KOSZULITY AND T-STRUCTURES

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Abstract. The aim of this paper is to prove the following theorem. Let D be a small triangulated category and $(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ be a t-structure on D with the core $\mathsf{E} = \mathsf{D}^{\leqslant 0} \cap \mathsf{D}^{\geqslant 0}$; so E is an abelian category. Suppose that every object of E has a finite length and let I_α denote the irreducible objects of E. Consider the "graded ring with many objects" $A = \bigoplus_n \bigoplus_{\alpha,\beta} \operatorname{Hom}_{\mathsf{D}}(I_\alpha, I_\beta[n])$. Assume that the ring A is Koszul. Then the t-structure $(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ is "of the derived type", i. e., for any objects X, $Y \in \mathsf{E}$ the natural maps $\operatorname{Ext}^n_{\mathsf{E}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathsf{D}}(X, Y[n])$ are isomorphisms for all n.

1. Introduction

1.1. The following theorem was proven in the paper [3].

Theorem 1.1. *Let C be a coaugmented coalgebra over a field k and* $\operatorname{Nilp} C = \bigcup_m \ker(C \to (C/k)^{\otimes m+1})$ *be its maximal conilpotent subcoalgebra. Assume that the cohomology algebra $H^*(C) = \bigoplus_n \operatorname{Ext}^n_C(k,k)$ is Koszul. Then the embedding $\operatorname{Nilp} C \longrightarrow C$ induces a cohomology isomorphism $H^*(\operatorname{Nilp} C) \simeq H^*(C)$.* □

Furthermore, this theorem has a version with an augmented algebra R replacing the coaugmented coalgebra C.

Theorem 1.2. *Let R be an augmented algebra over a field k and $R\hat{}$ be the coalgebra of its pronilpotent completion, defined as the inductive limit $\varinjlim_I (R/I)^*$ of the coalgebras dual to the finite-dimensional augmented quotient algebras R/I of R with nilpotent augmentation ideals. Assume that the cohomology algebra $H^*(R) = \bigoplus_n \operatorname{Ext}^n_R(k,k)$ is Koszul. Then the natural morphism of the cohomology algebras $H^*(R\hat{}) \longrightarrow H^*(R)$ is an isomorphism.* □

Theorems 1.1 and 1.2 are deduced from the following result about the cohomology of conilpotent coalgebras proven in [4].

Theorem 1.3. *Let E be a conilpotent coaugmented coalgebra, i. e.,* $\operatorname{Nilp} E = E$. *Assume that*

1. *the quadratic part $\mathrm{q}H^*(E)$ of the graded algebra $H^*(E)$ is Koszul; and*
2. *the morphism of graded algebras $\mathrm{q}H^*(E) \longrightarrow H^*(E)$ is an isomorphism in degree 2 and a monomorphism in degree 3.*

Then the graded algebra $H^*(E)$ is quadratic (and consequently, Koszul). □

Other versions of Theorem 1.1 were briefly discussed in [3], and a generalization to triangulated categories was announced. The goal of this paper is to prove this generalization.

1.2. The idea of the generalization to triangulated categories can be described as follows. Starting with a coaugmented coalgebra C, consider the conilpotent coalgebra $E = \operatorname{Nilp} C$ and the abelian category E of finite-dimensional left E-comodules. Consider the derived category of left C-comodules $\mathsf{D}(C\text{–}\mathsf{comod})$, and set D to denote the full triangulated subcategory of $\mathsf{D}(C\text{–}\mathsf{comod})$ generated by $\mathsf{E} \subset C\text{–}\mathsf{comod}$. Then E is the core of a t-structure on D.

Analogously, starting with an augmented algebra R, consider the conilpotent coalgebra $E = R^\frown$ and the abelian category E of finite-dimensional left E-comodules, that is finite-dimensional nilpotent left R-modules. Consider the derived category of left R-modules $\mathsf{D}(R\text{–}\mathsf{mod})$, and set D to denote the full triangulated subcategory of $\mathsf{D}(R\text{–}\mathsf{mod})$ generated by $\mathsf{E} \subset R\text{–}\mathsf{mod}$. Then E is the core of a t-structure on D.

In both cases, the assertions of Theorems 1.1 and 1.2 claim that the natural morphism $\operatorname{Ext}^n_{\mathsf{E}}(k,k) \longrightarrow \operatorname{Hom}_{\mathsf{D}}(k,k[n])$ is an isomorphism for all n, provided that the graded algebra $\bigoplus_n \operatorname{Hom}_{\mathsf{D}}(k,k[n])$ is Koszul. Here the trivial C-comodule or R-module k is the only irreducible object in E.

1.3. Let $S = \{\alpha\}$ be a set of indices. A "graded ring with many objects" (or *big graded ring*) A is a collection of abelian groups $A^n_{\alpha\beta}$ endowed with the multiplication maps $A^p_{\alpha\beta} \times A^q_{\beta\gamma} \longrightarrow A^{p+q}_{\alpha\gamma}$ and the unit elements $1_\alpha \in A^0_{\alpha\alpha}$ satisfying the obvious associativity and unit axioms. We will assume that $A^n_{\alpha\beta} = 0$ for $n < 0$ or $n = 0$ and $\alpha \neq \beta$, and that the rings $A^0_{\alpha\alpha}$ are semisimple.

The following theorem is the main result of this paper.

Theorem 1.4. *Let E be the core of a t-structure on a small triangulated category D. Suppose that every object of E has a finite length and let I_α be the irredicibles in E. Assume that the "graded ring with many objects" A with the components $A^n_{\alpha,\beta} = \operatorname{Hom}_{\mathsf{D}}(I_\alpha, I_\beta[n])$ is Koszul. Then for any two objects X, $Y \in \mathsf{E}$ and $n \geqslant 0$ the natural map $\operatorname{Ext}^n_{\mathsf{E}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathsf{D}}(X, Y[n])$ is an isomorphism.*

The above theorem is deduced from the following result about the Ext's between irreducible objects in abelian categories.

Theorem 1.5. *Let E be a small abelian category such that every object of E has a finite length; let I_α be the irreducibles of E. Consider the "graded ring with many objects" B with the components $B^n_{\alpha,\beta} = \operatorname{Ext}^n_{\mathsf{E}}(I_\alpha, I_\beta)$. Assume that*

(1) *the quadratic part $\mathrm{q}B$ of the big graded ring B is Koszul; and*
(2) *the morphism of big graded rings $\mathrm{q}B \longrightarrow B$ is an isomorphism in the degree $n = 2$ and a monomorphism in the degree $n = 3$.*

Then the big graded ring B is quadratic (and consequently, Koszul).

Actually, the following slightly stronger form of Theorem 1.4, generalizing both Theorems 1.4 and 1.5, can be obtained from Theorem 1.5.

Theorem 1.6. *Let E be the core of a t-structure on a small triangulated category D. Suppose that every object of E has a finite length and let I_α be the irredicibles in E. Consider the "graded rings with many objects" A and B with the components $A^n_{\alpha,\beta} = \mathrm{Ext}^n_{\mathsf{E}}(I_\alpha, I_\beta)$ and $B^n_{\alpha,\beta} = \mathrm{Hom}_{\mathsf{D}}(I_\alpha, I_\beta[n])$. Then whenever the big graded ring B satisfies the assumptions (1) and (2) of Theorem 1.5, the natural morphism of big graded rings $A \longrightarrow B$ induces an isomorphism $A \simeq \mathrm{q}B$.*

1.4. Let us clarify the following point, which otherwise might be a source of confusion. It is well-known [1] that for a t-structure with the core E on a triangulated category D the natural maps $\mathrm{Ext}^n_{\mathsf{E}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathsf{D}}(X,Y[n])$ are all isomorphisms for X, $Y \in \mathsf{E}$, $n \geqslant 0$ if and only if any element in $\mathrm{Hom}_{\mathsf{D}}(X,Y[n])$ can be decomposed into a product of n elements of the groups $\mathrm{Hom}_{\mathsf{D}}(U,V[1])$ with U, $V \in \mathsf{E}$. However, this is only true because one considers the one-generation condition for X, Y running over *all* objects of E and not just the irreducible objects. The Koszulity condition in Theorem 1.4 is much stronger than a one-generation condition, but it is applied to a much smaller algebra of Hom's between the irreducible objects of E.

1.5. For example, consider a discrete group Γ and set C to be the coalgebra of (functions on) its proalgebraic completion over a field k. Then the category E of finite-dimensional representations of Γ over k is isomorphic to the category of finite-dimensional left C-comodules. Let D denote the full triangulated subcategory of the derived category $\mathsf{D}(k[\Gamma]\text{–}\mathsf{mod})$ generated by $\mathsf{E} \subset k[\Gamma]\text{–}\mathsf{mod}$. Then E is the core of a t-structure on D, so Theorems 1.4–1.6 are applicable whenever the Koszulity assumption or the assumptions (1–2) are satisfied. This would allow to obtain a comparison between the cohomology of C and Γ with finite-dimensional coefficients.

On the other hand, let us emphasize that we do *not* assume the abelian category E to be linear over a field in the above theorems. E. g., the category of finite modules over a profinite group is fine, as is the category of modules of finite length over any complete commutative local ring, etc.

1.6. It is easy to deduce Theorems 1.4 and 1.6 from Theorem 1.5, but the proof of Theorem 1.5 requires some work.

Let us start with a comment on the proof of Theorem 1.3. It is based on two ingredients: the basic theory of quadratic and Koszul graded algebras and coalgebras over a field, and the spectral sequence connecting the cohomology of the coalgebra E with the cohomology of its associated graded coalgebra with respect to the coaugmentation filtration. In the case of Theorem 1.5, first of all one needs to develop the basic theory of quadratic and Koszul big graded rings, without which one cannot even explain what this theorem says. Then one has to work out the passage from the ungraded category E to its graded version. There are two approaches to this task, and accordingly two proofs of Theorem 1.5 given in this paper.

The most straightforward way is to associate a coalgebra-like structure to an abelian category consisting of objects of finite length, then filter this structure and pass to the associated graded one. It is explained in P. Gabriel's dissertation [2] how to do so. An abelian category consisting of objects of finite length is equivalent to the category of finitely generated discrete modules over a *pseudo-compact* topological ring. One needs to embed the category of finitely generated discrete modules into the category of pseudo-compact modules in order to do the cohomology computations with projective resolutions.

There is a more delicate approach, which is purely categorical. One associates with an abelian category E consisting of objects of finite length the exact category F whose objects are the objects of E endowed with a filtration for which all the successive quotient objects are semisimple. Then one needs to pass from the filtered exact category F to the associated graded abelian category G. The main property connecting the categories F and G is the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^n_{\mathsf{F}}(X, Y(-1)) \longrightarrow \operatorname{Ext}^n_{\mathsf{F}}(X, Y) \\ \longrightarrow \operatorname{Ext}^n_{\mathsf{G}}(\operatorname{gr} X,\ \operatorname{gr} Y) \longrightarrow \operatorname{Ext}^{n+1}_{\mathsf{F}}(X, Y(-1)) \longrightarrow \cdots$$

for any two objects X, $Y \in \mathsf{F}$, where $Z \longmapsto Z(m)$ denotes the filtration (or grading) shift and $\operatorname{gr}\colon \mathsf{F} \longrightarrow \mathsf{G}$ is the functor assigning to a filtered object its associated graded object. The main property connecting the categories E and F is the natural isomorphism

$$\operatorname{Ext}^n_{\mathsf{E}}(u(X), u(Y)) \simeq \varinjlim\nolimits_{m\to+\infty} \operatorname{Ext}^n_{\mathsf{F}}(X, Y(m)),$$

where $u\colon \mathsf{F} \longrightarrow \mathsf{E}$ denotes the functor of forgetting the filtration.

1.7. Big graded algebras and coalgebras are considered in Section 2. Pseudo-compact algebras, their associated graded algebras, and their cohomology are discussed in Section 3. The associated graded category to the exact category of filtered objects is constructed in Section 4. The proof of Theorem 1.5 is given in Section 5. Theorems 1.4 and 1.6 are proven in Section 6.

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