof: It is easy to check using the existence of direct sums in the category $\mathcal{E}$ that any element of $\mathcal{H}^1\otimes_{\mathcal{H}^0}\mathcal{H}^n$ has the form $\xi\otimes\eta$ for some $\xi\in\operatorname{Hom}_{\mathcal{D}}(Y,Z[1])$ and $\eta\in\operatorname{Hom}_{\mathcal{D}}(X,Y[n])$. Consider a distinguished triangle $Z\longrightarrow\operatorname{Cone}(\xi)[-1]\xrightarrow{f} Y\longrightarrow Z[1]$ corresponding to the morphism ξ in the category $\mathcal{D}$; by assumption, the object $T=\operatorname{Cone}(\xi)[-1]$ belongs to $\mathcal{E}$. Now if the composition $\xi\circ\eta\colon X\longrightarrow Z[n+1]$ is zero, there exists a morphism $\eta'[-n]\colon X\longrightarrow T[n+1]$ such that $\eta=f\circ\eta'$. Then we have $\xi\otimes\eta=\xi\otimes f\eta'=\xi f\otimes\eta'=0$ in the group $H^1\otimes_{\mathcal{H}^0}\mathcal{H}^n$. $\square$

Corollary 1. *For any exact category $\mathcal{E}$, the graded ring of Ext groups $\mathcal{H}=\mathcal{H}(\mathcal{E})$ is a free graded ring generated by the $\mathcal{H}^0$-bimodule $\mathcal{H}^1$, that is the multiplication morphisms $\mathcal{H}^1\otimes_{\mathcal{H}^0}\cdots\otimes_{\mathcal{H}^0}\mathcal{H}^1\longrightarrow\mathcal{H}^n$ are isomorphisms.*

Proof: This follows by induction from Proposition 2 and Corollary 3.2. $\square$

Corollary 2. *For any small exact subcategory $\mathcal{E}\subset\mathcal{D}$ in a triangulated category $\mathcal{D}$, there exists a unique sequence of morphisms of functors of two variables $\mathcal{E}^{\mathrm{op}}\times\mathcal{E}\longrightarrow\mathcal{A}b$*

$$\phi^n=\phi^n_{\mathcal{E},\mathcal{D}}\colon\ \operatorname{Ext}^n_{\mathcal{E}}(X,Y[n])\ \longrightarrow\ \operatorname{Hom}_{\mathcal{D}}(X,Y[n]),\qquad n\geqslant 0,$$

with the following properties: the maps ϕ^n are compatible with the composition, the map ϕ^0 is the identity, and the map ϕ^1 sends the Yoneda class of an extension $Y\longrightarrow T\longrightarrow X$ in the exact category $\mathcal{E}$ to the third arrow $X\longrightarrow Y[1]$ of the corresponding distinguished triangle in the triangulated category $\mathcal{D}$.

Proof: Accorging to Corollary 1, one only has to check that the morphisms of functors ϕ^0 and ϕ^1 defined in this way are compatible with the left and the right multiplication of the Ext elements of degree 1 on those of degree 0, which is straightforward. Alternatively, one can use the universal property of effaceable homological functors (see [3]). $\square$

Corollary 3. *For any small exact subcategory $\mathcal{E}$ of a triangulated category $\mathcal{D}$, the morphisms of functors $\phi^0_{\mathcal{E},\mathcal{D}}$ and $\phi^1_{\mathcal{E},\mathcal{D}}$ are isomorphisms. Furthermore, if the morphism of functors $\phi^n_{\mathcal{E},\mathcal{D}}$ is an isomorphism for a certain $n\geqslant 0$, then all the maps $\phi^{n+1}_{\mathcal{E},\mathcal{D}}$ are injective. In particular, the maps ϕ^2 are injective; if all the maps ϕ^i are surjective for $i\leqslant n$, then all of them are isomorphisms.*

Proof: This follows immediately from Proposition 2. $\square$

In view of the existence of (canonically defined and compatible with the compositition) morphisms of functors ϕ^n, it is natural to expect that there should exist an exact functor $\Phi\colon \mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}$. Unfortunately, it seems to be impossible to construct such a functor in general: the notion of a triangulated category is just not good enough for that. The following statement can be proven, though.

Proposition 3. *Assume that a triangulated category $\mathcal{D}$ can be constructed by passing several times to triangulated subcategories and quotient categories from the homotopy category of an additive category $\mathcal{K}(\mathcal{A})$. Then for any exact subcategory $\mathcal{E} \subset \mathcal{D}$ there exists an exact functor $\Phi\colon \mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}$ whose restriction on the subcategory $\mathcal{E}$ is the identity. Any such exact functor Φ induces the morphisms of functors $\phi^n_{\mathcal{E},\mathcal{D}}$.*

Proof: At least two ways to prove this result are known: a construction of A. Beilinson [1] using filtered triangulated categories and an argument of A. Neeman [8] using a refinement of the definition of a triangulated category. □

It is clear that if a functor Φ does exist then it is an equivalence of (triangulated) categories if and only if all the morphisms of functors ϕ^n are isomorphisms and the subcategory $\mathcal{E}$ generates the triangulated category $\mathcal{D}$. (One says that a full subcategory $\mathcal{A} \subset \mathcal{D}$ generates a triangulated category $\mathcal{D}$ if any object of $\mathcal{D}$ can be constructed from the objects of $\mathcal{A}$ using the operations of shift and cone.)

3.4. T-structures. A *t-structure* on a triangulated category $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}^{\leqslant 0}$, $\mathcal{D}^{\geqslant 0} \subset \mathcal{D}$ satisfying the following two conditions:

(1) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all objects $X \in \mathcal{D}^{\leqslant 0}$ and $Y \in \mathcal{D}^{\geqslant 1}$;
(2) for any object $X \in \mathcal{D}$ there exists a distinguished triangle

$$X_{\leqslant 0} \longrightarrow X \longrightarrow X_{\geqslant 1} \longrightarrow X_{\leqslant 0}[1]$$

with some objects $X_{\leqslant 0} \in \mathcal{D}^{\leqslant 0}$ and $X_{\geqslant 1} \in \mathcal{D}^{\geqslant 1}$;

where $\mathcal{D}^{\leqslant a} = \mathcal{D}^{\leqslant 0}[-a]$ and $\mathcal{D}^{\geqslant a} = \mathcal{D}^{\geqslant 0}[-a]$ for any $a \in \mathbb{Z}$. Denote also $\mathcal{D}^a = \mathcal{D}^{\leqslant a} \cap \mathcal{D}^{\geqslant a}$ and $\mathcal{D}^{[a,b]} = \mathcal{D}^{\geqslant a} \cap \mathcal{D}^{\leqslant b}$ for any interval $[a, b] \subset \mathbb{Z}$. It follows from the condition (1) that the distinguished triangle in (2) is defined uniquely up to a unique isomorphism and depends as a functor on the object X.

The subcategory $\mathcal{D}^0$ is called the *heart* of a t-structure; it turns out to be an abelian category. More precisely, the subcategory $\mathcal{D}^0$ is an exact subcategory of $\mathcal{D}$ in the sense of the definition from the previous subsection and the induced exact category structure on it is the canonical exact category structure on an abelian category. The converse statement is also essentially true. This is the result of a characterization theorem from the paper [2] by A. Beilinson, J. Bernstein, and P. Deligne. In order to formulate it, we have to give some definitions first.

A t-structure is called *bounded* if one has $\mathcal{D} = \bigcup_{a\in\mathbb{Z}} \mathcal{D}^{\leqslant a} = \bigcup_{a\in\mathbb{Z}} \mathcal{D}^{\geqslant a}$, or, equivalently, the triangulated category $\mathcal{D}$ is generated by the subcategory $\mathcal{D}^0$. For any t-structure on $\mathcal{D}$, the subcategory $\mathcal{D}^{[a,b]}$ coincides with the minimal subcategory containing $\mathcal{D}^0[-i]$ for all $a \leqslant i \leqslant b$ and closed under extensions; for a t-structure of bounded type, the same is true for $\mathcal{D}^{\leqslant a}$ and $\mathcal{D}^{\geqslant a}$. It follows that a bounded t-structure is uniquely determined by its heart $\mathcal{D}^0 \subset \mathcal{D}$.

Let $\mathcal{A}$, $\mathcal{B} \subset \mathcal{D}$ be two sets of isomorphism classes of objects of a triangulated category $\mathcal{D}$. Denote by $\mathcal{A} * \mathcal{B}$ the set of all objects $X \in \mathcal{D}$ which can be included into a distinguished triangle $A \longrightarrow X \longrightarrow B \longrightarrow A[1]$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Lemma. *The associativity identity $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$ is satisfied for any three sets of isomorphism classes of objects $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \subset \mathcal{D}$.*

Proof: Apply the octahedron axiom. □

Proposition. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A} \subset \mathcal{D}$ be an additive subcategory such that $\operatorname{Hom}_{\mathcal{D}}(X, Y[n]) = 0$ for any X, $Y \in \mathcal{A}$ and $n < 0$. Then the following two conditions are equivalent:*

(a) *The category $\mathcal{A}$ is an abelian category and for any exact triple $X \longrightarrow Y \longrightarrow Z$ in it there exists a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in the triangulated category $\mathcal{D}$.*

(b) *One has $\mathcal{A} * \mathcal{A}[1] \subset \mathcal{A}[1] * \mathcal{A}$.*

The subcategory $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ if and only if it is closed under extensions, the triangulated category $\mathcal{D}$ is generated by $\mathcal{A}$ and the above two equivalent conditions hold.

Moreover, if a subcategory $\mathcal{A}$ satisfying (a-b) generates the triangulated category $\mathcal{D}$, then the minimal subcategory $\mathcal{M} = \bigcup_m \mathcal{A}^{*m}$ in $\mathcal{D}$ containing $\mathcal{A}$ and closed under extensions is the heart of a bounded t-structure on $\mathcal{D}$.

Proof: To prove that (a) implies (b), consider an object $Z \in \mathcal{A} * \mathcal{A}[1]$; then $Z = \operatorname{Cone}(X \longrightarrow Y)$, where X, $Y \in \mathcal{A}$. By (a), we have distinguished triangles

$$K \longrightarrow X \longrightarrow H \longrightarrow K[1]$$
$$H \longrightarrow Y \longrightarrow C \longrightarrow H[1]$$

with K, H, $C \in \mathcal{A}$, which leads to the desired result by the octahedron axiom. To verifty the inverse implication, consider a morphism $f\colon X \longrightarrow Y$ in $\mathcal{A}$; by (b), we have exact triangles

$$X \longrightarrow Y \longrightarrow \operatorname{Cone}(f) \longrightarrow X[1]$$
$$K[1] \longrightarrow \operatorname{Cone}(f) \longrightarrow C \longrightarrow K[2]$$

with K, $C \in \mathcal{M}$. Then $K = \operatorname{Ker}(f)$ and $C = \operatorname{Coker}(f)$. Using the octahedron axiom, one can verify the isomorphism $\operatorname{Coker}(\ker(f)) = \operatorname{Ker}(\operatorname{coker}(f))$.

Assume that $\mathcal{A}$ is the heart of a t-structure on $\mathcal{D}$; then the condition $\operatorname{Hom}_{\mathcal{D}}(X, Y[n]) = 0$ for any X, $Y \in \mathcal{A}$ and $n < 0$ follows immediately from (1). Furthermore, $X \in \mathcal{D}^{\leqslant 0}$ iff $\operatorname{Hom}(X, \mathcal{D}^{\geqslant 1}) = 0$ and $Y \in \mathcal{D}^{\geqslant 1}$ iff $\operatorname{Hom}(\mathcal{D}^{\leqslant 0}, Y) = 0$. Therefore, the subcategories $\mathcal{D}^{\leqslant a}$ and $\mathcal{D}^{\geqslant a}$ are closed under extensions; it follows that $\mathcal{D}^{[a,b]} = \mathcal{D}^{\geqslant a} \cap \mathcal{D}^{\leqslant b}$ are also closed. Using the octahedron axiom, one can show that the cutting functors $X \longmapsto X_{\leqslant c}$ and $X \longmapsto X_{\geqslant d}$ preserve $\mathcal{D}^{[a,b]}$ and commute. Now $X \in \mathcal{D}^0 * \mathcal{D}^0[1] \subset \mathcal{D}^{[-1,0]}$ implies $X_{\leqslant -1} \in \mathcal{D}^{-1}$ and $X_{\geqslant 0} \in \mathcal{D}^0$, which proves (b).

Conversely, assume that $\mathcal{A}$ is closed under extensions, generates the triangulated category $\mathcal{D}$ and satisfies (b). Put $\mathcal{D}^{[a,b]}$ to be the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{A}[-i]$ for $a \leqslant i \leqslant b$ and closed under extensions; analogously for $\mathcal{D}^{\leqslant a}$ and $\mathcal{D}^{\geqslant a}$. It follows immediately from the condition that $\mathrm{Hom}_{\mathcal{D}}(X, Y[n]) = 0$ for any X, $Y \in \mathcal{A}$ and $n < 0$ that the subcategories $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 1}$ satisfy (1). Note that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, \mathcal{A}[-i]) = 0$ implies $\mathcal{A} * \mathcal{A}[i+1] = \mathcal{A} \oplus \mathcal{A}[i+1]$ for $i \geqslant 1$; therefore, we have $\mathcal{A} * \mathcal{A}[n] \subset \mathcal{A}[n] * \mathcal{A}$ for all $n > 0$. By associativity of the $*$-product and using the condition that $\mathcal{A}$ is closed under extensions, we get $\mathcal{D}^{[a,b]} = \mathcal{A}[-a] * \mathcal{A}[-a-1] * \cdots * \mathcal{A}[-b]$. In particular, $\mathcal{D}^{[a,b]} = \mathcal{D}^{[a,0]} * \mathcal{D}^{[1,b]}$ for $[0,1] \subset [a,b] \subset \mathbb{Z}$; since $\mathcal{A}$ generates $\mathcal{D}$, this proves (2). It remains to check that $\mathcal{D}^{[a,b]} = \mathcal{D}^{\geqslant a} \cap \mathcal{D}^{\leqslant b}$. Let $X \in \{Y_c\} * \{Y_{c+1}\} * \cdots * \{Y_d\}$, where $Y_i \in \mathcal{A}[-i]$; then $X \in \mathcal{D}^{\geqslant a}$ implies $\mathrm{Hom}(Y_c, X) = 0$ for $c < a$, whence $X \oplus Y_c[1] \in \{Y_{c+1}\} * \cdots * \{Y_d\}$ and $Y_c = 0$.

To check the last assertion, notice that $\mathcal{M}$ satisfies (b) whenever $\mathcal{A}$ does, which follows from associativity of the $*$-product. $\square$

Corollary. *Let $\mathcal{A} \subset \mathcal{D}$ be a subcategory generating the triangulated category $\mathcal{D}$ and such that $\mathrm{Hom}_{\mathcal{D}}(X, Y[n]) = 0$ for any X, $Y \in \mathcal{A}$ and $n < 0$. In this case, if $\mathcal{A}$ is a semisimple abelian category, then the category $\mathcal{M} = \bigcup_n \mathcal{A}^{*n} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$.* $\square$

3.5. Silly filtrations. The following results comment on the condition that the morphisms $\phi^n_{\mathcal{E},\mathcal{D}}$ are surjective (or, equivalently, isomorphisms) for an exact category $\mathcal{E} \subset \mathcal{D}$. It turns out that one does not even have to assume that the spaces $\mathrm{Hom}_{\mathcal{D}}(X, Y[n])$ with negative n are zero in order to discuss this condition.

Proposition 1. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{A} \subset \mathcal{D}$ be a full subcategory, and $\mathcal{M} = \bigcup_m \mathcal{A}^{*m} \subset \mathcal{D}$ be the minimal full subcategory containing $\mathcal{A}$ and closed under extensions. Then for any $n \geqslant 2$ the following three conditions are equivalent:*

(a) *Any morphism $X \longrightarrow Y[k]$ with $2 \leqslant k \leqslant n$ between two objects X, $Y \in \mathcal{M}$ can be presented as the composition of a chain of morphisms $Z_i \longrightarrow Z_{i-1}[1]$ with $Z_i \in \mathcal{M}$, $Z_0 = Y$, and $Z_k = X$.*

(b) *Any morphism $A \longrightarrow Y[k]$ with $2 \leqslant k \leqslant n$ between two objects $A \in \mathcal{A}$ and $Y \in \mathcal{M}$ can be presented as the composition of a morphism $A \longrightarrow Z[1]$ and a morphism $Z[1] \longrightarrow Y[k]$ with $Z \in \mathcal{M}$.*

(c) *Any morphism $A \longrightarrow Y[k]$ with $2 \leqslant k \leqslant n$ between two objects $A \in \mathcal{A}$ and $Y \in \mathcal{M}$ can be presented as the composition of a morphism $A \longrightarrow Z[k-1]$ and a morphism $Z[k-1] \longrightarrow Y[k]$ with $Z \in \mathcal{M}$.*

Furthermore, either of the next two conditions implies the previous three:

(d) *Any morphism $A \longrightarrow B[k]$ with $2 \leqslant k \leqslant n$ between two objects A, $B \in \mathcal{A}$ can be presented as the composition of a morphism $A \longrightarrow Z[1]$ and a morphism*

$Z[1] \longrightarrow B[k]$ *such that* $Z \in \mathcal{M}$ *and a cone of the morphism* $A \longrightarrow Z[1]$ *belongs to* $\mathcal{A}[1]$.

(e) *Any morphism* $A \longrightarrow Y[k]$ *with* $2 \leqslant k \leqslant n$ *between two objects* $A, B \in \mathcal{A}$ *can be presented as the composition of a morphism* $A \longrightarrow Z[k-1]$ *and a morphism* $Z[k-1] \longrightarrow B[k]$ *such that* $Z \in \mathcal{M}$ *and a cone of the morphism* $Z[k-1] \longrightarrow B[k]$ *belongs to* $\mathcal{A}[k]$.

Proof: A simple induction in n proves (c) $\Longrightarrow$ (b); including the equivalence of (a-c) for $k \leqslant n-1$ in the induction assumption, we obtain (b) $\Longrightarrow$ (c). It is obvious that (a) implies (b) and (c).

To prove (c) $\Longrightarrow$ (a), reformulate (c) as the assertion that for any morphism $A \longrightarrow Y[k]$ with $A \in \mathcal{A}$ and $Y \in \mathcal{M}$ there exists a distinguished triangle $Y[k] \longrightarrow W[k] \longrightarrow Z[k] \longrightarrow Y[k+1]$ with $Z \in \mathcal{M}$ such that the composition $A \longrightarrow Y[k] \longrightarrow W[k]$ vanishes. The condition (a) can be restated as the similar assertion with $A \in \mathcal{A}$ replaced by $X \in \mathcal{M}$. Let $A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow A_1[1]$ be a distinguished triangle in $\mathcal{D}$. Assuming that both objects A_1 and A_2 have the above property for any morphism $A_i \longrightarrow Y[k]$ with $Y \in \mathcal{M}$, we will show that the object A has the same property. Let $A \longrightarrow Y[k]$ be any morphism; consider the composition $A_1 \longrightarrow A \longrightarrow Y[k]$. Then there exists a distinguished triangle $Y[k] \longrightarrow W_1[k] \longrightarrow Z_1[k]$ with $Z_1 \in \mathcal{M}$ such that the composition $A_1 \longrightarrow A \longrightarrow Y[k] \longrightarrow W_1[k]$ vanishes. Hence the composition $A \longrightarrow Y[k] \longrightarrow W_1[k]$ factorizes through the morphism $A \longrightarrow A_2$. Consider the morphism $A_1 \longrightarrow W_1[k]$ that we have obtained. There exists a distinguished triangle $W_1[k] \longrightarrow W[k] \longrightarrow Z_2[k] \longrightarrow W_1[k+1]$ with $Z_2 \in \mathcal{M}$ such that the composition $A_2 \longrightarrow W_1[k] \longrightarrow W[k]$ vanishes. Then the composition $A \longrightarrow Y[k] \longrightarrow W_1[k] \longrightarrow W[k]$ also vanishes. By the octahedron axiom, the cone of the composition $Y[k] \longrightarrow W_1[k] \longrightarrow W[k]$ is an extension of $Z_2[k]$ and $Z_1[k]$, so it belongs to $\mathcal{M}[k]$.

To prove (d) $\Longrightarrow$ (b), reformulate (d) as the assertion that for any morphism $A \longrightarrow B[k]$ with $A, B \in \mathcal{A}$ there exists a distinguished triangle $Z \longrightarrow C \longrightarrow A \longrightarrow W[1]$ with $Z \in \mathcal{M}$ and $C \in \mathcal{A}$ such that the composition $C \longrightarrow A \longrightarrow B[k]$ vanishes. Then argue as above. Now we know that (d) $\Longrightarrow$ (a), the implication (e) $\Longrightarrow$ (a) follows by duality. $\square$

Proposition 2. *Let* $\mathcal{D}$ *be a triangulated category and* $\mathcal{M} \subset \mathcal{D}$ *be a full subcategory closed under extensions. Then any morphism* $X \longrightarrow Y[n]$ *with* $n \geqslant 2$ *between two objects* $X, Y \in \mathcal{D}$ *can be presented as the composition of a chain of morphisms* $Z_j \longrightarrow Z_{j-1}[1]$ *with* $Z_j \in \mathcal{M}$, $Z_0 = Y$, *and* $Z_n = X$ *if and only if the following two conditions hold:*

(i) *One has* $\mathcal{M}[n] * \mathcal{M} \subset \mathcal{M} * \mathcal{M}[1] * \cdots * \mathcal{M}[n]$ *for any* $n \geqslant 0$.

(ii) *Put* $\mathcal{M}^{[a,b]} = \mathcal{M}[-b] * \cdots * \mathcal{M}[-a]$ *for any* $a \leqslant b$, $\mathcal{M}^{\leqslant b} = \bigcup_a \mathcal{M}^{[a,b]}$, *and* $\mathcal{M}^{\geqslant a} = \bigcup_b \mathcal{M}^{[a,b]}$. *Then one should have* $\mathcal{M}^{\geqslant a} \cap \mathcal{M}^{\leqslant b} \subset (\mathcal{M}^{[a,b]})^{\mathrm{sat}}_{\mathcal{D}}$.

Besides, the triangulated subcategory generated by $\mathcal{M}$ *in* $\mathcal{D}$ *coincides with* $\bigcup_{a,b} \mathcal{M}^{[a,b]}$ *in this case.*

The key ideas of the proof are summarized in the following Lemma.

Lemma. *Let $\mathcal{D}$ be a triangulated category. Then*

(1) *Suppose that a morphism $f\colon A \longrightarrow B$ in $\mathcal{D}$ factorizes through an object E. Then $\{B\} * \{A[1]\} \ni \operatorname{Cone}(f) \in \{E\} * \{A[1]\} * \{B\} * \{E[1]\}$.*

(2) *Suppose that the cones of two morphisms $A \longrightarrow B$ and $C \longrightarrow D$ are isomorphic in $\mathcal{D}$. Then both morphisms factorize through an object E belonging to the intersection $\{A\} * \{D\} \cap \{C\} * \{B\}$.*

(3) *Suppose that an object W is a direct summand of an extension of objects A and D and also a direct summand of an extension of objects C and B. Then, in particular, there are morphisms $A \longrightarrow W$ and $W \longrightarrow B$; suppose moreover, that their composition factorizes through an object E. Then W is a direct summand of an object in $\{C\} * \{E\} * \{E\} * \{D\}$.*

Proof. Part (1): Consider the distinguished triangles

$$A \longrightarrow E \longrightarrow D \longrightarrow A[1]$$
$$B[-1] \longrightarrow C \longrightarrow E \longrightarrow B.$$

By the octahedron axiom, there are also distinguished triangles

$$A \longrightarrow B \longrightarrow K \longrightarrow A[1]$$
$$C \longrightarrow D \longrightarrow K \longrightarrow C[1].$$

So $D \in \{E\} * \{A[1]\}$ and $\mathcal{C}[1] \in \{B\} * \{E[1]\}$, hence

$$\{B\} * \{A[1]\} \ni K \in \{D\} * \{C[1]\} \subset \{E\} * \{A[1]\} * \{B\} * \{E[1]\}.$$

Part (2): By the octahedron axiom, there exist four distinguished triangles as above and the morphisms $A \longrightarrow B$ and $C \longrightarrow D$ factorize through E. Clearly, $E \in \{A\} * \{D\} \cap \{C\} * \{B\}$.

Part (3): By our assumption, there exist objects S, $T \in \mathcal{D}$ and distinguished triangles

$$A \longrightarrow S \oplus W \longrightarrow D \longrightarrow A[1]$$
$$B[-1] \longrightarrow C \longrightarrow T \oplus W \longrightarrow B.$$

Combining the given morphisms with appropriate signs, we get a pair of morphisms $A \longrightarrow S \oplus T \oplus W \oplus E \longrightarrow B$ with zero composition. There are distinguished triangles

$$A \longrightarrow S \oplus T \oplus W \oplus E \longrightarrow T \oplus K \longrightarrow A[1]$$
$$E \longrightarrow K \longrightarrow D \longrightarrow E[1]$$

and

$$S \oplus L \longrightarrow S \oplus T \oplus W \oplus E \longrightarrow B \longrightarrow S[1] \oplus L[1]$$
$$C \longrightarrow L \longrightarrow E \longrightarrow C[1]$$

and a pair of morphisms $A \longrightarrow S \oplus L$, $T \oplus K \longrightarrow B$ forming a morphism of distinguished triangles with a common vertex $S \oplus T \oplus W \oplus E$. Applying the octahedron axiom, we obtain distinguished triangles

$$A \longrightarrow S \oplus L \longrightarrow F \longrightarrow A[1]$$
$$F \longrightarrow T \oplus K \longrightarrow B \longrightarrow F[1]$$

Finally, there is a morphism $T \oplus K \longrightarrow S[1] \oplus L[1]$ that can be obtained as the composition $T \oplus K \longrightarrow A[1] \longrightarrow S[1] \oplus L[1]$ or $T \oplus K \longrightarrow B \longrightarrow S[1] \oplus L[1]$. This morphism can be included in the distinguished triangle

$$T[-1] \oplus K[-1] \longrightarrow S \oplus L \longrightarrow S \oplus T \oplus W \oplus E \oplus F \longrightarrow T \oplus K,$$

where the morphisms $S \longrightarrow S \oplus T \oplus W \oplus E \longrightarrow T$ are just the obvious embedding and projection, while the components $L \longrightarrow S$ and $T \longrightarrow K$ vanish. Hence we obtain the distinguished triangle

$$K[-1] \longrightarrow L \longrightarrow W \oplus E \oplus F \longrightarrow K.$$

Now $L \in \{C\} * \{E\}$ and $K \in \{E\} * \{D\}$, hence $W \oplus E \oplus F \in \{L\} * \{K\} \subset \{C\} * \{E\} * \{E\} * \{D\}$. $\square$

Proof of Proposition 2: First let us check the implication "if". Let X, $Y \in \mathcal{M}$, and $f\colon X \longrightarrow Y[n+1]$ be a morphism in $\mathcal{D}$, where $n \geqslant 1$. Consider the object $Z = \operatorname{Cone}(f[-1])$; then there is a distinguished triangle

$$Y[n] \longrightarrow Z \longrightarrow X \longrightarrow Y[n+1],$$

so $Z \in \mathcal{M}[n] * \mathcal{M}$. By (i), there exists a distinguished triangle

$$U \longrightarrow Z \longrightarrow V \longrightarrow U[1],$$

where $U \in \mathcal{M}$ and $V \in \mathcal{M}[1] * \cdots * \mathcal{M}[n]$. So Z is also a cone of the morphism $V[-1] \longrightarrow U$. According to part (2) of the above Lemma, the morphism $f[-1]$ factorizes through an object $E \in \{X[-1]\} * \{U\} \cap \{V[-1]\} * \{Y[n]\} \subset (\mathcal{M}[-1] * \mathcal{M}) \cap (\mathcal{M} * \cdots * \mathcal{M}[n])$. By (ii), $E \in \mathcal{M}_{\mathcal{D}}^{\mathrm{sat}}$. Now the morphism f factorizes through $E[1]$, thus it also factorizes through an object of $\mathcal{M}[1]$.

It is not difficult to prove by induction that the minimal subcategory of $\mathcal{D}$ containing $\mathcal{M}$, … , $\mathcal{M}[n]$ and closed under extensions coincides with $\mathcal{M} * \cdots * \mathcal{M}[n]$ for any $n \geqslant 0$ provided that (i) holds. Specifically, one proceeds by induction in n, and then in the number of factors $\mathcal{M}[i]$ with the largest possible value $i = n$ in an iterated extension. This proves the last assertion of the Proposition.

Now let us prove "only if". To verify (i), consider an object $Z \in \mathcal{M}[n] * \mathcal{M}$. As above, we have $Z = \operatorname{Cone}(f[-1])$ for some morphism $f\colon X \longrightarrow Y[n+1]$, where X, $Y \in \mathcal{M}$. Decompose the morphism f as $X \longrightarrow E[1] \longrightarrow Y[n+1]$, where $E \in \mathcal{M}$. By part (1) of Lemma, $Z \in \{E\} * \{X\} * \{Y[n]\} * \{E[1]\} \subset \mathcal{M} * \mathcal{M} * \mathcal{M}[n] * \mathcal{M}[1]$. A simple induction in n finishes the proof of (i).

To prove (ii), let us first check that

$$(\mathcal{M}[-1]*\cdots*\mathcal{M}[n-1])^{\mathrm{sat}}_{\mathcal{D}} \cap (\mathcal{M}*\cdots*\mathcal{M}[n])^{\mathrm{sat}}_{\mathcal{D}} \subset (\mathcal{M}*\cdots*\mathcal{M}[n-1])^{\mathrm{sat}}_{\mathcal{D}}.$$

Let W be an object in the intersection; then W is a direct summand of an extension of $A = X[-1] \in \mathcal{M}[-1]$ and $D \in \mathcal{M}*\cdots*\mathcal{M}[n-1]$ and also a direct summand of an extension of $B \in \mathcal{M}*\cdots*\mathcal{M}[n-1]$ and $D = Y[n] \in \mathcal{M}[n]$. The morphism $A \longrightarrow D$ factorizes through an object $E \in \mathcal{M}$. According to part (3) of Lemma, W is a direct summand of an object from $\{C\}*\{E\}*\{E\}*\{D\} \subset \mathcal{M}*\cdots*\mathcal{M}[n-1]* \mathcal{M}*\mathcal{M}*\mathcal{M}*\cdots*\mathcal{M}[n-1]$. Thus $W \in (\mathcal{M}*\cdots*\mathcal{M}[n-1])^{\mathrm{sat}}_{\mathcal{D}}$. A simple induction allows to deduce (ii). $\square$

In some cases the condition (ii) is satisfied automatically. First of all, it always holds when $\mathcal{M}$ is the heart of a t-structure on $\mathcal{D}$ (see below). Secondly, it holds in the filtered case described in the next statement.

Proposition 3. *Let $\mathcal{D}$ be a triangulated category endowed with a sequence of triangulated subcategories $\mathcal{D}_i \subset \mathcal{D}$, $i \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}}(X,Y) = 0$ for any $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_j$ with $i > j$. Let $\mathcal{M}_i \subset \mathcal{D}_i$ be full subcategories closed under extensions. Then the minimal full subcategory containing all $\mathcal{M}_i$ and closed under extensions $\mathcal{M} = \bigcup_{i\leqslant j} \mathcal{M}_j*\cdots*\mathcal{M}_i \subset \mathcal{D}$ satisfies the condition (ii) of Proposition 2 for $a=b$ if and only if all the subcategories $\mathcal{M}_i \subset \mathcal{D}_i$ do. Consequently, if $\mathcal{M}_i \subset \mathcal{D}_i$ satisfy (ii) for $a=b$ and $\mathcal{M} \subset \mathcal{D}$ satisfies (i), then $\mathcal{M}$ also satisfies (ii).*

Furthermore, if all $\mathcal{M}_i \subset \mathcal{D}_i$ satisfy the following stronger version of condition (ii), then so does $\mathcal{M} \subset \mathcal{D}$:

(ii′) *For any $a \leqslant b$ and $c \leqslant d$, the intersection of $\mathcal{M}^{[a,b]}$ with the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[-c], \dots, \mathcal{M}[-d]$ and closed under extensions is equal to $\mathcal{M}^{[a,b]\cap[c,d]}$, where $[a,b]\cap[c,d]$ is the intersection of the segments $[a,b]$ and $[c,d]$ in $\mathbb{Z}$.*

Proof: To verify the first assertion, if suffices to consider the associated graded object functors $\mathcal{D} \longrightarrow \mathcal{D}_i$. The second one follows from Proposition 2 and its proof in the "if" direction, which only uses (ii) for $a=b$. To prove the last assertion, let us show that the intersection of $\mathcal{M}[-1]*\cdots*\mathcal{M}[n]$ with the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[k]$ for $k \geqslant 0$ and closed under extensions coincides with $\mathcal{M}*\cdots* \mathcal{M}[n]$. Let $W \in \mathcal{M}[-1]*\cdots*\mathcal{M}[n]$; then there exists a distinguished triangle

$$V[-1] \longrightarrow U \longrightarrow W \longrightarrow V$$

with $U \in \mathcal{M}[-1]*\mathcal{M}$ and $V \in \mathcal{M}[1]*\cdots*\mathcal{M}[n]$. Assume that W belongs to the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[k]$ for $k \geqslant 0$ and closed under extensions; then, since $U \in \{V[-1]\}*\{W\}$, so does U. By our assumption, the images U_i of the object U under the functors of associated graded objects $\mathcal{D} \longrightarrow \mathcal{D}_i$ belong to $\mathcal{M}_i$. Thus $U \in \mathcal{M}$ and $W \in \mathcal{M}*\cdots*\mathcal{M}[n]$. The dual argument shows that the intersection of $\mathcal{M}[-n] * \cdots * \mathcal{M}[1]$ with the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[k]$ for $k \leqslant 0$ and closed under extensions coincides with $\mathcal{M}[-n] * \cdots * \mathcal{M}$. $\square$

Example: The following counterexample shows, however, that the condition (ii) does not always hold for an exact subcategory $\mathcal{M} \subset \mathcal{D}$, and moreover, (i) does not imply (ii) in this case. Let $\mathcal{B}$ be the abelian category of 3-term complexes of vector spaces $V^{(1)} \longrightarrow V^{(2)} \longrightarrow V^{(3)}$ (the composition of the two arrows must be zero). There are 5 indecomposable objects in this category, denoted E_1, E_2, E_3, E_{12}, E_{23} (see Example 1.5). Let $\mathcal{M} \subset \mathcal{B}$ be the full additive subcategory whose objects are the direct sums of all the indecomposables except E_2. Set $\mathcal{D} = \mathcal{D}^b(\mathcal{B})$. Then $\mathcal{M}$ is closed under extensions in $\mathcal{B}$ and $\mathcal{D}$ and inherits a trivial exact category structure, i. e., all the exact triples in $\mathcal{M}$ are split. One can check that $\mathcal{M}$ generates $\mathcal{D}$ and satisfies (i), but not (ii).

Furthermore, under the assumptions similar to those mentioned above the condition (i) allows a very simple reformulation.

Proposition 4. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{M} \subset \mathcal{D}$ be a full subcategory closed under extensions. Assume that we are in one of the following two situations:*

(a) *$\mathcal{M}$ is the heart of a bounded t-structure on $\mathcal{D}$.*

(b) *$\mathcal{D}$ is endowed with a sequence of triangulated subcategories $\mathcal{D}_i \subset \mathcal{D}$, $i \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for any $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_j$ with $i > j$. Each $\mathcal{D}_i$ is equivalent to the bounded derived category $\mathcal{D}^b(\mathcal{M}_i)$ of an exact category $\mathcal{M}_i$. The subcategory $\mathcal{M} \subset \mathcal{D}$ is the minimal full subcategory containing all $\mathcal{M}_i$ and closed under extensions, $\mathcal{M} = \bigcup_{i \leqslant j} \mathcal{M}_j * \cdots * \mathcal{M}_i$.*

Then the condition (ii) of Proposition 2 always holds and (i) is equivalent to its following weaker form:

(i′) *$\mathcal{D} = \bigcup_{a,b} \mathcal{M}^{[a,b]}$, where $\mathcal{M}^{[a,b]} = \mathcal{M}[-b] * \cdots * \mathcal{M}[-a]$ for any $a \leqslant b$.*

Proof: In the case (b), according to Proposition 3.1.3, $\mathcal{D}_i = \mathcal{D}^b(\mathcal{M}_i^{\mathrm{ss}})$. Clearly, the condition (ii) does not become weaker when one replaces $\mathcal{M}_i$ with $\mathcal{M}_i^{\mathrm{ss}}$ and $\mathcal{M}$ with $\mathcal{M}^{\mathrm{ss}}$. According to Proposition 2, the sum total of the conditions (i) and (ii) does not change, either. So we can assume that $\mathcal{M}_i$ are semi-saturated. With this assumption, in both cases (a) and (b) we will prove the condition (ii′) of Proposition 3. Clearly, it implies both (ii) and (i′) $\Longrightarrow$ (i).

According to Proposition 3, in the case (b) it suffices to consider the case when $\mathcal{D} = \mathcal{D}_i = \mathcal{D}^b(\mathcal{M}_i)$ and $\mathcal{M} = \mathcal{M}_i$.

As demonstrated in the proof of Proposition 3, it suffices to show that the intersection of $\mathcal{M}[-1] * \mathcal{M}$ with the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[k]$ for $k \geqslant 0$ and closed under extensions coincides with $\mathcal{M}$. Let $U \in \mathcal{M}[-1] * \mathcal{M}$; then there is a distinguished triangle

$$X[-1] \longrightarrow Y[-1] \longrightarrow U \longrightarrow X$$

with X, $Y \in \mathcal{M}$. Assume that U belongs to the minimal full subcategory of $\mathcal{D}$, containing $\mathcal{M}[k]$ for $k \geqslant 0$ and closed under extensions. In the case (a), we have $U \in \mathcal{D}^{\leqslant 0}$, and it follows immediately from the proof of Proposition 3.4 (b) $\Longrightarrow$ (a) that the morphism $X \longrightarrow Y$ is surjective. In the case (b), using, e. g., Corollary 3.1.2 and Proposition 2.4.1, we can conclude that the morphism $X \longrightarrow Y$ is an admissible epimorphism. Another way is to use Proposition 3.1.5 and Corollary 2.2.3. Alternatively, one can use Proposition 2 in order to conclude that $U \in \mathcal{M}^{\mathrm{sat}}_{\mathcal{D}}$, and then deduce the assertion that $X \longrightarrow Y$ is an admissible epimorphism in $\mathcal{M}$ from the facts that it is such in $\mathcal{M}^{\mathrm{sat}}$ and $\mathcal{M} = \mathcal{M}^{\mathrm{ss}}$. In both cases it follows that $U \in \mathcal{M}$. $\square$

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