# Semi-infinite algebraic geometry of quasi-coherent torsion sheaves

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## Introduction: Semi-Infinite Set Theory

A semi-infinite structure on a set $S$ is the datum of a subset $S^+ \subset S$ defined up to adjoining or removing a finite number of elements from $S$. The subset $S^+ \subset S$, as well any other subset $S^{+\prime} \subset S$ for which the symmetric difference $(S^+ \cup S^{+\prime}) \setminus (S^+ \cap S^{+\prime})$ is finite, is called a semi-infinite subset in $S$, while the complement $S \setminus S^+$ is called a co-semi-infinite subset.

Given a field $\Bbbk$ and a set $S$ with a semi-infinite structure, one can construct a topological $\Bbbk$-vector space

$$V_{S,S^+} = \bigoplus\nolimits_{t \in S \setminus S^+} \Bbbk t \oplus \prod\nolimits_{s \in S^+} \Bbbk s$$

which remains unchanged when one removes a finite number of elements from $S^+$ or adjoins to $S^+$ a finite number of elements from $S$. The set $S$ is a topological basis in the natural topology on $V_{S,S^+}$. Topological vector spaces of this form are called locally linearly compact, or Tate vector spaces.

## Introduction: Semi-Infinite Linear Algebra

More precisely, a complete, separated topological $\mathbb{k}$-vector space is called linearly compact (or "pseudocompact", or "pro-finite-dimensional") if it has a base of neighborhoods of zero consisting of vector subspaces of finite codimension. A topological vector space is called locally linearly compact if it has a linearly compact open subspace.

The standard example of a set with a semi-infinite structure is the set of all integers $S = \mathbb{Z}$ with the semi-infinite subset of positive integers $S^+ = \mathbb{Z}_{>0}$. The related topological vector space $V_{S,S^+}$ is the vector space of Laurent formal power series $V_{S,S^+} = \mathbb{k}((t))$.

# Introduction: Semi-Infinite Geometry

Semi-infinite geometry can be informally defined as a study of geometric shapes with local coordinates indexed by sets with semi-infinite structure. For a semi-infinite variety $Y$ with local coordinates $y_s$ indexed by a set $S$ with a semi-infinite subset $S^+ \subset S$, it makes sense to assume that, for every point $p \in Y$, the set of all indices $s \in S$ such that $y_s(p) \neq 0$ is contained in some semi-infinite subset $S^{+\prime} \subset S$ (depending on the point $p$).

The standard example of a semi-infinite algebraic variety is the underlying affine algebraic variety $Y$ of the vector space of Laurent formal power series $\Bbbk((t))$. Let us write $f(t) = \sum_{n\in\mathbb{Z}} y_n t^n$ for a generic element $f(t) \in \Bbbk((t))$. Then $y_n,\ n \in \mathbb{Z}$ is a global coordinate system on $Y$, indexed by the set $S = \mathbb{Z}$ with the standard semi-infinite structure. The condition above is satisfied: for every $f \in \Bbbk((t))$, the set of all $n \in \mathbb{Z}$ such that $y_n \neq 0$ is at most semi-infinite, i. e., it is contained in the union of $S^+ = \mathbb{Z}_{>0}$ with a finite set of nonpositive integers.

# Introduction: Summary for the Standard Example

Continuing with the standard example, consider the ring of polynomials $R = \Bbbk[\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots]$ in the doubly infinite sequence of variables $y_n$, $n \in \mathbb{Z}$. Let us say that an $R$-module $M$ is torsion if for every $m \in M$ there exists $\ell < 0$ such that $y_n m = 0$ for all $n < \ell$. Alternatively, one can think of torsion $R$-modules as of discrete modules over the topological ring

$$\mathfrak{R} = \varprojlim\nolimits_{\ell<0} \Bbbk[y_\ell, \dots, y_{-1}, y_0, y_1, y_2, \dots]$$

of functions on $\Bbbk((t)) = \{\, \sum_{n\in\mathbb{Z}} y_n t^n \mid y_n = 0 \text{ for } n \ll 0 \,\}$.

The category $\mathrm{A} = R\text{-}\mathrm{Mod}_{\mathrm{tors}}$ of torsion $R$-modules is abelian. The aim of this talk is to explain how to define a certain exotic derived category of the abelian category $\mathrm{A}$, called the semiderived category of torsion $R$-modules and denoted $\mathrm{D}^{\mathrm{si}}(\mathrm{A})$, so that $\mathrm{D}^{\mathrm{si}}(\mathrm{A})$ is naturally a tensor triangulated category. The tensor product operation $\Diamond$ on $\mathrm{D}^{\mathrm{si}}(\mathrm{A})$ is called the semitensor product.

## Introduction: Summary for the Standard Example

The triangulated category $\mathrm{D^{si}(A)}$ is constructed as the triangulated Verdier quotient category of the cochain homotopy category $\mathrm{K(A)}$ of unbounded complexes in $\mathrm{A}$ by a certain thick subcategory $\mathrm{Ac^{si}(A)}$; so $\mathrm{D^{si}(A) = K(A)/Ac^{si}(A)}$.

The thick (in fact, localizing) subcategory of complexes to be killed $\mathrm{Ac^{si}(A) \subset K(A)}$ is properly contained in the full subcategory $\mathrm{Ac(A) \subset K(A)}$ of acyclic complexes in $\mathrm{A}$, that is $\mathrm{Ac^{si}(A) \subsetneq Ac(A)}$.

So some acyclic complexes in the abelian category $\mathrm{A}$ represent nonzero objects in the semiderived category. In particular, the unit object of the tensor structure on $\mathrm{D^{si}(A)}$ turns out to be an acyclic complex in this example.

## Introduction: Semi-Infinite Exterior Forms

Consider the topological vector space $V = \mathbb{k}((t))$, and let $(e_i = t^i)_{i\in\mathbb{Z}}$ be the standard topological basis in it. We would like to define a $\mathbb{Z}$-graded (or rather, $(\infty/2+\mathbb{Z})$-graded) vector space of semi-infinite exterior forms $\bigwedge^{\infty/2+*}(V)$ consisting of infinite wedge products like

$$e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge \cdots \qquad (*)$$

One can delete a finite number $e_i$'s with $i \geq 0$ from the wedge product $(*)$ and adjoin a finite number of $e_j$'s with $j < 0$, obtaining other basis vectors of $\bigwedge^{\infty/2+*}(V)$.

Let the basis vector $(*)$ have grading $\infty/2+0$. Then deleting $m$ vectors $e_i$ from the wedge product $(*)$ and adjoining $n$ new vectors $e_j$ instead produces an infinite wedge product representing a basis vector of $\bigwedge^{\infty/2+*}(V)$ having grading $\infty/2+n-m$.

# Introduction: Semi-Infinite Exterior Forms

One would like to define the graded vector space $\bigwedge^{\infty/2+*}(V)$ in a more invariant way, not depending on the choice of a basis in $V$. But there is a problem: if both the expressions

$$e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge \cdots$$

and

$$2e_0 \wedge 2e_1 \wedge 2e_2 \wedge 2e_3 \wedge \cdots$$

represent some vectors in $\bigwedge^{\infty/2+*}(V)$, then these two vectors should naturally differ by multiplication with a scalar, but the scalar is undefined.

Similarly, one can write

$$e_1 \wedge e_0 \wedge e_3 \wedge e_2 \wedge e_5 \wedge e_4 \wedge e_7 \wedge e_6 \wedge \cdots ,$$

which should differ from $e_0 \wedge e_1 \wedge e_2 \wedge \cdots$ by a $\pm 1$ sign, but it is impossible to say whether it should be 1 or $-1$.

## Introduction: Semi-Infinite Exterior Forms

It turns out that the projectivization of the vector space $\bigwedge^{\infty/2+*}(V)$ is well-defined for any locally linearly compact $\Bbbk$-vector space $V$. Choosing a linearly compact open subspace $W \subset V$, one can construct a well-defined graded vector space of semi-infinite forms $\bigwedge_W^{\infty/2+*}(V)$.

Changing $W$ by $W'$, one has a natural isomorphism $\bigwedge_{W'}^{\infty/2+*}(V) \simeq \bigwedge^{\infty/2+d+*}(W) \otimes_{\Bbbk} D$, where $d \in \mathbb{Z}$ is an integer ("relative dimension") and $D$ is a one-dimensional $\Bbbk$-vector space ("relative determinant").

Specifically, the construction is

$$\bigwedge_W^{\infty/2+*}(V) = \varinjlim_{U \subset W} \bigwedge^*(V/U) \otimes_k \bigwedge^{\dim_{\Bbbk} W/U}(W/U)^*,$$

where the direct limit is taken over all open subspaces $U \subset W$.

## Introduction: Summary for the Standard Example Fin'd

Returning to the semitensor product operation on the semiderived category $\mathrm{D}^{\mathrm{si}}(R\text{-}\mathrm{Mod}_{\mathrm{tors}})$, consider the one-dimensional $R$-module $\Bbbk$ with the zero action of all the variables $y_i$.

Then the semitensor product $\Bbbk \lozenge \Bbbk$ in $\mathrm{D}^{\mathrm{si}}(R\text{-}\mathrm{Mod}_{\mathrm{tors}})$ is the doubly unbounded complex of $R$-modules $\bigwedge^{\infty/2+*}(\Bbbk((t)))$ with the zero action of the variables $y_i$ and zero differential,

$$\Bbbk \lozenge \Bbbk \simeq \bigwedge{}^{\infty/2+*}(\Bbbk((t))).$$

The dimensional shifts and determinantal twists arise from certain choices one has to make when defining the semitensor product operation on $\mathrm{D}^{\mathrm{si}}(R\text{-}\mathrm{Mod}_{\mathrm{tors}})$.

## Examples: Derived Category of Modules

Let us list some simpler examples which fit into the general theory as special cases.

**Example 1**

Let $B$ be a commutative ring. Then the unbounded derived category of $B$-modules $\mathrm{D}(B\text{-}\mathrm{Mod})$ is a tensor triangulated category with respect to the operation $\otimes_B^{\mathbb{L}}$ of left derived tensor product of complexes of $B$-modules. The unbounded left derived tensor product is constructed using homotopy flat or homotopy projective resolutions of complexes of $B$-modules. The $B$-module $B$ is the unit object of this tensor structure.

# Examples: Derived Category of Torsion Abelian Groups

## Example 2a

Let $\mathrm{A} = \mathbb{Z}\text{-}\mathrm{Mod}_{\mathrm{tors}}$ be the abelian category of torsion abelian groups. The abelian category $\mathrm{A}$ has a natural monoidal structure with the unit object $\mathbb{Q}/\mathbb{Z}$. The torsion product operation providing this monoidal structure can be defined as $A \circledT_{\mathbb{Q}/\mathbb{Z}} B = \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$.

The torsion product is a left exact functor. The right derived functor of torsion product $\circledT^{\mathbb{R}}_{\mathbb{Q}/\mathbb{Z}}$, constructed using injective coresolutions, makes the derived category of torsion abelian groups $\mathrm{D}(\mathrm{A})$ a tensor triangulated category.

The tensor triangulated category $\mathrm{D}(\mathrm{A})$ can be embedded into the derived category of abelian groups $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod})$ with the shifted tensor product operation $(A, B) \longmapsto (A \otimes_{\mathbb{Z}}^{\mathbb{L}} B)[-1]$, as a tensor triangulated subcategory with its own unit object. The unit object $\mathbb{Z}[1]$ of the monoidal structure $(A, B) \longmapsto (A \otimes_{\mathbb{Z}}^{\mathbb{L}} B)[-1]$ on $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod})$ is different from the unit object $\mathbb{Q}/\mathbb{Z}$ of $\mathrm{D}(\mathbb{Z}\text{-}\mathrm{Mod}_{\mathrm{tors}})$.

# Examples: Derived Category of Torsion $\Bbbk[x]$-Modules

This example is similar to the previous one.

## Example 2b

Let $\Bbbk[x]$ be the ring of polynomials in one variable $x$ over a field $\Bbbk$. Let us say that a $\Bbbk[x]$-module $M$ is $x$-torsion if for every $m \in M$ there exists $n \geq 1$ such that $x^n m = 0$.

Let $\mathrm{A} = \Bbbk[x]\text{-}\mathrm{Mod}_{x\text{-tors}}$ be the abelian category of $x$-torsion $\Bbbk[x]$-modules. The abelian category $\mathrm{A}$ has a natural monoidal structure with the Prüfer module $P_x = \Bbbk[x, x^{-1}]/\Bbbk[x]$ being the unit object. The torsion product operation providing this monoidal structure can be defined as $M \circledT_{P_x} N = \mathrm{Tor}_1^{\Bbbk[x]}(M, N)$.

Once again, the torsion product is a left exact functor. The right derived functor of torsion product $\circledT_{P_x}^{\mathbb{R}}$, constructed using injective coresolutions, makes the derived category of $x$-torsion $\Bbbk[x]$-modules $\mathrm{D}(\mathrm{A})$ a tensor triangulated category.

# Examples: Coderived Category of Comodules

The category of $x$-torsion $\Bbbk[x]$-modules is otherwise known as the category of comodules over the coalgebra $C$ dual to the topological algebra of Taylor formal power series $\Bbbk[[x]]$. The coalgebra $C$ is defined explicitly as the $\Bbbk$-vector space with the basis $1^*$, $x^*$, $x^{2*}$, $x^{3*}$, ..., the comultiplication $\mu(x^{n*}) = \sum_{p+q=n} x^{p*} \otimes x^{q*}$, and the counit $\epsilon(1^*) = 1$, $\epsilon(x^{n*}) = 0$ for $n > 0$.

Example 2b can be generalized to arbitrary cocommutative coalgebras over a field.

**Example 3**

Let $C$ be a coassociative, cocommutative, counital coalgebra over a field $\Bbbk$. Then the abelian category of $C$-comodules $C\text{-}\mathrm{Comod}$ has a natural monoidal structure with the unit object $C$. The cotensor product operation providing this monoidal structure is defined as follows: the cotensor product $M \,\square_C\, N$ of two comodules $M$ and $N$ is the kernel of the difference of two natural maps $M \otimes_\Bbbk N \rightrightarrows M \otimes_\Bbbk C \otimes_\Bbbk N$ induced by the coaction of $C$ in $M$, $N$.

# Examples: Coderived Category of Comodules

## Example 3 cont'd

The coderived category of $C$-comodules is simplest defined as the homotopy category of unbounded complexes of injective $C$-comodules, $\mathrm{D^{co}}(C\text{-}\mathrm{Comod}) = \mathrm{K}(C\text{-}\mathrm{Comod_{inj}})$. But this is not enough: one wants to be able to assign a coderived category object to an arbitrary complex of $C$-comodules.

A complex of $C$-comodules $A^\bullet$ is said to be coacyclic if the complex $\mathrm{Hom}_C(A^\bullet, J^\bullet)$ is acyclic for every complex of injective $C$-comodules $J^\bullet$. The coderived category can be then defined as the quotient category $\mathrm{D^{co}}(C\text{-}\mathrm{Comod}) = \mathrm{K}(C\text{-}\mathrm{Comod})/\mathrm{Ac^{co}}(C\text{-}\mathrm{Comod})$ of the homotopy category by the localizing subcategory of coacyclic complexes.

One can show that the composition of triangulated functors $\mathrm{K}(C\text{-}\mathrm{Comod_{inj}}) \longrightarrow \mathrm{K}(C\text{-}\mathrm{Comod}) \longrightarrow \mathrm{K}(C\text{-}\mathrm{Comod})/\mathrm{Ac^{co}}(C\text{-}\mathrm{Comod})$ is a triangulated equivalence. This is the coderived category of $C$-comodules.

## Example 3 fin'd

The definition of the coderived category of $C$-comodules, as per the previous slide, is valid for any coassociative coalgebra $C$ over a field $\Bbbk$.

When the coalgebra $C$ is cocommutative, one can define the right derived cotensor product operation on the coderived category $\mathrm{D^{co}}(C\text{-}\mathrm{Comod})$ by taking the cotensor products of complexes of injective $C$-comodules. In fact, for any complex of injective $C$-comodules $J^\bullet$ and any coacyclic complex of $C$-comodules $A^\bullet$, the complex of $C$-comodules $C^\bullet \mathbin{\square}_C A^\bullet$ is coacyclic. So it suffices to replace only one of the two complexes of $C$-comodules by a complex of injectives before taking their cotensor product.

The coderived category of $C$-comodules $\mathrm{D^{co}}(C\text{-}\mathrm{Comod})$, endowed with the derived cotensor product functor $\mathbin{\square}_C^{\mathbb{R}}$, becomes a tensor triangulated category. The $C$-comodule $C$ is the unit object.

## Examples: Noetherian Scheme with a Dualizing Complex

### Example 4

Let $X$ be a Noetherian scheme. The coderived category of quasi-coherent sheaves on $X$ is simplest defined as the homotopy category of unbounded complexes of injective quasi-coherent sheaves, $\mathrm{D^{co}}(X\text{-}\mathrm{Qcoh}) = \mathrm{K}(X\text{-}\mathrm{Qcoh_{inj}})$. Once again, one wants to be able to assign a coderived category object to an arbitrary complex of quasi-coherent sheaves. Hence the next definition.

A complex of quasi-coherent sheaves $\mathcal{A}^\bullet$ is said to be coacyclic if the complex of abelian groups $\mathrm{Hom}_X(\mathcal{A}^\bullet, \mathcal{J}^\bullet)$ is acyclic for every complex of injective quasi-coherent sheaves $\mathcal{J}^\bullet$. The coderived category can be alternatively defined as the quotient category $\mathrm{D^{co}}(X\text{-}\mathrm{Qcoh}) = \mathrm{K}(X\text{-}\mathrm{Qcoh})/\mathrm{Ac^{co}}(X\text{-}\mathrm{Qcoh})$ of the homotopy category by the localizing subcategory of coacyclic complexes.

## Examples: Noetherian Scheme with a Dualizing Complex

### Example 4 cont'd

One can show that the composition of triangulated functors $\mathrm{K}(X\text{-}\mathrm{Qcoh}_{\mathrm{inj}}) \longrightarrow \mathrm{K}(X\text{-}\mathrm{Qcoh}) \longrightarrow \mathrm{K}(X\text{-}\mathrm{Qcoh})/\mathrm{Ac}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is a triangulated equivalence. This is the coderived category of quasi-coherent sheaves on $X$.

A complex of injective quasi-coherent sheaves $\mathcal{D}^\bullet$ on $X$ is called a dualizing complex if:

1. $\mathcal{D}^\bullet$ is homotopy equivalent to a bounded complex of injective quasi-coherent sheaves;
2. the cohomology sheaves of $\mathcal{D}^\bullet$ are coherent sheaves;
3. the natural morphism of complexes of sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet, \mathcal{D}^\bullet)$ is a quasi-isomorphism of complexes of sheaves of $\mathcal{O}_X$-modules.

# Examples: Noetherian Scheme with a Dualizing Complex

Given a scheme $X$, one can consider the exact category $X\text{-}\mathrm{Qcoh}_{\mathrm{fl}}$ of flat quasi-coherent sheaves on $X$. As to any exact category, one can assign to $X\text{-}\mathrm{Qcoh}_{\mathrm{fl}}$ its unbounded derived category $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$. The triangulated category $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$ is the quotient category of $\mathrm{K}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$ by the localizing subcategory of acyclic complexes of flat sheaves with flat sheaves of cocycles.

**Theorem (Murfet '07)**

*Let $X$ be a semi-separated Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet$. Then there is a natural triangulated equivalence $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}}) \simeq \mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ depending on $\mathcal{D}^\bullet$.*

*To a complex of flat sheaves $\mathcal{F}^\bullet$, the complex of injective sheaves $\mathcal{D}^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet$ is assigned. To a complex of injective sheaves $\mathcal{J}^\bullet$, the complex of flat quasi-coherent sheaves $\mathcal{H}om_{X\text{-}\mathrm{qc}}(\mathcal{D}^\bullet, \mathcal{J}^\bullet)$ is assigned. Here $\mathcal{H}om_{X\text{-}\mathrm{qc}}$ denotes the quasi-coherent internal $\mathcal{H}om$ of quasi-coherent sheaves, which can be constructed by applying the coherator functor to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}$.*

## Examples: Noetherian Scheme with a Dualizing Complex

### Example 4 cont'd

On any scheme $X$, the derived category of flat quasi-coherent sheaves $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$ is a tensor triangulated category with respect to the tensor product functor $\otimes_{\mathcal{O}_X}$.

On a semi-separated Noetherian scheme $X$ with a dualizing complex $\mathcal{D}^\bullet$, one can transfer the tensor structure of $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}})$ along the triangulated equivalence $\mathrm{D}(X\text{-}\mathrm{Qcoh}_{\mathrm{fl}}) \simeq \mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$. This makes the coderived category $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ a tensor triangulated category with the unit object $\mathcal{D}^\bullet$.

The resulting operation on $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is called the cotensor product and denoted by $\square_{\mathcal{D}^\bullet}$. So $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$ is a tensor triangulated category with respect to the cotensor product over $\mathcal{D}^\bullet$.

# Examples: Noetherian Scheme with a Dualizing Complex

## Example 4 fin'd

Explicitly, let $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ be two complexes in $X\text{-}\mathrm{Qcoh}$, and let $\mathcal{M}^\bullet \longrightarrow \mathcal{J}^\bullet$ and $\mathcal{N}^\bullet \longrightarrow \mathcal{K}^\bullet$ be two morphisms with coacyclic cones, where $\mathcal{J}^\bullet$, $\mathcal{K}^\bullet \in \mathrm{K}(X\text{-}\mathrm{Qcoh}_{\mathrm{inj}})$. Then the three complexes

$$\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathrm{qc}}(\mathcal{D}^\bullet, \mathcal{K}^\bullet),$$
$$\mathcal{H}om_{X\text{-}\mathrm{qc}}(\mathcal{D}^\bullet, \mathcal{J}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet,$$
$$\mathcal{D}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathrm{qc}}(\mathcal{D}^\bullet, \mathcal{J}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathrm{qc}}(\mathcal{D}^\bullet, \mathcal{K}^\bullet)$$

are naturally isomorphic as objects of $\mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$.

The three complexes above represent the coderived category object $\mathcal{M}^\bullet \,\square_{\mathcal{D}^\bullet}\, \mathcal{N}^\bullet \in \mathrm{D}^{\mathrm{co}}(X\text{-}\mathrm{Qcoh})$.

# Posing the Problem

We are interested in a common generalization of Examples 1–4.

To be more precise, first of all we want a common generalization of Examples 2–4 (torsion modules, comodules, and coderived categories) to the context of ind-Noetherian ind-schemes $\mathfrak{X}$ with a dualizing complex $\mathscr{D}^\bullet$.

Secondly, we want to mount Example 1 (derived category of modules over a commutative ring) on top of the common generalization of Examples 2–4.

For this purpose, we will consider a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, where $\mathfrak{X}$ is an ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet = \mathscr{D}^\bullet_{\mathfrak{X}}$. The fibers of $\pi$ are arbitrary affine schemes.

In the language of the introduction above, one can say that the coordinates along $\mathfrak{X}$ are indexed by a co-semi-infinite subset $S^-$ of the set of all coordinates $S$ on $\mathfrak{Y}$, while the coordinates along the fiber of $\pi$ are indexed by a semi-infinite subset $S^+ \subset S$.

## Posing the Problem

The semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ (relative to $\mathfrak{X}$) is a mixture of the coderived category along the base $\mathfrak{X}$ and the conventional unbounded derived category along the fibers.

The desired operation of semitensor product on $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is a mixture of the cotensor product along $\mathfrak{X}$ and the conventional derived tensor product along the fibers.

The semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ with the semitensor product operation on it is a tensor triangulated category. The pullback (inverse image) $\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}$ on $\mathfrak{Y}$ of the dualizing complex $\mathscr{D}^\bullet_{\mathfrak{X}}$ on $\mathfrak{X}$ is the unit object of this tensor structure, which is denoted by $\lozenge_{\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}}$.

# Ind-Schemes

From now on, all schemes in this talk are presumed to be quasi-compact and quasi-separated. Then one can define an ind-scheme simply as an ind-object in the category of schemes.

So an ind-scheme $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ is represented by a directed diagram of schemes $(X_\gamma)_{\gamma\in\Gamma}$ indexed by a directed poset $\Gamma$.
The set of morphisms $\mathrm{Mor}(\mathfrak{Y}, \mathfrak{X})$ in the category of ind-schemes is defined by the rule

$$\mathrm{Mor}(\text{“}\varinjlim\text{”}_{\delta\in\Delta} Y_\delta,\ \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma) = \varprojlim_{\delta\in\Delta} \varinjlim_{\gamma\in\Gamma} \mathrm{Mor}(Y_\delta, X_\gamma).$$

So any morphism from a scheme $Y$ to an ind-scheme $\text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ factorizes through one of the schemes $X_\gamma$.

An ind-scheme $\mathfrak{X}$ is said to be strict if it can be represented by a direct system $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ such that all the morphisms $X_\beta \longrightarrow X_\gamma$, $\beta < \gamma \in \Gamma$, are closed immersions. We will assume all our ind-schemes to be strict.

# Ind-Schemes

A morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is said to be affine if, for any scheme $T$ and any morphism $T \longrightarrow \mathfrak{X}$, the ind-scheme $T \times_{\mathfrak{X}} \mathfrak{Y}$ is actually a scheme and the morphism of schemes $T \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow T$ is affine. Similarly, the morphism $\pi$ is said to be flat if, for any scheme $T$ and morphism $T \longrightarrow \mathfrak{X}$, the pullback $T \times_{\mathfrak{X}} \mathfrak{Y}$ is actually a scheme and the morphism of schemes $T \times_{\mathfrak{X}} \mathfrak{Y} \longrightarrow T$ is flat.

We will be interested in flat affine morphisms of schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Any such morphism can be represented by a diagram of flat affine morphisms of schemes $\pi_\gamma\colon Y_\gamma \longrightarrow X_\gamma$ indexed by a directed poset $\Gamma$; so $\pi =$ "$\varinjlim$"$_{\gamma\in\Gamma}\, \pi_\gamma$. Moreover, one can have $Y_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$.

An ind-scheme $\mathfrak{X}$ is said to be ind-Noetherian if it can be represented by a direct system of Noetherian schemes $(X_\gamma)_{\gamma\in\Gamma}$. Similarly, $\mathfrak{X}$ is said to be ind-semi-separated if it can be represented by a direct system of semi-separated schemes $X_\gamma$.

## Quasi-Coherent Torsion Sheaves

There are two notions of quasi-coherent sheaves on ind-schemes: quasi-coherent torsion sheaves and pro-quasi-coherent pro-sheaves.

Quasi-coherent torsion sheaves are a kind of sheaves $\mathscr{M}$ on an ind-scheme $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ such that every local section of $\mathscr{M}$ is supported in one of the closed subschemes $X_\gamma \subset \mathfrak{X}$. Under mild assumptions on $\mathfrak{X}$, quasi-coherent torsion sheaves form a Grothendieck abelian category $\mathfrak{X}\text{-}\mathrm{Tors}$; but working with it involves a subtlety which I will try to explain by example.

Suppose that we want to construct the category of torsion abelian groups, but we do not know what an abelian group is. All we have are the categories of $\mathbb{Z}/m\mathbb{Z}$-modules for $m \geq 2$.

# Quasi-Coherent Torsion Sheaves

Then we can describe an arbitrary torsion abelian group $A$ in terms of its subgroups ${}_mA$ of elements annihilated by $m$. So we say that a torsion abelian group $A$ is a collection of $\mathbb{Z}/m\mathbb{Z}$-modules ${}_mA$ together with embeddings ${}_mA \longrightarrow {}_nA$ for all $m \mid n$, satisfying suitable conditions.

What conditions? The main condition is that ${}_mA$ must be precisely the whole subgroup of elements annihilated by $m$ in ${}_nA$.

The subtlety is that the functor assigning to a torsion abelian group $A$ its subgroup ${}_mA$ is not exact. It is only left exact. So, if one defines the category of torsion abelian groups in this way, then constructing cokernels in this category becomes a nontrivial task.

So one has to develop some kind of "sheafification" theory, embedding the desired category of "sheaves" into a larger ambient category of "presheaves". The cokernel of a morphism of "sheaves" is then constructed as the "sheafification" of the cokernel of the same morphism taken in the category of "presheaves".

## Reasonable Ind-Schemes

A closed immersion of schemes $i\colon Z \longrightarrow X$ is said to be reasonable if the sheaf of ideals of $Z$ in $X$ (i. e., the kernel of the morphism $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$) is finitely generated as a quasi-coherent sheaf on $X$. An ind-scheme $\mathfrak{X}$ is said to be reasonable if it can be represented by a direct system of reasonable closed immersions of schemes.

Any ind-Noetherian ind-scheme is reasonable. Any ind-scheme flat or affine over a reasonable ind-scheme is reasonable.

# Quasi-Coherent Torsion Sheaves

Let $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme represented by a direct system of reasonable closed immersions $i_{\beta\gamma}\colon X_\beta \longrightarrow X_\gamma$, $\beta<\gamma\in\Gamma$. A quasi-coherent torsion sheaf $\mathscr{M}$ on $\mathfrak{X}$ can be defined as a set of data assigning

1. to every index $\gamma\in\Gamma$ a quasi-coherent sheaf $\mathscr{M}_\gamma$ on $X_\gamma$ and
2. to every pair of indices $\beta<\gamma\in\Gamma$ a morphism $i_{\beta\gamma*}\mathscr{M}_\beta \longrightarrow \mathscr{M}_\gamma$ of quasi-coherent sheaves on $X_\gamma$

such that

3. the corresponding morphism $\mathscr{M}_\beta \longrightarrow i_{\beta\gamma}^!\mathscr{M}_\gamma$ is an isomorphism for every $\beta<\gamma\in\Gamma$ and
4. the triangular diagram $i_{\alpha\gamma*}\mathscr{M}_\alpha \longrightarrow i_{\beta\gamma*}\mathscr{M}_\beta \longrightarrow \mathscr{M}_\gamma$ is commutative for every triple of indices $\alpha<\beta<\gamma\in\Gamma$.

Due to the isomorphism condition (iii), the category $\mathfrak{X}\text{-}\mathrm{Tors}$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ does not depend on the choice of a direct system of reasonable closed immersions $(X_\gamma)_{\gamma\in\Gamma}$ representing a given reasonable ind-scheme $\mathfrak{X}$.

## Quasi-Coherent Torsion Sheaves

Dropping the isomorphism condition (iii) from the definition of a quasi-coherent torsion sheaf, one obtains the definition of a Γ-system $\mathbb{M}$ of quasi-coherent sheaves on $\mathfrak{X} = \text{``}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$. The category of Γ-systems is obviously a Grothendieck abelian category, but it depends on a chosen direct system of reasonable closed immersions $(X_\gamma)_{\gamma\in\Gamma}$ representing a given reasonable ind-scheme $\mathfrak{X}$.

The category of Γ-systems plays the role of the category of "presheaves" in the analogy suggested above, while the category of quasi-coherent torsion sheaves is the desired category of "sheaves". The inclusion of the full subcategory of quasi-coherent torsion sheaves into the ambient abelian category of Γ-systems has a left adjoint functor $\mathbb{M} \longmapsto \mathbb{M}^+$, playing the role of the "sheafification" functor in this analogy. Using the inclusion into the category of Γ-systems and its left adoint functor $\mathbb{M} \longmapsto \mathbb{M}^+$, one can show that the category of quasi-coherent torsion sheaves is also a Grothendieck abelian category.

# Pro-Quasi-Coherent Pro-Sheaves

The definition of a pro-quasi-coherent pro-sheaf on an ind-scheme $\mathfrak{X}$ is technically simpler than that of a quasi-coherent torsion sheaf, but the resulting additive category is usually not abelian. One needs to impose some flatness condition on one's pro-quasi-coherent pro-sheaves in order to obtain a well-behaved exact subcategory in the poorly behaved ambient category of arbitrary pro-quasi-coherent pro-sheaves.

Let $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme. A pro-quasi-coherent pro-sheaf $\mathfrak{P}$ on $\mathfrak{X}$ is defined as a set of data assigning

1. to every index $\gamma \in \Gamma$ a quasi-coherent sheaf $\mathfrak{P}^\gamma$ on $X_\gamma$ and
2. to every pair of indices $\beta < \gamma \in \Gamma$ a morphism $\mathfrak{P}^\gamma \longrightarrow i_{\beta\gamma*}\mathfrak{P}^\beta$ of quasi-coherent sheaves on $X_\gamma$

such that

3. the corresponding morphism $i_{\beta\gamma}^*\mathfrak{P}^\gamma \longrightarrow \mathfrak{P}^\beta$ is an isomorphism for every $\beta < \gamma \in \Gamma$ and
4. the triangular diagrams $\mathfrak{P}^\gamma \longrightarrow i_{\beta\gamma*}\mathfrak{P}^\beta \longrightarrow i_{\alpha\gamma*}\mathfrak{P}^\alpha$ are commutative.

# Pro-Quasi-Coherent Pro-Sheaves

Due to the isomorphism condition (iii), the category $\mathfrak{X}\text{-}\mathrm{Pro}$ of pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ does not depend on the choice of a direct system of closed immersions $(X_\gamma)_{\gamma\in\Gamma}$ representing a given ind-scheme $\mathfrak{X}$.

The category of pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-}\mathrm{Pro}$ is naturally a monoidal category with respect to the tensor product operation given by the obvious rule $(\mathfrak{P}\otimes^{\mathfrak{X}}\mathfrak{Q})^\gamma=\mathfrak{P}^\gamma\otimes_{\mathcal{O}_{X_\gamma}}\mathfrak{Q}^\gamma$. Essentially, this works because the inverse image functors $i^*$ preserve tensor products of quasi-coherent sheaves. The "pro-structure pro-sheaf" $\mathfrak{O}_{\mathfrak{X}}$ with the components $(\mathfrak{O}_{\mathfrak{X}})^\gamma=\mathcal{O}_{X_\gamma}$ is the unit object of this monoidal structure.

The category of quasi-coherent torsion sheaves $\mathfrak{X}\text{-}\mathrm{Tors}$ is naturally a module category over the monoidal category $\mathfrak{X}\text{-}\mathrm{Pro}$. For any $\mathscr{M}\in\mathfrak{X}\text{-}\mathrm{Tors}$ and $\mathfrak{P}\in\mathfrak{X}\text{-}\mathrm{Pro}$, one considers the $\Gamma$-system $\mathbb{N}$ given by the obvious rule $\mathbb{N}_\gamma=\mathfrak{P}^\gamma\otimes_{\mathcal{O}_{X_\gamma}}\mathscr{M}_\gamma$. Then the quasi-coherent torsion sheaf $\mathfrak{P}\otimes_{\mathfrak{X}}\mathscr{M}$ is constructed as $\mathfrak{P}\otimes_{\mathfrak{X}}\mathscr{M}=\mathbb{N}^+$.

## Direct and Inverse Images (Brief Summary)

For any affine morphism of reasonable ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the direct image functors $f_*\colon \mathfrak{Y}\text{-}\mathrm{Tors} \longrightarrow \mathfrak{X}\text{-}\mathrm{Tors}$ and $f_*\colon \mathfrak{Y}\text{-}\mathrm{Pro} \longrightarrow \mathfrak{X}\text{-}\mathrm{Pro}$ can be defined, and the former one is exact. For any flat morphism of reasonable ind-schemes $f\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the inverse image functors $f^*\colon \mathfrak{X}\text{-}\mathrm{Tors} \longrightarrow \mathfrak{Y}\text{-}\mathrm{Tors}$ and $f^*\colon \mathfrak{X}\text{-}\mathrm{Pro} \longrightarrow \mathfrak{Y}\text{-}\mathrm{Pro}$ can be defined, and the former one is exact. The assumptions on $f$ actually can be relaxed, but these special cases are sufficient for the purposes of this talk.

For any closed immersion of reasonable ind-schemes $i\colon \mathfrak{Z} \longrightarrow \mathfrak{X}$, the functor of inverse image with supports $i^!\colon \mathfrak{X}\text{-}\mathrm{Tors} \longrightarrow \mathfrak{Z}\text{-}\mathrm{Tors}$ can be defined.

The functor $f^*$ is left adjoint to the functor $f_*$, and the functor $i_*$ is left adjoint to the functor $i^!$, whenever both are defined.

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be a reasonable ind-scheme and $i_\gamma\colon X_\gamma \longrightarrow \mathfrak{X}$ be the natural closed immersion. Then one has $i_\gamma^*\mathfrak{P} = \mathfrak{P}^\gamma$ and $i_\gamma^!\mathscr{M} = \mathscr{M}_\gamma$ for all $\mathfrak{P} \in \mathfrak{X}\text{-}\mathrm{Pro}$ and $\mathscr{M} \in \mathfrak{X}\text{-}\mathrm{Tors}$.

# Dualizing Complexes on Ind-Noetherian Ind-Schemes

Let $\mathfrak{X} = \text{“}\varinjlim\text{”}_{\gamma\in\Gamma} X_\gamma$ be an ind-Noetherian ind-scheme. Then a complex of injective quasi-coherent torsion sheaves $\mathscr{D}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}})$ is said to be a dualizing complex if, for every index $\gamma \in \Gamma$, the complex of injective quasi-coherent sheaves $\mathscr{D}^\bullet_\gamma = i_\gamma^! \mathscr{D}^\bullet$ is a dualizing complex on $X_\gamma$.

One can check that this condition on a complex of injective quasi-coherent torsion sheaves on $\mathfrak{X}$ does not depend on the choice of a direct system $(X_\gamma)_{\gamma\in\Gamma}$ representing $\mathfrak{X}$.

Notice that, by definition, a dualizing complex $\mathscr{D}^\bullet$ is an object of the homotopy category of unbounded complexes of injective quasi-coherent torsion sheaves on $\mathfrak{X}$. Another name for this category is the coderived category of quasi-coherent torsion sheaves, $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) = \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}})$. A dualizing complex on an ind-scheme cannot be viewed as an object of the conventional derived category $\mathrm{D}(\mathfrak{X}\text{-}\mathrm{Tors})$.

# Dualizing Complexes on Ind-Noetherian Ind-Schemes

## Example

Let us consider the ind-Noetherian ("negative") half of our standard example. Specifically, let $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\ell \le 0} X_\ell$, where $X_\ell = \operatorname{Spec} \Bbbk[x_\ell, \dots, x_{-1}]$. So $X_0 = \operatorname{Spec} \Bbbk$ is a point, $X_{-1} = \operatorname{Spec} \Bbbk[x_{-1}]$ is an affine line, $X_{-2} = \operatorname{Spec} \Bbbk[x_{-2}, x_{-1}]$ is an affine plane, etc. The closed immersions $X_0 \longrightarrow X_{-1} \longrightarrow X_{-2} \longrightarrow \cdots$ are the most obvious inclusions of coordinate hyperplanes.

What is a dualizing complex on $\mathfrak{X}$? To construct such a dualizing complex $\mathscr{D}^\bullet$, one needs to specify a dualizing complex $\mathcal{D}_\ell^\bullet$ on every scheme $X_\ell$ in such a way that, denoting by $i_\ell \colon X_\ell \longrightarrow X_{\ell-1}$ the coordinate closed immersion, one would have a homotopy equivalence $\mathcal{D}_\ell^\bullet \simeq i_\ell^! \mathcal{D}_{\ell-1}^\bullet$ for every $\ell \le 0$.

## Example fin'd

Any dualizing complex on $X_\ell$ is homotopy equivalent to an injective coresolution of the structure sheaf $\mathcal{O}_{X_\ell}$ shifted to some cohomological degree. The condition $\mathcal{D}_\ell^\bullet \simeq i_\ell^! \mathcal{D}_{\ell-1}^\bullet$ means that if the only cohomology sheaf of the complex $\mathcal{D}_\ell^\bullet$ sits in cohomological degree $n$, then the only cohomology sheaf of the complex $\mathcal{D}_{\ell-1}^\bullet$ sits in the cohomological degree $n-1$.

For example, if the complex $\mathcal{D}_0^\bullet$ is quasi-isomorphic to $\mathcal{O}_{X_0}$, then the complex $\mathcal{D}_\ell^\bullet$ is quasi-isomorphic to $\mathcal{O}_{X_0}[-\ell]$ for every $\ell \le 0$.

As $\ell$ tends to $-\infty$, the only cohomology sheaf of the complex $\mathcal{D}_\ell^\bullet$ moves to ever higher negative cohomological degrees and, in the direct limit, disappears at the cohomological degree $-\infty$. Consequently, the dualizing complex of quasi-coherent torsion sheaves $\mathscr{D}^\bullet$ on the ind-scheme $\mathfrak{X}$ is acyclic.

## Coderived Category of Quasi-Coherent Torsion Sheaves

Let $\mathfrak{X}$ be an ind-Noetherian ind-scheme. As above, a complex of quasi-coherent torsion sheaves $\mathscr{A}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-Tors})$ is said to be coacyclic if, for every complex of injective quasi-coherent torsion sheaves $\mathscr{J}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-Tors}_{\mathrm{inj}})$, the complex of abelian groups $\mathrm{Hom}_{\mathfrak{X}}(\mathscr{A}^\bullet, \mathscr{J}^\bullet)$ is acyclic. The coderived category $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$ is defined as the triangulated Verdier quotient category

$$\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}) = \mathrm{K}(\mathfrak{X}\text{-Tors})/\mathrm{Ac}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$$

of the homotopy category $\mathrm{K}(\mathfrak{X}\text{-Tors})$ by the localizing subcategory of coacyclic complexes.

One can show that the composition of triangulated functors $\mathrm{K}(\mathfrak{X}\text{-Tors}_{\mathrm{inj}}) \longrightarrow \mathrm{K}(\mathfrak{X}\text{-Tors}) \longrightarrow \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$ is a triangulated equivalence. So, in particular, a dualizing complex $\mathscr{D}^\bullet$ on $\mathfrak{X}$ is naturally viewed as an object of $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$.

# Flat Pro-Quasi-Coherent Pro-Sheaves

Let $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$ be an ind-scheme. A pro-quasi-coherent pro-sheaf $\mathfrak{F}$ on $\mathfrak{X}$ is said to be flat if the quasi-coherent sheaf $\mathfrak{F}^\gamma$ on $X_\gamma$ is flat for every $\gamma \in \Gamma$. One can check that this condition on a pro-quasi-coherent pro-sheaf on $\mathfrak{X}$ does not depend on the choice of a direct system $(X_\gamma)_{\gamma\in\Gamma}$ representing $\mathfrak{X}$.

The additive category $\mathfrak{X}\text{-Flat}$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$ has a natural exact category structure. A short sequence of flat pro-quasi-coherent pro-sheaves $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$ is said to be admissible exact in $\mathfrak{X}\text{-Flat}$ if the short sequence of quasi-coherent sheaves $0 \longrightarrow \mathfrak{F}^\gamma \longrightarrow \mathfrak{G}^\gamma \longrightarrow \mathfrak{H}^\gamma \longrightarrow 0$ is exact in the abelian category $X_\gamma\text{-Qcoh}$ for every $\gamma \in \Gamma$.

The full subcategory of flat pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-Flat}$ is a monoidal subcategory in the monoidal category of pro-quasi-coherent pro-sheaves $\mathfrak{X}\text{-Pro}$. The tensor product functor $\otimes^{\mathfrak{X}}$ is exact in $\mathfrak{X}\text{-Flat}$.

# Absolute Covariant Serre-Grothendieck Duality

**Theorem**

*Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$. Then there is a natural equivalence of triangulated categories*

$$\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) \simeq \mathrm{D}(\mathfrak{X}\text{-}\mathrm{Flat}),$$

*where $\mathrm{D}(\mathfrak{X}\text{-}\mathrm{Flat})$ is the conventional unbounded derived category of the exact category $\mathfrak{X}\text{-}\mathrm{Flat}$ of flat pro-quasi-coherent pro-sheaves.*

*The functor $\mathrm{D}(\mathfrak{X}\text{-}\mathrm{Flat}) \longrightarrow \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ is given by the rule $\mathfrak{F}^\bullet \longmapsto \mathscr{D}^\bullet \otimes_{\mathfrak{X}} \mathfrak{F}^\bullet$, where $\otimes_{\mathfrak{X}}$ denotes the action of the monoidal category $\mathfrak{X}\text{-}\mathrm{Flat}$ in the module category $\mathfrak{X}\text{-}\mathrm{Tors}$.*

*The inverse functor $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}(\mathfrak{X}\text{-}\mathrm{Flat})$ is denoted by $\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{X}\text{-}\mathrm{qc}}(\mathscr{D}^\bullet, -)$. It is a right derived functor constructed by applying a certain quasi-coherent internal $\mathfrak{Hom}$-type functor $\mathfrak{Hom}_{\mathfrak{X}\text{-}\mathrm{qc}}(\mathscr{D}^\bullet, -)$ to complexes of injective torsion sheaves.*

# The Cotensor Product

For any ind-scheme $\mathfrak{X}$, the tensor product of flat pro-quasi-coherent pro-sheaves induces a well-defined tensor product functor on the derived category $\mathrm{D}(\mathfrak{X}\text{-Flat})$,

$$\otimes^{\mathfrak{X}} \colon \mathrm{D}(\mathfrak{X}\text{-Flat}) \times \mathrm{D}(\mathfrak{X}\text{-Flat}) \longrightarrow \mathrm{D}(\mathfrak{X}\text{-Flat}).$$

For an ind-Noetherian ind-scheme $\mathfrak{X}$, the coderived category $\mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$ is a triangulated module category over the triangulated tensor category $\mathrm{D}(\mathfrak{X}\text{-Flat})$,

$$\otimes_{\mathfrak{X}} \colon \mathrm{D}(\mathfrak{X}\text{-Flat}) \times \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}) \longrightarrow \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}).$$

Now, given an ind-semi-separated ind-Noetherian ind-scheme $\mathfrak{X}$ with a dualizing complex $\mathscr{D}^\bullet$, one can transfer the tensor triangulated structure of $\mathrm{D}(\mathfrak{X}\text{-Flat})$ along the triangulated equivalence $\mathrm{D}(\mathfrak{X}\text{-Flat}) \simeq \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$. The resulting operation on coderived category is called the cotensor product and denoted by

$$\square_{\mathscr{D}^\bullet} \colon \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}) \times \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}) \longrightarrow \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors}).$$

## The Cotensor Product

Explicitly, let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ be two complexes in $\mathfrak{X}\text{-}\mathrm{Tors}$, and let $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{K}^\bullet$ be two morphisms with coacyclic cones, where $\mathscr{J}^\bullet, \mathscr{K}^\bullet \in \mathrm{K}(\mathfrak{X}\text{-}\mathrm{Tors}_{\mathrm{inj}})$. Then the three complexes

$$\mathscr{M}^\bullet \otimes_{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{K}^\bullet),$$
$$\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet) \otimes_{\mathfrak{X}} \mathscr{N}^\bullet,$$
$$\mathscr{D}^\bullet \otimes_{\mathfrak{X}} \big(\mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{J}^\bullet) \otimes^{\mathfrak{X}} \mathfrak{Hom}_{\mathfrak{X}\text{-qc}}(\mathscr{D}^\bullet, \mathscr{K}^\bullet)\big)$$

are naturally isomorphic as objects of $\mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$.

These three complexes represent the coderived category object $\mathscr{M}^\bullet \,\square_{\mathscr{D}^\bullet}\, \mathscr{N}^\bullet \in \mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$. The coderived category $\mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ endowed with the cotensor product operation $\square_{\mathscr{D}^\bullet}$ becomes a tensor triangulated category. The dualizing complex $\mathscr{D}^\bullet \in \mathrm{D^{co}}(\mathfrak{X}\text{-}\mathrm{Tors})$ is the unit object of this tensor structure.

# Semiderived Category

Now let us turn to the relative setting of a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Recall that if $\mathfrak{X} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} X_\gamma$, then $\mathfrak{Y} = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} Y_\gamma$, where $Y_\gamma = X_\gamma \times_{\mathfrak{X}} \mathfrak{Y}$. Moreover, $\pi = \text{“}\varinjlim\text{''}_{\gamma\in\Gamma} \pi_\gamma$, where $\pi_\gamma\colon Y_\gamma \longrightarrow X_\gamma$ are flat affine morphisms of schemes.

Assume that $\mathfrak{X}$ is ind-Noetherian. The semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ of quasi-coherent torsion sheaves on $\mathfrak{Y}$ relative to $\mathfrak{X}$ is defined as the triangulated Verdier quotient category of $\mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors})$ by the thick subcategory of all complexes $\mathscr{A}^\bullet \in \mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors})$ such that the complex $\pi_*\mathscr{A}^\bullet$ of quasi-coherent torsion sheaves on $\mathfrak{X}$ is coacyclic.

Informally, the semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is a mixture of the coderived category along the base ind-scheme $\mathfrak{X}$ and the conventional unbounded derived category along the fibers of the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$.

# Semiderived Category

The direct image functor $\pi_*\colon \mathrm{K}(\mathfrak{Y}\text{-Tors}) \longrightarrow \mathrm{K}(\mathfrak{X}\text{-Tors})$ takes coacyclic complexes to coacyclic complexes. It also preserves and reflects the conventional acyclicity of complexes. Consequently, the semiderived category is an intermediate Verdier quotient category between the coderived and the conventional derived category. In other words, there are natural Verdier quotient functors

$$\mathrm{D}^{\mathrm{co}}(\mathfrak{Y}\text{-Tors}) \twoheadrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors}) \twoheadrightarrow \mathrm{D}(\mathfrak{Y}\text{-Tors}).$$

When the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ is an isomorphism, i. e., $\mathfrak{Y} = \mathfrak{X}$, the semiderived category coincides with the coderived category: $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{X}\text{-Tors}) = \mathrm{D}^{\mathrm{co}}(\mathfrak{X}\text{-Tors})$.

If the homological dimension of the category $\mathfrak{X}\text{-Tors}$ is finite (e. g., $\mathfrak{X} = X$ is a regular scheme of finite Krull dimension), then the semiderived category coincides with the derived category: $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors}) = \mathrm{D}(\mathfrak{Y}\text{-Tors})$.

# Semiderived Category

A quasi-coherent torsion sheaf $\mathscr{K}$ on $\mathfrak{Y}$ is said to be $\mathfrak{X}$-injective if its direct image $\pi_*\mathscr{K}$ is an injective quasi-coherent torsion sheaf on $\mathfrak{X}$. The full subcategory of $\mathfrak{X}$-injective quasi-coherent torsion sheaves $\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}$ inherits an exact category structure from the ambient abelian category $\mathfrak{Y}\text{-}\mathrm{Tors}$.

For any ind-Noetherian ind-scheme $\mathfrak{X}$ and any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the composition of triangulated functors

$$\mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \longrightarrow \mathrm{K}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$$

induces a triangulated equivalence $\mathrm{D}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}) \simeq \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$. Here $\mathrm{D}(\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}})$ is the conventional unbounded derived category of the exact category $\mathfrak{Y}\text{-}\mathrm{Tors}_{\mathfrak{X}\text{-inj}}$.

# Pro-Sheaves Flat over the Base

A pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$ is said to be $\mathfrak{X}$-flat if the pro-quasi-coherent pro-sheaf $\pi_*\mathfrak{G}$ on $\mathfrak{X}$ is flat.

Just as the additive category $\mathfrak{X}\text{-}\mathrm{Flat}$ of flat pro-quasi-coherent pro-sheaves on $\mathfrak{X}$, the additive category $\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ has a natural exact category stucture. A short sequence of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow 0$ is said to be admissible exact in $\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}$ if the short sequence of flat pro-quasi-coherent pro-sheaves $0 \longrightarrow \pi_*\mathfrak{F} \longrightarrow \pi_*\mathfrak{G} \longrightarrow \pi_*\mathfrak{H} \longrightarrow 0$ on $\mathfrak{X}$ is admissible exact in $\mathfrak{X}\text{-}\mathrm{Flat}$.

For any flat pro-quasi-coherent pro-sheaf $\mathfrak{F}$ on $\mathfrak{Y}$ and any $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf $\mathfrak{G}$ on $\mathfrak{Y}$, the tensor product $\mathfrak{F} \otimes^{\mathfrak{Y}} \mathfrak{G}$ is an $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaf on $\mathfrak{Y}$. So there is a tensor product functor

$$\otimes^{\mathfrak{Y}} \colon \mathfrak{Y}\text{-}\mathrm{Flat} \times \mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat} \longrightarrow \mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}.$$

**Theorem**

*Let $\mathfrak{X}$ be an ind-semi-separated ind-Noetherian ind-scheme with a dualizing complex $\mathscr{D}^\bullet$, and let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. Then there is a natural equivalence of triangulated categories*

$$\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \simeq \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}),$$

*where $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is the conventional unbounded derived category of the exact category $\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}$ of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$.*

*The functor $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is given by the rule $\mathfrak{G}^\bullet \longmapsto (\pi^*\mathscr{D}^\bullet) \otimes_{\mathfrak{Y}} \mathfrak{G}^\bullet$. The inverse functor $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is denoted by $\mathbb{R}\,\mathfrak{Hom}_{\mathfrak{Y}\text{-}\mathrm{qc}}(\pi^*\mathscr{D}^\bullet, -)$. It is a right derived functor constructed by applying a certain quasi-coherent internal $\mathfrak{Hom}$-type functor $\mathfrak{Hom}_{\mathfrak{Y}\text{-}\mathrm{qc}}(\pi^*\mathscr{D}^\bullet, -)$ to complexes of $\mathfrak{X}$-injective quasi-coherent torsion sheaves on $\mathfrak{Y}$.*

# The Semitensor Product

Let $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat affine morphism of ind-schemes. First we need to construct the left derived functor of tensor product of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$,

$$\otimes^{\mathfrak{Y},\mathbb{L}}\colon \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-Flat}) \times \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-Flat}) \longrightarrow \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-Flat}).$$

$\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ are only flat along $\mathfrak{X}$, but need not be flat along the fibers of $\pi$, so their tensor product has to be derived.

For this purpose, we use relative bar-resolutions for the morphism $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$. Given a complex of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves $\mathfrak{G}^\bullet$ on $\mathfrak{Y}$, consider the bar bicomplex

$$\cdots \longrightarrow \pi^*\pi_*\pi^*\pi_*\pi^*\pi^*\mathfrak{G}^\bullet \longrightarrow \pi^*\pi_*\pi^*\pi_*\mathfrak{G}^\bullet \longrightarrow \pi^*\pi_*\mathfrak{G}^\bullet \longrightarrow 0$$

and denote by $\mathfrak{Bar}_\pi^\bullet(\mathfrak{G}^\bullet)$ its totalization constructed by taking coproducts along the diagonals.

## The Semitensor Product

Then $\mathfrak{Bar}_\pi^\bullet(\mathfrak{G}^\bullet)$ is a complex of flat pro-quasi-coherent pro-sheaves on $\mathfrak{Y}$ isomorphic to $\mathfrak{G}^\bullet$ in $\mathrm{D}(\mathfrak{Y}_\mathfrak{X}\text{-}\mathrm{Flat})$ and adjusted to the tensor product $\otimes^\mathfrak{Y}$.

For any two complexes $\mathfrak{F}^\bullet$ and $\mathfrak{G}^\bullet \in \mathrm{K}(\mathfrak{Y}_\mathfrak{X}\text{-}\mathrm{Flat})$, the three complexes

$$\mathfrak{Bar}_\pi^\bullet(\mathfrak{F}^\bullet) \otimes^\mathfrak{Y} \mathfrak{G}^\bullet,$$
$$\mathfrak{F}^\bullet \otimes^\mathfrak{Y} \mathfrak{Bar}_\pi^\bullet(\mathfrak{G}^\bullet),$$
$$\mathfrak{Bar}_\pi^\bullet(\mathfrak{F}^\bullet) \otimes^\mathfrak{Y} \mathfrak{Bar}_\pi^\bullet(\mathfrak{G}^\bullet)$$

are naturally isomorphic in $\mathrm{D}(\mathfrak{Y}_\mathfrak{X}\text{-}\mathrm{Flat})$. These three complexes represent the desired derived tensor product object $\mathfrak{F}^\bullet \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{G}^\bullet$ in $\mathrm{D}(\mathfrak{Y}_\mathfrak{X}\text{-}\mathrm{Flat})$.

# The Semitensor Product

Similarly, assuming $\mathfrak{X}$ to be ind-Noetherian, one constructs the left derived functor of tensor product of $\mathfrak{X}$-flat pro-quasi-coherent pro-sheaves and quasi-coherent torsion sheaves on $\mathfrak{Y}$,

$$\otimes_{\mathfrak{Y}}^{\mathbb{L}}\colon \mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-Flat}) \times \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors}).$$

Given two complexes $\mathfrak{G}^\bullet \in \mathrm{K}(\mathfrak{Y}_{\mathfrak{X}}\text{-Flat})$ and $\mathscr{M}^\bullet \in \mathrm{K}(\mathfrak{Y}\text{-Tors})$, one needs to either replace $\mathfrak{G}^\bullet$ by its bar-resolution $\mathfrak{Bar}^\bullet_\pi(\mathfrak{G}^\bullet)$, or replace $\mathscr{M}^\bullet$ by its bar-resolution $\mathscr{B}ar^\bullet_\pi(\mathscr{M}^\bullet)$, or both. Then the three complexes

$$\mathfrak{Bar}^\bullet_\pi(\mathfrak{G}^\bullet) \otimes_{\mathfrak{Y}} \mathscr{M}^\bullet,$$
$$\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}} \mathscr{B}ar^\bullet_\pi(\mathscr{M}^\bullet),$$
$$\mathfrak{Bar}^\bullet_\pi(\mathfrak{G}^\bullet) \otimes_{\mathfrak{Y}} \mathscr{B}ar^\bullet_\pi(\mathscr{M}^\bullet)$$

are naturally isomorphic in $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$. These three complexes represent the derived tensor product object $\mathfrak{G}^\bullet \otimes_{\mathfrak{Y}}^{\mathbb{L}} \mathscr{M}^\bullet$ in $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$.

## The Semitensor Product

For any flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, the derived category $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ is a tensor triangulated category with respect to the left derived tensor product functor $\otimes^{\mathfrak{Y},\mathbb{L}}$. Assuming $\mathfrak{X}$ to be ind-Noetherian, the semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is a triangulated module category over $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ with respect to the left derived tensor product functor $\otimes^{\mathbb{L}}_{\mathfrak{Y}}$.

Now, given an ind-semi-separated ind-Noetherian ind-scheme $\mathfrak{X}$ with a dualizing complex $\mathscr{D}^{\bullet}_{\mathfrak{X}}$ and a flat affine morphism of ind-schemes $\pi\colon \mathfrak{Y} \longrightarrow \mathfrak{X}$, one can transfer the tensor triangulated structure of $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat})$ along the triangulated equivalence $\mathrm{D}(\mathfrak{Y}_{\mathfrak{X}}\text{-}\mathrm{Flat}) \simeq \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$. The resulting operation on the semiderived category is called the semitensor product and denoted by

$$\lozenge_{\pi^*\mathscr{D}^{\bullet}_{\mathfrak{X}}}\colon \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \times \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}) \longrightarrow \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors}).$$

Explicitly, let $\mathscr{M}^\bullet$ and $\mathscr{N}^\bullet$ be two complexes in $\mathfrak{Y}\text{-Tors}$, and let $\mathscr{M}^\bullet \longrightarrow \mathscr{J}^\bullet$ and $\mathscr{N}^\bullet \longrightarrow \mathscr{K}^\bullet$ be two morphisms whose cones become coacyclic after applying $\pi_*$, while $\mathscr{J}^\bullet$, $\mathscr{K}^\bullet \in \mathrm{K}(\mathfrak{Y}\text{-Tors}_{\mathfrak{X}\text{-inj}})$. Then the three complexes

$$\mathscr{M}^\bullet \otimes^{\mathbb{L}}_{\mathfrak{Y}} \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}, \mathscr{K}^\bullet),$$

$$\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}, \mathscr{J}^\bullet) \otimes^{\mathbb{L}}_{\mathfrak{Y}} \mathscr{N}^\bullet,$$

$$\pi^*\mathscr{D}^\bullet_{\mathfrak{X}} \otimes_{\mathfrak{Y}} \Big(\mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}, \mathscr{J}^\bullet) \otimes^{\mathfrak{Y},\mathbb{L}} \mathfrak{Hom}_{\mathfrak{Y}\text{-qc}}(\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}, \mathscr{K}^\bullet)\Big)$$

are naturally isomorphic as objects of $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$.

These three complexes represent the semiderived category object $\mathscr{M}^\bullet \lozenge_{\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}} \mathscr{N}^\bullet \in \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$. The semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$ endowed with the semitensor product operation $\lozenge_{\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}}$ is a tensor triangulated category. The inverse image of the dualizing complex $\pi^*\mathscr{D}^\bullet_{\mathfrak{X}} \in \mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-Tors})$ is the unit object of this tensor structure.

## The Semitensor Product: Conclusion

The semitensor product operation $\lozenge_{\pi^*\mathscr{D}^\bullet_{\mathfrak{X}}}$ on the semiderived category $\mathrm{D}^{\mathrm{si}}_{\mathfrak{X}}(\mathfrak{Y}\text{-}\mathrm{Tors})$ is "semi-infinite" in that it produces doubly unbounded complexes as outputs even when given bounded complexes as inputs.

We have seen this in the introductory part of this talk, where, in the standard example of $\mathrm{D}^{\mathrm{si}}(\mathfrak{R}\text{-}\mathrm{Mod}_{\mathrm{tors}})$, for $\mathfrak{R} = \varprojlim_{\ell<0} \Bbbk[y_\ell, \dots, y_{-1}, y_0, y_1, y_2, \dots]$, we had

$$\Bbbk \lozenge \Bbbk \simeq \bigwedge\nolimits^{\infty/2+*}(\Bbbk((t))),$$

which is a doubly unbounded, $\mathbb{Z}$-graded or $(\infty/2+\mathbb{Z})$-graded complex.

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